INSTRUCTOR'S SOLUTIONS MANUAL

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CALCULUS
EARLY TRANSCENDENTALS
THIRD EDITION

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Chapter 1

Functions

1.1 Review of Functions

1.1.1 A function is a rule that assigns each to each value of the independent variable in the domain a unique value of the dependent variable in the range.

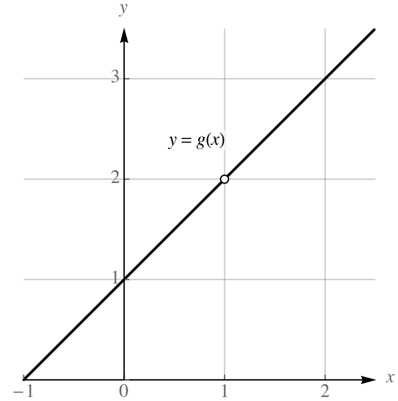
1.1.2 The independent variable belongs to the domain, while the dependent variable belongs to the range.

1.1.3 Graph A does not represent a function, while graph B does. Note that graph A fails the vertical line test, while graph B passes it.

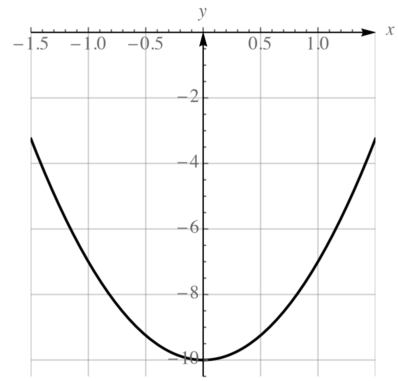
1.1.4 The domain of f is $[1, 4)$, while the range of f is $(1, 5]$. Note that the domain is the "shadow" of the graph on the x-axis, while the range is the "shadow" of the graph on the y-axis.

1.1.5 Item i. is true while item ii. isn't necessarily true. In the definition of function, item i. is stipulated. However, item ii. need not be true – for example, the function $f(x) = x^2$ has two different domain values associated with the one range value 4, because $f(2) = f(-2) = 4$.

1.1.6 $g(x) = \frac{x^2+1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1$, $x \neq 1$. The domain is $\{x : x \neq 1\}$ and the range is $\{x : x \neq 2\}$.



1.1.7 The domain of this function is the set of a real numbers. The range is $[-10, \infty)$.



1.1.8 The independent variable t is elapsed time and the dependent variable d is distance above the ground. The domain in context is $[0, 8]$

1.1.9 The independent variable h is the height of the water in the tank and the dependent variable V is the volume of water in the tank. The domain in context is $[0, 50]$

1.1.10 $f(2) = \dfrac{1}{2^3+1} = \dfrac{1}{9}$. $f(y^2) = \frac{1}{(y^2)^3+1} = \frac{1}{y^6+1}$.

1.1.11 $f(g(1/2)) = f(-2) = -3$; $g(f(4)) = g(9) = \frac{1}{8}$; $g(f(x)) = g(2x+1) = \dfrac{1}{(2x+1)-1} = \dfrac{1}{2x}$.

1.1.12 One possible answer is $g(x) = x^2 + 1$ and $f(x) = x^5$, because then $f(g(x)) = f(x^2+1) = (x^2+1)^5$. Another possible answer is $g(x) = x^2$ and $f(x) = (x+1)^5$, because then $f(g(x)) = f(x^2) = (x^2+1)^5$.

1.1.13 The domain of $f \circ g$ consists of all x in the domain of g such that $g(x)$ is in the domain of f.

1.1.14 $(f \circ g)(3) = f(g(3)) = f(25) = \sqrt{25} = 5$.
$(f \circ f)(64) = f(\sqrt{64}) = f(8) = \sqrt{8} = 2\sqrt{2}$.
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = x^{3/2} - 2$.
$(f \circ g)(x) = f(g(x)) = f(x^3 - 2) = \sqrt{x^3 - 2}$

1.1.15

a. $(f \circ g)(2) = f(g(2)) = f(2) = 4$.

b. $g(f(2)) = g(4) = 1$.

c. $f(g(4)) = f(1) = 3$.

d. $g(f(5)) = g(6) = 3$.

e. $f(f(8)) = f(8) = 8$.

f. $g(f(g(5))) = g(f(2)) = g(4) = 1$.

1.1.16

a. $h(g(0)) = h(0) = -1$.

b. $g(f(4)) = g(-1) = -1$.

c. $h(h(0)) = h(-1) = 0$.

d. $g(h(f(4))) = g(h(-1)) = g(0) = 0$.

e. $f(f(f(1))) = f(f(0)) = f(1) = 0$.

f. $h(h(h(0))) = h(h(-1)) = h(0) = -1$.

g. $f(h(g(2))) = f(h(3)) = f(0) = 1$.

h. $g(f(h(4))) = g(f(4)) = g(-1) = -1$.

i. $g(g(g(1))) = g(g(2)) = g(3) = 4$.

j. $f(f(h(3))) = f(f(0)) = f(1) = 0$.

1.1.17 $\dfrac{f(5) - f(0)}{5 - 0} = \dfrac{83 - 6}{5} = 15.4$; the radiosonde rises at an average rate of 15.4 ft/s during the first 5 seconds after it is released.

1.1.18 $f(0) = 0$. $f(34) = 127852.4 - 109731 = 18121.4$. $f(64) = 127852.4 - 75330.4 = 52522$.

$$\frac{f(64) - f(34)}{64 - 34} = \frac{52522 - 18121.4}{30} \approx 1146.69 \,\text{ft/s}.$$

1.1.19 $f(-2) = f(2) = 2$; $g(-2) = -g(2) = -(-2) = 2$; $f(g(2)) = f(-2) = f(2) = 2$; $g(f(-2)) = g(f(2)) = g(2) = -2$.

1.1.20 The graph would be the result of leaving the portion of the graph in the first quadrant, and then also obtaining a portion in the third quadrant which would be the result of reflecting the portion in the first quadrant around the y-axis and then the x-axis.

1.1.21 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin so is odd. Function C is symmetric about the y-axis, so is even.

1.1.22 Function A is symmetric about the y-axis, so is even. Function B is symmetric about the origin, so is odd. Function C is also symmetric about the origin, so is odd.

1.1.23 $f(x) = \dfrac{x^2 - 5x + 6}{x - 2} = \dfrac{(x-2)(x-3)}{x-2} = x - 3$, $x \neq 2$. The domain of f is $\{x : x \neq 2\}$. The range is $\{y : y \neq -1\}$.

1.1.24 $f(x) = \frac{x-2}{2-x} = \frac{x-2}{-(x-2)} = -1$, $x \neq 2$. The domain is $\{x : x \neq 2\}$. The range is $\{-1\}$.

1.1.25 The domain of the function is the set of numbers x which satisfy $7 - x^2 \geq 0$. This is the interval $[-\sqrt{7}, \sqrt{7}]$. Note that $f(\sqrt{7}) = 0$ and $f(0) = \sqrt{7}$. The range is $[0, \sqrt{7}]$.

1.1.26 The domain of the function is the set of numbers x which satisfy $25 - x^2 \geq 0$. This is the interval $[-5, 5]$. Note that $f(0) = -5$ and $f(5) = 0$. The range is $[-5, 0]$.

1.1.27 Because the cube root function is defined for all real numbrs, the domain is $\mathbb{R}$, the set of all real numbers.

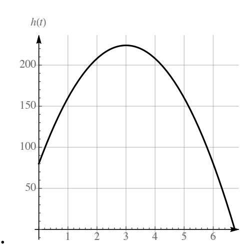
1.1.28 The domain consists of the set of numbers w for which $2 - w \geq 0$, so the interval $(-\infty, 2]$.

1.1.29 The domain consists of the set of numbers x for which $9 - x^2 \geq 0$, so the interval $[-3, 3]$.

1.1.30 Because $1 + t^2$ is never zero for any real numbered value of t, the domain of this function is $\mathbb{R}$, the set of all real numbers.

1.1.31

a. The formula for the height of the rocket is valid from $t = 0$ until the rocket hits the ground, which is the positive solution to $-16t^2 + 96t + 80 = 0$, which the quadratic formula reveals is $t = 3 + \sqrt{14}$. Thus, the domain is $[0, 3 + \sqrt{14}]$.

b. 

The maximum appears to occur at $t = 3$. The height at that time would be 224.

1.1.32

a. $d(0) = (10 - (2.2) \cdot 0)^2 = 100$.

b. The tank is first empty when $d(t) = 0$, which is when $10 - (2.2)t = 0$, or $t = 50/11$.

c. An appropriate domain would $[0, 50/11]$.

1.1.33 $g(1/z) = (1/z)^3 = \frac{1}{z^3}$

1.1.34 $F(y^4) = \frac{1}{y^4-3}$

1.1.35 $F(g(y)) = F(y^3) = \frac{1}{y^3-3}$

1.1.36 $f(g(w)) = f(w^3) = (w^3)^2 - 4 = w^6 - 4$

1.1.37 $g(f(u)) = g(u^2-4) = (u^2-4)^3$

1.1.38 $\dfrac{f(2+h)-f(2)}{h} = \dfrac{(2+h)^2-4-0}{h} = \dfrac{4+4h+h^2-4}{h} = \dfrac{4h+h^2}{h} = 4+h$

1.1.39 $F(F(x)) = F\left(\dfrac{1}{x-3}\right) = \dfrac{1}{\frac{1}{x-3}-3} = \dfrac{1}{\frac{1}{x-3}-\frac{3(x-3)}{x-3}} = \dfrac{1}{\frac{10-3x}{x-3}} = \dfrac{x-3}{10-3x}$

1.1.40 $g(F(f(x))) = g(F(x^2-4)) = g\left(\dfrac{1}{x^2-4-3}\right) = \left(\dfrac{1}{x^2-7}\right)^3$

1.1.41 $f(\sqrt{x+4}) = (\sqrt{x+4})^2 - 4 = x+4-4 = x.$

1.1.42 $F((3x+1)/x) = \dfrac{1}{\frac{3x+1}{x}-3} = \dfrac{1}{\frac{3x+1-3x}{x}} = \dfrac{x}{3x+1-3x} = x.$

1.1.43 $g(x) = x^3-5$ and $f(x) = x^{10}$.

1.1.44 $g(x) = x^6+x^2+1$ and $f(x) = \frac{2}{x^2}$.

1.1.45 $g(x) = x^4+2$ and $f(x) = \sqrt{x}$.

1.1.46 $g(x) = x^3-1$ and $f(x) = \frac{1}{\sqrt{x}}$.

1.1.47 $(f\circ g)(x) = f(g(x)) = f(x^2-4) = |x^2-4|$. The domain of this function is the set of all real numbers.

1.1.48 $(g\circ f)(x) = g(f(x)) = g(|x|) = |x|^2-4 = x^2-4$. The domain of this function is the set of all real numbers.

1.1.49 $(f\circ G)(x) = f(G(x)) = f\left(\frac{1}{x-2}\right) = \left|\frac{1}{x-2}\right| = \frac{1}{|x-2|}$. The domain of this function is the set of all real numbers except for the number 2.

1.1.50 $(f\circ g\circ G)(x) = f(g(G(x))) = f\left(g\left(\frac{1}{x-2}\right)\right) = f\left(\left(\frac{1}{x-2}\right)^2-4\right) = \left|\left(\frac{1}{x-2}\right)^2-4\right|$. The domain of this function is the set of all real numbers except for the number 2.

1.1.51 $(G\circ g\circ f)(x) = G(g(f(x))) = G(g(|x|)) = G(x^2-4) = \frac{1}{x^2-4-2} = \frac{1}{x^2-6}$. The domain of this function is the set of all real numbers except for the numbers $\pm\sqrt{6}$.

1.1.52 $(g\circ F\circ F)(x) = g(F(F(x))) = g(F(\sqrt{x})) = g(\sqrt{\sqrt{x}}) = \sqrt{x}-4$. The domain is $[0,\infty)$.

1.1.53 $(g\circ g)(x) = g(g(x)) = g(x^2-4) = (x^2-4)^2-4 = x^4-8x^2+16-4 = x^4-8x^2+12$. The domain is the set of all real numbers.

1.1.54 $(G\circ G)(x) = G(G(x)) = G(1/(x-2)) = \dfrac{1}{\frac{1}{x-2}-2} = \dfrac{1}{\frac{1-2(x-2)}{x-2}} = \dfrac{x-2}{1-2x+4} = \dfrac{x-2}{5-2x}$. Then $G\circ G$ is defined except where the denominator vanishes, so its domain is the set of all real numbers except for $x = \frac{5}{2}$.

1.1.55 Because $(x^2+3)-3 = x^2$, we may choose $f(x) = x-3$.

1.1.56 Because the reciprocal of x^2+3 is $\frac{1}{x^2+3}$, we may choose $f(x) = \frac{1}{x}$.

1.1.57 Because $(x^2+3)^2 = x^4+6x^2+9$, we may choose $f(x) = x^2$.

1.1.58 Because $(x^2+3)^2 = x^4+6x^2+9$, and the given expression is 11 more than this, we may choose $f(x) = x^2+11$.

1.1.59 Because $(x^2)^2+3 = x^4+3$, this expression results from squaring x^2 and adding 3 to it. Thus we may choose $f(x) = x^2$.

1.1.60 Because $x^{2/3}+3 = (\sqrt[3]{x})^2+3$, we may choose $f(x) = \sqrt[3]{x}$.

1.1.61

a. True. A real number z corresponds to the domain element $z/2+19$, because $f(z/2+19) = 2(z/2+19)-38 = z+38-38 = z$.

b. False. The definition of function does not require that each range element comes from a unique domain element, rather that each domain element is paired with a unique range element.

c. True. $f(1/x) = \frac{1}{1/x} = x$, and $\frac{1}{f(x)} = \frac{1}{1/x} = x$.

d. False. For example, suppose that f is the straight line through the origin with slope 1, so that $f(x) = x$. Then $f(f(x)) = f(x) = x$, while $(f(x))^2 = x^2$.

e. False. For example, let $f(x) = x+2$ and $g(x) = 2x-1$. Then $f(g(x)) = f(2x-1) = 2x-1+2 = 2x+1$, while $g(f(x)) = g(x+2) = 2(x+2)-1 = 2x+3$.

f. True. This is the definition of $f \circ g$.

g. True. If f is even, then $f(-z) = f(z)$ for all z, so this is true in particular for $z = ax$. So if $g(x) = cf(ax)$, then $g(-x) = cf(-ax) = cf(ax) = g(x)$, so g is even.

h. False. For example, $f(x) = x$ is an odd function, but $h(x) = x+1$ isn't, because $h(2) = 3$, while $h(-2) = -1$ which isn't $-h(2)$.

i. True. If $f(-x) = -f(x) = f(x)$, then in particular $-f(x) = f(x)$, so $0 = 2f(x)$, so $f(x) = 0$ for all x.

1.1.62 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{10-10}{h} = \dfrac{0}{h} = 0.$

1.1.63 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3(x+h)-3x}{h} = \dfrac{3x+3h-3x}{h} = \dfrac{3h}{h} = 3.$

1.1.64 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{4(x+h)-3-(4x-3)}{h} = \dfrac{4x+4h-3-4x+3}{h} = \dfrac{4h}{h} = 4.$

1.1.65 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^2-x^2}{h} = \dfrac{(x^2+2hx+h^2)-x^2}{h} = \dfrac{h(2x+h)}{h} = 2x+h.$

1.1.66 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h} =$
$\dfrac{2x^2+4xh+2h^2-3x-3h+1-2x^2+3x-1}{h} = \dfrac{4xh+2h^2-3h}{h} = \dfrac{h(4x+2h-3)}{h} = 4x+2h-3.$

1.1.67 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{2}{x+h}-\frac{2}{x}}{h} = \dfrac{\frac{2x-2(x+h)}{x(x+h)}}{h} = \dfrac{2x-2x-2h}{hx(x+h)} = -\dfrac{2h}{hx(x+h)} = -\dfrac{2}{x(x+h)}.$

1.1.68 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h} = \dfrac{\frac{(x+h)(x+1)-x(x+h+1)}{(x+1)(x+h+1)}}{h} = \dfrac{x^2+x+hx+h-x^2-xh-x}{h(x+1)(x+h+1)} =$
$\dfrac{h}{h(x+1)(x+h+1)} = \dfrac{1}{(x+1)(x+h+1)}$

1.1.69 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{x^2+x-(a^2+a)}{x-a} = \dfrac{(x^2-a^2)+(x-a)}{x-a} = \dfrac{(x-a)(x+a)+(x-a)}{x-a} = \dfrac{(x-a)(x+a+1)}{x-a} = x+a+1.$

1.1.70

$$\frac{f(x)-f(a)}{x-a} = \frac{4-4x-x^2-(4-4a-a^2)}{x-a} = \frac{-4(x-a)-(x^2-a^2)}{x-a} = \frac{-4(x-a)-(x-a)(x+a)}{x-a}$$
$$= \frac{(x-a)(-4-(x+a))}{x-a} = -4-x-a.$$

1.1.71 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{x^3-2x-(a^3-2a)}{x-a} = \dfrac{(x^3-a^3)-2(x-a)}{x-a} = \dfrac{(x-a)(x^2+ax+a^2)-2(x-a)}{x-a} = \dfrac{(x-a)(x^2+ax+a^2-2)}{x-a} = x^2+ax+a^2-2.$

1.1.72 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{x^4-a^4}{x-a} = \dfrac{(x^2-a^2)(x^2+a^2)}{x-a} = \dfrac{(x-a)(x+a)(x^2+a^2)}{x-a} = (x+a)(x^2+a^2).$

1.1.73 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{\frac{-4}{x^2}-\frac{-4}{a^2}}{x-a} = \dfrac{\frac{-4a^2+4x^2}{a^2x^2}}{x-a} = \dfrac{4(x^2-a^2)}{(x-a)a^2x^2} = \dfrac{4(x-a)(x+a)}{(x-a)a^2x^2} = \dfrac{4(x+a)}{a^2x^2}.$

1.1.74 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{\frac{1}{x}-x^2-(\frac{1}{a}-a^2)}{x-a} = \dfrac{\frac{1}{x}-\frac{1}{a}}{x-a} - \dfrac{x^2-a^2}{x-a} = \dfrac{\frac{a-x}{ax}}{x-a} - \dfrac{(x-a)(x+a)}{x-a} = -\dfrac{1}{ax}-(x+a).$

1.1.75

a. The slope is $\frac{12227-10499}{3-1} = 864$ ft/h. The hiker's elevation increases at an average rate of 874 feet per hour.

b. The slope is $\frac{12144-12631}{5-4} = -487$ ft/h. The hiker's elevation decreases at an average rate of 487 feet per hour.

c. The hiker might have stopped to rest during this interval of time or the trail is level on this portion of the hike.

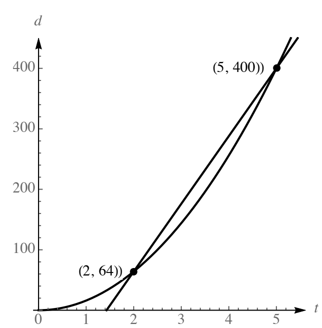
1.1.76

a. The slope is $\frac{11302-9954}{3-1} = 674$ ft/m. The elevation of the trail increases by an average of 674 feet per mile for $1 \le d \le 3$.

b. The slope is $\frac{12237-12357}{6-5} = -120$ ft/m. The elevation of the trail decreases by an average of 120 feet per mile for $5 \le d \le 6$.

c. The elevation of the trail doesn't change much for $4.5 \le d \le 5$.

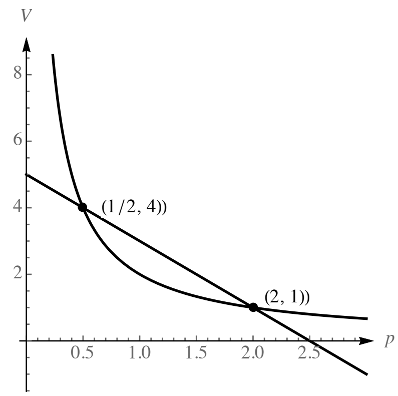
1.1.77



a.

b. The slope of the secant line is given by $\frac{400-64}{5-2} = \frac{336}{3} = 112$ feet per second. The object falls at an average rate of 112 feet per second over the interval $2 \le t \le 5$.

1.1.78

a. 

b. The slope of the secant line is given by $\frac{4-1}{.5-2} = \frac{3}{-1.5} = -2$ cubic centimeters per atmosphere. The volume decreases by an average of 2 cubic centimeters per atmosphere over the interval $0.5 \le p \le 2$.

1.1.79 This function is symmetric about the y-axis, because $f(-x) = (-x)^4 + 5(-x)^2 - 12 = x^4 + 5x^2 - 12 = f(x)$.

1.1.80 This function is symmetric about the origin, because $f(-x) = 3(-x)^5 + 2(-x)^3 - (-x) = -3x^5 - 2x^3 + x = -(3x^5 + 2x^3 - x) = f(x)$.

1.1.81 This function has none of the indicated symmetries. For example, note that $f(-2) = -26$, while $f(2) = 22$, so f is not symmetric about either the origin or about the y-axis, and is not symmetric about the x-axis because it is a function.

1.1.82 This function is symmetric about the y-axis. Note that $f(-x) = 2|-x| = 2|x| = f(x)$.

1.1.83 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $(-x)^{2/3} = ((-x)^2)^{1/3} = (x^2)^{1/3} = x^{2/3}$, and a similar fact holds for the term involving y.

1.1.84 This function is symmetric about the origin. Writing the function as $y = f(x) = x^{3/5}$, we see that $f(-x) = (-x)^{3/5} = -(x)^{3/5} = -f(x)$.

1.1.85 This function is symmetric about the origin. Note that $f(-x) = (-x)|(-x)| = -x|x| = -f(x)$.

1.1.86 This curve (which is not a function) is symmetric about the x-axis, the y-axis, and the origin. Note that replacing either x by $-x$ or y by $-y$ (or both) yields the same equation. This is due to the fact that $|-x| = |x|$ and $|-y| = |y|$.

1.1.87

a. $f(g(-2)) = f(-g(2)) = f(-2) = 4$

b. $g(f(-2)) = g(f(2)) = g(4) = 1$

c. $f(g(-4)) = f(-g(4)) = f(-1) = 3$

d. $g(f(5) - 8) = g(-2) = -g(2) = -2$

e. $g(g(-7)) = g(-g(7)) = g(-4) = -1$

f. $f(1 - f(8)) = f(-7) = 7$

1.1.88

a. $f(g(-1)) = f(-g(1)) = f(3) = 3$

b. $g(f(-4)) = g(f(4)) = g(-4) = -g(4) = 2$

c. $f(g(-3)) = f(-g(3)) = f(4) = -4$

d. $f(g(-2)) = f(-g(2)) = f(1) = 2$

e. $g(g(-1)) = g(-g(1)) = g(3) = -4$

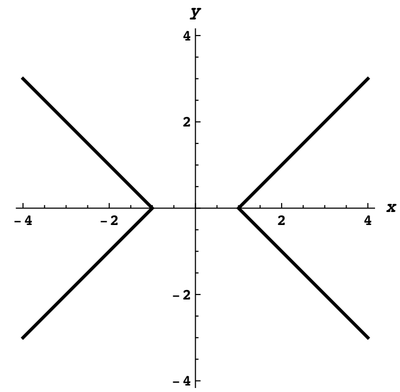
f. $f(g(0) - 1) = f(-1) = f(1) = 2$

g. $f(g(g(-2))) = f(g(-g(2))) = f(g(1)) = f(-3) = 3$

h. $g(f(f(-4))) = g(f(-4)) = g(-4) = 2$

i. $g(g(g(-1))) = g(g(-g(1))) = g(g(3)) = g(-4) = 2$

1.1.89 We will make heavy use of the fact that $|x|$ is x if $x > 0$, and is $-x$ if $x < 0$. In the first quadrant where x and y are both positive, this equation becomes $x - y = 1$ which is a straight line with slope 1 and y-intercept -1. In the second quadrant where x is negative and y is positive, this equation becomes $-x - y = 1$, which is a straight line with slope -1 and y-intercept -1. In the third quadrant where both x and y are negative, we obtain the equation $-x - (-y) = 1$, or $y = x + 1$, and in the fourth quadrant, we obtain $x + y = 1$. Graphing these lines and restricting them to the appropriate quadrants yields the following curve:



1.1.90 We have $y = 10 + \sqrt{-x^2 + 10x - 9}$, so by subtracting 10 from both sides and squaring we have $(y - 10)^2 = -x^2 + 10x - 9$, which can be written as

$$x^2 - 10x + (y - 10)^2 = -9.$$

To complete the square in x, we add 25 to both sides, yielding

$$x^2 - 10x + 25 + (y - 10)^2 = -9 + 25,$$

or

$$(x - 5)^2 + (y - 10)^2 = 16.$$

This is the equation of a circle of radius 4 centered at $(5, 10)$. Because $y \geq 10$, we see that the graph of f is the upper half of this circle. The domain of the function is $[1, 9]$ and the range is $[10, 14]$.

1.1.91 We have $y = 2 - \sqrt{-x^2 + 6x + 16}$, so by subtracting 2 from both sides and squaring we have $(y - 2)^2 = -x^2 + 6x + 16$, which can be written as

$$x^2 - 6x + (y - 2)^2 = 16.$$

To complete the square in x, we add 9 to both sides, yielding

$$x^2 - 6x + 9 + (y - 2)^2 = 16 + 9,$$

or

$$(x - 3)^2 + (y - 2)^2 = 25.$$

This is the equation of a circle of radius 5 centered at $(3, 2)$. Because $y \leq 2$, we see that the graph of f is the lower half of this circle. The domain of the function is $[-2, 8]$ and the range is $[-3, 2]$.

1.1.92

a. No. For example $f(x) = x^2 + 3$ is an even function, but $f(0)$ is not 0.

b. Yes. because $f(-x) = -f(x)$, and because $-0 = 0$, we must have $f(-0) = f(0) = -f(0)$, so $f(0) = -f(0)$, and the only number which is its own additive inverse is 0, so $f(0) = 0$.

1.1.93 Because the composition of f with itself has first degree, f has first degree as well, so let $f(x) = ax+b$. Then $(f \circ f)(x) = f(ax + b) = a(ax + b) + b = a^2x + (ab + b)$. Equating coefficients, we see that $a^2 = 9$ and $ab + b = -8$. If $a = 3$, we get that $b = -2$, while if $a = -3$ we have $b = 4$. So the two possible answers are $f(x) = 3x - 2$ and $f(x) = -3x + 4$.

1.1.94 Since the square of a linear function is a quadratic, we let $f(x) = ax+b$. Then $f(x)^2 = a^2x^2+2abx+b^2$. Equating coefficients yields that $a = \pm 3$ and $b = \pm 2$. However, a quick check shows that the middle term is correct only when one of these is positive and one is negative. So the two possible such functions f are $f(x) = 3x - 2$ and $f(x) = -3x + 2$.

1.1.95 Let $f(x) = ax^2 + bx + c$. Then $(f \circ f)(x) = f(ax^2 + bx + c) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$. Expanding this expression yields $a^3x^4 + 2a^2bx^3 + 2a^2cx^2 + ab^2x^2 + 2abcx + ac^2 + abx^2 + b^2x + bc + c$, which simplifies to $a^3x^4 + 2a^2bx^3 + (2a^2c + ab^2 + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c)$. Equating coefficients yields $a^3 = 1$, so $a = 1$. Then $2a^2b = 0$, so $b = 0$. It then follows that $c = -6$, so the original function was $f(x) = x^2 - 6$.

1.1.96 Because the square of a quadratic is a quartic, we let $f(x) = ax^2 + bx + c$. Then the square of f is $c^2 + 2bcx + b^2x^2 + 2acx^2 + 2abx^3 + a^2x^4$. By equating coefficients, we see that $a^2 = 1$ and so $a = \pm 1$. Because the coefficient on x^3 must be 0, we have that $b = 0$. And the constant term reveals that $c = \pm 6$. A quick check shows that the only possible solutions are thus $f(x) = x^2 - 6$ and $f(x) = -x^2 + 6$.

1.1.97 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \dfrac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}}.$$

1.1.98 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} = \dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h} \cdot \dfrac{\sqrt{1-2(x+h)}+\sqrt{1-2x}}{\sqrt{1-2(x+h)}+\sqrt{1-2x}} = \dfrac{1-2(x+h)-(1-2x)}{h(\sqrt{1-2(x+h)}+\sqrt{1-2x})} = -\dfrac{2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} = \frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} \cdot \frac{\sqrt{1-2x}+\sqrt{1-2a}}{\sqrt{1-2x}+\sqrt{1-2a}} = \frac{(1-2x)-(1-2a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = \frac{(-2)(x-a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} = -\frac{2}{(\sqrt{1-2x}+\sqrt{1-2a})}.$$

1.1.99 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{-3}{\sqrt{x+h}} - \frac{-3}{\sqrt{x}}}{h} = \dfrac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} = \dfrac{-3(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}} \cdot \dfrac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} = \dfrac{-3(x-(x+h))}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \dfrac{3}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$.

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{-3}{\sqrt{x}} - \frac{-3}{\sqrt{a}}}{x-a} = \frac{-3\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}\sqrt{x}}\right)}{x-a} = \frac{(-3)(\sqrt{a}-\sqrt{x})}{(x-a)\sqrt{a}\sqrt{x}} \cdot \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} = \frac{(3)(x-a)}{(x-a)(\sqrt{a}\sqrt{x})(\sqrt{a}+\sqrt{x})} = \frac{3}{\sqrt{ax}(\sqrt{a}+\sqrt{x})}.$$

1.1.100 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} = \dfrac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \cdot \dfrac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \dfrac{(x+h)^2+1-(x^2+1)}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \dfrac{x^2+2hx+h^2-x^2}{h(\sqrt{(x+h)^2+1}+\sqrt{x^2+1})} = \dfrac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$.

$$\frac{f(x)-f(a)}{x-a}=\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}=\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}\cdot\frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}}=$$
$$\frac{x^2+1-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=\frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=\frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1}}.$$

1.1.101 This would not necessarily have either kind of symmetry. For example, $f(x) = x^2$ is an even function and $g(x) = x^3$ is odd, but the sum of these two is neither even nor odd.

1.1.102 This would be an odd function, so it would be symmetric about the origin. Suppose f is even and g is odd. Then $(f \cdot g)(-x) = f(-x)g(-x) = f(x) \cdot (-g(x)) = -(f \cdot g)(x)$.

1.1.103 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $g(f(-x)) = g(f(x))$, because $f(-x) = f(x)$.

1.1.104 This would be an even function, so it would be symmetric about the y-axis. Suppose f is even and g is odd. Then $f(g(-x)) = f(-g(x)) = f(g(x))$.

1.2 Representing Functions

1.2.1 Functions can be defined and represented by a formula, through a graph, via a table, and by using words.

1.2.2 The domain of every polynomial is the set of all real numbers.

1.2.3 The slope of the line shown is $m = \frac{-3-(-1)}{3-0} = -2/3$. The y-intercept is $b = -1$. Thus the function is given by $f(x) = -\frac{2}{3}x - 1$.

1.2.4 Because it is to be parallel to a line with slope 2, it must also have slope 2. Using the point-slope form of the equation of the line, we have $y - 0 = 2(x - 5)$, or $y = 2x - 10$.

1.2.5 The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers for which $q(x) \neq 0$.

1.2.6 A piecewise linear function is one which is linear over intervals in the domain.

1.2.7 For $x < 0$, the graph is a line with slope 1 and y- intercept 3, while for $x > 0$, it is a line with slope $-1/2$ and y-intercept 3. Note that both of these lines contain the point $(0, 3)$. The function shown can thus be written

$$f(x) = \begin{cases} x+3 & \text{if } x < 0; \\ -\frac{1}{2}x+3 & \text{if } x \geq 0. \end{cases}$$

1.2.8 The transformed graph would have equation $y = \sqrt{x-2} + 3$.

1.2.9 Compared to the graph of $f(x)$, the graph of $f(x + 2)$ will be shifted 2 units to the left.

1.2.10 Compared to the graph of $f(x)$, the graph of $-3f(x)$ will be scaled vertically by a factor of 3 and flipped about the x axis.

1.2.11 Compared to the graph of $f(x)$, the graph of $f(3x)$ will be compressed horizontally by a factor of $\frac{1}{3}$.

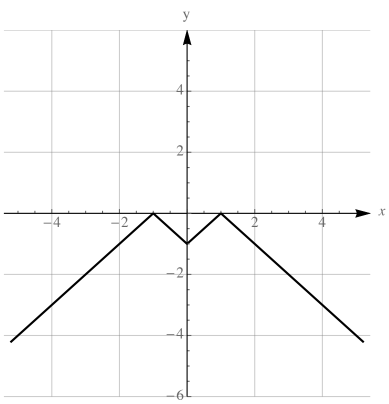
1.2.12 To produce the graph of $y = 4(x + 3)^2 + 6$ from the graph of x^2, one must

1. shift the graph horizontally by 3 units to left
2. scale the graph vertically by a factor of 4
3. shift the graph vertically up 6 units.

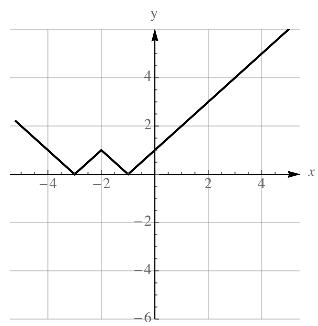
1.2.13 $f(x) = |x-2|+3$, because the graph of f is obtained from that of $|x|$ by shifting 2 units to the right and 3 units up.

$g(x) = -|x+2|-1$, because the graph of g is obtained from the graph of $|x|$ by shifting 2 units to the left, then reflecting about the x-axis, and then shifting 1 unit down.

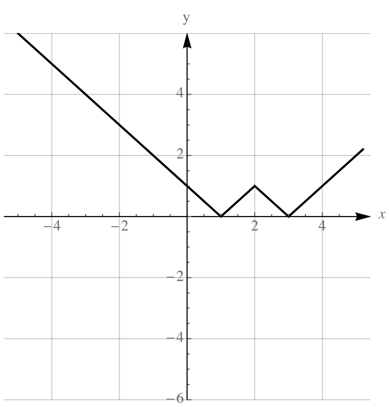
1.2.14



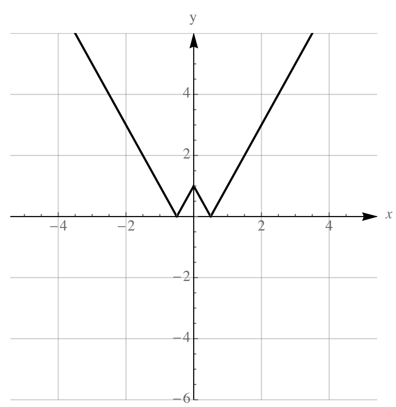
a.



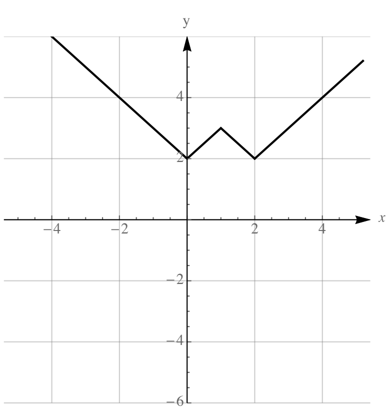
b.



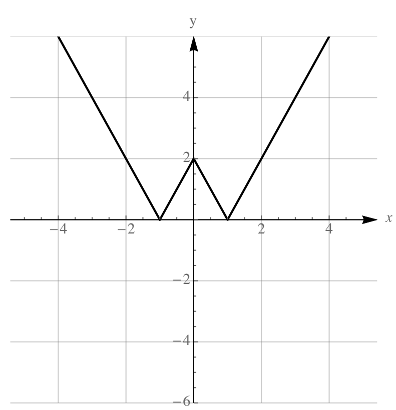
c.



d.



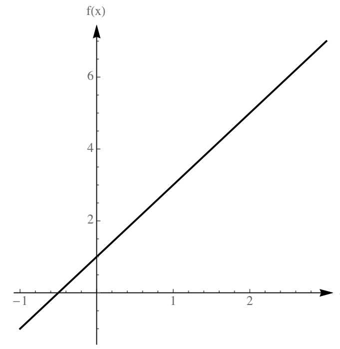
e.



f.

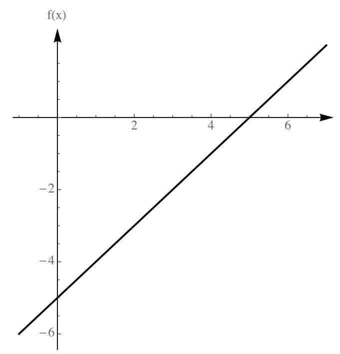
1.2.15

The slope is given by $\frac{5-3}{2-1} = 2$, so the equation of the line is $y - 3 = 2(x - 1)$, which can be written as $f(x) = 2x - 2 + 3$, or $f(x) = 2x + 1$.



1.2.16

The slope is given by $\frac{0-(-3)}{5-2} = 1$, so the equation of the line is $y - 0 = 1(x - 5)$, or $f(x) = x - 5$.



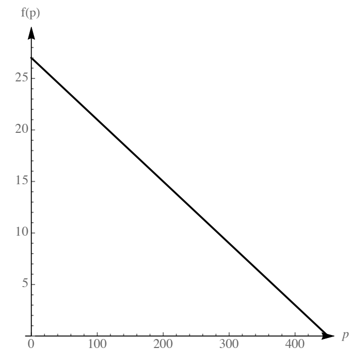
1.2.17 We are looking for the line with slope 3 that goes through the point $(3, 2)$. Using the point-slope form of the equation of a line, we have $y - 2 = 3(x - 3)$, which can be written as $y = 2 + 3x - 9$, or $y = 3x - 7$.

1.2.18 We are looking for the line with slope -4 which goes through the point $(-1, 4)$. Using the point-slope form of the equation of a line, we have $y - 4 = -4(x - (-1))$, which can be written as $y = 4 - 4x - 4$, or $y = -4x$.

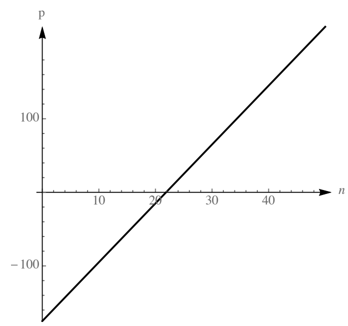
1.2.19 We have $571 = C_s(100)$, so $C_s = 5.71$. Therefore $N(150) = 5.71(150) = 856.5$ million.

1.2.20 We have $226 = C_s(100)$, so $C_s = 2.26$. Therefore $N(150) = 2.26(150) = 339$ million.

1.2.21 Using price as the independent variable p and the average number of units sold per day as the dependent variable d, we have the ordered pairs $(250, 12)$ and $(200, 15)$. The slope of the line determined by these points is $m = \frac{15-12}{200-250} = \frac{3}{-50}$. Thus the demand function has the form $d(p) = (-3/50)p + b$ for some constant b. Using the point $(200, 15)$, we find that $15 = (-3/50) \cdot 200 + b$, so $b = 27$. Thus the demand function is $d = (-3p/50) + 27$. While the domain of this linear function is the set of all real numbers, the formula is only likely to be valid for some subset of the interval $(0, 450)$, because outside of that interval either $p \leq 0$ or $d \leq 0$.



1.2.22 The profit is given by $p = f(n) = 8n - 175$. The break-even point is when $p = 0$, which occurs when $n = 175/8 = 21.875$, so they need to sell at least 22 tickets to not have a negative profit.

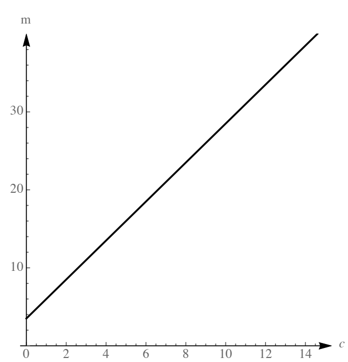


1.2.23

a. Using the points $(1986, 1875)$ and $(2000, 6471)$ we see that the slope is about 328.3. At $t = 0$, the value of p is 1875. Therefore a line which reasonably approximates the data is $p(t) = 328.3t + 1875$.

b. Using this line, we have that $p(9) = 4830$ breeding pairs.

1.2.24 The cost per mile is the slope of the desired line, and the intercept is the fixed cost of 3.5. Thus, the cost per mile is given by $c(m) = 2.5m + 3.5$. When $m = 9$, we have $c(9) = (2.5)(9) + 3.5 = 22.5 + 3.5 = 26$ dollars.



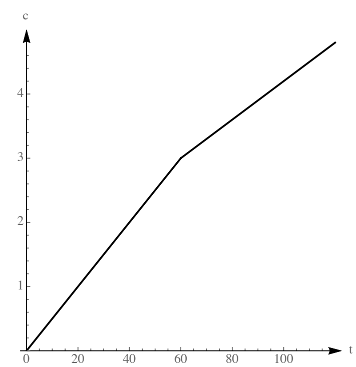
1.2.25 For $x \le 3$, we have the constant function 3. For $x \ge 3$, we have a straight line with slope 2 that contains the point $(3, 3)$. So its equation is $y - 3 = 2(x - 3)$, or $y = 2x - 3$. So the function can be written as $f(x) = \begin{cases} 3 & \text{if } x \le 3; \\ 2x - 3 & \text{if } x > 3 \end{cases}$

1.2.26 For $x < 3$ we have straight line with slope 1 and y-intercept 1, so the equation is $y = x+1$. For $x \geq 3$, we have a straight line with slope $-\frac{1}{3}$ which contains the point $(3, 2)$, so its equation is $y - 2 = -\frac{1}{3}(x - 3)$, or $y = -\frac{1}{3}x + 3$. Thus the function can be written as $f(x) = \begin{cases} x+1 & \text{if } x < 3; \\ -\frac{1}{3}x + 3 & \text{if } x \geq 3 \end{cases}$

1.2.27

The cost is given by

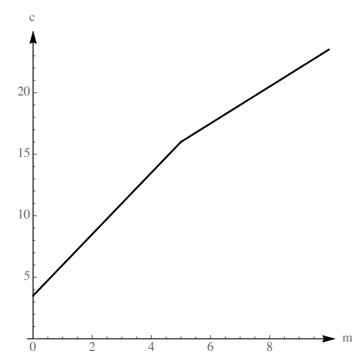
$$c(t) = \begin{cases} 0.05t & \text{for } 0 \leq t \leq 60 \\ 1.2 + 0.03t & \text{for } 60 < t \leq 120 \end{cases}.$$



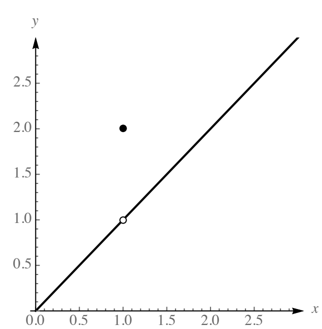
1.2.28

The cost is given by

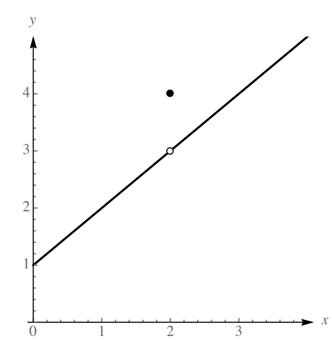
$$c(m) = \begin{cases} 3.5 + 2.5m & \text{for } 0 \leq m \leq 5 \\ 8.5 + 1.5m & \text{for } m > 5 \end{cases}.$$



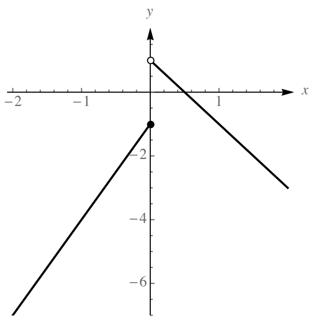
1.2.29



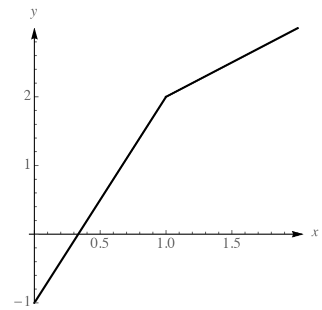
1.2.30



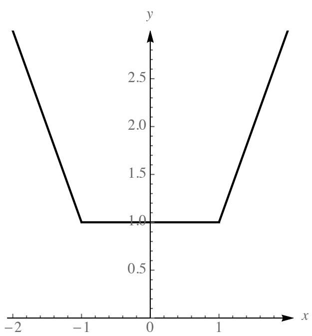
1.2.31



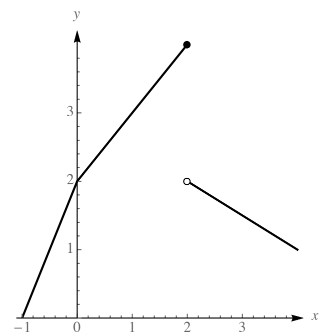
1.2.32



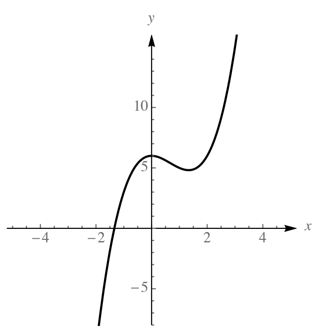
1.2.33



1.2.34



1.2.35

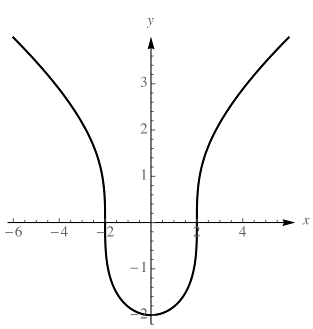
a. 

b. The function is a polynomial, so its domain is the set of all real numbers.

c. It has one peak near its y-intercept of $(0, 6)$ and one valley between $x = 1$ and $x = 2$. Its x-intercept is near $x = -4/3$.

1.2.36

a.

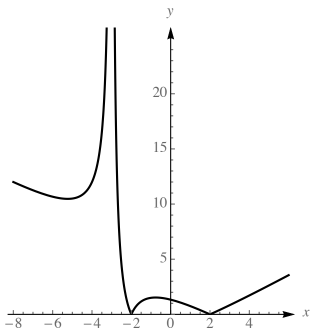


b. The function's domain is the set of all real numbers.

c. It has a valley at the y-intercept of $(0, -2)$, and is very steep at $x = -2$ and $x = 2$ which are the x-intercepts. It is symmetric about the y-axis.

1.2.37

a.

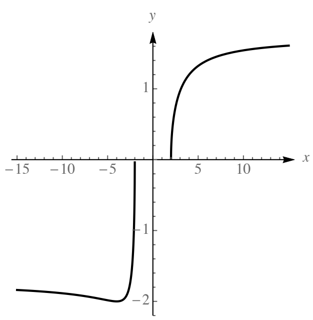


b. The domain of the function is the set of all real numbers except -3.

c. There is a valley near $x = -5.2$ and a peak near $x = -0.8$. The x-intercepts are at -2 and 2, where the curve does not appear to be smooth. There is a vertical asymptote at $x = -3$. The function is never below the x-axis. The y-intercept is $(0, 4/3)$.

1.2.38

a.

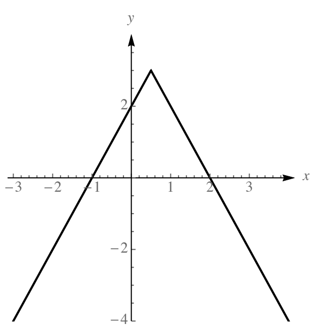


b. The domain of the function is $(-\infty, -2] \cup [2, \infty)$

c. x-intercepts are at -2 and 2. Because 0 isn't in the domain, there is no y-intercept. The function has a valley at $x = -4$.

1.2.39

a.

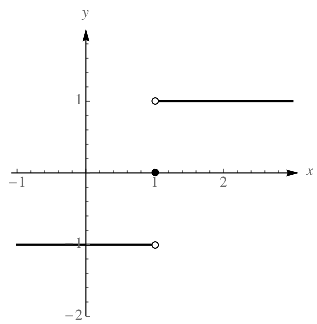


b. The domain of the function is $(-\infty, \infty)$

c. The function has a maximum of 3 at $x = 1/2$, and a y-intercept of 2.

1.2.40

a.



b. The domain of the function is $(-\infty, \infty)$

c. The function contains a jump at $x = 1$. The maximum value of the function is 1 and the minimum value is -1.

1.2.41

a. The zeros of f are the points where the graph crosses the x-axis, so these are points A, D, F, and I.

b. The only high point, or peak, of f occurs at point E, because it appears that the graph has larger and larger y values as x increases past point I and decreases past point A.

c. The only low points, or valleys, of f are at points B and H, again assuming that the graph of f continues its apparent behavior for larger values of x.

d. Past point H, the graph is rising, and is rising faster and faster as x increases. It is also rising between points B and E, but not as quickly as it is past point H. So the marked point at which it is rising most rapidly is I.

e. Before point B, the graph is falling, and falls more and more rapidly as x becomes more and more negative. It is also falling between points E and H, but not as rapidly as it is before point B. So the marked point at which it is falling most rapidly is A.

1.2.42

a. The zeros of g appear to be at $x = 0$, $x = 1$, $x = 1.6$, and $x \approx 3.15$.

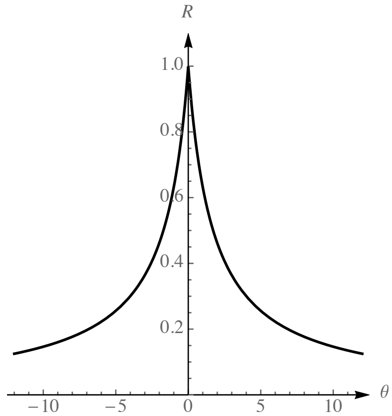
b. The two peaks of g appear to be at $x \approx 0.5$ and $x \approx 2.6$, with corresponding points $\approx (0.5, 0.4)$ and $\approx (2.6, 3.4)$.

c. The only valley of g is at $\approx (1.3, -0.2)$.

d. Moving right from $x \approx 1.3$, the graph is rising more and more rapidly until about $x = 2$, at which point it starts rising less rapidly (because, by $x \approx 2.6$, it is not rising at all). So the coordinates of the point at which it is rising most rapidly are approximately $(2.1, g(2)) \approx (2.1, 2)$. Note that while the curve is also rising between $x = 0$ and $x \approx 0.5$, it is not rising as rapidly as it is near $x = 2$.

e. To the right of $x \approx 2.6$, the curve is falling, and falling more and more rapidly as x increases. So the point at which it is falling most rapidly in the interval $[0, 3]$ is at $x = 3$, which has the approximate coordinates $(3, 1.4)$. Note that while the curve is also falling between $x \approx 0.5$ and $x \approx 1.3$, it is not falling as rapidly as it is near $x = 3$.

1.2.43

a. 

b. This appears to have a maximum when $\theta = 0$. Our vision is sharpest when we look straight ahead.

c. For $|\theta| \leq .19°$. We have an extremely narrow range where our eyesight is sharp.

1.2.44 Because the line is horizontal, the slope is constantly 0. So $S(x) = 0$.

1.2.45 The slope of this line is constantly 2, so the slope function is $S(x) = 2$.

1.2.46 The function can be written as $|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$.

The slope function is $S(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

1.2.47 The slope function is given by $S(x) = \begin{cases} 1 & \text{if } x < 0; \\ -1/2 & \text{if } x > 0. \end{cases}$

1.2.48 The slope function is given by $s(x) = \begin{cases} 1 & \text{if } x < 3; \\ -1/3 & \text{if } x > 3. \end{cases}$

1.2.49

a. Because the area under consideration is that of a rectangle with base 2 and height 6, $A(2) = 12$.

b. Because the area under consideration is that of a rectangle with base 6 and height 6, $A(6) = 36$.

c. Because the area under consideration is that of a rectangle with base x and height 6, $A(x) = 6x$.

1.2.50

a. Because the area under consideration is that of a triangle with base 2 and height 1, $A(2) = 1$.

b. Because the area under consideration is that of a triangle with base 6 and height 3, the $A(6) = 9$.

c. Because $A(x)$ represents the area of a triangle with base x and height $(1/2)x$, the formula for $A(x)$ is $\frac{1}{2} \cdot x \cdot \frac{x}{2} = \frac{x^2}{4}$.

1.2.51

a. Because the area under consideration is that of a trapezoid with base 2 and heights 8 and 4, we have $A(2) = 2 \cdot \frac{8+4}{2} = 12$.

b. Note that $A(3)$ represents the area of a trapezoid with base 3 and heights 8 and 2, so $A(3) = 3 \cdot \frac{8+2}{2} = 15$. So $A(6) = 15 + (A(6) - A(3))$, and $A(6) - A(3)$ represents the area of a triangle with base 3 and height 2. Thus $A(6) = 15 + 6 = 21$.

c. For x between 0 and 3, $A(x)$ represents the area of a trapezoid with base x, and heights 8 and $8 - 2x$. Thus the area is $x \cdot \frac{8+8-2x}{2} = 8x - x^2$. For $x > 3$, $A(x) = A(3) + A(x) - A(3) = 15 + 2(x - 3) = 2x + 9$. Thus

$$A(x) = \begin{cases} 8x - x^2 & \text{if } 0 \leq x \leq 3; \\ 2x + 9 & \text{if } x > 3. \end{cases}$$

1.2.52

a. Because the area under consideration is that of trapezoid with base 2 and heights 3 and 1, we have $A(2) = 2 \cdot \frac{3+1}{2} = 4$.

b. Note that $A(6) = A(2) + (A(6) - A(2)$, and that $A(6) - A(2)$ represents a trapezoid with base $6 - 2 = 4$ and heights 1 and 5. The area is thus $4 + \left(4 \cdot \frac{1+5}{2}\right) = 4 + 12 = 16$.

c. For x between 0 and 2, $A(x)$ represents the area of a trapezoid with base x, and heights 3 and $3 - x$. Thus the area is $x \cdot \frac{3+3-x}{2} = 3x - \frac{x^2}{2}$. For $x > 2$, $A(x) = A(2) + A(x) - A(2) = 4 + (A(x) - A(2))$. Note that $A(x) - A(2)$ represents the area of a trapezoid with base $x - 2$ and heights 1 and $x - 1$. Thus $A(x) = 4 + (x - 2) \cdot \frac{1+x-1}{2} = 4 + (x - 2)\left(\frac{x}{2}\right) = \frac{x^2}{2} - x + 4$. Thus

$$A(x) = \begin{cases} 3x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 2; \\ \frac{x^2}{2} - x + 4 & \text{if } x > 2. \end{cases}$$

1.2.53

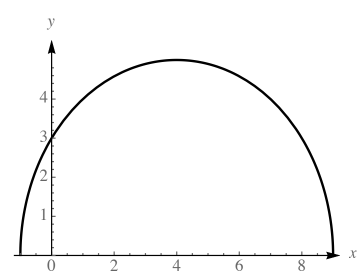
a. True. A polynomial $p(x)$ can be written as the ratio of polynomials $\frac{p(x)}{1}$, so it is a rational function. However, a rational function like $\frac{1}{x}$ is not a polynomial.

b. False. For example, if $f(x) = 2x$, then $(f \circ f)(x) = f(f(x)) = f(2x) = 4x$ is linear, not quadratic.

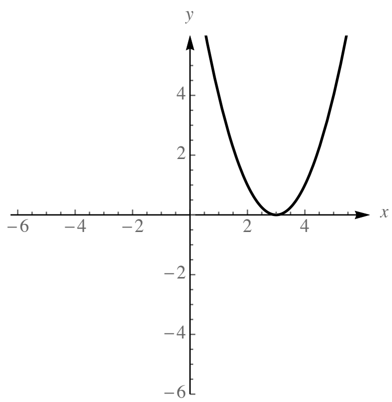
c. True. In fact, if f is degree m and g is degree n, then the degree of the composition of f and g is $m \cdot n$, regardless of the order they are composed.

d. False. The graph would be shifted two units to the left.

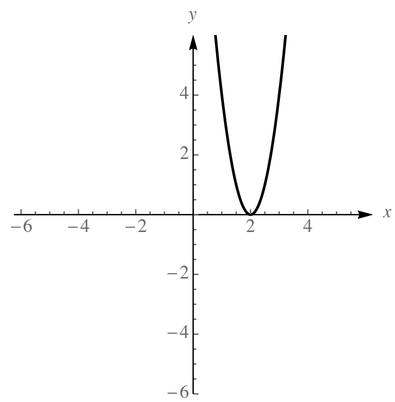
1.2.54 We complete the square for $-x^2 + 8x + 9$. Call this quantity z. Then $z = -(x^2 - 8x - 9)$, so $z = -(x^2 - 8x + 16 + (-16 - 9)) = -((x-4)^2 - 25 = 25 - (x-4)^2$. Thus $f(x)$ is obtained from the graph of $g(x) = \sqrt{25 - x^2}$ by shifting 4 units to the right. Thus the graph of f is the upper half of a circle of radius 5 centered at $(4, 0)$.



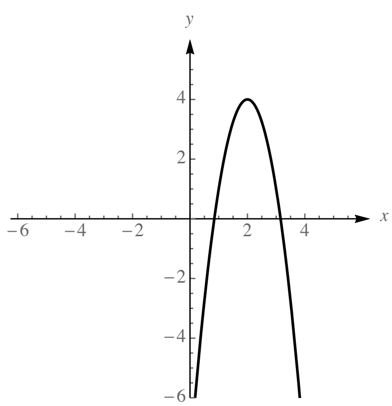
1.2.55



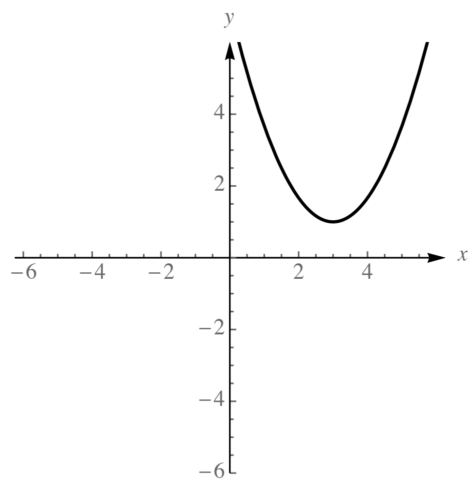
a. Shift 3 units to the right.



b. Horizontal compression by a factor of $\frac{1}{2}$, then shift 2 units to the right.

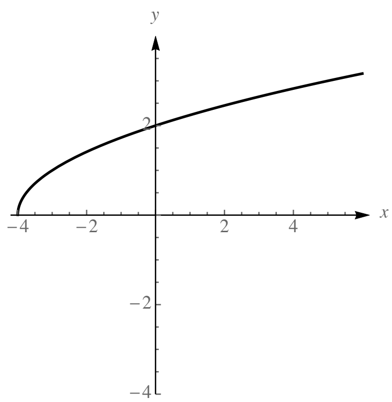


c. Shift to the right 2 units, vertically stretch by a factor of 3, reflect across the x-axis, and shift up 4 units.

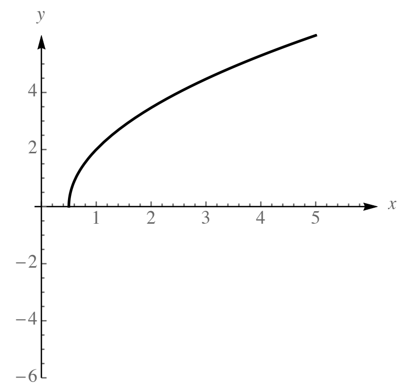


d. Horizontal stretch by a factor of 3, horizontal shift right 2 units, vertical stretch by a factor of 6, and vertical shift up 1 unit.

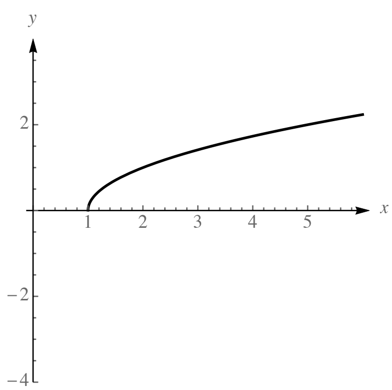
1.2.56



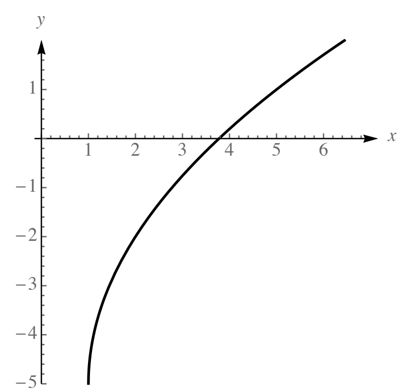
a.
Shift 4 units to the left.



b.
Horizontal compression by a factor of $\frac{1}{2}$, then shift $\frac{1}{2}$ units to the right. Then stretch vertically by a factor of 2.

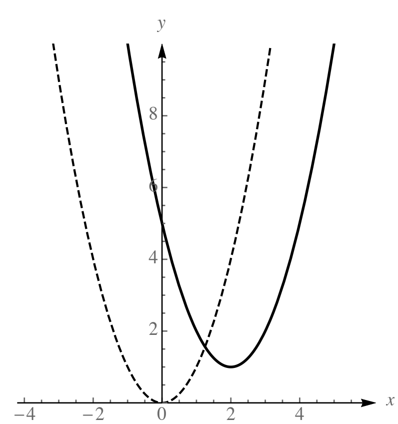


c.
Shift 1 unit to the right.

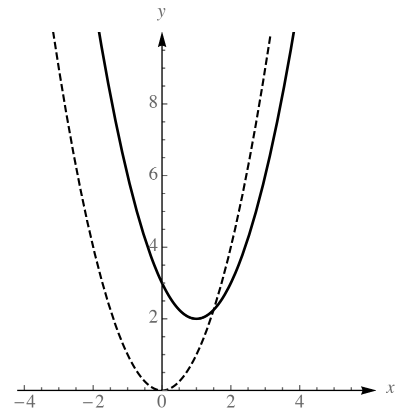


d.
Shift 1 unit to the right, then stretch vertically by a factor of 3, then shift down 5 units.

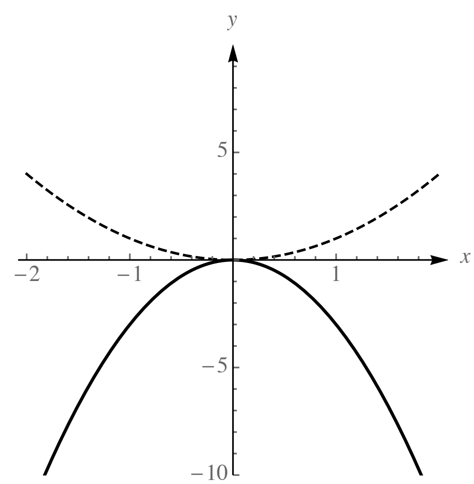
1.2.57 The graph is obtained by shifting the graph of x^2 two units to the right and one unit up.



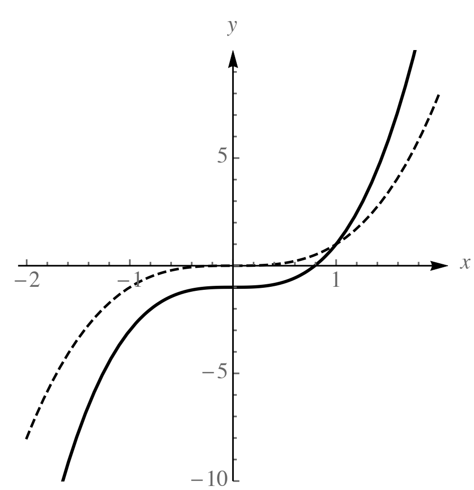
1.2.58 Write x^2-2x+3 as $(x^2-2x+1)+2=(x-1)^2+2$. The graph is obtained by shifting the graph of x^2 one unit to the right and two units up.



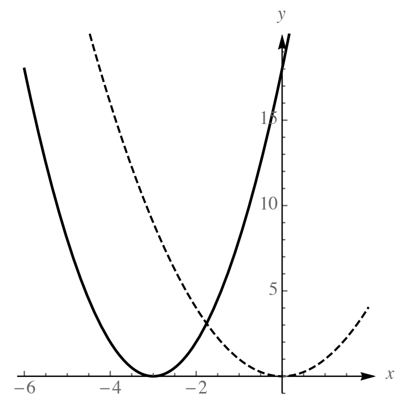
1.2.59 Stretch the graph of $y=x^2$ vertically by a factor of 3 and then reflect across the x-axis.



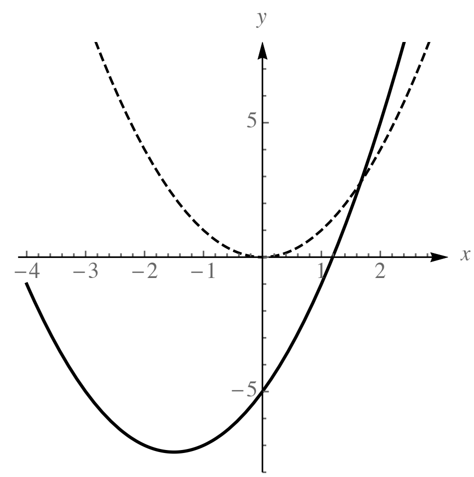
1.2.60 Scale the graph of $y=x^3$ vertically by a factor of 2, and then shift down 1 unit.



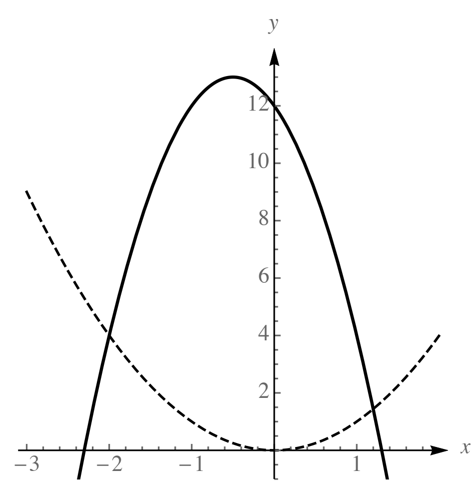
1.2.61 Shift the graph of $y = x^2$ left 3 units and stretch vertically by a factor of 2.



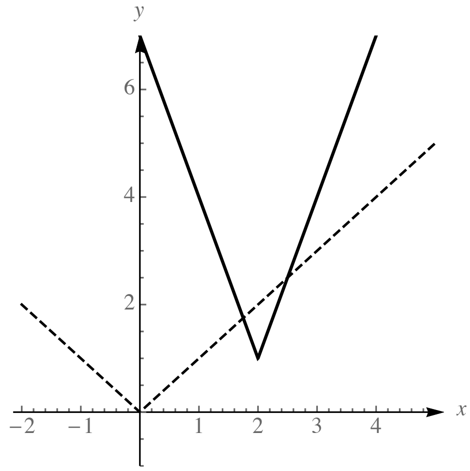
1.2.62 By completing the square, we have that $p(x) = x^2 + 3x - 5 = x^2 + 3x + \frac{9}{4} - 5 - \frac{9}{4} = (x + \frac{3}{2})^2 - \frac{29}{4}$. So it is $f(x + \frac{3}{2}) - (\frac{29}{4})$ where $f(x) = x^2$. The graph is shifted $\frac{3}{2}$ units to the left and then down $\frac{29}{4}$ units.



1.2.63 By completing the square, we have that $h(x) = -4(x^2 + x - 3) = -4\left(x^2 + x + \frac{1}{4} - \frac{1}{4} - 3\right) = -4(x + \frac{1}{2})^2 + 13$. So it is $-4f(x + (\frac{1}{2})) + 13$ where $f(x) = x^2$. The graph is shifted $\frac{1}{2}$ unit to the left, stretched vertically by a factor of 4, then reflected about the x-axis, then shifted up 13 units.



1.2.64 Because $|3x-6|+1 = 3|x-2|+1$, this is $3f(x-2)+1$ where $f(x) = |x|$. The graph is shifted 2 units to the right, then stretched vertically by a factor of 3, and then shifted up 1 unit.

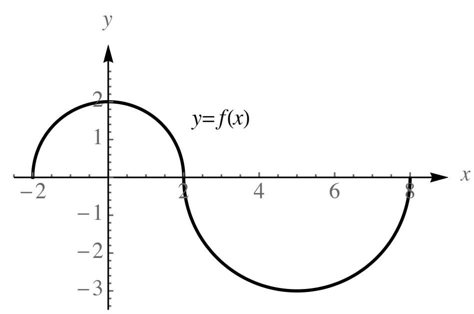


1.2.65 The curves intersect where $4\sqrt{2x} = 2x^2$. If we square both sides, we have $32x = 4x^4$, which can be written as $4x(8-x^3) = 0$, which has solutions at $x = 0$ and $x = 2$. So the points of intersection are $(0,0)$ and $(2,8)$.

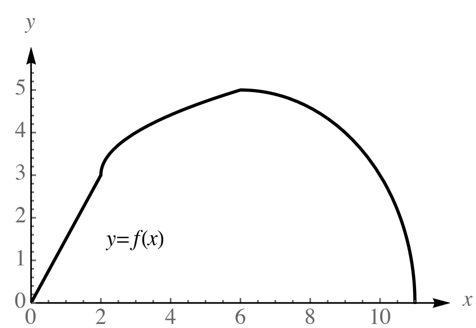
1.2.66 The points of intersection are found by solving $x^2+2 = x+4$. This yields the quadratic equation $x^2-x-2 = 0$ or $(x-2)(x+1) = 0$. So the x-values of the points of intersection are 2 and -1. The actual points of intersection are $(2,6)$ and $(-1,3)$.

1.2.67 The points of intersection are found by solving $x^2 = -x^2+8x$. This yields the quadratic equation $2x^2-8x = 0$ or $(2x)(x-4) = 0$. So the x-values of the points of intersection are 0 and 4. The actual points of intersection are $(0,0)$ and $(4,16)$.

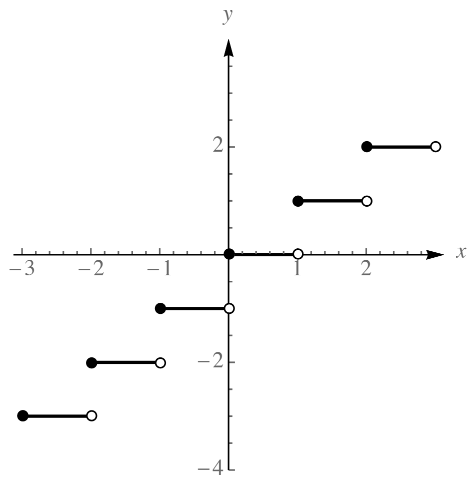
1.2.68 $f(x) = \begin{cases} \sqrt{4-x^2} & \text{if } -2 \le x \le 2 \\ -\sqrt{9-(x-5)^2} & \text{if } 2 < x \le 6. \end{cases}$.



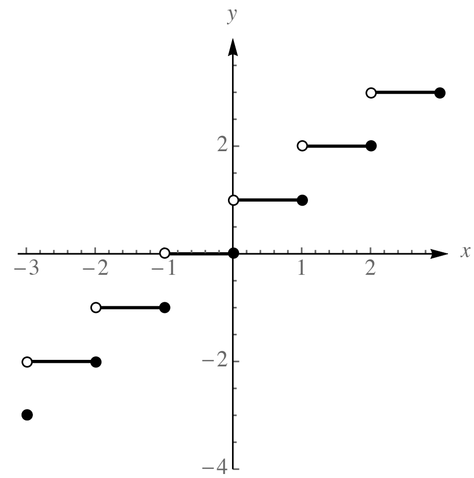
1.2.69



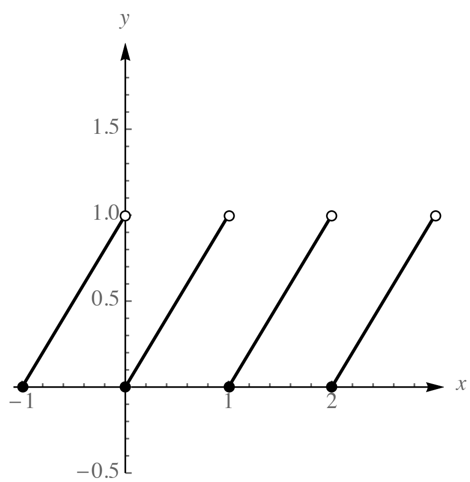
1.2.70



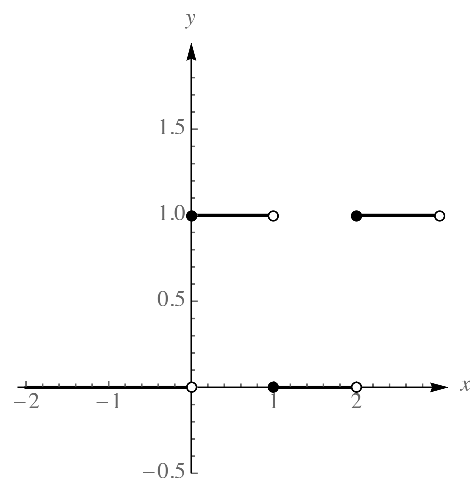
1.2.71



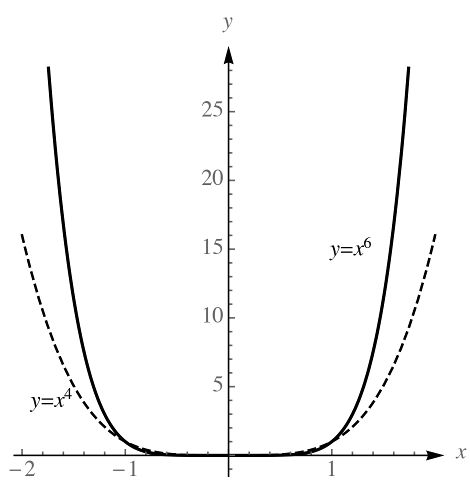
1.2.72



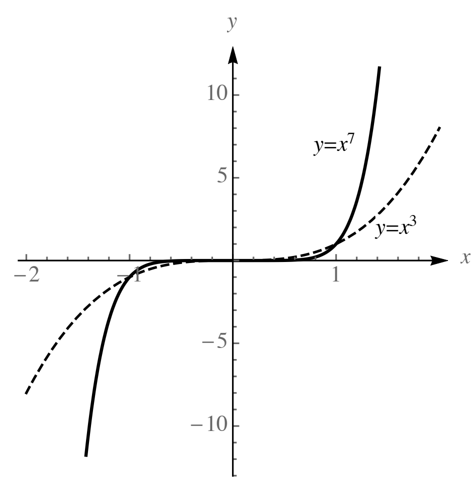
1.2.73



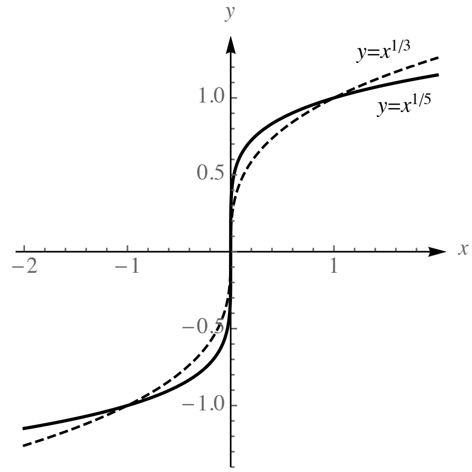
1.2.74



1.2.75



1.2.76



1.2.77

a. $f(0.75) = \frac{.75^2}{1-2(.75)(.25)} = .9$. There is a 90% chance that the server will win from deuce if they win 75% of their service points.

b. $f(0.25) = \frac{.25^2}{1-2(.25)(.75)} = .1$. There is a 10% chance that the server will win from deuce if they win 25% of their service points.

1.2.78

a. We know that the points $(32, 0)$ and $(212, 100)$ are on our line. The slope of our line is thus $\frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$. The function $f(F)$ thus has the form $C = (5/9)F + b$, and using the point $(32, 0)$ we see that $0 = (5/9)32 + b$, so $b = -(160/9)$. Thus $C = (5/9)F - (160/9)$

b. Solving the system of equations $C = (5/9)F-(160/9)$ and $C = F$, we have that $F = (5/9)F-(160/9)$, so $(4/9)F = -160/9$, so $F = -40$ when $C = -40$.

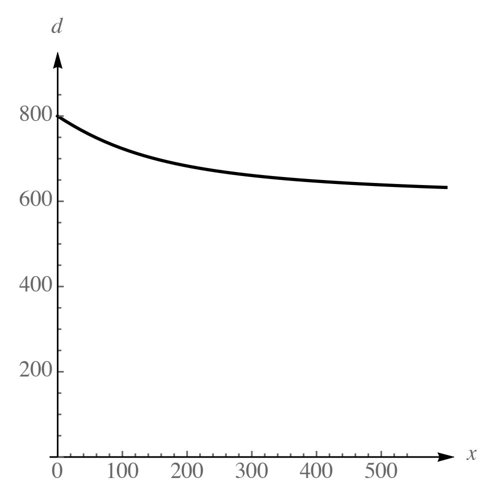
1.2.79

a. Because you are paying \$350 per month, the amount paid after m months is $y = 350m + 1200$.

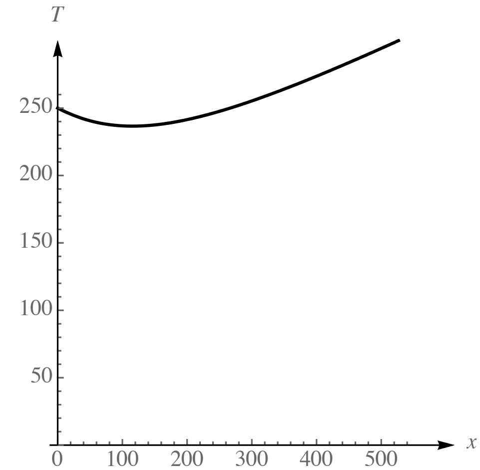
b. After 4 years (48 months) you have paid $350 \cdot 48 + 1200 = 18000$ dollars. If you then buy the car for \$10,000, you will have paid a total of \$28,000 for the car instead of \$25,000. So you should buy the car instead of leasing it.

1.2.80

a. Note that the island, the point P on shore, and the point down shore x units from P form a right triangle. By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{40000 + x^2}$. So Kelly must row this distance and then jog $600-x$ meters to get home. So her total distance $d(x) = \sqrt{40000 + x^2} + (600 - x)$.



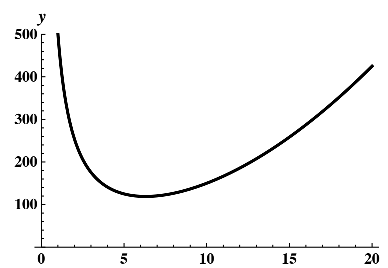
b. Because distance is rate times time, we have that time is distance divided by rate. Thus $T(x) = \frac{\sqrt{40000+x^2}}{2} + \frac{600-x}{4}$.



c. By inspection, it looks as though she should head to a point about 115 meters down shore from P. This would lead to a time of about 236.6 seconds.

1.2.81

a. The volume of the box is x^2h, but because the box has volume 125 cubic feet, we have that $x^2h = 125$, so $h = \frac{125}{x^2}$. The surface area of the box is given by x^2 (the area of the base) plus $4 \cdot hx$, because each side has area hx. Thus $S = x^2 + 4hx = x^2 + \frac{4\cdot 125\cdot x}{x^2} = x^2 + \frac{500}{x}$.



b. By inspection, it looks like the value of x which minimizes the surface area is about 6.3.

1.2.82 Let $f(x) = a_nx^n +$ smaller degree terms and let $g(x) = b_mx^m +$ some smaller degree terms.

a. The largest degree term in $f\cdot f$ is $a_nx^n \cdot a_nx^n = a_n^2x^{n+n}$, so the degree of this polynomial is $n+n = 2n$.

b. The largest degree term in $f \circ f$ is $a_n \cdot (a_nx^n)^n$, so the degree is n^2.

c. The largest degree term in $f \cdot g$ is $a_nb_mx^{m+n}$, so the degree of the product is $m + n$.

d. The largest degree term in $f \circ g$ is $a_n \cdot (b_mx^m)^n$, so the degree is mn.

1.2.83 Suppose that the parabola f crosses the x-axis at a and b, with $a < b$. Then a and b are roots of the polynomial, so $(x-a)$ and $(x-b)$ are factors. Thus the polynomial must be $f(x) = c(x-a)(x-b)$ for some non-zero real number c. So $f(x) = cx^2 - c(a+b)x + abc$. Because the vertex always occurs at the x value which is $\frac{-\text{coefficient on x}}{2\cdot\text{coefficient on } x^2}$ we have that the vertex occurs at $\frac{c(a+b)}{2c} = \frac{a+b}{2}$, which is halfway between a and b.

1.2.84

a. We complete the square to rewrite the function f. Write $f(x) = ax^2+bx+c$ as $f(x) = a(x^2 + \frac{b}{a}x + \frac{c}{a})$. Completing the square yields

$$a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a}\right) + \left(\frac{c}{a} - \frac{b^2}{4a}\right)\right) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

Thus the graph of f is obtained from the graph of x^2 by shifting $\frac{b}{2a}$ units horizontally (and then doing some scaling and vertical shifting) – moving the vertex from 0 to $-\frac{b}{2a}$. The vertex is therefore $\left(\frac{-b}{2a}, c - \frac{b^2}{4}\right)$.

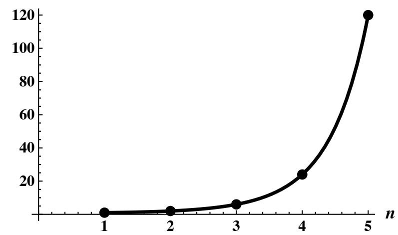
b. We know that the graph of f touches the x-axis twice if the equation $ax^2 + bx + c = 0$ has two real solutions. By the quadratic formula, we know that this occurs exactly when the discriminant $b^2 - 4ac$ is positive. So the condition we seek is for $b^2 - 4ac > 0$, or $b^2 > 4ac$.

1.2.85

a.

n	1	2	3	4	5
$n!$	1	2	6	24	120

b.



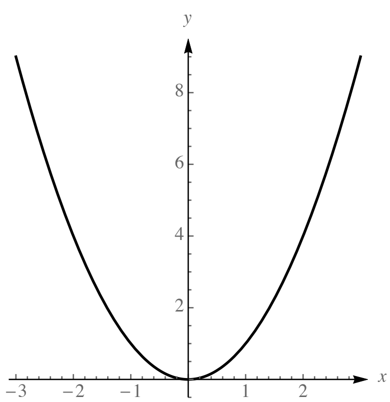
c. Using trial and error and a calculator yields that 10! is more than a million, but 9! isn't.

1.3 Inverse, Exponential and Logarithmic Functions

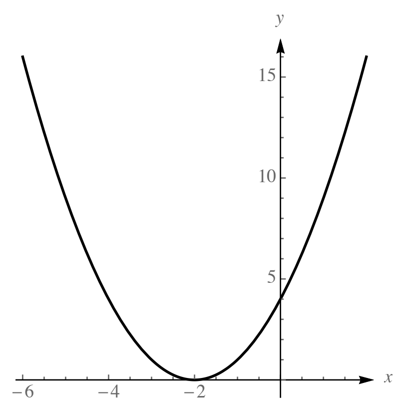
1.3.1 $D = \mathbb{R}, R = (0, \infty)$.

1.3.2 $f(x) = 2x + 1$ is one-to-one on all of $\mathbb{R}$. If $f(a) = f(b)$, then $2a + 1 = 2b + 1$, so it must follow that $a = b$.

1.3.3



1.3.4



1.3.5 f is one-to-one on $(-\infty, -1]$, on $[-1, 1]$, and on $[1, \infty)$.

1.3.6 f is one-to-one on $(-\infty, -2]$, on $[-2, 0]$, on $[0, 2]$, and on $[2, \infty)$.

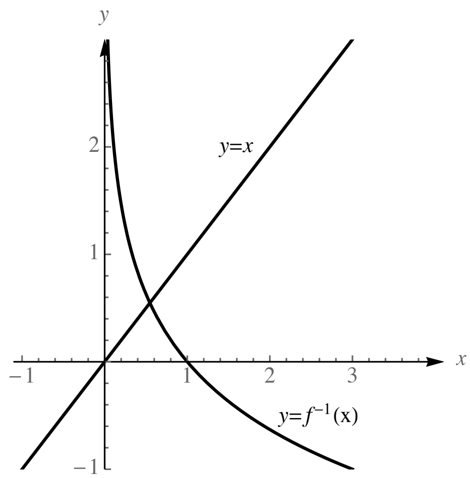
1.3.7 If a function f is not one-to-one, then there are domain values $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. If f^{-1} were to exist, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$ which would imply that $x_1 = x_2$, a contradiction.

1.3.8 Because $f(1) = 2$, $f^{-1}(2) = 1$. Because $f(5) = 9$, $f^{-1}(9) = 5$. Because $f(7) = 12$, $f^{-1}(12) = 7$.

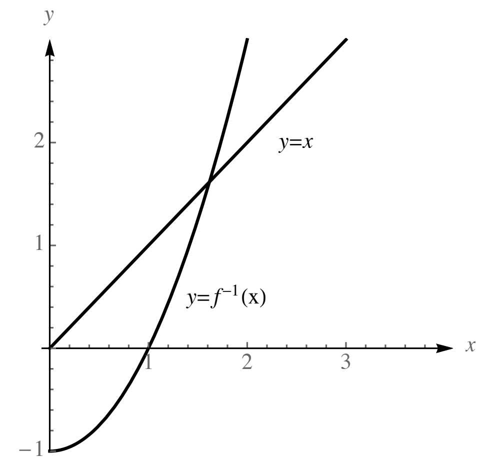
1.3.9 Suppose $x = 2y$, then $y = \frac{1}{2}x$, so the inverse of f is $f^{-1}(x) = \frac{1}{2}x$. Then $f(f^{-1}(x)) = f(\frac{1}{2}x) = 2 \cdot \frac{1}{2}x = x$. Also, $f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2} \cdot 2x = x$.

1.3.10 Suppose $x = \sqrt{y}$. Then $y = x^2$. So the inverse of f is $f^{-1}(x) = x^2$, $x \geq 0$. Then $f(f^{-1}(x)) = f(x^2) = \sqrt{x^2} = |x| = x$ for $x \geq 0$. Also, $f^{-1}(f(x)) = f^{-1}(\sqrt{x}) = (\sqrt{x})^2 = x$.

1.3.11



1.3.12



1.3.13 $g_1(x)$ is the right side of the standard parabola shifted up one unit. So $g_1(x) = x^2 + 1$, $x \geq 0$. The domain for g_1 is $[0, \infty)$ and the range is $[1, \infty)$. The inverse of g_1 is therefore the square root function shifted one unit to the right. So $g_1^{-1}(x) = \sqrt{x-1}$, and its domain is $[1, \infty)$ and its range is $[0, \infty)$.

1.3.14 $g_2(x)$ is the left side of the standard parabola shifted up one unit. So $g_2(x) = x^2 + 1$, $x \leq 0$. The domain for g_1 is $(\infty, 0]$ and the range is $[1, \infty)$. The inverse of g_2 is therefore the square root function shifted one unit to the right and reflected across the x-axis. So $g_2^{-1}(x) = -\sqrt{x-1}$, and its domain is $[1, \infty)$ and its range is $(\infty, 0]$.

1.3.15 $\log_b x$ represents the power to which b must be raised in order to obtain x. So, $b^{\log_b x} = x$.

1.3.16 The properties are related in that each can be used to derive the other. Assume $b^{x+y} = b^x b^y$, for all real numbers x and y. Then applying this rule to the numbers $\log_b x$ and $\log_b y$ gives $b^{\log_b x + \log_b y} = b^{\log_b x} b^{\log_b y} = xy$. Taking logs of the leftmost and rightmost sides of this equation yields $\log_b x + \log_b y = \log_b(xy)$.

Now assume that $\log_b(xy) = \log_b x + \log_b y$ for all positive numbers x and y. Applying this rule to the product $b^x b^y$, we have $\log_b(b^x b^y) = \log_b b^x + \log_b b^y = x + y$. Now looking at the leftmost and rightmost sides of this equality and applying the definition of logarithm yields $b^{x+y} = b^x b^y$, as was desired.

1.3.17 Because the domain of b^x is $\mathbb{R}$ and the range of b^x is $(0, \infty)$, and because $\log_b x$ is the inverse of b^x, the domain of $\log_b x$ is $(0, \infty)$ and the range is $\mathbb{R}$.

1.3.18 Let $2^5 = z$. Then $\ln(2^5) = \ln(z)$, so $\ln(z) = 5\ln(2)$. Taking the exponential function of both sides gives $z = e^{5\ln(2)}$. Therefore, $2^5 = e^{5\ln(2)}$.

1.3.19

a. Because $10^3 = 1000$, $\log_{10} 1000 = 3$.

b. Because $2^4 = 16$, $\log_2 16 = 4$.

c. Because $10^{-2} = \frac{1}{100} = 0.01$, $\log_{10} 0.01 = -2$.

d. Because e^x and $\ln x$ are inverses, $\ln e^3 = 3$.

e. Because e^x and $\ln x$ are inverses, $\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$.

1.3.20 $\log_2 a = \frac{\ln a}{\ln 2} \approx 5.4923$, and $\log_a 2 = \frac{\ln 2}{\ln a} \approx 0.1821$.

1.3.21 f is one-to-one on $(-\infty, \infty)$, so it has an inverse on $(-\infty, \infty)$.

1.3.22 f is one-to-one on $[-1/2, \infty)$, so it has an inverse on that set. (Alternatively, it is one-to-one on the interval $(-\infty, -1/2]$, so that interval could be used as well.)

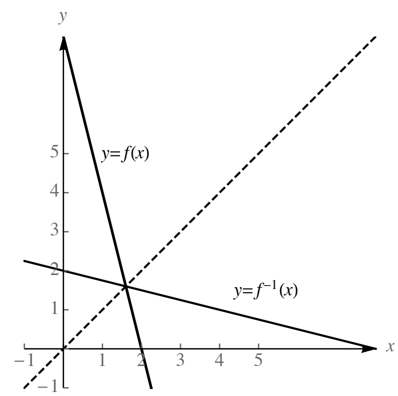
1.3.23 f is one-to-one on its domain, which is $(-\infty, 5) \cup (5, \infty)$, so it has an inverse on that set.

1.3.24 f is one-to-one on the set $(-\infty, 6]$, so it has an inverse on that set. (Alternatively, it is one-to-one on the interval $[6, \infty)$, so that interval could be used as well.)

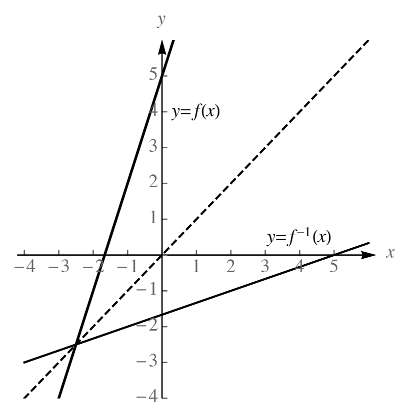
1.3.25 f is one-to-one on the interval $(0, \infty)$, so it has an inverse on that interval. (Alternatively, it is one-to-one on the interval $(-\infty, 0)$, so that interval could be used as well.)

1.3.26 Note that f can be written as $f(x) = x^2 - 2x + 8 = x^2 - 2x + 1 + 7 = (x-1)^2 + 7$. It is one-to-one on the interval $(1, \infty)$, so it has an inverse on that interval. (Alternatively, it is one-to-one on the interval $(-\infty, 1)$, so that interval could be used as well.)

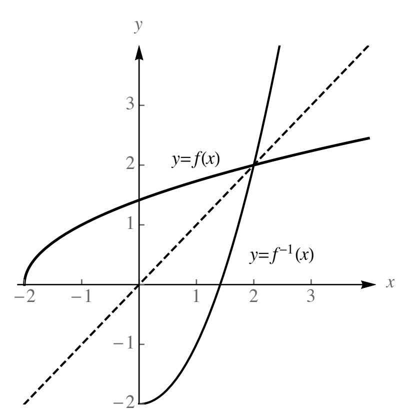
1.3.27 Switching x and y gives $x = 8 - 4y$. Solving this for y yields $y = f^{-1}(x) = \frac{8-x}{4}$.



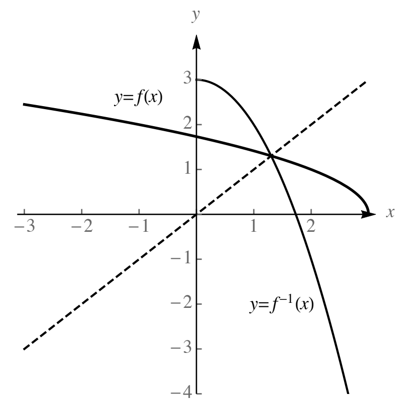
1.3.28 Switching x and y, we have $x = 3y+5$. Solving for y in terms of x we have $y = \frac{x-5}{3}$, so $y = f^{-1}(x) = \frac{x-5}{3}$.



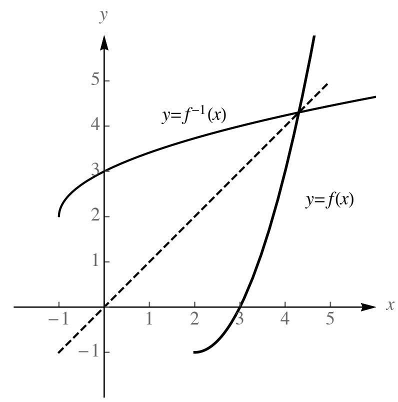
1.3.29 Switching x and y, we have $x = \sqrt{y+2}$. Solving for y in terms of x we have $y = f^{-1}(x) = x^2 - 2$. Note that because the range of f is $[0, \infty)$, that is also the domain of f^{-1}.



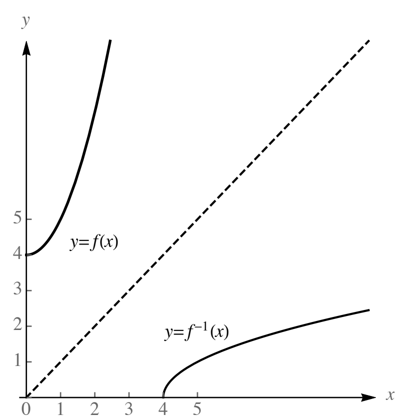
1.3.30 Switching x and y gives $x = \sqrt{3-y}$. Solving for y gives $3 - y = x^2$, or $y = 3 - x^2$. Since the domain of f is $(-\infty, 3]$ and the range of f is $[0, \infty)$, the domain of f^{-1} is $[0, \infty)$ and the range is $(-\infty, 3]$.



1.3.31 Switching x and y gives $x = (y-2)^2 - 1$. Then $x + 1 = (y-2)^2$, so $\sqrt{x+1} = |y-2|$, but because in the original function the variable is greater than or equal to 2, we choose the the positive portion of the graph of $|y-2|$. So we have $y = 2 + \sqrt{x+1}$.



1.3.32 Switching x and y, we have $x = y^2 + 4$. Solving for y in terms of x we have $y^2 = x - 4$, so $|y| = \sqrt{x-4}$. But because we are given that the domain of f is $\{x\colon x \geq 0\}$, we know that the range of f^{-1} is also non-negative. So $y = f^{-1}(x) = \sqrt{x-4}$.



1.3.33 Switching x and y, we have $x = \frac{2}{y^2+1}$. Then $y^2 + 1 = \frac{2}{x}$, so $y^2 = \frac{2}{x} - 1$. So $|y| = \sqrt{\frac{2}{x} - 1}$. We choose the positive portion, so that $y = \sqrt{\frac{2}{x} - 1}$. Note that the domain of f is $[0, \infty)$ while the range of f is $(0, 2]$. So the domain of f^{-1} is $(0, 2]$ and the range is $[0, \infty)$.

1.3.34 Switching x and y gives $x = \frac{6}{y^2-9}$. Solving yields $y^2 - 9 = \frac{6}{x}$, or $|y| = \sqrt{\frac{6}{x} + 9}$, but because the domain of f is positive, the range of f^{-1} must be positive as well, so we have $f^{-1}(x) = \sqrt{\frac{6}{x} + 9}$.

1.3.35 Switching x and y, we have $x = e^{2y+6}$. Then $\ln x = 2y + 6$, so $2y = \ln x - 6$, and $y = f^{-1}(x) = \frac{1}{2}\ln x - 3$.

1.3.36 Switching x and y, we have $x = 4e^{5y}$. Then $e^{5y} = \frac{x}{4}$, so $5y = \ln\frac{x}{4}$, and $y = f^{-1}(x) = .\frac{1}{5}\ln\frac{x}{4}$.

1.3.37 Switching x and y, we have $x = \ln(3y+1)$. Then $e^x = 3y+1$, so $3y = e^x - 1$ and $y = f^{-1}(x) = \frac{e^x-1}{3}$.

1.3.38 Switching x and y, we have $x = \log_{10} 4y$. Then $10^x = 4y$, so $y = f^{-1}(x) = \frac{10^x}{4}$.

1.3.39 Switching x and y, we have $x = 10^{-2y}$. Then $\log_{10} x = -2y$, so $y = f^{-1}(x) = -\frac{1}{2}\log_{10} x$.

1.3.40 Switching x and y, we have $x = \frac{1}{e^y+1}$. Then $e^y + 1 = \frac{1}{x}$. so $e^y = \frac{1}{x} - 1$, and $y = f^{-1}(x) = \ln\left(\frac{1}{x} - 1\right)$.

1.3.41 Switching x and y, we have $x = \frac{e^y}{e^y+2}$. Taking the reciprocal of both sides, we have $\frac{1}{x} = \frac{e^y+2}{e^y} = 1 + 2e^{-y}$. Then $2e^{-y} = \frac{1}{x} - 1$, and $e^{-y} = \frac{1}{2x} - \frac{1}{2}$. So $-y = \ln\left(\frac{1}{2x} - \frac{1}{2}\right)$, and $y = f^{-1}(x) = -\ln\left(\frac{1}{2x} - \frac{1}{2}\right) = -\ln\left(\frac{1-x}{2x}\right) = \ln\left(\frac{2x}{1-x}\right)$.

1.3.42 Switching x and y, we have $x = \frac{y}{y-2}$. Cross multiplying yields $xy - 2x = y$, so $xy - y = 2x$, and $y(x-1) = 2x$. Then $y = \frac{2x}{x-1}$. Note that the domain of f is given to be $(2, \infty)$, and the range is $(1, \infty)$. So the domain of f^{-1} must be restricted to be $(1, \infty)$.

1.3.43 First note that because the expression is symmetric, switching x and y doesn't change the expression. Solving for y gives $|y| = \sqrt{1-x^2}$. To get the four one-to-one functions, we restrict the domain and choose either the upper part or lower part of the circle as follows:

a. $f_1(x) = \sqrt{1-x^2},\ 0 \le x \le 1$
   $f_2(x) = \sqrt{1-x^2},\ -1 \le x \le 0$
   $f_3(x) = -\sqrt{1-x^2},\ -1 \le x \le 0$
   $f_4(x) = -\sqrt{1-x^2},\ 0 \le x \le 1$

b. Reflecting these functions across the line $y = x$ yields the following:
   $f_1^{-1}(x) = \sqrt{1-x^2},\ 0 \le x \le 1$
   $f_2^{-1}(x) = -\sqrt{1-x^2},\ 0 \le x \le 1$
   $f_3^{-1}(x) = -\sqrt{1-x^2},\ -1 \le x \le 0$
   $f_4^{-1}(x) = \sqrt{1-x^2},\ -1 \le x \le 0$

1.3.44 First note that because the expression is symmetric, switching x and y doesn't change the expression. Solving for y gives $|y| = \sqrt{2|x|}$. To get the four one-to-one functions, we restrict the domain and choose either the upper part or lower part of the parabola as follows:

a. $f_1(x) = \sqrt{2x},\ x \ge 0$
   $f_2(x) = \sqrt{-2x},\ x \le 0$
   $f_3(x) = -\sqrt{-2x},\ x \le 0$
   $f_4(x) = -\sqrt{2x},\ x \ge 0$

b. Reflecting these functions across the line $y = x$ yields the following:
   $f_1^{-1}(x) = x^2/2,\ x \ge 0$
   $f_2^{-1}(x) = -x^2/2,\ x \ge 0$
   $f_3^{-1}(x) = -x^2/2,\ x \le 0$
   $f_4^{-1}(x) = x^2/2,\ x \le 0$

1.3.45 $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y = 0.36 - .056 = -0.2.$

1.3.46 $\log_b x^2 = 2\log_b x = 2(0.36) = 0.72.$

1.3.47 $\log_b xz = \log_b x + \log_b z = 0.36 + 0.83 = 1.19.$

1.3.48 $\log_b \frac{\sqrt{xy}}{z} = \log_b(xy)^{1/2} - \log_b z = \frac{1}{2}(\log_b x + \log_b y) - \log_b z = (0.36)/2 + (0.56)/2 - 0.83 = -0.37.$

1.3.49 $\log_b \frac{\sqrt{x}}{\sqrt[3]{z}} = \log_b x^{1/2} - \log_b z^{1/3} = (1/2)\log_b x - (1/3)\log_b z = (0.36)/2 - (0.83)/3 = -.09\overline{6}.$

1.3.50 $\log_b \frac{b^2 x^{5/2}}{\sqrt{y}} = \log_b b^2 x^{5/2} - \log_b y^{1/2} = \log_b b^2 + (5/2)\log_b x - (1/2)\log_b y = 2 + (5/2)(0.36) - (1/2)(0.56) = 2.62.$

1.3.51 If $\log_{10} x = 3$, then $10^3 = x$, so $x = 1000$.

1.3.52 If $\log_5 x = -1$, then $5^{-1} = x$, so $x = 1/5$.

1.3.53 If $\log_8 x = 1/3$, then $x = 8^{1/3} = 2$.

1.3.54 If $\log_b 125 = 3$, then $b^3 = 125$, so $b = 5$ because $5^3 = 125$.

1.3.55 $\ln x = -1$, then $e^{-1} = x$, so $x = \frac{1}{e}$.

1.3.56 If $\ln y = 3$, then $y = e^3$.

1.3.57 Since $7^x = 21$, we have that $\ln 7^x = \ln 21$, so $x \ln 7 = \ln 21$, and $x = \frac{\ln 21}{\ln 7}$.

1.3.58 Since $2^x = 55$, we have that $\ln 2^x = \ln 55$, so $x \ln 2 = \ln 55$, and $x = \frac{\ln 55}{\ln 2}$.

1.3.59 Since $3^{3x-4} = 15$, we have that $\ln 3^{3x-4} = \ln 15$, so $(3x-4)\ln 3 = \ln 15$. Thus, $3x - 4 = \frac{\ln 15}{\ln 3}$, so $x = \frac{(\ln 15)/(\ln 3)+4}{3} = \frac{\ln 15 + 4\ln 3}{3\ln 3} = \frac{\ln 5 + \ln 3 + 4\ln 3}{3\ln 3} = \frac{\ln 5}{3\ln 3} + \frac{5}{3}$.

1.3.60 Since $5^{3x} = 29$, we have that $\ln 5^{3x} = \ln 29$, so $(3x)\ln 5 = \ln 29$. Solving for x gives $x = \frac{\ln 29}{3\ln 5}$.

1.3.61 We are seeking t so that $50 = 100e^{-t/650}$. This occurs when $e^{-t/650} = \frac{1}{2}$, which is when $-\frac{t}{650} = \ln(1/2)$, so $t = 650\ln 2 \approx 451$ years.

1.3.62 We need to solve $150 = 64e^{0.004t}$ for t. We have $\frac{150}{64} = e^{0.004t}$. So $0.004t = \ln \frac{150}{64}$, so $t = \frac{\ln \frac{150}{64}}{0.004} \approx 212.938$, so about 213 days.

1.3.63 We need to solve $1100 = 1000\left(1 + \dfrac{0.01}{12}\right)^{12t}$ for t. We have $1.1 = \left(1 + \dfrac{0.01}{12}\right)^{12t}$, so $\ln 1.1 = 12t \ln\left(1 + \dfrac{0.01}{12}\right)$. Then $t = \dfrac{\ln 1.1}{12\ln\left(1 + \frac{0.01}{12}\right)} \approx 9.53$ years.

1.3.64 We need to solve $22000 = 20000\left(1 + \dfrac{0.025}{12}\right)^{12t}$ for t. We have $1.1 = \left(1 + \dfrac{0.025}{12}\right)^{12t}$, so $\ln 1.1 = 12t \ln\left(1 + \dfrac{0.025}{12}\right)$. Then $t = \dfrac{\ln 1.1}{12\ln\left(1 + \frac{0.025}{12}\right)} \approx 3.82$ years.

1.3.65

a. No. The function takes on the values from 0 to 64 as t varies from 0 to 2, and then takes on the values from 64 to 0 as t varies from 2 to 4, so h is not one-to-one.

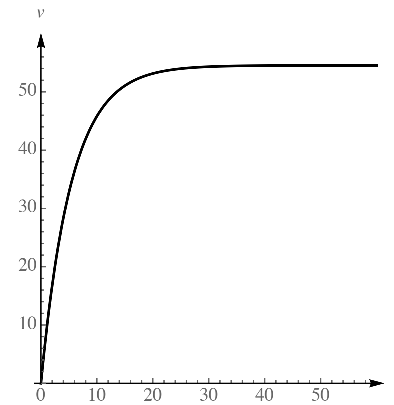
b. Solving for h in terms of t we have $h = 64t - 16t^2$, so (completing the square) we have $h - 64 = -16(t^2 - 4t + 4)$. Thus, $h - 64 = -16(t-2)^2$, and $(t-2)^2 = \frac{64-h}{16}$. Therefore $|t - 2| = \frac{\sqrt{64-h}}{4}$. When the ball is on the way up we know that $t < 2$, so the inverse of f is $f^{-1}(h) = 2 - \frac{\sqrt{64-h}}{4}$.

c. Using the work from the previous part of this problem, we have that when the ball is on the way down (when $t > 2$) we have that the inverse of f is $f^{-1}(h) = 2 + \frac{\sqrt{64-h}}{4}$.

d. On the way up, the ball is at a height of 30 ft at $2 - \frac{\sqrt{64-30}}{4} \approx 0.542$ seconds.

e. On the way down, the ball is at a height of 10 ft at $2 + \frac{\sqrt{64-10}}{4} \approx 3.837$ seconds.

1.3.66 The terminal velocity for $k = 11$ is $\frac{600}{11}$.



1.3.67 $\log_2 15 = \frac{\ln 15}{\ln 2} \approx 3.9069$.

1.3.68 $\log_3 30 = \frac{\ln 30}{\ln 3} \approx 3.0959$.

1.3.69 $\log_4 40 = \frac{\ln 40}{\ln 4} \approx 2.6610$.

1.3.70 $\log_6 60 = \frac{\ln 60}{\ln 6} \approx 2.2851$.

1.3.71 Let $2^x = z$. Then $\ln 2^x = \ln z$, so $x \ln 2 = \ln z$. Taking the exponential function of both sides gives $z = e^{x \ln 2}$.

1.3.72 Let $3^{\sin x} = z$. Then $\ln 3^{\sin x} = \ln z$, so $(\sin x) \ln 3 = \ln z$. Taking the exponential function of both sides gives $z = e^{(\sin x) \ln 3}$.

1.3.73 Let $z = \ln |x|$. Then $e^z = |x|$. Taking logarithms with base 5 of both sides gives $\log_5 e^z = \log_5 |x|$, so $z \cdot \log_5 e = \log_5 |x|$, and thus $z = \frac{\log_5 |x|}{\log_5 e}$.

1.3.74 Using the change of base formula, $\log_2(x^2 + 1) = \frac{\ln(x^2+1)}{\ln 2}$.

1.3.75 Let $z = a^{1/\ln a}$. Then $\ln z = \ln\left(a^{1/\ln a}\right) = \frac{1}{\ln a} \cdot \ln a = 1$. Thus $z = e$.

1.3.76 Let $z = a^{1/\log a}$. Then $\log z = \log\left(a^{1/\log a}\right) = \frac{1}{\log a} \cdot \log a = 1$. Thus $z = 10$.

1.3.77

a. False. For example, $3 = 3^1$, but $1 \neq \sqrt[3]{3}$.

b. False. For example, suppose $x = y = b = 2$. Then the left-hand side of the equation is equal to 1, but the right-hand side is 0.

c. False. $\log_5 4^6 = 6 \log_5 4 > 4 \log_5 6$.

d. True. This follows because 10^x and $\log_{10}$ are inverses of each other.

e. False. $\ln 2^e = e \ln 2 < 2$.

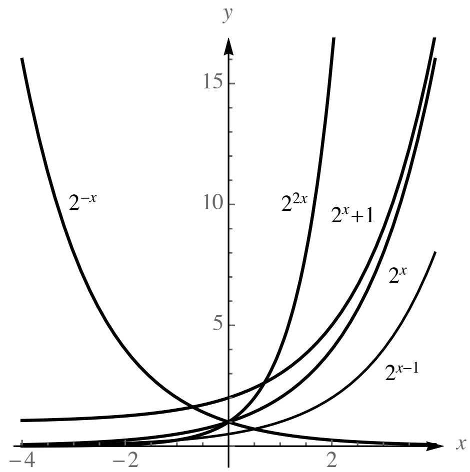
f. False. For example $f(0) = 1$, but the alleged inverse function evaluated at 1 is not 0 (rather, it has value 1/2.)

g. True. f is its own inverse because $f(f(x)) = f(1/x) = \frac{1}{1/x} = x$.

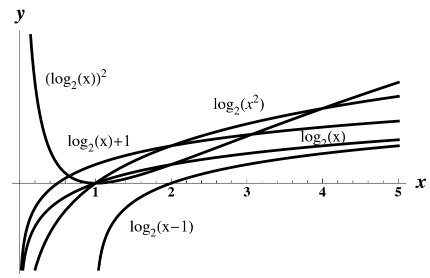
1.3.78 A is 2^{-x}, B is 3^{-x}, C is 3^x and D is 2^x.

1.3.79 A is $\log_2 x$, B is $\log_4 x$, C is $\log_{10} x$.

1.3.80



1.3.81



Note: need better pic from back of book

1.3.82

a. $p(0) = 150(2^{0/12}) = 150$.

b. At a given time t, let the population be $z = 150(2^{t/12})$. Then 12 hours later, the time is $12 + t$, and the population is $150(2^{(t+12)/12}) = 150(2^{(t/12)+1}) = 150(2^{t/12} \cdot 2) = 2z$.

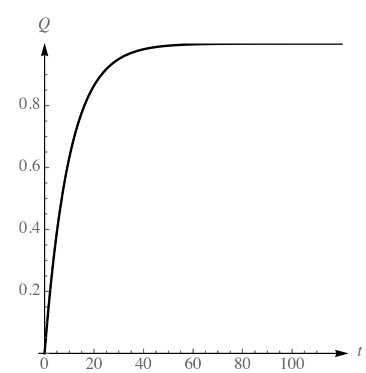
c. Since 4 days is 96 hours, we have $p(96) = 150(2^{96/12}) = 150(2^8) = 38{,}400$.

d. We can find the time to triple by solving $450 = 150(2^{t/12})$, which is equivalent to $3 = 2^{t/12}$. By taking logs of both sides we have $\ln 3 = \frac{t}{12} \cdot \ln 2$, so $t = \frac{12 \ln 3}{\ln 2} \approx 19.0$ hours.

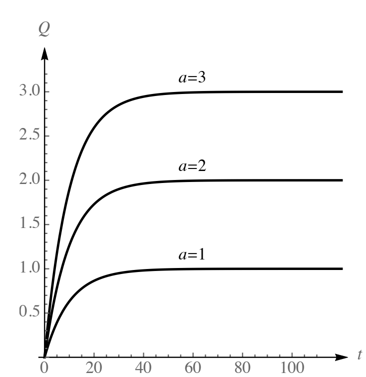
e. The population will reach 10,000 when $10,000 = 150(2^{12/t})$, which is equivalent to $\frac{200}{3} = 2^{t/12}$. By taking logs of both sides we have $\ln(200/3) = \frac{t}{12} \ln 2$, so $= \frac{12 \cdot \ln(200/3)}{\ln 2} \approx 72.7$ hours.

1.3.83

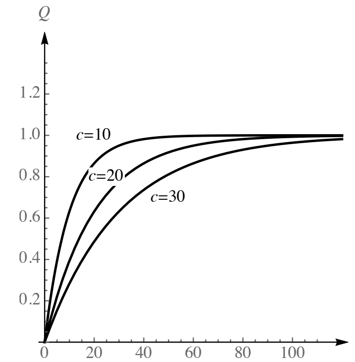
a. The relevant graph is:



b. Varying a while holding c constant scales the curve vertically. It appears that the steady-state charge is equal to a.



c. Varying c while holding a constant scales the curve horizontally. It appears that the steady-state charge does not vary with c.



d. As t grows large, the term $ae^{-t/c}$ approaches zero for any fixed c and a. So the steady-state charge for $a - ae^{-t/c}$ is a.

1.3.84 Since $e^x = x^{123}$, we have $x = \ln(x^{123})$, so $x = 123 \ln x$. Consider the function $f(x) = x - 123 \ln x$. Plotting this function using a computer or calculator reveals a graph which crosses the x axis twice, near $x = 1$ and near $x = 826$. (Try graphing it using the domain $(0, 900)$). Using a calculator and some trial and error reveals that the roots of f are approximately 1.0082 and 826.1659.

1.3.85 Begin by completing the square: $f(x) = x^2 - 2x + 6 = (x^2 - 2x + 1) + 5 = (x - 1)^2 + 5$. Switching x and y yields $x = (y - 1)^2 + 5$. Solving for y gives $|y - 1| = \sqrt{x - 5}$. Choosing the principal square root (because the original given interval has x positive) gives $y = f^{-1}(x) = \sqrt{x - 5} + 1$, $x \geq 5$.

1.3.86 Begin by completing the square: $f(x) = -x^2 - 4x - 3 = -(x^2 + 4x + 3) = -(x^2 + 4x + 4 - 1) = -((x + 2)^2 - 1) = 1 - (x + 2)^2$. Switching x and y yields $x = 1 - (y + 2)^2$, and solving for y gives $|y + 2| = \sqrt{1 - x}$. Since the given domain of f was negative, the range of f^{-1} must be negative, so we must have $y + 2 = -\sqrt{1 - x}$, so the inverse function is $f^{-1}(x) = -\sqrt{1 - x} - 2$.

1.3.87 Note that f is one-to-one, so there is only one inverse. Switching x and y gives $x = (y + 1)^3$. Then $\sqrt[3]{x} = y + 1$, so $y = f^{-1}(x) = \sqrt[3]{x} - 1$. The domain of f^{-1} is $\mathbb{R}$.

1.3.88 Note that to get a one-to-one function, we should restrict the domain to either $[4, \infty)$ or $(-\infty, 4]$. Switching x and y yields $x = (y - 4)^2$, so $\sqrt{x} = |y - 4|$. So $y = 4 \pm \sqrt{x}$. So the inverse of f when the domain of f is restricted to $[4, \infty)$ is $f^{-1}(x) = 4 + \sqrt{x}$, while if the domain of f is restricted to $(-\infty, 4]$ the inverse is $f^{-1}(x) = 4 - \sqrt{x}$. In either case, the domain of f^{-1} is $[0, \infty)$.

1.3.89 Note that to get a one-to-one function, we should restrict the domain to either $[0, \infty)$ or $(-\infty, 0]$. Switching x and y yields $x = \frac{2}{y^2+2}$, so $y^2 + 2 = (2/x)$. So $y = \pm\sqrt{(2/x) - 2}$. So the inverse of f when the domain of f is restricted to $[0, \infty)$ is $f^{-1}(x) = \sqrt{(2/x) - 2}$, while if the domain of f is restricted to $(-\infty, 0]$ the inverse is $f^{-1}(x) = -\sqrt{(2/x) - 2}$. In either case, the domain of f^{-1} is $(0, 1]$.

1.3.90 Note that f is one-to-one. Switching x and y yields $x = \frac{2y}{y+2}$, so $x(y + 2) = 2y$. Thus $xy + 2x = 2y$, so $2x = 2y - xy = y(2 - x)$. Thus, $y = \frac{2x}{2-x}$. The domain of $f^{-1}(x) = \frac{2x}{2-x}$ is $(-\infty, 2) \cup (2, \infty)$.

1.3.91 Using the change of base formula, we have $\log_{1/b} x = \frac{\ln x}{\ln 1/b} = \frac{\ln x}{\ln 1 - \ln b} = \frac{\ln x}{-\ln b} = -\frac{\ln x}{\ln b} = -\log_b x$.

1.3.92

a. Given $x = b^p$, we have $p = \log_b x$, and given $y = b^q$, we have $q = \log_b y$.

b. $xy = b^p b^q = b^{p+q}$.

c. $\log_b xy = \log_b b^{p+q} = p + q = \log_b x + \log_b y$.

1.3.93 Let $x = b^p$ and $y = b^q$. Then
$\frac{x}{y} = \frac{b^p}{b^q} = b^{p-q}$. Thus $\log_b \frac{x}{y} = \log_b b^{p-q} = p - q = \log_b x - \log_b y$.

1.3.94

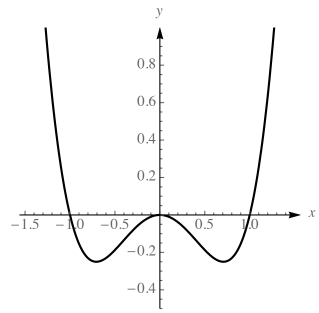
a. Given $x = b^p$, we have $p = \log_b x$.

b. $x^y = (b^p)^y = b^{yp}$.

c. $\log_b x^y = \log_b b^{yp} = yp = y \log_b x$.

1.3.95

a. f is one-to-one on $(-\infty, -\sqrt{2}/2]$, on $[-\sqrt{2}/2, 0]$, on $[0, \sqrt{2}/2]$, and on $[\sqrt{2}/2, \infty)$.



b. If $u = x^2$, then our function becomes $y = u^2 - u$. Completing the square gives $y + (1/4) = u^2 - u + (1/4) = (u - (1/2))^2$. Thus $|u - (1/2)| = \sqrt{y + (1/4)}$, so $u = (1/2) \pm \sqrt{y + (1/4)}$, with the "+" applying for $u = x^2 > (1/2)$ and the "−" applying when $u = x^2 < (1/2)$. Now letting $u = x^2$, we have $x^2 = (1/2) \pm \sqrt{y + (1/4)}$, so $x = \pm\sqrt{(1/2) \pm \sqrt{y + (1/4)}}$. Now switching the x and y gives the following inverses:

Domain of f	$(-\infty, -\sqrt{2}/2]$	$[-\sqrt{2}/2, 0]$	$[0, \sqrt{2}/2]$	$[\sqrt{2}/2, \infty)$
Range of f	$[-1/4, \infty)$	$[-1/4, 0]$	$[-1/4, 0]$	$[-1/4, \infty)$
Inverse of f	$-\sqrt{(1/2) + \sqrt{x + (1/4)}}$	$-\sqrt{(1/2) - \sqrt{x + (1/4)}}$	$\sqrt{(1/2) - \sqrt{x + (1/4)}}$	$\sqrt{(1/2) + \sqrt{x + (1/4)}}$

1.3.96

a. $f(x) = g(h(x)) = g(x^3) = 2x^3 + 3$. To find the inverse of f, we switch x and y to obtain $x = 2y^3 + 3$, so that $y^3 = \frac{x-3}{2}$, so $f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$. Note that $g^{-1}(x) = \frac{x-3}{2}$, and $h^{-1}(x) = \sqrt[3]{x}$, and so $f^{-1}(x) = h^{-1}(g^{-1}(x))$.

b. $f(x) = g(h(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$. so the inverse of f is $f^{-1}(x) = x - 1$. Note that $g^{-1}(x) = \sqrt{x - 1}$, and $h^{-1}(x) = x^2$, and so $f^{-1}(x) = h^{-1}(g^{-1}(x))$.

c. If h and g are one-to-one, then their inverses exist, and $f^{-1}(x) = h^{-1}(g^{-1}(x))$, because $f(f^{-1}(x)) = g(h(h^{-1}(g^{-1}(x))) = g(g^{-1}(x)) = x$ and likewise, $f^{-1}(f(x)) = h^{-1}(g^{-1}(g(h(x))) = h^{-1}(h(x)) = x$.

1.3.97 Using the change of base formulas $\log_b c = \frac{\ln c}{\ln b}$ and $\log_c b = \frac{\ln b}{\ln c}$ we have

$$(\log_b c) \cdot (\log_c b) = \frac{\ln c}{\ln b} \cdot \frac{\ln b}{\ln c} = 1.$$

1.4 Trigonometric Functions and Their Inverses

1.4.1 Let O be the length of the side opposite the angle x, let A be length of the side adjacent to the angle x, and let H be the length of the hypotenuse. Then $\sin x = \dfrac{O}{H}$, $\cos x = \dfrac{A}{H}$, $\tan x = \dfrac{O}{A}$, $\csc x = \dfrac{H}{O}$, $\sec x = \dfrac{H}{A}$, and $\cot x = \dfrac{A}{O}$.

1.4.2 Note that the distance from the origin to the point $(-4, -3)$ is $\sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$. Then we have $\sin\theta = -\dfrac{3}{5}$, $\cos\theta = -\dfrac{4}{5}$, $\tan\theta = \dfrac{3}{4}$, $\cot\theta = \dfrac{4}{3}$, $\sec\theta = -\dfrac{5}{4}$, $\csc\theta = -\dfrac{5}{3}$.

1.4.3 We have $t = \dfrac{v\sin\theta}{16} = \dfrac{96\sin\dfrac{\pi}{6}}{16} = \dfrac{96/2}{16} = \dfrac{48}{16} = 3$ seconds.

1.4.4

a. Because $\tan\theta = \dfrac{50}{d}$, we have $d = \frac{50}{\tan\theta}$.

b. Because $\sin\theta = \dfrac{50}{L}$, we have $L = \frac{50}{\sin\theta}$.

1.4.5 The radian measure of an angle θ is the length of the arc s on the unit circle associated with θ.

1.4.6 The period of a function is the smallest positive real number k so that $f(x+k) = f(x)$ for all x in the domain of the function. The sine, cosine, secant, and cosecant function all have period 2π. The tangent and cotangent functions have period π.

1.4.7 $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, and $\tan^2 x + 1 = \sec^2 x$.

1.4.8
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1/\sqrt{5}}{-2/\sqrt{5}} = -\dfrac{1}{2}$.
$\cot\theta = \frac{1}{\tan\theta} = -2$.
$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-2/\sqrt{5}} = -\frac{\sqrt{5}}{2}$.
$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{1/\sqrt{5}} = \sqrt{5}$.

1.4.9 The only point on the unit circle whose second coordinate is -1 is the point $(0, -1)$, which is the point associated with $\theta = \frac{3\pi}{2}$. So that is the only solution for $0 \le \theta < 2\pi$.

1.4.10 Note that if $0 \le \theta < 2\pi$, then $0 \le 2\theta < 4\pi$. So we must consider "two trips" around the unit circle. The second coordinate on the unit circle is 1 at the point $(0, 1)$, which is associated with $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. When $2\theta = \frac{\pi}{2}$ we have $\theta = \frac{\pi}{4}$, and when $2\theta = \frac{5\pi}{2}$ we have $\theta = \frac{5\pi}{4}$.

1.4.11 The tangent function is undefined where $\cos x = 0$, which is at all real numbers of the form $\dfrac{\pi}{2} + k\pi$, k an integer.

1.4.12 $\sec x$ is defined wherever $\cos x \neq 0$, which is $\{x \colon x \neq \dfrac{\pi}{2} + k\pi, k \text{ an integer}\}$.

1.4.13 The sine function is not one-to-one over its whole domain, so in order to define an inverse, it must be restricted to an interval on which it is one-to-one.

1.4.14 In order to define an inverse for the cosine function, we restricted the domain to $[0, \pi]$ in order to get a one-to-one function. Because the range of the inverse of a function is the domain of the function, we have that the values of $\cos^{-1} x$ lie in the interval $[0, \pi]$.

1.4.15 $\cos\dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$, so $\cos^{-1}\left(\cos\dfrac{5\pi}{4}\right) = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

1.4.16 $\sin\dfrac{11\pi}{6} = -\dfrac{1}{2}$, so $\sin^{-1}\left(\sin\dfrac{11\pi}{6}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$.

1.4.17 The numbers $\pm\pi/2$ are not in the range of $\tan^{-1} x$. The range is $(-\pi/2, \pi/2)$. However, it is true that as x increases without bound, the values of $\tan^{-1} x$ get close to $\pi/2$, and as x decreases without bound, the values of $\tan^{-1} x$ get close to $-\pi/2$.

1.4.18 The domain of $\sec^{-1} x$ is $\{x \colon |x| \ge 1\}$. The range is $[0, \pi/2) \cup (\pi/2, \pi]$.

1.4.19 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\cos(2\pi/3) = -1/2$.

1.4.20 The point on the unit circle associated with $2\pi/3$ is $(-1/2, \sqrt{3}/2)$, so $\sin(2\pi/3) = \sqrt{3}/2$.

1.4.21 The point on the unit circle associated with $-3\pi/4$ is $(-\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(-3\pi/4) = 1$.

1.4.22 The point on the unit circle associated with $15\pi/4$ is $(\sqrt{2}/2, -\sqrt{2}/2)$, so $\tan(15\pi/4) = -1$.

1.4.23 The point on the unit circle associated with $-13\pi/3$ is $(1/2, -\sqrt{3}/2)$, so $\cot(-13\pi/3) = -1/\sqrt{3} = -\sqrt{3}/3$.

1.4.24 The point on the unit circle associated with $7\pi/6$ is $(-\sqrt{3}/2, -1/2)$, so $\sec(7\pi/6) = -2/\sqrt{3} = -2\sqrt{3}/3$.

1.4.25 The point on the unit circle associated with $-17\pi/3$ is $(1/2, \sqrt{3}/2)$, so $\cot(-17\pi/3) = 1/\sqrt{3} = \sqrt{3}/3$.

1.4.26 The point on the unit circle associated with $16\pi/3$ is $(-1/2, -\sqrt{3}/2)$, so $\sin(16\pi/3) = -\sqrt{3}/2$.

1.4.27 Because the point on the unit circle associated with $\theta = 0$ is the point $(1, 0)$, we have $\cos 0 = 1$.

1.4.28 Because $-\pi/2$ corresponds to a quarter circle clockwise revolution, the point on the unit circle associated with $-\pi/2$ is the point $(0, -1)$. Thus $\sin(-\pi/2) = -1$.

1.4.29 Because $-\pi$ corresponds to a half circle clockwise revolution, the point on the unit circle associated with $-\pi$ is the point $(-1, 0)$. Thus $\cos(-\pi) = -1$.

1.4.30 Because 3π corresponds to one and a half counterclockwise revolutions, the point on the unit circle associated with 3π is $(-1, 0)$, so $\tan 3\pi = \dfrac{0}{-1} = 0$.

1.4.31 Because $5\pi/2$ corresponds to one and a quarter counterclockwise revolutions, the point on the unit circle associated with $5\pi/2$ is the same as the point associated with $\pi/2$, which is $(0, 1)$. Thus $\sec 5\pi/2$ is undefined.

1.4.32 Because π corresponds to one half circle counterclockwise revolution, the point on the unit circle associated with π is $(-1, 0)$. Thus $\cot \pi$ is undefined.

1.4.33 Using the fact that $\dfrac{\pi}{12} = \dfrac{\pi/6}{2}$ and the half-angle identity for cosine:

$$\cos^2(\pi/12) = \frac{1 + \cos(\pi/6)}{2} = \frac{1 + \sqrt{3}/2}{2} = \frac{2 + \sqrt{3}}{4}.$$

Thus, $\cos(\pi/12) = \sqrt{\dfrac{2 + \sqrt{3}}{4}}$.

1.4.34 Using the fact that $\dfrac{3\pi}{8} = \dfrac{3\pi/4}{2}$ and the half-angle identities for sine, we have:

$$\sin^2\left(\frac{3\pi}{8}\right) = \frac{1 - \cos(3\pi/4)}{2} = \frac{1 - (-\sqrt{2}/2)}{2} = \frac{2 + \sqrt{2}}{4},$$

and using the fact that $3\pi/8$ is in the first quadrant (and thus has positive value for sine) we deduce that $\sin\left(\dfrac{3\pi}{8}\right) = \dfrac{\sqrt{2 + \sqrt{2}}}{2}$.

1.4.35 First note that $\tan x = 1$ when $\sin x = \cos x$. Using our knowledge of the values of the standard angles between 0 and 2π, we recognize that the sine function and the cosine function are equal at $\pi/4$. Then, because we recall that the period of the tangent function is π, we know that $\tan(\pi/4 + k\pi) = \tan(\pi/4) = 1$ for every integer value of k. Thus the solution set is $\{\pi/4 + k\pi, \text{where } k \text{ is an integer}\}$.

1.4.36 Given that $2\theta\cos(\theta)+\theta=0$, we have $\theta(2\cos(\theta)+1)=0$. Which means that either $\theta=0$, or $2\cos(\theta)+1=0$. The latter leads to the equation $\cos\theta=-1/2$, which occurs at $\theta=2\pi/3$ and $\theta=4\pi/3$. Using the fact that the cosine function has period 2π the entire solution set is thus

$$\{0\}\cup\{2\pi/3+2k\pi, \text{where } k \text{ is an integer}\}\cup\{4\pi/3+2l\pi, \text{where } l \text{ is an integer}\}.$$

1.4.37 Given that $\sin^2\theta=\dfrac{1}{4}$, we have $|\sin\theta|=\dfrac{1}{2}$, so $\sin\theta=\dfrac{1}{2}$ or $\sin\theta=-\dfrac{1}{2}$. It follows that $\theta=\pi/6, 5\pi/6, 7\pi/6, 11\pi/6$.

1.4.38 Given that $\cos^2\theta=\dfrac{1}{2}$, we have $|\cos\theta|=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$. Thus $\cos\theta=\dfrac{\sqrt{2}}{2}$ or $\cos\theta=-\dfrac{\sqrt{2}}{2}$. We have $\theta=\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

1.4.39 The equation $\sqrt{2}\sin(x)-1=0$ can be written as $\sin x=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$. Standard solutions to this equation occur at $x=\pi/4$ and $x=3\pi/4$. Because the sine function has period 2π the set of all solutions can be written as:

$$\{\pi/4+2k\pi, \text{where } k \text{ is an integer}\}\cup\{3\pi/4+2l\pi, \text{where } l \text{ is an integer}\}.$$

1.4.40 $\sin^2(\theta)-1=0$ wherever $\sin^2(\theta)=1$, which is wherever $\sin(\theta)=\pm1$. This occurs for $\theta=\pi/2+k\pi$, where k is an integer.

1.4.41 If $\sin\theta\cos\theta=0$, then either $\sin\theta=0$ or $\cos\theta=0$. This occurs for $\theta=0,\pi/2,\pi,3\pi/2$.

1.4.42 Let $u=3x$. Note that because $0\le x<2\pi$, we have $0\le u<6\pi$. Because $\sin u=\sqrt{2}/2$ for $u=\pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, we must have that $\sin 3x=\sqrt{2}/2$ for $3x=\pi/4$, $3\pi/4$, $9\pi/4$, $11\pi/4$, $17\pi/4$, and $19\pi/4$, which translates into

$$x=\pi/12, \pi/4, 3\pi/4, 11\pi/12, 17\pi/12, \text{and } 19\pi/12.$$

1.4.43 Let $u=3x$. Then we are interested in the solutions to $\cos u=\sin u$, for $0\le u<6\pi$. This would occur for $u=3x=\pi/4, 5\pi/4, 9\pi/4, 13\pi/4, 17\pi/4, \text{and } 21\pi/4$. Thus there are solutions for the original equation at

$$x=\pi/12, 5\pi/12, 3\pi/4, 13\pi/12, 17\pi/12, \text{and } 7\pi/4.$$

1.4.44 If $\tan^2 2\theta=1$, then $\sin^2 2\theta=\cos^2 2\theta$, so we have either $\sin 2\theta=\cos 2\theta$ or $\sin 2\theta=-\cos 2\theta$. This occurs for $2\theta=\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ for $0\le 2\theta\le 2\pi$, so the corresponding values for θ are $\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$, $0\le\theta\le\pi$.

1.4.45 Using a computer algebra system or graphing calculator, we find that the roots are approximately 0.1007 and 1.4701.

1.4.46 Using a computer algebra system or graphing calculator, we find that the roots are approximately 0.375962, 1.71843, and 2.47036.

1.4.47 We are seeking solutions to the equation $400=\frac{150^2}{32}\sin 2\theta$, or $\sin 2\theta=0.56\overline{8}$. Using a computer algebra system or graphing calculator, we find that the solutions are about 0.30257 radians or about 17.3 degrees, and about 1.2682 radians which is about 72.7 degrees.

1.4.48 We are seeking solutions to the equation $350=\frac{160^2}{32}\sin 2\theta$, or $\sin 2\theta=0.4375$. Using a computer algebra system or graphing calculator, we find that the solutions are about 0.2264 radians which is about 13 degrees, and about 1.3444 radians which is about 77 degrees.

1.4.49 Let $z=\sin^{-1}1$. Then $\sin z=1$, and because $\sin\pi/2=1$, and $\pi/2$ is in the desired interval, $z=\pi/2$.

1.4.50 Let $z=\cos^{-1}(-1)$. Then $\cos z=-1$, and because $\cos\pi=-1$ and π is in the desired interval, $z=\pi$.

1.4.51 Let $z = \sin^{-1}(-1/2)$. Then $\sin z = 1/2$, and because $\sin(-\pi/6) = -1/2$, and $-\pi/6$ is in the desired interval, $z = -\pi/6$.

1.4.52 Let $z = \cos^{-1}(-\sqrt{2}/2)$. Then $\cos z = -\sqrt{2}/2$. Because $\cos 3\pi/4 = -\sqrt{2}/2$ and $3\pi/4$ is in the desired interval, we have $z = 3\pi/4$. (Note that $\cos(-\pi/4)$ is also equal to $-\sqrt{2}/2$, but $-\pi/4$ isn't in the desired interval $[0, \pi]$.)

1.4.53 $\sin^{-1}(\sqrt{3}/2) = \pi/3$, because $\sin(\pi/3) = \sqrt{3}/2$.

1.4.54 $\cos^{-1} 2$ does not exist, because 2 is not in the domain of the inverse cosine function (because 2 is not in the range of the cosine function.)

1.4.55 $\cos^{-1}(-1/2) = 2\pi/3$, because $\cos(2\pi/3) = -1/2$.

1.4.56 $\sin^{-1}(-1) = -\pi/2$, because $\sin(-\pi/2) = -1$.

1.4.57 $\cos(\cos^{-1}(-1)) = \cos(\pi) = -1$.

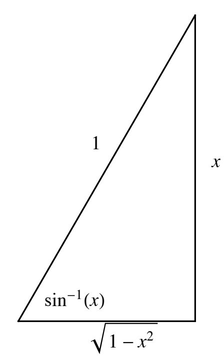
1.4.58 $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}(-\sqrt{3}/2) = 5\pi/6$. Note that the range of the inverse cosine function is $[0, \pi]$.

1.4.59 Because $\theta = \cos^{-1}(5/13)$, we know that $\cos\theta = 5/13$. The triangle in question has a leg of length 5 and a hypotenuse of length 13, so we can deduce using the Pythagorean theorem that the other leg as length 12. So $\sin\theta = 12/13$. Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12/13}{5/13} = \frac{12}{5}$.

1.4.60 Because $\theta = \tan^{-1}(4/3)$, we know that $\tan\theta = 4/3$. The triangle in question has legs of length 3 and 4, so the hypotenuse has length 5 by the Pythagorean theorem. Then $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{3/5} = \dfrac{5}{3}$. And $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{4/5} = \frac{5}{4}$.

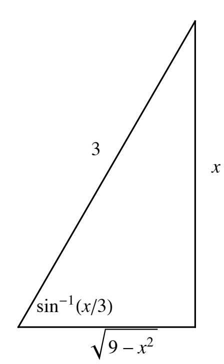
1.4.61

$$\cos(\sin^{-1}(x)) = \frac{\text{side adjacent to } \sin^{-1}(x)}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$



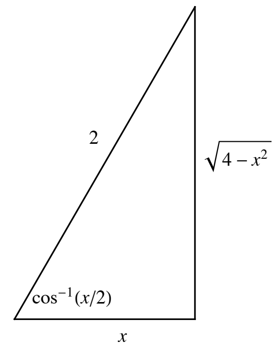
1.4.62

$$\cos(\sin^{-1}(x/3)) = \frac{\text{side adjacent to } \sin^{-1}(x/3)}{\text{hypotenuse}} = \frac{\sqrt{9-x^2}}{3}.$$



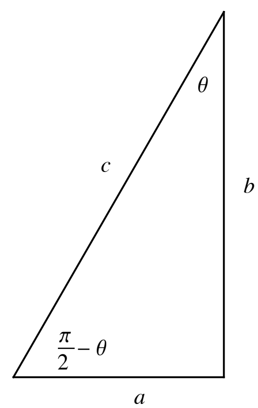
1.4.63

$$\sin(\cos^{-1}(x/2)) = \frac{\text{side opposite of } \cos^{-1}(x/2)}{\text{hypotenuse}} = \frac{\sqrt{4-x^2}}{2}.$$



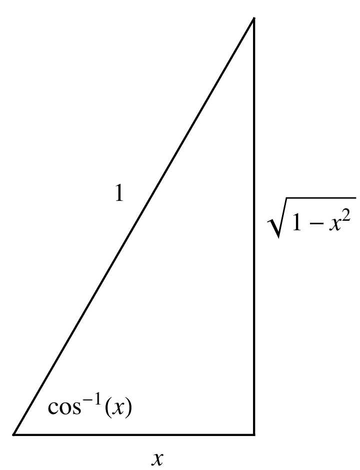
1.4.64

Note (from the triangle pictured) that $\cos\theta = \frac{b}{c} = \sin(\frac{\pi}{2} - \theta)$. Thus $\sin^{-1}(\cos\theta) = \sin^{-1}(\sin(\frac{\pi}{2} - \theta)) = \frac{\pi}{2} - \theta$.



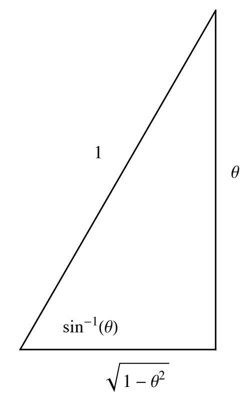
1.4.65

Using the identity given, we have $\sin(2\cos^{-1}(x)) = 2\sin(\cos^{-1}(x))\cos(\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x)) = 2x\sqrt{1-x^2}$.



1.4.66

First note that $\cos(\sin^{-1}(\theta)) = \sqrt{1-\theta^2}$, as indicated in the triangle shown.
Using the identity given, we have $\cos(2\sin^{-1}(x)) = \cos^2((\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = (\sqrt{1-x^2})^2 - x^2 = 1 - 2x^2$.



1.4.67 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\sec\theta = \dfrac{r}{x} = \dfrac{1}{x/r} = \dfrac{1}{\cos\theta}$.

1.4.68 From our definitions of the trigonometric functions via a point $P(x, y)$ on a circle of radius $r = \sqrt{x^2+y^2}$, we have $\tan\theta = \dfrac{y}{x} = \dfrac{y/r}{x/r} = \dfrac{\sin\theta}{\cos\theta}$.

1.4.69 We have already established that $\sin^2\theta + \cos^2\theta = 1$. Dividing both sides by $\cos^2\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$.

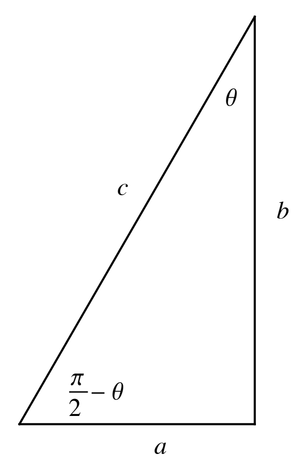
1.4.70 We have already established that $\sin^2\theta + \cos^2\theta = 1$. We can write this as $\frac{\sin\theta}{(1/\sin\theta)} + \frac{\cos\theta}{(1/\cos\theta)} = 1$, or $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$.

1.4.71

Using the triangle pictured, we see that

$$\sec(\pi/2 - \theta) = \frac{c}{a} = \csc\theta.$$

This also follows from the sum identity $\cos(a+b) = \cos a\cos b - \sin a \sin b$ as follows: $\sec(\pi/2 - \theta) = \dfrac{1}{\cos(\pi/2 + (-\theta))} = \dfrac{1}{\cos(\pi/2)\cos(-\theta) - \sin(\pi/2)\sin(-\theta)} = \dfrac{1}{0 - (-\sin(\theta))} = \csc(\theta)$.

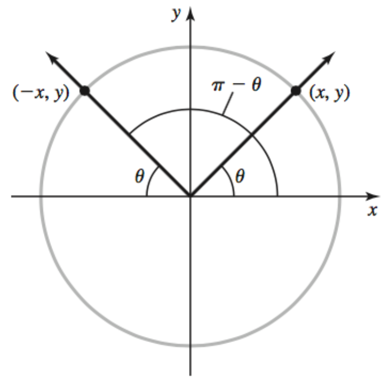


1.4.72 Using the trig identity for the cosine of a sum (mentioned in the previous solution) we have:

$$\sec(x+\pi) = \frac{1}{\cos(x+\pi)} = \frac{1}{\cos(x)\cos(\pi) - \sin(x)\sin(\pi)} = \frac{1}{\cos(x)\cdot(-1) - \sin(x)\cdot 0} = \frac{1}{-\cos(x)} = -\sec x.$$

1.4.73

Let $\theta = \cos^{-1}(x)$, and note from the diagram that it then follows that $\cos^{-1}(-x) = \pi - \theta$. So $\cos^{-1}(x) + \cos^{-1}(-x) = \theta + \pi - \theta = \pi$.



1.4.74 Let $\theta = \sin^{-1}(y)$. Then $\sin\theta = y$, and $\sin(-\theta) = -\sin(\theta) = -y$ (because the sine function is an odd function) and it then follows that $-\theta = \sin^{-1}(-y)$. Therefore, $\sin^{-1}(y) + \sin^{-1}(-y) = \theta + -\theta = 0$. It would be instructive for the reader to draw his or her own diagram like that in the previous solution.

1.4.75 $\tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\dfrac{\sqrt{3}/2}{1/2}\right) = \pi/3$, because $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$.

1.4.76 $\cot^{-1}(-1/\sqrt{3}) = \cot^{-1}\left(-\dfrac{1/2}{\sqrt{3}/2}\right) = 2\pi/3$, because $\sin(2\pi/3) = \sqrt{3}/2$ and $\cos(2\pi/3) = -1/2$.

1.4.77 $\sec^{-1}(2) = \sec^{-1}\left(\dfrac{1}{1/2}\right) = \pi/3$, because $\sec(\pi/3) = \dfrac{1}{\cos(\pi/3)} = \dfrac{1}{1/2} = 2$.

1.4.78 $\csc^{-1}(-1) = \sin^{-1}(-1) = -\pi/2$.

1.4.79 $\tan^{-1}(\tan(\pi/4)) = \tan^{-1}(1) = \pi/4$.

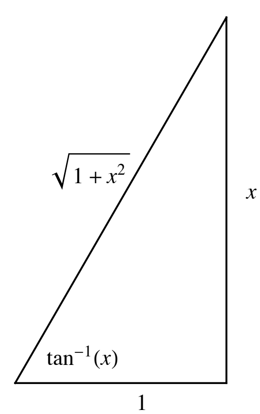
1.4.80 $\tan^{-1}(\tan(3\pi/4)) = \tan^{-1}(-1) = -\pi/4$.

1.4.81 Let $\csc^{-1}(\sec 2) = z$. Then $\csc z = \sec 2$, so $\sin z = \cos 2$. Now by applying the result of problem 64, we see that $z = \sin^{-1}(\cos 2) = \pi/2 - 2 = \dfrac{\pi - 4}{2}$.

1.4.82 $\tan(\tan^{-1}(1)) = \tan(\pi/4) = 1$.

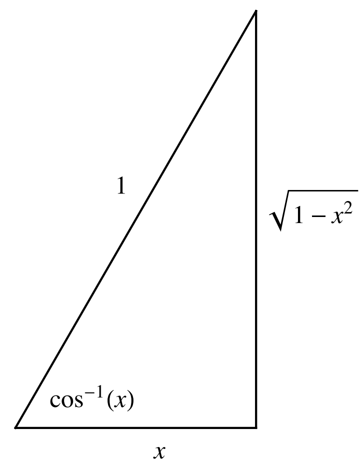
1.4.83

$$\cos(\tan^{-1}(x)) = \frac{\text{side adjacent to } \tan^{-1}(x)}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$



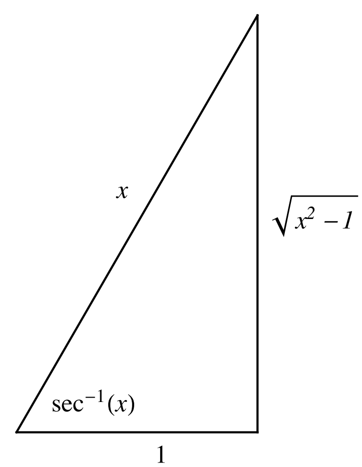
1.4.84

$$\tan(\cos^{-1}(x)) = \frac{\text{side opposite of } \cos^{-1}(x)}{\text{side adjacent to } \cos^{-1}(x)} = \frac{\sqrt{1-x^2}}{x}.$$



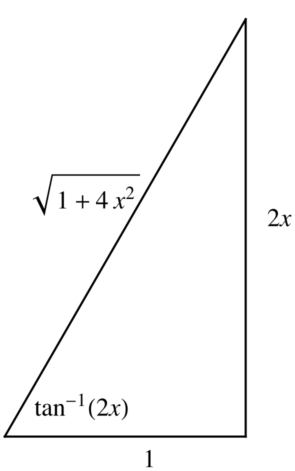
1.4.85

$$\cos(\sec^{-1}(x)) = \frac{\text{side adjacent to } \sec^{-1} x}{\text{hypotenuse}} = \frac{1}{x}.$$



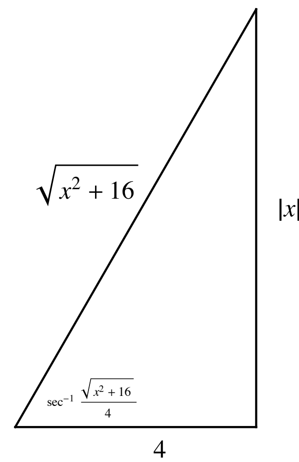
1.4.86

$$\cot(\tan^{-1} 2x) = \frac{\text{side adjacent to } \tan^{-1} 2x}{\text{side opposite of } \tan^{-1} 2x} = \frac{1}{2x}.$$



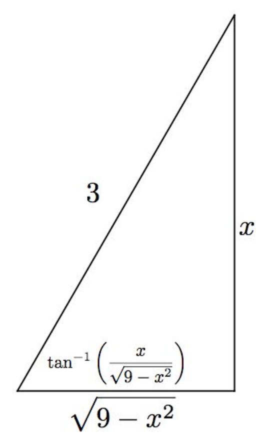
1.4.87

Assume $x > 0$. Then $\sin\left(\sec^{-1}\left(\frac{\sqrt{x^2+16}}{4}\right)\right) =$
$\dfrac{\text{side opposite of } \sec^{-1}\left(\frac{\sqrt{x^2+16}}{4}\right)}{\text{hypotenuse}} =$
$\dfrac{x}{\sqrt{x^2+16}}$.



1.4.88

$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right) =$
$\dfrac{\text{side adjacent to } \tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)}{\text{hypotenuse}} = \dfrac{\sqrt{9-x^2}}{3}$.

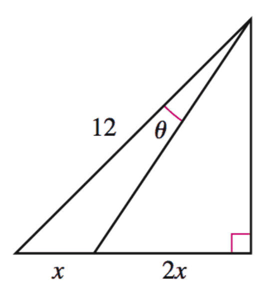


1.4.89 Because $\sin\theta = \frac{x}{6}$, $\theta = \sin^{-1}(x/6)$. Also, $\theta = \tan^{-1}\left(\frac{x}{\sqrt{36-x^2}}\right) = \sec^{-1}\left(\frac{6}{\sqrt{36-x^2}}\right)$.

1.4.90

Note that the vertical side in the adjacent diagram has length $\sqrt{144-9x^2}$ (by the Pythagorean theorem.) Let ψ be the angle in the adjacent diagram which is opposite the side labeled $2x$. First note that $\tan(\psi) = \frac{2x}{\sqrt{144-9x^2}}$, so $\psi = \tan^{-1}\left(\frac{2x}{\sqrt{144-9x^2}}\right)$. Also, $\sin(\theta+\psi) = \frac{3x}{12} = \frac{x}{4}$, so $\theta + \psi = \sin^{-1}(x/4)$. Therefore,

$$\theta = \sin^{-1}(x/4) - \psi = \sin^{-1}(x/4) - \tan^{-1}(2x/\sqrt{144-9x^2}).$$



1.4.91

a. False. For example, $\sin(\pi/2+\pi/2) = \sin(\pi) = 0 \neq \sin(\pi/2) + \sin(\pi/2) = 1 + 1 = 2$.

b. False. That equation has zero solutions, because the range of the cosine function is $[-1, 1]$.

c. False. It has infinitely many solutions of the form $\pi/6 + 2k\pi$, where k is an integer (among others.)

d. False. It has period $\dfrac{2\pi}{\pi/12} = 24$.

e. True. The others have a range of either $[-1, 1]$ or $(-\infty, -1] \cup [1, \infty)$.

f. False. For example, suppose $x = .5$. Then $\sin^{-1}(x) = \pi/6$ and $\cos^{-1}(x) = \pi/3$, so that $\dfrac{\sin^{-1}(x)}{\cos^{-1}(x)} = \dfrac{\pi/6}{\pi/3} = .5$. However, note that $\tan^{-1}(.5) \neq .5$,

g. True. Note that the range of the inverse cosine function is $[0, \pi]$.

h. False. For example, if $x = .5$, we would have $\sin^{-1}(.5) = \pi/6 \neq 1/\sin(.5)$.

1.4.92 If $\sin\theta = -4/5$, then the Pythagorean identity gives $|\cos\theta| = 3/5$. But if $\pi < \theta < 3\pi/2$, then the cosine of θ is negative, so $\cos\theta = -3/5$. Thus $\tan\theta = 4/3$, $\cot\theta = 3/4$, $\sec\theta = -5/3$, and $\csc\theta = -5/4$.

1.4.93 If $\cos\theta = 5/13$, then the Pythagorean identity gives $|\sin\theta| = 12/13$. But if $0 < \theta < \pi/2$, then the sine of θ is positive, so $\sin\theta = 12/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

1.4.94 If $\sec\theta = 5/3$, then $\cos\theta = 3/5$, and the Pythagorean identity gives $|\sin\theta| = 4/5$. But if $3\pi/2 < \theta < 2\pi$, then the sine of θ is negative, so $\sin\theta = -4/5$. Thus $\tan\theta = -4/3$, $\cot\theta = -3/4$, and $\csc\theta = -5/4$.

1.4.95 If $\csc\theta = 13/12$, then $\sin\theta = 12/13$, and the Pythagorean identity gives $|\cos\theta| = 5/13$. But if $0 < \theta < \pi/2$, then the cosine of θ is positive, so $\cos\theta = 5/13$. Thus $\tan\theta = 12/5$, $\cot\theta = 5/12$, and $\sec\theta = 13/5$.

1.4.96 The amplitude is 2, and the period is $\dfrac{2\pi}{2} = \pi$.

1.4.97 The amplitude is 3, and the period is $\dfrac{2\pi}{1/3} = 6\pi$.

1.4.98 The amplitude is 2.5, and the period is $\dfrac{2\pi}{1/2} = 4\pi$.

1.4.99 The amplitude is 3.6, and the period is $\dfrac{2\pi}{\pi/24} = 48$.

1.4.100 Using the given diagram, drop a perpendicular from the point $(b\cos\theta, b\sin\theta)$ to the x axis, and consider the right triangle thus formed whose hypotenuse has length c. By the Pythagorean theorem, $(b\sin\theta)^2 + (a - b\cos\theta)^2 = c^2$. Expanding the binomial gives $b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta = c^2$. Now because $b^2\sin^2\theta + b^2\cos^2\theta = b^2$, this reduces to $a^2 + b^2 - 2ab\cos\theta = c^2$.

1.4.101 Note that $\sin A = \dfrac{h}{c}$ and $\sin C = \dfrac{h}{a}$, so $h = c\sin A = a\sin C$. Thus

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Now drop a perpendicular from the vertex A to the line determined by $\overline{BC}$, and let h_2 be the length of this perpendicular. Then $\sin C = \dfrac{h_2}{b}$ and $\sin B = \dfrac{h_2}{C}$, so $h_2 = b\sin C = c\sin B$. Thus

$$\frac{\sin C}{c} = \frac{\sin B}{b}.$$

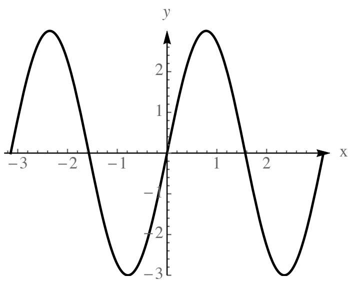
Putting the two displayed equations together gives

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

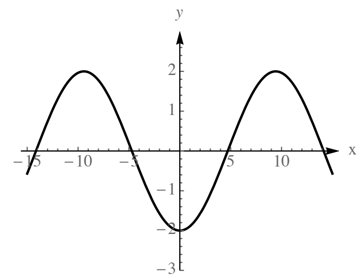
1.4.102 Consider the $\theta = \angle DOA$ where $D(300, 200)$ is the point where the Ditl is anchored, $O(0, 0)$ is the point where the observer is, and $A(300, 0)$ is the point on the x-axis closest to D. Then $\theta = \tan^{-1}(2/3)$. Then consider the angle $\phi = \angle WOB$ where $W(-100, 250)$ is the location of the Windborne and $B(-100, 0)$ is the point on the x-axis closest to W. Then $\phi = \tan^{-1}(250/100) = \tan^{-1}(5/2)$. The angle we are looking for has measure $\pi - \theta - \phi = \pi - \tan^{-1}(5/2) - \tan^{-1}(2/3) \approx 1.3633$ radians.

1.4.103 The area of the entire circle is πr^2. The ratio $\dfrac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\dfrac{\theta}{2\pi} \cdot \pi r^2 = \dfrac{r^2\theta}{2}$.

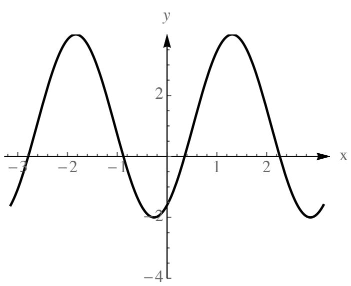
1.4.104



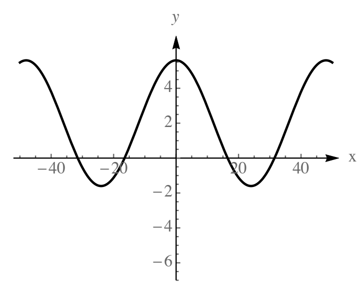
1.4.105



1.4.106



1.4.107



1.4.108 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 3$ (halfway between where the maximum and the minimum occur), we need to shift the function 3 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/6$ so that the new period is $\dfrac{2\pi}{\pi/6} = 12$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 4. Thus, the desired function is:

$$f(x) = 4\sin((\pi/6)(x - 3)) = 4\sin((\pi/6)x - \pi/2).$$

1.4.109 It is helpful to imagine first shifting the function horizontally so that the x intercept is where it should be, then stretching the function horizontally to obtain the correct period, and then stretching the function vertically to obtain the correct amplitude, and then shifting the whole graph up. Because the old x-intercept is at $x = 0$ and the new one should be at $x = 9$ (halfway between where the maximum and the minimum occur), we need to shift the function 9 units to the right. Then to get the right period, we need to multiply (before applying the sine function) by $\pi/12$ so that the new period is $\dfrac{2\pi}{\pi/12} = 24$. Finally, to get the right amplitude and to get the max and min at the right spots, we need to multiply on the outside by 3, and then shift the whole thing up 13 units. Thus, the desired function is:

$$f(x) = 3\sin((\pi/12)(x - 9)) + 13 = 3\sin((\pi/12)x - 3\pi/4) + 13.$$

1.4.110 Let C be the point on the end line so that segment $\overline{AC}$ is perpendicular to the endline. Then the distance $G_1C = 38.\overline{3}$, $G_2C = 15$, and $AC = 69$ and $BC = 84$, where all lengths are in feet. Thus

$$\mathrm{m}(\angle G_1AG_2) = \mathrm{m}(\angle G_1AC) - \mathrm{m}(\angle G_2AC) = \tan^{-1}\left(\frac{38.\overline{3}}{69}\right) - \tan^{-1}\left(\frac{15}{69}\right) \approx 16.79°,$$

while

$$\mathrm{m}(\angle G_1BG_2) = \mathrm{m}(\angle G_1BC) - \mathrm{m}(\angle G_2BC) = \tan^{-1}\left(\frac{38.\overline{3}}{84}\right) - \tan^{-1}\left(\frac{15}{84}\right) \approx 14.4°.$$

The kicking angle was not improved by the penalty.

1.4.111 Let C be the circumference of the earth. Then the first rope has radius $r_1 = \dfrac{C}{2\pi}$. The circle generated by the longer rope has circumference $C + 38$, so its radius is $r_2 = \dfrac{C+38}{2\pi} = \dfrac{C}{2\pi} + \dfrac{38}{2\pi} \approx r_1 + 6$, so the radius of the bigger circle is about 6 feet more than the smaller circle.

1.4.112

a. The period of this function is $\dfrac{2\pi}{2\pi/365} = 365$.

b. Because the maximum for the regular sine function is 1, and this function is scaled vertically by a factor of 2.8 and shifted 12 units up, the maximum for this function is $(2.8)(1) + 12 = 14.8$. Similarly, the minimum is $(2.8)(-1) + 12 = 9.2$. Because of the horizontal shift, the point at $t = 81$ is the midpoint between where the max and min occur. Thus the max occurs at $81 + (365/4) \approx 172$ and the min occurs approximately $(365/2)$ days later at about $t = 355$.

c. The solstices occur halfway between these points, at 81 and $81 + (365/2) \approx 264$.

1.4.113 We are seeking a function with amplitude 10 and period 1.5, and value 10 at time 0, so it should have the form $10\cos(kt)$, where $\dfrac{2\pi}{k} = 1.5$. Solving for k yields $k = \dfrac{4\pi}{3}$, so the desired function is $d(t) = 10\cos(4\pi t/3)$.

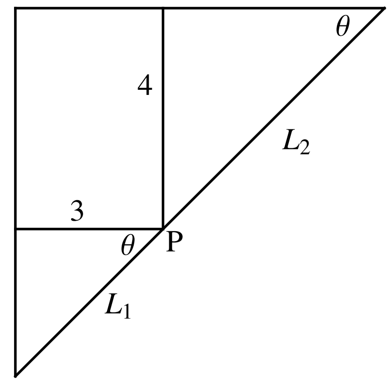
1.4.114 Let θ_1 be the viewing angle to the bottom of the television. Then $\theta_1 = \tan^{-1}\left(\dfrac{3}{x}\right)$. Now $\tan(\theta + \theta_1) = \dfrac{10}{x}$, so $\theta + \theta_1 = \tan^{-1}\left(\dfrac{10}{x}\right)$, so $\theta = \tan^{-1}\left(\dfrac{10}{x}\right) - \theta_1 = \tan^{-1}\left(\dfrac{10}{x}\right) - \tan^{-1}\left(\dfrac{3}{x}\right)$.

1.4.107 The area of the entire circle is πr^2. The ratio $\dfrac{\theta}{2\pi}$ represents the proportion of the area swept out by a central angle θ. Thus the area of a sector of a circle is this same proportion of the entire area, so it is $\dfrac{\theta}{2\pi} \cdot \pi r^2 = \dfrac{r^2\theta}{2}$.

1.4.115 Let L be the line segment connecting the tops of the ladders and let M be the horizontal line segment between the walls h feet above the ground. Now note that the triangle formed by the ladders and L is equilateral, because the angle between the ladders is 60 degrees, and the other two angles must be equal and add to 120, so they are 60 degrees as well. Now we can see that the triangle formed by L, M and the right wall is similar to the triangle formed by the left ladder, the left wall, and the ground, because they are both right triangles with one angle of 75 degrees and one of 15 degrees. Thus $M = h$ is the distance between the walls.

1.4.116

Let the corner point P divide the pole into two pieces, L_1 (which spans the 3-ft hallway) and L_2 (which spans the 4-ft hallway.) Then $L = L_1 + L_2$. Now $L_2 = \dfrac{4}{\sin\theta}$, and $\dfrac{3}{L_1} = \cos\theta$ (see diagram.) Thus $L = L_1 + L_2 = \dfrac{3}{\cos\theta} + \dfrac{4}{\sin\theta}$. When $L = 10$, $\theta \approx .9273$.



Chapter One Review

1

a. True. For example, $f(x) = x^2$ is such a function.

b. False. For example, $\cos(\pi/2 + \pi/2) = \cos(\pi) = -1 \neq \cos(\pi/2) + \cos(\pi/2) = 0 + 0 = 0$.

c. False. Consider $f(1+1) = f(2) = 2m + b \neq f(1) + f(1) = (m+b) + (m+b) = 2m + 2b$. (At least these aren't equal when $b \neq 0$.)

d. True. $f(f(x)) = f(1-x) = 1 - (1-x) = x$.

e. False. This set is the union of the disjoint intervals $(-\infty, -7)$ and $(1, \infty)$.

f. False. For example, if $x = y = 10$, then $\log_{10} xy = \log_{10} 100 = 2$, but $\log_{10} 10 \cdot \log_{10} 10 = 1 \cdot 1 = 1$.

g. True. $\sin^{-1}(\sin(2\pi)) = \sin^{-1}(0) = 0$.

2 B represents a function but A doesn't. A does not pass the vertical line test.

3 f is a one-to-one function, but g isn't (it fails the horizontal line test.)

4 Because the quantity under the radical must be non-zero, the domain of f is $[0, \infty)$. The range is also $[0, \infty)$.

5 The denominator must not be zero, so we must have $w \neq 2$. The domain is $\{w : w \neq 2\}$. Note that when $w \neq 2$, the function becomes $\dfrac{(w-2)(2w+1)}{w-2} = 2w + 1$. So the graph of f is a line of slope 2 with the point $(2, 5)$ missing, so the range is $\{y : y \neq 5\}$.

6 It is necessary that $x + 6 > 0$, so $x > -6$. The domain is $(-6, \infty)$ and the range is $(-\infty, \infty)$.

7 Because h can be written $h(z) = \sqrt{(z-3)(z+1)}$, we see that the domain is $(-\infty, -1] \cup [3, \infty)$. The range is $[0, \infty)$. (Note that as z gets large, $h(z)$ gets large as well.)

8 $f(g(2)) = f(-2) = f(2) = 2$, and $g(f(-2) = g(f(2)) = g(2) = -2$.

9 Yes, $\tan(\tan^{-1} x) = x$ because the range of the inverse tangent function is a subset of the domain of the tangent function, and the functions are inverses. However, $\tan^{-1}(\tan x)$ does not always equal x, for example: $\tan^{-1}(\tan \pi) = \tan^{-1} 0 = 0$.

10 $f(g(4)) = f(9) = 11.$

11 $g(f(4)) = g(5) = 8.$

12 $f^{-1}(10) = 8$ (Because $f(8) = 10.$)

13 $g^{-1}(5) = 7$ (Because $g(7) = 5.$)

14 $f^{-1}(g^{-1}(4) = f^{-1}(8) = 6.$

15 $g^{-1}(f(3)) = g^{-1}(4) = 8.$

16 Note that $f(6) = 8$ and recall that f is an odd function. Then $f(-6) = -f(6) = -8$, so $f^{-1}(-8) = -6.$

17 $f^{-1}(1 + f(-3)) = f^{-1}(1 - f(3)) = f^{-1}(1 - 4) = f^{-1}(-3) = -f(3) = -2.$

18 $g(1 - f(f^{-1}(-7))) = g(1 - (-7)) = g(8) = 4.$

19

a. $h(g(\pi/2)) = h(1) = 1$

b. $h(f(x)) = h(x^3) = x^{3/2}.$

c. $f(g(h(x))) = f(g(\sqrt{x})) = f(\sin(\sqrt{x})) = (\sin(\sqrt{x}))^3.$

d The domain of $g(f(x))$ is $\mathbb{R}$, because the domain of both functions is the set of all real numbers.

e. The range of $f(g(x))$ is $[-1, 1]$. This is because the range of g is $[-1, 1]$, and on the restricted domain $[-1, 1]$, the range of f is also $[-1, 1]$.

20

a. If $g(x) = x^2 + 1$ and $f(x) = \sin x$, then $f(g(x)) = f(x^2 + 1) = \sin(x^2 + 1).$

b. If $g(x) = x^2 - 4$ and $f(x) = x^{-3}$ then $f(g(x)) = f(x^2 - 4) = (x^2 - 4)^{-3}.$

c. If $g(x) = \cos 2x$ and $f(x) = e^x$, then $f(g(x)) = f(\cos 2x) = e^{\cos 2x}.$

21

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h} = \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \frac{2hx + h^2 - 2h}{h} = 2x + h - 2.\end{aligned}$$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - 2x - (a^2 - 2a)}{x - a} = \frac{(x^2 - a^2) - 2(x - a)}{x - a} = \frac{(x - a)(x + a) - 2(x - a)}{x - a} = x + a - 2.$$

22 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{4 - 5(x+h) - (4 - 5x)}{h} = \dfrac{4 - 5x - 5h - 4 + 5x}{h} = -\dfrac{5h}{h} = -5.$

$\dfrac{f(x) - f(a)}{x - a} = \dfrac{4 - 5x - (4 - 5a)}{x - a} = -\dfrac{5(x - a)}{x - a} = -5.$

23

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 2 - (x^3 + 2)}{h} = \frac{x^2 + 3x^2h + 3xh^2 + h^3 + 2 - x^3 - 2}{h} \\ &= \frac{h(3x^2 + 3xh + h^2)}{h} = 3x^2 + 3xh + h^2.\end{aligned}$$

$$\frac{f(x) - f(a)}{x - a} = \frac{x^3 + 2 - (a^3 + 2)}{x - a} = \frac{x^3 - a^3}{x - a} = \frac{(x - a)(x^2 + ax + a^2)}{x - a} = x^2 + ax + a^2.$$

24

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{7}{x+h+3}-\frac{7}{x+3}}{h} = \frac{\frac{7x+21-(7x+7h+21)}{(x+3)(x+h+3)}}{h}$$
$$= -\frac{7h}{(h)(x+3)(x+h+3)} = -\frac{7}{(x+3)(x+h+3)}.$$

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{7}{x+3}-\frac{7}{a+3}}{x-a} = \frac{\frac{7a+21-(7x+21)}{(x+3)(a+3)}}{x-a} = -\frac{7(x-a)}{(x-a)(x+3)(a+3)} = -\frac{7}{(x+3)(a+3)}.$$

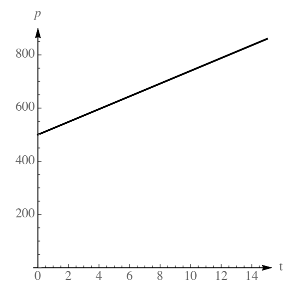
25

a. This line has slope $\frac{2-(-3)}{4-2} = \frac{5}{2}$. Therefore the equation of the line is $y-2 = \frac{5}{2}(x-4)$, so $y = \frac{5}{2}x - 8$.

b. This line has the form $y = \frac{3}{4}x + b$, and because $(-4, 0)$ is on the line, $0 = (3/4)(-4) + b$, so $b = 3$. Thus the equation of the line is given by $y = \frac{3}{4}x + 3$.

c. This line has slope $\frac{0-(-2)}{4-0} = \frac{1}{2}$, and the y-intercept is given to be -2, so the equation of this line is $y = \frac{1}{2}x - 2$.

26 If t is the number of years **after** 2018, then $p(t) = 24t + 500$. Because the year 2033 is 15 years after 2018, the population is predicted by $p(15) = 24 \cdot 15 + 500 = 360 + 500 = 860$.



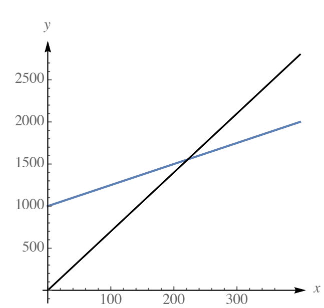
27 We are looking for the line between the points $(0, 212)$ and $(6000, 200)$. The slope is $\frac{212-200}{0-6000} = -\frac{12}{6000} = -\frac{1}{500}$. Because the intercept is given, we deduce that the line is $B = f(a) = -\frac{1}{500}a + 212$.

28

a. The cost of producing x books is $C(x) = 1000 + 2.5x$.

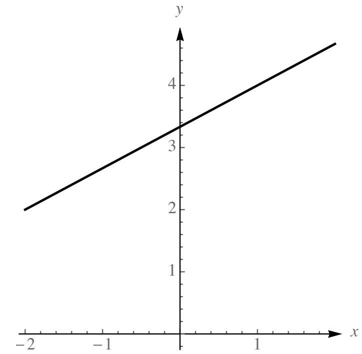
b. The revenue generated by selling x books is $R(x) = 7x$.

c. The break-even point is where $R(x) = C(x)$. This is where $7x = 1000 + 2.5x$, or $4.5x = 1000$. So $x = \frac{1000}{4.5} \approx 222$.

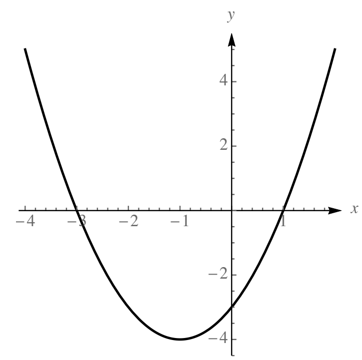


29

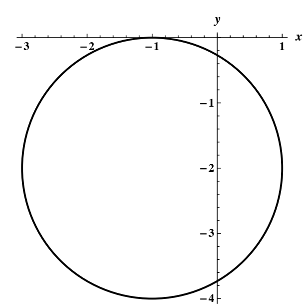
a. This is a straight line with slope 2/3 and y-intercept 10/3.



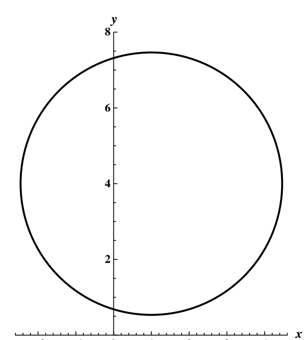
b. Completing the square gives $y = (x^2 + 2x + 1) - 4$, or $y = (x+1)^2 - 4$, so this is the standard parabola shifted one unit to the left and down 4 units.



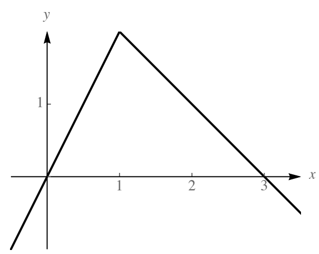
c. Completing the square, we have $x^2 + 2x + 1 + y^2 + 4y + 4 = -1 + 1 + 4$, so we have $(x+1)^2 + (y+2)^2 = 4$, a circle of radius 2 centered at $(-1, -2)$.



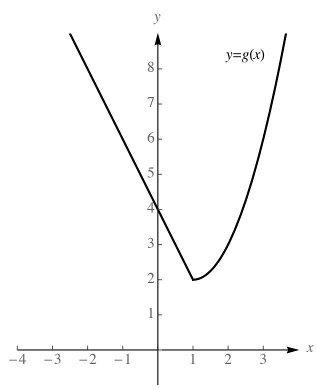
d. Completing the square, we have $x^2 - 2x + 1 + y^2 - 8y + 16 = -5 + 1 + 16$, or $(x-1)^2 + (y-4)^2 = 12$, which is a circle of radius $\sqrt{12}$ centered at $(1, 4)$.



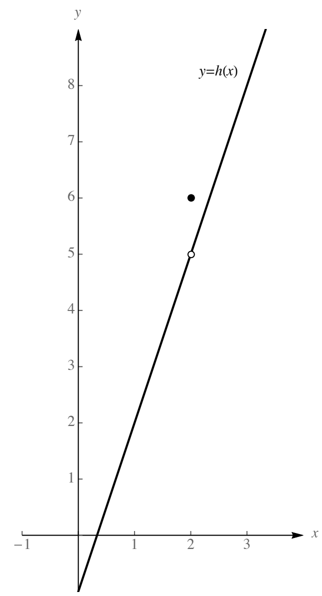
30



31

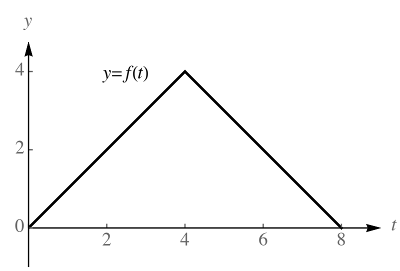


32



33

a.



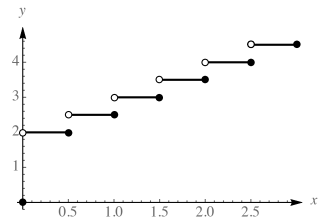
b. $A(2) = \dfrac{1}{2} \cdot 2 \cdot 2 = 2$ and $A(6) = 16 - \dfrac{1}{2} \cdot 2 \cdot 2 = 14$.

c. Note that for $0 \leq x \leq 4$, the area is that of a triangle with base x and height x. For $4 \leq x \leq 8$, the area is given by the difference of 16 (the total area under the curve from 0 to 8) minus the area of a triangle with base $8 - x$ and height $8 - x$. So the area for x in that range is $16 - \dfrac{(8-x)^2}{2} = 16 - 32 + 8x - \dfrac{x^2}{2}$. Therefore,

$$A(x) = \begin{cases} \frac{x^2}{2} & \text{if } 0 \leq x \leq 4 \\ -\frac{x^2}{2} + 8x - 16 & \text{if } 4 \leq x \leq 8. \end{cases}$$

34

The function is a piecewise step function which jumps up by one every half-hour step.

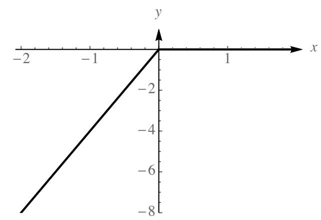


35

Because $|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x \geq 0, \end{cases}$

we have

$2(x - |x|) = \begin{cases} 2(x - (-x)) = 4x & \text{if } x < 0; \\ 2(x - x) = 0 & \text{if } x \geq 0. \end{cases}$



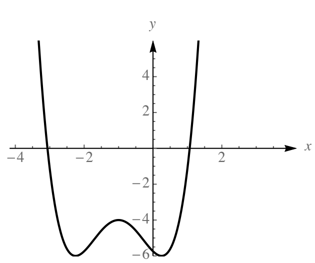
36 To solve $x^{1/3} = x^{1/4}$ we raise each side to the 12th power, yielding $x^4 = x^3$. This gives $x^4 - x^3 = 0$, or $x^3(x-1) = 0$, so the only solutions are $x = 0$ and $x = 1$ (which can be easily verified as solutions.) Between 0 and 1, $x^{1/4} > x^{1/3}$, but for $x > 1$, $x^{1/3} > x^{1/4}$.

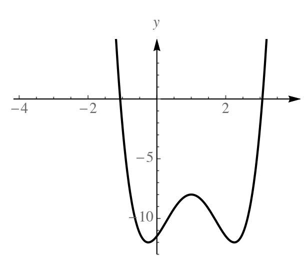
37 The domain of $x^{1/7}$ is the set of all real numbers, as is its range. The domain of $x^{1/4}$ is the set of non-negative real numbers, as is its range.

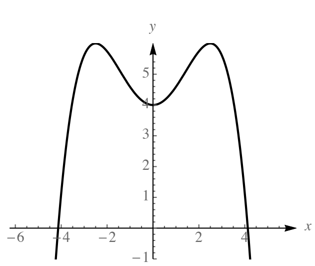
38 Completing the square in the second equation, we have $x^2 + y^2 - 7y + \frac{49}{4} = -8 + \frac{49}{4}$, which can be written as $x^2 + (y - (7/2))^2 = \frac{17}{4}$. Thus we have a circle of radius $\sqrt{17}/2$ centered at $(0, 7/2)$, along with the standard parabola. These intersect when $y = 7y - y^2 - 8$, which occurs for $y^2 - 6y + 8 = 0$, so for $y = 2$ and $y = 4$, with corresponding x values of ± 2 and $\pm\sqrt{2}$.

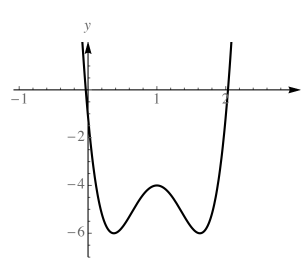
39 Completing the square, we can write $x^2 + 6x - 3 = x^2 + 6x + 9 - 3 - 9 = (x+3)^2 - 12$, so the graph is obtained by shifting $y = x^2$ 3 units right and 12 units down.

40

a.


b.


c.


d.


41

a. Because $f(-x) = \cos -3x = \cos 3x = f(x)$, this is an even function, and is symmetric about the y-axis.

b. Because $f(-x) = 3(-x)^4 - 3(-x)^2 + 1 = 3x^4 - 3x^2 + 1 = f(x)$, this is an even function, and is symmetric about the y-axis.

c. Because replacing x by $-x$ and/or replacing y by $-y$ gives the same equation, this represents a curve which is symmetric about the y-axis and about the origin and about the x-axis.

42 We have $8 = e^{4k}$, and so $\ln 8 = 4k$, so $k = \frac{\ln 8}{4}$.

43 If $\log x^2 + 3\log x = \log 32$, then $\log(x^2 \cdot x^3) = \log(32)$, so $x^5 = 32$ and $x = 2$. The answer does not depend on the base of the log.

44 $\ln 3x + \ln(x+2) = \ln(3x(x+2))$. This is zero when $3x(x+2) = 1$, or $3x^2 + 6x - 1 = 0$. By the quadratic formula, we have $\frac{-6\pm\sqrt{36-4\cdot 3\cdot(-1)}}{6} = \frac{-6\pm\sqrt{48}}{6} = -1 \pm \frac{2\sqrt{3}}{3}$. The original equation isn't defined for $-1 - \frac{2\sqrt{3}}{3}$ so the only solution is $-1 + \frac{2\sqrt{3}}{3}$.

45 If $3\ln(5t+4) = 12$, then $\ln(5t+4) = 4$, and $e^4 = 5t+4$. It then follows that $5t = e^4 - 4$ and so $t = \frac{e^4-4}{5}$.

46 If $7^{y-3} = 50$, then $\ln(7^{y-3}) = \ln 50$, so $(y-3)\ln 7 = \ln 50$ and $y - 3 = \frac{\ln 50}{\ln 7}$. Then $y = 3 + \frac{\ln 50}{\ln 7}$.

47 If $1 - 2\sin^2\theta = 0$, then $\sin^2\theta = \frac{1}{2}$, so $|\sin\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So $\theta = \frac{\pi}{4}, \frac{3\pi}{4} \frac{5\pi}{4} \frac{7\pi}{4}$.

48 First note that if θ is between $-\pi/2$ and $\pi/2$, that 2θ is then between $-\pi$ and π. If $\sin^2 2\theta = \frac{1}{2}$, then $|\sin 2\theta| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So $2\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$. Thus $\theta = -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$

49 First note that if θ is between $-\pi/2$ and $\pi/2$, that 2θ is then between $-\pi$ and π. If $4\cos^2 2\theta = 3$, then $\cos^2 2\theta = \frac{3}{4}$, and $|\cos 2\theta| = \frac{\sqrt{3}}{2}$. Thus $2\theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$, and $\theta = \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}$.

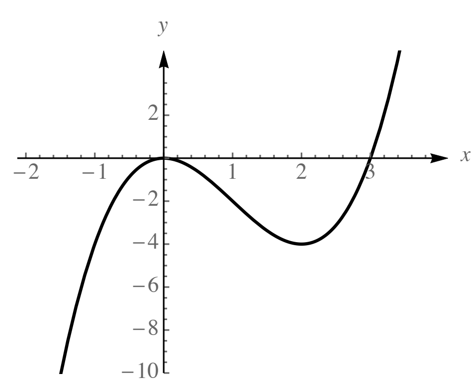
50 First note that if θ is between 0 and π that 3θ is then between 0 and 3π. If $\sqrt{2}\sin 3\theta + 1 = 2$, then $\sin 3\theta = \frac{1}{\sqrt{2}}$. Then $3\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$, so $\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$.

51 In 2010 (when $t = 0$), the population is $P(0) = 100$. So we are seeking t so that $200 = 100e^{t/50}$, or $e^{t/50} = 2$. Taking the natural logarithm of both sides yields $\frac{t}{50} = \ln 2$, or $t = 50\ln 2 \approx 35$ years.

52 Curve A is $y = 3^{-x}$, curve B is $y = 2^x$, and curve C is $y = -\ln x$.

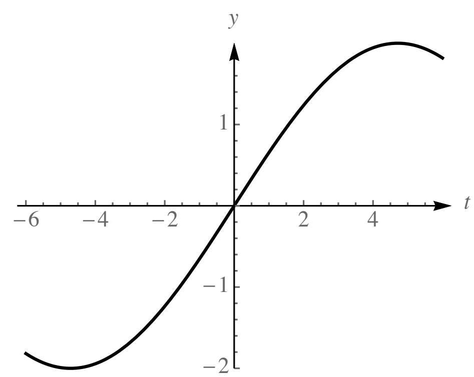
53

By graphing, it is clear that this function is not one-to-one on its whole domain, but it is one-to-one on the interval $(-\infty, 0]$, on the interval $[0, 2]$, and on the interval $[2, \infty)$, so it would have an inverse if we restricted it to any of these particular intervals.



54

This function is a stretched version of the sine function, it is one-to-one on the interval $[-3\pi/2, 3\pi/2]$ (and on other intervals as well ...)



55 Switching x and y gives $x = 6 - 4y$. Then $x - 6 = -4y$, so

$$y = \frac{x-6}{-4} = \frac{6-x}{4} = -\frac{1}{4}x + \frac{3}{2}.$$

56 Switching x and y gives $x = 3y - 4$, so $x + 4 = 3y$ and $y = \frac{x+4}{3}$.

57 Completing the square gives $f(x) = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$. Switching the x and y and solving for y yields $(y-2)^2 = x - 1$, so $|y - 2| = \sqrt{x-1}$, and thus $y = f^{-1}(x) = 2 + \sqrt{x-1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 2$, so the range of f^{-1} must also consist of numbers greater than or equal to 2).

58 Switching x and y gives $x = \frac{4y^2}{y^2+10}$. Then $x(y^2+10) = 4y^2$, so $xy^2 - 4y^2 = -10x$. Then $y^2(x-4) = -10x$, so $y^2 = \frac{10x}{4-x}$, and $|y| = \sqrt{\frac{10x}{4-x}}$, so $y = \sqrt{\frac{10x}{4-x}}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0).

59 Switching x and y gives $x = 3y^2 + 1$, so $3y^2 = x - 1$, and $y^2 = \frac{x-1}{3}$. Then $|y| = \sqrt{\frac{x-1}{3}}$, and $y = -\sqrt{\frac{x-1}{3}}$ (we choose the "−" rather than the "+" because the domain of f is $x \leq 0$, so the range of f^{-1} must also consist of numbers less than or equal to 0).

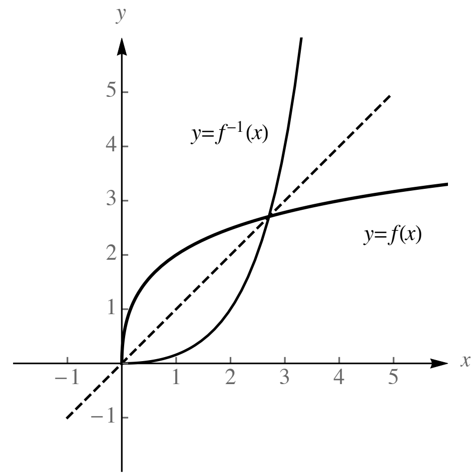
60 If $y = 1/x^2$, then switching x and y gives $x = 1/y^2$, so $y = f^{-1}(x) = 1/\sqrt{x}$.

61 Switching x and y gives $x = e^{y^2+1}$. Then $\ln x = y^2 + 1$, so $y^2 = \ln x - 1$, and $|y| = \sqrt{\ln x - 1}$, so $y = \sqrt{\ln x - 1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0).

62 Switching x and y gives $x = \ln(y^2 + 1)$, so $e^x = y^2 + 1$, and $y^2 = e^x - 1$, so $|y| = \sqrt{e^x - 1}$, so $y = \sqrt{e^x - 1}$ (we choose the "+" rather than the "−" because the domain of f is $x \geq 0$, so the range of f^{-1} must also consist of numbers greater than or equal to 0.)

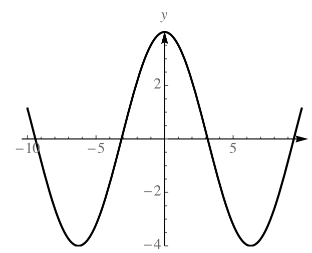
63

Switching x and y gives $x = \frac{6\sqrt{y}}{\sqrt{y}+2}$. Then $x\sqrt{y} + 2x = 6\sqrt{y}$, and so $x\sqrt{y} - 6\sqrt{y} = -2x$. Then $\sqrt{y}(x-6) = -2x$, so $\sqrt{y} = \frac{2x}{6-x}$, and $y = \left(\frac{2x}{6-x}\right)^2 = \frac{4x^2}{(6-x)^2}$. Because the range of f must match the domain of f^{-1}, we must restrict our inverse function to $[0, 6]$.

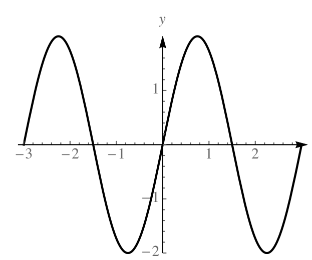


64

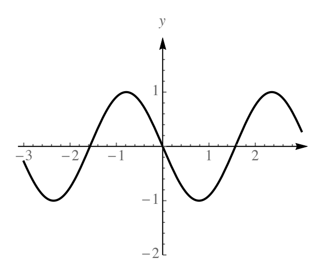
a. This function has period $\frac{2\pi}{1/2} = 4\pi$ and amplitude 4.



b. This function has period $\frac{2\pi}{2\pi/3} = 3$ and amplitude 2.



c. This function has period $\frac{2\pi}{2} = \pi$ and amplitude 1. Compared to the ordinary cosine function it is compressed horizontally, flipped about the x-axis, and shifted $\pi/4$ units to the right.



65

a. We need to scale the ordinary cosine function so that its period is 6, and then shift it 3 units to the right, and multiply it by 2. So the function we seek is $y = 2\cos((\pi/3)(t-3)) = -2\cos(\pi t/3)$.

b. We need to scale the ordinary cosine function so that its period is 24, and then shift it to the right 6 units. We then need to change the amplitude to be half the difference between the maximum and minimum, which would be 5. Then finally we need to shift the whole thing up by 15 units. The function we seek is thus $y = 15 + 5\cos((\pi/12)(t-6)) = 15 + 5\sin(\pi t/12)$.

66 The pictured function has a period of π, an amplitude of 2, and a maximum of 3 and a minimum of -1. It can be described by $y = 1 + 2\cos(2(x - \pi/2))$.

67

a. $-\sin x$ is pictured in F.

b. $\cos 2x$ is pictured in E.

c. $\tan(x/2)$ is pictured in D.

d. $-\sec x$ is pictured in B.

e. $\cot 2x$ is pictured in C.

f. $\sin^2 x$ is pictured in A.

68 If $\sec x = 2$, then $\cos x = \frac{1}{2}$. This occurs for $x = -\pi/3$ and $x = \pi/3$, so the intersection points are $(-\pi/3, 2)$ and $(\pi/3, 2)$.

69 $\sin x = -\frac{1}{2}$ for $x = 7\pi/6$ and for $x = 11\pi/6$, so the intersection points are $(7\pi/6, -1/2)$ and $(11\pi/6, -1/2)$.

70 Note that $\frac{5\pi}{8} = \frac{5\pi/4}{2}$. Using a half-angle identity,

$$\sin\left(\frac{5\pi/4}{2}\right) = \sqrt{\frac{1-\cos 5\pi/4}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

71 Note that $\frac{7\pi}{8} = \frac{7\pi/4}{2}$. Using the half-angle identity,

$$\cos\left(\frac{7\pi/4}{2}\right) = -\sqrt{\frac{1+\cos 7\pi/4}{2}} = -\sqrt{\frac{1+\sqrt{2}/2}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}} = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

72 Because $\sin(\pi/3) = \sqrt{3}/2$, $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

73 Because $\cos(\pi/6) = \sqrt{3}/2$, $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

74 Because $\cos(2\pi/3) = -1/2$, $\cos^{-1}(-1/2) = 2\pi/3$.

75 Because $\sin(-\pi/2) = -1$, $\sin^{-1}(-1) = -\pi/2$.

76 $\cos(\cos^{-1}(-1)) = \cos(\pi) = -1$.

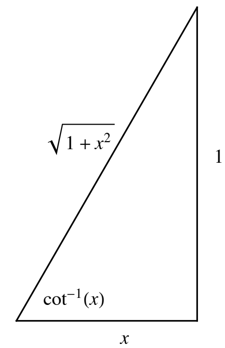
77 $\sin(\sin^{-1}(x)) = x$, for all x in the domain of the inverse sine function.

78 $\cos^{-1}(\sin 3\pi) = \cos^{-1}(0) = \pi/2$.

79 If $\theta = \sin^{-1}(12/13)$, then $0 < \theta < \pi/2$, and $\sin\theta = 12/13$. Then (using the Pythagorean identity) we can deduce that $\cos\theta = 5/13$. It must follow that $\tan\theta = 12/5$, $\cot\theta = 5/12$, $\sec\theta = 13/5$, and $\csc\theta = 13/12$.

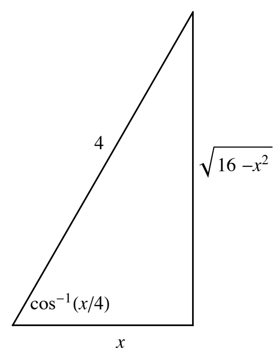
80

$$\csc(\cot^{-1} x) = \frac{\text{hypotenuse}}{\text{side opposite of } \cot^{-1} x} = \sqrt{1+x^2}$$



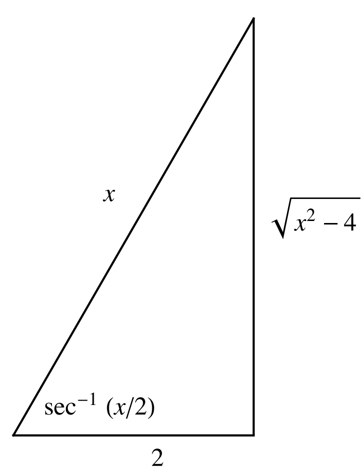
81

$$\sin(\cos^{-1}(x/4)) = \frac{\text{side opposite of } \cos^{-1}(x/4)}{\text{hypotenuse}} = \frac{\sqrt{16-x^2}}{4}.$$



82

$$\tan(\sec^{-1}(x/2)) = \frac{\text{side opposite of } \sec^{-1}(x/2)}{\text{side adjacent to } \sec^{-1}(x/2)} = \frac{\sqrt{x^2-4}}{2}.$$



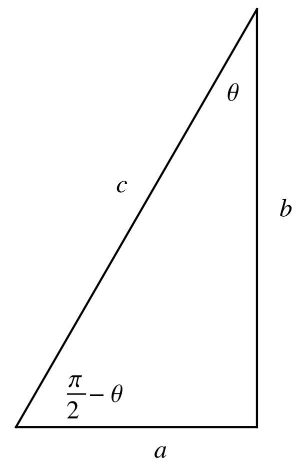
83

Note that

$$\tan\theta = \frac{a}{b} = \cot(\pi/2-\theta).$$

Thus,

$$\cot^{-1}(\tan\theta) = \cot^{-1}(\cot(\pi/2-\theta)) = \pi/2-\theta.$$



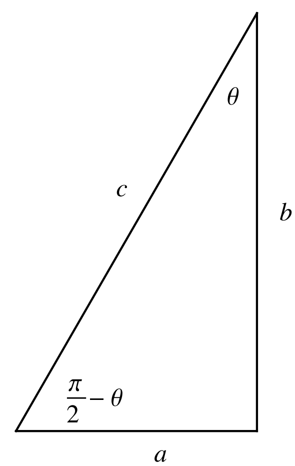
84

Note that

$$\sec\theta = \frac{c}{b} = \csc(\pi/2-\theta).$$

Thus,

$$\csc^{-1}(\sec\theta) = \csc^{-1}(\csc(\pi/2-\theta)) = \pi/2-\theta.$$



85 Let $\theta = \sin^{-1}(x)$. Then $\sin\theta = x$ and note that then $\sin(-\theta) = -\sin\theta = -x$, so $-\theta = \sin^{-1}(-x)$. Then $\sin^{-1}(x) + \sin^{-1}(-x) = \theta + -\theta = 0$.

86 We multiply the quantity $\frac{\sin\theta}{1+\cos\theta}$ by the conjugate of the denominator over itself:

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

87 Using the definition of the tangent function in terms of sine and cosine, we have:

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}.$$

If we divide both the numerator and denominator of this last expression by $\cos^2\theta$, we obtain

$$\frac{2\tan\theta}{1-\tan^2\theta}.$$

88 Let N be the north pole, and C the center of the given circle, and consider the angle CNP. This angle measures $\frac{\pi-\varphi}{2}$. (Note that the triangle CNP is isosceles.) Now consider the triangle NOX where O is the origin and X is the point $(x, 0)$. Using triangle NOX, we have $\tan\left(\frac{\pi-\varphi}{2}\right) = \frac{x}{2R}$, so $x = 2R\tan\left(\frac{\pi-\varphi}{2}\right)$.

89

a.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	5	14	30	55	91	140	204	285	385

b. The domain of this function consists of the positive integers.

c. Using trial and error and a calculator yields that $T(n) > 1000$ for the first time for $n = 14$.

90

a.

n	1	2	3	4	5	6	7	8	9	10
$S(n)$	1	3	6	10	15	21	28	36	45	55

b. The domain of this function consists of the positive integers. The range is a subset of the set of positive integers.

c. Using trial and error and a calculator yields that $S(n) > 1000$ for the first time for $n = 45$.

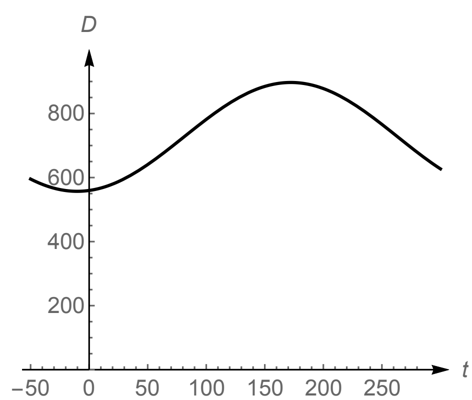
91 To find $s(t)$ note that we are seeking a periodic function with period 365, and with amplitude 87.5 (which is half of the number of minutes between 7:25 and 4:30). We need to shift the function 4 days plus one fourth of 365, which is about 95 days so that the max and min occur at $t = 4$ days and at half a year later. Also, to get the right value for the maximum and minimum, we need to multiply by negative one and add 117.5 (which represents 30 minutes plus half the amplitude, because $s = 0$ corresponds to 4:00 AM.) Thus we have

$$s(t) = 117.5 - 87.5 \sin\left(\frac{\pi}{182.5}(t - 95)\right).$$

A similar analysis leads to the formula

$$S(t) = 844.5 + 87.5 \sin\left(\frac{\pi}{182.5}(t - 67)\right).$$

The graph pictured shows $D(t) = S(t) - s(t)$, the length of day function, which has its max at the summer solstice which is about the 172nd day of the year, and its min at the winter solstice.



Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time $t = a$ and $t = b$ is the change in position divided by the elapsed time: $v_{\text{av}} = \dfrac{s(b) - s(a)}{b - a}$.

2.1.2 In order to compute the instantaneous velocity of the object at time $t = a$, we compute the average velocity over smaller and smaller time intervals of the form $[a, t]$, using the formula: $v_{\text{av}} = \dfrac{s(t) - s(a)}{t - a}$. We let t approach a. If the quantity $\dfrac{s(t) - s(a)}{t - a}$ approaches a limit as $t \to a$, then that limit is called the instantaneous velocity of the object at time $t = a$.

2.1.3 The average velocity is $\dfrac{s(3) - s(2)}{3 - 2} = 156 - 136 = 20$.

2.1.4 The average velocity is $\dfrac{s(4) - s(1)}{4 - 1} = \dfrac{144 - 84}{3} = \dfrac{60}{3} = 20$.

2.1.5

a. $\dfrac{s(2) - s(0)}{2 - 0} = \dfrac{72 - 0}{2} = 36$.

b. $\dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{66 - 0}{1.5} = 44$.

c. $\dfrac{s(1) - s(0)}{1 - 0} = \dfrac{52 - 0}{1} = 52$.

d. $\dfrac{s(.5) - s(0)}{.5 - 0} = \dfrac{30 - 0}{.5} = 60$.

2.1.6

a. $\dfrac{s(2.5) - s(.5)}{2.5 - .5} = \dfrac{150 - 46}{2} = 52$.

b. $\dfrac{s(2) - s(.5)}{2 - .5} = \dfrac{136 - 46}{1.5} = 60$.

c. $\dfrac{s(1.5) - s(.5)}{1.5 - .5} = \dfrac{114 - 46}{1} = 68$.

d. $\dfrac{s(1) - s(.5)}{1 - .5} = \dfrac{84 - 46}{.5} = 76$.

2.1.7 $\dfrac{s(1.01)-s(1)}{.01} = 47.84$, while $\dfrac{s(1.001)-s(1)}{.001} = 47.984$ and $\dfrac{s(1.0001)-s(1)}{.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.8 $\dfrac{s(2.01)-s(2)}{.01} = -4.16$, while $\dfrac{s(2.001)-s(2)}{.001} = -4.016$ and $\dfrac{s(2.0001)-s(2)}{.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.9 The slope of the secant line between points $(a, f(a))$ and $(b, f(b))$ is the ratio of the differences $f(b) - f(a)$ and $b - a$. Thus $m_{\text{sec}} = \dfrac{f(b)-f(a)}{b-a}$.

2.1.10 In order to compute the slope of the tangent line to the graph of $y = f(t)$ at $(a, f(a))$, we compute the slope of the secant line over smaller and smaller time intervals of the form $[a, t]$. Thus we consider $\dfrac{f(t)-f(a)}{t-a}$ and let $t \to a$. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve $y = f(t)$ at $t = a$.

2.1.11 Both problems involve the same mathematics, namely finding the limit as $t \to a$ of a quotient of differences of the form $\dfrac{g(t)-g(a)}{t-a}$ for some function g.

2.1.12 Note that $f(2) = 64$.

a. i. $f(0.5) = 28$. So the slope of the secant line is $\frac{28-64}{0.5-2} = \frac{-36}{-3/2} = 24$.

ii. $f(1.9) \approx 63.84$. So the slope of the secant line is about $\frac{63.84-64}{1.9-2} = 1.6$.

iii. $f(1.99) \approx 63.9984$. So the slope of the secant line is about $\frac{63.9984-64}{1.99-2} = 0.16$.

iv. $f(1.999) \approx 63.999984$. So the slope of the secant line is about $\frac{63.999984-64}{1.999-2} = 0.016$.

v. $f(1.9999) \approx 63.99999984$. So the slope of the secant line is about $\frac{63.99999984-64}{1.9999-2} = 0.0016$.

b. A good guess is that the limit is 0.

c. The slope of the tangent line is the limit of the slopes of the secant lines, so it is also 0.

2.1.13

a. Over $[1, 4]$, we have $v_{\text{av}} = \dfrac{s(4)-s(1)}{4-1} = \dfrac{256-112}{3} = 48$.

b. Over $[1, 3]$, we have $v_{\text{av}} = \dfrac{s(3)-s(1)}{3-1} = \dfrac{240-112}{2} = 64$.

c. Over $[1, 2]$, we have $v_{\text{av}} = \dfrac{s(2)-s(1)}{2-1} = \dfrac{192-112}{1} = 80$.

d. Over $[1, 1+h]$, we have

$$v_{\text{av}} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{-16(1+h)^2 + 128(1+h) - (112)}{h} = \frac{-16h^2 - 32h + 128h}{h}$$
$$= \frac{h(-16h+96)}{h} = 96 - 16h = 16(6-h).$$

2.1.14

a. Over $[0, 3]$, we have $v_{\text{av}} = \dfrac{s(3)-s(0)}{3-0} = \dfrac{65.9-20}{3} = 15.3$.

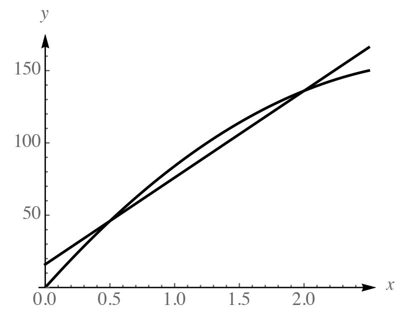
b. Over $[0, 2]$, we have $v_{\text{av}} = \dfrac{s(2)-s(0)}{2-0} = \dfrac{60.4-20}{2} = 20.2$.

c. Over $[0, 1]$, we have $v_{\text{av}} = \dfrac{s(1) - s(0)}{1 - 0} = \dfrac{45.1 - 20}{1} = 25.1.$

d. Over $[0, h]$, we have

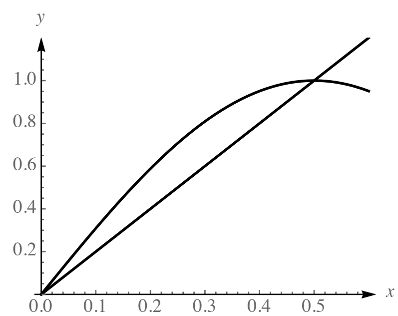
$$\begin{aligned} v_{\text{av}} &= \frac{s(h) - s(0)}{h - 0} = \frac{-4.9h^2 + 30h + 20 - 20}{h} \\ &= \frac{(h)(-4.9h + 30)}{h} = -4.9h + 30. \end{aligned}$$

2.1.15



The slope of the secant line is given by $\dfrac{s(2) - s(0.5)}{2 - 0.5} = \dfrac{136 - 46}{1.5} = 60$. This represents the average velocity of the object over the time interval $[0.5, 2]$.

2.1.16



The slope of the secant line is given by $\dfrac{s(0.5) - s(0)}{0.5 - 0} = \dfrac{1}{0.5} = 2$. This represents the average velocity of the object over the time interval $[0, 0.5]$.

2.1.17

Time Interval	$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.18

Time Interval	$[2, 3]$	$[2, 2.25]$	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.19

Time Interval	$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[2.99999, 3]$
Average Velocity	20	5.6	4.16	4.016	4.002	4.0002

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	$[\pi/2, \pi]$	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-1.90986	-0.149875	-0.0149999	-0.0015	-0.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	$[3, 3.1]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
Average Velocity	-17.6	-16.16	-16.016	-16.002

The instantaneous velocity appears to be -16 ft/s.

2.1.22

Time Interval	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Average Velocity	-19.9667	-19.9997	-20.0000	-20.0000

The instantaneous velocity appears to be -20 ft/s.

2.1.23

Time Interval	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Average Velocity	79.468	79.995	80.000	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	$[0, 1]$	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$
Average Velocity	-10	-18.1818	-19.802	-19.98

The instantaneous velocity appears to be -20 ft/s.

2.1.25

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x **Interval**	$[\pi/2, \pi/2 + .1]$	$[\pi/2, \pi/2 + .01]$	$[\pi/2, \pi/2 + .001]$	$[\pi/2, \pi/2 + .0001]$
Slope of Secant Line	-2.995	-2.99995	-3.0000	-3.0000

The slope of the tangent line appears to be -3.

2.1.27

x **Interval**	$[0, 0.1]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
Slope of the Secant Line	1.05171	1.00502	1.0005	1.00005

The slope of the tangent line appears to be 1.

2.1.28

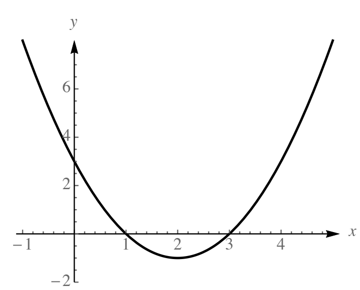
x **Interval**	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex $(2, -1)$.

b. At $(2, -1)$ the function has tangent line with slope 0.



c.

x **Interval**	$[2, 2.1]$	$[2, 2.01]$	$[2, 2.001]$	$[2, 2.0001]$
Slope of the Secant Line	0.1	0.01	0.001	0.0001

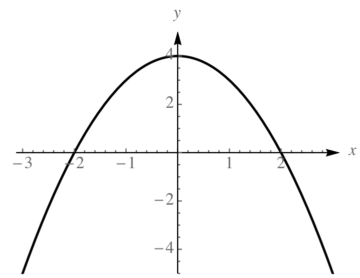
The slope of the tangent line at $(2, -1)$ appears to be 0.

2.1.30

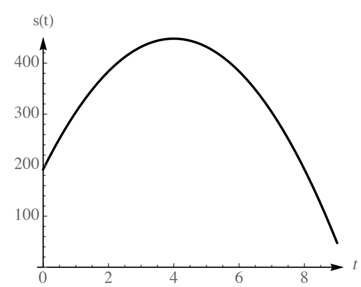
a. Note that the graph is a parabola with vertex $(0, 4)$.

b. At $(0, 4)$ the function has a tangent line with slope 0.

c. This is true for this function – because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus $f(0+h) = 4-(0+h)^2 = 4-(-h)^2 = f(0-h)$. So the slope of any such secant line is $\dfrac{4-h^2-(4-h^2)}{h-(-h)} = \dfrac{0}{2h} = 0$.



2.1.31



a. Note that the graph is a parabola with vertex $(4, 448)$.

b. At $(4, 448)$ the function has tangent line with slope 0, so $a = 4$.

c.

x **Interval**	$[4, 4.1]$	$[4, 4.01]$	$[4, 4.001]$	$[4, 4.0001]$
Slope of the Secant Line	-1.6	$-.16$	$-.016$	$-.0016$

The slopes of the secant lines appear to be approaching zero.

d. On the interval $[0, 4)$ the instantaneous velocity of the projectile is positive.

e. On the interval $(4, 9]$ the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when $s(t) = 96$. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

b.

t **Interval**	$[\sqrt{6}-.1, \sqrt{6}]$	$[\sqrt{6}-.01, \sqrt{6}]$	$[\sqrt{6}-.001, \sqrt{6}]$	$[\sqrt{6}-.0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx 0.63662.$$

For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + .5) - f(\pi/2)}{(\pi/2 + .5) - (\pi/2)} = -\frac{\cos(\pi/2 + .5)}{.5} \approx 0.958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + .05) - f(\pi/2)}{(\pi/2 + .05) - (\pi/2)} = -\frac{\cos(\pi/2 + .05)}{.05} \approx 0.999583.$$

Computing one more slope of a secant line:

$$m_{\text{sec}} = \frac{f(\pi/2 + .01) - f(\pi/2)}{(\pi/2 + .01) - (\pi/2)} = -\frac{\cos(\pi/2 + .01)}{.01} \approx 0.999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definition of a Limit

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3

a. $h(2) = 5$.

b. $\lim_{x\to 2} h(x) = 3$.

c. $h(4)$ does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.4

a. $g(0) = 0$.

b. $\lim_{x\to 0} g(x) = 1$.

c. $g(1) = 2$.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.5

a. $f(1) = -1$.

b. $\lim_{x\to 1} f(x) = 1$.

c. $f(0) = 2$.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.6

a. $f(2) = 2$.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.7

a.

x	1.9	1.99	1.999	1.9999	2.1	2.01	2.001	2.0001
$f(x) = \dfrac{x^2 - 4}{x - 2}$	3.9	3.99	3.999	3.9999	4.1	4.01	4.001	4.0001

b. $\lim_{x\to 2} f(x) = 4$.

2.2.8

a.

x	0.9	0.99	0.999	0.9999	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	3.31	3.30301	3.003	3.0003

b. $\displaystyle\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

2.2.9

a.

t	8.9	8.99	8.999	9.1	9.01	9.001
$g(t) = \dfrac{t-9}{\sqrt{t}-3}$	5.98329	5.99833	5.99983	6.01662	6.00167	6.00017

b. $\displaystyle\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = 6.$

2.2.10

a.

x	0.01	0.001	0.0001	0.00001
$f(x) = (1+x)^{1/x}$	2.70481	2.71692	2.71815	2.71827

x	−0.01	−0.001	−0.0001	−0.00001
$f(x) = (1+x)^{1/x}$	2.732	2.71964	2.71842	2.71830

b. $\displaystyle\lim_{x\to 0}(1+x)^{1/x} \approx 2.718.$

c. $\displaystyle\lim_{x\to 0}(1+x)^{1/x} = e.$

2.2.11 Suppose the function f is defined for all x near a but greater than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then we write $\displaystyle\lim_{x\to a^+} f(x) = L$.

2.2.12 Suppose the function f is defined for all x near a but less than a. If $f(x)$ is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then we write $\displaystyle\lim_{x\to a^-} f(x) = L$.

2.2.13 It must be true that $L = M$.

2.2.14

a.

x	1.9	1.99	1.999	1.9999
$g(x) = \frac{x^3-4x}{8\|x-2\|}$	-0.92625	-0.99251	-0.99925	-0.99993

x	2.1	2.01	2.001	2.0001
$g(x) = \frac{x^3-4x}{8\|x-2\|}$	1.07625	1.00751	1.00075	1.00008

b. $\displaystyle\lim_{x\to 2^-} g(x) = -1$. $\displaystyle\lim_{x\to 2^+} g(x) = 1$. $\displaystyle\lim_{x\to 2} g(x)$ does not exist.

2.2.15

a. $f(1) = 0.$

b. $\displaystyle\lim_{x\to 1^-} f(x) = 1.$

c. $\displaystyle\lim_{x\to 1^+} f(x) = 0.$

d. $\displaystyle\lim_{x\to 1} f(x)$ does not exist, since the two one-sided limits aren't equal.

2.2.16 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

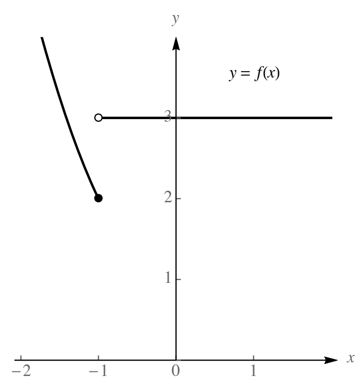
2.2.17

a. $f(1) = 3$.
b. $\lim_{x\to 1^-} f(x) = 2$.
c. $\lim_{x\to 1^+} f(x) = 2$.
d. $\lim_{x\to 1} f(x) = 2$.
e. $f(3) = 2$.
f. $\lim_{x\to 3^-} f(x) = 4$.
g. $\lim_{x\to 3^+} f(x) = 1$.
h. $\lim_{x\to 3} f(x)$ does not exist.
i. $f(2) = 3$.
j. $\lim_{x\to 2^-} f(x) = 3$.
k. $\lim_{x\to 2^+} f(x) = 3$.
l. $\lim_{x\to 2} f(x) = 3$.

2.2.18

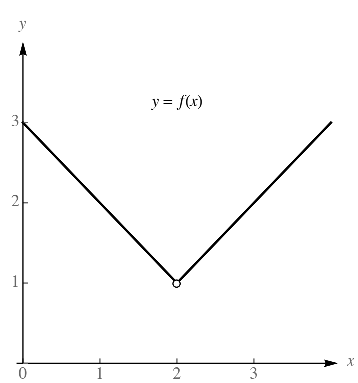
a. $g(2) = 3$.
b. $\lim_{x\to 2^-} g(x) = 2$.
c. $\lim_{x\to 2^+} g(x) = 3$.
d. $\lim_{x\to 2} g(x)$ does not exist.
e. $g(3) = 2$.
f. $\lim_{x\to 3^-} g(x) = 3$.
g. $\lim_{x\to 3^+} g(x) = 2$.
h. $g(4) = 3$.
i. $\lim_{x\to 4} g(x) = 3$.

2.2.19



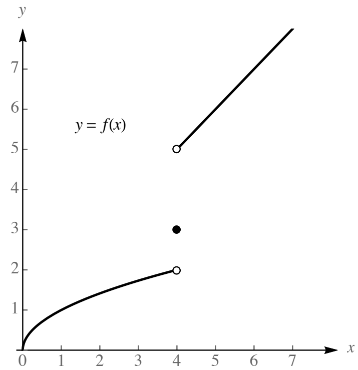
$f(-1) = 2$, $\lim_{x\to -1^-} f(x) = 2$, $\lim_{x\to -1^+} f(x) = 3$, $\lim_{x\to -1} f(x)$ does not exist.

2.2.20



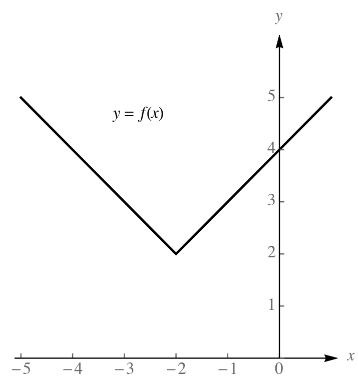
$f(2)$ is undefined. $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 1$, and $\lim_{x\to 2} f(x) = 1$.

2.2.21



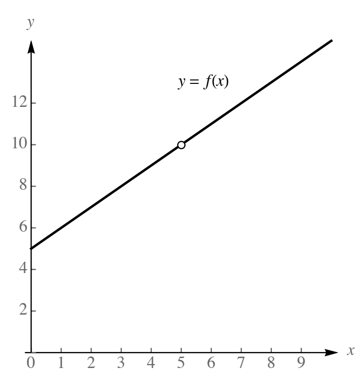
$f(4) = 3$, $\lim_{x\to 4^-} f(x) = 2$, $\lim_{x\to 4^+} f(x) = 5$, $\lim_{x\to 4} f(x)$ does not exist.

2.2.22



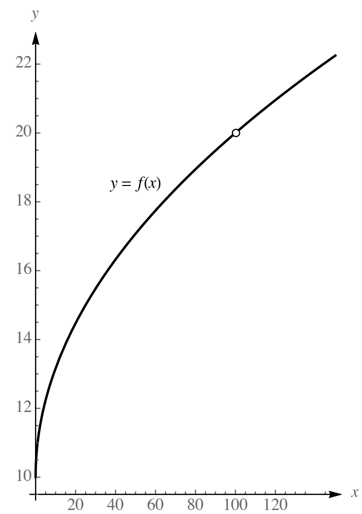
$f(-2) = 2$, $\lim_{x\to -2^-} f(x) = 2$, $\lim_{x\to -2^+} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$.

2.2.23



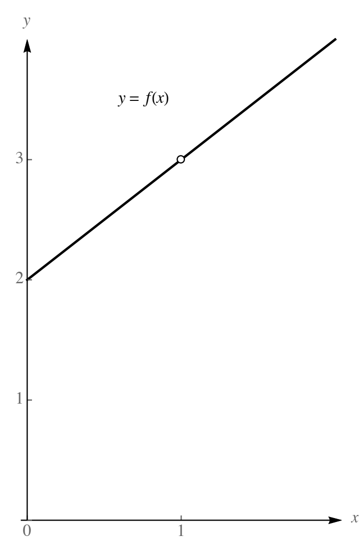
$f(5)$ does not exist. $\lim_{x\to 5^-} f(x) = 10$, $\lim_{x\to 5^+} f(x) = 10$, $\lim_{x\to 5} f(x) = 10$.

2.2.24



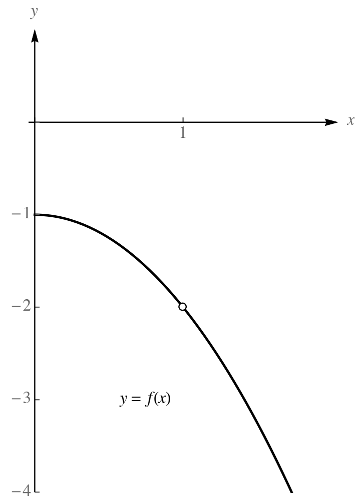
$f(100)$ does not exist. $\lim_{x\to 100^-} f(x) = 20$, $\lim_{x\to 100^+} f(x) = 20$, $\lim_{x\to 100} f(x) = 20$.

2.2.25



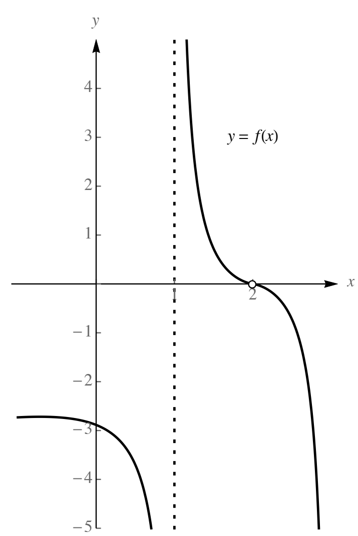
$f(1)$ does not exist. $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 3$, $\lim_{x\to 1} f(x) = 3$.

2.2.26



$f(1)$ does not exist. $\lim\limits_{x\to 1^-} f(x) = -2$, $\lim\limits_{x\to 1^+} f(x) = -2$, $\lim\limits_{x\to 1} f(x) = -2$.

2.2.27

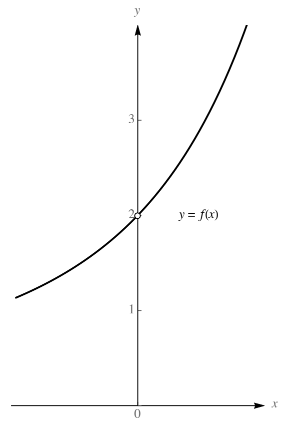


x	1.99	1.999	1.9999
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	0.0021715	0.00014476	0.000010857

x	2.0001	2.001	2.01
$f(x) = \frac{x-2}{\ln\lvert x-2\rvert}$	−0.000010857	−0.00014476	−0.0021715

From the graph and the table, the limit appears to be 0.

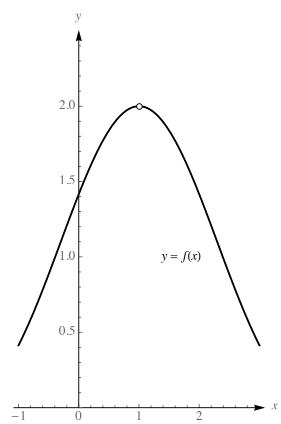
2.2.28



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.8731	1.98673	1.9987	2.0013	2.0134	2.1403

From both the graph and the table, the limit appears to be 2.

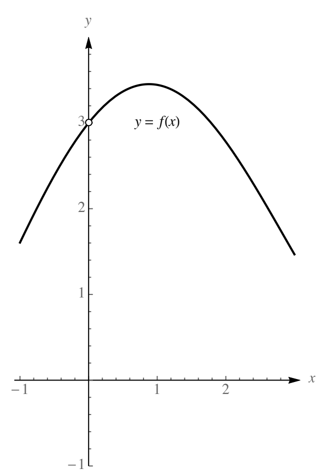
2.2.29



x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.993342	1.999933	1.999999	1.999999	1.999933	1.993342

From both the graph and the table, the limit appears to be 2.

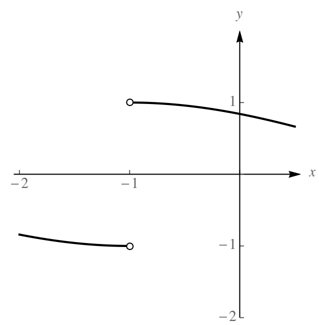
2.2.30



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.8951	2.99	2.999	3.001	3.0099	3.0949

From both the graph and the table, the limit appears to be 3.

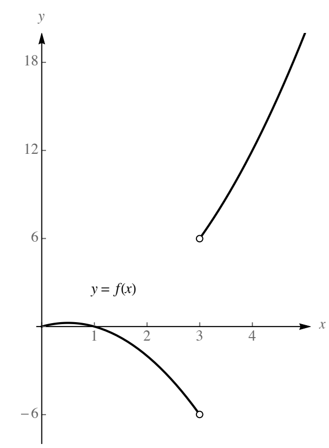
2.2.31



x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$	-0.9983342	-0.9999833	-0.9999998	0.9999998	0.9999833	0.9983342

From both the graph and the table, it appears that the limit does not exist.

2.2.32



x	2.9	2.99	2.999	3.001	3.01	3.1
$g(x)$	-5.51	-5.9501	-5.995001	6.005001	6.0501	6.51

From both the graph and the table, it appears that the limit does not exist.

2.2.33

a. False. In fact $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3}(x+3) = 6$.

b. False. For example, if $f(x) = \begin{cases} x^2 & \text{if } x \neq 0; \\ 5 & \text{if } x = 0 \end{cases}$ and if $a = 0$ then $f(a) = 5$ but $\lim\limits_{x\to a} f(x) = 0$.

c. False. For example, the limit in part a of this problem exists, even though the corresponding function is undefined at $a = 3$.

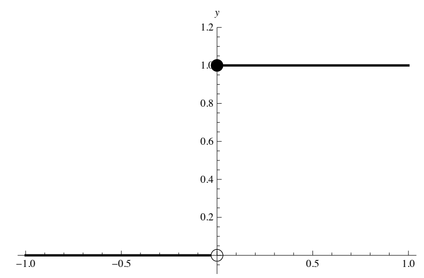
d. False. It is true that the limit of $\sqrt{x}$ as x approaches zero from the right is zero, but because the domain of $\sqrt{x}$ does not include any numbers to the left of zero, the two-sided limit doesn't exist.

e. True. Note that $\lim_{x\to\pi/2} \cos x = 0$ and $\lim_{x\to\pi/2} \sin x = 1$, so $\lim_{x\to\pi/2} \frac{\cos x}{\sin x} = \frac{0}{1} = 0$.

2.2.34

a. Note that H is piecewise constant.

b. $\lim_{x\to 0^-} H(x) = 0$, $\lim_{x\to 0^+} H(x) = 1$, and so $\lim_{x\to 0} H(x)$ does not exist.

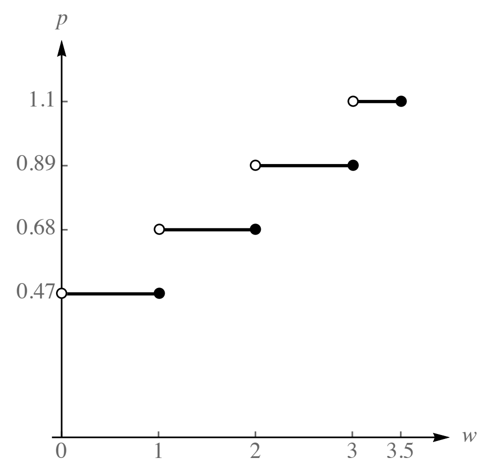


2.2.35

a. Note that the function is piecewise constant.

b. $\lim_{w\to 2.3} f(w) = .89$.

c. $\lim_{w\to 3^+} f(w) = 1.1$ corresponds to the fact that for any piece of mail that weighs slightly over 3 ounces, the postage will cost \$1.1 cents. $\lim_{w\to 3^-} f(w) = \$0.89$ corresponds to the fact that for any piece of mail that weighs slightly less than 3 ounces, the postage will cost 89 cents. Because the two one-sided limits are not equal, $\lim_{w\to 3} f(w)$ does not exist.



2.2.36

h	0.01	0.001	0.0001	−0.0001	−0.001	−0.01
$\frac{(1+2h)^{1/h}}{2e^{2+h}}$	0.48535	0.498504	0.49985	0.50015	0.501504	0.515367

$\lim_{h\to 0} \frac{(1+2h)^{1/h}}{2e^{2+h}} = \frac{1}{2}$.

2.2.37

x	1.37	1.47	1.57	1.58	1.68	1.78
$\frac{\cot 3x}{\cos x}$	3.44773	3.10016	3.00001	3.0008	3.11834	3.49316

$\lim_{x\to\pi/2} \frac{\cot 3x}{\cos x} = 3$.

2.2.38

x	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}}$	2.29222	2.02691	2.00267	1.99734	1.97357	1.75541

$\lim_{x\to 1} \frac{18(\sqrt[3]{x}-1)}{x^3-1} = 2$.

2.2.39

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$\frac{18(\sqrt[3]{x}-1)}{x^3-1}$	15.5803	15.9574	15.9957	16.0043	16.0427	16.4339

$$\lim_{x\to 1} \frac{9\sqrt{2x-x^4}-\sqrt[3]{x}}{1-x^{3/4}} = 16.$$

2.2.40

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\frac{6^x-3^x}{x\ln 16}$	0.24614	0.249639	0.249964	0.250036	0.250362	0.25364

$$\lim_{x\to 0} \frac{6^x-3^x}{x\ln 16} = \frac{1}{4}.$$

2.2.41

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{\ln(1+h)}{h}$	0.995033	0.9995	0.99995	1.00005	1.0005	1.00503

$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1.$$

2.2.42

h	0.01	0.001	0.0001	-0.0001	-0.001	-0.01
$\frac{4^h-1}{h\ln(h+2)}$	1.99954	1.99994	1.99999	2.00001	2.00006	2.00067

$$\lim_{h\to 0} \frac{4^h-1}{h\ln(h+2)} = 2.$$

2.2.43

a.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$
$f(x)=\sin(1/x)$	1	-1	1	-1	1	-1

If $x_n = \frac{2}{(2n+1)\pi}$, then $f(x_n) = (-1)^n$ where n is a non-negative integer.

b. As $x \to 0$, $1/x \to \infty$. So the values of $f(x)$ oscillate dramatically between -1 and 1.

c. $\lim_{x\to 0} \sin(1/x)$ does not exist.

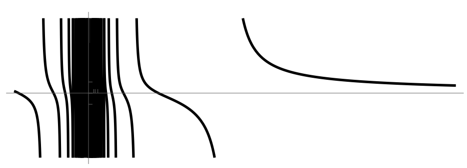
2.2.44

a.

x	$\frac{12}{\pi}$	$\frac{12}{3\pi}$	$\frac{12}{5\pi}$	$\frac{12}{7\pi}$	$\frac{12}{9\pi}$	$\frac{12}{11\pi}$
$f(x)=\tan(3/x)$	1	-1	1	-1	1	-1

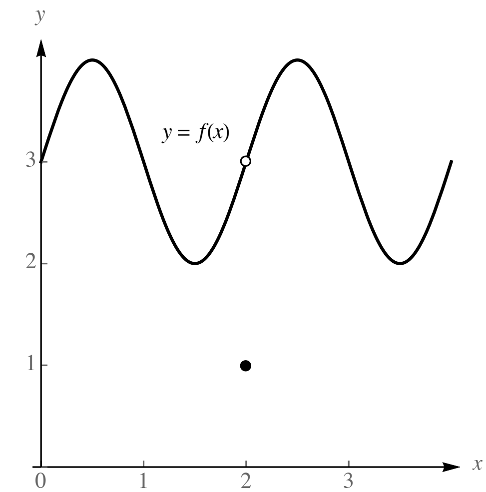
We have alternating 1's and -1's.

b. $\tan 3x$ alternates between 1 and -1 infinitely many times on $(0,h)$ for any $h>0$.

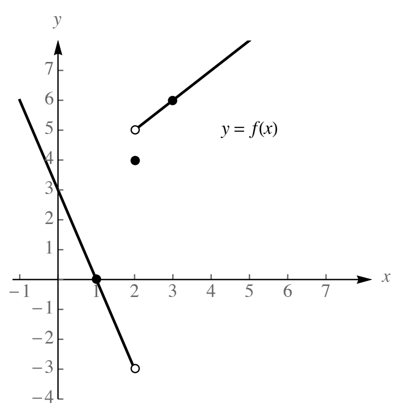


c. $\lim_{x\to 0} \tan(3/x)$ does not exist.

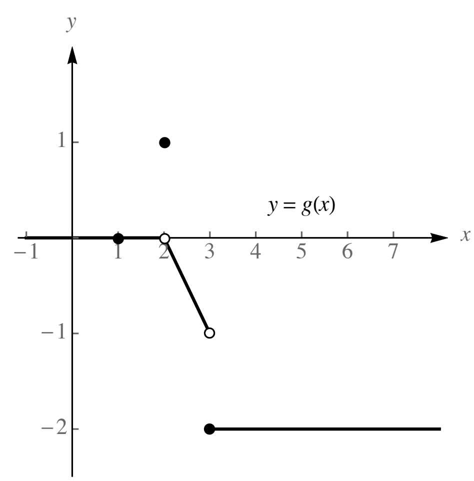
2.2.45



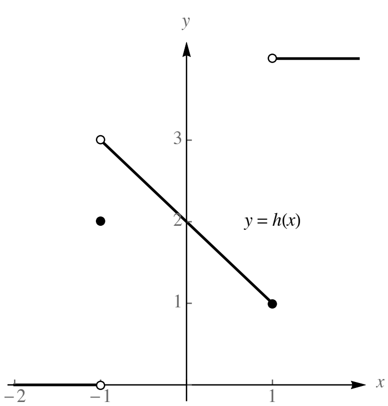
2.2.46



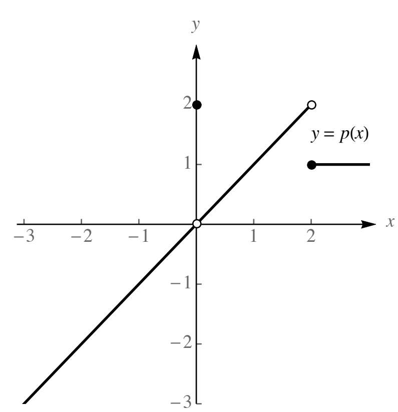
2.2.47



2.2.48



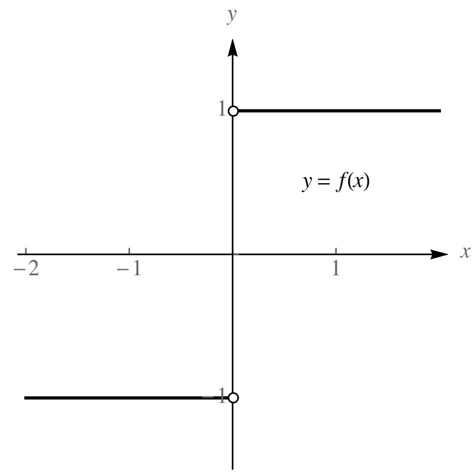
2.2.49



2.2.50

a. Note that $f(x) = \frac{|x|}{x}$ is undefined at 0, and $\lim_{x\to0^-} f(x) = -1$ and $\lim_{x\to0^+} f(x) = 1$.

b. $\lim_{x\to0} f(x)$ does not exist, since the two one-side limits aren't equal.



2.2.51

a. $\lim_{x\to-1^-} \lfloor x\rfloor = -2$, $\lim_{x\to-1^+} \lfloor x\rfloor = -1$, $\lim_{x\to2^-} \lfloor x\rfloor = 1$, $\lim_{x\to2^+} \lfloor x\rfloor = 2$.

b. $\lim_{x\to2.3^-} \lfloor x\rfloor = 2$, $\lim_{x\to2.3^+} \lfloor x\rfloor = 2$, $\lim_{x\to2.3} \lfloor x\rfloor = 2$.

c. In general, for an integer a, $\lim_{x\to a^-} \lfloor x\rfloor = a-1$ and $\lim_{x\to a^+} \lfloor x\rfloor = a$.

d. In general, if a is not an integer, $\lim_{x\to a^-} \lfloor x\rfloor = \lim_{x\to a^+} \lfloor x\rfloor = \lfloor a\rfloor$.

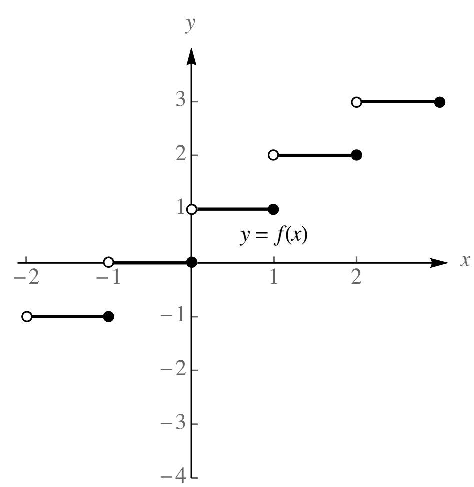
e. $\lim_{x\to a} \lfloor x\rfloor$ exists and is equal to $\lfloor a\rfloor$ for non-integers a.

2.2.52

a. Note that the graph is piecewise constant.

b. $\lim_{x\to2^-} \lceil x\rceil = 2$, $\lim_{x\to1^+} \lceil x\rceil = 2$, $\lim_{x\to1.5} \lceil x\rceil = 2$.

c. $\lim_{x\to a} \lceil x\rceil$ exists and is equal to $\lceil a\rceil$ for non-integers a.



2.2.53

a. Because of the symmetry about the y axis, we must have $\lim_{x\to-2^+} f(x) = 8$.

b. Because of the symmetry about the y axis, we must have $\lim_{x\to-2^-} f(x) = 5$.

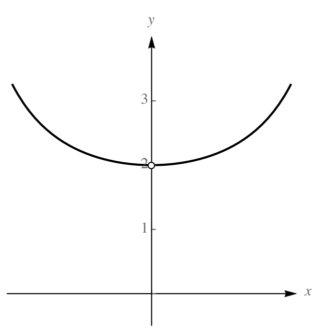
2.2.54

a. Because of the symmetry about the origin, we must have $\lim_{x\to -2^+} g(x) = -8$.

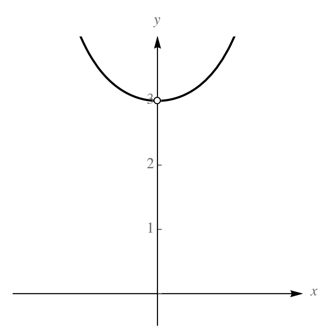
b. Because of the symmetry about the origin, we must have $\lim_{x\to -2^-} g(x) = -5$.

2.2.55

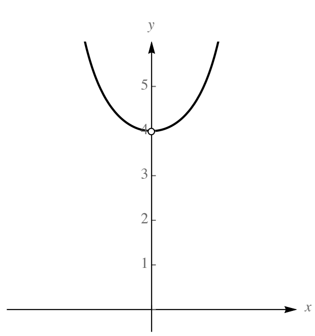
a.



$$\lim_{x\to 0}\frac{\tan 2x}{\sin x} = 2.$$



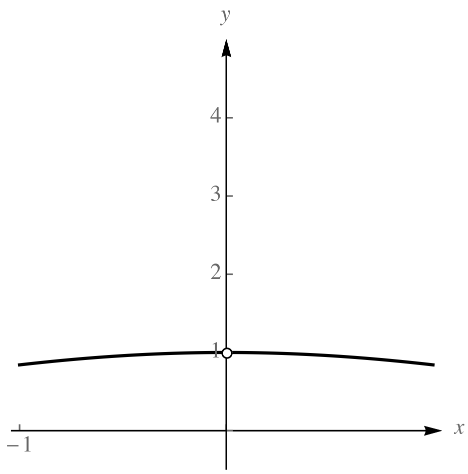
$$\lim_{x\to 0}\frac{\tan 3x}{\sin x} = 3.$$



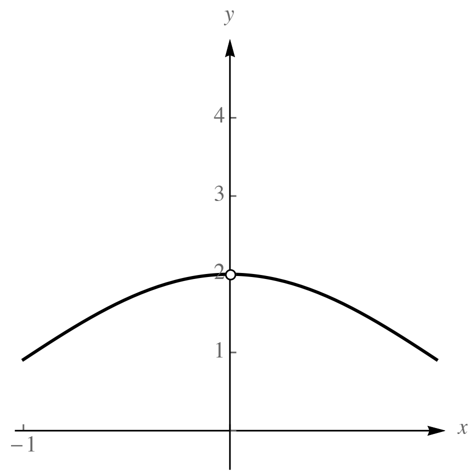
$$\lim_{x\to 0}\frac{\tan 4x}{\sin x} = 4.$$

b. It appears that $\lim_{x\to 0}\frac{\tan(px)}{\sin x} = p$.

2.2.56

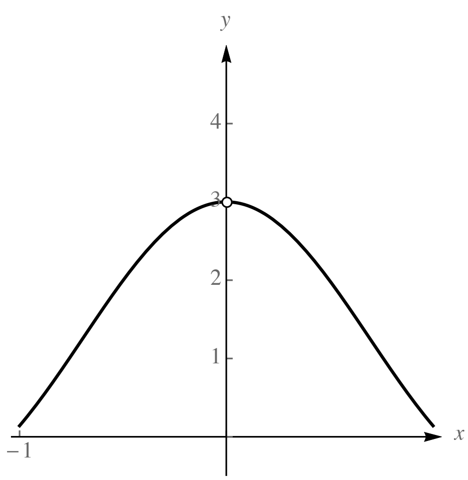


$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

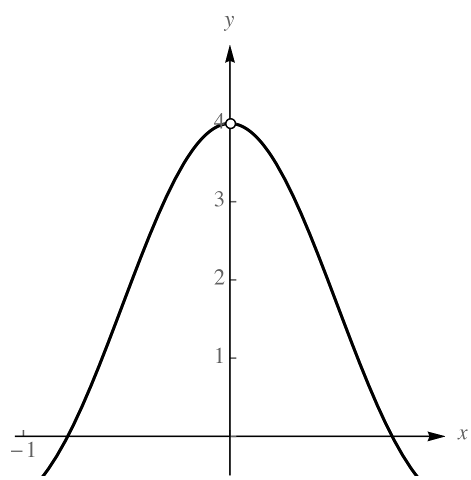


$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2.$$

a.



$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3.$$

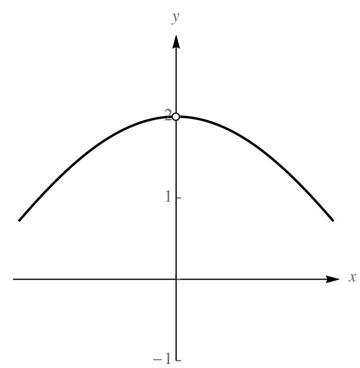


$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4.$$

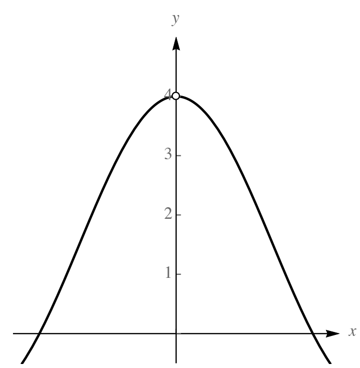
b. It appears that $\displaystyle\lim_{x\to 0} \frac{\sin(px)}{x} = p$.

2.2.57

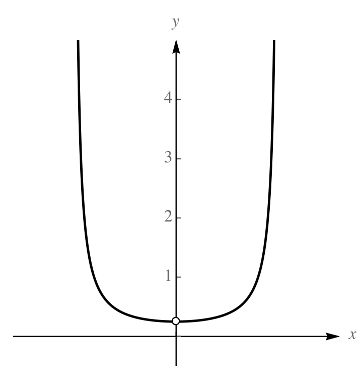
For $p = 8$ and $q = 2$, it appears that the limit is 4.



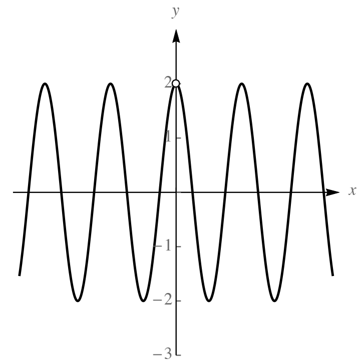
For $p = 12$ and $q = 3$, it appears that the limit is 4.



For $p = 4$ and $q = 16$, it appears that the limit is 1/4.



For $p = 100$ and $q = 50$, it appears that the limit is 2.



Conjecture: $\lim_{x\to 0} \dfrac{\sin px}{\sin qx} = \dfrac{p}{q}$.

2.3 Techniques for Computing Limits

2.3.1 If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $\lim_{x\to a} p(x) = \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$
$= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1 \lim_{x\to a} x + \lim_{x\to a} a_0$
$= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0. = p(a)$.

2.3.2 $\lim_{x\to 1}(x^3 + 3x^2 - 3x + 1) = 1 + 3 - 3 + 1 = 2$.

2.3.3 For a rational function $r(x)$, we have $\lim_{x\to a} r(x) = r(a)$ exactly for those numbers a which are in the domain of r. (Which are those for which the denominator isn't zero.)

2.3.4 $\lim_{x\to 4}\left(\dfrac{x^2 - 4x - 1}{3x - 1}\right) = \dfrac{16 - 16 - 1}{12 - 1} = -\dfrac{1}{11}$.

2.3.5 Because $\frac{x^2-7x+12}{x-3} = \frac{(x-3)(x-4)}{x-3} = x - 4$ (for $x \neq 3$), we can see that the graphs of these two functions are the same except that one is undefined at $x = 3$ and the other is a straight line that is defined everywhere. Thus the function $\frac{x^2-7x+12}{x-3}$ is a straight line except that it has a "hole" at $(3, -1)$. The two functions have the same limit as $x \to 3$, namely $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x - 3} = \lim_{x\to 3}(x - 4) = -1$.

2.3.6 $\lim_{x\to 5} \dfrac{4x^2 - 100}{x - 5} = \lim_{x\to 5} \dfrac{4(x-5)(x+5)}{x - 5} = \lim_{x\to 5} 4(x + 5) = 40$.

2.3.7 $\lim_{x\to 1} 4f(x) = 4\lim_{x\to 1} f(x) = 4 \cdot 8 = 32$. This follows from the Constant Multiple Law.

2.3.8 $\lim_{x\to 1} \dfrac{f(x)}{h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} h(x)} = \dfrac{8}{2} = 4$. This follows from the Quotient Law.

2.3.9 $\lim_{x\to 1}(f(x) - g(x)) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 8 - 3 = 5$. This follows from the Difference Law.

2.3.10 $\lim_{x\to 1} f(x)h(x) = \lim_{x\to 1} f(x) \cdot \lim_{x\to 1} h(x) = 8 \cdot 2 = 16$. This follows from the Product Law.

2.3.11 $\lim_{x\to 1} \dfrac{f(x)}{g(x) - h(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1}[g(x) - h(x)]} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - \lim_{x\to 1} h(x)} = \dfrac{8}{3 - 2} = 8$. This follows from the Quotient and Difference Laws.

2.3.12 $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3} = \sqrt[3]{\lim_{x\to 1}(f(x)g(x)+3)} = \sqrt[3]{\lim_{x\to 1} f(x)\cdot \lim_{x\to 1} g(x) + \lim_{x\to 1} 3} = \sqrt[3]{8\cdot 3+3} = \sqrt[3]{27} = 3$. This follows form the Root, Product, Sum and Constant Laws.

2.3.13 $\lim_{x\to 1} f(x)^{2/3} = \left(\lim_{x\to 1} f(x)\right)^{2/3} = 8^{2/3} = 2^2 = 4$. This follows from the Root and Power Laws.

2.3.14 If $p(x)$ is a polynomial, then $\lim_{x\to a^-} p(x) = \lim_{x\to a^+} p(x) = p(a)$.

2.3.15 $\lim_{x\to 0} g(x) = \lim_{x\to 0}(2x+1) = 1$, while $g(0) = 5$.

2.3.16 $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 4 = 4$, and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x+2) = 5$. Because the two one-sided limits differ, the two-sided limit doesn't exist.

2.3.17 If p and q are polynomials then $\lim_{x\to 0} \frac{p(x)}{q(x)} = \frac{\lim_{x\to 0} p(x)}{\lim_{x\to 0} q(x)} = \frac{p(0)}{q(0)}$. Because this quantity is given to be equal to 10, we have $\frac{p(0)}{2} = 10$, so $p(0) = 20$.

2.3.18 By a direct application of the squeeze theorem, $\lim_{x\to 2} g(x) = 5$.

2.3.19 $\lim_{x\to 4}(3x-7) = 3\lim_{x\to 4} x - 7 = 3\cdot 4 - 7 = 5$.

2.3.20 $\lim_{x\to 1}(-2x+5) = -2\lim_{x\to 1} x + 5 = -2\cdot 1 + 5 = 3$.

2.3.21 $\lim_{x\to -9}(5x) = 5\lim_{x\to -9} x = 5\cdot -9 = -45$.

2.3.22 $\lim_{x\to 6} 4 = 4$.

2.3.23 $\lim_{x\to 1}(2x^3-3x^2+4x+5) = \lim_{x\to 1} 2x^3 - \lim_{x\to 1} 3x^2 + \lim_{x\to 1} 4x + \lim_{x\to 1} 5 = 2(\lim_{x\to 1} x)^3 - 3(\lim_{x\to 1} x)^2 + 4(\lim_{x\to 1} x) + 5 = 2(1)^3 - 3(1)^2 + 4\cdot 1 + 5 = 8$.

2.3.24 $\lim_{t\to -2}(t^2+5t+7) = \lim_{t\to -2} t^2 + \lim_{t\to -2} 5t + \lim_{t\to -2} 7 = \left(\lim_{t\to -2} t\right)^2 + 5\lim_{t\to -2} t + 7 = (-2)^2 + 5\cdot(-2) + 7 = 1$.

2.3.25 $\lim_{x\to 1} \frac{5x^2+6x+1}{8x-4} = \frac{\lim_{x\to 1}(5x^2+6x+1)}{\lim_{x\to 1}(8x-4)} = \frac{5(\lim_{x\to 1} x)^2 + 6\lim_{x\to 1} x + \lim_{x\to 1} 1}{8\lim_{x\to 1} x - \lim_{x\to 1} 4} = \frac{5(1)^2+6\cdot 1+1}{8\cdot 1-4} = 3$.

2.3.26 $\lim_{t\to 3} \sqrt[3]{t^2-10} = \sqrt[3]{\lim_{t\to 3}(t^2-10)} = \sqrt[3]{\lim_{t\to 3} t^2 - \lim_{t\to 3} 10} = \sqrt[3]{\left(\lim_{t\to 3} t\right)^2 - 10} = \sqrt[3]{(3)^2-10} = -1$.

2.3.27 $\lim_{p\to 2} \frac{3p}{\sqrt{4p+1}-1} = \frac{\lim_{p\to 2} 3p}{\lim_{p\to 2}(\sqrt{4p+1}-1)} = \frac{3\lim_{p\to 2} p}{\lim_{p\to 2}\sqrt{4p+1} - \lim_{p\to 2} 1} = \frac{3\cdot 2}{\sqrt{\lim_{p\to 2}(4p+1)} - 1} = \frac{6}{3-1} = 3$.

2.3.28 $\lim_{x\to 2}(x^2-x)^5 = \left(\lim_{x\to 2}(x^2-x)\right)^5 = \left(\lim_{x\to 2} x^2 - \lim_{x\to 2} x\right)^5 = (4-2)^5 = 32$.

2.3.29 $\lim_{x\to 3} -\frac{5x}{\sqrt{4x-3}} = \frac{\lim_{x\to 3} -5x}{\lim_{x\to 3}\sqrt{4x-3}} = \frac{-5\lim_{x\to 3} x}{\sqrt{\lim_{x\to 3}(4x-3)}} = -\frac{5\cdot 3}{\sqrt{4\lim_{x\to 3} x - \lim_{x\to 3} 3}} = -\frac{15}{\sqrt{4\cdot 3-3}} = -5$.

2.3.30

$$\lim_{h\to 0}\frac{3}{\sqrt{16+3h}+4}=\frac{\lim\limits_{h\to 0}3}{\lim\limits_{h\to 0}(\sqrt{16+3h}+4)}=\frac{3}{\sqrt{\lim\limits_{h\to 0}(16+3h)}+\lim\limits_{h\to 0}4}=\frac{3}{\sqrt{\lim\limits_{h\to 0}16+\lim\limits_{h\to 0}3h}+4}$$
$$=\frac{3}{\sqrt{16+3\cdot 0}+4}=\frac{3}{4+4}=\frac{3}{8}.$$

2.3.31 $\lim\limits_{x\to 2}(5x-6)^{3/2}=(5\cdot 2-6)^{3/2}=4^{3/2}=2^3=8.$

2.3.32 $\lim\limits_{h\to 0}\dfrac{100}{(10h-1)^{11}+2}=\dfrac{100}{(-1)^{11}+2}=\dfrac{100}{1}=100.$

2.3.33 $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=\lim\limits_{x\to 1}\dfrac{(x+1)(x-1)}{x-1}=\lim\limits_{x\to 1}(x+1)=2.$

2.3.34 $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x-3}=\lim\limits_{x\to 3}\dfrac{(x-3)(x+1)}{x-3}=\lim\limits_{x\to 3}(x+1)=4.$

2.3.35 $\lim\limits_{x\to 4}\dfrac{x^2-16}{4-x}=\lim\limits_{x\to 4}\dfrac{(x+4)(x-4)}{-(x-4)}=\lim\limits_{x\to 4}[-(x+4)]=-8.$

2.3.36 $\lim\limits_{t\to 2}\dfrac{3t^2-7t+2}{2-t}=\lim\limits_{t\to 2}\dfrac{(t-2)(3t-1)}{-(t-2)}=\lim\limits_{t\to 2}[-(3t-1)]=-5.$

2.3.37 $\lim\limits_{x\to b}\dfrac{(x-b)^{50}-x+b}{x-b}=\lim\limits_{x\to b}\dfrac{(x-b)^{50}-(x-b)}{x-b}=\lim\limits_{x\to b}\dfrac{(x-b)((x-b)^{49}-1)}{x-b}=$
$\lim\limits_{x\to b}[(x-b)^{49}-1]=-1.$

2.3.38 $\lim\limits_{x\to -b}\dfrac{(x+b)^7+(x+b)^{10}}{4(x+b)}=\lim\limits_{x\to -b}\dfrac{(x+b)((x+b)^6+(x+b)^9)}{4(x+b)}=\lim\limits_{x\to -b}\dfrac{(x+b)^6+(x+b)^9}{4}=\dfrac{0}{4}=0.$

2.3.39

$$\lim_{x\to -1}\frac{(2x-1)^2-9}{x+1}=\lim_{x\to -1}\frac{(2x-1-3)(2x-1+3)}{x+1}$$
$$=\lim_{x\to -1}\frac{2(x-2)2(x+1)}{x+1}=\lim_{x\to -1}4(x-2)=4\cdot(-3)=-12.$$

2.3.40

$$\lim_{h\to 0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}=\lim_{h\to 0}\frac{\left(\frac{1}{5+h}-\frac{1}{5}\right)\cdot 5\cdot(5+h)}{h\cdot 5\cdot(5+h)}$$
$$=\lim_{h\to 0}\frac{5-(5+h)}{5h(5+h)}=\lim_{h\to 0}-\frac{h}{5h(5+h)}=\lim_{h\to 0}-\frac{1}{5(5+h)}=-\frac{1}{25}.$$

2.3.41 $\lim\limits_{x\to 9}\dfrac{\sqrt{x}-3}{x-9}=\lim\limits_{x\to 9}\dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}=\lim\limits_{x\to 9}\dfrac{x-9}{(x-9)(\sqrt{x}+3)}=\lim\limits_{x\to 9}\dfrac{1}{\sqrt{x}+3}=\dfrac{1}{6}.$

2.3.42 $\lim\limits_{w\to 1}\left(\dfrac{1}{w^2-w}-\dfrac{1}{w-1}\right)=\lim\limits_{w\to 1}\left(\dfrac{1}{w(w-1)}-\dfrac{w}{w(w-1)}\right)=\lim\limits_{w\to 1}\dfrac{1-w}{w(w-1)}=-\lim\limits_{w\to 1}\dfrac{1}{w}=-1.$

2.3.43

$$\lim_{t\to 5}\left(\frac{1}{t^2-4t-5}-\frac{1}{6(t-5)}\right)=\lim_{t\to 5}\left(\frac{1}{(t-5)(t+1)}-\frac{1}{6(t-5)}\right)$$
$$=\lim_{t\to 5}\left(\frac{6}{6(t-5)(t+1)}-\frac{t+1}{6(t-5)(t+1)}\right)$$
$$=\lim_{t\to 5}\frac{5-t}{6(t-5)(t+1)}=-\lim_{t\to 5}\frac{1}{6(t+1)}=-\frac{1}{36}.$$

2.3.44 Expanding gives

$$\begin{aligned}\lim_{t\to 3}\left(\left(4t-\frac{2}{t-3}\right)(6+t-t^2)\right) &= \lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(6+t-t^2)}{t-3}\right)\\ &= \lim_{t\to 3}\left(4t(6+t-t^2)-\frac{2(3-t)(2+t)}{t-3}\right).\end{aligned}$$

Now because $t-3=-(3-t)$, we have

$$\lim_{t\to 3}\left(4t(6+t-t^2)+2(2+t)\right)=12(6+3-9)+2(2+3)=10.$$

2.3.45 $\displaystyle\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{x-a}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}=\lim_{x\to a}(\sqrt{x}+\sqrt{a})=2\sqrt{a}.$

2.3.46 $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}=\lim_{x\to a}\frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{(x-a)(x+a)(\sqrt{x}+\sqrt{a})}{x-a}=$
$(a+a)(\sqrt{a}+\sqrt{a})=4a^{3/2}$.

2.3.47

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{16+h}-4}{h} &= \lim_{h\to 0}\frac{(\sqrt{16+h}-4)(\sqrt{16+h}+4)}{h(\sqrt{16+h}+4)}=\lim_{h\to 0}\frac{(16+h)-16}{h(\sqrt{16+h}+4)}\\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{16+h}+4)}=\lim_{h\to 0}\frac{1}{(\sqrt{16+h}+4)}=\frac{1}{8}.\end{aligned}$$

2.3.48 $\displaystyle\lim_{x\to c}\frac{x^2-2cx+c^2}{x-c}=\lim_{x\to c}\frac{(x-c)^2}{x-c}=\lim_{x\to c}x-c=c-c=0.$

2.3.49 $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}=\lim_{x\to 4}\frac{\frac{4-x}{4x}}{x-4}=\lim_{x\to 4}\frac{4-x}{4x(x-4)}=-\lim_{x\to 4}\frac{1}{4x}=-\frac{1}{16}.$

2.3.50

$$\begin{aligned}\lim_{x\to 3}\frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3} &= \lim_{x\to 3}\frac{\frac{15-(x^2+2x)}{15(x^2+2x)}}{x-3}=\lim_{x\to 3}\frac{15-(x^2+2x)}{15(x^2+2x)(x-3)}=\lim_{x\to 3}\frac{15-2x-x^2}{15(x^2+2x)(x-3)}\\ &= \lim_{x\to 3}\frac{(3-x)(5+x)}{15(x^2+2x)(x-3)}=\lim_{x\to 3}-\frac{(5+x)}{15(x^2+2x)}=-\frac{8}{225}.\end{aligned}$$

2.3.51

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{10x-9}-1}{x-1} &= \lim_{x\to 1}\frac{(\sqrt{10x-9}-1)(\sqrt{10x-9}+1)}{(x-1)(\sqrt{10x-9}+1)}=\lim_{x\to 1}\frac{(10x-9)-1}{(x-1)(\sqrt{10x-9}+1)}\\ &= \lim_{x\to 1}\frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)}=\lim_{x\to 1}\frac{10}{(\sqrt{10x-9}+1)}=\frac{10}{2}=5.\end{aligned}$$

2.3.52 $\displaystyle\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{2}{x^2-2x}\right)=\lim_{x\to 2}\left(\frac{x}{x(x-2)}-\frac{2}{x(x-2)}\right)=\lim_{x\to 2}\left(\frac{x-2}{x(x-2)}\right)=\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}.$

2.3.53 $\displaystyle\lim_{h\to 0}\frac{(5+h)^2-25}{h}=\lim_{h\to 0}\frac{25+10h+h^2-25}{h}=\lim_{h\to 0}\frac{h(10+h)}{h}=\lim_{h\to 0}(10+h)=10.$

2.3.54 We have

$$\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to -k}\frac{(w+4k)(w+k)}{(w)(w+k)}=\lim_{w\to -k}\frac{w+4k}{w}=\frac{-k+4k}{-k}=-3.$$

If $k=0$, we have $\displaystyle\lim_{w\to -k}\frac{w^2+5kw+4k^2}{w^2+kw}=\lim_{w\to 0}\frac{w^2}{w^2}=1.$

2.3.55 $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}=\lim_{x\to 1}(\sqrt{x}+1)=2.$

2.3.56

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{4x+5}-3}&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{(\sqrt{4x+5}-3)(\sqrt{4x+5}+3)}=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9}\\&=\lim_{x\to 1}\frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)}=\lim_{x\to 1}\frac{(\sqrt{4x+5}+3)}{4}=\frac{6}{4}=\frac{3}{2}.\end{aligned}$$

2.3.57

$$\begin{aligned}\lim_{x\to 4}\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{(3-\sqrt{x+5})(3+\sqrt{x+5})}=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{9-(x+5)}\\&=\lim_{x\to 4}\frac{3(x-4)(\sqrt{x+5})(3+\sqrt{x+5})}{-(x-4)}\\&=\lim_{x\to 4}[-3(\sqrt{x+5})(3+\sqrt{x+5})]=(-3)(3)(3+3)=-54\end{aligned}$$

2.3.58 Assume $c\neq 0$.

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{cx+1}-1}&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(\sqrt{cx+1}-1)(\sqrt{cx+1}+1)}=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{(cx+1)-1}\\&=\lim_{x\to 0}\frac{x(\sqrt{cx+1}+1)}{cx}=\lim_{x\to 0}\frac{(\sqrt{cx+1}+1)}{c}=\frac{2}{c}.\end{aligned}$$

2.3.59 $\lim_{x\to 0}x\cos x=0\cdot 1=0.$

2.3.60 $\lim_{x\to 0}\frac{\sin 2x}{\sin x}=\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}=\lim_{x\to 0}2\cos x=2.$

2.3.61 $\lim_{x\to 0}\frac{1-\cos x}{\cos^2 x-3\cos x+2}=\lim_{x\to 0}\frac{1-\cos x}{(\cos x-2)(\cos x-1)}=-\lim_{x\to 0}\frac{1}{\cos x-2}=-\frac{1}{1-2}=1.$

2.3.62 $\lim_{x\to 0}\frac{\cos x-1}{\cos^2 x-1}=\lim_{x\to 0}\frac{\cos x-1}{(\cos x-1)(\cos x+1)}=\lim_{x\to 0}\frac{1}{\cos x+1}=\frac{1}{2}.$

2.3.63 $\lim_{x\to 0^-}\frac{x^2-x}{|x|}=\lim_{x\to 0^-}\frac{x(x-1)}{-x}=-\lim_{x\to 0^-}(x-1)=1.$

2.3.64 $\lim_{w\to 3^-}\frac{|w-3|}{w^2-7w+12}=\lim_{w\to 3^-}\frac{3-w}{(w-3)(w-4)}=-\lim_{w\to 3^-}\frac{1}{w-4}=1.$

2.3.65 $\lim_{t\to 2^+}\frac{|2t-4|}{t^2-4}=\lim_{t\to 2^+}\frac{2(t-2)}{(t-2)(t+2)}=\frac{2}{4}=\frac{1}{2}.$

2.3.66 $\lim_{x\to -1^-}g(x)=\lim_{x\to -1^-}\frac{x^2-1}{x+1}=\lim_{x\to -1^-}\frac{(x+1)(x-1)}{x+1}=\lim_{x\to -1^-}(x-1)=-2$. Also, $\lim_{x\to -1^+}g(x)=\lim_{x\to -1^+}(-2)=-2$. Therefore, $\lim_{x\to -1}g(x)=-2$.

2.3.67 $\lim_{x\to 3^+}\frac{x-3}{|x-3|}=\lim_{x\to 3^+}\frac{x-3}{x-3}=\lim_{x\to 3^+}1=1$. On the other hand, $\lim_{x\to 3^-}\frac{x-3}{|x-3|}=\lim_{x\to 3^-}\frac{x-3}{3-x}=\lim_{x\to 3^-}(-1)=-1$. Therefore, $\lim_{x\to 3}\frac{x-3}{|x-3|}$ does not exist.

2.3.68 $\lim_{x\to 5^+} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$. On the other hand, $\lim_{x\to 5^-} \frac{|x-5|}{x^2-25} = \lim_{x\to 5^-} \frac{5-x}{(x-5)(x+5)} = -\lim_{x\to 5^+} \frac{1}{x+5} = -\frac{1}{10}$. Therefore, $\lim_{x\to 5} \frac{|x-5|}{x^2-25}$ does not exist.

2.3.69 Because the domain of $f(x) = \frac{x^3+1}{\sqrt{x-1}}$ is the interval $(1,\infty)$, the limit doesn't exist.

2.3.70 $\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim_{x\to 1^+} \frac{x-1}{(x-1)^{1/2}(x+1)^{1/2}} = \lim_{x\to 1^+} \frac{(x-1)^{1/2}}{(x+1)^{1/2}} = \frac{0}{\sqrt{2}} = 0$.

2.3.71

a. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ then $\lim_{x\to 1} f(x) = 1$ but $f(1) = 4$.

b. False. For example, if $f(x) = \begin{cases} x+1 & \text{if } x \leq 1; \\ x-6 & \text{if } x > 1, \end{cases}$ then $\lim_{x\to 1^-} f(x) = 2$ but $\lim_{x\to 1^+} f(x) = -5$.

c. False. For example, if $f(x) = \begin{cases} x & \text{if } x \neq 1; \\ 4 & \text{if } x = 1, \end{cases}$ and $g(x) = 1$, then f and g both have limit 1 as $x \to 1$, but $f(1) = 4 \neq g(1)$.

d. False. For example $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exists and is equal to 4.

e. False. For example, it would be possible for the domain of f to be $[1,\infty)$, so that the one-sided limit exists but the two-sided limit doesn't even make sense. This would be true, for example, if $f(x) = x-1$.

2.3.72

a. $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (5x-15) = 5$.

b. $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \sqrt{6x+1} = 5$.

c. Because the two one-sided limits both exist and are equal to 5, $\lim_{x\to 4} g(x) = 5$.

2.3.73

a. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+1) = (-1)^2 + 1 = 2$.

b. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = \sqrt{-1+1} = 0$.

c. Because the two one-sided limits differ, $\lim_{x\to -1} f(x)$ does not exist.

2.3.74

a. $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} 0 = 0$.

b. $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \sqrt{25-x^2} = \sqrt{25-25} = 0$.

c. $\lim_{x\to -5} f(x) = 0$.

d. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{25-x^2} = \sqrt{25-25} = 0$.

e. $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 3x = 15$.

f. $\lim_{x\to 5} f(x)$ does not exist.

2.3.75

a. $\lim\limits_{x\to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0$.

b. The domain of $f(x) = \sqrt{x-2}$ is $[2,\infty)$. Thus, any question about this function that involves numbers less than 2 doesn't make any sense, because those numbers aren't in the domain of f.

2.3.76

a. Note that the domain of $f(x) = \sqrt{\frac{x-3}{2-x}}$ is $(2,3]$. $\lim\limits_{x\to 3^-} \sqrt{\dfrac{x-3}{2-x}} = 0$.

b. Because the numbers to the right of 3 aren't in the domain of this function, the limit as $x \to 3^+$ of this function doesn't make any sense.

2.3.77 $\lim\limits_{x\to 10} E(x) = \lim\limits_{x\to 10} \dfrac{4.35}{x\sqrt{x^2+0.01}} = \dfrac{4.35}{10\sqrt{100.01}} \approx 0.0435$ N/C.

2.3.78 $\lim\limits_{t\to 200^-} d(t) = \lim\limits_{t\to 200^-} (3-0.015t)^2 = (3-(0.015)(200))^2 = (3-3)^2 = 0$. As time approaches 200 seconds, the depth of the water in the tank is approaching 0.

2.3.79 $\lim\limits_{S\to 0^+} r(S) = \lim\limits_{S\to 0^+} (1/2)\left(\sqrt{100+\dfrac{2S}{\pi}} - 10\right) = 0$.

The radius of the circular cylinder approaches zero as the surface area approaches zero.

2.3.80

a. $L(c/2) = L_0\sqrt{1-\frac{(c/2)^2}{c^2}} = L_0\sqrt{1-(1/4)} = \sqrt{3}L_0/2$.

b. $L(3c/4) = L_0\sqrt{1-(1/c^2)(3c/4)^2} = L_0\sqrt{1-(9/16)} = \sqrt{7}L_0/4$.

c. It appears that that the observed length L of the ship decreases as the ship speed increases.

d. $\lim\limits_{x\to c^-} L_0\sqrt{1-(\nu^2/c^2)} = L_0 \cdot 0 = 0$. As the speed of the ship approaches the speed of light, the observed length of the ship shrinks to 0.

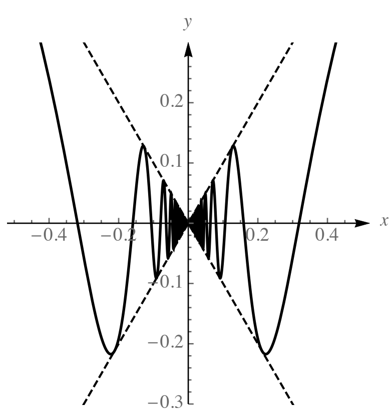
2.3.81

a. The statement we are trying to prove can be stated in cases as follows: For $x > 0$, $-x \le x\sin(1/x) \le x$, and for $x < 0$, $x \le x\sin(1/x) \le -x$.

Now for all $x \neq 0$, note that $-1 \le \sin(1/x) \le 1$ (because the range of the sine function is $[-1,1]$). We will consider the two cases $x > 0$ and $x < 0$ separately, but in each case, we will multiply this inequality through by x, switching the inequalities for the $x < 0$ case.

For $x > 0$ we have $-x \le x\sin(1/x) \le x$, and for $x < 0$ we have $-x \ge x\sin(1/x) \ge x$, which are exactly the statements we are trying to prove.

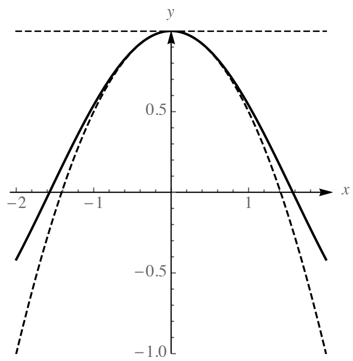
b.



c. Because $\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$, and because $-|x| \le x\sin(1/x) \le |x|$, the Squeeze Theorem assures us that $\lim\limits_{x\to 0}[x\sin(1/x)] = 0$ as well.

2.3.82

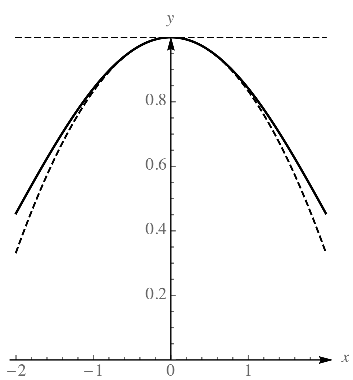
a.



b. Note that $\lim_{x\to 0}\left[1-\dfrac{x^2}{2}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{2} \le \cos x \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\cos x = 1$ as well.

2.3.83

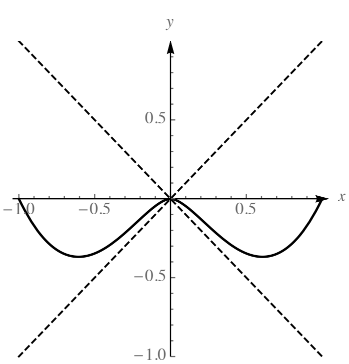
a.



b. Note that $\lim_{x\to 0}\left[1-\dfrac{x^2}{6}\right] = 1 = \lim_{x\to 0} 1$. So because $1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1$, the squeeze theorem assures us that $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$ as well.

2.3.84

a.



b. Note that $\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|$. So because $-|x| \le x^2\ln x^2 \le |x|$, the squeeze theorem assures us that $\lim_{x\to 0}(x^2\ln x^2) = 0$ as well.

2.3.85 Using the definition of $|x|$ given, we have $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = -0 = 0$. Also, $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$. Because the two one-sided limits are both 0, we also have $\lim_{x\to 0}|x| = 0$.

2.3.86

If $a > 0$, then for x near a, $|x| = x$. So in this case, $\lim_{x\to a} |x| = \lim_{x\to a} x = a = |a|$.

If $a < 0$, then for x near a, $|x| = -x$. So in this case, $\lim_{x\to a} |x| = \lim_{x\to a}(-x) = -a = |a|$, (because $a < 0$).

If $a = 0$, we have already seen in a previous problem that $\lim_{x\to 0} |x| = 0 = |0|$.

Thus in all cases, $\lim_{x\to a} |x| = |a|$.

2.3.87 $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2 - 5x + 6}{x - 3} = \lim_{x\to 3} \dfrac{(x-3)(x-2)}{x-3} = \lim_{x\to 3}(x - 2) = 1$. So $a = 1$.

2.3.88 In order for $\lim_{x\to 2} f(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3x + b) = 6 + b$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x - 2) = 0$. So we need $6 + b = 0$, so we require that $b = -6$. Then $\lim_{x\to 2} f(x) = 0$.

2.3.89 In order for $\lim_{x\to -1} g(x)$ to exist, we need the two one-sided limits to exist and be equal. We have $\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-}(x^2 - 5x) = 6$, and $\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+}(ax^3 - 7) = -a - 7$. So we need $-a - 7 = 6$, so we require that $a = -13$. Then $\lim_{x\to -1} f(x) = 6$.

2.3.90 $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x\to 2}(x^4 + 2x^3 + 4x^2 + 8x + 16) = 16 + 16 + 16 + 16 + 16 = 80$.

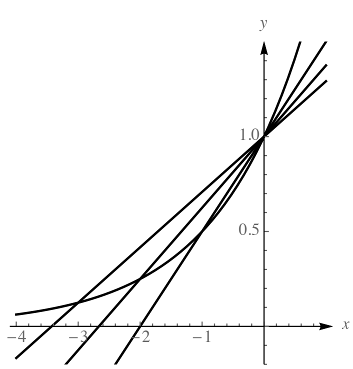
2.3.91 $\lim_{x\to 1} \dfrac{x^6 - 1}{x - 1} = \lim_{x\to 1} \dfrac{(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)}{x - 1} = \lim_{x\to 1}(x^5 + x^4 + x^3 + x^2 + x + 1) = 6$.

2.3.92 $\lim_{x\to -1} \dfrac{x^7 + 1}{x + 1} = \lim_{x\to -1} \dfrac{(x+1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)}{x + 1} = \lim_{x\to -1}(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) = 7$.

2.3.93 $\lim_{x\to a} \dfrac{x^5 - a^5}{x - a} = \lim_{x\to a} \dfrac{(x-a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{x - a} = \lim_{x\to a}(x^4 + ax^3 + a^2x^2 + a^3x + a^4) = 5a^4$.

2.3.94 $\lim_{x\to a} \dfrac{x^n - a^n}{x - a} = \lim_{x\to a} \dfrac{(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})}{x - a} = \lim_{x\to a}(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}) = na^{n-1}$.

2.3.95

a. 

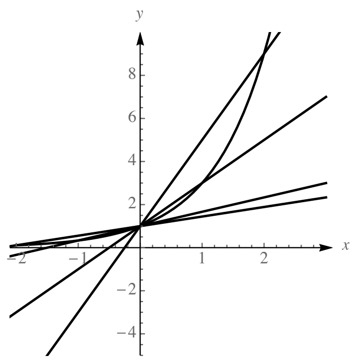
b. The slope of the secant line between $(0, 1)$ and $(x, 2^x)$ is $\frac{2^x - 1}{x}$.

c.

x	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
$\frac{2^x-1}{x}$	0.5	0.66967	0.69075	0.692907	0.693123	0.693145

It appears that $\lim_{x\to 0^-} \frac{2^x-1}{x} \approx 0.693$.

2.3.96



a.

b. The slope of the secant line between $(0,1)$ and $(x,3^x)$ is $\frac{3^x-1}{x}$.

c.

x	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$\frac{3^x-1}{x}$	1.04042	1.0926	1.09801	1.09855	1.09867	1.09922	1.10467	1.16123

It appears that $\lim_{x\to 0}\dfrac{3^x-1}{x}\approx 1.099$.

2.3.97 $\lim_{x\to -1^-} f(x) = \lim_{x\to 1^+} f(x) = 6$. $\lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = 5$.

2.3.98 $\lim_{x\to -1^-} g(x) = -\lim_{x\to 1^+} g(x) = -6$. $\lim_{x\to -1^+} g(x) = -\lim_{x\to 1^-} g(x) = -5$.

2.3.99 $\lim_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x-1} = \lim_{x\to 1}\dfrac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \lim_{x\to 1}\dfrac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1} = \dfrac{1}{3}$.

2.3.100 $\lim_{x\to 16}\dfrac{\sqrt[4]{x}-2}{x-16} = \lim_{x\to 16}\dfrac{\sqrt[4]{x}-2}{(\sqrt[4]{x}-2)(\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8)} = \lim_{x\to 16}\dfrac{1}{\sqrt[4]{x^3}+2\sqrt[4]{x^2}+4\sqrt[4]{x}+8} = \dfrac{1}{32}$.

2.3.101 Let $f(x) = x-1$ and $g(x) = \frac{5}{x-1}$. Then $\lim_{x\to 1} f(x) = 0$, $\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1}\dfrac{5(x-1)}{x-1} = \lim_{x\to 1} 5 = 5$.

2.3.102 Let $f(x) = x^2-1$. Then $\lim_{x\to 1}\dfrac{f(x)}{x-1} = \lim_{x\to 1}\dfrac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$.

2.3.103 Let $p(x) = x^2+2x-8$. Then $\lim_{x\to 2}\dfrac{p(x)}{x-2} = \lim_{x\to 2}\dfrac{(x-2)(x+4)}{x-2} = \lim_{x\to 2}(x+4) = 6$.

The constants are unique. We know that 2 must be a root of p (otherwise the given limit couldn't exist), so it must have the form $p(x) = (x-2)q(x)$, and q must be a degree 1 polynomial with leading coefficient 1 (otherwise p wouldn't have leading coefficient 1.) So we have $p(x) = (x-2)(x+d)$, but because $\lim_{x\to 2}\dfrac{p(x)}{x-2} = \lim_{x\to 2}(x+d) = 2+d = 6$, we are forced to realize that $d = 4$. Therefore, we have deduced that the only possibility for p is $p(x) = (x-2)(x+4) = x^2+2x-8$.

2.3.104 Because $\lim_{x\to 1} f(x) = 4$, we know that f is near 4 when x is near 1 (but not equal to 1). It follows that $\lim_{x\to -1} f(x^2) = 4$ as well, because when x is near but not equal to -1, x^2 is near 1 but not equal to 1. Thus $f(x^2)$ is near 4 when x is near -1.

2.3.105 As $x\to 0^+$, $(1-x)\to 1^-$. So $\lim_{x\to 0^+} g(x) = \lim_{(1-x)\to 1^-} f(1-x) = \lim_{z\to 1^-} f(z) = 6$. (Where $z = 1-x$.)

As $x\to 0^-$, $(1-x)\to 1^+$. So $\lim_{x\to 0^-} g(x) = \lim_{(1-x)\to 1^+} f(1-x) = \lim_{z\to 1^+} f(z) = 4$. (Where $z = 1-x$.)

2.3.106

a. Suppose $0 < \theta < \pi/2$. Note that $\sin\theta > 0$, so $|\sin\theta| = \sin\theta$. Also, $\sin\theta = \frac{|AC|}{1}$, so $|AC| = |\sin\theta|$.

Now suppose that $-\pi/2 < \theta < 0$. Then $\sin\theta$ is negative, so $|\sin\theta| = -\sin\theta$. We have $\sin\theta = \frac{-|AC|}{1}$, so $|AC| = -\sin\theta = |\sin\theta|$.

b. Suppose $0 < \theta < \pi/2$. Because AB is the hypotenuse of triangle ABC, we know that $|AB| > |AC|$. We have $|\sin\theta| = |AC| < |AB| <$ the length of arc $AB = \theta = |\theta|$.

If $-\pi/2 < \theta < 0$, we can make a similar argument. We have

$$|\sin\theta| = |AC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

c. If $0 < \theta < \pi/2$, we have $\sin\theta = |\sin\theta| < |\theta|$, and because $\sin\theta$ is positive, we have $-|\theta| \le 0 < \sin\theta$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

If $-\pi/2 < \theta < 0$, then $|\sin\theta| = -\sin\theta$. From the previous part, we have $|\sin\theta| = -\sin\theta < |\theta|$. Therefore, $-|\theta| < \sin\theta$. Now because $\sin\theta$ is negative on this interval, we have $\sin\theta < 0 \le |\theta|$. Putting these together gives $-|\theta| < \sin\theta < |\theta|$.

d. If $0 < \theta < \pi/2$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = \theta = |\theta|.$$

For $-\pi/2 < \theta < 0$, we have

$$0 \le 1 - \cos\theta = |OB| - |OC| = |BC| < |AB| < \text{the length of arc } AB = -\theta = |\theta|.$$

e. Using the result of part d, we multiply through by -1 to obtain $-|\theta| \le \cos\theta - 1 \le 0$, and then add 1 to all parts, obtaining $1 - |\theta| \le \cos\theta \le 1$, as desired.

2.3.107

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\
&= \lim_{x\to a}(a_n x^n) + \lim_{x\to a}(a_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(a_1 x) + \lim_{x\to a} a_0 \\
&= a_n \lim_{x\to a} x^n + a_{n-1}\lim_{x\to a} x^{n-1} + \cdots + a_1 \lim_{x\to a} x + a_0 \\
&= a_n(\lim_{x\to a} x)^n + a_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + a_1(\lim_{x\to a} x) + a_0 \\
&= a_n a^n + a_{n-1}a^{n-1} + \cdots + a_1 a + a_0 = p(a).
\end{aligned}$$

2.4 Infinite Limits

2.4.1 As x approaches a from the right, the values of $f(x)$ are negative and become arbitrarily large in magnitude.

2.4.2 As x approaches a (from either side), the values of $f(x)$ are positive and become arbitrarily large in magnitude.

2.4.3 A vertical asymptote for a function f is a vertical line $x = a$ so that one or more of the following are true: $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$.

2.4.4 No. For example, if $f(x) = x^2 - 4$ and $g(x) = x - 2$ and $a = 2$, we would have $\lim_{x\to 2}\dfrac{f(x)}{g(x)} = 4$, even though $g(2) = 0$.

2.4.5

x	$\frac{x+1}{(x-1)^2}$	x	$\frac{x+1}{(x-1)^2}$
1.1	210	.9	190
1.01	20,100	.99	19,900
1.001	2,001,000	.999	1,999,000
1.0001	200,010,000	.9999	199,990,000

From the data given, it appears that $\lim_{x\to 1} f(x) = \infty$.

2.4.6 $\lim_{x\to 3} f(x) = \infty$, and $\lim_{x\to -1} f(x) = -\infty$.

2.4.7

a. $\lim_{x\to 1^-} f(x) = \infty$.
b. $\lim_{x\to 1^+} f(x) = \infty$.
c. $\lim_{x\to 1} f(x) = \infty$.
d. $\lim_{x\to 2^-} f(x) = \infty$.
e. $\lim_{x\to 2^+} f(x) = -\infty$.
f. $\lim_{x\to 2} f(x)$ does not exist.

2.4.8

a. $\lim_{x\to 2^-} g(x) = \infty$.
b. $\lim_{x\to 2^+} g(x) = -\infty$.
c. $\lim_{x\to 2} g(x)$ does not exist.
d. $\lim_{x\to 4^-} g(x) = -\infty$.
e. $\lim_{x\to 4^+} g(x) = -\infty$.
f. $\lim_{x\to 4} g(x) = -\infty$.

2.4.9

a. $\lim_{x\to -2^-} h(x) = -\infty$.
b. $\lim_{x\to -2^+} h(x) = -\infty$.
c. $\lim_{x\to -2} h(x) = -\infty$.
d. $\lim_{x\to 3^-} h(x) = \infty$.
e. $\lim_{x\to 3^+} h(x) = -\infty$.
f. $\lim_{x\to 3} h(x)$ does not exist.

2.4.10

a. $\lim_{x\to -2^-} p(x) = -\infty$.
b. $\lim_{x\to -2^+} p(x) = -\infty$.
c. $\lim_{x\to -2} p(x) = -\infty$.
d. $\lim_{x\to 3^-} p(x) = -\infty$.
e. $\lim_{x\to 3^+} p(x) = -\infty$.
f. $\lim_{x\to 3} p(x) = -\infty$.

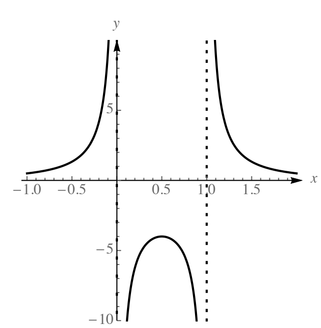
2.4.11

a. $\lim_{x\to 0^-} \dfrac{1}{x^2 - x} = \infty$.

b. $\lim_{x\to 0^+} \dfrac{1}{x^2 - x} = -\infty$.

c. $\lim_{x\to 1^-} \dfrac{1}{x^2 - x} = -\infty$.

d. $\lim_{x\to 1^+} \dfrac{1}{x^2 - x} = \infty$.



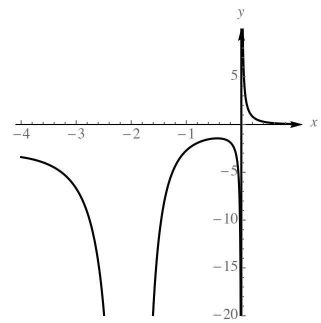
2.4.12

a. $\displaystyle\lim_{x\to -2^+} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

b. $\displaystyle\lim_{x\to -2} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

c. $\displaystyle\lim_{x\to 0^-} \frac{e^{-x}}{x(x+2)^2} = -\infty.$

d. $\displaystyle\lim_{x\to 0^+} \frac{e^{-x}}{x(x+2)^2} = \infty.$



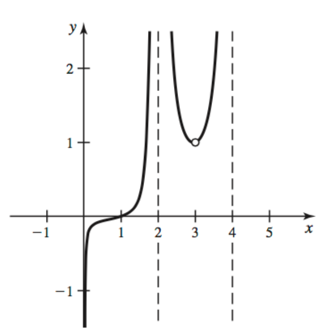
2.4.13 Because the numerator is approaching a non-zero constant while the denominator is approaching zero, the quotient of these numbers is getting big – at least the absolute value of the quotient is getting big. The quotient is actually always negative, because a number near 100 divided by a negative number is always negative. Thus $\displaystyle\lim_{x\to 2} \frac{f(x)}{g(x)} = -\infty$.

2.4.14 Using the same sort of reasoning as in the last problem – as $x \to 3$ the numerator is fixed at 1, but the denominator is getting small, so the quotient is getting big. It remains to investigate the sign of the quotient. As $x \to 3^-$, the quantity $x-3$ is negative, so the quotient of the positive number 1 and this small negative number is negative. On the other hand, as $x \to 3^+$, the quantity $x-3$ is positive, so the quotient of 1 and this number is positive. Thus: $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$, and $\displaystyle\lim_{x\to 3^+} \frac{1}{x-3} = \infty$.

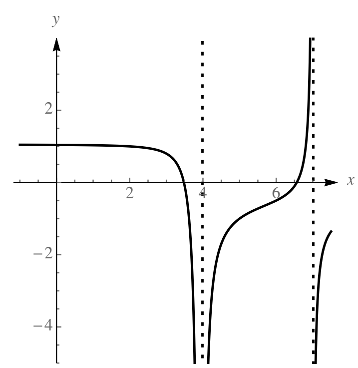
2.4.15 Note that $\displaystyle\lim_{x\to 1} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 1} \frac{x-3}{x-2} = \frac{-2}{-1} = 2$. So there is *not* a vertical asymptote at $x=1$. On the other hand, $\displaystyle\lim_{x\to 2^+} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 2^+} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 2^+} \frac{x-3}{x-2} = -\infty$, so there is a vertical asymptote at $x=2$.

2.4.16 Note the at $x \to 0$ the numerator has limit 1 while the denominator has limit 0, so the quotient is growing without bound. Note also that the denominator is always positive, because for all x, $\cos x \le 1$ so $1-\cos x \ge 0$.

2.4.17



2.4.18



2.4.19 Both a and b are true statements.

2.4.20 Both a and c are true statements.

2.4.21

a. $\lim_{x\to 2^+} \frac{1}{x-2} = \infty.$

b. $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.$

c. $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.

2.4.22

a. $\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \infty.$

b. $\lim_{x\to 3^-} \frac{2}{(x-3)^3} = -\infty.$

c. $\lim_{x\to 3} \frac{2}{(x-3)^3}$ does not exist.

2.4.23

a. $\lim_{x\to 4^+} \frac{x-5}{(x-4)^2} = -\infty.$

b. $\lim_{x\to 4^-} \frac{x-5}{(x-4)^2} = -\infty.$

c. $\lim_{x\to 4} \frac{x-5}{(x-4)^2} = -\infty.$

2.4.24

a. $\lim_{x\to 1^+} \frac{x}{|x-1|} = \infty.$

b. $\lim_{x\to 1^-} \frac{x}{|x-1|} = \infty.$

c. $\lim_{x\to 1} \frac{x}{|x-1|} = \infty.$

2.4.25

a. $\lim_{x\to 3^+} \frac{(x-1)(x-2)}{(x-3)} = \infty.$

b. $\lim_{x\to 3^-} \frac{(x-1)(x-2)}{(x-3)} = -\infty.$

c. $\lim_{x\to 3} \frac{(x-1)(x-2)}{(x-3)}$ does not exist.

2.4.26

a. $\lim_{x\to -2^+} \frac{(x-4)}{x(x+2)} = \infty.$

b. $\lim_{x\to -2^-} \frac{(x-4)}{x(x+2)} = -\infty.$

c. $\lim_{x\to -2} \frac{(x-4)}{x(x+2)}$ does not exist.

2.4.27

a. $\lim\limits_{x\to 2^+} \dfrac{x^2-4x+3}{(x-2)^2} = -\infty.$

b. $\lim\limits_{x\to 2^-} \dfrac{x^2-4x+3}{(x-2)^2} = -\infty.$

c. $\lim\limits_{x\to 2} \dfrac{x^2-4x+3}{(x-2)^2} = -\infty.$

2.4.28

a. $\lim\limits_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to -2^+} \dfrac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim\limits_{x\to -2^+} \dfrac{x-3}{x(x+2)} = \infty.$

b. $\lim\limits_{x\to -2^-} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to -2^-} \dfrac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim\limits_{x\to -2^-} \dfrac{x-3}{x(x+2)} = -\infty.$

c. Because the two one-sided limits differ, $\lim\limits_{x\to -2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$ does not exist.

d. $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+6x}{x^4-4x^2} = \lim\limits_{x\to 2} \dfrac{x-3}{x(x+2)} = \dfrac{-1}{8}.$

2.4.29

a. $\lim\limits_{x\to 2^+} \dfrac{1}{\sqrt{x(x-2)}} = \infty.$

b. $\lim\limits_{x\to 2^-} \dfrac{1}{\sqrt{x(x-2)}}$ does not exist. Note that the domain of the function is $(-\infty,0)\cup(2,\infty)$.

c. $\lim\limits_{x\to 2} \dfrac{1}{\sqrt{x(x-2)}}$ does not exist.

2.4.30

a. $\lim\limits_{x\to 1^+} \dfrac{x-3}{\sqrt{x^2-5x+4}}$ does not exist. Note that $x^2-5x+4=(x-4)(x-1)$ so the domain of the function is $(-\infty,1)\cup(4,\infty)$.

b.] $\lim\limits_{x\to 1^-} \dfrac{x-3}{\sqrt{x^2-5x+4}} = -\infty.$

c. $\lim\limits_{x\to 1} \dfrac{x-3}{\sqrt{x^2-5x+4}}$ does not exist.

2.4.31

a. $\lim\limits_{x\to 0} \dfrac{x-3}{x^4-9x^2} = \lim\limits_{x\to 0} \dfrac{x-3}{x^2(x-3)(x+3)} = \lim\limits_{x\to 0} \dfrac{1}{x^2(x+3)} = \infty.$

b. $\lim\limits_{x\to 3} \dfrac{x-3}{x^4-9x^2} = \lim\limits_{x\to 3} \dfrac{x-3}{x^2(x-3)(x+3)} = \lim\limits_{x\to 3} \dfrac{1}{x^2(x+3)} = \dfrac{1}{54}.$

c. $\lim\limits_{x\to -3} \dfrac{x-3}{x^4-9x^2} = \lim\limits_{x\to -3} \dfrac{x-3}{x^2(x-3)(x+3)} = \lim\limits_{x\to -3} \dfrac{1}{x^2(x+3)}$, which does not exist.

2.4.32

a. $\lim\limits_{x\to 0} \dfrac{x-2}{x^5-4x^3} = \lim\limits_{x\to 0} \dfrac{x-2}{x^3(x-2)(x+2)} = \lim\limits_{x\to 0} \dfrac{1}{x^3(x+2)}$, which does not exist.

b. $\displaystyle\lim_{x\to 2}\frac{x-2}{x^5-4x^3}=\lim_{x\to 2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x^3(x+2)}=\frac{1}{32}$.

c. $\displaystyle\lim_{x\to -2}\frac{x-2}{x^5-4x^3}=\lim_{x\to -2}\frac{x-2}{x^3(x-2)(x+2)}=\lim_{x\to -2}\frac{1}{x^3(x+2)}$, which does not exist.

2.4.33 $\displaystyle\lim_{x\to 0}\frac{x^3-5x^2}{x^2}=\lim_{x\to 0}\frac{x^2(x-5)}{x^2}=\lim_{x\to 0}(x-5)=-5$.

2.4.34 $\displaystyle\lim_{t\to 5}\frac{4t^2-100}{t-5}=\lim_{t\to 5}\frac{4(t-5)(t+5)}{t-5}=\lim_{t\to 5}[4(t+5)]=40$.

2.4.35 $\displaystyle\lim_{x\to 1^+}\frac{x^2-5x+6}{x-1}=\lim_{x\to 1^+}\frac{(x-2)(x-3)}{x-1}=\infty$. (Note that as $x\to 1^+$, the numerator is near 2, while the denominator is near zero, but is positive. So the quotient is positive and large.)

2.4.36 $\displaystyle\lim_{z\to 4}\frac{z-5}{(z^2-10z+24)^2}=\lim_{z\to 4}\frac{z-5}{(z-4)^2(z-6)^2}=-\infty$. (Note that as $z\to 4$, the numerator is near -1 while the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.37 $\displaystyle\lim_{x\to 6^+}\frac{x-7}{\sqrt{x-6}}=-\infty$. (Note that as $x\to 6^+$ the numerator is near -1 and the denominator is near zero but is positive. So the quotient is negative with large absolute value.)

2.4.38 $\displaystyle\lim_{x\to 2^-}\frac{x-1}{\sqrt{(x-3)(x-2)}}=\infty$. Note that as $x\to 2^-$ the numerator is near 1 and the denominator is near zero but is positive. So the quotient is positive with large absolute value.)

2.4.39 $\displaystyle\lim_{\theta\to 0^+}\csc\theta=\lim_{\theta\to 0^+}\frac{1}{\sin\theta}=\infty$.

2.4.40 $\displaystyle\lim_{x\to 0^-}\csc x=\lim_{x\to 0^-}\frac{1}{\sin x}=-\infty$.

2.4.41 $\displaystyle\lim_{x\to 0^+}-10\cot x=\lim_{x\to 0^+}\frac{-10\cos x}{\sin x}=-\infty$. (Note that as $x\to 0^+$, the numerator is near -10 and the denominator is near zero, but is positive. Thus the quotient is a negative number whose absolute value is large.)

2.4.42 $\displaystyle\lim_{\theta\to(\pi/2)^+}\frac{1}{3}\tan\theta=\lim_{\theta\to(\pi/2)^+}\frac{\sin\theta}{3\cos\theta}=-\infty$. (Note that as $\theta\to(\pi/2)^+$, the numerator is near 1 and the denominator is near 0, but is negative. Thus the quotient is a negative number whose absolute value is large.)

2.4.43 $\displaystyle\lim_{\theta\to 0}\frac{2+\sin\theta}{1-\cos^2\theta}=\infty$. (Note that as $\theta\to 0$, the numerator is near 2 and the denominator is near 0, but is positive. Thus the quotient is a positive number whose absolute value is large.)

2.4.44 $\displaystyle\lim_{\theta\to 0^-}\frac{\sin\theta}{\cos^2\theta-1}=\lim_{\theta\to 0^-}\frac{\sin\theta}{-\sin^2\theta}=-\lim_{\theta\to 0^-}\frac{1}{\sin\theta}=\infty$.

2.4.45

a. $\displaystyle\lim_{x\to 5}\frac{x-5}{x^2-25}=\lim_{x\to 5}\frac{1}{x+5}=\frac{1}{10}$, so there isn't a vertical asymptote at $x=5$.

b. $\displaystyle\lim_{x\to -5^-}\frac{x-5}{x^2-25}=\lim_{x\to -5^-}\frac{1}{x+5}=-\infty$, so there is a vertical asymptote at $x=-5$.

c. $\displaystyle\lim_{x\to -5^+}\frac{x-5}{x^2-25}=\lim_{x\to -5^+}\frac{1}{x+5}=\infty$. This also implies that $x=-5$ is a vertical asymptote, as we already noted in part b.

2.4.46

a. $\displaystyle\lim_{x\to7^-}\frac{x+7}{x^4-49x^2}=\lim_{x\to7^-}\frac{x+7}{x^2(x+7)(x-7)}=\lim_{x\to7^-}\frac{1}{x^2(x-7)}=-\infty$, so there is a vertical asymptote at $x=7$.

b. $\displaystyle\lim_{x\to7^+}\frac{x+7}{x^4-49x^2}=\lim_{x\to7^+}\frac{x+7}{x^2(x+7)(x-7)}=\lim_{x\to7^+}\frac{1}{x^2(x-7)}=\infty$. This also implies that there is a vertical asymptote at $x=7$, as we already noted in part a.

c. $\displaystyle\lim_{x\to-7}\frac{x+7}{x^4-49x^2}=\lim_{x\to-7}\frac{x+7}{x^2(x+7)(x-7)}=\lim_{x\to-7}\frac{1}{x^2(x-7)}=\frac{1}{-686}$. So there is not a vertical asymptote at $x=7$.

d. $\displaystyle\lim_{x\to0}\frac{x+7}{x^4-49x^2}=\lim_{x\to0}\frac{x+7}{x^2(x+7)(x-7)}=\lim_{x\to0}\frac{1}{x^2(x-7)}=-\infty$. So there is a vertical asymptote at $x=0$.

2.4.47 $f(x)=\dfrac{x^2-9x+14}{x^2-5x+6}=\dfrac{(x-2)(x-7)}{(x-2)(x-3)}$. Note that $x=3$ is a vertical asymptote, while $x=2$ appears to be a candidate but isn't one. We have $\displaystyle\lim_{x\to3^+}f(x)=\lim_{x\to3^+}\frac{x-7}{x-3}=-\infty$ and $\displaystyle\lim_{x\to3^-}f(x)=\lim_{x\to3^-}\frac{x-7}{x-3}=\infty$, and thus $\displaystyle\lim_{x\to3}f(x)$ doesn't exist. Note that $\displaystyle\lim_{x\to2}f(x)=5$.

2.4.48 $f(x)=\dfrac{\cos x}{x(x+2)}$ has vertical asymptotes at $x=0$ and at $x=-2$. Note that $\cos x$ is near 1 when x is near 0 , and $\cos x$ is near -0.4 when x is near -2. Thus, $\displaystyle\lim_{x\to0^+}f(x)=+\infty$, $\displaystyle\lim_{x\to0^-}f(x)=-\infty$, $\displaystyle\lim_{x\to-2^+}f(x)=\infty$, and $\displaystyle\lim_{x\to-2^-}f(x)=-\infty$.

2.4.49 $f(x)=\dfrac{x+1}{x^3-4x^2+4x}=\dfrac{x+1}{x(x-2)^2}$. There are vertical asymptotes at $x=0$ and $x=2$. We have $\displaystyle\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{x+1}{x(x-2)^2}=-\infty$, while $\displaystyle\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{x+1}{x(x-2)^2}=\infty$, and thus $\displaystyle\lim_{x\to0}f(x)$ doesn't exist.

Also we have $\displaystyle\lim_{x\to2^-}f(x)=\lim_{x\to2^-}\frac{x+1}{x(x-2)^2}=\infty$, while $\displaystyle\lim_{x\to2^+}f(x)=\lim_{x\to2^+}\frac{x+1}{x(x-2)^2}=\infty$, and thus $\displaystyle\lim_{x\to2}f(x)=\infty$ as well.

2.4.50 $g(x)=\dfrac{x^3-10x^2+16x}{x^2-8x}=\dfrac{x(x-2)(x-8)}{x(x-8)}$. This function has no vertical asymptotes.

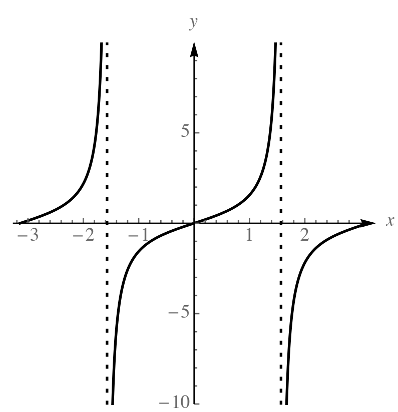
2.4.51

a. $\displaystyle\lim_{x\to(\pi/2)^+}\tan x=-\infty$.

b. $\displaystyle\lim_{x\to(\pi/2)^-}\tan x=\infty$.

c. $\displaystyle\lim_{x\to(-\pi/2)^+}\tan x=-\infty$.

d. $\displaystyle\lim_{x\to(-\pi/2)^-}\tan x=\infty$.



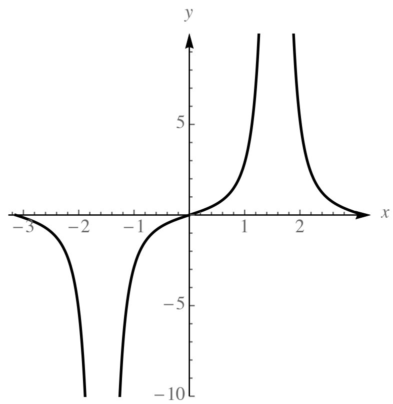
2.4.52

a. $\displaystyle\lim_{x\to(\pi/2)^+} \sec x \tan x = \infty.$

b. $\displaystyle\lim_{x\to(\pi/2)^-} \sec x \tan x = \infty.$

c. $\displaystyle\lim_{x\to(-\pi/2)^+} \sec x \tan x = -\infty.$

d. $\displaystyle\lim_{x\to(-\pi/2)^-} \sec x \tan x = -\infty.$



2.4.53

a. False. $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x-6)}{(x-1)(x+1)} = -\frac{5}{2}.$

b. True. For example, $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{(x-1)(x-6)}{(x-1)(x+1)} = -\infty.$

c. False. For example $g(x) = \frac{1}{x-1}$ has $\displaystyle\lim_{x\to 1^+} g(x) = \infty$, but $\displaystyle\lim_{x\to 1^-} g(x) = -\infty$.

2.4.54

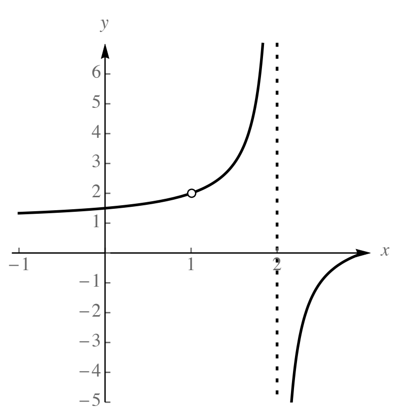
function	a	b	c	d	e	f
graph	D	C	F	B	A	E

2.4.55 We are seeking a function with a factor of $x-1$ in the denominator, but there should be more factors of $x-1$ in the numerator, and there should be a factor of $(x-2)^2$ in the denominator. This will accomplish the desired results. So

$$r(x) = \frac{(x-1)^2}{(x-1)(x-2)^2}.$$

2.4.56

One such function is $f(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-1)(x-3)}{(x-1)(x-2)}$.



2.4.57 One example is $f(x) = \frac{1}{x-6}$.

2.4.58 $f(x) = \dfrac{x^2-1}{(x^2-1)(x^2+1)} = \dfrac{1}{x^2+1}$ (for $x \neq \pm 1$). There are no vertical asymptotes, because for all a, $\lim_{x\to a} f(x) = \dfrac{1}{a^2+1}$.

2.4.59 $f(x) = \dfrac{x^2-3x+2}{x^{10}-x^9} = \dfrac{(x-2)(x-1)}{x^9(x-1)}$. f has a vertical asymptote at $x=0$, because $\lim_{x\to 0^+} f(x) = -\infty$ (and $\lim_{x\to 0^-} f(x) = \infty$.) Note that $\lim_{x\to 1} f(x) = -1$, so there isn't a vertical asymptote at $x=1$.

2.4.60 $g(x) = 2 - \ln x^2$ has a vertical asymptote at $x=0$, because $\lim_{x\to 0}(2-\ln x^2) = \infty$.

2.4.61 $h(x) = \dfrac{e^x}{(x+1)^3}$ has a vertical asymptote at $x=-1$, because

$$\lim_{x\to -1^+} \frac{e^x}{(x+1)^3} = \infty \quad \text{and} \quad \lim_{x\to -1^-} h(x) = -\infty.$$

2.4.62 $p(x) = \sec(\pi x/2) = \dfrac{1}{\cos(\pi x/2)}$ has a vertical asymptote on $(-2,2)$ at $x = \pm 1$.

2.4.63 $g(\theta) = \tan(\pi\theta/10) = \dfrac{\sin(\pi\theta/10)}{\cos(\pi\theta/10)}$ has a vertical asymptote at each $\theta = 10n+5$ where n is an integer. This is due to the fact that $\cos(\pi\theta/10) = 0$ when $\pi\theta/10 = \pi/2 + n\pi$ where n is an integer, which is the same as $\{\theta\colon \theta = 10n+5, n \text{ an integer}\}$. Note that at all of these numbers which make the denominator zero, the numerator isn't zero.

2.4.64 $q(s) = \dfrac{\pi}{s - \sin s}$ has a vertical asymptote at $s=0$. Note that this is the only number where $\sin s = s$.

2.4.65 $f(x) = \frac{1}{\sqrt{x}\sec x} = \frac{\cos x}{\sqrt{x}}$ has a vertical asymptote at $x=0$.

2.4.66 $g(x) = e^{1/x}$ has a vertical asymptote at $x=0$, because $\lim_{x\to 0^+} e^{1/x} = \infty$. (Note that as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty$ as well.)

2.4.67

a. Note that the numerator of the given expression factors as $(x-3)(x-4)$. So if $a=3$ or if $a=4$ the limit would be a finite number. In fact, $\lim_{x\to 3} \dfrac{(x-3)(x-4)}{x-3} = -1$ and $\lim_{x\to 4} \dfrac{(x-3)(x-4)}{x-4} = 1$.

b. For any number other than 3 or 4, the limit would be either $\pm\infty$. Because $x-a$ is always positive as $x \to a^+$, the limit would be $+\infty$ exactly when the numerator is positive, which is for a in the set $(-\infty, 3) \cup (4, \infty)$.

c. The limit would be $-\infty$ for a in the set $(3,4)$.

2.4.68

a. The slope of the secant line is given by $\dfrac{f(h)-f(0)}{h} = \dfrac{h^{1/3}}{h} = h^{-2/3}$.

b. $\lim_{h\to 0} \dfrac{1}{\sqrt[3]{h^2}} = \infty$. This tells us that the slope of the tangent line is infinite – which means that the tangent line at $(0,0)$ is vertical.

2.4.69

a. The slope of the secant line is $\dfrac{f(h)-f(0)}{h} = \dfrac{h^{2/3}}{h} = h^{-1/3}$.

b. $\lim_{h\to 0^+} \dfrac{1}{h^{1/3}} = \infty$, and $\lim_{h\to 0^-} \dfrac{1}{h^{1/3}} = -\infty$. The tangent line is infinitely steep at the origin (i.e., it is a vertical line.)

2.5 Limits at Infinity

2.5.1 As $x < 0$ becomes arbitrarily large in absolute value, the corresponding values of f approach 10.

2.5.2 $\lim_{x\to\infty} f(x) = -2$ and $\lim_{x\to-\infty} f(x) = 4$.

2.5.3 $\lim_{x\to\infty} x^{12} = \infty$. Note that x^{12} is positive when $x > 0$.

2.5.4 $\lim_{x\to-\infty} 3x^{11} = -\infty$. Note that x^{11} is negative when $x < 0$.

2.5.5 $\lim_{x\to\infty} x^{-6} = \lim_{x\to\infty} \frac{1}{x^6} = 0$.

2.5.6 $\lim_{x\to-\infty} x^{-11} = \lim_{x\to-\infty} \frac{1}{x^{11}} = 0$.

2.5.7 $\lim_{t\to\infty} (-12t^{-5}) = \lim_{t\to\infty} -\frac{12}{t^5} = 0$.

2.5.8 $\lim_{x\to-\infty} 2x^{-8} = \lim_{x\to-\infty} \frac{2}{x^8} = 0$.

2.5.9 $\lim_{x\to\infty} (3 + 10/x^2) = 3 + \lim_{x\to\infty} (10/x^2) = 3 + 0 = 3$.

2.5.10 $\lim_{x\to\infty} (5 + 1/x + 10/x^2) = 5 + \lim_{x\to\infty} (1/x) + \lim_{x\to\infty} (10/x^2) = 5 + 0 + 0 = 5$.

2.5.11 If $f(x) \to 100,000$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, then the ratio $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. (Because *eventually* the values of f are small compared to the values of g.)

2.5.12 $\lim_{x\to\infty} \frac{3 + 2x + 4x^2}{x^2} = \lim_{x\to\infty} \frac{3}{x^2} + \lim_{x\to\infty} \frac{2x}{x^2} + \lim_{x\to\infty} \frac{4x^2}{x^2} = 0 + \lim_{x\to\infty} \frac{2}{x} + \lim_{x\to\infty} 4 = 0 + 0 + 4 = 4$.

2.5.13 $\lim_{t\to\infty} e^t = \infty$, $\lim_{t\to-\infty} e^t = 0$, and $\lim_{t\to\infty} e^{-t} = 0$.

2.5.14 As $x \to \infty$, we note that $e^{-2x} \to 0$, while as $x \to -\infty$, we have $e^{-2x} \to \infty$.

2.5.15 Because $\lim_{x\to\infty} 3 - \frac{1}{x^2} = 3$ and $\lim_{x\to\infty} 3 + \frac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to\infty} g(x) = 3$.

Similarly, because $\lim_{x\to-\infty} 3 - \frac{1}{x^2} = 3$ and $\lim_{x\to-\infty} 3 + \frac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim_{x\to-\infty} g(x) = 3$.

2.5.16 $\lim_{x\to-\infty} g(x) = 3$, $\lim_{x\to\infty} g(x) = -1$, $\lim_{x\to-2^-} g(x) = \infty$, $\lim_{x\to 2^+} g(x) = -\infty$.

2.5.17 $\lim_{\theta\to\infty} \frac{\cos\theta}{\theta^2} = 0$. Note that $-1 \le \cos\theta \le 1$, so $-\frac{1}{\theta^2} \le \frac{\cos\theta}{\theta^2} \le \frac{1}{\theta^2}$. The result now follows from the Squeeze Theorem.

2.5.18 Note that $\frac{5t^2 + t\sin t}{t^2}$ can be written as $5 + \frac{\sin t}{t}$. Also, note that because $-1 \le \sin t \le 1$, we have $-\frac{1}{t} \le \frac{\sin t}{t} \le \frac{1}{t}$, so $\frac{\sin t}{t} \to 0$ as $t \to \infty$ by the Squeeze Theorem. Therefore,

$$\lim_{t\to\infty} \frac{5t^2 + t\sin t}{t^2} = \lim_{t\to\infty} \left(5 + \frac{\sin t}{t}\right) = 5 + 0 = 5.$$

2.5.19 $\displaystyle\lim_{x\to\infty}\frac{\cos x^5}{\sqrt{x}}=0$. Note that $-1\le\cos x^5\le 1$, so $\frac{-1}{\sqrt{x}}\le\frac{\cos x^5}{\sqrt{x}}\le\frac{1}{\sqrt{x}}$. Because $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x}}=\lim_{x\to\infty}\frac{-1}{\sqrt{x}}=0$, we have $\displaystyle\lim_{x\to\infty}\frac{\cos x^5}{\sqrt{x}}=0$ by the Squeeze Theorem.

2.5.20 $\displaystyle\lim_{x\to-\infty}\left(5+\frac{100}{x}+\frac{\sin^4(x^3)}{x^2}\right)=5+0+0=5$. For this last limit, note that $0\le\sin^4(x^3)\le 1$, so $\displaystyle 0\le\frac{\sin^4(x^3)}{x^2}\le\frac{1}{x^2}$. The result now follows from the Squeeze Theorem.

2.5.21 $\displaystyle\lim_{x\to\infty}(3x^{12}-9x^7)=\infty$.

2.5.22 $\displaystyle\lim_{x\to-\infty}(3x^7+x^2)=-\infty$.

2.5.23 $\displaystyle\lim_{x\to-\infty}(-3x^{16}+2)=-\infty$.

2.5.24 $\displaystyle\lim_{x\to-\infty}(2x^{-8}+4x^3)=0+\lim_{x\to-\infty}4x^3=-\infty$.

2.5.25 $\displaystyle\lim_{x\to\infty}\frac{(14x^3+3x^2-2x)}{(21x^3+x^2+2x+1)}\cdot\frac{1/x^3}{1/x^3}=\lim_{x\to\infty}\frac{14+(3/x)-(2/x^2)}{21+(1/x)+(2/x^2)+(1/x^3)}=\frac{14}{21}=\frac{2}{3}$.

2.5.26 $\displaystyle\lim_{x\to\infty}\frac{(9x^3+x^2-5)}{(3x^4+4x^2)}\cdot\frac{1/x^4}{1/x^4}=\lim_{x\to\infty}\frac{(9/x)+(1/x^2)-(5/x^4)}{3+(4/x^2)}=\frac{0}{3}=0$.

2.5.27 $\displaystyle\lim_{x\to-\infty}\frac{(3x^2+3x)}{(x+1)}\cdot\frac{1/x}{1/x}=\lim_{x\to-\infty}\frac{3x+3}{1+(1/x)}=-\infty$.

2.5.28 $\displaystyle\lim_{x\to\infty}\frac{(x^4+7)}{(x^5+x^2-x)}\cdot\frac{1/x^5}{1/x^5}=\lim_{x\to\infty}\frac{(1/x)+(7/x^5)}{1+(1/x^3)-(1/x^4)}=\frac{0+0}{1+0-0}=0$.

2.5.29 Note that for $w>0$, $w^2=\sqrt{w^4}$. We have

$$\lim_{w\to\infty}\frac{(15w^2+3w+1)}{\sqrt{9w^4+w^3}}\cdot\frac{1/w^2}{1/\sqrt{w^4}}=\lim_{w\to\infty}\frac{15+(3/w)+(1/w^2)}{\sqrt{9+(1/w)}}=\frac{15}{\sqrt{9}}=5.$$

2.5.30 Note that $\sqrt{x^8}=x^4$ (even for $x<0$). We have

$$\lim_{x\to-\infty}\frac{(40x^4+x^2+5x)}{\sqrt{64x^8+x^6}}\cdot\frac{1/x^4}{1/\sqrt{x^8}}=\lim_{x\to-\infty}\frac{40+(1/x^2)+(5/x^3)}{\sqrt{64+(1/x^2)}}=\frac{40}{\sqrt{64}}=\frac{40}{8}=5.$$

2.5.31 Note that for $x<0$, $\sqrt{x^2}=-x$. We have

$$\lim_{x\to-\infty}\frac{\sqrt{16x^2+x}}{x}\cdot\frac{\sqrt{1/x^2}}{-1/x}=\lim_{x\to-\infty}\frac{\sqrt{16+(1/x)}}{-1}=-\sqrt{16}=-4.$$

2.5.32 Note that $x^2=\sqrt{x^4}$ for all x. We have

$$\lim_{x\to\infty}\frac{6x^2}{(4x^2+\sqrt{16x^4+x^2})}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{6}{(4+\sqrt{16+(1/x^2)})}=\frac{6}{4+\sqrt{16}}=\frac{3}{4}.$$

2.5.33 $\displaystyle\lim_{x\to\infty}\frac{(x^2-\sqrt{x^4+3x^2})}{1}\cdot\frac{(x^2+\sqrt{x^4+3x^2})}{(x^2+\sqrt{x^4+3x^2})}=\lim_{x\to\infty}\frac{x^4-(x^4+3x^2)}{x^2+\sqrt{x^4+3x^2}}=\lim_{x\to\infty}\frac{-3x^2}{x^2+\sqrt{x^4+3x^2}}$. Now divide the numerator and denominator by x^2 to give

$$\lim_{x\to\infty}\frac{-3x^2}{x^2+\sqrt{x^4+3x^2}}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{-3}{1+\sqrt{1+(3/x^2)}}=-\frac{3}{2}.$$

2.5.34 $\lim_{x\to-\infty} \frac{(x+\sqrt{x^2-5x})}{1}\cdot\frac{(x-\sqrt{x^2-5x})}{(x-\sqrt{x^2-5x})} = \lim_{x\to-\infty} \frac{x^2-(x^2-5x)}{x-\sqrt{x^2-5x}} = \lim_{x\to-\infty} \frac{5x}{x-\sqrt{x^2-5x}}$. Now divide the numerator and denominator by x (and recall that for $x<0$ we have $-\sqrt{x^2}=x$) giving

$$\lim_{x\to-\infty} \frac{5x}{(x-\sqrt{x^2-5x})}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{5}{1+\sqrt{1-(5/x)}} = \frac{5}{2}.$$

2.5.35 Note that because $-1\le \sin x\le 1$, we have $-\frac{1}{e^x}\le\frac{\sin x}{e^x}\le\frac{1}{e^x}$. Then because $\lim_{x\to\infty}\frac{\pm 1}{e^x}=0$, the Squeeze Theorem tells us that $\lim_{x\to\infty}\frac{\sin x}{e^x}=0$.

2.5.36 Note that because $-1\le\cos x\le 1$, we have $-e^x\le e^x\cos x\le e^x$. Then $3-e^x\le e^x\cos x+3\le e^x+3$. Because $\lim_{x\to-\infty} 3-e^x=3$ and $\lim_{x\to-\infty} e^x+3=3$, the Squeeze Theorem tells us that $\lim_{x\to-\infty} e^x\cos x+3=3$.

2.5.37 $\lim_{x\to\infty}\frac{4x}{20x+1} = \lim_{x\to\infty}\frac{4x}{20x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{4}{20+1/x} = \frac{4}{20} = \frac{1}{5}$. Thus, the line $y=\frac{1}{5}$ is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{4x}{20x+1} = \lim_{x\to-\infty}\frac{4x}{20x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty}\frac{4}{20+1/x} = \frac{4}{20} = \frac{1}{5}$. This shows that the curve is also asymptotic to the asymptote in the negative direction.

2.5.38 $\lim_{x\to\infty}\frac{3x^2-7}{x^2+5x} = \lim_{x\to\infty}\frac{3x^2-7}{x^2+5x}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{3-(7/x^2)}{1+(5/x)} = \frac{3-0}{1+0} = 3$. Thus, the line $y=3$ is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty}\frac{3x^2-7}{x^2+5x}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to-\infty}\frac{3-(7/x^2)}{1+(5/x)} = \frac{3-0}{1+0} = 3$. Thus, the curve is also asymptotic to the asymptote in the negative direction.

2.5.39 $\lim_{x\to\infty}\frac{(6x^2-9x+8)}{(3x^2+2)}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{6-9/x+8/x^2}{3+2/x^2} = \frac{6-0+0}{3+0} = 2$. Similarly $\lim_{x\to-\infty} f(x)=2$. The line $y=2$ is a horizontal asymptote.

2.5.40 $\lim_{x\to\infty}\frac{(12x^8-3)}{(3x^8-2x^7)}\cdot\frac{1/x^8}{1/x^8} = \lim_{x\to\infty}\frac{12-3/x^8}{3-2/x} = \frac{12-0}{3-0} = 4$. Similarly $\lim_{x\to-\infty} f(x)=4$. The line $y=4$ is a horizontal asymptote.

2.5.41 $\lim_{x\to\infty}\frac{3x^3-7}{x^4+5x^2} = \lim_{x\to\infty}\frac{3x^3-7}{x^4+5x^2}\cdot\frac{3/x^4}{1/x^4} = \lim_{x\to\infty}\frac{1/x-(7/x^4)}{1+(5/x^2)} = \frac{0-0}{1+0} = 0$. Thus, the line $y=0$ (the x-axis) is a horizontal asymptote.

$\lim_{x\to-\infty}\frac{3x^3-7}{x^4+5x^2} = \lim_{x\to-\infty}\frac{3x^3-7}{x^4+5x^2}\cdot\frac{3/x^4}{1/x^4} = \lim_{x\to-\infty}\frac{1/x-(7/x^4)}{1+(5/x^2)} = \frac{0-0}{1+0} = 0$. Thus, the curve is asymptotic to the x-axis in the negative direction as well.

2.5.42 $\lim_{x\to\infty}\frac{(2x+1)}{(3x^4-2)}\cdot\frac{1/x^4}{1/x^4} = \lim_{x\to\infty}\frac{2/x^3+1/x^4}{3-2/x^4} = \frac{0+0}{3-0} = 0$. Similarly $\lim_{x\to-\infty} f(x)=0$. The line $y=0$ is a horizontal asymptote.

2.5.43 $\lim_{x\to\infty}\frac{(40x^5+x^2)}{(16x^4-2x)}\cdot\frac{1/x^4}{1/x^4} = \lim_{x\to\infty}\frac{40x+1/x^2}{16-2/x^3} = \infty$. Similarly $\lim_{x\to-\infty} f(x) = -\infty$. There are no horizontal asymptotes.

2.5.44 Note that for all x, $\sqrt{x^4}=x^2$. Then

$$\lim_{x\to\pm\infty}\frac{(6x^2+1)}{\sqrt{4x^4+3x+1}}\cdot\frac{1/x^2}{\sqrt{1/x^4}} = \lim_{x\to\pm\infty}\frac{6+(1/x^2)}{\sqrt{4+(3/x^3)+(1/x^4)}} = \frac{6}{\sqrt{4}} = 3.$$

So $y=3$ is the only horizontal asymptote.

2.5.45 Note that for all x, $\sqrt{x^8} = x^4$. Then $\lim_{x\to\pm\infty} \frac{1}{(2x^4 - \sqrt{4x^8 - 9x^4}} \cdot \frac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(2x^4 + \sqrt{4x^8 - 9x^4})}$
$= \lim_{x\to\pm\infty} \frac{(2x^4 + \sqrt{4x^8 - 9x^4})}{(4x^8 - (4x^8 - 9x^4))} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\pm\infty} \frac{2 + \sqrt{4 - (9/x^4)}}{9} = \frac{4}{9}$.
So $y = \frac{4}{9}$ is the only horizontal asymptote.

2.5.46 First note that $\sqrt{x^2} = x$ for $x > 0$, while $\sqrt{x^2} = -x$ for $x < 0$. Then $\lim_{x\to\infty} f(x)$ can be written as

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{1/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2+1/x} = \frac{1}{2}.$$

However, $\lim_{x\to-\infty} f(x)$ can be written as

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{1+1/x^2}}{-2-1/x} = -\frac{1}{2}.$$

2.5.47 First note that $\sqrt{x^6} = x^3$ if $x > 0$, but $\sqrt{x^6} = -x^3$ if $x < 0$. We have $\lim_{x\to\infty} \frac{4x^3+1}{(2x^3 + \sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} =$
$\lim_{x\to\infty} \frac{4 + 1/x^3}{2 + \sqrt{16 + 1/x^6}} = \frac{4+0}{2+\sqrt{16+0}} = \frac{2}{3}$.

However, $\lim_{x\to-\infty} \frac{4x^3+1}{(2x^3 + \sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to-\infty} \frac{4+1/x^3}{2 - \sqrt{16+1/x^6}} = \frac{4+0}{2-\sqrt{16+0}} = \frac{4}{-2} = -2$.

So $y = \frac{2}{3}$ is a horizontal asymptote (as $x \to \infty$) and $y = -2$ is a horizontal asymptote (as $x \to -\infty$).

2.5.48 First note that for $x > 0$ we have $\sqrt{x^2} = x$, but for $x < 0$, we have $-x = \sqrt{x^2}$. Then we have $\lim_{x\to\infty} x - \sqrt{x^2 - 9x} = \lim_{x\to\infty} \frac{(x - \sqrt{x^2-9x})(x + \sqrt{x^2-9x})}{x + \sqrt{x^2-9x}} = \lim_{x\to\infty} \frac{9x}{(x+\sqrt{x^2-9x})} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{9}{1+\sqrt{1-9/x}} = \frac{9}{2}$.
On the other hand, $\lim_{x\to-\infty} x - \sqrt{x^2-9x} = \lim_{x\to-\infty} \frac{(x-\sqrt{x^2-9x})(x+\sqrt{x^2-9x})}{x+\sqrt{x^2-9x}} =$
$\lim_{x\to-\infty} \frac{9x}{(x+\sqrt{x^2-9x})} \cdot \frac{-1/x}{-1/x} = \lim_{x\to-\infty} \frac{-9}{-1+\sqrt{1-9/x}} = -\infty$. The last equal sign follows because $\sqrt{1-9/x} > 1$ but is approaching 1 as $x \to -\infty$. We can therefore conclude that $y = \frac{9}{2}$ is the only horizontal asymptote, and is an asymptote as $x \to \infty$.

2.5.49 First note that $\sqrt[3]{x^6} = x^2$ and $\sqrt{x^4} = x^2$ for all x (even when $x < 0$.) We have $\lim_{x\to\infty} \frac{\sqrt[3]{x^6+8}}{(4x^2+\sqrt{3x^4+1})} \cdot$
$\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt[3]{1+8/x^6}}{4+\sqrt{3+1/x^4}} = \frac{1}{4+\sqrt{3+0}} = \frac{1}{4\sqrt{3}}$.

The calculation as $x \to -\infty$ is similar. So $y = \frac{1}{4\sqrt{3}}$ is a horizontal asymptote.

2.5.50 First note that $\sqrt{x^2} = x$ for $x > 0$ and $\sqrt{x^2} = -x$ for $x < 0$.

We have

$$\begin{aligned}
\lim_{x\to\infty} 4x(3x - \sqrt{9x^2+1}) &= \lim_{x\to\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\
&= \lim_{x\to\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})} \cdot \frac{1/x}{1/x} \\
&= \lim_{x\to\infty} -\frac{4}{3+\sqrt{9+1/x^2}} = -\frac{4}{6} = -\frac{2}{3}.
\end{aligned}$$

Moreover, as $x \to -\infty$ we have

$$\begin{aligned}\lim_{x\to-\infty} 4x(3x-\sqrt{9x^2+1}) &= \lim_{x\to-\infty} \frac{4x(3x-\sqrt{9x^2+1})(3x+\sqrt{9x^2+1})}{3x+\sqrt{9x^2+1}} \\ &= \lim_{x\to-\infty} \frac{(4x)(-1)}{(3x+\sqrt{9x^2+1})}\cdot\frac{1/x}{1/x} \\ &= \lim_{x\to-\infty} -\frac{4}{3-\sqrt{9+1/x^2}} = \infty.\end{aligned}$$

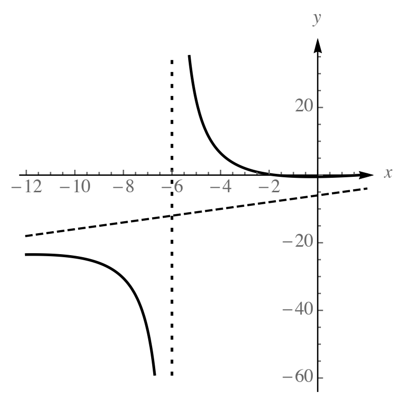
Note that this last equality is due to the fact that the numerator is the constant -4 and the denominator is approaching zero (from the left) so the quotient is positive and is getting large.

So $y = -\frac{2}{3}$ is the only horizontal asymptote.

2.5.51

a. $f(x) = \dfrac{x^2-3}{x+6} = x - 6 + \dfrac{33}{x+6}$. The oblique asymptote of f is $y = x - 6$.

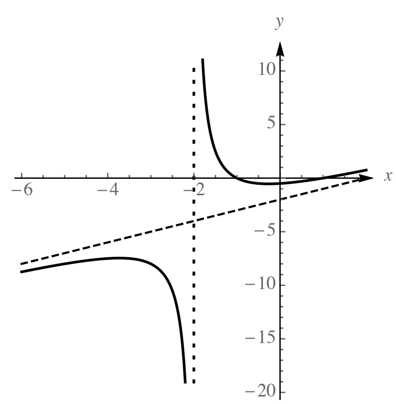
b. Because $\lim_{x\to-6^+} f(x) = \infty$, there is a vertical asymptote at $x = -6$. Note also that $\lim_{x\to-6^-} f(x) = -\infty$.

c. 

2.5.52

a. $f(x) = \dfrac{x^2-1}{x+2} = x - 2 + \dfrac{3}{x+2}$. The oblique asymptote of f is $y = x - 2$.

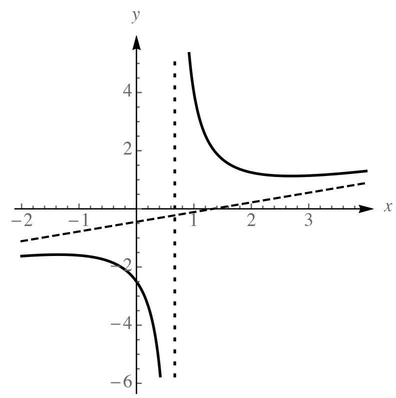
b. Because $\lim_{x\to-2^+} f(x) = \infty$, there is a vertical asymptote at $x = -2$. Note also that $\lim_{x\to-2^-} f(x) = -\infty$.

c. 

2.5.53

a. $f(x) = \dfrac{x^2-2x+5}{3x-2} = (1/3)x - 4/9 + \dfrac{37}{9(3x-2)}$. The oblique asymptote of f is $y = (1/3)x - 4/9$.

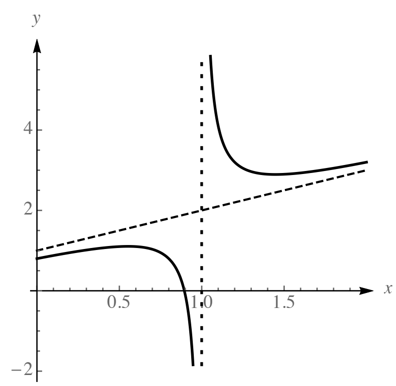
b. Because $\lim_{x\to(2/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = 2/3$. Note also that $\lim_{x\to(2/3)^-} f(x) = -\infty$.

c. 

2.5.54

a. $f(x) = \dfrac{5x^2-4}{5x-5} = x+1+\dfrac{1}{5x-5}$. The oblique asymptote of f is $y = x+1$.

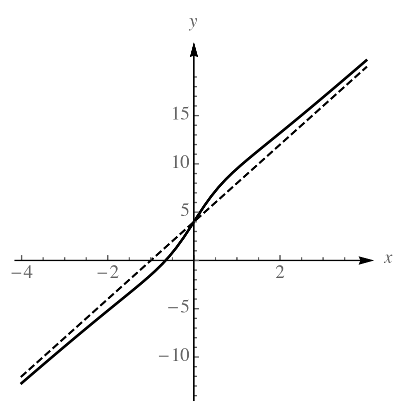
b. Because $\lim_{x\to1^+} f(x) = \infty$, there is a vertical asymptote at $x = 1$. Note also that $\lim_{x\to1^-} f(x) = -\infty$.

c. 

2.5.55

a. $f(x) = \dfrac{4x^3+4x^2+7x+4}{1+x^2} = 4x+4+\dfrac{3x}{1+x^2}$. The oblique asymptote of f is $y = 4x+4$.

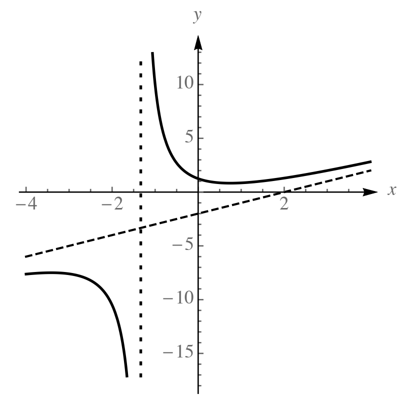
b. There are no vertical asymptotes.

c. 

2.5.56

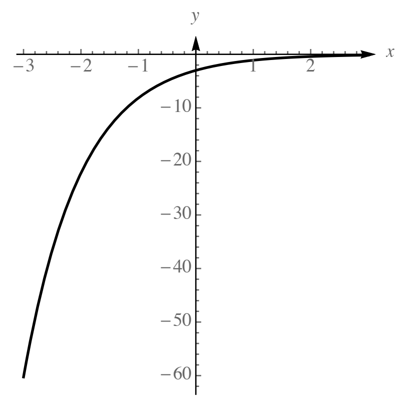
a. $f(x) = \dfrac{3x^2-2x+5}{3x+4} = x-2+\dfrac{13}{3x+4}$. The oblique asymptote of f is $y = x-2$.

b. Because $\lim_{x\to(-4/3)^+} f(x) = \infty$, there is a vertical asymptote at $x = -4/3$. Note also that $\lim_{x\to(-4/3)^-} f(x) = -\infty$.

c. 

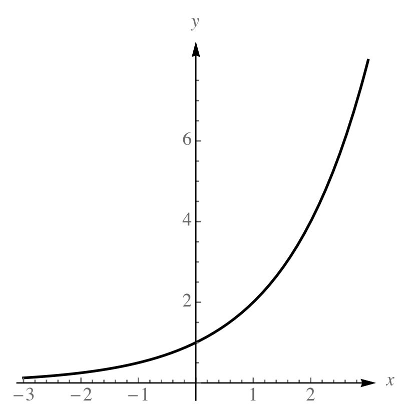
2.5.57

$\lim_{x\to\infty}(-3e^{-x}) = -3\cdot 0 = 0$. $\lim_{x\to-\infty}(-3e^{-x}) = -\infty$.



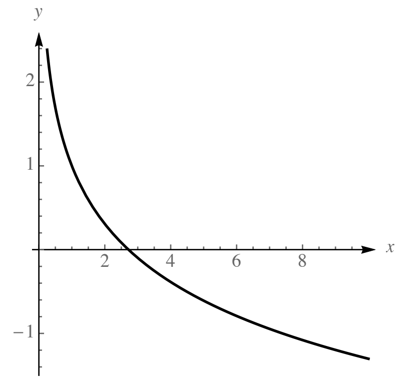
2.5.58

$\lim_{x\to\infty} 2^x = \infty$. $\lim_{x\to-\infty} 2^x = 0$.



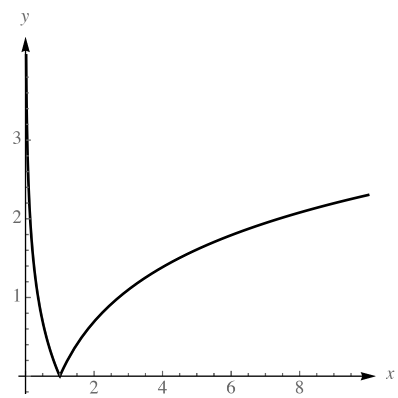
2.5.59

$\lim_{x\to\infty}(1-\ln x) = -\infty$. $\lim_{x\to 0^+}(1-\ln x) = \infty$.



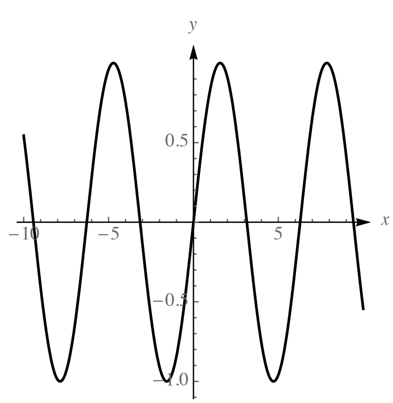
2.5.60

$\lim_{x\to\infty}|\ln x| = \infty$. $\lim_{x\to 0^+}|\ln x| = \infty$.



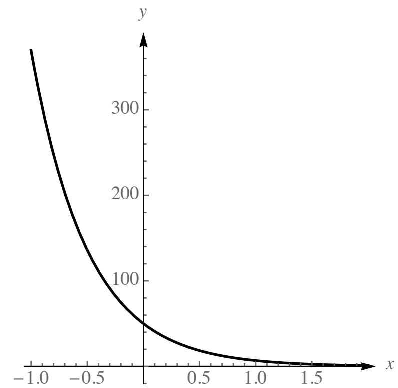
2.5.61

$y = \sin x$ has no asymptotes. $\lim_{x\to\infty}\sin x$ and $\lim_{x\to-\infty}\sin x$ do not exist.



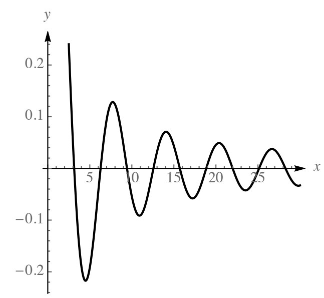
2.5.62

$\lim_{x\to\infty} \frac{50}{e^{2x}} = 0$. $\lim_{x\to-\infty} \frac{50}{e^{2x}} = \infty$.



2.5.63

a. False. For example, the function $y = \frac{\sin x}{x}$ on the domain $[1, \infty)$ has a horizontal asymptote of $y = 0$, and it crosses the x-axis infinitely many times.



b. False. If f is a rational function, and if $\lim_{x\to\infty} f(x) = L \neq 0$, then the degree of the polynomial in the numerator must equal the degree of the polynomial in the denominator. In this case, both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x) = \frac{a_n}{b_n}$ where a_n is the leading coefficient of the polynomial in the numerator and b_n is the leading coefficient of the polynomial in the denominator. In the case where $\lim_{x\to\infty} f(x) = 0$, then the degree of the numerator is strictly less than the degree of the denominator. This case holds for $\lim_{x\to-\infty} f(x) = 0$ as well.

c. True. There are only two directions which might lead to horizontal asymptotes: there could be one as $x \to \infty$ and there could be one as $x \to -\infty$, and those are the only possibilities.

d. False. The limit of the difference of two functions can be written as the difference of the limits only when both limits exist. It is the case that $\lim_{x\to\infty}(x^3 - x) = \infty$.

2.5.64 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{2500}{t+1} = 0$. The steady state exists. The steady state value is 0.

2.5.65 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{3500t}{t+1} = 3500$. The steady state exists. The steady state value is 3500.

2.5.66 $\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 200(1 - 2^{-t}) = 200$. The steady state exists. The steady state value is 200.

2.5.67 $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} 1000e^{0.065t} = \infty$. The steady state does not exist.

2.5.68 $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1500}{3 + 2e^{-.1t}} = \frac{1500}{3} = 500$. The steady state exists. The steady state value is 500.

2.5.69 $\lim_{t\to\infty} a(t) = \lim_{t\to\infty} 2\left(\frac{t+\sin t}{t}\right) = \lim_{t\to\infty} 2\left(1+\frac{\sin t}{t}\right) = 2$. The steady state exists. The steady state value is 2.

2.5.70

a. $\lim_{x\to\infty} \frac{x^2-4x+3}{x-1} = \infty$, and $\lim_{x\to-\infty} \frac{x^2-4x+3}{x-1} = -\infty$. There are no horizontal asymptotes.

b. It appears that $x=1$ is a candidate to be a vertical asymptote, but note that $f(x) = \frac{x^2-4x+3}{x-1} = \frac{(x-1)(x-3)}{x-1}$. Thus $\lim_{x\to1} f(x) = \lim_{x\to1}(x-3) = -2$. So f has no vertical asymptotes.

2.5.71

a. $\lim_{x\to\infty} \frac{2x^3+10x^2+12x}{x^3+2x^2}\cdot\frac{(1/x^3)}{(1/x^3)} = \lim_{x\to\infty}\frac{2+10/x+12/x^2}{1+2/x} = 2$. Similarly, $\lim_{x\to-\infty} f(x) = 2$. Thus, $y=2$ is a horizontal asymptote.

b. Note that $f(x) = \frac{2x(x+2)(x+3)}{x^2(x+2)}$. So $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{2(x+3)}{x} = \infty$, and similarly, $\lim_{x\to0^-} f(x) = -\infty$. There is a vertical asymptote at $x=0$. Note that there is no asymptote at $x=-2$ because $\lim_{x\to-2} f(x) = -1$.

2.5.72

a. We have $\lim_{x\to\infty}\frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}\cdot\frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty}\frac{\sqrt{16+64/x^2}+1}{2-4/x^2} = \frac{5}{2}$. Similarly, $\lim_{x\to-\infty} f(x) = \frac{5}{2}$. So $y=\frac{5}{2}$ is a horizontal asymptote.

b. $\lim_{x\to\sqrt{2}^+} f(x) = \lim_{x\to-\sqrt{2}^-} f(x) = \infty$, and $\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to-\sqrt{2}^+} f(x) = -\infty$ so there are vertical asymptotes at $x=\pm\sqrt{2}$.

2.5.73

a. We have $\lim_{x\to\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}\cdot\frac{(1/x^4)}{(1/x^4)} = \lim_{x\to\infty}\frac{3+3/x-36/x^2}{1-25/x^2+144/x^4} = 3$. Similarly, $\lim_{x\to-\infty} f(x) = 3$. So $y=3$ is a horizontal asymptote.

b. Note that $f(x) = \frac{3x^2(x+4)(x-3)}{(x+4)(x-4)(x+3)(x-3)}$. Thus, $\lim_{x\to-3^+} f(x) = -\infty$ and $\lim_{x\to-3^-} f(x) = \infty$. Also, $\lim_{x\to4^-} f(x) = -\infty$ and $\lim_{x\to4^+} f(x) = \infty$. Thus there are vertical asymptotes at $x=-3$ and $x=4$.

2.5.74

a. First note that

$$f(x) = x^2(4x^2-\sqrt{16x^4+1})\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}} = -\frac{x^2}{4x^2+\sqrt{16x^4+1}}.$$

We have $\lim_{x\to\infty} -\frac{x^2}{4x^2+\sqrt{16x^4+1}}\cdot\frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} -\frac{1}{4+\sqrt{16+1/x^4}} = -\frac{1}{8}$. Similarly, the limit as $x\to-\infty$ of $f(x)$ is $-\frac{1}{8}$ as well. So $y=-\frac{1}{8}$ is a horizontal asymptote.

b. f has no vertical asymptotes.

2.5.75

a. $\lim_{x\to\infty} \frac{x^2-9}{x^2-3x}\cdot\frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty}\frac{1-9/x^2}{1-3/x} = 1$. A similar result holds as $x \to -\infty$. So $y = 1$ is a horizontal asymptote.

b. Because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{x+3}{x} = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$, there is a vertical asymptote at $x = 0$.

2.5.76

a. $\lim_{x\to\pm\infty}\frac{x^4-1}{x^2-1} = \lim_{x\to\pm\infty}\frac{(x^2-1)(x^2+1)}{x^2-1} = \lim_{x\to\pm\infty} x^2+1 = \infty$. There are no horizontal or slant asymptotes.

b. It appears that $x = \pm 1$ may be candidates for vertical asymptotes, but because

$$\frac{x^4-1}{x^2-1} = \frac{(x^2-1)(x^2+1)}{x-1} = x^2+1$$

for $x \neq \pm 1$ there are no vertical asymptotes either.

2.5.77

a. First note that $f(x)$ can be written s

$$\frac{\sqrt{x^2+2x+6}-3}{x-1}\cdot\frac{\sqrt{x^2+2x+6}+3}{\sqrt{x^2+2x+6}+3} = \frac{x^2+2x+6-9}{(x-1)(\sqrt{x^2+2x+6}+3)} = \frac{(x-1)(x+3)}{(x-1)(\sqrt{x^2+2x+6}+3)}.$$

Thus

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+3}{\sqrt{x^2+2x+6}+3}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{1+3/x}{\sqrt{1+2/x+6/x^2}+3/x} = 1.$$

Using the fact that $\sqrt{x^2} = -x$ for $x < 0$, we have $\lim_{x\to-\infty} f(x) = -1$. Thus the lines $y = 1$ and $y = -1$ are horizontal asymptotes.

b. f has no vertical asymptotes.

2.5.78

a. Note that when x is large $|1-x^2| = x^2-1$. We have $\lim_{x\to\infty}\frac{|1-x^2|}{x^2+x} = \lim_{x\to\infty}\frac{x^2-1}{x^2+x} = 1$. Likewise $\lim_{x\to-\infty}\frac{|1-x^2|}{x^2+x} = \lim_{x\to-\infty}\frac{x^2-1}{x^2+x} = 1$. So there is a horizontal asymptote at $y = 1$.

b. Note that when x is near 0, we have $|1-x^2| = 1-x^2 = (1-x)(1+x)$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{1-x}{x} = \infty$. Similarly, $\lim_{x\to 0^-} f(x) = -\infty$. There is a vertical asymptote at $x = 0$.

2.5.79

a. Note that when $x > 1$, we have $|x| = x$ and $|x-1| = x-1$. Thus

$$f(x) = (\sqrt{x}-\sqrt{x-1})\cdot\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}+\sqrt{x-1}}.$$

Thus $\lim_{x\to\infty} f(x) = 0$.

When $x < 0$, we have $|x| = -x$ and $|x-1| = 1-x$. Thus

$$f(x) = (\sqrt{-x}-\sqrt{1-x})\cdot\frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}+\sqrt{1-x}} = -\frac{1}{\sqrt{-x}+\sqrt{1-x}}.$$

Thus, $\lim_{x\to-\infty} f(x) = 0$. There is a horizontal asymptote at $y = 0$.

b. f has no vertical asymptotes.

2.5.80

a. $\lim_{x\to\infty} \frac{(3e^x+10)}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{3+(10/e^x)}{1} = 3$. On the other hand, $\lim_{x\to-\infty} \frac{3e^x+10}{e^x} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{3+(10e^{-x})}{1} = \infty$. $y=3$ is a horizontal asymptote as $x\to\infty$.

b. f has no vertical asymptotes.

2.5.81

a. $\lim_{x\to\infty} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \lim_{x\to\infty} \left(2+\frac{\cos x}{\sqrt{x}}\right) = 2$. $y=2$ is a horizontal asymptote.

b. $\lim_{x\to 0^+} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}} = \infty$. and $\lim_{x\to 0^-} \frac{\cos x + 2\sqrt{x}}{\sqrt{x}}$ does not exist. $x=0$ is a vertical asymptote.

2.5.82

a. $\lim_{x\to\infty} \cot^{-1} x = 0$.

b. $\lim_{x\to-\infty} \cot^{-1} x = \pi$.

2.5.83

a. $\lim_{x\to\infty} \sec^{-1} x = \pi/2$.

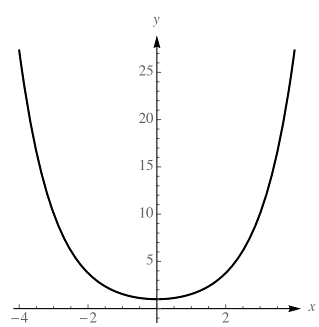
b. $\lim_{x\to-\infty} \sec^{-1} x = \pi/2$.

2.5.84

a. $\lim_{x\to\infty} \frac{e^x+e^{-x}}{2} = \infty$.

$\lim_{x\to-\infty} \frac{e^x+e^{-x}}{2} = \infty$.

b. $\cosh(0) = \frac{e^0+e^0}{2} = \frac{1+1}{2} = 1$.

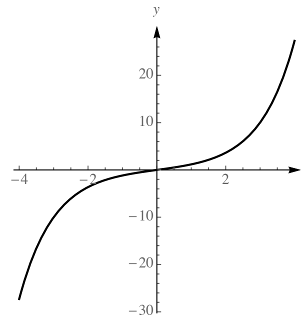


2.5.85

a. $\lim_{x\to\infty} \frac{e^x-e^{-x}}{2} = \infty$.

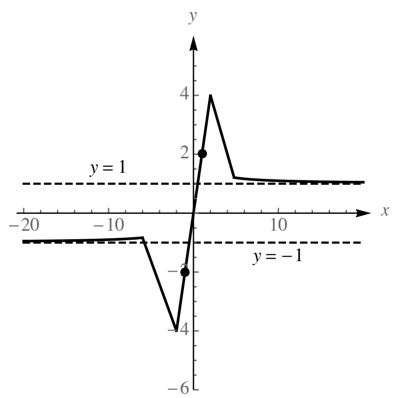
$\lim_{x\to-\infty} \frac{e^x-e^{-x}}{2} = -\infty$.

b. $\sinh(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0.$



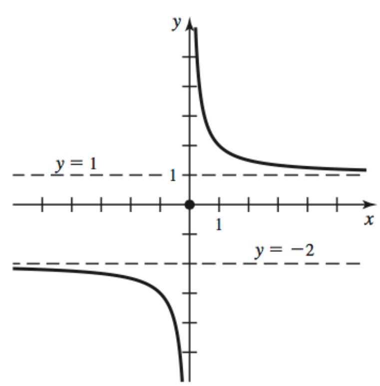
2.5.86

One possible such graph is:



2.5.87

One possible such graph is:



2.5.88 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{4}{n} = 0.$

2.5.89 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} [1 - (1/n)] = 1.$

2.5.90 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{n}{1+1/n} = \infty$, so the limit does not exist.

2.5.91 $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} [1/n + 1/n^2] = 0.$

2.5.92

a. Suppose $m = n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{a_n}{b_n}.\end{aligned}$$

b. Suppose $m < n$.

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{p(x)}{q(x)} \cdot \frac{1/x^n}{1/x^n} \\ &= \lim_{x\to\pm\infty} \frac{a_n/x^{n-m} + a_{n-1}/x^{n-m+1} + \cdots + a_1/x^{n-1} + a_0/x^n}{b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n} \\ &= \frac{0}{b_n} = 0.\end{aligned}$$

2.5.93

a. No. If $m = n$, there will be a horizontal asymptote, and if $m = n + 1$, there will be a slant asymptote.

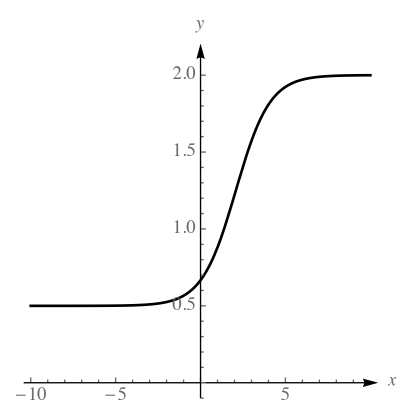
b. Yes. For example, $f(x) = \frac{x^4}{\sqrt{x^6+1}}$ has slant asymptote $y = x$ as $x \to \infty$ and slant asymptote $y = -x$ as $x \to -\infty$.

2.5.94

a. $\lim_{x\to\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^{2x}}{1/e^{2x}} = \lim_{x\to\infty} \frac{(4/e^x) + 2}{(8/e^x) + 1} = 2.$

b. $\lim_{x\to-\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to-\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{4 + 2e^x}{8 + e^x} = \frac{1}{2}.$

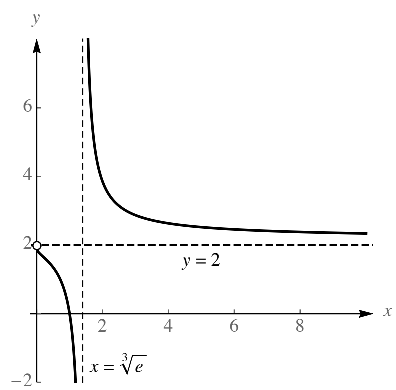
c. The lines $y = 2$ and $y = \frac{1}{2}$ are horizontal asymptotes.

d. 

2.5.95 $\lim\limits_{x\to\infty} \dfrac{2e^x+3}{e^x+1} = \lim\limits_{x\to\infty} \dfrac{(2e^x+3)}{(e^x+1)} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{2+3/e^x}{1+1/e^x} = \dfrac{2+0}{1+0} = 2$. Thus the line $y = 2$ is a horizontal asymptote. Also $\lim\limits_{x\to-\infty} \dfrac{2e^x+3}{e^x+1} = \dfrac{0+3}{0+1} = 3$, so $y = 3$ is a horizontal asymptote.

2.5.96 $\lim\limits_{x\to\infty} \dfrac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim\limits_{x\to\infty} \dfrac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})} \cdot \dfrac{1/e^{6x}}{1/e^{6x}} = \lim\limits_{x\to\infty} \dfrac{3e^{-x}+7}{9e^{-x}+14} = \dfrac{7}{14} = \dfrac{1}{2}$. So $y = \dfrac{1}{2}$ is a horizontal asymptote. Also, $\lim\limits_{x\to-\infty} \dfrac{3e^{5x}+7e^{6x}}{9e^{5x}+14e^{6x}} = \lim\limits_{x\to-\infty} \dfrac{(3e^{5x}+7e^{6x})}{(9e^{5x}+14e^{6x})} \cdot \dfrac{1/e^{5x}}{1/e^{5x}} = \lim\limits_{x\to-\infty} \dfrac{3+7e^x}{9+14e^x} = \dfrac{3}{9} = \dfrac{1}{3}$. So $y = \dfrac{1}{3}$ is a horizontal asymptote.

2.5.97 Using the rules of logarithms, $f(x) = \dfrac{6\ln x}{3\ln x - 1}$. The domain of f is $(0, \sqrt[3]{e}) \cup (\sqrt[3]{e}, \infty)$. We first examine the end behavior of the function. Observe that $\lim\limits_{x\to\infty} \dfrac{6\ln x}{3\ln x - 1} = \lim\limits_{x\to\infty} \dfrac{6}{3-(1/\ln x)} = \dfrac{6}{3} = 2$ and $\lim\limits_{x\to 0^+} \dfrac{6\ln x}{3\ln x - 1} = \lim\limits_{x\to 0^+} \dfrac{6}{3-(1/\ln x)} = \dfrac{6}{3} = 2$. So the function has a horizontal asymptote of $y = 2$ and it is undefined at $x = 0$ but has limit 2 as x approaches 0 from the right. Notice also that as $x \to \sqrt[3]{e}^+$, $6\ln x \to 2$ and $3\ln x - 1$ is positive and approaches 0. Therefore, $\lim\limits_{x\to \sqrt[3]{e}^+} \dfrac{6\ln x}{3\ln x - 1} = \infty$ and by a similar argument, $\lim\limits_{x\to \sqrt[3]{e}^-} \dfrac{6\ln x}{3\ln x - 1} = -\infty$



2.6 Continuity

2.6.1

a. $a(t)$ is a continuous function during the time period from when she jumps from the plane and when she touches down on the ground, because her position is changing continuously with time.

b. $n(t)$ is not a continuous function of time. The function "jumps" at the times when a quarter must be added.

c. $T(t)$ is a continuous function, because temperature varies continuously with time.

d. $p(t)$ is not continuous – it jumps by whole numbers when a player scores a point.

2.6.2 In order for f to be continuous at $x = a$, the following conditions must hold:

- f must be defined at a (i.e. a must be in the domain of f),
- $\lim\limits_{x\to a} f(x)$ must exist, and

- $\lim_{x \to a} f(x)$ must equal $f(a)$.

2.6.3 A function f is continuous on an interval I if it is continuous at all points in the interior of I, and it must be continuous from the right at the left endpoint (if the left endpoint is included in I) and it must be continuous from the left at the right endpoint (if the right endpoint is included in I.)

2.6.4 The words "hole" and "break" are not mathematically precise, so a strict mathematical definition can not be based on them.

2.6.5 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $f(1)$ is not defined (so the first condition is violated). At $x = 2$, $f(2)$ is defined and $\lim_{x \to 2} f(x)$ exists, but $\lim_{x \to 2} f(x) \neq f(2)$ (so condition 3 is violated). At $x = 3$, $\lim_{x \to 3} f(x)$ does not exist (so condition 2 is violated).

2.6.6 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x) \neq f(1)$ (so condition 3 is violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.7 f is discontinuous at $x = 1$, at $x = 2$, and at $x = 3$. At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist, and $f(1)$ is not defined (so conditions 1 and 2 are violated). At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined (so condition 1 is violated).

2.6.8 f is discontinuous at $x = 2$, at $x = 3$, and at $x = 4$. At $x = 2$, $\lim_{x \to 2} f(x)$ does not exist (so condition 2 is violated). At $x = 3$, $f(3)$ is not defined and $\lim_{x \to 3} f(x)$ does not exist (so conditions 1 and 2 are violated). At $x = 4$, $\lim_{x \to 4} f(x) \neq f(4)$ (so condition 3 is violated).

2.6.9

a. A function f is continuous from the left at $x = a$ if a is in the domain of f, and $\lim_{x \to a^-} f(x) = f(a)$.

b. A function f is continuous from the right at $x = a$ if a is in the domain of f, and $\lim_{x \to a^+} f(x) = f(a)$.

2.6.10 If f is right-continuous at $x = 3$, then $f(3) = \lim_{x \to 3^+} f(x) = 6$, so $f(3) = 6$.

2.6.11 f is continuous on $(0, 1)$, on $(1, 2)$, on $(2, 3]$, and on $(3, 4)$. It is continuous from the left at 3.

2.6.12 f is continuous on $(0, 1)$, on $(1, 2]$, on $(2, 3)$, and on $(3, 4)$. It is continuous from the left at 2.

2.6.13 f is continuous on $[0, 1)$, on $(1, 2)$, on $[2, 3)$, and on $(3, 5)$. It is continuous from the right at 2.

2.6.14 f is continuous on $(0, 2]$, on $(2, 3)$, on $(3, 4)$, and on $(4, 5)$. It is continuous from the left at 2.

2.6.15 The domain of $f(x) = \dfrac{e^x}{x}$ is $(-\infty, 0) \cup (0, \infty)$, and f is continuous everywhere on this domain.

2.6.16 The function is continuous on $(0, 15]$, on $(15, 30]$, on $(30, 45]$, and on $(45, 60]$.

2.6.17 The number -5 is not in the domain of f, because the denominator is equal to 0 when $x = -5$. Thus, the function is not continuous at -5.

2.6.18 The function is defined at 5, in fact $f(5) = \frac{50+15+1}{25+25} = \frac{66}{50} = \frac{33}{25}$. Also, $\lim_{x \to 5} f(x) = \lim_{x \to 5} \dfrac{2x^2 + 3x + 1}{x^2 + 5x} = \dfrac{33}{25} = f(5)$. The function is continuous at $a = 5$.

2.6.19 f is discontinuous at 1, because 1 is not in the domain of f; $f(1)$ is not defined.

2.6.20 g is discontinuous at 3 because 3 is not in the domain of g; $g(3)$ is not defined.

2.6.21 f is discontinuous at 1, because $\lim_{x\to 1} f(x) \neq f(1)$. In fact, $f(1) = 3$, but $\lim_{x\to 1} f(x) = 2$.

2.6.22 f is continuous at 3, because $\lim_{x\to 3} f(x) = f(3)$. In fact, $f(3) = 2$ and $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x-1) = 2$.

2.6.23 f is discontinuous at 4, because 4 is not in the domain of f; $f(4)$ is not defined.

2.6.24 f is discontinuous at -1 because $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x(x+1)}{x+1} = \lim_{x\to -1} x = -1 \neq f(-1) = 2$.

2.6.25 Because p is a polynomial, it is continuous on all of $\mathbb{R} = (-\infty, \infty)$.

2.6.26 Because g is a rational function, it is continuous on its domain, which is all of $\mathbb{R} = (-\infty, \infty)$. (Because $x^2 + x + 1$ has no real roots.)

2.6.27 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

2.6.28 Because s is a rational function, it is continuous on its domain. Its domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

2.6.29 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.30 Because f is a rational function, it is continuous on its domain. Its domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2.6.31 Because $f(x) = \left(x^8 - 3x^6 - 1\right)^{40}$ is a polynomial, it is continuous everywhere, including at 0. Thus $\lim_{x\to 0} f(x) = f(0) = (-1)^{40} = 1$.

2.6.32 Because $f(x) = \left(\frac{3}{2x^5 - 4x^2 - 50}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 2$. So $\lim_{x\to 2} f(x) = f(2) = \frac{81}{16}$.

2.6.33 Because $x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x-4)(x+2)$, we have (as long as $x \neq 4$)

$$\sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \sqrt{x(x+2)}.$$

Thus, $\lim_{x\to 4} \sqrt{\frac{x^3 - 2x^2 - 8x}{x-4}} = \lim_{x\to 4} \sqrt{x(x+2)} = \sqrt{24}$, using Theorem 2.12 and the fact that the square root is a continuous function.

2.6.34 Note that $t - 4 = (\sqrt{t} - 2)(\sqrt{t} + 2)$, so for $t \neq 4$, we have

$$\frac{t-4}{\sqrt{t}-2} = \sqrt{t} + 2.$$

Thus, $\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2} = \lim_{t\to 4}(\sqrt{t} + 2) = 4$. Then using Theorem 2.12 and the fact that the tangent function is continuous at 4, we have $\lim_{t\to 4} \tan\left(\frac{t-4}{\sqrt{t}-2}\right) = \tan\left(\lim_{t\to 4} \frac{t-4}{\sqrt{t}-2}\right) = \tan 4$.

2.6.35 Because $f(x) = \left(\frac{x+5}{x+2}\right)^4$ is a rational function, it is continuous at all points in its domain, including at $x = 1$. Thus $\lim\limits_{x\to 1} f(x) = f(1) = 16$.

2.6.36 $\lim\limits_{x\to\infty} \left(\dfrac{2x+1}{x}\right)^3 = \lim\limits_{x\to\infty} (2 + (1/x))^3 = 2^3 = 8$.

2.6.37 Note that

$$\lim_{x\to 5} \frac{6(\sqrt{x^2-16}-3)}{5(x-5)} = \lim_{x\to 5} \frac{6(\sqrt{x^2-16}-3)}{5(x-5)} \cdot \frac{(\sqrt{x^2-16}+3)}{(\sqrt{x^2-16}+3)} = \lim_{x\to 5} \frac{6(x^2-25)}{5(x-5)(\sqrt{x^2-16}+3)}$$
$$= \lim_{x\to 5} \frac{6(x+5)}{5(\sqrt{x^2-16}+3)} = \frac{60}{30} = 2.$$

2.6.38 First note that

$$\lim_{x\to 0} \frac{x}{\sqrt{16x+1}-1} = \lim_{x\to 0} \frac{x}{(\sqrt{16x+1}-1)} \cdot \frac{(\sqrt{16x+1}+1)}{(\sqrt{16x+1}+1)} = \lim_{x\to 0} \frac{x(\sqrt{16x+1}+1)}{16x} = \frac{2}{16} = \frac{1}{8}.$$

Then because $f(x) = x^{1/3}$ is continuous at $1/8$, we have $\lim\limits_{x\to 0} \left(\dfrac{x}{\sqrt{16x+1}-1}\right)^{1/3} = \left(\dfrac{1}{8}\right)^{1/3} = \dfrac{1}{2}$, by Theorem 2.12.

2.6.39

a. f is defined at 1. We have $f(1) = 1^2 + (3)(1) = 4$. To see whether or not $\lim\limits_{x\to 1} f(x)$ exists, we investigate the two one-sided limits. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x = 2$, and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x^2 + 3x) = 4$, so $\lim\limits_{x\to 1} f(x)$ does not exist. Thus f is discontinuous at $x = 1$.

b. f is continuous from the right, because $\lim\limits_{x\to 1^+} f(x) = 4 = f(1)$.

c. f is continuous on $(-\infty, 1)$ and on $[1, \infty)$.

2.6.40

a. f is defined at 0, in fact $f(0) = 1$. However, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x^3 + 4x + 1) = 1$, while $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x^3 = 0$. So $\lim\limits_{x\to 0} f(x)$ does not exist.

b. f is continuous from the left at 0, because $\lim\limits_{x\to 0^-} f(x) = f(0) = 1$.

c. f is continuous on $(-\infty, 0]$ and on $(0, \infty)$.

2.6.41 f is defined and is continuous on $(-\infty, 5]$. It is continuous from the left at 5.

2.6.42 f is defined and is continuous on $[-5, 5]$. It is continuous from the right at -5 and is continuous from the left at 5.

2.6.43 f is continuous on $(-\infty, -\sqrt{8}]$ and on $[\sqrt{8}, \infty)$. It is continuous from the left at $-\sqrt{8}$ and from the right at $\sqrt{8}$.

2.6.44 $g(x) = \sqrt{x^2 - 3x + 2} = \sqrt{(x-1)(x-2)}$ is defined and is continuous on $(-\infty, 1]$ and on $[2, \infty)$. It is continuous from the left at 1 and from the right at 2.

2.6.45 Because f is the composition of two functions which are continuous on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$.

2.6.46 f is continuous on $(-\infty, -1]$ and on $[1, \infty)$. It is continuous from the left at -1 and from the right at 1.

2.6.47 Because f is the composition of two functions which are continuous on $(-\infty, \infty)$, it is continuous on $(-\infty, \infty)$.

2.6.48 f is continuous on $[1, \infty)$. It is continuous from the right at 1.

2.6.49 $\lim_{x\to 2}\sqrt{\dfrac{4x+10}{2x-2}} = \sqrt{\dfrac{18}{2}} = 3.$

2.6.50 $\lim_{x\to -1}\left(x^2 - 4 + \sqrt[3]{x^2-9}\right) = (-1)^2 - 4 + \sqrt[3]{(-1)^2-9} = -3 + \sqrt[3]{-8} = -3 + -2 = -5.$

2.6.51 $\lim_{x\to\pi}\dfrac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \lim_{x\to\pi}\dfrac{(\cos x+1)(\cos x+2)}{\cos x+1} = \lim_{x\to\pi}(\cos x + 2) = 1.$

2.6.52 $\lim_{x\to 3\pi/2}\dfrac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \lim_{x\to 3\pi/2}\dfrac{(\sin x+5)(\sin x+1)}{(\sin x-1)(\sin x+1)} = \lim_{x\to 3\pi/2}\dfrac{\sin x+5}{\sin x-1} = \dfrac{4}{-2} = -2.$

2.6.53 $\lim_{x\to 3}\sqrt{x^2+7} = \sqrt{9+7} = 4.$

2.6.54 $\lim_{t\to 2}\dfrac{t^2+5}{1+\sqrt{t^2+5}} = \dfrac{9}{1+\sqrt{9}} = \dfrac{9}{4}.$

2.6.55 $\lim_{x\to\pi/2}\dfrac{\sin x - 1}{\sqrt{\sin x} - 1} = \lim_{x\to\pi/2}(\sqrt{\sin x}+1) = 2.$

2.6.56 $\lim_{\theta\to 0}\dfrac{\frac{1}{2+\sin\theta} - \frac{1}{2}}{\sin\theta}\cdot\dfrac{(2)(2+\sin\theta)}{(2)(2+\sin\theta)} = \lim_{\theta\to 0}\dfrac{2-(2+\sin\theta)}{(\sin\theta)(2)(2+\sin\theta)} = \lim_{\theta\to 0} -\dfrac{1}{2(2+\sin\theta)} = -\dfrac{1}{4}.$

2.6.57 $\lim_{x\to 0}\dfrac{\cos x - 1}{\sin^2 x} = \lim_{x\to 0}\dfrac{\cos x-1}{1-\cos^2 x} = \lim_{x\to 0}\dfrac{\cos x - 1}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0} -\dfrac{1}{1+\cos x} = -\dfrac{1}{2}.$

2.6.58 $\lim_{x\to 0^+}\dfrac{1-\cos^2 x}{\sin x} = \lim_{x\to 0^+}\dfrac{\sin^2 x}{\sin x} = \lim_{x\to 0^+}\sin x = 0.$

2.6.59 $\lim_{x\to 0}\dfrac{e^{4x}-1}{e^x-1} = \lim_{x\to 0}\dfrac{(e^{2x}+1)(e^{2x}-1)}{e^x-1} = \lim_{x\to 0}\dfrac{(e^{2x}+1)(e^x-1)(e^x+1)}{e^x-1} =$
$\lim_{x\to 0}(e^{2x}+1)(e^x+1) = 2\cdot 2 = 4.$

2.6.60 $\lim_{x\to e^2}\dfrac{\ln^2 x - 5\ln x + 6}{\ln x - 2} = \lim_{x\to e^2}\dfrac{(\ln x-2)(\ln x-3)}{\ln x-2} = \lim_{x\to e^2}(\ln x - 3) = -1.$

2.6.61 $f(x) = \csc x$ isn't defined at $x = k\pi$ where k is an integer, so it isn't continuous at those points. So it is continuous on intervals of the form $(k\pi, (k+1)\pi)$ where k is an integer. $\lim_{x\to\pi/4}\csc x = \sqrt{2}$. $\lim_{x\to 2\pi^-}\csc x = -\infty$.

2.6.62 f is defined on $[0, \infty)$, and it is continuous there, because it is the composition of continuous functions defined on that interval. $\lim_{x\to 4} f(x) = e^2$. $\lim_{x\to 0} f(x)$ does not exist—but $\lim_{x\to 0^+} f(x) = e^0 = 1$, because f is continuous from the right.

2.6.63 f isn't defined for any number of the form $\pi/2 + k\pi$ where k is an integer, so it isn't continuous there. It is continuous on intervals of the form $(\pi/2 + k\pi, \pi/2 + (k+1)\pi)$, where k is an integer.

$\lim_{x\to\pi/2^-} f(x) = \infty$. $\lim_{x\to 4\pi/3} f(x) = \dfrac{1-\sqrt{3}/2}{-1/2} = \sqrt{3} - 2.$

2.6.64 The domain of f is $(0,1]$, and f is continuous on this interval because it is the quotient of two continuous functions and the function in the denominator isn't zero on that interval.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\ln x}{\sin^{-1}(x)} = \frac{\ln 1}{\sin^{-1}(1)} = \frac{0}{\pi/2} = 0.$$

2.6.65 This function is continuous on its domain, which is $(-\infty,0)\cup(0,\infty)$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x}{1-e^x} = \infty, \text{ while } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^x}{1-e^x} = -\infty.$$

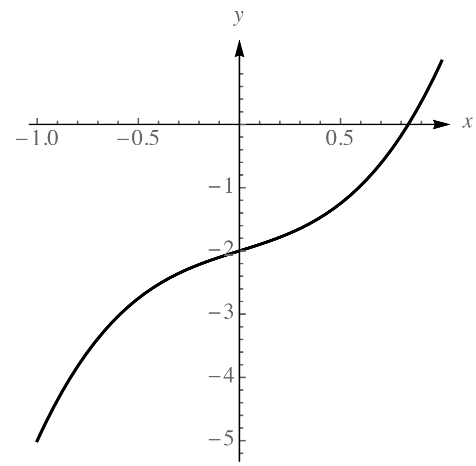
2.6.66 This function is continuous on its domain, which is $(-\infty,0)\cup(0,\infty)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0} \frac{(e^x+1)(e^x-1)}{e^x-1} = \lim_{x\to 0}(e^x+1) = 2.$$

2.6.67

a. Note that $f(x) = 2x^3 + x - 2$ is continuous everywhere, so in particular it is continuous on $[-1,1]$. Note that $f(-1) = -5 < 0$ and $f(1) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between -1 and 1 where $f(c) = 0$.

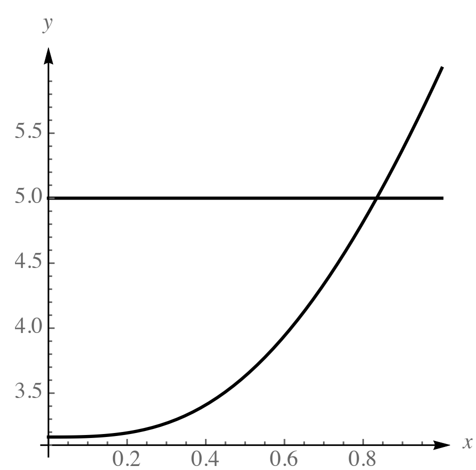
b. Using a graphing calculator and a computer algebra system, we see that the root of f is about 0.835.

c. 

2.6.68

a. Note that $f(x) = \sqrt{x^4 + 25x^3 + 10} - 5$ is continuous on its domain, so in particular it is continuous on $[0,1]$. Note that $f(0) = \sqrt{10} - 5 < 0$ and $f(1) = 6 - 5 = 1 > 0$. Because 0 is an intermediate value between $f(0)$ and $f(1)$, the Intermediate Value Theorem guarantees a number c between 0 and 1 where $f(c) = 0$.

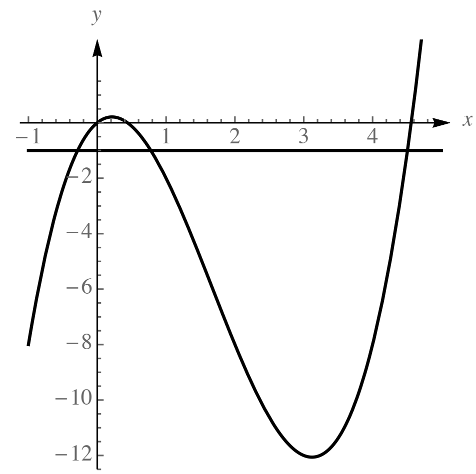
b. Using a graphing calculator and a computer algebra system, we see the root of $f(x)$ is at about .834.

c. 

2.6.69

a. Note that $f(x) = x^3 - 5x^2 + 2x$ is continuous everywhere, so in particular it is continuous on $[-1, 5]$. Note that $f(-1) = -8 < -1$ and $f(5) = 10 > -1$. Because -1 is an intermediate value between $f(-1)$ and $f(5)$, the Intermediate Value Theorem guarantees a number c between -1 and 5 where $f(c) = -1$.

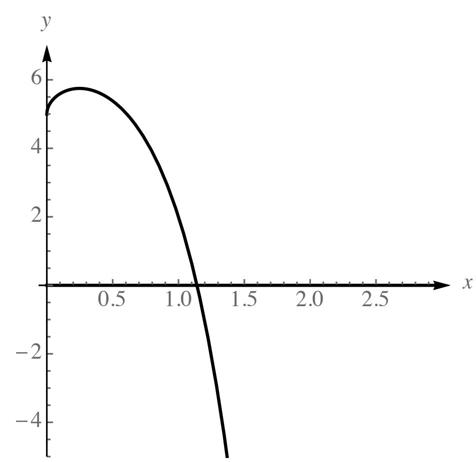
b. Using a graphing calculator and a computer algebra system, we see that there are actually three different values of c between -1 and 5 for which $f(c) = -1$. They are $c \approx -0.285$, $c \approx 0.778$, and $c \approx 4.507$.

c. 

2.6.70

a. Note that $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$ is continuous on its domain, so in particular it is continuous on $[0, 3]$. Note that $f(0) = 5 > 0$ and $f(3) \approx -270.5 < 0$. Because 0 is an intermediate value between $f(0)$ and $f(3)$, the Intermediate Value Theorem guarantees a number c between 0 and 3 where $f(c) = 0$.

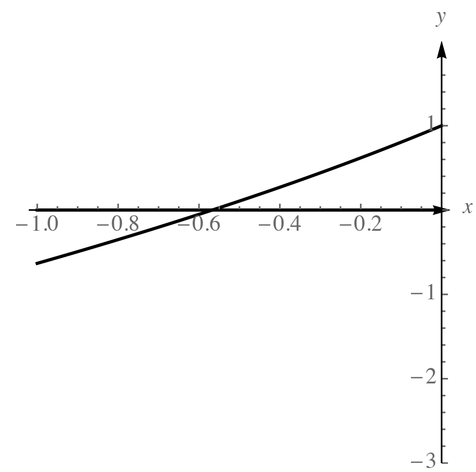
b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.141.

c. 

2.6.71

a. Note that $f(x) = e^x + x$ is continuous on its domain, so in particular it is continuous on $[-1, 0]$. Note that $f(-1) = \frac{1}{e} - 1 < 0$ and $f(0) = 1 > 0$. Because 0 is an intermediate value between $f(-1)$ and $f(0)$, the Intermediate Value Theorem guarantees a number c between -1 and 0 where $f(c) = 0$.

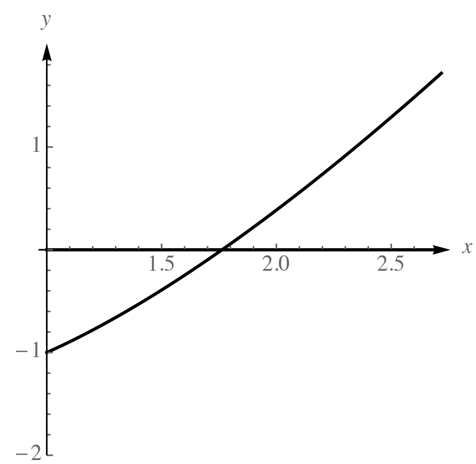
b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about -0.567.

c. 

2.6.72

a. Note that $f(x) = x \ln x - 1$ is continuous on its domain, so in particular it is continuous on $[1, e]$. Note that $f(1) = \ln 1 - 1 = -1 < 0$ and $f(e) = e - 1 > 0$. Because 0 is an intermediate value between $f(1)$ and $f(e)$, the Intermediate Value Theorem guarantees a number c between 1 and e where $f(c) = 0$.

b. Using a graphing calculator and a computer algebra system, we see that the value of c guaranteed by the theorem is about 1.76322.

c. 

2.6.73

a. True. If f is right continuous at a, then $f(a)$ exists and the limit from the right at a exists and is equal to $f(a)$. Because it is left continuous, the limit from the left exists — so we now know that the limit as $x \to a$ of $f(x)$ exists, because the two one-sided limits are both equal to $f(a)$.

b. True. If $\lim\limits_{x\to a} f(x) = f(a)$, then $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to a^-} f(x) = f(a)$.

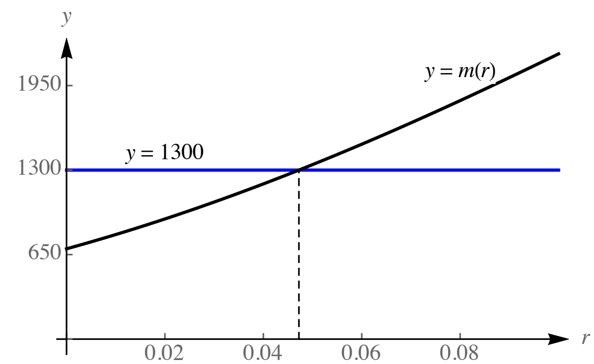
c. False. The statement would be true if f were continuous. However, if f isn't continuous, then the statement doesn't hold. For example, suppose that $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } 1 \le x \le 2, \end{cases}$ Note that $f(0) = 0$ and $f(2) = 1$, but there is no number c between 0 and 2 where $f(c) = 1/2$.

d. False. Consider $f(x) = x^2$ and $a = -1$ and $b = 1$. Then f is continuous on $[a, b]$, but $\frac{f(1)+f(-1)}{2} = 1$, and there is no c on (a, b) with $f(c) = 1$.

2.6.74

a. Because m is a continuous function of r on $[0.04, 0.05]$, and because $m(0.04) \approx 1193.54$ and $m(0.05) \approx 1342.05$, (and 1300 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0.04 and 0.05 where $m(r) = 1300$.

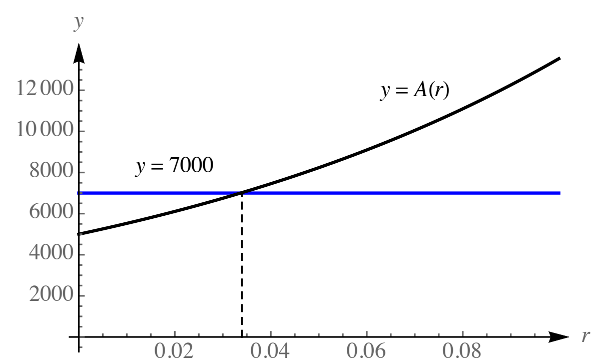
b. Using a computer algebra system, we see that the required interest rate is about 0.047.



2.6.75

a. Because A is a continuous function of r on $[0, 0.08]$, and because $A(0) = 5000$ and $A(0.08) \approx 11098.2$, (and 7000 is an intermediate value between these two numbers) the Intermediate Value Theorem guarantees a value of r between 0 and 0.08 where $A(r) = 7000$.

b. Solving $5000(1 + (r/12))^{120} = 7000$ for r, we see that $(1+(r/12))^{120} = 7/5$, so $1+r/12 = \sqrt[120]{7/5}$, so $r = 12(\sqrt[120]{7/5} - 1) \approx 0.034$.



2.6.76

a. Note that $A(0.01) \approx 2615.55$ and $A(0.1) \approx 3984.36$. By the Intermediate Value Theorem, there must be a number r_0 between 0.01 and 0.1 so that $A(r_0) = 3500$.

b. The desired value is $r_0 \approx 0.0728$ or 7.28%.

2.6.77 Consider the function $f(x) = \cos x - 2x$ on the interval $[0, 1]$. Note that $f(0) = 1$ and $f(\pi/2) = -\pi < 0$. So by the Intermediate Value Theorem, there must be a root of f on the interval $[0, \pi/2]$. Using a computer algebra system, we find a root of approximately 0.45.

2.6.78 Let $f(x) = |x|$.

For values of a other than 0, it is clear that $\lim_{x\to a} |x| = |a|$ because f is defined to be either the polynomial x (for values greater than 0) or the polynomial $-x$ (for values less than 0.) For the value of $a = 0$, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0)$. Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = -0 = 0$. Thus $\lim_{x\to 0} f(x) = f(0)$, so f is continuous at 0.

2.6.79 Because $f(x) = x^3 + 3x - 18$ is a polynomial, it is continuous on $(-\infty, \infty)$, and because the absolute value function is continuous everywhere, $|f(x)|$ is continuous everywhere.

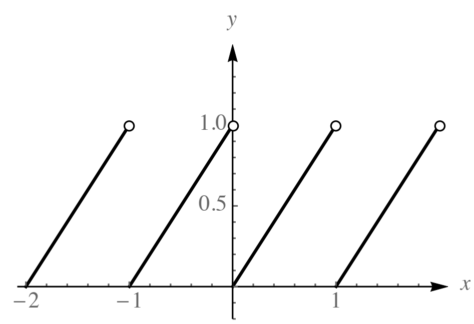
2.6.80 Let $f(x) = \dfrac{x+4}{x^2-4}$. Then f is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. So $g(x) = |f(x)|$ is also continuous on this set.

2.6.81 Let $f(x) = \dfrac{1}{\sqrt{x}-4}$. Then f is continuous on $[0, 16) \cup (16, \infty)$. So $h(x) = |f(x)|$ is continuous on this set as well.

2.6.82 Because $x^2 + 2x + 5$ is a polynomial, it is continuous everywhere, as is $|x^2 + 2x + 5|$. So $h(x) = |x^2 + 2x + 5| + \sqrt{x}$ is continuous on its domain, namely $[0, \infty)$.

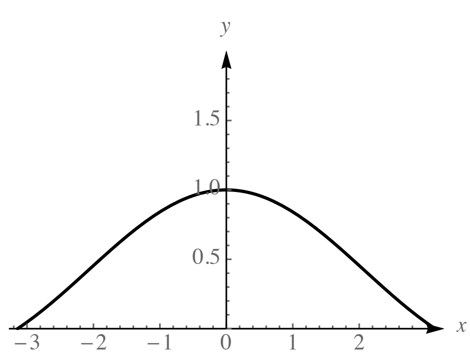
2.6.83

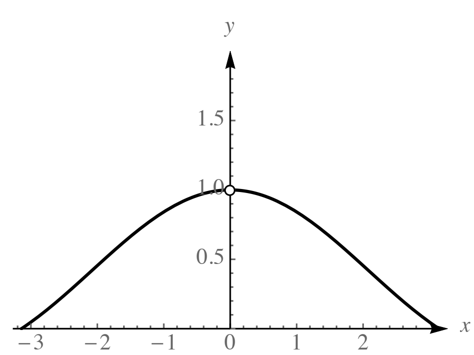
The graph shown isn't drawn correctly at the integers. At an integer a, the value of the function is 0, whereas the graph shown appears to take on all the values from 0 to 1.
Note that in the correct graph, $\lim_{x \to a^-} f(x) = 1$ and $\lim_{x \to a^+} f(x) = 0$ for every integer a.



2.6.84

The graph as drawn on most graphing calculators appears to be continuous at $x = 0$, but it isn't, of course (because the function isn't defined at $x = 0$). A better drawing would show the "hole" in the graph at $(0, 1)$.

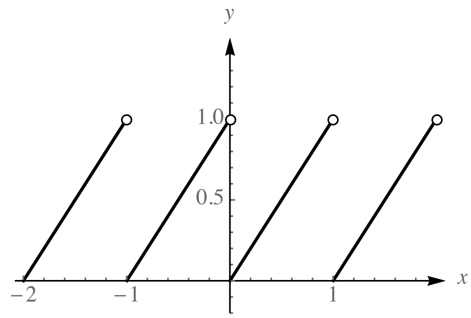
a. 

b. 

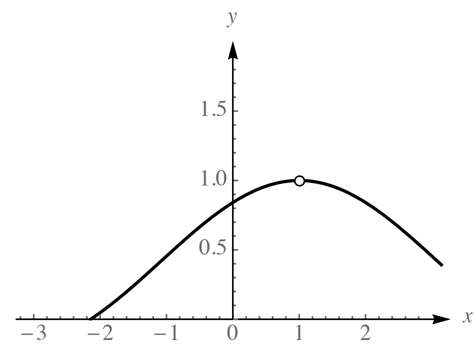
c. It appears that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

2.6.85 With slight modifications, we can use the examples from the previous two problems.

a. The function $y = x - \lfloor x \rfloor$ is defined at $x = 1$ but isn't continuous there.



b. The function $y = \frac{\sin(x-1)}{x-1}$ has a limit at $x = 1$, but isn't defined there, so isn't continuous there.



2.6.86 In order for this function to be continuous at $x = -1$, we require $\lim_{x\to-1} f(x) = f(-1) = a$. So the value of a must be equal to the value of $\lim_{x\to-1} \frac{x^2+3x+2}{x+1} = \lim_{x\to-1} \frac{(x+2)(x+1)}{x+1} = \lim_{x\to-1}(x+2) = 1$. Thus we must have $a = 1$.

2.6.87

a. In order for g to be continuous from the left at $x = 1$, we must have $\lim_{x\to1^-} g(x) = g(1) = a$. We have $\lim_{x\to1^-} g(x) = \lim_{x\to1^-}(x^2+x) = 2$. So we must have $a = 2$.

b. In order for g to be continuous from the right at $x = 1$, we must have $\lim_{x\to1^+} g(x) = g(1) = a$. We have $\lim_{x\to1^+} g(x) = \lim_{x\to1^+}(3x+5) = 8$. So we must have $a = 8$.

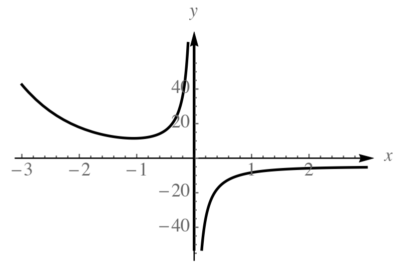
c. Because the limit from the left and the limit from the right at $x = 1$ don't agree, there is no value of a which will make the function continuous at $x = 1$.

2.6.88 $\lim_{x\to0^-} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to0^-} \frac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = \infty$.

$\lim_{x\to0^+} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to0^+} \frac{2e^x+5e^{3x}}{e^{2x}(1-e^x)} = -\infty$.

$\lim_{x\to-\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to-\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \frac{e^{-2x}}{e^{-2x}} = \lim_{x\to-\infty} \frac{2e^{-x}+5e^x}{1-e^x} = \infty$.

$\lim_{x\to\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to\infty} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} \cdot \frac{e^{-3x}}{e^{-3x}} = \lim_{x\to\infty} \frac{2e^{-2x}+5}{e^{-x}-1} = -5$.

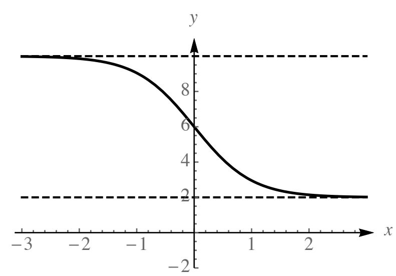


There is a vertical asymptote at $x = 0$, and the line $y = -5$ is a horizontal asymptote.

2.6.89 $\lim_{x\to 0} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \dfrac{12}{2} = 6.$

$\lim_{x\to -\infty} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to -\infty} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^x}{e^x} = \lim_{x\to -\infty} \dfrac{2e^{2x} + 10}{e^{2x} + 1} = \dfrac{10}{1} = 10.$

$\lim_{x\to \infty} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x\to \infty} \dfrac{2e^x + 10e^{-x}}{e^x + e^{-x}} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim_{x\to \infty} \dfrac{2 + 10e^{-2x}}{1 + e^{-2x}} = \dfrac{2}{1} = 2.$



There are no vertical asymptotes. The lines $y = 2$ and $y = 10$ are horizontal asymptotes.

2.6.90 Let $f(x) = x^3 + 10x^2 - 100x + 50$. Note that $f(-20) < 0$, $f(-5) > 0$, $f(5) < 0$, and $f(10) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(-20, -5)$, at least one on $(-5, 5)$, and at least one on $(5, 10)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 \approx -16.32$, $x_2 \approx 0.53$ and $x_3 \approx 5.79$.

2.6.91 Let $f(x) = 70x^3 - 87x^2 + 32x - 3$. Note that $f(0) < 0$, $f(0.2) > 0$, $f(0.55) < 0$, and $f(1) > 0$. Because the given polynomial is continuous everywhere, the Intermediate Value Theorem guarantees us a root on $(0, 0.2)$, at least one on $(0.2, 0.55)$, and at least one on $(0.55, 1)$. Because there can be at most 3 roots and there are at least 3 roots, there must be exactly 3 roots. The roots are $x_1 = 1/7$, $x_2 = 1/2$ and $x_3 = 3/5$.

2.6.92

a. We have $f(0) = 0$, $f(2) = 3$, $g(0) = 3$ and $g(2) = 0$.

b. $h(t) = f(t) - g(t)$, $h(0) = -3$ and $h(2) = 3$.

c. By the Intermediate Value Theorem, because h is a continuous function and 0 is an intermediate value between -3 and 3, there must be a time c between 0 and 2 where $h(c) = 0$. At this point $f(c) = g(c)$, and at that time, the distance from the car is the same on both days, so the hiker is passing over the exact same point at that time.

2.6.93 We can argue essentially like the previous problem, or we can imagine an identical twin to the original monk, who takes an identical version of the original monk's journey up the winding path while the monk is taking the return journey down. Because they must pass somewhere on the path, that point is the one we are looking for.

2.6.94

a. Because $|-1| = 1$, $|g(x)| = 1$, for all x.

b. The function g isn't continuous at $x = 0$, because $\lim_{x\to 0^+} g(x) = 1 \neq -1 = \lim_{x\to 0^-} g(x)$.

c. This constant function is continuous everywhere, in particular at $x = 0$.

d. This example shows that in general, the continuity of $|g|$ does not imply the continuity of g.

2.6.95 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is a jump discontinuity.

2.6.96 The discontinuity is not removable, because $\lim_{x\to a} f(x)$ does not exist. The discontinuity pictured is an infinite discontinuity.

2.6.97 Note that $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x-5)}{x-2} = \lim_{x\to 2}(x-5) = -3$. Because this limit exists, the discontinuity is removable.

2.6.98 Note that $\lim_{x\to 1} \dfrac{x^2 - 1}{1 - x} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{1-x} = \lim_{x\to 1}[-(x+1)] = -2$. Because this limit exists, the discontinuity is removable.

2.6.99 Note that $h(x) = \dfrac{x^3 - 4x^2 + 4x}{x(x-1)} = \dfrac{x(x-2)^2}{x(x-1)}$. Thus $\lim_{x\to 0} h(x) = -4$, and the discontinuity at $x = 0$ is removable. However, $\lim_{x\to 1} h(x)$ does not exist, and the discontinuity at $x = 1$ is not removable (it is infinite.)

2.6.100 This is a jump discontinuity, because $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$.

2.6.101

a. Note that $-1 \le \sin(1/x) \le 1$ for all $x \neq 0$, so $-x \le x\sin(1/x) \le x$ (for $x > 0$. For $x < 0$ we would have $x \le x\sin(1/x) \le -x$.) Because both $x \to 0$ and $-x \to 0$ as $x \to 0$, the Squeeze Theorem tells us that $\lim_{x\to 0} x\sin(1/x) = 0$ as well. Because this limit exists, the discontinuity is removable.

b. Note that as $x \to 0^+$, $1/x \to \infty$, and thus $\lim_{x\to 0^+} \sin(1/x)$ does not exist. So the discontinuity is not removable.

2.6.102 Because g is continuous at a, as $x \to a$, $g(x) \to g(a)$. Because f is continuous at $g(a)$, as $z \to g(a)$, $f(z) \to f(g(a))$. Let $z = g(x)$, and suppose $x \to a$. Then $g(x) = z \to g(a)$, so $f(z) = f(g(x)) \to f(g(a))$, as desired.

2.6.103

a. Consider $g(x) = x + 1$ and $f(x) = \frac{|x-1|}{x-1}$. Note that both g and f are continuous at $x = 0$. However $f(g(x)) = f(x+1) = \frac{|x|}{x}$ is not continuous at 0.

b. The previous theorem says that the composition of f and g is continuous at a if g is continuous at a and f is continuous at $g(a)$. It does not say that if g and f are both continuous at a that the composition is continuous at a.

2.6.104 The Intermediate Value Theorem requires that our function be continuous on the given interval. In this example, the function f is not continuous on $[-2, 2]$ because it isn't continuous at 0.

2.6.105

a. Using the hint, we have

$$\sin x = \sin(a + (x - a)) = \sin a \cos(x - a) + \sin(x - a) \cos a.$$

Note that as $x \to a$, we have that $\cos(x - a) \to 1$ and $\sin(x - a) \to 0$.

So,

$$\lim_{x\to a} \sin x = \lim_{x\to a} \sin(a+(x-a)) = \lim_{x\to a} (\sin a \cos(x - a) + \sin(x - a) \cos a) = (\sin a)\cdot 1+0\cdot \cos a = \sin a.$$

b. Using the hint, we have

$$\cos x = \cos(a + (x - a)) = \cos a \cos(x - a) - \sin a \sin(x - a).$$

So,

$$\begin{aligned}\lim_{x\to a} \cos x = \lim_{x\to a} \cos(a + (x - a)) &= \lim_{x\to a} ((\cos a) \cos(x - a) - (\sin a) \sin(x - a)) \\ &= (\cos a) \cdot 1 - (\sin a) \cdot 0 = \cos a.\end{aligned}$$

2.7 Precise Definitions of Limits

2.7.1 Note that all the numbers in the interval $(1, 3)$ are within 1 unit of the number 2. So $|x - 2| < 1$ is true for all numbers in that interval. In fact, $\{x\colon 0 < |x - 2| < 1\}$ is exactly the set $(1, 3)$ with $x \neq 2$.

2.7.2 Note that all the numbers in the interval $(2, 6)$ are within 2 units of the number 4. So $|f(x) - 4| < \varepsilon$ for $\varepsilon = 2$ (or any number greater than 2).

2.7.3

a. This is symmetric about $x = 5$, because $\dfrac{1+9}{2} = 5$.

b. This is symmetric about $x = 5$, because $\dfrac{4+6}{2} = 5$.

c. This is not symmetric about $x = 5$, because $\dfrac{3+8}{2} \neq 5$.

d. This is symmetric about $x = 5$, because $\dfrac{4.5+5.5}{2} = 5$.

2.7.4 The set $\{x : |x - a| < \delta\}$ is the interval $(a - \delta, a + \delta)$ and $\{x : 0 < |x - a| < \delta\}$ is the set of all points in the interval $(a - \delta, a + \delta)$ excluding the point a.

2.7.5 $\lim_{x\to a} f(x) = L$ if for any arbitrarily small positive number ε, there exists a number δ, so that $f(x)$ is within ε units of L for any number x within δ units of a (but not including a itself).

2.7.6 The set of all x for which $|f(x) - L| < \varepsilon$ is the set of numbers so that the value of the function f at those numbers is within ε units of L.

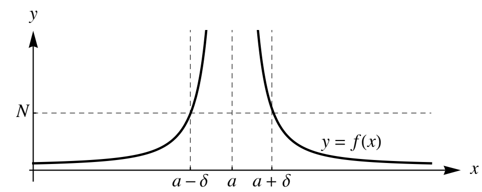
2.7.7 We are given that $|f(x) - 5| < 0.1$ for values of x in the interval $(0, 5)$, so we need to ensure that the set of x values we are allowing fall in this interval.

Note that the number 0 is two units away from the number 2 and the number 5 is three units away from the number 2. In order to be sure that we are talking about numbers in the interval $(0, 5)$ when we write $|x - 2| < \delta$, we would need to have $\delta = 2$ (or a number less than 2). In fact, the set of numbers for which $|x - 2| < 2$ is the interval $(0, 4)$ which is a subset of $(0, 5)$.

If we were to allow δ to be any number greater than 2, then the set of all x so that $|x - 2| < \delta$ would include numbers less than 0, and those numbers aren't on the interval $(0, 5)$.

2.7.8

$\lim_{x\to a} f(x) = \infty$, if for any $N > 0$, there exists $\delta > 0$ so that if $0 < |x - a| < \delta$ then $f(x) > N$.



2.7.9

a. In order for f to be within 2 units of 5, it appears that we need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1 unit of 5, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 0.5$.

2.7.10

a. In order for f to be within 1 unit of 4, it appears that we would need x to be within 1 unit of 2. So $\delta = 1$.

b. In order for f to be within 1/2 unit of 4, it appears that we would need x to be within 1/2 unit of 2. So $\delta = 1/2$.

2.7.11

a. In order for f to be within 3 units of 6, it appears that we would need x to be within 2 units of 3. So $\delta = 2$.

b. In order for f to be within 1 unit of 6, it appears that we would need x to be within 1/2 unit of 3. So $\delta = 1/2$.

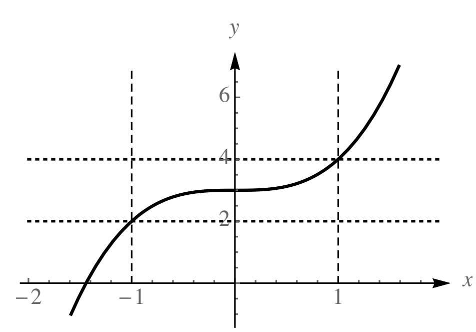
2.7.12

a. In order for f to be within 1 unit of 5, it appears that we would need x to be within 3 units of 4. So $\delta = 3$.

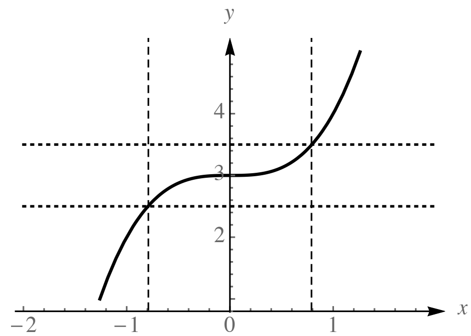
b. In order for f to be within 1/2 unit of 5, it appears that we would need x to be within 2 units of 4. So $\delta = 2$.

2.7.13

a. If $\varepsilon = 1$, we need $|x^3 + 3 - 3| < 1$. So we need $|x| < \sqrt[3]{1} = 1$ in order for this to happen. Thus $\delta = 1$ will suffice.

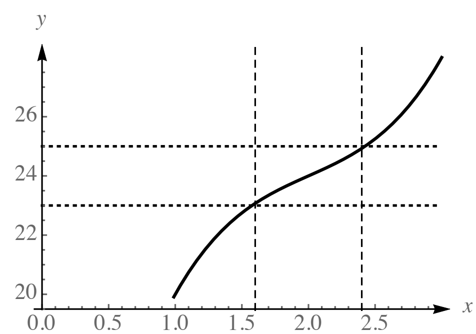


b. If $\varepsilon = 0.5$, we need $|x^3 + 3 - 3| < 0.5$. So we need $|x| < \sqrt[3]{0.5}$ in order for this to happen. Thus $\delta = \sqrt[3]{0.5} \approx 0.79$ will suffice.

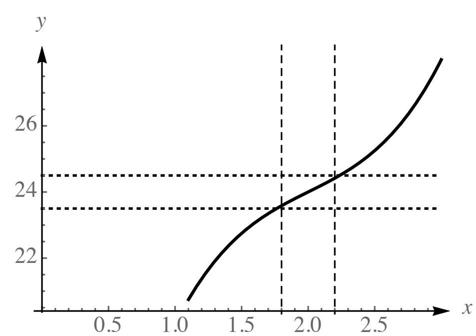


2.7.14

a. By looking at the graph, it appears that for $\varepsilon = 1$, we would need δ to be about 0.4 or less.



b. By looking at the graph, it appears that for $\varepsilon = 0.5$, we would need δ to be about 0.2 or less.



2.7.15

a. For $\varepsilon = 1$, the required value of δ would also be 1. A larger value of δ would work to the right of 2, but this is the largest one that would work to the left of 2.

b. For $\varepsilon = 1/2$, the required value of δ would also be $1/2$.

c. It appears that for a given value of ε, it would be wise to take $\delta = \min(\varepsilon, 2)$. This assures that the desired inequality is met on both sides of 2.

2.7.16

a. For $\varepsilon = 2$, the required value of δ would be 1 (or smaller). This is the largest value of δ that works on either side.

b. For $\varepsilon = 1$, the required value of δ would be $1/2$ (or smaller). This is the largest value of δ that works on the right of 4.

c. It appears that for a given value of ε, the corresponding value of $\delta = \min(5/2, \varepsilon/2)$.

2.7.17

a. For $\varepsilon = 2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of $\delta = \varepsilon/2$ or smaller appears to suffice.

2.7.18

a. For $\varepsilon = 1/2$, it appears that a value of $\delta = 1$ (or smaller) would work.

b. For $\varepsilon = 1/4$, it appears that a value of $\delta = 1/2$ (or smaller) would work.

c. For an arbitrary ε, a value of 2ε or smaller appears to suffice.

2.7.19 For any $\varepsilon > 0$, let $\delta = \varepsilon/8$. Then if $0 < |x-1| < \delta$, we would have $|x-1| < \varepsilon/8$. Then $|8x-8| < \varepsilon$, so $|(8x+5)-13| < \varepsilon$. This last inequality has the form $|f(x) - L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 1}(8x+5) = 13$.

2.7.20 For any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \varepsilon/2$. Then $|2x-6| < \varepsilon$, so $|-2x+6| < \varepsilon$, so $|(-2x+8)-2| < \varepsilon$. This last inequality has the form $|f(x) - L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 3}(-2x+8) = 2$.

2.7.21 First note that if $x \neq 4$, $f(x) = \frac{x^2-16}{x-4} = x+4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-4| < \delta$. Then $x \neq 4$, so the function $f(x)$ can be described by $x+4$. Also, because $|x-4| < \delta$, we have $|x-4| < \varepsilon$. Thus $|(x+4)-8| < \varepsilon$. This last inequality has the form $|f(x) - L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4} = 8$.

2.7.22 First note that if $x \neq 3$, $f(x) = \frac{x^2-7x+12}{x-3} = \frac{(x-4)(x-3)}{x-3} = x-4$.

Now if $\varepsilon > 0$ is given, let $\delta = \varepsilon$. Now suppose $0 < |x-3| < \delta$. Then $x \neq 3$, so the function $f(x)$ can be described by $x-4$. Also, because $|x-3| < \delta$, we have $|x-3| < \varepsilon$. Thus $|(x-4)-(-1)| < \varepsilon$. This last inequality has the form $|f(x) - L| < \varepsilon$, which is what we were attempting to show. Thus, $\lim\limits_{x\to 3} f(x) = -1$.

2.7.23 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \varepsilon$. It follows that $||x|-0| = |x| = |x-0| < \delta = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \varepsilon$.

2.7.24 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \frac{\varepsilon}{5}$. It follows that $||5x|-0| = 5|x-0| < 5\delta = 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$. We have shown that for any $\varepsilon > 0$, $||5x|-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \frac{\varepsilon}{5}$.

2.7.25 Let $\varepsilon > 0$ be given and assume that $0 < |x-7| < \delta$ where $\delta = \varepsilon/3$. If $x < 7$, $|f(x)-9| = |3x-12-9| = 3|x-7| < 3\delta = 3(\varepsilon/3) = \varepsilon$; if $x > 7$, then $|f(x)-9| = |x+2-9| = |x-7| < \delta = \varepsilon/3 < \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-9| < \varepsilon$ whenever $0 < |x-7| < \delta$, provided $0 < \delta \le \varepsilon/3$.

2.7.26 Let $\varepsilon > 0$ be given and assume that $0 < |x-5| < \delta$ where $\delta = \varepsilon/4$. If $x < 5$, $|f(x)-4| = |2x-6-4| = 2|x-5| < 2\delta = 2(\varepsilon/4) = \varepsilon/2 < \varepsilon$; if $x > 5$, then $|f(x)-4| = |-4x+24-4| = |-4(x-5)| = 4|x-5| < 4\delta = 4(\varepsilon/4) = \varepsilon$. We've shown that for any $\varepsilon > 0$, $|f(x)-4| < \varepsilon$ whenever $0 < |x-5| < \delta$, provided $0 < \delta \le \varepsilon/4$.

2.7.27 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-0| < \delta$, we would have $|x| < \sqrt{\varepsilon}$. But then $|x^2| < \varepsilon$, which has the form $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to 0} f(x) = 0$.

2.7.28 Let $\varepsilon > 0$ be given. Let $\delta = \sqrt{\varepsilon}$. Then if $0 < |x-3| < \delta$, we would have $|x-3| < \sqrt{\varepsilon}$. But then $|(x-3)^2| < \varepsilon$, which has the form $|f(x) - L| < \varepsilon$. Thus, $\lim_{x\to 3} f(x) = 0$.

2.7.29 Let $\varepsilon > 0$ be given and assume that $0 < |x-2| < \delta$ where $\delta = \min\{1, \varepsilon/8\}$. By factoring $x^2+3x-10$, we find that $|x^2+3x-10| = |x-2||x+5|$. Because $|x-2| < \delta$ and $\delta \le 1$, we have $|x-2| < 1$, which implies that $-1 < x-2 < 1$, or $1 < x < 3$. It follows that $|x+5| = x+5 \le 8$. We also know that $|x-2| < \varepsilon/8$ because $0 < |x-2| < \delta$ and $\delta \le \varepsilon/8$. Therefore $|x^2+3x-10| = |x-2||x+5| < (\varepsilon/8)\cdot 8 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2+3x-10| < \varepsilon$ whenever $0 < |x-2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/8\}$.

2.7.30 Let $\varepsilon > 0$ be given and assume that $0 < |x-4| < \delta$ where $\delta = \min\{1, \varepsilon/14\}$. Observe that $|2x^2-4x+1-17| = |2x^2-4x-16| = 2|x-4||x+2|$. Because $|x-4| < \delta$ and $\delta \le 1$, we have $|x-4| < 1$ which implies that $-1 < x-4 < 1$ or $3 < x < 5$. It follows that $|x+2| = x+2 \le 7$. We also know that $|x-4| < \varepsilon/14$ because $0 < |x-4| < \delta$ and $\delta \le \varepsilon/14$. Therefore $|2x^2-4x+1| = 2|x-4||x+2| < 2(\varepsilon/14)\cdot 7 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|2x^2-4x+1-17| < \varepsilon$ provided $0 < \delta \le \min\{1, \varepsilon/14\}$.

2.7.31 Let $\varepsilon > 0$ be given and assume that $0 < |x-(-3)| < \delta$ where $\delta = \varepsilon/2$. Using the inequality $||a|-|b|| \le |a-b|$ with $a = 2x$ and $b = -6$, it follows that $||2x|-6| = ||2x|-|-6|| \le |2x-(-6)| = 2|x-(-3)| < 2\delta = 2(\varepsilon/2) = \varepsilon$ and therefore $||2x|-6| < \varepsilon$. We have shown that for any $\varepsilon > 0$, $||2x|-6| < \varepsilon$ whenever $0 < |x-(-3)| < \delta$, provided $0 < \delta \le \varepsilon/2$.

2.7.32 Let $\varepsilon > 0$ be given and assume that $0 < |x-25| < \delta$ where $\delta = \min\{25, 5\varepsilon\}$. Because $|x-25| < \delta$ and $\delta \le 25$, we have $|x-25| < 25$, which implies that $-25 < x-25 < 25$ or $0 < x < 50$. Because $x > 0$, we have $\sqrt{x}+5 > 5$ and it follows that $\frac{1}{\sqrt{x}+5} < \frac{1}{5}$. Therefore

$$|\sqrt{x}-5| = \frac{|x-25|}{\sqrt{x}+5} \le \frac{|x-25|}{5} < \frac{\delta}{5} \le \frac{5\varepsilon}{5} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|\sqrt{x}-5| < \varepsilon$, provided $0 < \delta \le \min\{25, 5\varepsilon\}$.

2.7.33 Assume $|x-3| < 1$, as indicated in the hint. Then $2 < x < 4$, so $\frac{1}{4} < \frac{1}{x} < \frac{1}{2}$, and thus $\left|\frac{1}{x}\right| < \frac{1}{2}$.

Also note that the expression $\left|\frac{1}{x} - \frac{1}{3}\right|$ can be written as $\left|\frac{x-3}{3x}\right|$.

Now let $\varepsilon > 0$ be given. Let $\delta = \min(6\varepsilon, 1)$. Now assume that $0 < |x-3| < \delta$. Then

$$|f(x) - L| = \left|\frac{x-3}{3x}\right| < \left|\frac{x-3}{6}\right| < \frac{6\varepsilon}{6} = \varepsilon.$$

Thus we have established that $\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$ whenever $0 < |x-3| < \delta$.

2.7.34 Note that for $x \ne 4$, the expression $\frac{x-4}{\sqrt{x}-2} = \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \sqrt{x}+2$. Also note that if $|x-4| < 1$, then x is between 3 and 5, so $\sqrt{x} > 0$. Then it follows that $\sqrt{x}+2 > 2$, and therefore $\frac{1}{\sqrt{x}+2} < \frac{1}{2}$. We will use this fact below.

Let $\varepsilon > 0$ be given. Let $\delta = \min(2\varepsilon, 1)$. Suppose that $0 < |x-4| < \delta$, so $|x-4| < 2\varepsilon$. We have

$$\begin{aligned}|f(x) - L| &= |\sqrt{x}+2-4| = |\sqrt{x}-2| = \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{|x-4|}{2} < \frac{2\varepsilon}{2} = \varepsilon.\end{aligned}$$

2.7.35 Assume $|x-(1/10)| < (1/20)$, as indicated in the hint. Then $1/20 < x < 3/20$, so $\frac{20}{3} < \frac{1}{x} < \frac{20}{1}$, and thus $\left|\frac{1}{x}\right| < 20$.

Also note that the expression $\left|\frac{1}{x} - 10\right|$ can be written as $\left|\frac{10x-1}{x}\right|$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(\varepsilon/200, 1/20)$. Now assume that $0 < |x-(1/10)| < \delta$. Then

$$\begin{aligned}|f(x) - L| &= \left|\frac{10x-1}{x}\right| < |(10x-1)\cdot 20| \\ &\le |x-(1/10)|\cdot 200 < \frac{\varepsilon}{200}\cdot 200 = \varepsilon.\end{aligned}$$

Thus we have established that $\left|\frac{1}{x} - 10\right| < \varepsilon$ whenever $0 < |x-(1/10)| < \delta$.

2.7.36 Multiplying both sides of the inequality $|\sin\frac{1}{x}| \le 1$ by $|x|$, we have $|x\sin\frac{1}{x}| \le |x|$. Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \varepsilon$. We have $|x\sin\frac{1}{x} - 0| = |x\sin\frac{1}{x}| \le |x| \le |x-0| < \delta = \varepsilon$. Therefore it as been shown that for any $\varepsilon > 0$, $|x\sin\frac{1}{x} - 0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \varepsilon$.

2.7.37 Let $\varepsilon > 0$ be given and assume that $0 < |x-0| < \delta$ where $\delta = \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$. Because $|x-0| < \delta$, we have $|x| < 1$ and $|x| < \sqrt{\varepsilon}/\sqrt{2}$, which implies that $x^2 < 1$ and $x^2 < \varepsilon/2$. It follows that $|x^2 + x^4 - 0| = x^2+x^4 = x^2(1+x^2) \le \frac{\varepsilon}{2}\cdot 2 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^2+x^4-0| < \varepsilon$ whenever $0 < |x-0| < \delta$, provided $0 < \delta \le \min\{1, \sqrt{\varepsilon}/\sqrt{2}\}$.

2.7.38 Let $f(x) = b$. Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x) - b| = |b-b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b-b| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided δ equals any positive number.

2.7.39 Let $m = 0$, then the proof is as follows: Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = 1$ (or any other positive number). Then $|f(x)-b| = |b-b| = 0 < \varepsilon$. We have shown that for any $\varepsilon > 0$, $|b-b| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided δ equals any positive number.

Now assume that $m \ne 0$. Let $\varepsilon > 0$ be given and assume that $0 < |x-a| < \delta$ where $\delta = \varepsilon/|m|$. Then

$$|(mx+b) - (ma+b)| = |m||x-a| < |m|\delta = |m|(\varepsilon/|m|) = \varepsilon.$$

Therefore it has been shown that for any $\varepsilon > 0$, $|(mx+b)-(ma+b)| < \varepsilon$ whenever $0 < |x-a| < \delta$, provided $\delta = \varepsilon/|m|$.

2.7.40 Let $\varepsilon > 0$ be given and assume that $0 < |x-3| < \delta$ where $\delta = \min\{1, \varepsilon/37\}$. By factoring x^3-27, we find that $|x^3-27| = |x-3||x^2+3x+9|$. Because $|x-3| < \delta$ and $\delta \le 1$, we have $|x-3| \le 1$, which implies that $-1 < x-3 < 1$ or $2 < x < 4$. It follows that $|x^2+3x+9| = x^2+3x+9 \le 4^2+3(4)+9 = 37$. We also know that $|x-3| < \varepsilon/37$ because $0 < |x-3| < \delta$ and $\delta \le \varepsilon/37$. Therefore $|x^3-27| = |x-3||x^2+3x+9| \le (\varepsilon/37)\cdot 37 = \varepsilon$. We have shown that for any $\varepsilon > 0$, $|x^3-27| < \varepsilon$ whenever $0 < |x-3| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/37\}$.

2.7.41 Let $\varepsilon > 0$ be given and assume that $0 < |x-1| < \delta$ where $\delta = \min\left\{\dfrac{1}{2}, \dfrac{8\varepsilon}{65}\right\}$. Observe that $|x^4-1| = |(x^2-1)(x^2+1)| = |x-1||x+1||x^2+1|$. Because $|x-1| < \delta$ and $\delta \le \dfrac{1}{2}$, we have $|x-1| < \dfrac{1}{2}$, which implies that $-\dfrac{1}{2} < x-1 < \dfrac{1}{2}$, or $\dfrac{1}{2} < x < \dfrac{3}{2}$. It follows that $|x+1| = x+1 \le \dfrac{5}{2}$. Also $x^2 < \dfrac{9}{4}$, so $|x^2+1| = x^2+1 \le \dfrac{13}{4}$. We also know that $|x-1| < \dfrac{8\varepsilon}{65}$ because $|x-1| < \delta$ and $\delta \le \dfrac{8\varepsilon}{65}$. Therefore

$$|x^4-1| = |x-1||x+1||x^2+1| \le \frac{8\varepsilon}{65}\cdot\frac{5}{2}\cdot\frac{13}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|x^4-1| < \varepsilon$ whenever $0 < |x-1| < \delta$, provided $0 < \delta = \min\left\{\dfrac{1}{2}, \dfrac{8\varepsilon}{65}\right\}$.

2.7.42 Note that if $|x-5| < 1$, then $4 < x < 6$, so that $9 < x+5 < 11$, so $|x+5| < 11$. Note also that $16 < x^2 < 36$, so $\frac{1}{x^2} < \frac{1}{16}$.

Let $\varepsilon > 0$ be given. Let $\delta = \min(1, \frac{400}{11}\varepsilon)$. Assume that $0 < |x-5| < \delta$. Then

$$\begin{aligned}|f(x) - L| &= \left|\frac{1}{x^2} - \frac{1}{25}\right| = \frac{|x+5||x-5|}{25x^2} \\ &< \frac{11|x-5|}{25x^2} < \frac{11}{25\cdot 16}|x-5| < \frac{11}{400}\frac{400\varepsilon}{11} = \varepsilon.\end{aligned}$$

2.7.43 Let $\varepsilon > 0$ be given.

Because $\lim_{x\to a} f(x) = L$, we know that there exists a $\delta_1 > 0$ so that $|f(x)-L| < \varepsilon/2$ when $0 < |x-a| < \delta_1$. Also, because $\lim_{x\to a} g(x) = M$, there exists a $\delta_2 > 0$ so that $|g(x) - M| < \varepsilon/2$ when $0 < |x-a| < \delta_2$.

Now let $\delta = \min(\delta_1, \delta_2)$.

Then if $0 < |x-a| < \delta$, we would have $|f(x) - g(x) - (L-M)| = |(f(x) - L) + (M - g(x))| \le |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$. Note that the key inequality in this sentence follows from the triangle inequality.

2.7.44 First note that the theorem is trivially true if $c = 0$. So assume $c \neq 0$.

Let $\varepsilon > 0$ be given. Because $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ so that if $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon/|c|$. But then $|c||f(x) - L| = |cf(x) - cL| < \varepsilon$, as desired. Thus, $\lim_{x\to a} cf(x) = cL$.

2.7.45 Let $N > 0$ be given. Let $\delta = 1/\sqrt{N}$. Then if $0 < |x - 4| < \delta$, we have $|x - 4| < 1/\sqrt{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x-4|} > \sqrt{N}$, and squaring both sides of this inequality yields $\dfrac{1}{(x-4)^2} > N$. Thus $\lim_{x\to 4} f(x) = \infty$.

2.7.46 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Then if $0 < |x-(-1)| < \delta$, we have $|x+1| < 1/\sqrt[4]{N}$. Taking the reciprocal of both sides, we have $\dfrac{1}{|x+1|} > \sqrt[4]{N}$, and raising both sides to the 4th power yields $\dfrac{1}{(x+1)^4} > N$. Thus $\lim_{x\to -1} f(x) = \infty$.

2.7.47 Let $N > 1$ be given. Let $\delta = 1/\sqrt{N-1}$. Suppose that $0 < |x - 0| < \delta$. Then $|x| < 1/\sqrt{N-1}$, and taking the reciprocal of both sides, we see that $1/|x| > \sqrt{N-1}$. Then squaring both sides yields $dsfrac1x^2 > N - 1$, so $\dfrac{1}{x^2} + 1 > N$. Thus $\lim_{x\to 0} f(x) = \infty$.

2.7.48 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N+1}$. Then if $0 < |x - 0| < \delta$, we would have $|x| < 1/\sqrt[4]{N+1}$. Taking the reciprocal of both sides yields $\dfrac{1}{|x|} > \sqrt[4]{N+1}$, and then raising both sides to the 4th power gives $\dfrac{1}{x^4} > N+1$, so $\dfrac{1}{x^4} - 1 > N$. Now because $-1 \le \sin x \le 1$, we can surmise that $\frac{1}{x^4} - \sin x > N$ as well, because $\dfrac{1}{x^4} - \sin x \ge \dfrac{1}{x^4} - 1$. Hence $\lim_{x\to 0}\left(\dfrac{1}{x^4} - \sin x\right) = \infty$.

2.7.49

a. False. In fact, if the statement is true for a specific value of δ_1, then it would be true for any value of $\delta < \delta_1$. This is because if $0 < |x - a| < \delta$, it would automatically follow that $0 < |x - a| < \delta_1$.

b. False. This statement is not equivalent to the definition – note that it says "for an arbitrary δ there exists an ε" rather than "for an arbitrary ε there exists a δ."

c. True. This is the definition of $\lim_{x\to a} f(x) = L$.

d. True. Both inequalities describe the set of x's which are within δ units of a.

2.7.50

a. We want it to be true that $|f(x)-2| < 0.25$. So we need $|x^2-2x+3-2| = |x^2-2x+1| = (x-1)^2 < 0.25$. Therefore we need $|x - 1| < \sqrt{0.25} = 0.5$. Thus we should let $\delta = 0.5$.

b. We want it to be true that $|f(x) - 2| < \varepsilon$. So we need $|x^2 - 2x + 3 - 2| = |x^2 - 2x + 1| = (x - 1)^2 < \varepsilon$. Therefore we need $|x - 1| < \sqrt{\varepsilon}$. Thus we should let $\delta = \sqrt{\varepsilon}$.

2.7.51 Because we are approaching a from the right, we are only considering values of x which are close to, but a little larger than a. The numbers x to the right of a which are within δ units of a satisfy $0 < x-a < \delta$.

2.7.52 Because we are approaching a from the left, we are only considering values of x which are close to, but a little smaller than a. The numbers x to the left of a which are within δ units of a satisfy $0 < a-x < \delta$.

2.7.53

a. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/2$. Suppose that $0 < x < \delta$. Then $0 < x < \varepsilon/2$ and

$$\begin{aligned}|f(x) - L| &= |2x - 4 - (-4)| = |2x| = 2|x| \\ &= 2x < \varepsilon.\end{aligned}$$

b. Let $\varepsilon > 0$ be given. let $\delta = \varepsilon/3$. Suppose that $0 < 0 - x < \delta$. Then $-\delta < x < 0$ and $-\varepsilon/3 < x < 0$, so $\varepsilon > -3x$. We have

$$\begin{aligned}|f(x) - L| &= |3x - 4 - (-4)| = |3x| = 3|x| \\ &= -3x < \varepsilon.\end{aligned}$$

c. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/3$. Because $\varepsilon/3 < \varepsilon/2$, we can argue that $|f(x) - L| < \varepsilon$ whenever $0 < |x| < \delta$ exactly as in the previous two parts of this problem.

2.7.54

a. This statement holds for $\delta = 2$ (or any number less than 2).

b. This statement holds for $\delta = 2$ (or any number less than 2).

c. This statement holds for $\delta = 1$ (or any number less than 1).

d. This statement holds for $\delta = .5$ (or any number less than 0.5).

2.7.55 Let $\varepsilon > 0$ be given, and let $\delta = \varepsilon^2$. Suppose that $0 < x < \delta$, which means that $x < \varepsilon^2$, so that $\sqrt{x} < \varepsilon$. Then we have

$$|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \varepsilon.$$

as desired.

2.7.56

a. Suppose that $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$. Let $\varepsilon > 0$ be given. There exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta_1$, and there exists a number δ_2 so that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. It immediately follows that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$, as desired.

b. Suppose $\lim\limits_{x\to a} f(x) = L$, and let $\varepsilon > 0$ be given. We know that a δ exists so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In particular, it must be the case that $|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$ and also that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$. Thus $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

2.7.57

a. We say that $\lim\limits_{x\to a^+} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a < x < a + \delta.$$

b. We say that $\lim\limits_{x\to a^-} f(x) = -\infty$ if for each negative number N, there exists $\delta > 0$ such that

$$f(x) < N \quad \text{whenever} \quad a - \delta < x < a.$$

c. We say that $\lim\limits_{x\to a^-} f(x) = \infty$ if for each positive number N, there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad a - \delta < x < a.$$

2.7.58 Let $N < 0$ be given. Let $\delta = -1/N$, and suppose that $1 < x < 1 + \delta$. Then $1 < x < \dfrac{N-1}{N}$, so $\dfrac{1-N}{N} < -x < -1$, and therefore $1 + \dfrac{1-N}{N} < 1 - x < 0$, which can be written as $\dfrac{1}{N} < 1 - x < 0$. Taking reciprocals yields the inequality $N > \dfrac{1}{1-x}$, as desired.

2.7.59 Let $N > 0$ be given. Let $\delta = 1/N$, and suppose that $1 - \delta < x < 1$. Then $\dfrac{N-1}{N} < x < 1$, so $\dfrac{1-N}{N} > -x > -1$, and therefore $1 + \dfrac{1-N}{N} > 1 - x > 0$, which can be written as $\dfrac{1}{N} > 1 - x > 0$. Taking reciprocals yields the inequality $N < \dfrac{1}{1-x}$, as desired.

2.7.60 Let $M < 0$ be given. Let $\delta = \sqrt{-2/M}$. Suppose that $0 < |x-1| < \delta$. Then $(x-1)^2 < -2/M$, so $\dfrac{1}{(x-1)^2} > \dfrac{M}{-2}$, and $\dfrac{-2}{(x-1)^2} < M$, as desired.

2.7.61 Let $M < 0$ be given. Let $\delta = \sqrt[4]{-10/M}$. Suppose that $0 < |x+2| < \delta$. Then $(x+2)^4 < -10/M$, so $\dfrac{1}{(x+2)^4} > \dfrac{M}{-10}$, and $-\dfrac{10}{(x+2)^4} < M$, as desired.

2.7.62 Let $N > 0$ be given and let $N_1 = \max\{1, N - c\}$. Because $\lim\limits_{x\to a} f(x) = \infty$ there exists $\delta > 0$ such that $f(x) > N_1$ whenever $0 < |x-a| < \delta$. It follows that $f(x) + c > N_1 + c \geq N - c + c = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + c > N$ whenever $0 < |x-a| < \delta$.

2.7.63 Let $N > 0$ be given. Because $\lim\limits_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $f(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_1$. Similarly, because $\lim\limits_{x\to a} g(x) = \infty$, there exists $\delta_2 > 0$ such that $g(x) > \frac{N}{2}$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and assume that $0 < |x-a| < \delta$. Because $\delta = \min\{\delta_1, \delta_2\}$, $\delta \leq \delta_1$ and $\delta \leq \delta_2$. It follows that $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ and therefore $f(x) + g(x) > \frac{N}{2} + \frac{N}{2} = N$. So for any $N > 0$, there exists $\delta > 0$ such that $f(x) + g(x) > N$ whenever $0 < |x-a| < \delta$.

2.7.64 Let $\varepsilon > 0$ be given. Let $N = \dfrac{10}{\varepsilon}$. Suppose that $x > N$. Then $x > \dfrac{10}{\varepsilon}$ so $0 < \dfrac{10}{x} < \varepsilon$. Thus, $|\dfrac{10}{x} - 0| < \varepsilon$, as desired.

2.7.65 Let $\varepsilon > 0$ be given. Let $N = 1/\varepsilon$. Suppose that $x > N$. Then $\dfrac{1}{x} < \varepsilon$, and so

$$|f(x) - L| = |2 + \frac{1}{x} - 2| < \varepsilon.$$

2.7.66 Let $M > 0$ be given. Let $N = 100M$. Suppose that $x > N$. Then $x > 100M$, so $\dfrac{x}{100} > M$, as desired.

2.7.67 Let $M > 0$ be given. Let $N = M - 1$. Suppose that $x > N$. Then $x > M - 1$, so $x + 1 > M$, and thus $\dfrac{x^2 + x}{x} > M$, as desired.

2.7.68 Let $\varepsilon > 0$ be given. Because $\lim\limits_{x\to a} f(x) = L$, there exists a number δ_1 so that $|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_1$. And because $\lim\limits_{x\to a} h(x) = L$, there exists a number δ_2 so that $|h(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$, and suppose that $0 < |x-a| < \delta$. Because $f(x) \leq g(x) \leq h(x)$ for x near a, we also have that $f(x) - L \leq g(x) - L \leq h(x) - L$. Now whenever x is within δ units of a (but $x \neq a$), we also note that $-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon$. Therefore $|g(x) - L| < \varepsilon$, as desired.

2.7.69 Let $\varepsilon > 0$ be given. Let $N = \lfloor(1/\varepsilon)\rfloor + 1$. By assumption, there exists an integer $M > 0$ so that $|f(x) - L| < 1/N$ whenever $|x - a| < 1/M$. Let $\delta = 1/M$.

Now assume $0 < |x - a| < \delta$. Then $|x - a| < 1/M$, and thus $|f(x) - L| < 1/N$. But then

$$|f(x) - L| < \frac{1}{\lfloor(1/\varepsilon)\rfloor + 1} < \varepsilon,$$

as desired.

2.7.70 Suppose that $\varepsilon = 1$. Then no matter what δ is, there are numbers in the set $0 < |x - 2| < \delta$ so that $|f(x) - 2| > \varepsilon$. For example, when x is only slightly greater than 2, the value of $|f(x) - 2|$ will be 2 or more.

2.7.71 Let $f(x) = \dfrac{|x|}{x}$ and suppose $\lim\limits_{x\to 0} f(x)$ does exist and is equal to L. Let $\varepsilon = 1/2$. There must be a value of δ so that when $0 < |x| < \delta$, $|f(x) - L| < 1/2$. Now consider the numbers $\delta/3$ and $-\delta/3$, both of which are within δ of 0. We have $f(\delta/3) = 1$ and $f(-\delta/3) = -1$. However, it is impossible for both $|1 - L| < 1/2$ and $|-1 - L| < 1/2$, because the former implies that $1/2 < L < 3/2$ and the latter implies that $-3/2 < L < -1/2$. Thus $\lim\limits_{x\to 0} f(x)$ does not exist.

2.7.72 Suppose that $\lim\limits_{x\to a} f(x)$ exists and is equal to L. Let $\varepsilon = 1/2$. By the definition of limit, there must be a number δ so that $|f(x) - L| < \frac{1}{2}$ whenever $0 < |x - a| < \delta$. Now in every set of the form $(a, a + \delta)$ there are both rational and irrational numbers, so there will be value of f equal to both 0 and 1. Thus we have $|0 - L| < 1/2$, which means that L lies in the interval $(-1/2, 1/2)$, and we have $|1 - L| < 1/2$, which means that L lies in the interval $(1/2, 3/2)$. Because these both can't be true, we have a contradiction.

2.7.73 Because f is continuous at a, we know that $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a) > 0$. Let $\varepsilon = f(a)/3$. Then there is a number $\delta > 0$ so that $|f(x) - f(a)| < f(a)/3$ whenever $|x - a| < \delta$. Then whenever x lies in the interval $(a - \delta, a + \delta)$ we have $-f(a)/3 \le f(x) - f(a) \le f(a)/3$, so $2f(a)/3 \le f(x) \le 4f(a)/3$, so f is positive in this interval.

2.7.74 Using the triangle inequality, we have $|a| = |(a-b)+b| \le |a-b|+|b|$. This implies that $|a| \le |a-b|+|b|$ or $|a| - |b| \le |a - b|$. A similar argument shows that $|b| - |a| \le |a - b|$. Because the expression $||a| - |b||$ is equal to either $|a| - |b|$ or $|b| - |a|$, it follows that $||a| - |b|| \le |a - b|$.

Chapter Two Review

1

a. False. Because $\lim\limits_{x\to 1} \dfrac{x - 1}{x^2 - 1} = \lim\limits_{x\to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$, f doesn't have a vertical asymptote at $x = 1$.

b. False. In general, these methods are too imprecise to produce accurate results.

c. False. For example, the function $f(x) = \begin{cases} 2x & \text{if } x < 0; \\ 1 & \text{if } x = 0; \\ 4x & \text{if } x > 0 \end{cases}$ has a limit of 0 as $x \to 0$, but $f(0) = 1$.

d. True. When we say that a limit exists, we are saying that there is a real number L that the function is approaching. If the limit of the function is ∞, it is still the case that there is no real number that the function is approaching. (There is no real number called "infinity.")

e. False. It could be the case that $\lim\limits_{x\to a^-} f(x) = 1$ and $\lim\limits_{x\to a^+} f(x) = 2$.

f. False.

g. False. For example, the function $f(x) = \begin{cases} 2 & \text{if } 0 < x < 1; \\ 3 & \text{if } 1 \le x < 2, \end{cases}$ is continuous on $(0,1)$, and on $[1,2)$, but isn't continuous on $(0,2)$.

h. True. $\lim_{x\to a} f(x) = f(a)$ if and only if f is continuous at a.

2 $s(1) = 48$ and $s(1.5) = 60$, so the average velocity over the time period is

$$\frac{s(1.5)-s(1)}{1.5-1} = \frac{60-48}{0.5} = 24\,\text{ft/s}.$$

3 For various values of b, we calculate $v_{\text{avg}} = \dfrac{s(b)-s(1.5)}{b-1.5}$.

b	1.6	1.51	1.501	1.5001	1.50001
v_{avg}	10.4	11.84	11.984	11.9984	11.9998

We estimate that the instantaneous velocity is 12.

4

a. $f(-1) = 1$

b. $\lim_{x\to -1^-} f(x) = 3.$

c. $\lim_{x\to -1^+} f(x) = 1.$

d. $\lim_{x\to -1} f(x)$ does not exist.

e. $f(1) = 5.$

f. $\lim_{x\to 1} f(x) = 5.$

g. $\lim_{x\to 2} f(x) = 4.$

h. $\lim_{x\to 3^-} f(x) = 3.$

i. $\lim_{x\to 3^+} f(x) = 5.$

j. $\lim_{x\to 3} f(x)$ does not exist.

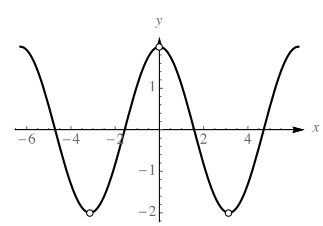
5 This function is discontinuous at $x = -1$, at $x = 1$, and at $x = 3$. At $x = -1$ it is discontinuous because $\lim_{x\to -1} f(x)$ does not exist. At $x = 1$, it is discontinuous because $\lim_{x\to 1} f(x) \ne f(1)$. At $x = 3$, it is discontinuous because $f(3)$ does not exist, and because $\lim_{x\to 3} f(x)$ does not exist.

6

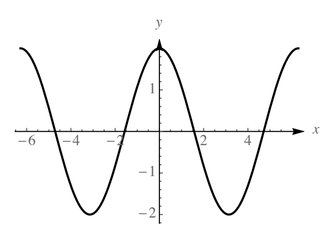
a. The graph drawn by most graphing calculators and computer algebra systems doesn't show the discontinuities where $\sin\theta = 0$.

b. It appears to be equal to 2

c. Using a trigonometric identity, $\lim_{\theta\to 0} \dfrac{\sin 2\theta}{\sin\theta} = \lim_{\theta\to 0} \dfrac{2\sin\theta\cos\theta}{\sin\theta}$. This can then be seen to be $\lim_{\theta\to 0} 2\cos\theta = 2$.



True graph, showing discontinuities where $\sin\theta = 0$.



Graph shown without discontinuities.

7

a.

x	$0.9\pi/4$	$0.99\pi/4$	$0.999\pi/4$	$0.9999\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

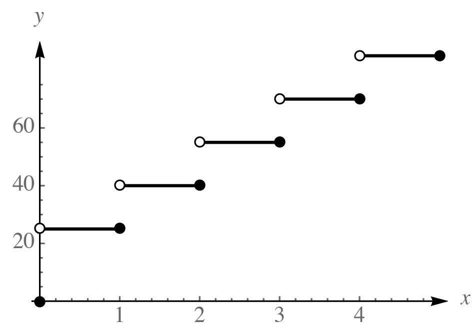
x	$1.1\pi/4$	$1.01\pi/4$	$1.001\pi/4$	$1.0001\pi/4$
$f(x)$	1.4098	1.4142	1.4142	1.4142

The limit appears to be approximately 1.4142.

b. $\displaystyle\lim_{x\to\pi/4} \frac{\cos 2x}{\cos x - \sin x} = \lim_{x\to\pi/4} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \lim_{x\to\pi/4}(\cos x + \sin x) = \sqrt{2}$.

8

a.



b. $\displaystyle\lim_{t\to 2.9} f(t) = 55$.

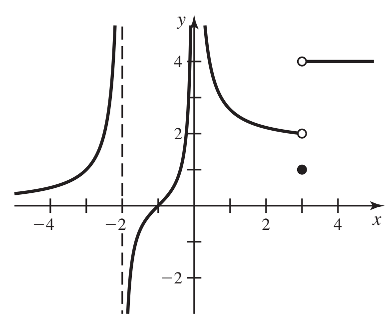
c. $\displaystyle\lim_{t\to 3^-} f(t) = 55$ and $\displaystyle\lim_{t\to 3^+} f(t) = 70$.

d. The cost of the rental jumps by \$15 exactly at $t = 3$. A rental lasting slightly less than 3 days cost \$55 and rentals lasting slightly more than 3 days cost \$70.

e. The function f is continuous everywhere except at the integers. The cost of the rental jumps by \$15 at each integer.

9

There are infinitely many different correct functions which you could draw. One of them is:



10 $\displaystyle\lim_{x\to 1000} 18\pi^2 = 18\pi^2$.

11 $\displaystyle\lim_{x\to 1} \sqrt{5x+6} = \sqrt{11}$.

12

$$\lim_{h\to 0} \frac{\sqrt{5x+5h}-\sqrt{5x}}{h} \cdot \frac{\sqrt{5x+5h}+\sqrt{5x}}{\sqrt{5x+5h}+\sqrt{5x}} = \lim_{h\to 0} \frac{(5x+5h)-5x}{h(\sqrt{5x+5h}+\sqrt{5x})} = \lim_{h\to 0} \frac{5}{\sqrt{5x+5h}+\sqrt{5x}} = \frac{5}{2\sqrt{5x}}.$$

13

$$\lim_{h\to 0} \frac{(h+6)^2+(h+6)-42}{h} = \lim_{h\to 0} \frac{h^2+12h+36+h+6-42}{h} = \lim_{h\to 0} \frac{h^2+13h}{h} = \lim_{h\to 0}(h+13) = 13.$$

14 Factoring the numerator as the difference of squares, we have

$$\begin{aligned}\lim_{x\to a}\frac{(3x+1)^2-(3a+1)^2}{x-a} &= \lim_{x\to a}\frac{((3x+1)-(3a+1))((3x+1)+(3a+1))}{x-a}\\ &= \lim_{x\to a}\frac{(3x-3a)(3x+3a+2)}{x-a}\\ &= 3\lim_{x\to a}(3x+3a+2)\\ &= 3(3a+3a+2)=18a+6.\end{aligned}$$

15 $\lim\limits_{x\to 1}\dfrac{x^3-7x^2+12x}{4-x}=\dfrac{1-7+12}{4-1}=\dfrac{6}{3}=2.$

16 $\lim\limits_{x\to 4}\dfrac{x^3-7x^2+12x}{4-x}=\lim\limits_{x\to 4}\dfrac{x(x-3)(x-4)}{4-x}=\lim\limits_{x\to 4}x(3-x)=-4.$

17 $\lim\limits_{x\to 1}\dfrac{1-x^2}{x^2-8x+7}=\lim\limits_{x\to 1}\dfrac{(1-x)(1+x)}{(x-7)(x-1)}=\lim\limits_{x\to 1}\dfrac{-(x+1)}{x-7}=\dfrac{1}{3}.$

18 $\lim\limits_{x\to 3}\dfrac{\sqrt{3x+16}-5}{x-3}\cdot\dfrac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}=\lim\limits_{x\to 3}\dfrac{3(x-3)}{(x-3)(\sqrt{3x+16}+5)}=\lim\limits_{x\to 3}\dfrac{3}{\sqrt{3x+16}+5}=\dfrac{3}{10}.$

19

$$\begin{aligned}\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) &= \lim_{x\to 3}\frac{2-\sqrt{x+1}}{2(x-3)\sqrt{x+1}}\cdot\frac{(2+\sqrt{x+1})}{(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{4-(x+1)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}\frac{-(x-3)}{2(x-3)(\sqrt{x+1})(2+\sqrt{x+1})}\\ &= \lim_{x\to 3}-\frac{1}{2\sqrt{x+1}(2+\sqrt{x+1})}=-\frac{1}{16}.\end{aligned}$$

20 $\lim\limits_{t\to 1/3}\dfrac{t-\frac{1}{3}}{(3t-1)^2}=\lim\limits_{t\to 1/3}\dfrac{3t-1}{3(3t-1)^2}=\lim\limits_{t\to 1/3}\dfrac{1}{3(3t-1)}$, which does not exist.

21 $\lim\limits_{x\to 3}\dfrac{x^4-81}{x-3}=\lim\limits_{x\to 3}\dfrac{(x-3)(x+3)(x^2+9)}{x-3}=\lim\limits_{x\to 3}(x+3)(x^2+9)=108.$

22 Note that $\dfrac{p^5-1}{p-1}=p^4+p^3+p^2+p+1$. (Use long division.)
$\lim\limits_{p\to 1}\dfrac{p^5-1}{p-1}=\lim\limits_{p\to 1}(p^4+p^3+p^2+p+1)=5.$

23 $\lim\limits_{x\to 81}\dfrac{\sqrt[4]{x}-3}{x-81}=\lim\limits_{x\to 81}\dfrac{\sqrt[4]{x}-3}{(\sqrt{x}+9)(\sqrt[4]{x}+3)(\sqrt[4]{x}-3)}=\lim\limits_{x\to 81}\dfrac{1}{(\sqrt{x}+9)(\sqrt[4]{x}+3)}=\dfrac{1}{108}.$

24 $\lim\limits_{\theta\to\pi/2}\dfrac{\sin^2\theta-5\sin\theta+4}{\sin^2\theta-1}=\lim\limits_{\theta\to\pi/2}\dfrac{(\sin\theta-4)(\sin\theta-1)}{(\sin\theta-1)(\sin\theta+1)}=\lim\limits_{\theta\to\pi/2}\dfrac{\sin\theta-4}{\sin\theta+1}=\dfrac{1-4}{1+1}=-\dfrac{3}{2}.$

25 $\lim\limits_{x\to\pi/2}\dfrac{\frac{1}{\sqrt{\sin x}}-1}{x+\pi/2}=\dfrac{0}{\pi}=0.$

26 The domain of $f(x)=\sqrt{\dfrac{x-1}{x-3}}$ is $(-\infty,1]$ and $(3,\infty)$, so $\lim\limits_{x\to 1^+}f(x)$ doesn't exist.
However, we have $\lim\limits_{x\to 1^-}f(x)=0.$

27 $\lim\limits_{x\to 5} \dfrac{x-7}{x(x-5)^2} = -\infty.$

28 $\lim\limits_{x\to -5^+} \dfrac{x-5}{x+5} = -\infty.$

29 $\lim\limits_{x\to 3^-} \dfrac{x-4}{x^2-3x} = \lim\limits_{x\to 3^-} \dfrac{x-4}{x(x-3)} = \infty.$

30 $\lim\limits_{x\to 0^+} \dfrac{u-1}{\sin u} = -\infty.$

31 $\lim\limits_{x\to 1^+} \dfrac{4x^3-4x^2}{|x-1|} = \lim\limits_{x\to 1^+} \dfrac{4x^2(x-1)}{x-1} = \lim\limits_{x\to 1^+} 4x^2 = 4.$

32 The expression $2x-4 = 2(x-2)$ is negative for $x < 2$, so $|2x-4| = -2(x-2)$. Therefore, $\lim\limits_{x\to 2^-} \dfrac{|2x-4|}{x^2-5x+6} = \lim\limits_{x\to 2^-} \dfrac{-2(x-2)}{(x-2)(x-3)} = \lim\limits_{x\to 2^-} \dfrac{-2}{x-3} = 2.$

33 $\lim\limits_{x\to 0^-} \dfrac{2}{\tan x} = -\infty.$

34 First note that for all x, $\sqrt{x^4} = x^2$. Then we have

$$\lim_{x\to\infty} \frac{(4x^2+3x+1)}{\sqrt{8x^4+2}} \cdot \frac{1/x^2}{1/\sqrt{x^4}} = \lim_{x\to\infty} \frac{4+(3/x)+(1/x^2)}{\sqrt{8+(2/x^4)}} = \frac{4}{\sqrt{8}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

35 $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+10} = \lim\limits_{x\to\infty} \dfrac{2-(3/x)}{4+(10/x)} = \dfrac{2}{4} = \dfrac{1}{2}.$

36 $\lim\limits_{x\to\infty} \dfrac{x^4-1}{x^5+2} = \lim\limits_{x\to\infty} \dfrac{(1/x)-(1/x^5)}{1+(2/x^5)} = \dfrac{0-0}{1+0} = 0.$

37 Note that for $x < 0$, $x = -\sqrt{x^2}$. Then we have

$$\lim_{x\to-\infty} \frac{(3x+1)}{\sqrt{ax^2+2}} \cdot \frac{1/x}{-1/\sqrt{x^2}} = -\lim_{x\to-\infty} \frac{3+(1/x)}{\sqrt{a+(2/x^2)}} = -\frac{3}{\sqrt{a}}.$$

38 We have

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2-b} &= \lim_{x\to\infty} \frac{(\sqrt{x^2+ax}-\sqrt{x^2-b})}{1} \cdot \frac{(\sqrt{x^2+ax}+\sqrt{x^2-b})}{(\sqrt{x^2+ax}+\sqrt{x^2-b})} \\ &= \lim_{x\to\infty} \frac{ax+b}{\sqrt{x^2+ax}+\sqrt{x^2-b}}.\end{aligned}$$

Now noting that $x = \sqrt{x^2}$ for $x > 0$ we have

$$\lim_{x\to\infty} \frac{(ax+b)}{(\sqrt{x^2+ax}+\sqrt{x^2-b})} \cdot \frac{1/x}{1/\sqrt{x^2}} = \lim_{x\to\infty} \frac{a+b/x}{\sqrt{1+a/x}+\sqrt{1-b/x}} = \frac{a}{2}.$$

39 We multiply the numerator and denominator by the conjugate of the denominator (i.e., the expression $\sqrt{x^2-ax}+\sqrt{x^2-x}$). This gives

$$\lim_{x\to\infty} \frac{\sqrt{x^2-ax}+\sqrt{x^2-x}}{(x^2-ax)-(x^2-x)} = \lim_{x\to\infty} \frac{\sqrt{x^2-ax}+\sqrt{x^2-x}}{-ax+x}.$$

We now multiply by $\dfrac{1/\sqrt{x^2}}{1/x}$ to obtain

$$\lim_{x\to\infty} \frac{\sqrt{1-(a/x)}+\sqrt{1-(1/x)}}{-a+1} = \frac{2}{1-a}.$$

40 $\lim_{z\to\infty}\left(e^{-2z}+\frac{2}{z}\right)=0+0=0.$

41 $\lim_{x\to\infty}(3\tan^{-1}x+2)=\frac{3\pi}{2}+2.$

42 $\lim_{x\to-\infty}(-3x^3+5)=\infty.$

43 For $x<1$, $|x-1|+x=-(x-1)+x=1$. Therefore $\lim_{x\to-\infty}(|x-1|+x)=\lim_{x\to-\infty}1=1$.
For $x>1$, $|x-1|+x=x-1+x=2x-1$. Therefore $\lim_{x\to\infty}(|x-1|+x)=\lim_{x\to\infty}2x-1=\infty$.

44 For $x<2$, $|x-2|+x=-(x-2)+x=2$. Therefore $\lim_{x\to-\infty}\frac{|x-2|+x}{x}=\lim_{x\to-\infty}\frac{2}{x}=0$.
For $x>2$, $|x-2|+x=x-2+x=2x-2$. Therefore $\lim_{x\to\infty}\frac{|x-2|+x}{x}=\lim_{x\to\infty}\frac{2x-2}{x}=2$.

45 $\lim_{w\to\infty}\frac{\ln w^2}{\ln w^3+1}=\lim_{w\to\infty}\frac{2\ln w}{(3\ln w+1)}\cdot\frac{1/\ln w}{1/\ln w}=\lim_{w\to\infty}\frac{2}{3+(1/\ln w)}=\frac{2}{3}.$

46 $\lim_{r\to-\infty}\frac{1}{2+e^r}=\frac{1}{2+0}=\frac{1}{2}.$
$\lim_{r\to\infty}\frac{1}{2+e^r}=0.$

47 $\lim_{r\to\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{5r}}{1/e^{5r}}=\lim_{r\to\infty}\frac{(2/e^r)+3}{(7/e^r)-9}=\frac{3}{-9}=-\frac{1}{3}.$
$\lim_{r\to-\infty}\frac{(2e^{4r}+3e^{5r})}{(7e^{4r}-9e^{5r})}\cdot\frac{1/e^{4r}}{1/e^{4r}}=\lim_{r\to-\infty}\frac{2+3e^r}{7-9e^r}=\frac{2}{7}.$

48 Because $-1\le\sin x\le1$, $-e^x\le e^x\sin x\le e^x$. Because $\lim_{x\to-\infty}-e^x=\lim_{x\to-\infty}e^x=0$, we can conclude that $\lim_{x\to-\infty}e^x\sin x=0$ by the Squeeze Theorem.

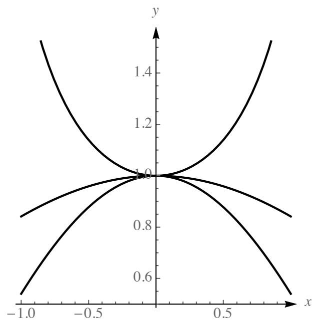
49 We know that $0\le\cos^4x\le1$. Dividing each part of this inequality by x^2+x+1 and then adding 5, we have $5\le5+\frac{\cos^4x}{x^2+x+1}\le5+\frac{1}{x^2+x+1}$. Note that $\lim_{x\to\infty}5=5$ and $\lim_{x\to\infty}\left(5+\frac{1}{x^2+x+1}\right)=5$, so by the Squeeze Theorem we can conclude that $\lim_{x\to\infty}\left(5+\frac{\cos^4x}{x^2+x+1}\right)=5$.

50 Recall that $-1\le\cos t\le1$, and that $e^{3t}>0$ for all t. Thus $-\frac{1}{e^{3t}}\le\frac{\cos t}{e^{3t}}\le\frac{1}{e^{3t}}$. Because $\lim_{t\to\infty}\frac{1}{e^{3t}}=\lim_{t\to\infty}-\frac{1}{e^{3t}}=0$, we can conclude $\lim_{t\to\infty}\frac{\cos t}{e^{3t}}=0$ by the Squeeze Theorem.

51 $\lim_{x\to1^-}\frac{x}{\ln x}=-\infty.$

52 Note that $\lim_{x\to0}(\sin^2x+1)=1$. Thus if $1\le g(x)\le\sin^2x+1$, the Squeeze Theorem assures us that $\lim_{x\to0}g(x)=1$ as well.

53

a. 

b. Because $\lim_{x\to 0}\cos x = \lim_{x\to 0}\frac{1}{\cos x} = 1$, the Squeeze Theorem assures us that $\lim_{x\to 0}\frac{\sin x}{x} = 1$ as well.

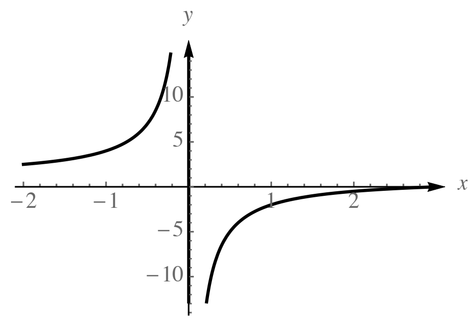
54

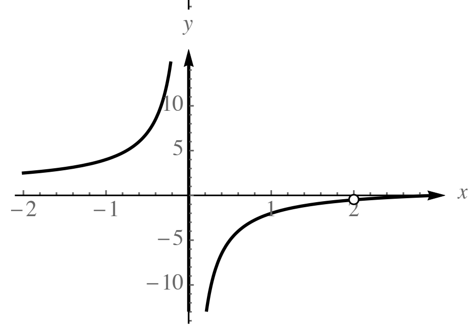
First note that $f(x) = \frac{x^2-5x+6}{x^2-2x} = \frac{(x-3)(x-2)}{x(x-2)}$.

a. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{(x-3)(x-2)}{x(x-2)} = \infty.$
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{(x-3)(x-2)}{x(x-2)} = -\infty.$
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{x-3}{x} = -\frac{1}{2}.$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{x-3}{x} = -\frac{1}{2}.$

b. By the above calculations and the definition of vertical asymptote, f has a vertical asymptote at $x = 0$.

c. Note that the actual graph has a "hole" at the point $(2, -1/2)$, because $x = 2$ isn't in the domain, but $\lim_{x\to 2} f(x) = -1/2$.





55 $\lim_{x\to\infty}\frac{4x^3+1}{1-x^3} = \lim_{x\to\infty}\frac{4+(1/x^3)}{(1/x^3)-1} = \frac{4+0}{0-1} = -4$. A similar result holds as $x\to-\infty$. Thus, $y=-4$ is a horizontal asymptote as $x\to\infty$ and as $x\to-\infty$.

56 Note that $\sqrt{x^{12}} = x^6$ for all x. We have $\lim_{x\to\pm\infty}\frac{(x^6+1)}{\sqrt{16x^{14}+1}}\cdot\frac{1/x^6}{1/\sqrt{x^{12}}} = \lim_{x\to\pm\infty}\frac{1+(1/x^6)}{\sqrt{16x^2+(1/x^{12})}} = 0.$

57 $\lim_{x\to\infty}(1-e^{-2x}) = 1$, while $\lim_{x\to-\infty}(1-e^{-2x}) = -\infty$.
$y = 1$ is a horizontal asymptote as $x\to\infty$.

58 $\lim_{x\to\infty}\frac{1}{\ln x^2} = 0$, and $\lim_{x\to-\infty}\frac{1}{\ln x^2} = 0$, so $y=0$ is a horizontal asymptote as $x\to\infty$ and as $x\to-\infty$.

59 $\lim_{x\to\infty}\frac{(6e^x+20)}{(3e^x+4)}\cdot\frac{1/e^x}{1/e^x} = \lim_{x\to\infty}\frac{6+(20/e^x)}{3+(4/e^x)} = \frac{6}{3} = 2.$
$\lim_{x\to-\infty}\frac{6e^x+20}{3e^x+4} = \frac{0+20}{0+4} = 5.$

60 First note that $\sqrt{\dfrac{1}{x^2}} = \left|\dfrac{1}{x}\right| = \begin{cases} \frac{1}{x} & \text{if } x > 0; \\ -\frac{1}{x} & \text{if } x < 0. \end{cases}$.

$\lim_{x\to\infty} \dfrac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \dfrac{1+(1/x)}{\sqrt{9+\frac{1}{x}}} = \dfrac{1}{3}$.

On the other hand, $\lim_{x\to-\infty} \dfrac{x+1}{\sqrt{9x^2+x}} = \lim_{x\to\infty} \dfrac{1+(1/x)}{-\sqrt{9+\frac{1}{x}}} = -\dfrac{1}{3}$.

So $y = \frac{1}{3}$ is a horizontal asymptote as $x \to \infty$, and $y = -\frac{1}{3}$ is a horizontal asymptote as $x \to -\infty$.

61

a. $\lim_{x\to\infty} \dfrac{(3x^2+5x+7)}{(x+1)} \cdot \dfrac{1/x}{1/x} = \lim_{x\to\infty} \dfrac{3x+5+(7/x)}{1+(1/x)} = \infty$.

$\lim_{x\to-\infty} \dfrac{(3x^2+5x+7)}{(x+1)} \cdot \dfrac{1/x}{1/x} = \lim_{x\to-\infty} \dfrac{3x+5+(7/x)}{1+(1/x)} = -\infty$.

b. By long division, we see that $\frac{3x^2+5x+7}{x+1} = 3x+2+\frac{5}{x+1}$, so the line $y = 3x+2$ is a slant asymptote.

62

a. $\lim_{x\to\infty} \dfrac{9x^2+4}{(2x-1)^2} = \lim_{x\to\infty} \dfrac{9x^2+4}{4x^2-4x+1} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to\infty} \dfrac{9+4/x^2}{4-4/x+1/x^2} = \dfrac{9}{4}$.

$\lim_{x\to-\infty} \dfrac{9x^2+4}{(2x-1)^2} = \lim_{x\to-\infty} \dfrac{9x^2+4}{4x^2-4x+1} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \dfrac{9+4/x^2}{4-4/x+1/x^2} = \dfrac{9}{4}$.

b. Because there is a horizontal asymptote, there is not a slant asymptote.

63

a. $\lim_{x\to\infty} \dfrac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to\infty} \dfrac{1+x-2x^2-x^3}{x^2+1} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to\infty} \dfrac{1/x^2+1/x-2-x}{1+1/x^2} = -\infty$.

$\lim_{x\to-\infty} \dfrac{1+x-2x^2-x^3}{x^2+1} = \lim_{x\to-\infty} \dfrac{1+x-2x^2-x^3}{x^2+1} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \dfrac{1/x^2+1/x-2-x}{1+1/x^2} = \infty$.

b. By long division, we can write $f(x)$ as $f(x) = -x-2+\frac{2x+3}{x^2+1}$, so the line $y = -x-2$ is the slant asymptote.

64

a. $\lim_{x\to\infty} \dfrac{x(x+2)^3}{3x^2-4x} = \lim_{x\to\infty} \dfrac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to\infty} \dfrac{x^2+6x+12+8/x}{3-4/x} = \infty$.

$\lim_{x\to-\infty} \dfrac{x(x+2)^3}{3x^2-4x} = \lim_{x\to-\infty} \dfrac{x^4+6x^3+12x^2+8x}{3x^2-4x} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \dfrac{x^2+6x+12+8/x}{3-4/x} = \infty$.

b. Because the degree of the numerator of this rational function is two more than the degree of the denominator, there is no slant asymptote.

65

a. $\lim_{x\to\infty} \dfrac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to\infty} \dfrac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = \infty$.

$\lim_{x\to-\infty} \dfrac{(4x^3+x^2+7)}{(x^2-x+1)} \cdot \dfrac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \dfrac{4x+1+(7/x)}{1-(1/x)+(1/x^2)} = -\infty$.

b. By long division, we can write $\dfrac{4x^3+x^2+7}{x^2-x+1} = 4x+5+\dfrac{x+2}{x^2-x+1}$. Therefore $y = 4x+5$ is a slant asymptote.

66 Note that $f(x) = \dfrac{2x^2+6}{2x^2+3x-2} = \dfrac{2(x^2+3)}{(2x-1)(x+2)}$.

We have $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{2+6/x^2}{2+3/x-2/x^2} = 1$. A similar result holds as $x \to -\infty$.
$\lim\limits_{x\to 1/2^-} f(x) = -\infty$. $\lim\limits_{x\to 1/2^+} f(x) = \infty$.
$\lim\limits_{x\to -2^-} f(x) = \infty$. $\lim\limits_{x\to -2^+} f(x) = -\infty$.
Thus, $y = 1$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$. Also, $x = \frac{1}{2}$ and $x = -2$ are vertical asymptotes.

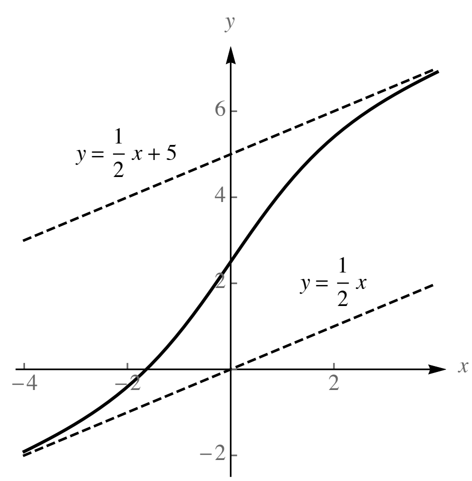
67 Recall that $\tan^{-1} x = 0$ only for $x = 0$. The only vertical asymptote is $x = 0$.
$\lim\limits_{x\to\infty} \dfrac{1}{\tan^{-1} x} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi}$.
$\lim\limits_{x\to-\infty} \dfrac{1}{\tan^{-1} x} = \dfrac{1}{-\pi/2} = -\dfrac{2}{\pi}$. So $y = \frac{2}{\pi}$ is a horizontal asymptote as $x \to \infty$ and $y = -\frac{2}{\pi}$ is a horizontal asymptote as $x \to -\infty$.

68 By long division, we can write $\dfrac{2x^2-7}{x-2} = 2x+4+\dfrac{1}{x-2}$, so $y = 2x+4$ is a slant asymptote. Also, $\lim\limits_{x\to 2^+} \dfrac{2x^2-7}{x-2} = \infty$, so $x = 2$ is a vertical asymptote.

69 Observe that

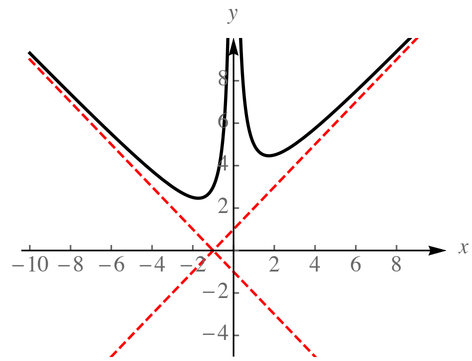
$$f(x) = \frac{x+xe^x+10e^x}{2(e^x+1)} = \frac{x(1+e^x)+10e^x}{2(e^x+1)} = \frac{1}{2}x + \frac{5e^x}{e^x+1}.$$

Because $\lim\limits_{x\to\infty} \dfrac{5e^x}{e^x+1} = \lim\limits_{x\to\infty} \dfrac{5}{1+(1/e^x)} = 5$, the graph of f and the line $y = \frac{1}{2}x+5$ approach each other as $x \to \infty$. Similarly, $\lim\limits_{x\to-\infty} \dfrac{5e^x}{e^x+1} = \dfrac{0}{0+1} = 0$ and therefore the graph of f and the line $y = \frac{1}{2}x$ approach each other as $x \to -\infty$.



70 Observe that $\lim\limits_{x\to 0^+} \dfrac{x^2+x+3}{|x|} = \lim\limits_{x\to 0^+} \dfrac{x^2+x+3}{x} = \lim\limits_{x\to 0^+} x+1+(3/x) = \infty$ and $\lim\limits_{x\to 0^-} \dfrac{x^2+x+3}{|x|} = \lim\limits_{x\to 0^-} \dfrac{x^2+x+3}{-x} = -\lim\limits_{x\to 0^-} (x+1+(3/x)) = -\infty$. For $x > 0$, we have $f(x) = \dfrac{x^2+x+3}{x} = x+1+\dfrac{3}{x}$, so $y = x+1$ is a slant asymptote as $x \to \infty$. For $x < 0$, we have $f(x) = \dfrac{x^2+x+3}{-x} = -x-1-\dfrac{3}{x}$, so

$y = -x - 1$ is a slant asymptote as $x \to -\infty$. So the function has one vertical asymptote $x = 0$ and two slant asymptotes, $y = x + 1$ and $y = -x - 1$.



71 The function f is not continuous at 5 because $f(5)$ is not defined.

72 g is discontinuous at 4 because $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{(x+4)(x-4)}{x-4} = 8 \neq g(4)$.

73 Observe that $h(5) = -2(5) + 14 = 4$. Because $\lim_{x\to 5^-} h(x) = \lim_{x\to 5^-} (-2x + 14) = 4$ and $\lim_{x\to 5^+} h(x) = \lim_{x\to 5^+} \sqrt{x^2 - 9} = \sqrt{25 - 9} = 4$, we have $\lim_{x\to 5} h(x) = 4$. Thus f is continuous at $x = 5$.

74 Observe that $g(2) = -2$ and $\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^3 - 5x^2 + 6x}{x-2} = \lim_{x\to 2} \frac{x(x-2)(x-3)}{x-2} = \lim_{x\to 2} x(x-3) = -2$. Therefore g is continuous at $x = 2$.

75 The domain of f is $(-\infty, -\sqrt{5}]$ and $[\sqrt{5}, \infty)$, and f is continuous on that domain. It is left continuous at $-\sqrt{5}$ and right continuous at $\sqrt{5}$.

76 The domain of g is $[2, \infty)$, and it is continuous on that domain. It is continuous from the right at $x = 2$.

77 The domain of h is $(-\infty, -5)$, $(-5, 0)$, $(0, 5)$, $(5, \infty)$, and like all rational functions, it is continuous on its domain.

78 g is the composition of two functions which are defined and continuous on $(-\infty, \infty)$, so g is continuous on that interval as well.

79 In order for g to be left continuous at 1, it is necessary that $\lim_{x\to 1^-} g(x) = g(1)$, which means that $a = 3$. In order for g to be right continuous at 1, it is necessary that $\lim_{x\to 1^+} g(x) = g(1)$, which means that $a + b = 3 + b = 3$, so $b = 0$.

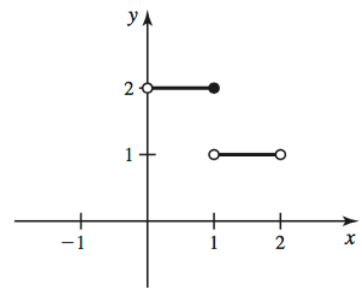
80

a. Because the domain of h is $(-\infty, -3]$ and $[3, \infty)$, there is no way that h can be left continuous at 3.

b. h is right continuous at 3, because $\lim_{x\to 3^+} h(x) = 0 = h(3)$.

81

One such possible graph is pictured to the right.



82

a. Consider the function $f(x) = x^5 + 7x + 5$. f is continuous everywhere, and $f(-1) = -3 < 0$ while $f(0) = 5 > 0$. Therefore, 0 is an intermediate value between $f(-1)$ and $f(0)$. By the Intermediate Value Theorem, there must a number c between 0 and 1 so that $f(c) = 0$.

b. Using a computer algebra system, one can find that $c \approx -0.691671$ is a root.

83

a. Rewrite The equation as $x - \cos x = 0$ and let $f(x) = x - \cos x$. Because x and $\cos x$ are continuous on the given interval, so is f. Because $f(0) = -1 < 0$ and $f(\pi/2) = \pi/2 > 0$, it follows from the Intermediate Value Theorem that the equation has a solution on $(0, \pi/2)$.

b. Using a computer algebra system, one can find that $c \approx 0.739085$ is a root.

84 Temperature changes gradually, so it is reasonable to assume that T is a continuous function and therefore f is also continuous. Because$f(0) = -33 < 0$ and $f(12) = 33 > 0$, it follows from the Intermediate Value Theorem that there is a value t_0 in $(0, 12)$ satisfying $f(t_0) = 0$. . Therefore, $T(t_0) - T(t_0 + 12) = 0$, or $T(t_0) = T(t_0 + 12)$.

85

a. Note that $m(0) = 0$ and $m(5) \approx 38.34$ and $m(15) \approx 21.2$. Thus, 30 is an intermediate value between both $m(0)$ and $m(5)$, and $m(5)$ and $m(15)$. Note also that m is a continuous function. By the IVT, there must be a number c_1 between 0 and 5 with $m(c_1) = 30$, and a number c_2 between 5 and 15 with $m(c_2) = 30$.

b. A little trial and error leads $c_1 \approx 2.4$ and $c_2 \approx 10.8$.

c. No. The graph of the function on a graphing calculator suggests that it peaks at about 38.5

86 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/5$. Now suppose that $0 < |x - 1| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= |(5x - 2) - 3| = |5x - 5| \\ &= 5|x - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

87 Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Now suppose that $0 < |x - 5| < \delta$.
Then

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 - 25}{x - 5} - 10 \right| = \left| \frac{(x-5)(x+5)}{x - 5} - 10 \right| = |x + 5 - 10| \\ &= |x - 5| < \varepsilon. \end{aligned}$$

88 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{4}$ and assume that $0 < |x-3| < \delta$. For $x < 3$,

$$|f(x) - 5| = |3x - 4 - 5| = 3|x - 3| < 3\delta = 3 \cdot \frac{\varepsilon}{4} < \varepsilon.$$

For $x > 3$,

$$|f(x) - 5| = |-4x + 17 - 5| = 4|x - 3| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

We have shown that for any $\varepsilon > 0$, $|f(x) - 5| < \varepsilon$ whenever $0 < |x - 2| < \delta$, provided $0 < \delta \le \frac{\varepsilon}{4}$.

89 Let $\varepsilon > 0$ be given. Let $\delta = \min\{1, \varepsilon/15\}$ and assume that $0 < |x - 2| < \delta$. Then $|3x^2 - 4 - 8| = 3|x^2 - 4| = 3|x - 2||x + 2|$. Because $0 < |x - 2| < \delta$ and $\delta \le 1$, $-1 < x - 2 < 1$ and so $1 < x < 3$. It follows that $x + 2 < 5$. Therefore $|3x^2 - 4 - 8| = 3|x - 2||x + 2| < 3 \cdot \frac{\varepsilon}{15} \cdot 5 = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|3x^2 - 4 - 8| < \varepsilon$ whenever $0 < |x - 2| < \delta$, provided $0 < \delta \le \min\{1, \varepsilon/15\}$.

90 Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon^2}{4}$ and assume that $0 < x - 2 < \delta$. Then $|\sqrt{4x - 8} - 0| = 2\sqrt{x - 2} < 2\sqrt{\delta} = 2\sqrt{\frac{\varepsilon^2}{4}} = \varepsilon$. So we've shown that for any $\varepsilon > 0$, $|\sqrt{4x - 8} - 0| < \varepsilon$ whenever $0 < x - 2 < \delta$, provided $0 < \delta \le \frac{\varepsilon^2}{4}$.

91 Let $N > 0$ be given. Let $\delta = 1/\sqrt[4]{N}$. Suppose that $0 < |x-2| < \delta$. Then $|x-2| < \dfrac{1}{\sqrt[4]{N}}$, so $\dfrac{1}{|x-2|} > \sqrt[4]{N}$, and $\dfrac{1}{(x-2)^4} > N$, as desired.

92

a. Assume $L > 0$. (If $L = 0$, the result follows immediately because that would imply that the function f is the constant function 0, and then $f(x)g(x)$ is also the constant function 0.) Assume that δ_1 is a number so that $|f(x)| \le L$ for $|x - a| < \delta_1$.

   Let $\varepsilon > 0$ be given. Because $\lim_{x \to a} g(x) = 0$, we know that there exists a number $\delta_2 > 0$ so that $|g(x)| < \varepsilon/L$ whenever $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$.

   Then
   $$|f(x)g(x) - 0| = |f(x)||g(x)| < L \cdot \frac{\varepsilon}{L} = \varepsilon,$$
   whenever $0 < |x - a| < \delta$.

b. Let $f(x) = \dfrac{x^2}{x-2}$. Then
   $$\lim_{x \to 2} f(x)(x - 2) = \lim_{x \to 2} \frac{x^2(x-2)}{x-2} = \lim_{x \to 2} x^2 = 4 \ne 0.$$
   This doesn't violate the previous result because the given function f is not bounded near $x = 2$.

c. Because $|H(x)| \le 1$ for all x, the result follows directly from part a) of this problem (using $L = 1$, $a = 0$, $f(x) = H(x)$, and $g(x) = x$).

Chapter 3

Derivatives

3.1 Introducing the Derivative

3.1.1 The secant line through the points $(a, f(a))$ and $(x, f(x))$ for x near a, of the graph of f, is given by $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. As x approaches a, we obtain the limit $m_{\text{tan}} = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim_{x\to a} m_{\text{sec}}$.

3.1.2 The slope of the secant line through the points $(a, f(a))$ and $(x, f(x))$ for x near a, of the graph of f, is given by $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. So the slope is the change of f divided by the length of the interval $[a, x]$ over which the change occurs, that is, the average rate of change of f over $[a, x]$.

3.1.3 The average rate of change of f over $[a, x]$ is the slope of the secant line $m_{\text{sec}} = \dfrac{f(x)-f(a)}{x-a}$. As x approaches a, the length of the interval $x-a$ goes to zero, and in the limit we obtain the instantaneous rate of change of f at a given by $m_{\text{tan}} = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

3.1.4 The slope of the tangent line, the instantaneous rate of change, and the value of the derivative of a function at a given point are all the same.

3.1.5 $f'(a)$ is the value of the derivative of f at a. Also, $f'(a)$ is the slope of the tangent line to the graph of f at $(a, f(a))$. Furthermore, $f'(a)$ is the instantaneous rate of change of f at a.

3.1.6 $f(6)$ is the y-value of the point on the graph associated with $x = 6$, so $f(6) = 5$. $f'(6)$ is the slope of the tangent line at the point $(6, 5)$, so is $f'(6) = -2$.

3.1.7 $f(2) = 4(2) - 1 = 7$. $f'(2)$ is the slope of the tangent line, so $f'(2) = 4$.

3.1.8 $g(3) = 5(3) + 4 = 19$. $g'(3)$ is the slope of the tangent line, so $g'(3) = 5$.

3.1.9 Using the point-slope form of the equation of a line, we have $y - 2 = 3(x-1)$, or $y = 3x - 1$.

3.1.10 Using the point-slope form of the equation of a line, we have $y - 4 = 7(x - (-2))$, or $y = 7x + 18$.

3.1.11 $m_{\text{tan}} = \lim_{x\to 1} \dfrac{-5x+1+4}{x-1} = \lim_{x\to 1} \dfrac{-5x+5}{x-1} = \lim_{x\to 1} \left(-5\left(\dfrac{x-1}{x-1}\right)\right) = -5.$

3.1.12 $m_{\text{tan}} = \lim_{x\to 1} \dfrac{5-5}{x-1} = 0.$

3.1.13

$$\begin{aligned} s'(1) &= \lim_{t\to 1} \frac{-16t^2+100t-84}{t-1} = -4\lim_{t\to 1} \frac{4t^2-25t+21}{t-1} = -4\lim_{t\to 1} \frac{(4t-21)(t-1)}{t-1} \\ &= -4\lim_{t\to 1}(4t-21) = 68\,\text{ft/s}. \end{aligned}$$

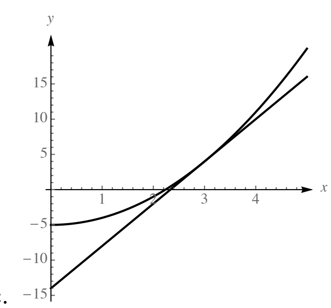
3.1.14

$$\begin{aligned} s'(2) &= \lim_{t\to 2} \frac{-16t^2 + 128t + 192 - 384}{t-2} \\ &= -16 \lim_{t\to 2} \frac{t^2 - 8t + 12}{t-2} \\ &= -16 \lim_{t\to 2} \frac{(t-2)(t-6)}{t-2} \\ &= -16 \lim_{t\to 2} (t-6) = 64\,\text{ft/s}. \end{aligned}$$

3.1.15

a. $m_{\tan} = \lim_{x\to 3} \frac{x^2-5-4}{x-3} = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$

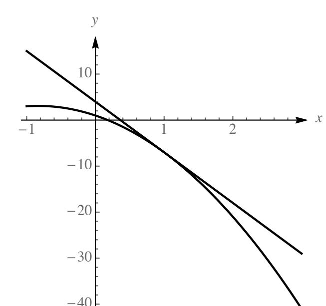
b. Using the point-slope form of the equation of a line, we obtain $y - 4 = 6(x-3)$, or $y = 6x - 14$.

c. 

3.1.16

a. $m_{\tan} = \lim_{x\to 1} \frac{-3x^2-5x+1-(-7)}{x-1} = \lim_{x\to 1} \frac{-3x^2-5x+8}{x-1} = \lim_{x\to 1} \frac{-(3x+8)(x-1)}{x-1} = \lim_{x\to 1}(-3x-8) = -11.$

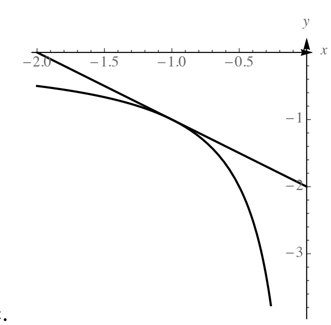
b. Using the point-slope form of the equation of a line, we get $y + 7 = -11(x-1)$, or $y = -11x + 4$.

c. 

3.1.17

a. $m_{\tan} = \lim_{x\to -1} \frac{\frac{1}{x}+1}{x+1} = \lim_{x\to -1} \frac{\frac{1+x}{x}}{x+1} = \lim_{x\to -1} \frac{1}{x} = -1.$

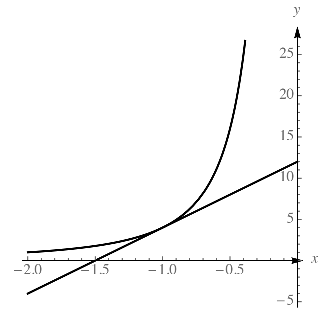
b. $y - (-1) = -1(x+1)$, or $y = -x - 2$.

c. 

3.1.18

a. $m_{\tan} = \lim_{x\to -1} \frac{\frac{4}{x^2}-4}{x+1} = \lim_{x\to -1} \frac{4(1-x)(x+1)}{x^2(x+1)} = \lim_{x\to -1} \frac{4(1-x)}{x^2} = 8.$

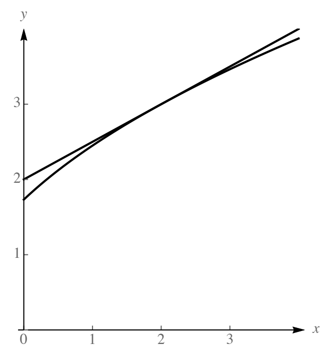
b. $y - 4 = 8(x+1)$, or $y = 8x + 12$.

c. 

3.1.19

a. $m_{\tan} = \lim_{x\to 2} \frac{\sqrt{3x+3}-3}{(x-2)} \cdot \frac{\sqrt{3x+3}+3}{\sqrt{3x+3}+3} = \lim_{x\to 2} \frac{3(x-2)}{(x-2)(\sqrt{3x+3}+3)} = \lim_{x\to 2} \frac{3}{\sqrt{3x+3}+3} = \frac{1}{2}.$

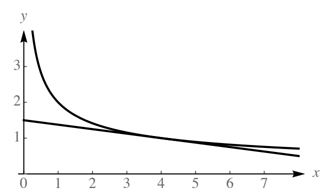
b. $y - 3 = \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x + 2$.

c. 

3.1.20

a. $m_{\tan} = \lim_{x\to 4} \frac{\frac{2}{\sqrt{x}}-1}{(x-4)} = \lim_{x\to 4} \frac{(2-\sqrt{x})}{\sqrt{x}(x-4)} \cdot \frac{(2+\sqrt{x})}{(2+\sqrt{x})} = \lim_{x\to 4} \frac{4-x}{\sqrt{x}(x-4)(2+\sqrt{x})} = -\lim_{x\to 4} \frac{1}{\sqrt{x}(2+\sqrt{x})} = -\frac{1}{2(2+2)} = -\frac{1}{8}.$

b. $y - 1 = -\frac{1}{8}(x-4)$ or $y = -\frac{1}{8}x + \frac{3}{2}$.

c. 

3.1.21

a. $m_{\tan} = \lim_{h\to 0} \frac{2(0+h)+1-1}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$

b. $y - 1 = 2x$, or $y = 2x + 1$.

3.1.22

a. $m_{\tan} = \lim_{h\to 0} \frac{-7(-1+h)-7}{h} = \lim_{h\to 0} -\frac{7h}{h} = \lim_{h\to 0} -7 = -7.$

b. $y - 7 = -7(x+1)$ or $y = -7x$.

3.1.23

a. $m_{\tan} = \lim_{h\to 0} \frac{3(1+h)^2-4(1+h)+1}{h} = \lim_{h\to 0} \frac{3+6h+3h^2-4-4h+1}{h} = \lim_{h\to 0} \frac{3h^2+2h}{h} = \lim_{h\to 0}(3h+2) = 2.$

b. $y + 1 = 2(x-1)$, or $y = 2x - 3$.

3.1.24

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{8 - 2h^2 - 8}{h} = \lim\limits_{h\to 0} -\dfrac{2h^2}{h} = \lim\limits_{h\to 0} -2h = 0.$

b. $y - 8 = 0(x - 0)$ or $y = 8$.

3.1.25

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{(2+h)^2 - 4 - 0}{h} = \lim\limits_{h\to 0} \dfrac{4 + 4h + h^2 - 4}{h} = \lim\limits_{h\to 0}(4 + h) = 4.$

b. $y - 0 = 4(x - 2)$, or $y = 4x - 8$.

3.1.26

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{1+h} - 1}{h} = \lim\limits_{h\to 0} \dfrac{1 - (1+h)}{(1+h)h} = \lim\limits_{h\to 0} \dfrac{-1}{1+h} = -1.$

b. $y - 1 = -1(x - 1)$, or $y = -x + 2$.

3.1.27

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{(1+h)^3 - 1}{h} = \lim\limits_{h\to 0} \dfrac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim\limits_{h\to 0} \dfrac{h(3 + 3h + h^2)}{h} = \lim\limits_{h\to 0}(3 + 3h + h^2) = 3.$

b. $y - 1 = 3(x - 1)$, or $y = 3x - 2$.

3.1.28

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{2h+1} - 1}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1-2h-1}{2h+1}}{h} = \lim\limits_{h\to 0} -\dfrac{2h}{h(2h+1)} = \lim\limits_{h\to 0} -\dfrac{2}{2h+1} = -2\ .$

b. $y - 1 = -2x$, or $y = -2x + 1$.

3.1.29

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\frac{1}{3-2(h-1)} - \frac{1}{5}}{h} = \lim\limits_{h\to 0} \dfrac{\frac{5-(3-2h+2)}{15-10(h-1)}}{h} = \lim\limits_{h\to 0} \dfrac{2}{15 - 10(h-1)} = \dfrac{2}{25}.$

b. $y - \frac{1}{5} = \frac{2}{25}(x + 1)$, or $y = \dfrac{2}{25}x + \dfrac{7}{25}$.

3.1.30

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\sqrt{h+2-1} - 1}{h} = \lim\limits_{h\to 0} \dfrac{(\sqrt{h+1} - 1)(\sqrt{h+1} + 1)}{h(\sqrt{h+1} + 1)} = \lim\limits_{h\to 0} \dfrac{h + 1 - 1}{h(\sqrt{h+1} + 1)} =$
$\lim\limits_{h\to 0} \dfrac{1}{\sqrt{h+1} + 1} = \dfrac{1}{2}.$

b. $y - 1 = \frac{1}{2}(x - 2)$, or $y = \frac{1}{2}x$.

3.1.31

a. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{\sqrt{1+h+3} - 2}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{4+h} - 2}{h} \cdot \dfrac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \lim\limits_{h\to 0} \dfrac{4 + h - 4}{h(\sqrt{4+h} + 2)} =$
$\lim\limits_{h\to 0} \dfrac{1}{\sqrt{4+h} + 2} = \dfrac{1}{4}.$

b. $y - 2 = \frac{1}{4}\,(x - 1)$, or $y = \dfrac{1}{4}x + \dfrac{7}{4}$.

3.1.32

a. $m_{\tan} = \lim_{h\to 0} \dfrac{\frac{-2+h}{-2+h+1} - 2}{h} = \lim_{h\to 0} \dfrac{-2+h-2(-2+h+1)}{(h)(-2+h+1)} = \lim_{h\to 0} -\dfrac{1}{-2+h+1} = 1.$

b. $y - 2 = (x - (-2))$ or $y = x + 4$.

3.1.33

a. $f'(-3) = \lim_{h\to 0} \dfrac{8(-3+h)+24}{h} = \lim_{h\to 0} \dfrac{8h}{h} = 8.$

b. $y - (-24) = 8(x+3)$, or $y = 8x$.

3.1.34

a. $f'(3) = \lim_{h\to 0} \dfrac{(3+h)^2 - 9}{h} = \lim_{h\to 0} \dfrac{(9+6h+h^2)-9}{h} = \lim_{h\to 0} \dfrac{6h+h^2}{h} = 6.$

b. $y - 9 = 6(x-3)$, or $y = 6x - 9$.

3.1.35

a. $f'(-2) = \lim_{h\to 0} \dfrac{4(-2+h)^2 + 2(-2+h) - 12}{h} = \lim_{h\to 0} \dfrac{16 - 16h + 4h^2 - 4 + 2h - 12}{h} =$
$\lim_{h\to 0} \dfrac{-14h + 4h^2}{h} = -14.$

b. $y - 12 = -14(x+2)$, or $y = -14x - 16$.

3.1.36

a. $f'(10) = \lim_{h\to 0} \dfrac{2(10+h)^3 - 2000}{h} = \lim_{h\to 0} \dfrac{2(1000 + 300h + 30h^2 + h^3) - 2000}{h} =$
$\lim_{h\to 0} (600 + 60h + 2h^2) = 600.$

b. $y - 2000 = 600(x - 10)$, or $y = 600x - 4000$.

3.1.37

a. $f'\left(\dfrac{1}{4}\right) = \lim_{h\to 0} \dfrac{\frac{1}{\sqrt{\frac{1}{4}+h}} - 2}{h} = \lim_{h\to 0} \dfrac{1 - 2\sqrt{\frac{1}{4}+h}}{h\sqrt{\frac{1}{4}+h}} = \lim_{h\to 0} \dfrac{\left(1 - 2\sqrt{\frac{1}{4}+h}\right)\left(1 + 2\sqrt{\frac{1}{4}+h}\right)}{h\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} =$
$\lim_{h\to 0} \dfrac{1 - 4\left(\frac{1}{4}+h\right)}{h\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} = \lim_{h\to 0} -\dfrac{4}{\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} = -4.$

b. $y - 2 = -4\left(x - \frac{1}{4}\right)$, or $y = -4x + 3$.

3.1.38

a. $f'(1) = \lim_{h\to 0} \dfrac{\frac{1}{(1+h)^2} - 1}{h} = \lim_{h\to 0} \dfrac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h\to 0} \dfrac{1 - 1 - 2h - h^2}{h(1+h)^2} = \lim_{h\to 0} \dfrac{-2-h}{(1+h)^2} = -2.$

b. $y - 1 = -2(x-1)$, or $y = -2x + 3$.

3.1.39

a. $f'(4) = \lim_{h\to 0} \dfrac{\sqrt{2(4+h)+1} - 3}{h} = \lim_{h\to 0} \dfrac{\sqrt{9+2h} - 3}{h} \cdot \dfrac{\sqrt{9+2h}+3}{\sqrt{9+2h}+3} = \lim_{h\to 0} \dfrac{9 + 2h - 9}{h(\sqrt{9+2h}+3)} =$
$\lim_{h\to 0} \dfrac{2}{\sqrt{9+2h}+3} = \dfrac{1}{3}.$

b. $y - 3 = \frac{1}{3}(x-4)$, or $y = \frac{1}{3}x + \frac{5}{3}$.

3.1.40

a. $f'(12) = \lim_{h\to 0} \dfrac{\sqrt{3(12+h)}-6}{h} = \lim_{h\to 0} \dfrac{\sqrt{36+3h}-6}{h} \cdot \dfrac{\sqrt{36+3h}+6}{\sqrt{36+3h}+6} = \lim_{h\to 0} \dfrac{36+3h-36}{h(\sqrt{36+3h}+6)} =$
$\lim_{h\to 0} \dfrac{3}{\sqrt{36+3h}+6} = \dfrac{1}{4}$.

b. $y - 6 = \frac{1}{4}(x-12)$, or $y = \frac{1}{4}x + 3$.

3.1.41

a. $f'(5) = \lim_{h\to 0} \dfrac{\frac{1}{5+h+5} - \frac{1}{10}}{h} = \lim_{h\to 0} \dfrac{10-(10+h)}{10h(10+h)} = \lim_{h\to 0} \dfrac{-1}{10(10+h)} = -\dfrac{1}{100}$.

b. $y - \frac{1}{10} = -\frac{1}{100}(x-5)$, or $y = -\frac{1}{100}x + \frac{3}{20}$.

3.1.42

a. $f'(2) = \lim_{h\to 0} \dfrac{\frac{1}{3(2+h)-1} - \frac{1}{5}}{h} = \lim_{h\to 0} \dfrac{5-(3(2+h)-1)}{h(3(2+h)-1)5} = \lim_{h\to 0} -\dfrac{3h}{h(5+3h)5} = \lim_{h\to 0} -\dfrac{3}{(5+3h)5} = -\dfrac{3}{25}$.

b. $y - \frac{1}{5} = -\frac{3}{25}(x-2)$, or $y = -\frac{3}{25}x + \frac{11}{25}$.

3.1.43 $m_{\tan} = \lim_{h\to 0} \dfrac{\frac{1}{1+h+1} - \frac{1}{2}}{h} = \lim_{h\to 0} \dfrac{2-(2+h)}{h(2+h)2} = \lim_{h\to 0} \dfrac{-1}{(2+h)2} = -\dfrac{1}{4}$.

3.1.44 $m_{\tan} = \lim_{h\to 0} \dfrac{2+h-(2+h)^2-(2-4)}{h} = \lim_{h\to 0} \dfrac{2+h-4-4h-h^2+2}{h} = \lim_{h\to 0} \dfrac{-3h-h^2}{h} =$
$\lim_{h\to 0}(-3-h) = -3$.

3.1.45 $m_{\tan} = \lim_{h\to 0} \dfrac{2\sqrt{25+h}-1-(2\sqrt{25}-1)}{h} = \lim_{h\to 0} \dfrac{2(\sqrt{25+h}-\sqrt{25})}{h} =$
$\lim_{h\to 0} \dfrac{2(\sqrt{25+h}-\sqrt{25})(\sqrt{25+h}+\sqrt{25})}{h(\sqrt{25+h}+\sqrt{25})} = \lim_{h\to 0} \dfrac{2(25+h-25)}{h(\sqrt{25+h}+\sqrt{25})} = \lim_{h\to 0} \dfrac{2}{\sqrt{25+h}+\sqrt{25}} = \dfrac{1}{5}$.

3.1.46 $m_{\tan} = \lim_{h\to 0} \dfrac{\pi(3+h)^2-9\pi}{h} = \lim_{h\to 0} \dfrac{\pi(9+6h+h^2)-9\pi}{h} = \lim_{h\to 0} \dfrac{6\pi h+\pi h^2}{h} = \lim_{h\to 0}(6\pi+\pi h) = 6\pi$.

3.1.47

a. True. Because the graph is a line, any secant line has the same graph as the function and thus the same slope.

b. False. For example, take $f(x) = x^2$, $P = (0,0)$ and $Q = (1,1)$. Then the secant line has slope $m_{\sec} = \frac{1-0}{1-0} = 1$, but the the graph has a horizontal tangent at P so $m_{\tan} = 0$ and $m_{\sec} > m_{\tan}$.

c. True. $m_{\sec} = \dfrac{(x+h)^2-x^2}{h} = \dfrac{2xh+h^2}{h} = 2x+h$, while $m_{\tan} = \lim_{h\to 0}(2x+h) = 2x$. Because we assume that $h > 0$, we have $m_{\sec} = 2x+h > 2x = m_{\tan}$.

3.1.48 $V'(12) = \lim_{t\to 12} \dfrac{3t-36}{t-12} = 3 \lim_{t\to 12} \dfrac{t-12}{t-12} = 3\,\text{gal/min}$. The instantaneous rate of change of the volume of water in the tub is 3 gal/min 12 minutes after the faucet was turned on.

3.1.49 $d'(4) = \lim_{t\to 4} \dfrac{16t^2-256}{t-4} = 16 \lim_{t\to 4} \dfrac{(t-4)(t+4)}{t-4} = 16 \lim_{t\to 4}(t+4) = 16 \cdot 8 = 128\,\text{ft/s}$. The object is falling with an instantaneous speed of 128 ft/s 4 seconds after being dropped.

3.1.50 $F'(10) = \lim_{x\to 10} \dfrac{\frac{k}{x^2} - \frac{k}{100}}{x-10} = k \lim_{x\to 10} \dfrac{100 - x^2}{100x^2(x-10)} = -k \lim_{x\to 10} \dfrac{(x-10)(x+10)}{100x^2(x-10)} = -k \lim_{x\to 10} \dfrac{x+10}{100x^2} =$ $-0.002k$ N/m. The instantaneous rate of change in the force between the two masses is $-0.002k$ N/m at a distance of separation of 10 m.

3.1.51 $v'(3) = \lim_{t\to 3} \dfrac{(t-5)^2 - 4}{t-3} = \lim_{t\to 3} \dfrac{(t-5-2)(t-5+2)}{t-3} = \lim_{t\to 3}(t-7) = -4$ m/s per second. The instantaneous rate of change in the car's speed is $-4\,\text{m/s}^2$ at time 3 seconds.

3.1.52

a. The average rate of growth is the slope of the secant line between $t = 20$ and $t = 30$ so $m_{\text{sec}} = \dfrac{528,000 - 304,744}{30 - 20} \approx 22,326$, which means that the average population growth is a bit over 22,000 people per year (Census data only provide estimates; there is no use in calculating more accurately).

b. Drawing the secant line from part a) and the approximate tangent line for 1975 (corresponding to $t = 25$), we see that they have about the same slope.

c. The average rate of growth is given by $m_{\text{sec}} = \dfrac{1,563,282 - 304,744}{50 - 20} \approx 41,951$, which means that the average rate of growth of Las Vegas from 1970 to 2000 was about 41,951 people/year. This is an underestimate of the rate of growth in 2000 because the slope of the line tangent to the curve at $t = 50$ is steeper. The approximate slope of the tangent line at $t = 50$ is 60,000, which means the population was growing at about 60,000 people/year in 2000.

3.1.53

a. $L'(15) \approx \dfrac{6.75 - 3.5}{2 - 1.25} \approx 4.3$ mm/week. At 1.5 weeks, the talon is growing at a rate of about 4.3 mm/week.

b. $L'(a) \approx 0$ for $a \geq 4$. At 4 weeks the talons have stopped growing.

3.1.54 $A'(7) \approx -5$ mg/hour. $A'(15) \approx -2$ mg/hour. The caffeine levels are dropping at about 5 mg/hour after 7 hours and at 2 mg/hr after 15 hours.

3.1.55 $D'(60) \approx 0.05$ hr/day. 60 days after January 1, the daylight hours are increasing at about 0.05 hours per day (3 minutes per day). $D'(170) \approx 0$ hr/day. The number of daylight hours is neither increasing nor decreasing 170 days after January 1.

3.1.56 Consider $a = 1$ and $f(x) = 3x^2 + 4x$.

Then $f'(1) = \lim_{x\to 1} \dfrac{3x^2 + 4x - 7}{x - 1}$, as desired.

We have $f'(1) = \lim_{x\to 1} \dfrac{3x^2 + 4x - 7}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(3x+7)}{x-1} = \lim_{x\to 1}(3x+7) = 10$.

3.1.57 Consider $a = 2$ and $f(x) = 5x^2$.

Then $f'(2) = \lim_{x\to 2} \dfrac{5x^2 - 20}{x-2}$ as desired.

We have $f'(2) = \lim_{x\to 2} \dfrac{5x^2 - 20}{x - 2} = \lim_{x\to 2} \dfrac{5(x-2)(x+2)}{x-2} = \lim_{x\to 2} 5(x+2) = 20$.

3.1.58 Consider $a = 2$ and $f(x) = \frac{1}{x+1}$.

Then $f'(2) = \lim_{x\to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x-2}$ as desired.

We have $f'(2) = \lim_{x\to 2} \dfrac{\frac{1}{x+1} - \frac{1}{3}}{x-2} = \lim_{x\to 2} \dfrac{3 - (x+1)}{(x-2)3(x+1)} = \lim_{x\to 2} \dfrac{-(x-2)}{(x-2)3(x+1)} = \lim_{x\to 2} \dfrac{-1}{3(x+1)} = -\dfrac{1}{9}$.

3.1.59 Consider $a = 2$ and $f(x) = x^4$.

Then $f'(2) = \lim_{h\to 0} \dfrac{(2+h)^4 - 16}{h}$ as desired.

We have

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^4 - 16}{h} = \lim_{h\to 0} \frac{16 + 32h + 24h^2 + 8h^3 + h^4 - 16}{h}$$
$$= \lim_{h\to 0} \frac{h\left(32 + 24h + 8h^2 + h^3\right)}{h} = \lim_{h\to 0} \left(32 + 24h + 8h^2 + h^3\right) = 32.$$

3.1.60 Consider $a = 2$ and $f(x) = \sqrt{x}$.

Then $f'(2) = \lim_{h\to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$, as desired.

We have

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{2+h} - \sqrt{2}\right)\left(\sqrt{2+h} + \sqrt{2}\right)}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$
$$= \lim_{h\to 0} \frac{2 + h - 2}{h\left(\sqrt{2+h} + \sqrt{2}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

3.1.61 Consider $a = -1$ and $f(x) = |x|$.

Then $f'(-1) = \lim_{h\to 0} \dfrac{|-1+h| - 1}{h}$ as desired.

We have $f'(-1) = \lim_{h\to 0} \dfrac{|-1+h| - 1}{h} = \lim_{h\to 0} \dfrac{1 - h - 1}{h} = \lim_{h\to 0} -1 = -1.$

3.1.62

a. $f'(4) = \lim_{x\to 4} \dfrac{\sqrt{x} - \sqrt{4}}{x - 4} = \lim_{x\to 4} \dfrac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x\to 4} \dfrac{1}{\sqrt{x} + 2} = \dfrac{1}{4}.$

b. For $|h|$ near zero we have $f'(4) \approx \frac{f(4+h) - f(4)}{h} = \frac{\sqrt{4+h} - 2}{h}$.

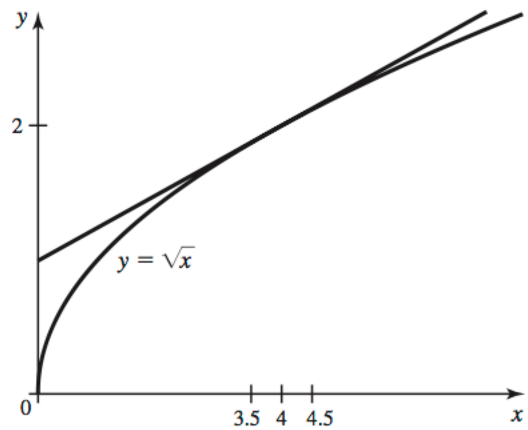
c.

h	$\frac{\sqrt{4+h}-2}{h}$	Error	h	$\frac{\sqrt{4+h}-2}{h}$	Error
0.1	0.248457	−0.001543	−0.1	0.251582	.001582
0.01	0.249844	−0.000156	−0.01	0.250156	.000156
0.001	0.249984	−0.000016	−0.001	0.250016	.000016
0.0001	0.249998	−0.000002	−0.0001	0.250002	.000002

d. The error approaches zero as h approaches zero.

3.1.63

a. Note that the slope generated by the centered difference quotient is $\dfrac{f(4.5) - f(3.5)}{2(0.5)} = \sqrt{4.5} - \sqrt{3.5} \approx$ 0.250492. The centered difference quotient line is very close to the tangent line, which closely approximates the function near the point of tangency.



b.

h	Approximation	Error
0.1	0.25002	2.0×10^{-5}
0.01	≈ 0.25000	2.0×10^{-7}
0.001	≈ 0.25000	2.0×10^{-9}

c. The centered difference quotient is symmetric about zero, so using the corresponding negative values yields the same results.

d. The centered difference quotient appears to be more accurate than the approximation in the previous problem.

3.1.64

a. $\dfrac{f(2.5) - f(2)}{2.5 - 2} = \dfrac{81 - 55}{0.5} = 52.$

b. $\dfrac{f(2.5) - f(1.5)}{2(0.5)} = 81 - 33 = 48.$

3.1.65

a. Forward:

$$\frac{\operatorname{erf}(1.05) - \operatorname{erf}(1)}{0.05} = \frac{0.862436 - 0.842701}{0.05} = 0.3947.$$

Centered:

$$\frac{\operatorname{erf}(1.05) - \operatorname{erf}(0.95)}{2(0.05)} = \frac{0.862436 - 0.820891}{0.1} = 0.41545.$$

b. Forward:

$$\left|0.3947 - \frac{2}{e\sqrt{\pi}}\right| \approx 0.02.$$

Centered:

$$\left|0.41545 - \frac{2}{e\sqrt{\pi}}\right| \approx 0.003$$

3.2 The Derivative as a Function

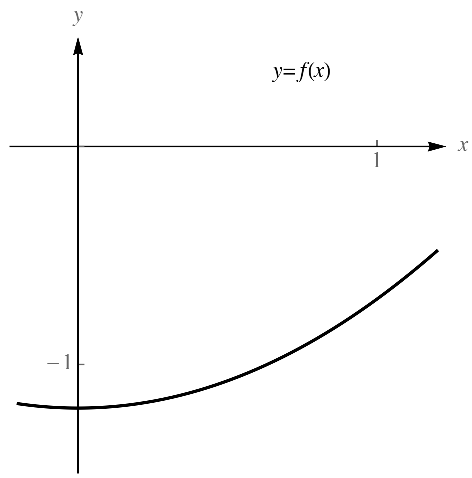
3.2.1 It represents the slope function of f.

3.2.2 $f'(1) = 5,\ f'(2) = 8,\ f'(3) = 11$.

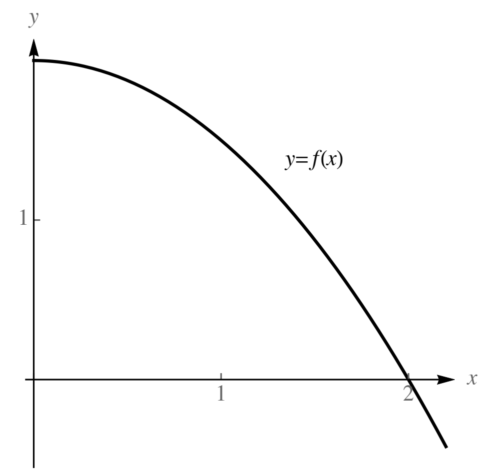
3.2.3 $\dfrac{dy}{dx}$ is the limit of $\dfrac{\Delta y}{\Delta x}$ and is the rate of change of y with respect to x.

3.2.4 The derivative of f with respect to x can be written as $f'(x)$ or $\dfrac{df}{dx}$ or $D_x(f)$ or $\frac{d}{dx}(f)$.

3.2.5



3.2.6



3.2.7 Yes, differentiable functions are continuous by Theorem 3.1.

3.2.8 No, there are continuous functions which are not differentiable. For example $f(x) = |x|$ is continuous everywhere but the graph of f has a corner at 0, and thus f is not differentiable at 0.

3.2.9 The graph is a line with y-intercept of 1 and a slope of 3.

3.2.10 $f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$

3.2.11 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{7(x+h) - 7x}{h} = \lim\limits_{h\to 0} \dfrac{7h}{h} = \lim\limits_{h\to 0} 7 = 7$.

3.2.12 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{-3(x+h) - (-3x)}{h} = \lim\limits_{h\to 0} \dfrac{-3h}{h} = -\lim\limits_{h\to 0} 3 = -3$.

3.2.13 $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{x^2 + 2hx + h^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{h(2x+h)}{h} = \lim\limits_{h\to 0} 2x + h = 2x$. Therefore, $\left.\dfrac{dy}{dx}\right|_{x=3} = 6$ and $\left.\dfrac{dy}{dx}\right|_{x=-2} = -4$.

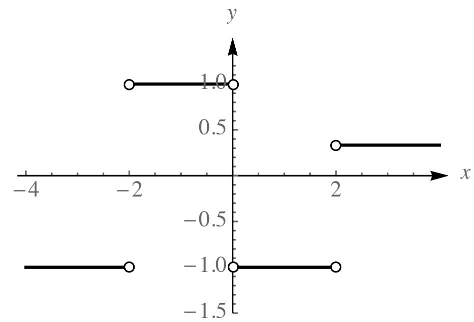
3.2.14 $w'(x) = \lim\limits_{h\to 0} \dfrac{1.5(x+h) - 35.8 - (1.5x - 35.8)}{h} = \lim\limits_{h\to 0} \dfrac{1.5h}{h} = \lim\limits_{h\to 0} 1.5 = 1.5$. The weight of an Atlantic salmon increases at a rate of 1.5 lb/in when its length is between 33 and 48 inches long.

3.2.15 (c) is the only line with negative slope, so it corresponds to derivative (A). Since (d) contains the points $(2, 0)$ and $(0, 1)$, it has slope $\frac{1}{2}$, so it corresponds to derivative (B). Finally, lines (a) and (b) are parallel; since (b) contains the points $(0, 1)$ and $(-1, 0)$, it has slope 1, so that (a) has slope 1 as well. They both correspond to derivative (C).

3.2.16 Note that (A) and (C) have positive slope, while (B) and (D) have negative slope. Since (a) is the largest positive derivative, it corresponds to (A), which has larger slope than (B). Thus (b), the other positive derivative, corresponds to (C). Since (d) is the negative derivative of largest magnitude, it corresponds to (D), since the slope of (D) has larger magnitude than that of (B). So (c), the other negative derivative, corresponds to (B).

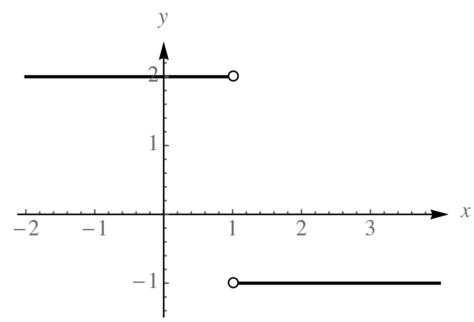
3.2.17

The function f is not differentiable at $x = -2, 0, 2$, so f' is not defined at those points. Elsewhere, the slope is constant.



3.2.18

The function f is not differentiable at $x = 1$ so f' is not defined there. Elsewhere, the slope is constant.



3.2.19

a. f is not continuous at $x = 1$.

b. f is not differentiable at $x = 1$ and at $x = 0$.

3.2.20

a. g is not continuous at $x = -1$ and $x = 0$.

b. g is not differentiable at $x = -1$ and at $x = 0$ and at $x = 1$.

3.2.21

a. $f'(x) = \lim_{h \to 0} \dfrac{5(x+h) + 2 - (5x+2)}{h} = \lim_{h \to 0} \dfrac{5h}{h} = \lim_{h \to 0} 5 = 5.$

b. $f'(1) = 5$ and $f'(2) = 5$.

3.2.22

a. $f'(x) = \lim_{h \to 0} \dfrac{7-7}{h} = \lim_{h \to 0} 0 = 0.$

b. $f'(-1) = 0$ and $f'(2) = 0$.

3.2.23

a. $f'(x) = \lim_{h\to 0} \dfrac{4(x+h)^2+1-(4x^2+1)}{h} = \lim_{h\to 0} \dfrac{4x^2+8hx+4h^2+1-4x^2-1}{h} = \lim_{h\to 0} \dfrac{h(8x+4h)}{h} = \lim_{h\to 0} 8x+4h = 8x.$

b. $f'(2) = 16$ and $f'(4) = 32$.

3.2.24

a. $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^2+3(x+h)-(x^2+3x)}{h} = \lim_{h\to 0} \dfrac{x^2+2hx+h^2+3x+3h-x^2-3x}{h} = \lim_{h\to 0} \dfrac{h(2x+h+3)}{h} = \lim_{h\to 0}(2x+h+3) = 2x+3.$

b. $f'(-1) = 1$ and $f'(4) = 11$.

3.2.25

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} = \lim_{h\to 0} \dfrac{(x+1)-(x+h+1)}{h(x+1)(x+h+1)} = \lim_{h\to 0} \dfrac{-h}{h(x+1)(x+h+1)} = -\lim_{h\to 0} \dfrac{1}{(x+1)(x+h+1)} = -\dfrac{1}{(x+1)^2}.$

b. $f'(-1/2) = -4$ and $f'(5) = -\dfrac{1}{36}$.

3.2.26

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{x+h}{x+h+2}-\frac{x}{x+2}}{h} = \lim_{h\to 0} \dfrac{(x+2)(x+h)-(x+h+2)(x)}{h(x+h+2)(x+2)} = \lim_{h\to 0} \dfrac{x^2+hx+2x+2h-x^2-hx-2x}{h(x+h+2)(x+2)} = \lim_{h\to 0} \dfrac{2}{(x+h+2)(x+2)} = \dfrac{2}{(x+2)^2}.$

b. $f'(-1) = 2$ and $f'(0) = \frac{1}{2}$.

3.2.27

a. $f'(t) = \lim_{h\to 0} \dfrac{\frac{1}{\sqrt{t+h}}-\frac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \dfrac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}} = \lim_{h\to 0} \dfrac{(\sqrt{t}-\sqrt{t+h})}{h\sqrt{t}\sqrt{t+h}} \cdot \dfrac{(\sqrt{t}+\sqrt{t+h})}{(\sqrt{t}+\sqrt{t+h})} = \lim_{h\to 0} \dfrac{t-(t+h)}{h\sqrt{t}\sqrt{t+h}(\sqrt{t}+\sqrt{t+h})} = -\lim_{h\to 0} \dfrac{1}{\sqrt{t}\sqrt{t+h}(\sqrt{t}+\sqrt{t+h})} = -\dfrac{1}{2t^{3/2}}.$

b. $f'(9) = -\frac{1}{54}$ and $f'(1/4) = -4$.

3.2.28

a. $f'(w) = \lim_{h\to 0} \dfrac{(\sqrt{4(w+h)-3}-\sqrt{4w-3})}{h} \cdot \dfrac{(\sqrt{4(w+h)-3}+\sqrt{4w-3})}{(\sqrt{4(w+h)-3}+\sqrt{4w-3})} = \lim_{h\to 0} \dfrac{4(w+h)-3-(4w-3)}{h(\sqrt{4(w+h)-3}+\sqrt{4w-3})} = \lim_{h\to 0} \dfrac{4}{\sqrt{4(w+h)-3}+\sqrt{4w-3}} = \dfrac{2}{\sqrt{4w-3}}.$

b. $f'(1) = 2$ and $f'(3) = \frac{2}{3}$.

3.2.29

a.

$$\begin{aligned}
f'(s) &= \lim_{h\to 0} \frac{4(s+h)^3 + 3(s+h) - (4s^3+3s)}{h} \\
&= \lim_{h\to 0} \frac{4s^3 + 12s^2h + 12sh^2 + 4h^3 + 3s + 3h - 4s^3 - 3s}{h} \\
&= \lim_{h\to 0} \frac{h(12s^2 + 12sh + 4h^2 + 3)}{h} = \lim_{h\to 0}(12s^2 + 12sh + 4h^2 + 3) = 12s^2 + 3.
\end{aligned}$$

b. $f'(-3) = 111$ and $f'(-1) = 15$.

3.2.30

a.

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{3(t+h)^4 - 3t^4}{h} \\
&= 3\lim_{h\to 0} \frac{t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4 - t^4}{h} \\
&= 3\lim_{h\to 0}(4t^3 + 6t^2h + 4th^2 + h^3) = 12t^3
\end{aligned}$$

b. $f'(-2) = -96$ and $f'(2) = 96$.

3.2.31

a. $v(t) = s'(t) = \lim\limits_{h\to 0} \dfrac{-16(t+h)^2 + 100(t+h) - (-16t^2 + 100t)}{h} =$
$\lim\limits_{h\to 0} \dfrac{-16t^2 - 32th - 16h^2 + 100t + 100h + 16t^2 - 100t}{h} = \lim\limits_{h\to 0} \dfrac{h(-32t - 16h + 100)}{h} = -32t + 100.$

b. $v(1) = 68$ ft/s and $v(2) = 36$ ft/s.

3.2.32

a. $v(t) = s'(t) = \lim\limits_{h\to 0} \dfrac{-16(t+h)^2 + 128(t+h) + 192 - (-16t^2 + 128t + 192)}{h} =$
$\lim\limits_{h\to 0} \dfrac{-16t^2 - 32th - 16h^2 + 128t + 128h + 192 + 16t^2 - 128t - 192}{h} =$
$\lim\limits_{h\to 0} \dfrac{h(-32t - 16h + 128)}{h} = -32t + 128.$

b. $v(1) = 96$ ft/s and $v(2) = 64$ ft/s.

3.2.33

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\frac{x+h+1}{x+h+2} - \frac{x+1}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(x+2)(x+h+1) - (x+1)(x+h+2)}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{x^2 + xh + x + 2x + 2h + 2 - (x^2 + xh + 2x + x + h + 2)}{h(x+h+2)(x+2)} \\
&= \lim_{h\to 0} \frac{h}{h(x+h+2)(x+2)} = \frac{1}{(x+2)^2}
\end{aligned}$$

Therefore $\left.\dfrac{dy}{dx}\right|_2 = \dfrac{1}{16}$.

3.2.34

$$\begin{aligned}\frac{ds}{dt} &= \lim_{h\to 0}\frac{11(t+h)^3+t+h+1-(11t^3+t+1)}{h}\\ &= \lim_{h\to 0}\frac{11t^3+33t^2h+33th^2+11h^3+t+h+1-11t^3-t-1}{h}\\ &= \lim_{h\to 0}\frac{h(33t^2+33th+11h^2+1)}{h}\\ &= \lim_{h\to 0}(33t^2+33th+11h^2+1) = 33t^2+1.\end{aligned}$$

Therefore $\left.\frac{ds}{dt}\right|_{-1} = 34.$

3.2.35

a.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{3(x+h)^2+2(x+h)-10-(3x^2+2x-10)}{h}\\ &= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h-10-3x^2-2x+10}{h}\\ &= \lim_{h\to 0}\frac{6xh+2h+3h^2}{h} = \lim_{h\to 0}(6x+2+3h) = 6x+2.\end{aligned}$$

b. We have $f'(1) = 8$, and the tangent line is given by $y+5 = 8(x-1)$, or $y = 8x-13$.

3.2.36

a. $f'(x) = \lim_{h\to 0}\frac{5(x+h)^2-6(x+h)+1-(5x^2-6x+1)}{h} =$
$\lim_{h\to 0}\frac{5x^2+10xh+5h^2-6x-6h-5x^2+6x}{h} = \lim_{h\to 0}\frac{10xh+5h^2-6h}{h} = \lim_{h\to 0}(10x+5h-6) = 10x-6.$

b. We have $f'(2) = 14$, so the tangent line is given by $y-9 = 14(x-2)$, or $y = 14x-19$.

3.2.37

a.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\\ &= \lim_{h\to 0}\frac{3x+3h+1-3x-1}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})} = \lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}.\end{aligned}$$

b. We have $f'(8) = \frac{3}{10}$. Using the point-slope form, we get that the tangent line has equation $y-5 = \frac{3}{10}(x-8)$, which can be written as $y = \frac{3}{10}x+\frac{13}{5}$.

3.2.38

a.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}\\ &= \lim_{h\to 0}\frac{x+h+2-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}.\end{aligned}$$

b. We have $f'(7) = \frac{1}{6}$. Using the point-slope form, we get that the tangent line has equation $y - 3 = \frac{1}{6}(x-7)$, or $y = \frac{1}{6}x + \frac{11}{6}$.

3.2.39

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{2}{3(x+h)+1} - \frac{2}{3x+1}}{h} = \lim_{h\to 0} \dfrac{6x+2-(6x+6h+2)}{h(3x+1)(3x+3h+1)} = \lim_{h\to 0} \dfrac{-6h}{h(3x+1)(3x+3h+1)} = -\dfrac{6}{(3x+1)^2}$.

b. We have $f'(-1) = -\frac{3}{2}$. Using the point-slope form, we get that the tangent line has equation $y + 1 = -\frac{3}{2}(x+1)$, which can be written as $y = -\frac{3}{2}x - \frac{5}{2}$.

3.2.40

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \dfrac{x - x - h}{h(x+h)x} = \lim_{h\to 0} \dfrac{-1}{x^2+xh} = -\dfrac{1}{x^2}$.

b. We have $f'(-5) = -\frac{1}{25}$. Using the point-slope form, we get that the tangent line has equation $y + \frac{1}{5} = -\frac{1}{25}(x+5)$, which can be written as $y = \frac{-1}{25}x - \frac{2}{5}$.

3.2.41

a. From the graph we approximate the derivative by the slope of a secant line: For example we see that $E(6) = 250\,\text{kWh}$ and $E(18) = 350$ kWh, so the power after 10 hours is approximately the slope of the secant line through these points, so $P(10) \approx m_{sec} = \dfrac{E(18) - E(6)}{18 - 6} = \dfrac{350\,\text{kWh} - 250\,\text{kWh}}{12\text{h}} \approx 8.3\,\text{kW}$. Similarly, after 20 hours, using 18 hours and 25 hours, that $P(20) \approx m_{\text{sec}} = \dfrac{E(22) - E(18)}{22 - 18} = \dfrac{325\,\text{kWh} - 350\,\text{kWh}}{4\text{h}} \approx -6.25\text{kW}$.

b. The power is zero where the graph of $E(t)$ has a horizontal tangent line, which happens approximately at $t = 6$ hours and $t = 18$ hours.

c. The power has a maximum where the graph of $E(t)$ has the steepest increase, which is approximately at $t = 12$ hours.

3.2.42 $m_{\tan} = \lim_{h\to 0} \dfrac{m(x+h) + b - (mx+b)}{h} = \lim_{h\to 0} \dfrac{mh}{h} = m$. Thus the derivative has the same value as the slope of the line and the graph and formula of the tangent line are the same as those of the function, namely $mx + b$.

3.2.43

a. $\dfrac{d}{dx}\left(ax^2+bx+c\right)=$
$\displaystyle\lim_{h\to 0}\frac{a(x+h)^2+b(x+h)+c-\left(ax^2+bx+c\right)}{h}=\lim_{h\to 0}\frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}$
$\displaystyle=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}=\lim_{h\to 0}(2ax+ah+b)=2ax+b.$

b. With $a=4, b=-3, c=10$ we have $\frac{d}{dx}(4x^2-3x+10)=2\cdot 4\cdot x+(-3)=8x-3$.

c. From part (b), $f'(1)=8\cdot 1-3=5$.

3.2.44

a. $\displaystyle\frac{d}{dx}\sqrt{ax+b}=\lim_{h\to 0}\frac{\sqrt{a(x+h)+b}-\sqrt{ax+b}}{h}=$
$\displaystyle\lim_{h\to 0}\frac{\left(\sqrt{a(x+h)+b}-\sqrt{ax+b}\right)\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=$
$\displaystyle\lim_{h\to 0}\frac{a(x+h)+b-(ax+b)}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=\lim_{h\to 0}\frac{ax+ah+b-ax-b}{h\left(\sqrt{a(x+h)+b}+\sqrt{ax+b}\right)}=$
$\displaystyle\lim_{h\to 0}\frac{a}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}=\frac{a}{2\sqrt{ax+b}}$, provided $ax+b>0$.

b. With $a=5, b=9$, we have $\frac{d}{dx}\sqrt{5x+9}=\frac{5}{2\sqrt{5x+9}}$.

c. From part (b), $f'(-1)=\frac{5}{2\sqrt{-5+9}}=\frac{5}{2\sqrt{4}}=\frac{5}{4}$.

3.2.45

a. At C and D, the slope of the tangent line (and thus of the curve) is negative.

b. At A, B, and E, the slope of the curve is positive.

c. The graph is in its steepest ascent at A followed by B. At E it barely increases, at D it slightly decreases and at C it is decreasing the most, so the points in decreasing order of slope are A, B, E, D, C.

3.2.46

a. The graph of the function has negative slope to the right of the vertical axis so the slope is negative at D and E.

b. The graph of the function has positive slope to the left of the vertical axis so the slope is positive at A, B, C.

c. The slope at D and E is negative, with a slightly larger absolute value at D. The slope at A and C is about equal and positive, and the slope is steepest at B. So the order is B, A, C, E, D, where A and C could be switched.

3.2.47

a. The function has non-negative slope everywhere, and as there is a horizontal tangent at $x=0$, so the derivative has to be zero at zero. The graph of the derivative has to be above the x-axis and touching it at $x=0$, so (D) is the graph of the derivative.

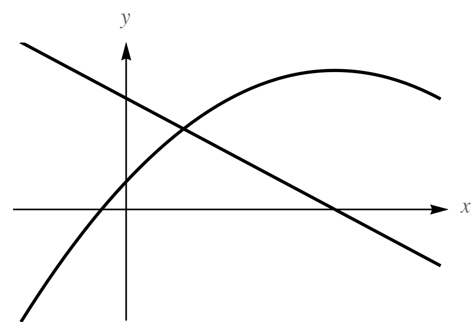
b. The graph of this function has three horizontal tangent lines, at $x=-1,0,1$, and the matching graph of the derivative with three zeros is (C).

c. The function has negative slope on $(-1, 0)$, and positive slope on $(0, 1)$ and has a horizontal tangent at $x = 0$, so the derivative has to be negative on $(-1, 0)$, positive on $(0, 1)$ and zero at $x = 0$; the graph is (B).

d. The function has negative slope everywhere so the graph of the derivative has to be negative everywhere, which is graph (A).

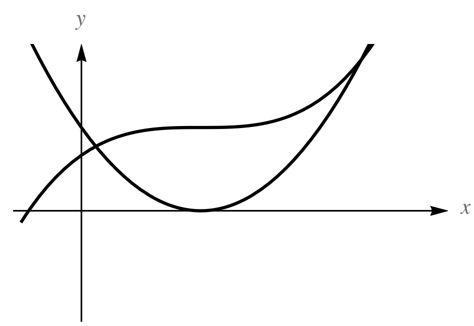
3.2.48

The function has a positive slope for $x < 1$ and a negative slope for $x > 1$, and a horizontal tangent line at $x = 1$.



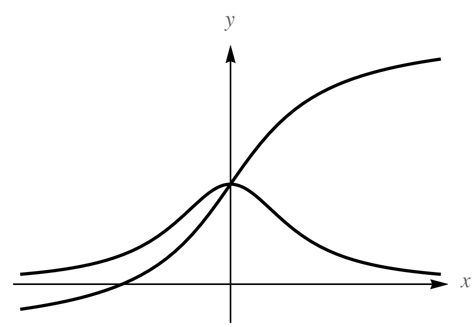
3.2.49

The function always has non-negative slope, so the derivative is never below the x axis.
However, it does have slope zero at about $x = 2$.

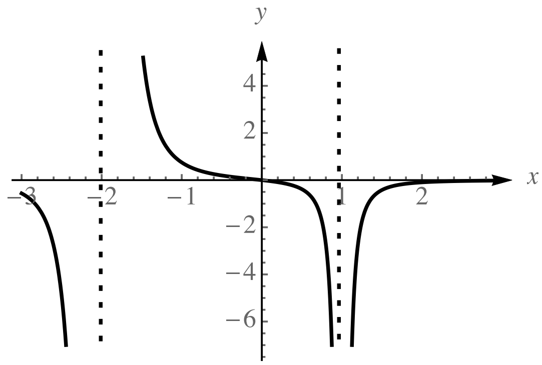


3.2.50

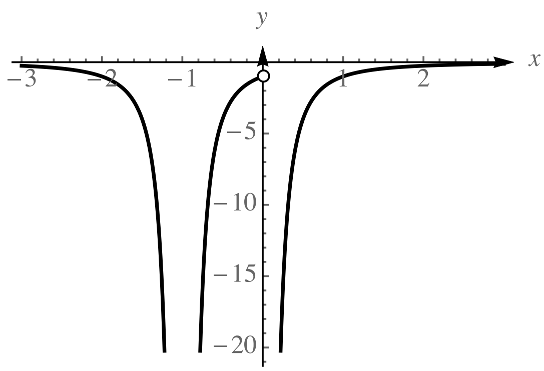
The slope increases until the function crosses the y axis at $x = 0$, and then the slope is still positive, but decreases.



3.2.51 Note that f is undefined at $x = -2$ and $x = 1$, but is differentiable elsewhere. It is decreasing, and increasingly rapidly, as x increases towards $x = -2$. It decreases, but increasingly slowly, and towards a zero slope, as x increases from 1. Finally, between $x = -2$ and $x = 1$, the function increases, but increasingly slowly, until x =0 and then decreases, but increasingly rapidly, as x approaches 1. A graph of the derivative is



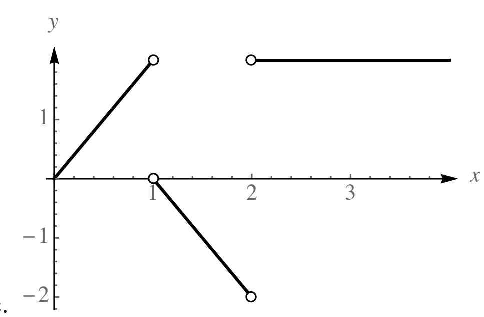
3.2.52 Note that f is undefined at $x = -1$ and $x = 0$, but is differentiable elsewhere. It is decreasing everywhere it is defined. On $(-\infty, -1)$, it decreases increasingly rapidly as $x \to -1$, while on $(-1, 0)$ it decreases more and more slowly from a very large negative slope just to the right of -1 until it is almost flat near $x = 0$. Finally, on $(0, \infty)$ it decreases more and more slowly from a very large negative slope and approaches a zero slope as $x \to \infty$. A graph of the derivative is



3.2.53

a. The function f is not continuous at $x = 1$, because the graph has a jump there.

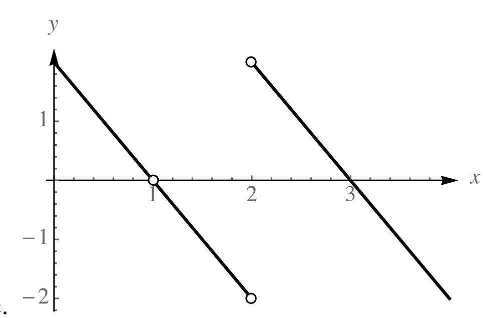
b. The function f is not differentiable at $x = 1$ because it is not continuous at that point (Theorem 3.1 Alternate Version), and it is also not differentiable at $x = 2$ because the graph has a corner there.

c. 

3.2.54

a. The function g is not defined at $x = 1$, because the graph has a hole there.

b. The function g is not differentiable at $x = 1$ because it is not defined at that point, and it is also not differentiable at $x = 2$ because the graph has a cusp there.

c. 

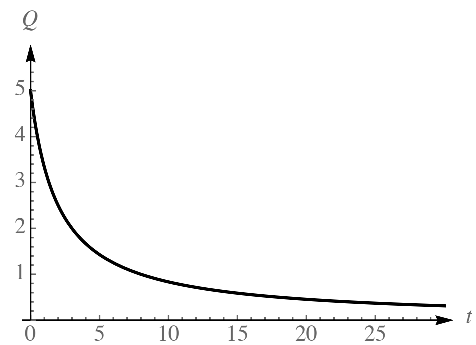
3.2.55

a. The tangent line for Q looks to be the steepest at $t = 0$.

b. All tangent lines for Q have positive slope, so Q' is positive for $t \geq 0$.

c. The tangent lines appear to be getting less steep as t increases, so Q' is decreasing.

d.



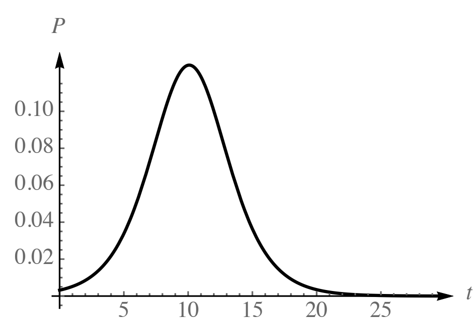
3.2.56

a. The tangent line appears to be the steepest at $t = 10$.

b. The tangent lines to P all have positive slope, so $P' > 0$ for $t \geq 0$.

c. P' appears to increase from $t = 0$ to $t = 10$, and then decrease after that.

d.



3.2.57

a. True. Differentiability implies continuity, by Theorem 3.1.

b. True. Because the absolute value function is continuous, and $y = x + 1$ is continuous, and the composition of continuous functions is continuous, we know that this function is continuous. Note that it is not differentiable at $x = -1$ because the absolute value function is not differentiable at $x = 0$.

c. False. In order for f to be differentiable on $[a, b]$, f would need to be defined at a and at b. Because the domain of f doesn't include these endpoints, this situation is not possible.

3.2.58

a. $f'(x) = 2x$

b. $f'(x) = 3x^2$

c. $f'(x) = 4x^3$

d. $f'(x) = nx^{n-1}$.

3.2.59 In order for f to be differentiable at $x = 1$, it would need to be continuous there. Thus, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2 = 2 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax - 2 = a - 2$, so the only possible value for a is 4. Now checking the differentiability at 1, we have (from the left)

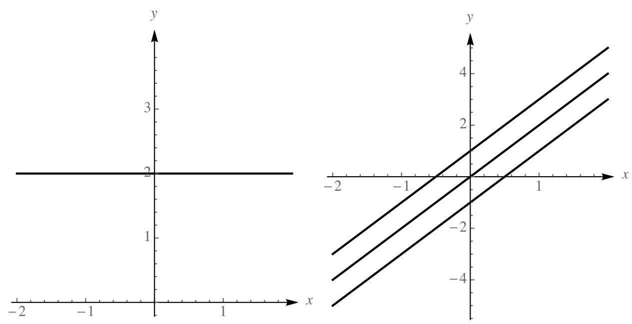
$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{2x^2 - 2}{x - 1} = \lim_{x\to 1^-} 2(x + 1) = 4.$$

Also, from the right we have

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{4x - 2 - 2}{x - 1} = \lim_{x\to 1^+} 4 = 4,$$

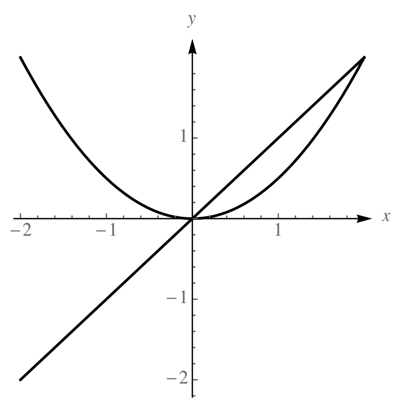
so f is differentiable at 1 for $a = 4$.

3.2.60 The graph of $f'(x) = 2$ is a horizontal line. The possible graphs of f are all lines with slope 2 (and are thus parallel).



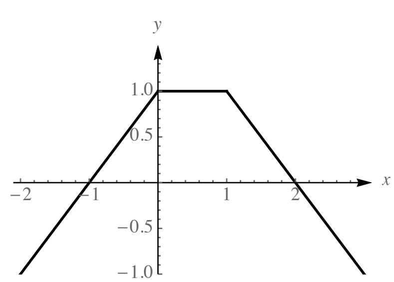
3.2.61

Because $f'(x) = x$ is negative for $x < 0$ and positive for $x > 0$, we have that the graph of f has to have negative slope on $(-\infty, 0)$ and positive slope on $(0, \infty)$ and has to have a horizontal tangent at $x = 0$. Because f' only gives us the slope of the tangent line and not the actual value of f, there are infinitely many graphs possible, they all have the same shape, but are shifted along the $y-$axis.



3.2.62

Because the derivative is constant on $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$, the graph of f has to consist of pieces of straight lines on these intervals. There are infinitely many possible functions f that have f' for its derivative. Because f is assumed to be continuous, each possible f is a shift, up or down, of another possible f.



3.2.63 With $f(x) = 3x - 4$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - 4 - (3x-4)}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3.$$

The slope of the tangent line at $(1, -1)$ is 3, so the slope of the normal line is $-\frac{1}{3}$. The equation of the normal line is thus $y - (-1) = -\frac{1}{3}(x-1)$, or $y = -\frac{1}{3}x - \frac{2}{3}$.

3.2.64 With $f(x) = \sqrt{x}$, we have $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} =$ $\lim_{h\to 0} \dfrac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \dfrac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$. Thus, the slope of the tangent line at $(4, 2)$ is $\frac{1}{4}$, so the slope of the normal line is -4. The equation of the normal line is $y - 2 = -4(x-4)$, or $y = -4x + 18$.

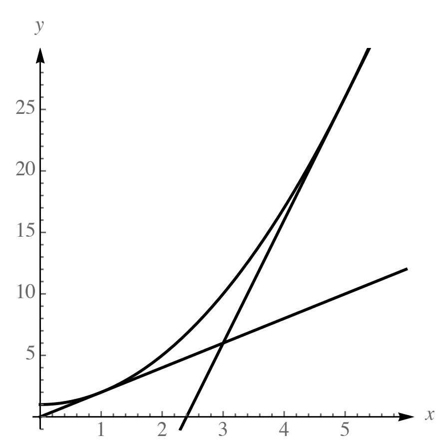
3.2.65 With $f(x) = \frac{2}{x}$, we have $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{2}{x+h} - \frac{2}{x}}{h} = \lim_{h\to 0} \dfrac{\frac{2x-(2(x+h))}{x(x+h)}}{h} =$ $\lim_{h\to 0} \dfrac{-2h}{hx(x+h)} = \lim_{h\to 0} \dfrac{-2}{x(x+h)} = -\dfrac{2}{x^2}$. At the point $(1, 2)$ the slope of the tangent line is -2, so the slope of the normal line is $\frac{1}{2}$. The equation of the normal line is $y - 2 = \frac{1}{2}(x-1)$ or $y = \frac{x}{2} + \frac{3}{2}$.

3.2.66 With $f(x) = x^2 - 3x$, we have $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} =$ $\lim_{h\to 0} \dfrac{x^2 + 2hx + h^2 - 3x - 3h - x^2 + 3x}{h} = \lim_{h\to 0} \dfrac{2hx + h^2 - 3h}{h} = \lim_{h\to 0} \dfrac{h(2x+h-3)}{h} == 2x - 3$. At the point $(3, 0)$ the slope of the tangent line is 3, so the slope of the normal line is $-\frac{1}{3}$. The equation of the normal line is $y - 0 = -\frac{1}{3}(x-3)$ or $y = -\frac{x}{3} + 1$.

3.2.67 With $f(x) = x^2 + 1$, we have $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 + 1 - (x^2+1)}{h} =$ $\lim_{h\to 0} \dfrac{x^2 + 2hx + h^2 + 1 - x^2 - 1}{h} = \lim_{h\to 0} \dfrac{2hx + h^2}{h} = \lim_{h\to 0} \dfrac{h(2x+h)}{h} = 2x$. We are looking for points (x, x^2+1) where the slope of the line between this point and $Q(3, 6)$ is equal to $2x$. So we seek solutions to the equation

$$\frac{6 - (x^2+1)}{3 - x} = 2x,$$

which can be written as $5 - x^2 = 2x(3-x)$, or $x^2 - 6x + 5 = 0$. Factoring, we obtain $(x-5)(x-1) = 0$, so the solutions are $x = 5$ and $x = 1$. Note that at the point $(5, 26)$ on the curve the tangent line is $y - 26 = 10(x-5)$ which does contain the point $Q(3, 6)$ and at the point $(1, 2)$ the equation of the tangent line is $y - 2 = 2(x-1)$, which also contains the point $Q(3, 6)$.



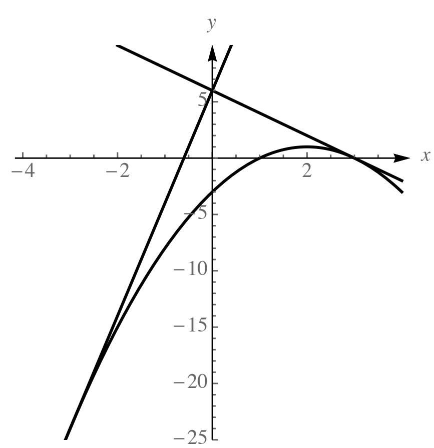
3.2.68 With $f(x) = -x^2 + 4x - 3$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{-(x+h)^2 + 4(x+h) - 3 - (-x^2+4x-3)}{h} \\ &= \lim_{h\to 0} \frac{-x^2 - 2hx - h^2 + 4x + 4h - 3 + x^2 - 4x + 3}{h} \\ &= \lim_{h\to 0} \frac{-2hx - h^2 + 4h}{h} = \lim_{h\to 0} \frac{h(-2x-h+4)}{h} \\ &= \lim_{h\to 0} (-2x - h + 4) = -2x + 4. \end{aligned}$$

We are looking for points $(x, -x^2+4x-3)$ where the slope of the line between this point and $Q(0,6)$ is equal to $-2x+4$. So we seek solutions to the equation

$$\frac{-x^2+4x-3-6}{x-0} = -2x+4,$$

which can be written as $-x^2+4x-9 = -2x^2+4x$, or $x^2-9=0$. This factors as $(x-3)(x+3)=0$, so the solutions are $x=\pm 3$. Note that at the point $(3,0)$ the equation of the tangent line is $y = -2(x-3)$ which does contain the point $Q(0,6)$, and at the point $(-3,-24)$ the equation of the tangent line is $y+24 = 10(x+3)$, which also contains the point $Q(0,6)$.



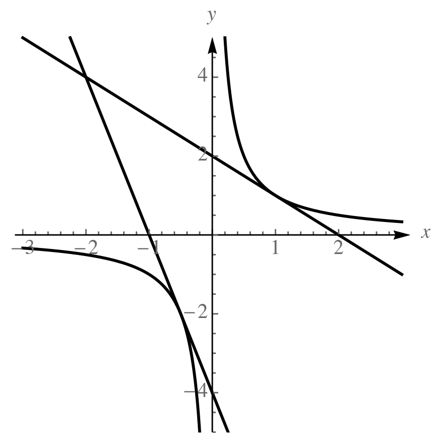
3.2.69 With $f(x) = \frac{1}{x}$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}. \end{aligned}$$

We are looking for points $(x, 1/x)$ where the slope of the line between this point and $Q(-2,4)$ is equal to $\frac{-1}{x^2}$. So we seek solutions to the equation

$$\frac{1/x - 4}{x+2} = -\frac{1}{x^2},$$

which can be written as $x - 4x^2 = -x - 2$, or $4x^2 - 2x - 2 = 0$, or $2x^2 - x - 1 = 0$. This factors as $(2x+1)(x-1) = 0$, so the solutions are $x = 1$ and $x = -1/2$. Note that at $x = 1$ the equation of the tangent line is $y - 1 = -1(x-1)$ which does contain the point $Q(-2,4)$, and at $x = -1/2$ the equation of the tangent line is $y + 2 = -4\left(x + \frac{1}{2}\right)$ or $y = -4x - 4$ which also contains the point $Q(-2,4)$.



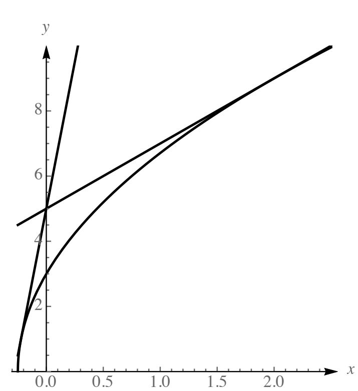
3.2.70 With $f(x) = 3\sqrt{4x+1}$, we have

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{3\sqrt{4(x+h)+1} - 3\sqrt{4x+1}}{h} \\ &= 3\lim_{h\to 0} \frac{(\sqrt{4(x+h)+1} - \sqrt{4x+1})}{h} \cdot \frac{(\sqrt{4(x+h)+1} + \sqrt{4x+1})}{(\sqrt{4(x+h)+1} + \sqrt{4x+1})} \\ &= 3\lim_{h\to 0} \frac{4(x+h)+1-(4x+1)}{h(\sqrt{4(x+h)+1} + \sqrt{4x+1})} \\ &= 12\lim_{h\to 0} \frac{1}{\sqrt{4(x+h)+1} + \sqrt{4x+1}} = \frac{6}{\sqrt{4x+1}} \end{aligned}$$

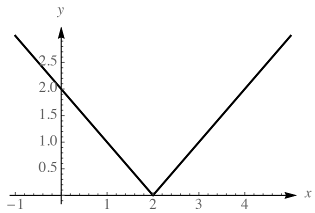
We are looking for points $(x, 3\sqrt{4x+1})$ where the slope of the line between this point and $Q(0,5)$ is equal to $\frac{6}{\sqrt{4x+1}}$. So we seek solutions to the equation

$$\frac{3\sqrt{4x+1}-5}{x} = \frac{6}{\sqrt{4x+1}}.$$

Cross multiplying gives $6x = 3(4x+1) - 5\sqrt{4x+1}$. This can be written as $5\sqrt{4x+1} = 6x+3$. Squaring both sides and gathering like terms gives $36x^2 - 64x - 16 = 0$ which can be written as $4(9x+2)(x-2) = 0$, so the solutions are $x = 2$ and $x = -\frac{2}{9}$. At $x = 2$ the tangent line is $y = 2(x-2)+9$ or $y = 2x+5$ and at $x = -\frac{2}{9}$ the tangent line is $y = 18x+5$.



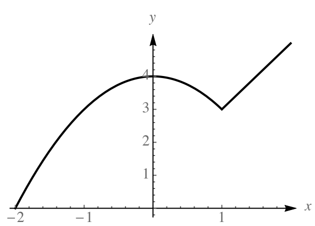
3.2.71

a. 

b. $f'_+(2) = \lim_{h\to 0^+} \frac{|2+h-2|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$, because for $h > 0$, we have $|h| = h$. Similarly, $f'_-(2) = \lim_{h\to 0^-} \frac{|2+h-2|-0}{h} = \lim_{h\to 0^-} -\frac{h}{h} = -1$, because for $h < 0$, we have $|h| = -h$.

c. Because f is defined at $a = 2$ and the graph of f does not jump, f is continuous at $a = 2$. Because the left-hand and right-hand derivatives are not equal, f is not differentiable at $a = 2$.

3.2.72

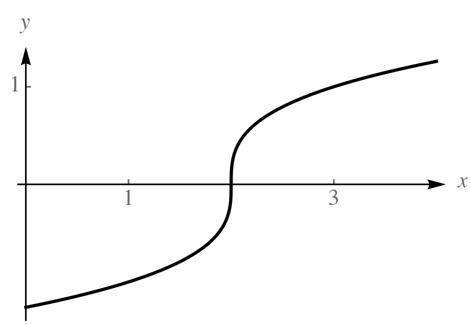
a. 

b. Assuming the point $a = 1$, we get $f'_+(1) = \lim_{h\to 0^+} \frac{2(1+h)+1-3}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$, because for $h > 0$, we have $1 + h > 1$. Similarly, $f'_-(1) = \lim_{h\to 0^-} \frac{4-(1+h)^2-3}{h} = \lim_{h\to 0^-} \frac{-2h-h^2}{h} = -2$, because for $h < 0$, we have $1 + h < 1$.

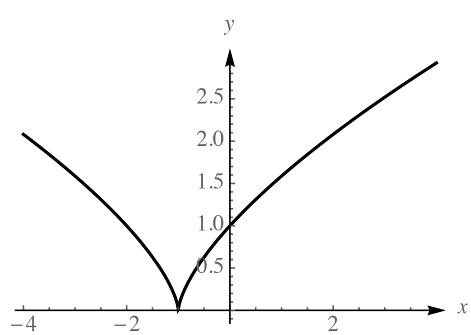
c. Because f is defined at $a = 1$ and the graph of f does not jump, f is continuous at $a = 1$. Because the left-hand and right-hand derivatives are not equal, f is not differentiable at $a = 1$.

3.2.73

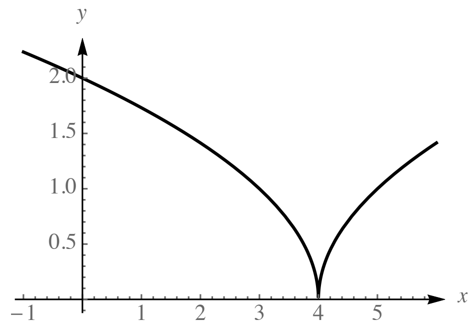
a. The graph has a vertical tangent at $x = 2$.



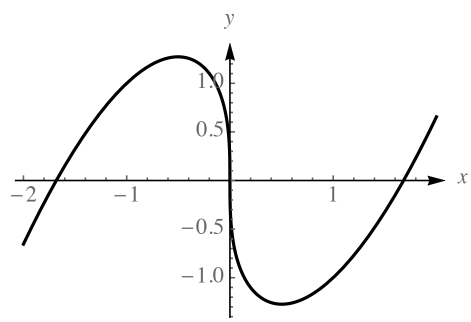
b. The graph has a vertical tangent at $x = -1$.



c. The graph has a vertical tangent at $x = 4$.

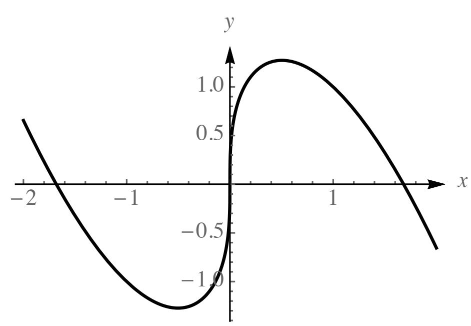


d. The graph has a vertical tangent at $x = 0$.

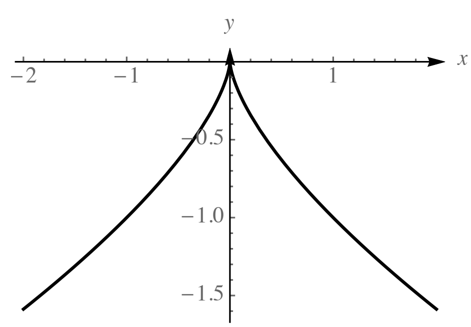


3.2.74

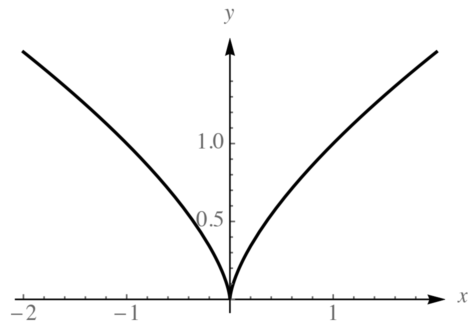
a. This graph has $\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x) = +\infty$, where $a = 0$.



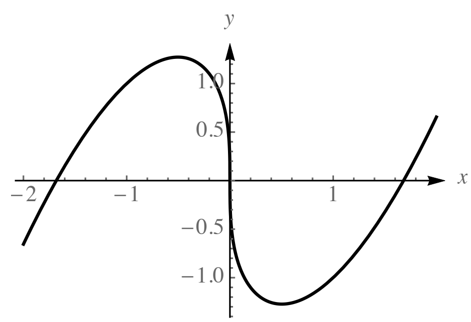
b. This graph has $\lim_{x \to a^-} f'(x) = +\infty$ and $\lim_{x \to a^+} f'(x) = -\infty$, where $a = 0$.



c. This graph has $\lim_{x\to a^-} f'(x) = -\infty$ and $\lim_{x\to a^+} f'(x) = +\infty$, where $a = 0$.



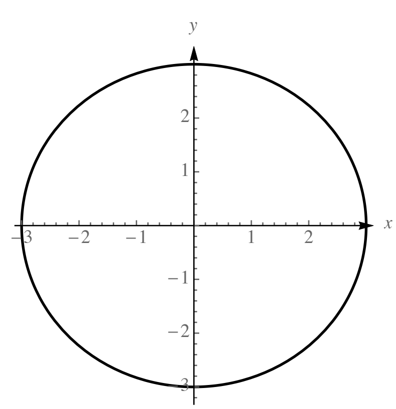
d. This graph has $\lim_{x\to a^-} f'(x) = -\infty$ and $\lim_{x\to a^+} f'(x) = -\infty$, where $a = 0$.



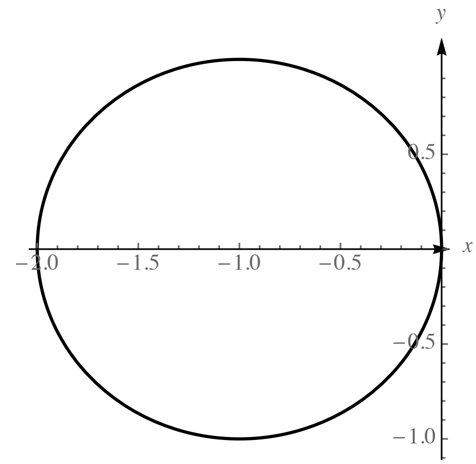
3.2.75 $f'(0) = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$. Thus the graph of f has a vertical tangent at $x = 0$.

3.2.76

a. This circle has vertical tangents at $x = 3$ and $x = -3$.

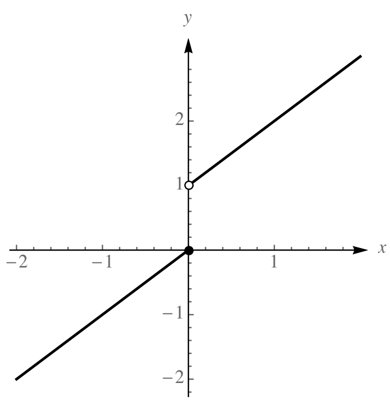


b. This circle has vertical tangents at $x = -2$ and $x = 0$.



3.2.77

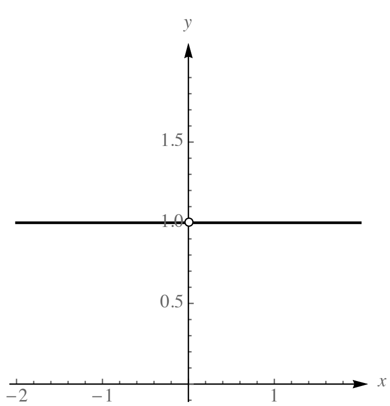
a.



b. For $x < 0$, we have $f'(x) = \lim_{h\to 0} \dfrac{x+h-x}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1.$

c. For $x > 0$, we have $f'(x) = \lim_{h\to 0} \dfrac{x+h+1-(x+1)}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1.$

d.



e. f is not differentiable at $x = 0$ as it is not continuous there. Also, if we were to compute the derivative of f from the right at 0 we would have

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h+1-0}{h} = \lim_{h \to 0^+} \frac{h+1}{h},$$

which does not exist.

3.2.78 It is not differentiable at $x = 2$. The denominator of f is zero when $x = 2$, so f is not defined at $x = 2$, and is therefore not differentiable there.

3.3 Rules of Differentiation

3.3.1 Often the limit definition of f' is difficult to compute, especially for functions which are reasonably complicated. The rules for differentiation allow us to easily compute the derivatives of complex functions.

3.3.2 It is shown to be valid for all positive integers n. In future sections we will see that it holds for all real numbers.

3.3.3 The function $f(x) = e^x$ is an example of a function with this property.

3.3.4 The sum rule tells us that the derivative of $f + g$ is $f' + g'$. That is, the derivative of the sum of two functions is the sum of the derivatives of those functions.

3.3.5 By the constant multiple rule, the derivative of the function cf where c is a constant and f is a function is cf'. That is, the derivative of a constant times a function is that same constant times the derivative of the function.

3.3.6 The 5th derivative of a function is found by differentiating the 4th derivative of the function. The 4th derivative is found by differentiating the 3rd derivative of the function, and so on. Thus, one would need to compute f' and then four more derivatives to arrive at the 5th derivative of f.

3.3.7 $(f + g)'(3) = f'(3) + g'(3) = 6 + (-2) = 4$.

3.3.8 $g'(x) = f'(x) + e^x$, so $g'(0) = f'(0) + e^0 = 6 + 1 = 7$.

3.3.9 $F'(2) = f'(2) + g'(2) = -1 + \frac{1}{2} = -\frac{1}{2}$.

3.3.10 $G'(6) = f'(6) - g'(6) = -1 - \frac{1}{2} = -\frac{3}{2}$.

3.3.11 $H'(2) = 3f'(2) + 2g'(2) = 3(-1) + 2(\frac{1}{2}) = -2$.

3.3.12 $\dfrac{d}{dx}\left[f(x) + g(x)\right]_{x=1} = f'(1) + g'(1) = 3 + 2 = 5$.

3.3.13 $\dfrac{d}{dx}\left[1.5f(x)\right]_{x=2} = 1.5f'(2) = 1.5 \cdot 5 = 7.5$.

3.3.14 $\dfrac{d}{dx}\left[2x - 3g(x)\right]_{x=4} = 2 - 3g'(4) = 2 - (3 \cdot 1) = -1$.

3.3.15 $f'(t) = 10t^9$, $f''(t) = 90t^8$, and $f'''(t) = 720t^7$.

3.3.16 $y(0) = e^0 = 1$ and $y'(0) = e^0 = 1$. The equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

3.3.17 $\dfrac{d}{dx}\ 4f(x))\Big|_{x=5} = 4f'(5) = \dfrac{4}{10} = \dfrac{2}{5}$.

3.3.18 $(f + g)(3) = f(3) + g(3) = 10 - 14 = -4$ and $(f + g)'(3) = f'(3) + g'(3) = 4 - 5 = -1$.

3.3.19 By the power rule, $y' = 5x^{5-1} = 5x^4$.

3.3.20 By the power rule $f'(t) = 1t^{1-1} = t^0 = 1$.

3.3.21 By the constant rule, $f'(x) = 0$.

3.3.22 By the constant rule, $g'(x) = 0$. (Note that e^3 is a constant, its value does not depend on x.)

3.3.23 By the constant multiple rule and the power rule, $f'(x) = 5 \cdot \frac{d}{dx}x^3 = 5 \cdot 3x^2 = 15x^2$.

3.3.24 By the constant multiple and power rules, $g'(w) = \frac{5}{6} \cdot \frac{d}{dw}w^{12} = \frac{5}{6} \cdot 12w^{11} = 10w^{11}$.

3.3.25 By the power rule, the sum rule, the constant multiplier rule, and the constant rule, $h'(x) = \frac{2t}{2} + 0 = t$.

3.3.26 By the power rule, the constant rule, and the sum rule, $f'(v) = 100v^{99} + e^v$.

3.3.27 By the constant multiple and power rules, $p'(x) = 8 \cdot \frac{d}{dx}x = 8 \cdot 1 = 8$.

3.3.28 By the constant multiple rule and the power rule, $g'(t) = 6 \cdot \frac{d}{dt}\sqrt{t} = 6 \cdot \frac{1}{2\sqrt{t}} = \frac{3}{\sqrt{t}}$.

3.3.29 By the constant multiple and power rules, $g'(t) = 100\frac{d}{dt}t^2 = 100 \cdot 2t = 200t$.

3.3.30 By the constant multiple rule and the power rule, $f'(s) = \frac{1}{4} \cdot \frac{d}{ds}\sqrt{s} = \frac{1}{4} \cdot \frac{1}{2\sqrt{s}} = \frac{1}{8\sqrt{s}}$.

3.3.31 $f'(x) = \frac{d}{dx}(3x^4 + 7x) = \frac{d}{dx}(3x^4) + \frac{d}{dx}(7x) = 12x^3 + 7$.

3.3.32 $g'(x) = \frac{d}{dx}(6x^5 - \frac{5}{2}x^2 + x + 5) = \frac{d}{dx}(6x^5) - \frac{d}{dx}\left(\frac{5}{2}x^2\right) + \frac{d}{dx}(x) + \frac{d}{dx}(5) = 30x^4 - 5x + 1$.

3.3.33 $f'(x) = \frac{d}{dx}(10x^4 - 32x + e^2) = \frac{d}{dx}(10x^4) - \frac{d}{dx}(32x) + \frac{d}{dx}(e^2) = 40x^3 - 32 - 0 = 40x^3 - 32$.

3.3.34 $f'(t) = \frac{d}{dt}(6\sqrt{t} - 4t^3 + 9) = \frac{d}{dt}(6\sqrt{t}) - \frac{d}{dt}(4t^3) + \frac{d}{dt}(9) = \frac{6}{2\sqrt{t}} - 12t^2 + 0 = \frac{3}{\sqrt{t}} - 12t^2$.

3.3.35 $g'(w) = \frac{d}{dw}(2w^3 + 3w^2 + 10w) = 2\frac{d}{dw}(w^3) + 3\frac{d}{dw}(w^2) + 10\frac{d}{dw}(w) = 2(3w^2) + 3(2w) + 10(1) = 6w^2 + 6w + 10$.

3.3.36 $s'(t) = \frac{d}{dt}(4\sqrt{t} - \frac{1}{4}t^4 + t + 1) = \frac{d}{dt}(4\sqrt{t}) - \frac{d}{dt}\left(\frac{1}{4}t^4\right) + \frac{d}{dt}(t) + \frac{d}{dt}(1) = \frac{4}{2\sqrt{t}} - t^3 + 1 + 0 = \frac{2}{\sqrt{t}} - t^3 + 1$.

3.3.37 $f'(x) = \frac{d}{dx}(3e^x + 5x + 5) = 3\frac{d}{dx}(e^x) + 5\frac{d}{dx}(x) + \frac{d}{dx}(5) = 3e^x + 5$.

3.3.38 $g'(w) = \frac{d}{dw}(e^w - e^2 + 8) = \frac{d}{dw}(e^w) - \frac{d}{dw}(e^2) + \frac{d}{dw}(8) = e^w - 0 + 0 = e^w$.

3.3.39 $f'(x) = \begin{cases} 2x & \text{if } x < 0 \\ 4x + 1 & \text{if } x > 0. \end{cases}$

3.3.40 $g'(w) = \begin{cases} 1 + 5e^w & \text{if } w < 1 \\ 6w^2 + 4 & \text{if } w > 1. \end{cases}$

3.3.41

a. $d'(t) = 32t$ is the velocity of the stone after t seconds, measured in feet per second.

b. The stone travels $d(6) = 16 \cdot 6^2 = 576$ feet and strikes the ground with a velocity of $32 \cdot 6 = 192$ feet per second. Converting to miles per hour, we have $192 \cdot \frac{3600}{5280} \approx 130.9$ miles per hour.

3.3.42

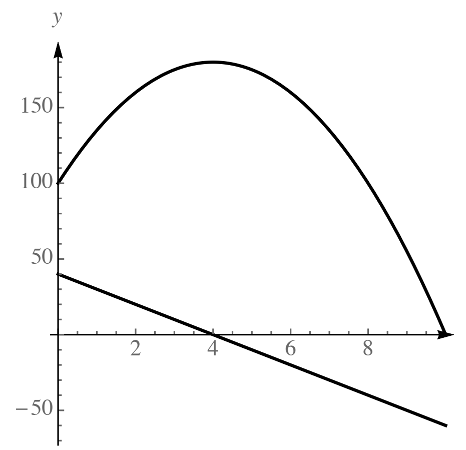
a. The instantaneous velocity is given by $v(t) = \frac{d}{dt}s(t) = -10t + 40$, $0 \le t \le 10$.

b. $v(t) = 0$ when $-10t + 40 = 0$, which occurs at $t = 4$.

c. The magnitude of the velocity is $|v(t)|$. Note that $v(t) \ge 0$ for $0 \le t \le 4$, and $v(t) < 0$ for $4 < t \le 10$. Thus

$$|v(t)| = \begin{cases} -10t + 40 & \text{for } 0 \le t \le 4, \\ 10t - 40 & \text{for } 4 < t \le 10. \end{cases}$$

Note that $|v(0)| = 40$ and $|v(10)| = 60$. The greatest magnitude over this time interval is 60 meters per second.

d. 

3.3.43

a. $A'(t) = -\frac{1}{25}t + 2$ square miles per year.

b. First we must find when $A(t) = 38$. This occurs when $-\frac{t^2}{50} + 2t + 28 = 38$, which can be written as $t^2 - 100t + 900 = 0$. This factors as $(t - 90)(t - 10) = 0$, so the only solution on the given domain is $t = 10$. At this time, we have $A'(10) = -0.4 + 2 = 1.6$ square miles per year.

c. Note that $A'(20) = -0.8 + 2 = 1.2$ square miles per year. In order to maintain a density of 1000 people per square mile, we must multiply the density of people per square mile times the number of square miles per year in order to obtain the rate of people per year required to maintain that density. Thus the population growth rate must be $1000 \cdot 1.2 = 1200$ people per year.

3.3.44

a. Because $p(t) = 1200e^t$, we get $p'(t) = 1200 \cdot e^t$ cells per hour.

b. Because $p'(t)$ is also exponential, it is smallest when $t = 0$, and largest when $t = 4$.

3.3.45 $w'(x) = \begin{cases} 0.4 & \text{if } 19 < x < 21 \\ 0.8 & \text{if } 21 < x < 32 \\ 1.5 & \text{if } x > 32. \end{cases}$ The derivative measures the rate of change (in lb/inch) of Atlantic salmon. Longer salmon put on more weight per inch than shorter salmon.

3.3.46 Expanding the product gives $f(x) = x - 1$, so $f'(x) = 1$.

3.3.47 Expanding the product yields $f(x) = 6x^3 + 3x^2 + 4x + 2$. So

$$\begin{aligned} f'(x) &= \frac{d}{dx}(6x^3 + 3x^2 + 4x + 2) = \frac{d}{dx}(6x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(2) \\ &= 18x^2 + 6x + 4. \end{aligned}$$

3.3.48 Expanding the product yields $g(r) = 5r^5 + 18r^3 + r^2 + 9r + 3$. So

$$g'(r) = \frac{d}{dr}(5r^5 + 18r^3 + r^2 + 9r + 3) = \frac{d}{dr}(5r^5) + \frac{d}{dr}(18r^3) + \frac{d}{dr}(r^2) + \frac{d}{dr}(9r) + \frac{d}{dr}(3)$$
$$= 25r^4 + 54r^2 + 2r + 9.$$

3.3.49 f simplifies as $f(w) = w^2 - 1$, so $f'(w) = 2w$ for $w \neq 0$.

3.3.50 y simplifies as $y = \dfrac{(4s)(3s^2 - 2s + 3)}{4s} = 3s^2 - 2s + 3$. Thus $y' = 6s - 2$, for $s \neq 0$.

3.3.51 Expanding the product yields $h(x) = x^4 + 2x^2 + 1$. So

$$h'(x) = \frac{d}{dx}(x^4 + 2x^2 + 1) = \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(1)$$
$$= 4x^3 + 4x.$$

3.3.52 Expanding the product gives $h(x) = x - x^2$, so $h'(x) = 1 - 2x$.

3.3.53 g simplifies as $g(x) = \dfrac{(x-1)(x+1)}{x-1} = x + 1$. Thus $g'(x) = 1$ for $x \neq 1$.

3.3.54 h simplifies as $h(x) = \dfrac{(x)(x-2)(x-4)}{(x)(x-2)} = x - 4$. Thus, $h'(x) = 1$ for $x \neq 0, 2$.

3.3.55 y simplifies as $y = \dfrac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} - \sqrt{a}} = \sqrt{x} + \sqrt{a}$. Thus $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$ for $x \neq a$.

3.3.56 y simplifies as $y = \dfrac{(x-a)^2}{x-a} = x - a$. Thus $\dfrac{dy}{dx} = 1$ for $x \neq a$.

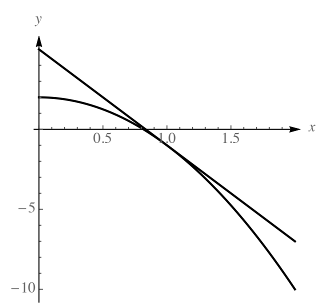
3.3.57 g simplifies as $g(w) = \dfrac{e^{2w}}{e^w} + \dfrac{e^w}{e^w} = e^w + 1$. So $g'(w) = e^w$.

3.3.58 r simplifies as $r(t) = \dfrac{(e^t + 2)(e^t + 1)}{e^t + 2} = e^t + 1$, so $r'(t) = e^t$.

3.3.59

a. $y' = -6x$, so the slope of the tangent line at $a = 1$ is -6. Thus, the tangent line at the point $(1, -1)$ is $y + 1 = -6(x - 1)$, or $y = -6x + 5$.

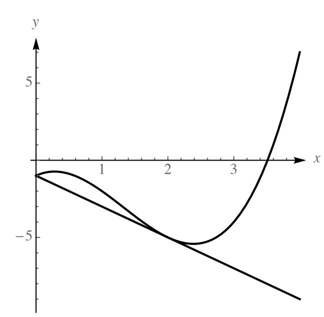
b.



3.3.60

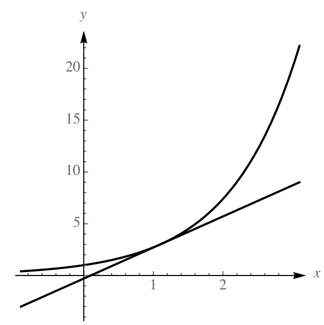
a. $y' = 3x^2 - 8x + 2$, so the slope of the tangent line at $a = 2$ is -2. Thus, the tangent line at the point $(2, -5)$ is $y + 5 = -2(x - 2)$, or $y = -2x - 1$.

b.



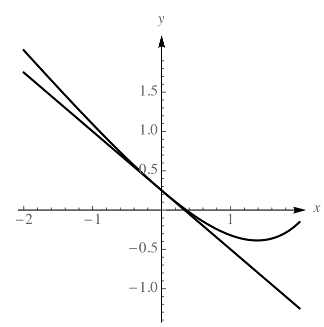
3.3.61

a. $y' = e^x$, so the slope of the tangent line at $a = \ln 3$ is 3. Thus, the tangent line at the point $(\ln 3, 3)$ is $y - 3 = 3(x - \ln 3)$, or $y = 3x + 3 - 3\ln 3$.

b. 

3.3.62

a. $y' = \frac{e^x}{4} - 1$, so the slope of the tangent line at $a = 0$ is $-\frac{3}{4}$. Thus, the tangent line at the point $(0, 1/4)$ is $y - 1/4 = -\frac{3}{4} \cdot (x - 0)$, or $y = -\frac{3x}{4} + \frac{1}{4}$.

b. 

3.3.63

a. $f'(x) = 2x - 6$, so the slope is zero when $2x - 6 = 0$, which is at $x = 3$.

b. The slope is 2 when $2x - 6 = 2$ which is at $x = 4$.

3.3.64

a. $f'(t) = 3t^2 - 27$, so the slope of the tangent line is zero when $3t^2 - 27 = 0$, which is at $t = 3$ and $t = -3$.

b. The slope is 21 where $f'(t) = 3t^2 - 27 = 21$, or when $t^2 = 16$, so at $t = 4$ and $t = -4$.

3.3.65

a. The slope of the tangent line is given by $f'(x) = 6x^2 - 6x - 12$, and this quantity is zero when $x^2 - x - 2 = 0$, or $(x - 2)(x + 1) = 0$. The two solutions are thus $x = -1$ and $x = 2$, so the points on the graph are $(-1, 11)$ and $(2, -16)$.

b. The slope of the tangent line is 60 when $6x^2 - 6x - 12 = 60$, which is when $6x^2 - 6x - 72 = 0$. Simplifying this quadratic expression yields the equation $x^2 - x - 12 = 0$, which has solutions $x = -3$ and $x = 4$, so the points on the graph are $(-3, -41)$, and $(4, 36)$.

3.3.66

a. The slope of the tangent line is given by $f'(x) = 2e^x - 6$. This is equal to zero when $2e^x = 6$, which occurs for $x = \ln 3$. The point on the graph is therefore $(\ln 3, 6 - 6\ln 3)$.

b. The slope of the tangent line is 12 when $2e^x - 6 = 12$, or $e^x = 9$. This occurs for $x = \ln 9$. The point on the graph is therefore $(\ln 9, 18 - 6\ln 9)$.

3.3.67

a. The slope of the tangent line is given by $\dfrac{2}{\sqrt{x}} - 1$. This is equal to zero when $\sqrt{x} = 2$, or $x = 4$. The point on the graph is $(4, 4)$.

b. The slope of the tangent line is $-\frac{1}{2}$ when $\dfrac{2}{\sqrt{x}} - 1 = -\dfrac{1}{2}$. Solving for x gives $x = 16$. The point on the graph is $(16, 0)$.

3.3.68 $f'(x) = 9x^2 + 10x + 6$, $f''(x) = 18x + 10$, and $f^{(3)}(x) = 18$.

3.3.69 $f'(x) = 20x^3 + 30x^2 + 3$, $f''(x) = 60x^2 + 60x$, and $f^{(3)}(x) = 120x + 60$.

3.3.70 $f'(x) = 6x + 5e^x$, $f''(x) = 6 + 5e^x$, and $f^{(3)}(x) = 5e^x$.

3.3.71 f simplifies as $f(x) = \frac{(x-8)(x+1)}{x+1} = x - 8$. So for $x \neq -1$, $f'(x) = 1$, $f''(x) = 0$, and $f^{(3)}(x) = 0$.

3.3.72 $f'(x) = f''(x) = f^{(3)}(x) = 10e^x$.

3.3.73

a. False. 10^5 is a constant, so the constant rule assures us that $\frac{d}{dx}(10^5) = 0$.

b. True. This follows because the slope is given by $f'(x) = e^x > 0$ for all x.

c. False. $\frac{d}{dx}(e^3) = 0$.

d. False. $\frac{d}{dx}(e^x) = e^x$, not xe^{x-1}.

e. False. We have $\frac{d}{dx}(5x^3 + 2x + 5) = 15x^2 + 2$. Thus we have $\frac{d^2}{dx^2}(5x^3 + 2x + 5) = 30x$, and $\frac{d^3}{dx^3}(5x^3 + 2x + 5) = 30$. It is true that $\frac{d^n}{dx^n}(5x^3 + 2x + 5) = 0$ for $n \geq 4$.

3.3.74

a. The slope of the tangent line to g at x is given by $g'(x) = 2x + f'(x)$, so $g'(3) = 6 + f'(3) = 10$. The point on the curve $y = g(x)$ at $x = 3$ is $(3, 9 + f(3)) = (3, 9 + 1) = (3, 10)$. Thus the equation of the tangent line at this point is $y - 10 = 10(x - 3)$, or $y = 10x - 20$.

b. The slope of the tangent line to h at x is given by $h'(x) = 3f'(x)$, so $h'(3) = 3f'(3) = 3 \cdot 4 = 12$. The point on the curve $y = h(x)$ at $x = 3$ is $(3, 3 \cdot f(3)) = (3, 3)$. Thus the equation of the tangent line at this point is $y - 3 = 12(x - 3)$, or $y = 12x - 33$.

3.3.75 First note that because the slope of $4x + 1$ is 4, it must be the case that $f'(2) = 4$. Also, at $x = 2$, we have $y = 4 \cdot 2 + 1 = 9$, so $f(2) = 9$. Because the line tangent to the graph of g at 2 has slope 3, we know that $g'(2) = 3$. The tangent line to g at $x = 2$ must be $y - (-2) = 3(x - 0)$, so the value of the tangent line at 2 (which must also be the value of $g(2)$) is 4. So $g(2) = 4$.

a. $y'(2) = f'(2) + g'(2) = 4 + 3 = 7$. The line contains the point $(2, f(2) + g(2)) = (2, 13)$. Thus, the equation of the tangent line is $y - 13 = 7(x - 2)$, or $y = 7x - 1$.

b. $y'(2) = f'(2) - 2g'(2) = 4 - 2 \cdot 3 = -2$. The line contains the point $(2, f(2) - 2g(2)) = (2, 1)$. Thus, the equation of the tangent line is $y - 1 = -2(x - 2)$, or $y = -2x + 5$.

c. $y'(2) = 4f'(2) = 4 \cdot 4 = 16$. The line contains the point $(2, 4f(2)) = (2, 36)$. Thus, the equation of the tangent line is $y - 36 = 16(x - 2)$, or $y = 16x + 4$.

3.3.76 For $y = x + \sqrt{x}$ we have $y' = 1 + \frac{1}{2}\left(x^{-1/2}\right) = 1 + \frac{1}{2\sqrt{x}}$. Setting this equal to 2 yields $1 + \frac{1}{2\sqrt{x}} = 2$, or $\sqrt{x} = \frac{1}{2}$. Thus the tangent line has slope 2 for $x = 1/4$.

3.3.77 For $f(x) = x^2 + bx + c$ we have $f'(x) = 2x + b$, so $f'(1) = 2 + b$. Because the slope of $4x + 2$ is 4, we require $2+b = 4$, so $b = 2$. Also, because the value of $4x+2$ at $x = 1$ is 6, we must have $f(1) = 1+2+c = 6$, so $c = 3$. Thus the curve $f(x) = x^2 + 2x + 3$ has $y = 4x + 2$ as its tangent line at $x = 1$.

3.3.78 $F'(2) = f'(2) + g'(2) = -3 + 1 = -2.$

3.3.79 $G'(2) = 3f'(2) - g'(2) = 3(-3) - 1 = -10.$

3.3.80 $F'(5) = f'(5) + g'(5) = 1 - 1 = 0.$

3.3.81 $G'(5) = 3f'(5) - g'(5) = 3 \cdot 1 - (-1) = 4.$

3.3.82

a. Let $f(x) = e^x$ and $a = 0$. Then $\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{e^x - 1}{x} = f'(0)$.

b. $f'(x) = e^x$, so $f'(0) = e^0 = 1$, so $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

3.3.83

a. Let $f(x) = x + e^x$ and $a = 0$. Then $\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{x + e^x - 1}{x} = f'(0)$.

b. $f'(x) = 1 + e^x$, so $f'(0) = 1 + e^0 = 2$, so $\lim_{x\to 0} \frac{x + e^x - 1}{x} = 2$.

3.3.84

a. Let $f(x) = x^{100}$ and $a = 1$. Then $\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^{100} - 1}{x - 1} = f'(1)$.

b. Because $f'(x) = 100x^{99}$, we have $f'(1) = 100$, so this is the value of the original limit.

3.3.85

a. Let $f(x) = \sqrt{x}$ and $a = 9$. Then $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h} = f'(9)$.

b. Because $f'(x) = \frac{1}{2\sqrt{x}}$, we have $f'(9) = \frac{1}{6}$, so this is the value of the original limit.

3.3.86

a. Let $f(x) = x^8 + x^3$, and $a = 1$. Then $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(1+h)^8 + (1+h)^3 - 2}{h} = f'(1)$.

b. Because $f'(x) = 8x^7 + 3x^2$, we have $f'(1) = 8 + 3 = 11$, so this is the value of the original limit.

3.3.87

a. Let $f(x) = e^x$, and $a = 3$. Then $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{e^{3+h} - e^3}{h} = f'(3)$.

b. Because $f'(x) = e^x$, we have $f'(3) = e^3$, so this is the value of the original limit.

3.3.88

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$
-1.0	0.5	0.666667
-0.1	.66967	1.040415
-0.01	.69075	1.0926
-0.001	.692907	1.098009
-0.0001	.693123	1.098552
$-.00001$	.693145	1.098606

It appears that $\lim_{h\to 0^-} \frac{2^h - 1}{h} \approx .6931$ and $\lim_{h\to 0^-} \frac{3^h - 1}{h} \approx 1.0986$.

3.3.89 $\lim_{x\to 0}\dfrac{e^{3x}-1}{x}=3.$

3.3.90 $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n\approx 2.7183.$

3.3.91 $\lim_{x\to 0^+} x^x=1.$

3.3.92 $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{c-c}{h}=\lim_{h\to 0}\dfrac{0}{h}=\lim_{h\to 0}0=0.$

3.3.93

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0}\left(\frac{x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n-x^n}{h}\right)\\ &= \lim_{h\to 0}\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right)=nx^{n-1}+0+0+\cdots+0=nx^{n-1}\end{aligned}$$

3.3.94

a. Let $m=-n$ so that $m>0$.

$$\begin{aligned}f'(a) &= \lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}=\lim_{x\to a}\frac{1}{a^mx^m}\cdot\frac{a^m-x^m}{x-a}\\ &= \left(-\lim_{x\to a}\frac{1}{a^mx^m}\right)\left(\lim_{x\to a}\frac{x^m-a^m}{x-a}\right)\\ &= \left(-\lim_{x\to a}\frac{1}{a^mx^m}\right)\left(\lim_{x\to a}\frac{(x-a)(x^{m-1}+ax^{m-2}+a^2x^{m-3}+\cdots+a^{m-1})}{x-a}\right)\\ &= \left(-\lim_{x\to a}\frac{1}{a^mx^m}\right)\left(\lim_{x\to a}\left(x^{m-1}+ax^{m-2}+a^2x^{m-3}+\cdots+a^{m-1}\right)\right)\\ &= -\frac{1}{a^{2m}}\cdot ma^{m-1}=-ma^{-m-1}=na^{n-1}.\end{aligned}$$

b. $\dfrac{d}{dx}(x^{-7})=-7x^{-8}$. Also, $\dfrac{d}{dx}(x^{-10})=-10x^{-11}=-\dfrac{10}{x^{11}}$.

3.3.95

a. $\dfrac{d}{dx}(\sqrt{x})=\dfrac{d}{dx}x^{1/2}=\dfrac{1}{2}\cdot x^{-1/2}=\dfrac{1}{2\sqrt{x}}.$

b.

$$\begin{aligned}\frac{d}{dx}x^{3/2} &= \lim_{h\to 0}\frac{(x+h)^{3/2}-x^{3/2}}{h}=\lim_{h\to 0}\frac{((x+h)^{3/2}-x^{3/2})((x+h)^{3/2}+x^{3/2})}{h((x+h)^{3/2}+x^{3/2})}\\ &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h((x+h)^{3/2}+x^{3/2})}=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h((x+h)^{3/2}+x^{3/2})}\\ &= \lim_{h\to 0}\frac{3x^2+3xh+h^2}{((x+h)^{3/2}+x^{3/2})}=\frac{3x^2+0+0}{x^{3/2}+x^{3/2}}=\frac{3x^2}{2x^{3/2}}=\frac{3}{2}x^{1/2}.\end{aligned}$$

c.

$$\begin{aligned}\frac{d}{dx}x^{5/2} &= \lim_{h\to 0}\frac{(x+h)^{5/2}-x^{5/2}}{h}=\lim_{h\to 0}\frac{((x+h)^{5/2}-x^{5/2})((x+h)^{5/2}+x^{5/2})}{h((x+h)^{5/2}+x^{5/2})}\\ &= \lim_{h\to 0}\frac{(x+h)^5-x^5}{h((x+h)^{5/2}+x^{5/2})}=\lim_{h\to 0}\frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h((x+h)^{5/2}+x^{5/2})}\\ &= \lim_{h\to 0}\frac{5x^4+10x^3h+10x^2h^2+5xh^3+h^4}{((x+h)^{5/2}+x^{5/2})}\\ &= \frac{5x^4+0+0+0+0}{x^{5/2}+x^{5/2}}=\frac{5x^4}{2x^{5/2}}=\frac{5}{2}x^{3/2}.\end{aligned}$$

d. It appears that $\dfrac{d}{dx}x^{n/2} = \dfrac{n}{2} \cdot x^{(n/2)-1}$.

3.3.96

a. $\dfrac{d}{dx}(e^{-x}) = \lim\limits_{h\to 0} \dfrac{e^{-(x+h)} - e^{-x}}{h} = \lim\limits_{h\to 0} \dfrac{e^{-x}(e^{-h}-1)}{h} = e^{-x} \lim\limits_{h\to 0} \dfrac{e^{-h}-1}{h}$.

b. Let $w = -h$. Then $\lim\limits_{h\to 0} \dfrac{e^{-h}-1}{h} = \lim\limits_{w\to 0} \dfrac{e^{w}-1}{-w} = -\lim\limits_{w\to 0} \dfrac{e^{w}-1}{w} = -1$.

c. $\dfrac{d}{dx}(e^{-x}) = e^{-x} \lim\limits_{h\to 0} \dfrac{e^{-h}-1}{h} = -e^{-x}$.

3.3.97

a. $\dfrac{d}{dx}(e^{2x}) = \lim\limits_{h\to 0} \dfrac{e^{2(x+h)} - e^{2x}}{h} = \lim\limits_{h\to 0} \dfrac{e^{2x}(e^{2h}-1)}{h} = e^{2x} \lim\limits_{h\to 0} \dfrac{e^{2h}-1}{h}$.

b. Let $z = 2h$. Then $\lim\limits_{h\to 0} \dfrac{e^{2h}-1}{h} = \lim\limits_{z\to 0} \dfrac{e^{z}-1}{z/2} = 2\lim\limits_{z\to 0} \dfrac{e^{z}-1}{z} = 2$.

c. $\dfrac{d}{dx}(e^{2x}) = e^{2x} \lim\limits_{h\to 0} \dfrac{e^{2h}-1}{h} = 2e^{2x}$.

3.3.98

a.

$$\begin{aligned}
\frac{d}{dx}(x^2e^x) &= \lim_{h\to 0} \frac{(x+h)^2e^{x+h} - x^2e^x}{h} \\
&= \lim_{h\to 0} e^x \cdot \frac{(x^2+2xh+h^2)e^h - x^2}{h} = e^x \cdot \lim_{h\to 0} \frac{(x^2+2xh+h^2)e^h - x^2}{h}.
\end{aligned}$$

b.

$$\begin{aligned}
&e^x \cdot \lim_{h\to 0} \frac{(x^2+2xh+h^2)e^h - x^2}{h} = e^x \left(\lim_{h\to 0} \frac{x^2e^h + 2xhe^h + h^2e^h - x^2}{h} \right) \\
&= e^x \left(\lim_{h\to 0} \frac{x^2(e^h-1) + 2xhe^h + h^2e^h}{h} \right) = e^x \left(x^2 \lim_{h\to 0} \frac{e^h-1}{h} + 2x \lim_{h\to 0} e^h + \lim_{h\to 0} he^h \right) \\
&= e^x \left(x^2 + 2x \right).
\end{aligned}$$

3.4 The Product and Quotient Rules

3.4.1 The derivative of the product fg with respect to x is given by $f'(x)g(x) + f(x)g'(x)$.

3.4.2 The derivative of the quotient $\dfrac{f}{g}$ with respect to x is given by $\dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

3.4.3 $\dfrac{d}{dx}((x+1)(3x+2)) = \left(\dfrac{d}{dx}(x+1)\right)(3x+2) + (x+1)\dfrac{d}{dx}(3x+2) = 1(3x+2) + (x+1)3 = 6x+5$.

3.4.4 $f'(x) = 4x^3e^x + x^4e^x = e^x(4x^3 + x^4) = x^3e^x(x+4)$. Thus $f'(1) = 5e$.

3.4.5 $\dfrac{d}{dx}\left(\dfrac{x-1}{3x+2}\right) = \dfrac{(3x+2)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(3x+2)}{(3x+2)^2} = \dfrac{3x+2-3(x-1)}{(3x+2)^2} = \dfrac{5}{(3x+2)^2}$.

3.4.6 $g'(x) = \dfrac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \dfrac{x(x+2)}{(x+1)^2}$, so $g'(1) = \dfrac{3}{4}$.

3.4.7

a. Using the product rule: $f'(x) = 1(x-1) + x(1) = 2x - 1$.

b. By expanding first: $f(x) = x^2 - x$ so $f'(x) = 2x - 1$.

3.4.8

a. By the product rule: $g'(t) = 1(t^2 - t + 1) + (t+1)(2t-1) = t^2 - t + 1 + 2t^2 - t + 2t - 1 = 3t^2$.

b. By expanding first: $g(t) = t^3 - t^2 + t + t^2 - t + 1 = t^3 + 1$, so $g'(t) = 3t^2$.

3.4.9

a. By the product rule: $y' = (2t+7)(3t-4) + (t^2 + 7t) \cdot 3 = 9t^2 + 34t - 28$.

b. By expanding first: $y' = \frac{d}{dt}(3t^3 + 17t^2 - 28t) = 9t^2 + 34t - 28$.

3.4.10

a. By the product rule: $h'(z) = (3z^2 + 8z + 1)(z-1) + (z^3 + 4z^2 + z)(1) = 3z^3 + 8z^2 + z - 3z^2 - 8z - 1 + z^3 + 4z^2 + z = 4z^3 + 9z^2 - 6z - 1$.

b. By expanding first: $h'(z) = \dfrac{d}{dz}(z^4 + 4z^3 + z^2 - z^3 - 4z^2 - z) = \dfrac{d}{dz}(z^4 + 3z^3 - 3z^2 - z) = 4z^3 + 9z^2 - 6z - 1$.

3.4.11

a. By the quotient rule:

$$f'(w) = \frac{w(3w^2 - 1) - (w^3 - w) \cdot 1}{w^2} = \frac{2w^3}{w^2} = 2w \text{ for } w \neq 0.$$

b. For $w \neq 0$ this simplifies as $w^2 - 1$. $f'(w) = \dfrac{d}{dw}(w^2 - 1) = 2w$.

3.4.12

a. By the quotient rule:

$$\begin{aligned} g'(s) &= \frac{4s(12s^2 - 16s + 4) - (4s^3 - 8s^2 + 4s) \cdot 4}{(4s)^2} = \frac{48s^3 - 64s^2 + 16s - 16s^3 + 32s^2 - 16s}{16s^2} \\ &= \frac{32s^3 - 32s^2}{16s^2} = 2s - 2 \end{aligned}$$

for $s \neq 0$.

b. For $s \neq 0$, the function simplifies to $g(s) = s^2 - 2s + 1$. Then $g'(s) = \dfrac{d}{ds}(s^2 - 2s + 1) = 2s - 2$.

3.4.13

a. By the quotient rule: $y' = \dfrac{(x-a)(2x) - (x^2 - a^2)}{(x-a)^2} = \dfrac{2x^2 - 2ax - x^2 + a^2}{(x-a)^2} = \dfrac{x^2 - 2ax + a^2}{(x-a)^2} = \dfrac{(x-a)^2}{(x-a)^2} = 1$ for $x \neq a$.

b. For $x \neq a$, this simplifies as $y = \dfrac{(x+a)(x-a)}{x-a} = x + a$. $y' = \dfrac{d}{dx}(x+a) = 1$.

3.4.14

a. By the quotient rule: $y' = \dfrac{(x-a)(2x-2a) - (x^2 - 2ax + a^2) \cdot 1}{(x-a)^2} = \dfrac{x^2 - 2ax + a^2}{(x-a)^2} = \dfrac{(x-a)^2}{(x-a)^2} = 1$ for $x \neq a$.

b. For $x \neq a$, this simplifies as $y = \dfrac{(x-a)^2}{(x-a)} = x - a$. $y' = \dfrac{d}{dx}(x-a) = 1$.

3.4.15 $\dfrac{d}{dx}(f(x)g(x))\bigg|_{x=1} = f'(1)g(1) + f(1)g'(1) = 4 \cdot 2 + 5 \cdot 3 = 23$.
$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\bigg|_{x=1} = \dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{2 \cdot 4 - 5 \cdot 3}{2^2} = -\dfrac{7}{4}$.

3.4.16 By the extended power rule, $\dfrac{d}{dx}\dfrac{1}{x^{10}} = \dfrac{d}{dx}x^{-10} = -10x^{-11} = -\dfrac{10}{x^{11}}$.
By the quotient rule,

$$\frac{d}{dx}\frac{1}{x^{10}} = \frac{x^{10} \cdot 0 - 1 \cdot 10x^9}{(x^{10})^2} = -\frac{10x^9}{x^{20}} = -\frac{10}{x^{11}}.$$

3.4.17 By the quotient rule, $f'(x) = \dfrac{(x+6) \cdot 1 - x \cdot 1}{(x+6)^2} = \dfrac{6}{(x+6)^2}$, so $f'(3) = \dfrac{6}{81} = \dfrac{2}{27}$ and $f'(-2) = \dfrac{3}{8}$.

3.4.18 We have $f'(x) = 0 + 1 \cdot e^x + xe^x = e^x(1+x)$, so $f'(0) = 1$.

3.4.19 $f'(x) = 12x^3(2x^2-1) + 3x^4 \cdot 4x = 24x^5 - 12x^3 + 12x^5 = 36x^5 - 12x^3$.

3.4.20 $g'(x) = 6 - (2e^x + 2xe^x)$.

3.4.21 $f'(x) = \dfrac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2} = \dfrac{1}{(x+1)^2}$.

3.4.22 $f'(x) = \dfrac{(x-2)(3x^2-8x+1) - (x^3-4x^2+x)(1)}{(x-2)^2} = \dfrac{2x^3 - 10x^2 + 16x - 2}{(x-2)^2}$.

3.4.23 $f'(t) = \dfrac{5}{3}t^{2/3}e^t + t^{5/3}e^t = e^t t^{2/3}(\dfrac{5}{3} + t)$.

3.4.24 $g'(w) = (10w+3)e^w + (5w^2+3w+1)e^w = e^w(5w^2+13w+4)$.

3.4.25 $f'(x) = \dfrac{(e^x+1)e^x - e^x(e^x)}{(e^x+1)^2} = \dfrac{e^x}{(e^x+1)^2}$.

3.4.26 $f'(x) = \dfrac{(2e^x+1)(2e^x) - (2e^x-1)2e^x}{(2e^x+1)^2} = \dfrac{2e^x(2e^x+1-(2e^x-1))}{(2e^x+1)^2} = \dfrac{4e^x}{(2e^x+1)^2}$.

3.4.27 $f'(x) = (1)e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$.

3.4.28 $f'(x) = e^x \cdot \sqrt[3]{x} + e^x \cdot \dfrac{1}{3}x^{-2/3} = e^x\left(\dfrac{3x+1}{3x^{2/3}}\right)$.

3.4.29 $y' = \dfrac{d}{dt}\left(\dfrac{3t-1}{2t-2}\right) = \dfrac{(2t-2) \cdot 3 - (3t-1) \cdot 2}{(2t-2)^2} = -\dfrac{4}{(2t-2)^2} = -\dfrac{1}{(t-1)^2}$.

3.4.30 $h'(w) = \dfrac{(w^2+1)(2w) - (w^2-1)(2w)}{(w^2+1)^2} = \dfrac{2w^3 + 2w - 2w^3 + 2w}{(w^2+1)^2} = \dfrac{4w}{(w^2+1)^2}$.

3.4.31 $h'(x) = (1)(x^3+x^2+x+1) + (x-1)(3x^2+2x+1) = x^3+x^2+x+1+3x^3+2x^2+x-3x^2-2x-1 = 4x^3$.

3.4.32 $f'(x) = -\dfrac{2}{x^3} \cdot (x^2+1) + \left(1 + \dfrac{1}{x^2}\right)(2x) = -\dfrac{2}{x} - \dfrac{2}{x^3} + 2x + \dfrac{2}{x} = 2x - \dfrac{2}{x^3}$.

3.4.33 $g'(w) = e^w(w^3-1) + e^w \cdot 3w^2 = e^w(w^3+3w^2-1)$.

3.4.34 $s'(t) = \dfrac{e^t \cdot \frac{4}{3}t^{1/3} - t^{4/3}e^t}{e^{2t}} = \dfrac{t^{1/3}e^t(\frac{4}{3} - t)}{e^{2t}} = t^{1/3}\left(\dfrac{4-3t}{3e^t}\right).$

3.4.35 $f'(t) = e^t(t^2 - 2t + 2) + e^t(2t - 2) = e^t(t^2 - 2t + 2 + 2t - 2) = t^2e^t.$

3.4.36 $f'(x) = e^x(x^3 - 3x^2 + 6x - 6) + e^x(3x^2 - 6x + 6) = e^x(x^3 - 3x^2 + 6x - 6 + 3x^2 - 6x + 6) = x^3e^x.$

3.4.37 $g'(x) = \dfrac{(x^2-1)\cdot e^x - e^x \cdot 2x}{(x^2-1)^2} = \dfrac{e^x(x^2 - 2x - 1)}{(x^2-1)^2}.$

3.4.38

$$\begin{aligned} y' = \frac{d}{dx}\left(\frac{2\sqrt{x}-1}{4x+1}\right) &= \frac{(4x+1)\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x}-1)4}{(4x+1)^2} \\ &= \frac{4\sqrt{x} + \frac{1}{\sqrt{x}} - 8\sqrt{x} + 4}{(4x+1)^2} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{-4x + 1 + 4\sqrt{x}}{\sqrt{x}(4x+1)^2}. \end{aligned}$$

3.4.39 $f'(x) = (-9)\cdot 3 \cdot x^{-9-1} = -27x^{-10}.$

3.4.40 $y' = \dfrac{d}{dp}(4p^{-3}) = -12p^{-4}.$

3.4.41 $g'(t) = \dfrac{d}{dt}(3t^2 + 6t^{-7}) = 6t - 42t^{-8}.$

3.4.42 $y' = \dfrac{d}{dw}(w^2 + 5 + w^{-1}) = 2w - w^{-2}.$

3.4.43 $g'(t) = \dfrac{d}{dt}(1 + 3t^{-1} + t^{-2}) = -3t^{-2} - 2t^{-3}.$

3.4.44 $p'(x) = \dfrac{d}{dx}(2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}) = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}.$

3.4.45 $g'(x) = \dfrac{(x-2)((x+1)e^x + e^x) - (x+1)e^x}{(x-2)^2} = \dfrac{e^x}{(x-2)^2} \cdot \dfrac{(x-2)(x+2) - (x+1)}{1} = \dfrac{e^x}{(x-2)^2} \cdot \big(x^2 - x - 5\big).$

3.4.46 First note that $x^3 - 1 = (x-1)(x^2+x+1)$. So we can simplify $h(x)$ as $h(x) = \dfrac{2x^2-1}{x^2+x+1}$. Thus

$$h'(x) = \frac{(x^2+x+1)(4x) - (2x^2-1)(2x+1)}{(x^2+x+1)^2} = \frac{(4x^3+4x^2+4x) - (4x^3+2x^2-2x-1)}{(x^2+x+1)^2} = \frac{2x^2+6x+1}{(x^2+x+1)^2}.$$

3.4.47 $h'(x) = \dfrac{(x+1)e^x - xe^x}{(x+1)^2} = \dfrac{e^x}{(x+1)^2}.$

3.4.48 $h'(x) = \dfrac{x^2e^x \cdot 1 - (x+1)(2xe^x + x^2e^x)}{x^4e^{2x}} = \dfrac{x^2e^x - 2x^2e^x - x^3e^x - 2xe^x - x^2e^x}{x^4e^{2x}} = -\dfrac{x^2+2x+2}{x^3e^x}.$

3.4.49

$$\begin{aligned} g'(w) &= \frac{(\sqrt{w}-w)(\frac{1}{2\sqrt{w}}+1) - (\sqrt{w}+w)(\frac{1}{2\sqrt{w}}-1)}{(\sqrt{w}-w)^2} = \frac{\frac{1}{2} + \sqrt{w} - \frac{1}{2}\sqrt{w} - w - (\frac{1}{2} - \sqrt{w} + \frac{1}{2}\sqrt{w} - w)}{(\sqrt{w}-w)^2} \\ &= \frac{2\sqrt{w}-\sqrt{w}}{(\sqrt{w}-w)^2} = \frac{\sqrt{w}}{(\sqrt{w}-w)^2}. \end{aligned}$$

3.4.50 $f'(x) = \frac{d}{dx}\left(\frac{(2-x)(2+x)}{x-2}\right) = \frac{d}{dx}(-(2+x)) = \frac{d}{dx}(-2-x) = -1.$

3.4.51 $h'(w) = \frac{(w^{5/3}+1)\frac{5}{3}w^{2/3} - w^{5/3}(\frac{5}{3}w^{2/3})}{(w^{5/3}+1)^2} = \frac{\frac{5}{3}w^{2/3}}{(w^{5/3}+1)^2} = \frac{5w^{2/3}}{3(w^{5/3}+1)^2}.$

3.4.52 $g'(x) = \frac{(x^{4/3}+1)\frac{4}{3}x^{1/3} - (x^{4/3}-1)\frac{4}{3}x^{1/3}}{(x^{4/3}+1)^2} = \frac{\frac{8}{3}x^{1/3}}{(x^{4/3}+1)^2} = \frac{8x^{1/3}}{3(x^{4/3}+1)^2}.$

3.4.53 $f'(x) = \frac{d}{dx}\left(4x^2 - \frac{2x}{5x+1}\right) = 8x - \frac{(5x+1)2-(2x)(5)}{(5x+1)^2} = 8x - \frac{2}{(5x+1)^2}.$

3.4.54

$$\begin{aligned} f'(z) &= \left(\frac{z(2z)-(z^2+1)}{z^2}\right)e^z + \left(\frac{z^2+1}{z}\right)e^z = e^z\left(\left(\frac{z^2-1}{z^2}\right) + \left(\frac{z^2+1}{z}\right)\right) \\ &= e^z\left(\frac{z^3+z^2+z-1}{z^2}\right). \end{aligned}$$

3.4.55

$$\begin{aligned} h'(r) &= \frac{(r+1)(-1-\frac{1}{2\sqrt{r}}) - (2-r-\sqrt{r})\cdot 1}{(r+1)^2} = \frac{-r-\frac{\sqrt{r}}{2} - 1 - \frac{1}{2\sqrt{r}} - 2 + r + \sqrt{r}}{(r+1)^2} \\ &= \frac{\frac{\sqrt{r}}{2} - \frac{1}{2\sqrt{r}} - 3}{(r+1)^2}\cdot\frac{2\sqrt{r}}{2\sqrt{r}} = \frac{r-1-6\sqrt{r}}{2\sqrt{r}(r+1)^2}. \end{aligned}$$

3.4.56

$$\begin{aligned} y' &= \frac{(\sqrt{x}-\sqrt{a})\cdot 1 - (x-a)\frac{1}{2\sqrt{x}}}{(\sqrt{x}-\sqrt{a})^2} = \left(\frac{(\sqrt{x}-\sqrt{a})\cdot 1 - (x-a)\frac{1}{2\sqrt{x}}}{(\sqrt{x}-\sqrt{a})^2}\right)\cdot\frac{2\sqrt{x}}{2\sqrt{x}} \\ &= \frac{2x - 2\sqrt{ax} - x + a}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{x-2\sqrt{ax}+a}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{(\sqrt{x}-\sqrt{a})^2}{2\sqrt{x}(\sqrt{x}-\sqrt{a})^2} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

3.4.57 $h'(x) = (35x^6+5)(6x^3+3x^2+3) + (5x^7+5x)(18x^2+6x) = 15((7x^6+1)(2x^3+x^2+1) + (x^7+x)(6x^2+2x)) = 300x^9 + 135x^8 + 105x^6 + 120x^3 + 45x^2 + 15.$

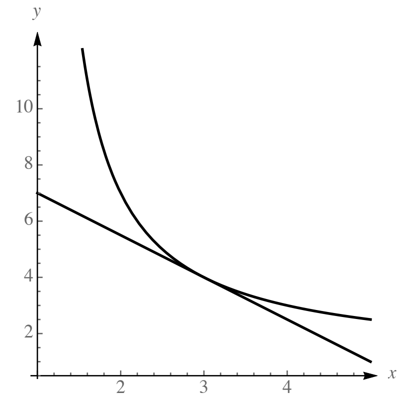
3.4.58 Consider $s(t)$ as the product of $(t+1)(t+2)$ and $(t+3)$. Then by the product rule, $s'(t) = ((1)(t+2)+(t+1)(1))(t+3) + (t+1)(t+2)(1) = (t+2)(t+3) + (t+1)(t+3) + (t+1)(t+2) = t^2+5t+6+t^2+4t+3+t^2+3t+2 = 3t^2+12t+11$. Alternatively, one could multiply out $s(t)$ from the outset giving $s(t) = t^3+6t^2+11t+6$ and so $s'(t) = 3t^2+12t+11$.

3.4.59 $f(x) = \sqrt{(e^x+4x^2)^2} = |e^x+4x^2| = e^x+4x^2$. Therefore, $f'(x) = e^x + 8x$.

3.4.60 $g(x) = \frac{(e^x-1)(e^x+1)}{e^x-1} = e^x + 1$. Therefore, $g'(x) = e^x$.

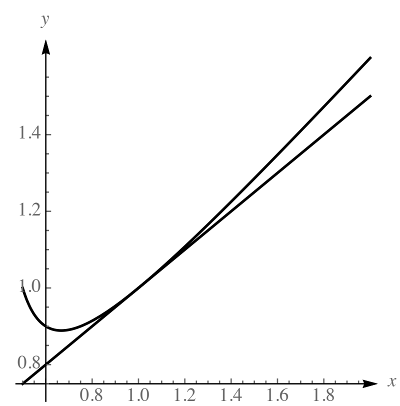
3.4.61

a. $y' = \dfrac{(x-1)-(x+5)}{(x-1)^2} = -\dfrac{6}{(x-1)^2}$.
At $a = 3$ we have $y' = -\dfrac{6}{4} = -\dfrac{3}{2}$ and $y = 4$, so the equation of the tangent line is $y - 4 = -\dfrac{3}{2} \cdot (x-3)$, or $y = -\dfrac{3}{2}x + \dfrac{17}{2}$.

b. 

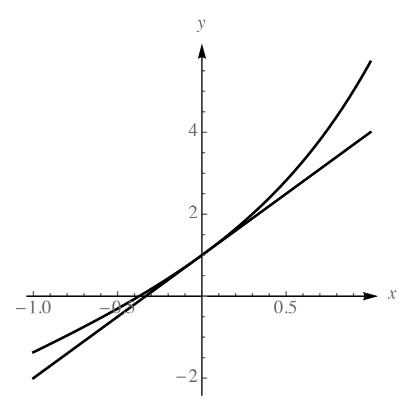
3.4.62

a. $y' = \dfrac{(3x-1)4x-(2x^2)3}{(3x-1)^2} = \dfrac{6x^2-4x}{(3x-1)^2}$.
At $a = 1$ we have $y' = \dfrac{1}{2}$ and $y = 1$, so the equation of the tangent line is $y - 1 = \dfrac{1}{2} \cdot (x-1)$, or $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

b. 

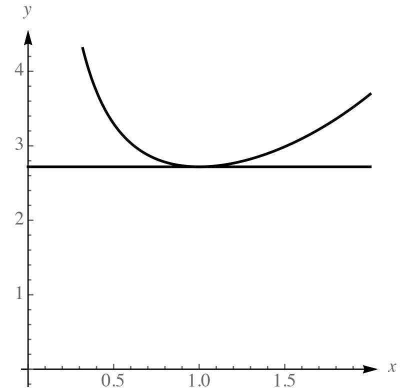
3.4.63

a. $y' = 2 + (1)e^x + xe^x$.
At $a = 0$ we have $y' = 2+1+0 = 3$ and $y = 1$. So the equation of the tangent line is $y - 1 = 3(x-0)$, or $y = 3x + 1$.

b. 

3.4.64

a. $y' = \dfrac{xe^x - e^x}{x^2}$.
At $a = 1$ we have $y' = \dfrac{e-e}{1} = 0$, and $y = e$. Thus, the equation of the tangent line is $y - e = 0$, or $y = e$.

b. 

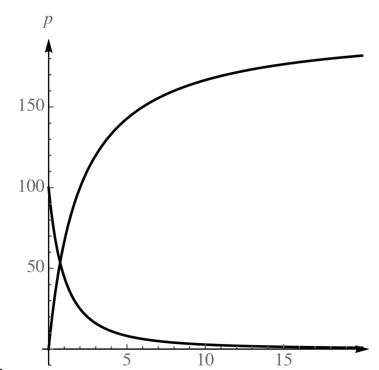
3.4.65

a. $p'(t) = \dfrac{(t+2)200 - 200t}{(t+2)^2} = \dfrac{400}{(t+2)^2}$.

b. $p'(5) = \dfrac{400}{49} \approx 8.16$.

c. The value of p' is as large as possible when its denominator is as small as possible, which is when $t = 0$. The value of $p'(0)$ is 100.

d. $\lim_{t\to\infty} p'(t) = \lim_{t\to\infty} \dfrac{400}{(t+2)^2} = 0$. This means that the population eventually has a growth rate of 0, which means that the population approaches a steady state.

e. 

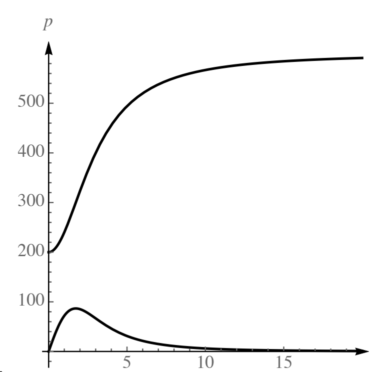
3.4.66

a. The instantaneous growth rate is simply the derivative of the function, so $p'(t) = 600\dfrac{(t^2+9)(2t) - 2t(t^2+3)}{(t^2+9)^2} = 600\dfrac{12t}{(t^2+9)^2}$.

b. $p'(5) = 600\left(\dfrac{60}{34^2}\right) = \dfrac{9000}{289} \approx 31.14$.

c. Consider the graph of $y = p'(t)$, shown in part (e). It appears that $p'(t)$ is at a maximum somewhere between $t = 1$ and $t = 2$; in fact, the value is $t = \sqrt{3} \approx 1.732$.

d. We have $\lim_{t\to\infty} p(t) = 600 \lim_{t\to\infty} \dfrac{1+3/t^2}{1+9/t^2} = 600$. This means that the population eventually approaches a steady state of 600.

e. 

3.4.67

a. The instantaneous rate of change is $\dfrac{d}{dx}F(x) = -\dfrac{2kQq}{x^3}$ N/m $= -\dfrac{1.8\times 10^{10}Qq}{x^3}$ N/m.

b. $\left[\frac{d}{dx}F(x)\right]\Big|_{x=0.001} = -\frac{2(9\times 10^9)}{(0.001)^3} = -\frac{18\times 10^9}{10^{-9}} = -18\times 10^{18} = -1.8\times 10^{19}$ Newtons per meter.

c. Because the distance x appears in the denominator of $F'(x)$, the absolute value of the instantaneous rate of change decreases with the separation.

3.4.68

a. The instantaneous rate of change if $\frac{d}{dx}F(x) = \frac{2GMm}{x^3}$ Newtons per meter.

b. $\left[\frac{d}{dx}F(x)\right]\Big|_{x=0.01} = \frac{13.4\times 10^{-11}\cdot (0.1)^2}{(0.01)^3} = 1.34\times 10^{-6}$ Newtons per meter.

c. Because the distance x appears in the denominator of $F'(x)$, the instantaneous rate of change decreases with the separation.

3.4.69

a. False. In fact, because e^5 is a constant, its derivative is zero.

b. False. It is certainly a reasonable way to proceed, but one could also write the given quantity as $x+3+2x^{-1}$, and then proceed using the sum rule and the power rule and the extended power rule.

c. False. $\frac{d}{dx}\left(\frac{1}{x^5}\right) = \frac{d}{dx}\left(x^{-5}\right) = -5x^{-6} = -\frac{5}{x^6}$.

d. False. The derivative is $3x^2e^x + x^3e^x = x^2e^x(3+x)$.

3.4.70 $f'(x) = -x^{-2} = -\frac{1}{x^2}$. $f''(x) = 2x^{-3} = \frac{2}{x^3}$. $f'''(x) = -6x^{-4} = -\frac{6}{x^4}$.

3.4.71 $f'(x) = 2x(2+x^{-3}) + x^2(-3x^{-4}) = 4x + 2x^{-2} - 3x^{-2} = 4x - x^{-2} = 4x - \frac{1}{x^2}$.
$f''(x) = 4 + 2x^{-3} = 2(2+\frac{1}{x^3})$.
$f'''(x) = -6x^{-4} = -\frac{6}{x^4}$.

3.4.72

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x+2}\right) = \frac{(x+2)-x}{(x+2)^2} = \frac{2}{(x+2)^2} = \frac{2}{x^2+4x+4}.$$

$$\begin{aligned} f''(x) = \frac{d}{dx}\left(\frac{2}{x^2+4x+4}\right) &= \frac{(x^2+4x+4)\cdot 0 - 2(2x+4)}{(x+2)^4} \\ &= \frac{-4(x+2)}{(x+2)^4} = -\frac{4}{(x+2)^3} = -\frac{4}{x^3+6x^2+12x+8}. \end{aligned}$$

3.4.73

$$f'(x) = \frac{d}{dx}\left(\frac{x^2-7x}{x+1}\right) = \frac{(x+1)(2x-7)-(x^2-7x)\cdot 1}{(x+1)^2} = \frac{x^2+2x-7}{x^2+2x+1} = \frac{x^2+2x-7}{(x+1)^2}.$$

$$\begin{aligned} f''(x) = \frac{d}{dx}\left(\frac{x^2+2x-7}{x^2+2x+1}\right) &= \frac{(x^2+2x+1)(2x+2)-(x^2+2x-7)(2x+2)}{(x+1)^4} \\ &= \frac{(2x^2+4x+2)-(2x^2+4x-14)}{(x+1)^3} = \frac{16}{(x+1)^3}. \end{aligned}$$

3.4.74

a. $g'(x) = 2xf(x) + x^2f'(x)$, so $g'(2) = 2 \cdot 2 \cdot f(2) + 4 \cdot f'(2) = 8 + 12 = 20$. Thus, the tangent line is given by $y - 8 = 20(x - 2)$, or $y = 20x - 32$.

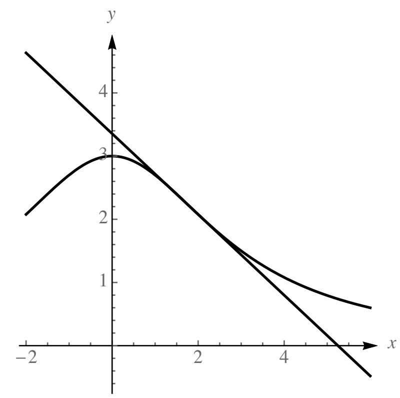
b. $h'(x) = \dfrac{(x-3)f'(x) - f(x)}{(x-3)^2}$, so $h'(2) = \dfrac{(-1) \cdot 3 - 2}{(-1)^2} = -5$. Also, $h(2) = -2$. Thus, the tangent line is given by $y + 2 = (-5)(x - 2)$, or $y = -5x + 8$.

3.4.75

a. $y' = -\dfrac{54x}{(x^2+9)^2}$. At $x = 2$, $y' = -\dfrac{108}{169}$ and $y = \dfrac{27}{13}$. Thus the tangent line is given by

$$y - \frac{27}{13} = -\frac{108}{169}(x - 2),$$

or $y = -\dfrac{108}{169}x + \dfrac{567}{169}$.

b. 

3.4.76 $\dfrac{d}{dx}\left[f(x)g(x)\right]\Big|_{x=1} = f'(1)g(1) + f(1)g'(1) = 3 \cdot 4 + 5 \cdot 2 = 22.$

3.4.77 $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\Bigg|_{x=2} = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{2 \cdot 5 - 4 \cdot 4}{4} = -\dfrac{3}{2}.$

3.4.78 $\dfrac{d}{dx}\left[xf(x)\right]\Big|_{x=3} = f(3) + 3 \cdot f'(3) = 3 + 3 \cdot 2 = 9.$

3.4.79 $\dfrac{d}{dx}\left[\dfrac{f(x)}{x+2}\right]\Bigg|_{x=4} = \dfrac{(4+2)f'(4) - f(4)}{36} = \dfrac{6-2}{36} = \dfrac{1}{9}.$

3.4.80 $\dfrac{d}{dx}\left[\dfrac{xf(x)}{g(x)}\right]\Bigg|_{x=4} = \dfrac{g(4)(f(4) + 4f'(4)) - 4f(4)g'(4)}{(g(4))^2} = \dfrac{3(2 + 4 \cdot 1) - 4 \cdot 2 \cdot 1}{9} = \dfrac{10}{9}.$

3.4.81 $\dfrac{d}{dx}\left[\dfrac{f(x)g(x)}{x}\right]\Bigg|_{x=4} = \dfrac{4(f'(4)g(4) + f(4)g'(4)) - f(4)g(4)}{16} = \dfrac{4(1 \cdot 3 + 2 \cdot 1) - (2 \cdot 3)}{16} = \dfrac{14}{16} = \dfrac{7}{8}.$

3.4.82

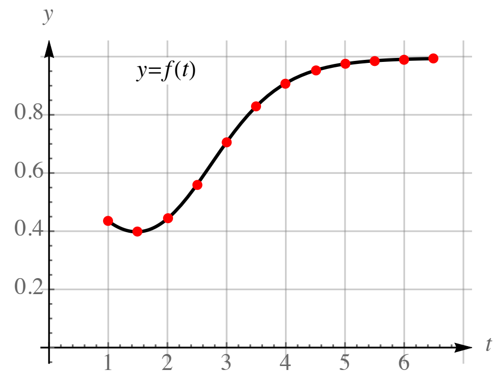
a. The derivative $m'(t)$ has units of g/week and measures the rate at which the owlets are increasing in mass.

b. The derivative $w'(t)$ has units of mm/week and measures the rate at which the wing chord is increasing.

c. The derivative $f'(t)$ has units of mm/g per week, which measures the rate at which the ratio of wing chord length to mass is changing per week.

3.4.83

a.



b. For $t \approx 3$.

c. $f'(t) = \dfrac{115.48(41.38) - 12.48(81.55)}{115.48^2} \approx 0.28\ mm/g$ per week. At a young age, the bird's wings are growing quickly relative to its age.

d. It appears that $f'(t) \approx 0$ for $t \geq 6.5$. $f'(6.5) = \dfrac{121.45(0.38) - 0.01(120.61)}{121.45^2} \approx 0.003\ mm/g$ per week. As the bird matures the bird's growth rate slows down, so the ratio of wing chord length to mass doesn't change much, which implies that the rate of change of wing chord length to mass, f', is almost 0.

3.4.84 $p'(x) = f'(x)g(x) + f(x)g'(x)$, so $p'(3) = f'(3)g(3) + f(3)g'(3) = 5(2) + 2(-10) = -10$.

3.4.85 $q'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, so $q'(3) = \dfrac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \dfrac{5(2) - 2(-10)}{4} = \dfrac{30}{4} = \dfrac{15}{2}$.

3.4.86 $y' = 1 \cdot f(x) + xf'(x)$, so at $x = 1$ we have $y'(1) = f(1) + f'(1) = 2 + 2 = 4$.

3.4.87 $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=4} = f'(4)g(4) + f(4)g'(4) = \dfrac{1}{2}(1) + 3(-1) = -\dfrac{5}{2}$.

3.4.88 $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)\Big|_{x=4} = \dfrac{g(4)f'(4) - f(4)g'(4)}{g(4)^2} = \dfrac{1\left(\dfrac{1}{2}\right) - 3(-1)}{1^2} = \dfrac{7}{2}$.

3.4.89 $\dfrac{d}{dx}(xg(x))\Big|_{x=2} = g(2) + 2g'(2) = 3 + 2(-1) = 1$.

3.4.90 $\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\Big|_{x=2} = \dfrac{f(2)(4) - 4f'(2)}{f(2)^2} = \dfrac{2(4) - 4(1/2)}{4} = \dfrac{3}{2}$.

3.4.91

a. $y' = (2x - 3)h(x) + (x^2 - 3x)h'(x)$ so $y'(4) = 5(2) + 4(-3) = -2$. Also, $y(4) = 4(2) = 8$. So the equation is $y - 8 = -2(x - 4)$ or $y = -2x + 16$.

b. $y' = \dfrac{(x+2)h'(x) - h(x)}{(x+2)^2}$, so $y'(4) = \dfrac{6(-3) - 2}{36} = -\dfrac{20}{36} = -\dfrac{5}{9}$. Also, $y(4) = \dfrac{1}{3}$. So the equation of the tangent line is $y - \dfrac{1}{3} = -\dfrac{5}{9}(x - 4)$ or $y = -\dfrac{5}{9}x + \dfrac{23}{9}$.

3.4.92

a. Because the slope of f at 2 is $f'(2) = 4$ and the slope of g at 2 is $g'(2) = 3$ and $f(2) = 4 \cdot 2 + 1 = 9$, $g(2) = 3 \cdot 2 - 2 = 4$ we have $y'(2) = f'(2)g(2) + f(2)g'(2) = 4 \cdot 4 + 9 \cdot 3 = 43$. Thus, the tangent line at this point is $y - 36 = 43(x - 2)$, or $y = 43x - 50$.

b. $y'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{4 \cdot 4 - 9 \cdot 3}{16} = -\dfrac{11}{16}$. So the equation of the tangent line at this point is $y - \dfrac{9}{4} = -\dfrac{11}{16}(x - 2)$, or $y = -\dfrac{11}{16}x + \dfrac{29}{8}$.

3.4.93 Following the hint, we note that $f(0) = 100$ and we have $f'(x) = 40x^7 + 30x^4 + 20x^3 + 6x + 20$, so $f'(0) = 20$. Also note that $g(0) = 2$ and $g'(x) = 100x^9 + 72x^8 + 30x^4 + 12x + 4$, so $g'(0) = 4$. Then

$$q'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{g(0)^2} = \frac{2(20) - 100(4)}{4} = -90.$$

3.4.94 Following the hint in the previous problem, let $p(x) = f(x)g(x)$. Then $f'(x) = 5e^x + 50x^4 + 60x^2 + 200x + 5$ and $g'(x) = 50x^4 + 120x^2 + 40x + 4$. We have $f(0) = 25$ and $f'(0) = 10$ and $g(0) = 10$ and $g'(0) = 4$. Then

$$p'(0) = f(0)g'(0) + f'(0)g(0) = 25(4) + 10(10) = 200.$$

3.4.95

a. The tangent line at $x = a$ is $y - a^2 = 2a(x - a)$ and at $x = b$ is $y - b^2 = 2b(x - b)$.

These intersect when $a^2 + 2ax - 2a^2 = b^2 + 2bx - 2b^2$, or $(2a - 2b)x = a^2 - b^2$, which is met when $x = \dfrac{a+b}{2}$. So $c = \dfrac{a+b}{2}$.

b. The tangent line at $x = a$ is $y - \sqrt{a} = \dfrac{1}{2\sqrt{a}}(x - a)$ and at $x = b$ is $y - \sqrt{b} = \dfrac{1}{2\sqrt{b}}(x - b)$.

These intersect when $\sqrt{a} + \dfrac{1}{2\sqrt{a}}(x - a) = \sqrt{b} + \dfrac{1}{2\sqrt{b}}(x - b)$, or $\left(\dfrac{1}{2\sqrt{a}} - \dfrac{1}{2\sqrt{b}}\right) x = \dfrac{\sqrt{b} - \sqrt{a}}{2}$, which is met when $x = \sqrt{ab}$. So $c = \sqrt{ab}$.

c. The tangent line at $x = a$ is $y - \dfrac{1}{a} = -\dfrac{1}{a^2}(x - a)$ and at $x = b$ is $y - \dfrac{1}{b} = -\dfrac{1}{b^2}(x - b)$.

These intersect when $\dfrac{1}{a} - \dfrac{1}{a^2}(x - a) = \dfrac{1}{b} - \dfrac{1}{b^2}(x - b)$, or $\left(\dfrac{2}{a} - \dfrac{x}{a^2}\right) = \left(\dfrac{2}{b} - \dfrac{x}{b^2}\right)$, which is met when $x\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right) = \dfrac{2}{b} - \dfrac{2}{a}$, or $x \cdot \left(\dfrac{a^2 - b^2}{a^2b^2}\right) = \dfrac{2(a-b)}{ab}$. Thus we arrive at $x = \dfrac{2ab}{a+b}$. So $c = \dfrac{2ab}{a+b}$.

d. The tangent line at $x = a$ is $y - f(a) = f'(a)(x - a)$ and at $x = b$ is $y - f(b) = f'(b)(x - b)$.

These intersect when $f(a) + f'(a)(x - a) = f(b) + f'(b)(x - b)$, or $(f'(a) - f'(b))x = f(b) - f(a) - f'(b)b + f'(a)a$. Solving for x yields $x = \dfrac{f(b) - f(a) - f'(b)b + f'(a)a}{f'(a) - f'(b)}$ provided $f'(a) \neq f'(b)$ (which occurs when the tangent lines are parallel and don't intersect.)

3.4.96

a.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)}.$$

b.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h \cdot g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{g(x)\left(\dfrac{f(x+h)-f(x)}{h}\right) - f(x)\left(\dfrac{g(x+h)-g(x)}{h}\right)}{g(x+h)g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

c. F' exists provided that f and g are differentiable, and $g(x) \neq 0$. Note that we used the fact that $\lim_{h\to 0} g(x+h) = g(x)$, which is true because g is continuous (because it is differentiable).

3.4.97

$$\begin{aligned}
\frac{d^2}{dx^2}(f(x)g(x)) &= \frac{d}{dx}(f'(x)g(x) + f(x)g'(x)) = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) \\
&= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).
\end{aligned}$$

3.4.98

a. $(fg)^{(2)} = (f'g + fg')' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$.

b. We proceed by induction. For $n = 1$, we have that

$$(fg)' = \sum_{k=0}^{1} f^{(k)} g^{(1-k)} = f'g + fg'.$$

Now suppose that the rule holds for $n = m$. We will show that the rule holds for $n = m + 1$.

$$\begin{aligned}
(fg)^{(m+1)} &= \left((fg)^{(m)}\right)' = \sum_{k=0}^{n} \binom{n}{k} (f^{(k)} g^{(n-k)})' = \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n+1-k)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n+1-(k+1))} + f^{(k)} g^{(n+1-k)}\right) \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)},
\end{aligned}$$

because $\binom{n}{0} = \binom{n}{n+1} = 1$ and $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

c. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ follows a similar pattern.

3.4.99

a.

$$\begin{aligned}
\frac{d}{dx}[(f(x)g(x))h(x)] &= \frac{d}{dx}[f(x)g(x)] \cdot h(x) + f(x)g(x) \cdot \frac{d}{dx}h(x) \\
&= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\
&= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).
\end{aligned}$$

b. $\frac{d}{dx}[e^x(x-1)(x+3)] = e^x(x-1)(x+3) + e^x(x+3) + e^x(x-1) = e^x(x^2+2x-3+x+3+x-1) = e^x(x^2+4x-1)$.

3.5 Derivatives of Trigonometric Functions

3.5.1 A direct substitution would yield the quotient of zero with itself, which isn't defined

3.5.2 It is an important ingredient in the derivation of the formula $\dfrac{d}{dx}\sin x = \cos x$.

3.5.3 Because $\tan x = \dfrac{\sin x}{\cos x}$, and $\cot x = \dfrac{\cos x}{\sin x}$, we can use the quotient rule to compute these derivatives, because we know the derivatives of $\sin x$ and of $\cos x$.

3.5.4 Remember the rule that the derivative of a "co" function can be obtained from the derivative of a function by changing all of the functions in the formula to their cofunctions, and introducing a factor of negative one. Thus, for example, because $\dfrac{d}{dx}\tan x = \sec^2 x$, we would have $\dfrac{d}{dx}\cot x = -\csc^2 x$.

3.5.5 $f'(x) = \cos x$ and $f'(\pi) = \cos\pi = -1$.

3.5.6 $\dfrac{d}{dx}(\tan x)\Big|_{x=\pi/3} = \sec^2(\pi/3) = 4$.

3.5.7 $y' = \cos x$, so $y'(0) = \cos 0 = 1$. So the equation of the tangent line is $y - 0 = 1(x-0)$ or $y = x$.

3.5.8 Because $\dfrac{d}{dx}\sin x = \cos x$, the graph of $\sin x$ will have a horizontal tangent line where the cosine function is 0. This happens at all real numbers of the form $x = \dfrac{2n+1}{2}\cdot\pi$ where n is an integer.

3.5.9 $\dfrac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$, so $\dfrac{d^2}{dx^2}(\sin x + \cos x) = \dfrac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x = -(\sin x + \cos x)$.

3.5.10 $\dfrac{d^2}{dx^2}(\sec x) = \dfrac{d}{dx}(\sec x\tan x) = \sec x\tan x\tan x + \sec x\sec^2 x = \sec x(\tan^2 x + \sec^2 x)$.

3.5.11 $\displaystyle\lim_{x\to0}\frac{\sin 3x}{x} = \lim_{x\to0}\frac{3\sin 3x}{3x} = 3\lim_{x\to0}\frac{\sin 3x}{3x} = 3\cdot 1 = 3$.

3.5.12 $\displaystyle\lim_{x\to0}\frac{\sin 5x}{3x} = \frac{1}{3}\lim_{x\to0}\frac{5\sin 5x}{5x} = \frac{5}{3}\lim_{x\to0}\frac{\sin 5x}{5x} = \frac{5}{3}\cdot 1 = \frac{5}{3}$.

3.5.13 $\displaystyle\lim_{x\to0}\frac{\sin 7x}{\sin 3x} = \lim_{x\to0}\frac{\frac{7\sin 7x}{7x}}{\frac{3\sin 3x}{3x}} = \frac{7}{3}\cdot\lim_{x\to0}\frac{\frac{\sin 7x}{7x}}{\frac{\sin 3x}{3x}} = \frac{7}{3}\cdot\frac{1}{1} = \frac{7}{3}$.

3.5.14 $\displaystyle\lim_{x\to0}\frac{\sin 3x}{\tan 4x} = \lim_{x\to0}\frac{\sin 3x\cos 4x}{\sin 4x} = \lim_{x\to0}\frac{\frac{3\sin 3x}{3x}\cdot\cos 4x}{\frac{4\sin 4x}{4x}} = \frac{3\cdot1\cdot1}{4\cdot1} = \frac{3}{4}$.

3.5.15 $\displaystyle\lim_{x\to0}\frac{\tan 5x}{x} = \lim_{x\to0}\frac{5\sin 5x}{5x\cos 5x} = 5\lim_{x\to0}\frac{\sin 5x}{5x}\cdot\lim_{x\to0}\frac{1}{\cos 5x} = 5\cdot1\cdot1 = 5$.

3.5.16 $\displaystyle\lim_{\theta\to0}\frac{\cos^2\theta - 1}{\theta} = \left(\lim_{\theta\to0}(\cos\theta + 1)\right)\cdot\left(\lim_{\theta\to0}\frac{\cos\theta - 1}{\theta}\right) = 2\cdot 0 = 0$.

3.5.17 $\displaystyle\lim_{x\to0}\frac{\tan 7x}{\sin x} = \lim_{x\to0}\frac{\sin 7x}{\cos 7x\cdot\sin x} = \lim_{x\to0}\left(\frac{1}{\cos 7x}\cdot\frac{x}{\sin x}\cdot\frac{7\sin 7x}{7x}\right) =$
$\displaystyle 7\cdot\lim_{x\to0}\frac{1}{\cos 7x}\cdot\lim_{x\to0}\frac{x}{\sin x}\cdot\lim_{x\to0}\frac{\sin 7x}{7x} = 7\cdot1\cdot1\cdot1 = 7$.

3.5.18 $\lim_{\theta\to 0}\frac{\sec\theta-1}{\theta}=\lim_{\theta\to 0}\frac{\frac{1}{\cos\theta}-1}{\theta}=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\cos\theta}=\lim_{\theta\to 0}\frac{1}{\cos\theta}\cdot\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=1\cdot 0=0.$

3.5.19 $\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}=\lim_{x\to 2}\left(\frac{1}{x+2}\cdot\frac{\sin(x-2)}{x-2}\right)=\lim_{x\to 2}\frac{1}{x+2}\cdot\lim_{x\to 2}\frac{\sin(x-2)}{x-2}=\frac{1}{4}\cdot 1=\frac{1}{4}.$

3.5.20 $\lim_{x\to -3}\frac{\sin(x+3)}{x^2+8x+15}=\lim_{x\to -3}\frac{\sin(x+3)}{(x+5)(x+3)}=\lim_{x\to -3}\frac{1}{x+5}\cdot\lim_{x\to -3}\frac{\sin(x+3)}{(x+3)}=\frac{1}{2}\cdot 1=\frac{1}{2}.$

3.5.21 $\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{a\sin ax}{ax}\cdot\frac{bx}{b\sin bx}=\frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\lim_{x\to 0}\frac{bx}{\sin bx}=\frac{a}{b}\cdot 1\cdot 1=\frac{a}{b}.$

3.5.22 $\lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}=\frac{a}{b}\cdot 1=\frac{a}{b}.$

3.5.23 $\frac{dy}{dx}=\cos x-\sin x.$

3.5.24 $\frac{dy}{dx}=10x-\sin x.$

3.5.25 $\frac{dy}{dx}=-e^{-x}\sin x+e^{-x}\cos x=e^{-x}(\cos x-\sin x).$

3.5.26 $\frac{dy}{dx}=\cos x+4e^x.$

3.5.27 $\frac{dy}{dx}=\sin x+x\cos x.$

3.5.28 $\frac{dy}{dx}=e^x(-\sin x+\cos x)+e^x(\cos x+\sin x)=2e^x\cos x.$

3.5.29

$$\begin{aligned}\frac{dy}{dx}&=\frac{(\sin x+1)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}=\frac{-1(\sin^2 x+\cos^2 x)-\sin x}{(1+\sin x)^2}\\&=\frac{-1(1+\sin x)}{(1+\sin x)^2}=-\frac{1}{1+\sin x}.\end{aligned}$$

3.5.30 $\frac{dy}{dx}=\frac{(1+\sin x)(-\cos x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}=\frac{-2\cos x}{(1+\sin x)^2}.$

3.5.31 $\frac{dy}{dx}=\cos x\cos x+\sin x\cdot(-\sin x)=\cos^2 x-\sin^2 x=\cos(2x).$

3.5.32

$$\begin{aligned}&\frac{d}{dx}\left(\frac{a\sin x+b\cos x}{a\sin x-b\cos x}\right)=\frac{(a\sin x-b\cos x)(a\cos x-b\sin x)-(a\sin x+b\cos x)(a\cos x+b\sin x)}{(a\sin x-b\cos x)^2}\\&=\frac{a^2\sin x\cos x-ab\sin^2 x-ab\cos^2 x+b^2\cos x\sin x-a^2\sin x\cos x-ab\sin^2 x-ab\cos^2 x-b^2\sin x\cos x}{(a\sin x-b\cos x)^2}\\&=\frac{-2ab(\sin^2 x+\cos^2 x)}{(a\sin x-b\cos x)^2}=-\frac{2ab}{(a\sin x-b\cos x)^2}.\end{aligned}$$

3.5.33 $\frac{dy}{dx}=-\sin x\cos x+\cos x(-\sin x)=-2\sin x\cos x=-\sin(2x).$

3.5.34

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x)(\sin x + x\cos x) - x\sin x(-\sin x)}{(1+\cos x)^2} = \frac{\sin x + x\cos x + \sin x\cos x + x\cos^2 x + x\sin^2 x}{(1+\cos x)^2} \\ &= \frac{\sin x + x\cos x + \sin x\cos x + x}{(1+\cos x)^2} = \frac{\sin x(1+\cos x) + x(1+\cos x)}{(1+\cos x)^2} = \frac{\sin x + x}{1+\cos x}.\end{aligned}$$

3.5.35 $\dfrac{dy}{dw} = 2w\sin w + w^2\cos w + 2\cos w - 2w\sin w - 2\cos w = w^2\cos w.$

3.5.36 $\dfrac{dy}{dx} = -3x^2\cos x + x^3\sin x + 6x\sin x + 3x^2\cos x + 6\cos x - 6x\sin x - 6\cos x = x^3\sin x.$

3.5.37 $\dfrac{dy}{dx} = \cos x\sin x + x(-\sin x)\sin x + x\cos x\cos x = \sin x\cos x - x\sin^2 x + x\cos^2 x = \dfrac{1}{2}\sin 2x + x\cos 2x.$

3.5.38 $\dfrac{dy}{dx} = \dfrac{0 - 1\cdot\cos x}{(2+\sin x)^2} = -\dfrac{\cos x}{(2+\sin x)^2}.$

3.5.39 $\dfrac{dy}{dx} = \dfrac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}.$

3.5.40 $\dfrac{dy}{dx} = \dfrac{e^x(\cos x - \sin x) - e^x(\sin x + \cos x)}{e^{2x}} = \dfrac{-2\sin x}{e^x} = -2e^{-x}\sin x.$

3.5.41 $\dfrac{dy}{dx} = \dfrac{(1+\cos x)\sin x - (1-\cos x)(-\sin x)}{(1+\cos x)^2} = \dfrac{2\sin x}{(1+\cos x)^2}.$

3.5.42 $\dfrac{dy}{dx} = \sec^2 x - \csc^2 x.$

3.5.43 $\dfrac{dy}{dx} = \sec x\tan x - \csc x\cot x.$

3.5.44 $\dfrac{dy}{dx} = \sec x\tan x\tan x + \sec x\sec^2 x = \sec x(\tan^2 x + \sec^2 x).$

3.5.45 $\dfrac{dy}{dx} = e^x\csc x + e^x(-\csc x\cot x) = e^x\csc x(1 - \cot x).$

3.5.46 $\dfrac{dy}{dw} = \dfrac{(1+\tan w)\sec^2 w - \tan w\sec^2 w}{(1+\tan w)^2} = \dfrac{\sec^2 w}{(1+\tan w)^2}.$

3.5.47

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\csc x)(-\csc^2 x) - \cot x(-\csc x\cot x)}{(1+\csc x)^2} = \frac{-\csc^2 x - \csc^3 x + \csc x(\csc^2 x - 1)}{(1+\csc x)^2} \\ &= \frac{-\csc x(1+\csc x)}{(1+\csc x)^2} = -\frac{\csc x}{1+\csc x}\end{aligned}$$

3.5.48

$$\begin{aligned}\frac{dy}{dt} &= \frac{(1+\sec t)\sec^2 t - \tan t(\sec t\tan t)}{(1+\sec t)^2} = \frac{\sec^2 t + \sec^3 t - \sec t(\sec^2 t - 1)}{(1+\sec t)^2} \\ &= \frac{\sec t(1+\sec t)}{(1+\sec t)^2} = \frac{\sec t}{1+\sec t}.\end{aligned}$$

3.5.49

$$\frac{dy}{dz} = \frac{0-(\sec z\tan z\csc z-\sec z\csc z\cot z)}{\sec^2 z\csc^2 z} = \frac{\sec z\csc z(\cot z-\tan z)}{\sec^2 z\csc^2 z}$$
$$= \frac{\cot z-\tan z}{\sec z\csc z} = \cos^2 z-\sin^2 z = \cos(2z).$$

3.5.50 Because $\csc^2\theta - 1 = \cot^2\theta$, we have $\frac{dy}{d\theta} = \frac{d}{d\theta}\cot^2\theta = -\csc^2\theta\cot\theta + \cot\theta(-\csc^2\theta) = -2\csc^2\theta\cot\theta$.

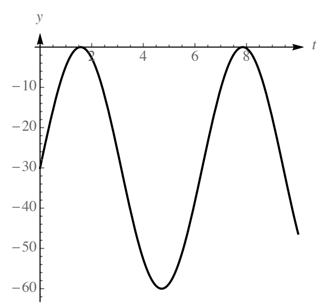
3.5.51 $\frac{dy}{dx} = 1-(-\sin x\sin x+\cos x\cos x) = 1+\sin^2 x-\cos^2 x = 1+\sin^2 x-(1-\sin^2 x) = 2\sin^2 x$.

3.5.52 $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$.

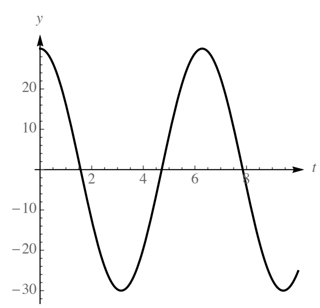
3.5.53 $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0-(-\sin x)}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\tan x$.

3.5.54 $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{0-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\cot x$.

3.5.55

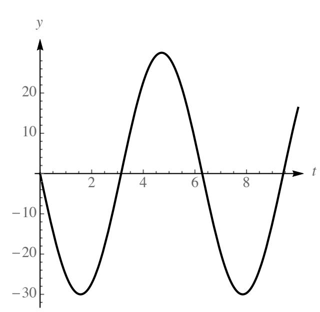
a. 

b. $v(t) = y'(t) = 30\cos t$ cm per second.

c. 

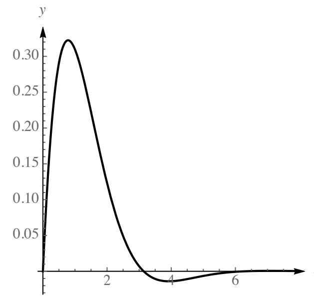
d. $v(t) = 30\cos t = 0$ when $t = \frac{2n+1}{2}\cdot\pi$ where n is a non-negative integer. At those times, the position is given by $\begin{cases} 0 & \text{if } n \text{ is even} \\ -60 & \text{if } n \text{ is odd.} \end{cases}$

e. The maximum velocity is 30 cm per second because $|\cos t| \le 1$ for all t. We have $\cos t = 1$ for $t = 2n\pi$ for a positive integer n. At those times, $y(2n\pi) = -30$.

f. 

$a(t) = v'(t) = -30\sin t$.

3.5.56

a. 

The graph of $f(t) = e^{-k}\sin t$ oscillates between $-e^{-t}$ and e^{-t} because $-1 \le \sin t \le 1$.

b. We have $f'(t) = -e^{-t}\sin t + e^{-t}\cos t$, which is zero when $\cos t - \sin t = 0$, which occurs for $t = \frac{\pi}{4} + n\pi$ where n is any integer.

c. Because $-1 \le \sin t \le 1$ and $e^{-t} > 0$ for all t, we have $-e^{-t} \le e^{-t}\sin t \le e^{-t}$. And because $\lim_{t\to\infty} e^{-t} = 0$, we have that $\lim_{t\to\infty} e^{-t}\sin t = 0$ by the Squeeze Theorem. This means that the vibrations approach zero in the long run.

3.5.57 $y' = \sin x + x\cos x$, so $y'' = \cos x + \cos x + -x\sin x = 2\cos x - x\sin x$.

3.5.58 $y' = 2x\cos x + x^2(-\sin x) = x(2\cos x - x\sin x)$, so $y'' = (2\cos x - x\sin x) + x(-2\sin x - (\sin x + x\cos x)) = 2\cos x - x\sin x - 2x\sin x - x\sin x - x^2\cos x = 2\cos x - 4x\sin x - x^2\cos x$.

3.5.59 $y' = e^x\sin x + e^x\cos x$, so $y'' = e^x\sin x + e^x\cos x + e^x\cos x + e^x(-\sin x) = 2e^x\cos x$.

3.5.60 $y' = \dfrac{1}{2}e^x\cos x + \dfrac{1}{2}e^x(-\sin x)$, so $y'' = \dfrac{1}{2}e^x\cos x + \dfrac{1}{2}e^x(-\sin x) + \dfrac{1}{2}e^x(-\sin x) + \dfrac{1}{2}e^x(-\cos x) = -e^x\sin x$.

3.5.61 $y' = -\csc^2 x$ and $y'' = -((-\csc x\cot x)\csc x + \csc x(-\csc x\cot x)) = 2\cot x\csc^2 x$.

3.5.62 $y' = \sec^2 x$ and $y'' = \sec x\tan x(\sec x) + \sec x(\sec x\tan x) = 2\tan x\sec^2 x$.

3.5.63

$$y' = \sec x\tan x\csc x - \sec x\csc x\cot x = \sec x\csc x(\tan x - \cot x) = \sec^2 x - \csc^2 x.$$

$$\begin{aligned} y'' &= \sec x(\sec x\tan x) + (\sec x\tan x)\sec x - ((-\csc x\cot x)\csc x + \csc x(-\csc x\cot x)) \\ &= 2\sec^2 x\tan x + 2\csc^2 x\cot x. \end{aligned}$$

3.5.64 $y' = (-\sin x)\sin x + \cos x\cos x = \cos^2 x - \sin^2 x = \cos(2x)$.
$y'' = 2\cos x(-\sin x) - 2\sin x\cos x = -2(2\sin x\cos x) = -2\sin(2x)$.

3.5.65

a. False. $\dfrac{d}{dx}\sin^2 x = \sin x\cos x + \cos x\sin x = 2\sin x\cos x \ne \cos^2 x$.

b. False. $\dfrac{d^2}{dx^2}\sin x = \dfrac{d}{dx}\cos x = -\sin x \ne \sin x$.

c. True. $\dfrac{d^4}{dx^4}\cos x = \dfrac{d^3}{dx^3}(-\sin x) = \dfrac{d^2}{dx^2}(-\cos x) = \dfrac{d}{dx}\sin x = \cos x$.

d. True. In fact, $\pi/2$ isn't even in the domain of $\sec x$.

3.5.66 $\lim_{x\to\pi/2}\dfrac{\cos x}{x-\pi/2} == \lim_{x\to\pi/2}\dfrac{\cos x-\cos\pi/2}{x-\pi/2} = \dfrac{d}{dx}\cos x\Big|_{x=\pi/2} = -\sin\pi/2 = -1.$

3.5.67 $\lim_{x\to\pi/4}\dfrac{\tan x-1}{x-\pi/4} = \lim_{x\to\pi/4}\dfrac{\tan x-\tan\pi/4}{x-\pi/4} = \dfrac{d}{dx}\tan x\Big|_{x=\pi/4} = \sec^2\pi/4 = 2.$

3.5.68 $\lim_{h\to 0}\dfrac{\sin(\pi/6+h)-(1/2)}{h} = \lim_{h\to 0}\dfrac{\sin(\pi/6+h)-\sin\pi/6}{h} = \dfrac{d}{dx}\sin x\Big|_{x=\pi/6} = \cos(\pi/6) = \sqrt{3}/2.$

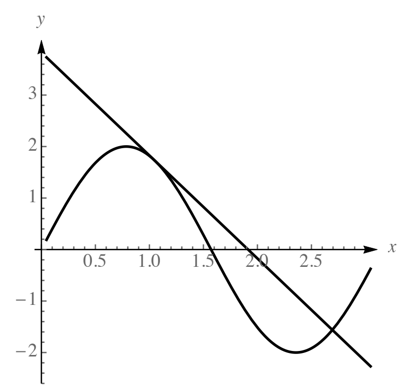
3.5.69 $\lim_{h\to 0}\dfrac{\cos(\pi/6+h)-(\sqrt{3}/2)}{h} = \lim_{h\to 0}\dfrac{\cos(\pi/6+h)-(\cos\pi/6)}{h} = \dfrac{d}{dx}\cos x\Big|_{x=\pi/6} = -\sin(\pi/6) = -1/2.$

3.5.70 $\lim_{x\to\pi/4}\dfrac{\cot(x)-1}{x-\pi/4} = \lim_{x\to\pi/4}\dfrac{\cot(x)-\cot\pi/4}{x-\pi/4} = \dfrac{d}{dx}\cot x\Big|_{x=\pi/4} = -\csc^2(\pi/4) = -2.$

3.5.71 $\lim_{h\to 0}\dfrac{\tan(5\pi/6+h)+(1/\sqrt{3})}{h} = \lim_{h\to 0}\dfrac{\tan(5\pi/6+h)-(\tan 5\pi/6)}{h} = \dfrac{d}{dx}\tan x\Big|_{x=\pi/6} =$
$\sec^2(5\pi/6) = 4/3.$

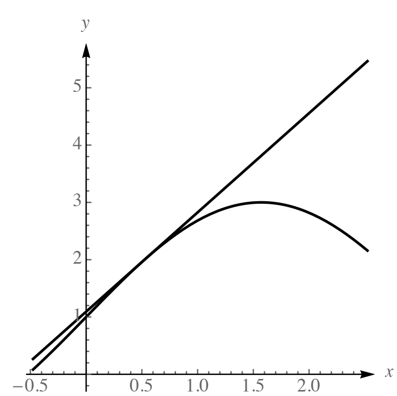
3.5.72

a. $y' = 4\cos^2 x - 4\sin^2 x$, so $y'(\pi/3) = 4\left(\dfrac{1}{4}-\dfrac{3}{4}\right) = -2$. $y(\pi/3) = 4\cdot(\sqrt{3}/2)\cdot(1/2) = \sqrt{3}$. The tangent line is thus given by $y-\sqrt{3} = -2(x-\pi/3)$, or $y = -2x+\sqrt{3}+\frac{2\pi}{3}$.

b. 

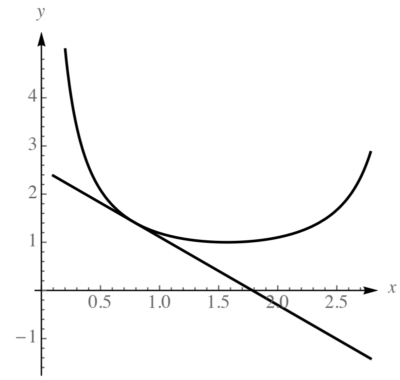
3.5.73

a. $y' = 2\cos x$, so $y'(\pi/6) = \sqrt{3}$. $y(\pi/6) = 2$. The tangent line is thus given by $y-2 = \sqrt{3}(x-\pi/6)$, or $y = \sqrt{3}x+2-\frac{\pi\sqrt{3}}{6}$.

b. 

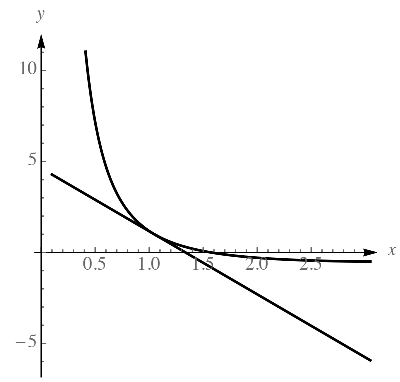
3.5.74

a. $y' = -\csc x \cot x$, so $y'(\pi/4) = -\sqrt{2}$. $y(\pi/4) = \sqrt{2}$. The tangent line is thus given by $y - \sqrt{2} = -\sqrt{2}(x - \pi/4)$, or $y = -\sqrt{2}x + \sqrt{2} + \frac{\sqrt{2}\pi}{4}$.

b. 

3.5.75

a. $y' = \dfrac{(1-\cos x)(-\sin x) - \cos x \sin x}{(1-\cos x)^2} = -\dfrac{\sin x}{(1-\cos x)^2}$, so $y'(\pi/3) = -2\sqrt{3}$. $y(\pi/3) = 1$. The tangent line is thus given by $y - 1 = -2\sqrt{3}(x - \pi/3)$, or $y = -2\sqrt{3}x + \frac{2\sqrt{3}\pi}{3} + 1$.

b. 

3.5.76

a. A horizontal tangent line occurs when $g'(x) = 1 - \cos x = 0$, which is when $\cos x = 1$. This occurs when $x = 2n\pi$, where n is any integer.

b. A slope of 1 occurs when $g'(x) = 1 - \cos x = 1$, which is when $\cos x = 0$. This occurs when $x = \dfrac{2n+1}{2} \cdot \pi$, where n is any integer.

3.5.77 For a horizontal tangent line we need $f'(x) = 1 + 2\sin x = 0$, or $\sin x = -\dfrac{1}{2}$. This occurs for $x = \frac{7\pi}{6} + 2n\pi$ where n is any integer, or for $x = \frac{11\pi}{6} + 2n\pi$ where n is any integer.

3.5.78

a. The derivative of graph (a) is graph (D), because graph (a) has a positive slope everywhere, its derivative must be positive everywhere, and graph (D) is the only one with this property.

b. The derivative of graph (b) is graph (B), because graph (b) has negative slope everywhere, its derivative must be negative everywhere, and graph (B) is the only one with this property.

c. The derivative of graph (c) is graph (A), because graph (c) has horizontal tangents at 0 and $\pm\pi$, its derivative needs to be 0 at these points, and only graph (A) has this property.

d. The derivative of graph (d) is graph (C), because graph (d) has horizontal tangents at $\pm\pi/2$ and $\pm 3\pi/2$, its derivative needs to be 0 at these points, and only graph (C) has this property.

3.5.79

a. $y'(t) = A\cos t$, $y''(t) = -A\sin t$, so $y''(t) + y(t) = -A\sin t + A\sin t = 0$ for all A and all t.

b. $y'(t) = -B\sin t$, $y''(t) = -B\cos t$, so $y''(t) + y(t) = -B\cos t + B\cos t = 0$ for all B and all t.

c. $y' = A\cos t - B\sin t$, $y'' = -A\sin t - B\cos t$, so $y''(t) + y(t) = -A\sin t - B\cos t + A\sin t + B\cos t = 0$ for all A, B, t.

3.5.80 $\frac{d}{dx}(\sin 2x) = \frac{d}{dx}(2\sin x\cos x) = (2\cos x)\cos x + (2\sin x)(-\sin x) = 2(\cos^2 x - \sin^2 x) = 2\cos 2x$.

3.5.81

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x\to 0} -\frac{\sin^2 x}{x(\cos x + 1)}$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0} -\frac{\sin x}{(\cos x + 1)} = 1\cdot\frac{0}{2} = 0.$$

3.5.82 We will use this version of the half-angle formula: $\frac{1-\cos x}{2} = \sin^2(x/2)$.

We have $\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0} -\frac{1-\cos x}{2x/2} = \lim_{x\to 0} -\frac{\sin^2(x/2)}{x/2} = -\lim_{x\to 0}\sin(x/2)\cdot\lim_{x\to 0}\frac{\sin x/2}{x/2} = -1\cdot 0\cdot 1 = 0.$

3.5.83

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= \cos x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) - \sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = \cos x\cdot 0 - \sin x\cdot 1 = -\sin x.$$

3.5.84 f is continuous at 0 if and only if $\lim_{x\to 0} f(x) = f(0)$. Because $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{3\sin x}{x} = 3$, we require $a = 3$ in order for f to be continuous.

3.5.85 g is continuous at 0 if and only if $\lim_{x\to 0} g(x) = g(0)$. Because $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{1-\cos x}{2x} = \frac{1}{2}\cdot 0 = 0$, we require $a = 0$ in order for g to be continuous.

3.5.86

a. The unit circle consists of 360 degrees and 2π radians, so each degree corresponds to $\frac{2\pi}{360} = \frac{\pi}{180}$ radians.

b. $\lim_{x\to 0}\frac{s(x)}{x} = \lim_{x\to 0}\frac{\sin(\pi x/180)}{x} = \frac{\pi}{180}\lim_{x\to 0}\frac{\sin(\pi x/180)}{\pi x/180} = \frac{\pi}{180}\cdot 1 = \frac{\pi}{180}$.

3.5.87

a. $\frac{d}{dx}\sin^2 x = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$.

b. $\frac{d}{dx}\sin^3 x = \frac{d}{dx}(\sin^2 x)(\sin x) = (2\sin x\cos x)\sin x + \sin^2 x\cdot\cos x = 3\sin^2 x\cos x$.

c. $\frac{d}{dx}\sin^4 x = \frac{d}{dx}(\sin^3 x)(\sin x) = (3\sin^2 x\cos x)(\sin x) + (\sin^3 x)(\cos x) = 4\sin^3 x\cos x$.

d. We guess that $\dfrac{d}{dx}\sin^n x = n\sin^{n-1} x\cos x$.

We have already seen that the claim is valid for $n = 2$. Suppose our guess is valid for a given positive integer n. Then

$$\frac{d}{dx}\sin^{n+1} x = \frac{d}{dx}(\sin^n x)(\sin x) = (n\sin^{n-1} x\cos x)(\sin x) + \sin^n x\cos x = (n+1)\sin^n x\cos x.$$

Thus by induction, the result holds for all n.

3.5.88 Consider the statement $\dfrac{d^{2n}}{dx^{2n}}(\sin x) = (-1)^n \sin x$. This statement is valid for $n = 1$ because

$$\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\cos x = -\sin x.$$

Now suppose the statement is valid for some positive integer n. Then

$$\frac{d^{2n+2}}{dx^{2n+2}}\sin x = \frac{d^2}{dx^2}\left(\frac{d^{2n}}{dx^{2n}}\sin x\right) = \frac{d^2}{dx^2}\left((-1)^n\sin x\right) = (-1)^n\cdot(-1)\sin x = (-1)^{n+1}\sin x,$$

which completes the proof.

Similarly, consider the statement $\dfrac{d^{2n}}{dx^{2n}}(\cos x) = (-1)^n \cos x$ This statement is valid for $n = 1$ because

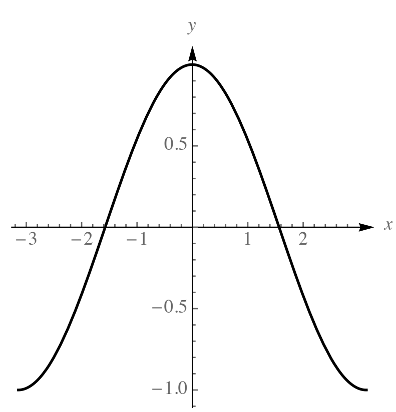
$$\frac{d^2}{dx^2}\cos x = \frac{d}{dx}(-\sin x) = -\cos x.$$

Now suppose the statement is valid for some positive integer n. Then

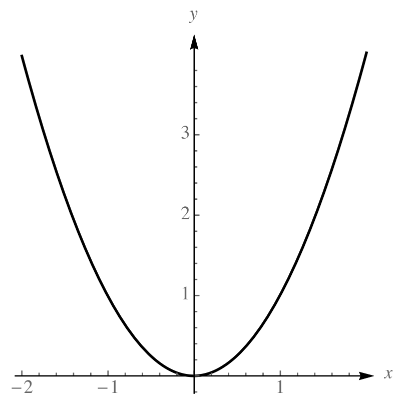
$$\frac{d^{2n+2}}{dx^{2n+2}}\cos x = \frac{d^2}{dx^2}\left(\frac{d^{2n}}{dx^{2n}}\cos x\right) = \frac{d^2}{dx^2}\left((-1)^n\cos x\right) = (-1)^n\cdot(-1)\cos x = (-1)^{n+1}\cos x,$$

which completes the proof.

3.5.89 Because D is a difference quotient, and because $h = 0.01$ is small, D is a good approximation to f'. Therefore, the graph of D is nearly indistinguishable from the graph of $f'(x) = \cos x$.



3.5.90 Because D is a difference quotient, and because $h = 0.01$ is small, D is a good approximation to f'. Therefore, the graph of D is nearly indistinguishable from the graph of $f'(x) = x^2$.



3.6 Derivatives as Rates of Change

3.6.1 The average rate of change is $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, whereas the instantaneous rate of change is the limit of this quotient as $\Delta x \to 0$.

3.6.2 If $\dfrac{dy}{dx}$ is large, then small changes in x will result in relatively large changes in the value of y.

3.6.3 If $\dfrac{dy}{dx}$ is small, then small changes in x will result in relatively small changes in the value of y.

3.6.4

a. It is negative – it is equal to the force of gravity which is -32 ft/sec^2.

b. At its highest point it is not moving – so its velocity is 0.

3.6.5 At 15 weeks, the puppy grows at a rate of 1.75 lb/week.

3.6.6 The speed of an object is the absolute value of its velocity. Thus, velocity encompasses the direction that the object is moving, while speed does not (it is always positive).

3.6.7 Acceleration is the instantaneous rate of change of the velocity; that is, if $s(t)$ is the position of an object at time t, then $s''(t) = \dfrac{d}{dt}(v(t)) = a(t)$ is the acceleration of the object at time t.

3.6.8 If the object is moving in the positive direction, the velocity will decrease. If it is moving in the negative direction, the velocity will increase.

3.6.9 $V'(T) = 0.6$. The speed of sound increases by 0.6 m/s for each increase of 1° in temperature.

3.6.10 $\dfrac{dV}{dt}$ represents the rate in ft^3/hr at which the pool is being filled.

3.6.11

a. $v_{\text{avg}} = \dfrac{f(0.75)-f(0)}{0.75} = \dfrac{30-0}{0.75} = 40$ mph.

b. $v_{\text{avg}} = \dfrac{f(0.75)-f(0.25)}{0.75-0.25} = \dfrac{30-10}{0.5} = 40$ mph.

This is a pretty good estimate, since the graph is nearly linear over that time interval.

c. $v_{\text{avg}} = \dfrac{f(2.25) - f(1.75)}{2.25 - 1.75} = \dfrac{-14 - 16}{0.5} = -60$ mph.

At 11 a.m. the velocity is $v(2) \approx -60$ mph. The car is moving south with a speed of approximately 60 mph.

d. From 9 a.m. until about 10:08 a.m., the car moves north, away from the station. Then it moves south, passing the station at approximately 11:02 a.m., and continues south until about 11:40 a.m. Then the car drives north until 12:00 noon stopping south of the station.

3.6.12

a. $v_{\text{avg}} = \dfrac{s(1.5) - s(0)}{1.5 - 0} = \dfrac{600 - 0}{1.5} = 400$ mph.

b. $v_{\text{avg}} = \dfrac{s(8.5) - s(7.5)}{8.5 - 7.5} = \dfrac{0 - 300}{1} = -300$ mph.

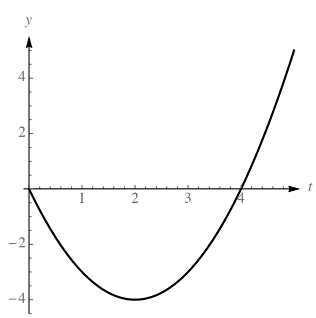
c. The velocity is zero from about 9 a.m. until 11:10 a.m. when the plane is at the gate in Minneapolis.

d. $v(6) \approx \dfrac{800 - 1400}{1} = -600$ mph. The velocity is negative as the plane returns to Seattle.

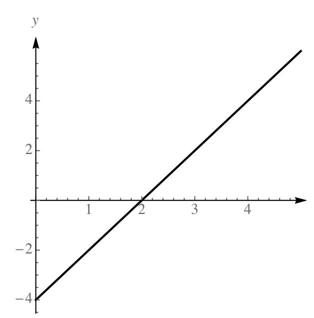
3.6.13 Each of the first 200 stoves cost on average \$70 to produce, while the 201st stove costs \$65 to produce.

3.6.14 If $D(p)$ is decreasing, then $\dfrac{dD}{dp}$ is negative. Both p and D are positive, so $E(p) = \dfrac{dD}{dp}\dfrac{p}{D}$ is negative.

3.6.15

a. 

$f(t) = t^2 - 4t$

b. 

$f'(t) = 2t - 4$

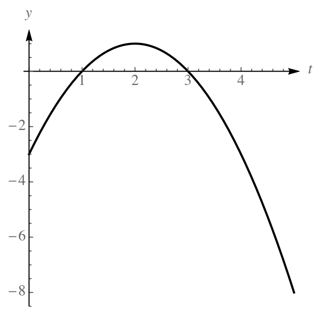
b. $f'(t) = 0$ when $t = 2$ – that is when the object is stationary. For $0 \le t < 2$ we have $f'(t) < 0$ so the object is moving to the left. For $2 < t \le 5$ we have $f'(t) > 0$ so the object is moving to the right.

c. $f'(1) = -2$ ft/sec and $f''(t) = 2\,\text{ft/sec}^2$, so in particular, $f''(1) = 2\,\text{ft/sec}^2$.

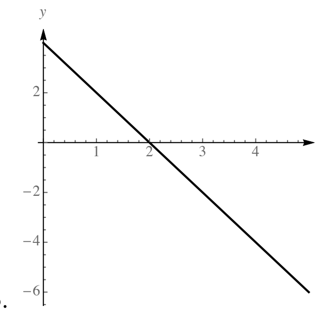
d. $f'(t) = 0$ when $t = 2$ and $f''(2) = 2\,\text{ft/sec}^2$.

e. On the interval $(2, 5]$ the velocity and acceleration are both positive, so the object's speed is increasing.

3.6.16

a. 

$f(t) = -t^2 + 4t - 3$

b. 

$f'(t) = -2t + 4$

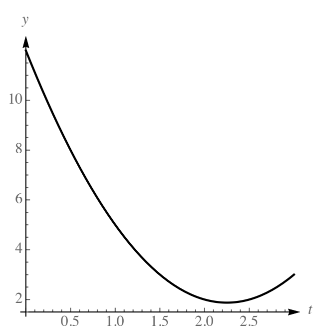
b. $f'(t) = 0$ when $t = 2$ – that is when the object is stationary. For $0 \le t < 2$ we have $f'(t) > 0$ so the object is moving to the right. For $2 < t \le 5$ we have $f'(t) < 0$ so the object is moving to the left.

c. $f'(1) = 2$ ft/sec and $f''(t) = -2\,\text{ft/sec}^2$, so in particular, $f''(1) = -2\,\text{ft/sec}^2$.

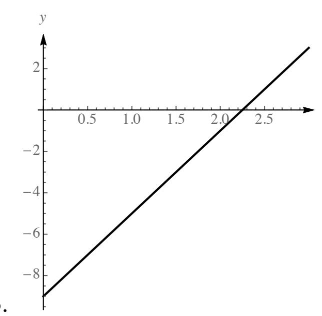
d. $f'(t) = 0$ when $t = 2$ and $f''(2) = -2\,\text{ft/sec}^2$.

e. On the interval $(2, 5]$ the velocity and the acceleration are both negative, so the object's speed is increasing.

3.6.17

a. 

$f(t) = 2t^2 - 9t + 12$

b. 

$f'(t) = 4t - 9$

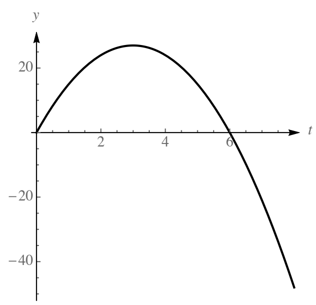
b. $f'(t) = 0$ when $t = 9/4$ – that is when the object is stationary. For $0 \le t < 9/4$ we have $f'(t) < 0$ so the object is moving to the left. For $9/4 < t \le 3$ we have $f'(t) > 0$ so the object is moving to the right.

c. $f'(1) = -5$ ft/sec and $f''(t) = 4\,\text{ft/sec}^2$, so in particular, $f''(1) = 4\,\text{ft/sec}^2$.

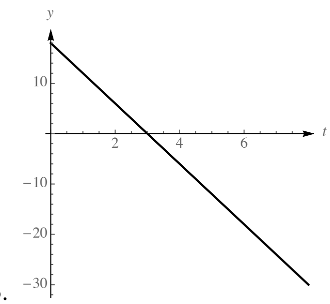
d. $f'(t) = 0$ when $t = 9/4$ and $f''(9/4) = 4\,\text{ft/sec}^2$.

e. On the interval $(9/4, 3]$ both the velocity and acceleration are positive, so the object's speed is increasing.

3.6.18

a. 

$f(t) = 18t - 3t^2$

b. 

$f'(t) = 18 - 6t$

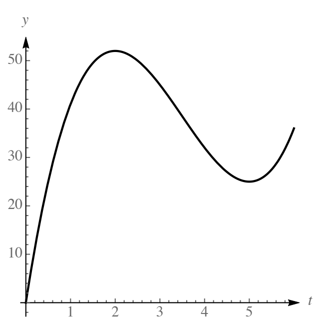
b. $f'(t) = 0$ when $t = 3$ – that is when the object is stationary. For $0 \le t < 3$ we have $f'(t) > 0$ so the object is moving to the right. For $3 < t \le 8$ we have $f'(t) < 0$ so the object is moving to the left.

c. $f'(1) = 12$ ft/sec and $f''(t) = -6\,\text{ft/sec}^2$, so in particular, $f''(1) = -6\,\text{ft/sec}^2$.

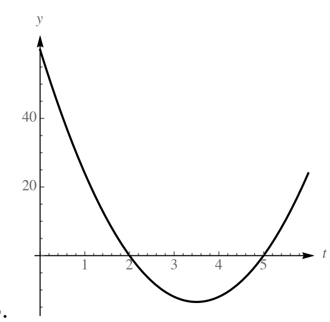
d. $f'(t) = 0$ when $t = 3$ and $f''(3) = -6\,\text{ft/sec}^2$.

e. On the interval $(3, 8]$ both the velocity and acceleration are negative, so the object's speed is increasing on that interval.

3.6.19

a. 

$f(t) = 2t^3 - 21t^2 + 60t$

b. 

$f'(t) = 6t^2 - 42t + 60$

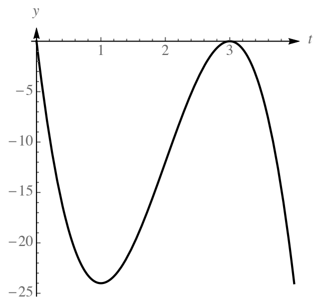
b. $f'(t) = 0$ when $6(t-2)(t-5) = 0$, which is at $t = 2$ and $t = 5$ – that is when the object is stationary. For $0 \le t < 2$ we have $f'(t) > 0$ so the object is moving to the right. For $2 < t < 5$ we have $f'(t) < 0$ so the object is moving to the left. For $5 < t \le 8$ we have $f'(t) > 0$, so the object is moving to the right again.

c. $f'(1) = 24$ ft/sec and $f''(t) = 12t - 42$, so $f''(1) = -30\,\text{ft/sec}^2$.

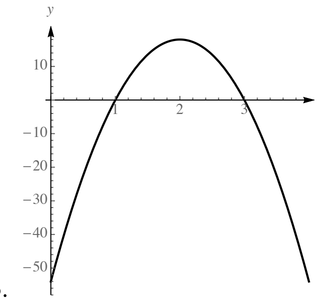
d. $f'(t) = 0$ when $t = 2$ and $t = 5$. We have $f''(2) = -18\,\text{ft/sec}^2$ and $f''(5) = 18\,\text{ft/sec}^2$.

e. $f''(t) = 12t - 42$ is positive for $t > \frac{42}{12} = \frac{7}{2}$ and negative for $t < \frac{7}{2}$. So f' and f'' are both positive on $(5, 6]$ and are both negative on $(2, 3.5)$, so that is where the object is speeding up.

3.6.20

a. 

$f(t) = -6t^3 + 36t^2 - 54t$

b. 

$f'(t) = -18t^2 + 72t - 54$

b. $f'(t) = 0$ when $-18(t-3)(t-1) = 0$, which is at $t = 1$ and $t = 3$ – that is when the object is stationary. For $0 \le t < 1$ we have $f'(t) < 0$ so the object is moving to the left. For $1 < t < 3$ we have $f'(t) > 0$ so the object is moving to the right. For $3 < t \le 4$ we have $f'(t) < 0$, so the object is moving to the left again.

c. $f'(1) = 0$ ft/sec and $f''(t) = -36t + 72$, so $f''(1) = 36\,\text{ft/sec}^2$.

d. $f'(t) = 0$ when $t = 1$ and $t = 3$. We have $f''(1) = 36\,\text{ft/sec}^2$ and $f''(3) = -36\,\text{ft/sec}^2$.

e. $f''(t) = -36t + 72$ is positive on $[0, 2)$, so the object's velocity and acceleration are both negative on $(1, 2)$ and they are both negative on $(3, 4]$, so that is where the object is speeding up.

3.6.21 To find out when it hits the water, we set $s(t) = -16t^2 + 64$ equal to 0. This gives $-16t^2 = -64$, or $t^2 = 4$, so $t = 2$ is the time the stone hits the water. The velocity at time t is $v(t) = s'(t) = -32t$, so $v(2) = -64$, so the speed is $|-64| = 64$ ft/sec.

3.6.22 To determine when it hits the ground, we set $s(t) = -6t^2 + 54$ equal to 0. This gives $-6t^2 = -54$ or $t^2 = 9$, so $t = 3$ is the time the stone hits the ground. The velocity at time t is $v(t) = s'(t) = -12t$, so $v(3) = -36$, so the speed is $|-36| = 36$ ft/sec.

3.6.23

a. $v(t) = s'(t) = -32t + 32$.

b. $v(t) = 0$ when $-32t + 32 = 0$, so at $t = 1$ s.

c. $s(1) = -16 + 32 + 48 = 64$ ft.

d. $s(t) = 0$ when $-16t^2 + 32t + 48 = 0$, or $-16(t + 1)(t - 3) = 0$, so at $t = 3$ s.

e. $v(3) = -96 + 32 = -64$ ft/s.

f. Note that the acceleration is always negative, so the stone will be speeding up when its velocity is also negative, which occurs on the interval $(1, 3)$. This corresponds to the downward portion of the stone's trip.

3.6.24

a. $v(t) = s'(t) = -9.8t + 19.6$.

b. $v(t) = 0$ when $-9.8t + 19.6 = 0$, so at $t = 2$ s.

c. $s(2) = -4.9 \cdot 4 + 19.6(2) + 24.5 = 44.1$ m.

d. $s(t) = 0$ when $-4.9t^2 + 19.6t + 24.5 = 0$, or $t^2 - 4t - 5 = (t + 1)(t - 5) = 0$, so at $t = 5$ s.

e. $v(5) = -9.8(5) + 19.6 = -29.4$ ft/s.

f. Note that the acceleration is always negative, so the stone will be speeding up when its velocity is also negative, which occurs on the interval $(2, 5)$. This corresponds to the downward portion of the stone's trip.

3.6.25

a. $v(t) = s'(t) = -32t + 64$ ft/sec.

b. $v(t) = 0$ when $-32t + 64 = 0$, which is at $t = 2$.

c. $s(2) = -16 \cdot 4 + 64 \cdot 2 + 32 = 96$ feet.

d. $s(t) = 0$ when $-16t^2 + 64t + 32 = 0$. Using the quadratic formula we see that this occurs when $t = 2 + \sqrt{6} \approx 4.45$ seconds.

e. The velocity when the stone hits the ground is $v(2+\sqrt{6}) = -32(2+\sqrt{6}) + 64 = -32\sqrt{6} \approx -78.38$ feet per second.

f. The acceleration due to gravity is always negative, so the object is speeding up when its velocity is negative; that is, on its downward journey during the interval $(2, 2+\sqrt{6})$.

3.6.26

a. The ball is at its maximum height when $v(t) = -32t + v_0 = 0$, so $t = \dfrac{v_0}{32}$. Its height at this time is $s(v_0/32) = -16\left(\dfrac{v_0}{32}\right)^2 + v_0\left(\dfrac{v_0}{32}\right) = \dfrac{v_0^2}{64}$.

b. The ball hits the ground when $s(t) = -16t^2 + v_0 t = 0$, so at $t = \dfrac{v_0}{16}$. Its velocity at this time is $v\left(\dfrac{v_0}{16}\right) = -32\left(\dfrac{v_0}{16}\right) + v_0 = -v_0$.

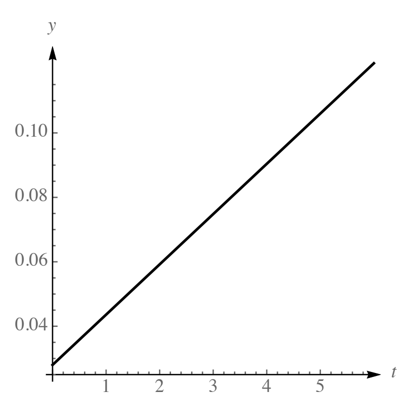
3.6.27 Because the maximum height is $\dfrac{v_0^2}{64}$ (see the previous problem), we need $\dfrac{v_0^2}{64} = 128$, so $v_0 = \sqrt{128 \cdot 64} \approx 90.5$ ft/s.

3.6.28

a. $\dfrac{p(6) - p(0)}{6 - 0} = 0.0748$ million people/year or 74,800 people/year.

b. $p'(1) = 0.0436$ million people/year or 43,60 people/year. $p'(5) = 0.106$ million people/year or 106,000 people/year.

c. The rate of growth in Washington is increasing over this time period.



3.6.29

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{1000}{x} + .1$. The marginal cost function is given by $M(x) = C'(x) = .1$.

b. At $a = 2000$ we have $\overline{C}(2000) = \dfrac{1000}{2000} + .1 = .6$, and $M(2000) = .1$.

c. The average cost per item when producing 2000 items is \$0.60. The cost of producing the next item is \$0.10.

3.6.30

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{500}{x} + .02$. The marginal cost function is given by $M(x) = C'(x) = .02$.

b. At $a = 1000$ we have $\overline{C}(1000) = \dfrac{500}{1000} + .02 = .52$, and $M(1000) = .02$.

c. The average cost per item when producing 1000 items is \$0.52. The cost of producing the next item is \$0.02.

3.6.31

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{100}{x} + 40 - 0.01x$. The marginal cost function is given by $M(x) = C'(x) = 40 - 0.02x$.

b. At $a = 1000$ we have $\overline{C}(1000) = \dfrac{100}{1000} + 40 - (.01)(1000) = 30.1$, and $M(1000) = 20$.

c. The average cost per item when producing 1000 items is \$30.10. The cost of producing the next item is \$20.00.

3.6.32

a. The average cost function is given by $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{800}{x} + 100 - 0.04x$. The marginal cost function is given by $M(x) = C'(x) = 100 - 0.08x$.

b. At $a = 500$ we have $\overline{C}(500) = \dfrac{800}{500} + 100 - (.04)(500) = 81.6$, and $M(500) = 60$.

c. The average cost per item when producing 500 items is \$81.60. The cost of producing the next item is \$60.00.

3.6.33

a. $D(10) = 40 - 20 = 20$ DVDs per day.

b. Demand is zero when $D(p) = 40 - 2p = 0$, which occurs for $p = 20$ dollars.

c. The elasticity is $E(p) = \dfrac{dD}{dp}\dfrac{p}{D} = -2\left(\dfrac{p}{40-2p}\right) = \dfrac{p}{p-20}$.

d. This quantity satisfies $-1 < E(p) < 0$ when $-1 < \frac{p}{p-20} < 0$ which occurs when $p < 20 - p$, or $p < 10$. So for prices in the interval $(0, 10)$ the demand is inelastic, while for prices in the interval $(10, 20)$ the demand is elastic.

e. If the price goes up from 10 to 10.25, that is a $\frac{.25}{10} = .025 = 2.5\%$ increase in price.

f. If the price goes up from 10 to 10.25, the demand goes from $D(10) = 40 - 20 = 20$ to $D(10.25) = 40 - 20.5 = 19.5$, which is a $\frac{.5}{20} = 2.5\%$ decrease.

3.6.34

a. The domain of the demand function is $(40, \infty)$.

b. $D(60) = \dfrac{1800}{60-40} = \dfrac{1800}{20} = 90$.

c. $E(p) = \dfrac{dD}{dp}\dfrac{p}{D} = \dfrac{-1800}{(p-40)^2} \cdot \dfrac{p(p-40)}{1800} = \dfrac{p}{40-p}$.

d. Note that $E(p) = \dfrac{p}{40-p}$ is always less than -1 on the interval $(40, \infty)$. (Note that the equation $\dfrac{p}{40-p} = -1$ has no solutions). So the demand is always elastic.

e. $E(60) = \dfrac{60}{40-60} = \dfrac{60}{-20} = -3$. So a $\frac{2}{60} = 3.\overline{33}\%$ change in price will decrease the demand by about $3 \cdot 3.\overline{33}\% = 10\%$.

3.6.35

a. False. For example, when a ball is thrown up in the air near the surface of the earth, its acceleration is constant (due to gravity) but its velocity changes during its trip.

b. True. If the rate of change of velocity is zero, then velocity must be constant.

c. False. If the velocity is constant over an interval, then the average velocity is equal to the instantaneous velocity over the interval.

d. True. For example, a ball dropped from a tower has negative acceleration and increasing speed as it falls toward the earth.

3.6.36 The velocity is $v(t) = s'(t) = -1.6t$. The feather strikes the surface of the moon when $s(t) = 40 - 0.8t^2 = 0$. This occurs when $t = \sqrt{50} \approx 7.07$ seconds. The velocity at this time is $v(\sqrt{50}) = -1.6\sqrt{50} \approx -11.31$ meters per second, and $a(\sqrt{50}) = -1.6$ meters per second^2.

3.6.37 In each case, the stone reaches its maximum height when its velocity is zero.

On Mars, this occurs when $v(t) = s'(t) = 96 - 12t = 0$, or when $t = 8$ seconds. So the maximum height on Mars is $s(8) = 384$ feet.

On Earth, this occurs when $v(t) = s'(t) = 96 - 32t = 0$, or when $t = 3$ seconds. So the maximum height on Earth is $s(3) = 144$ feet.

The stone will travel $384 - 144 = 240$ feet higher on Mars.

3.6.38

a. Both stones reach their highest points when the derivative of their position functions are 0. Note, however that $f'(t) = -32t + 48 = g'(t)$. Thus both stones reach their maximum height at $t = \dfrac{48}{32} = \dfrac{3}{2}$.

b. The height of the stone thrown from the bridge at $t = 1.5$ seconds is $f(1.5) = 68$ feet, while the other stone reaches $g(1.5) = 36$ feet, so the one thrown from the bridge goes 32 feet higher.

c. The stone from ground level hits the ground when $g(t) = 0$, which occurs when $t = 3$. The velocity at this time is $g'(3) = -48$ feet per second. The stone thrown from the bridge hits the ground when $f(t) = 0$, which occurs when $-16t^2 + 48t + 32 = 0$, or $t^2 - 3t - 2 = 0$, or $t = \dfrac{3 + \sqrt{9 - (4)(-2)}}{2} = \dfrac{3 + \sqrt{17}}{2} \approx 3.56$. At that time, the velocity is approximately -65.97 feet per second.

3.6.39 The first stone reaches its maximum height when $f'(t) = -32t + 32 = 0$, so after 1 second, and its maximum height is therefore $f(1) = -16 + 32 + 48 = 64$ feet.

The second stone reaches its maximum height when $g'(t) = -32t + v_0 = 0$, so when $t = \dfrac{v_0}{32}$. Its height at that time is $g(v_0/32) = -16(v_0/32)^2 + (v_0^2/32) = \dfrac{v_0^2}{64}$. This is equal to 64 when $v_0 = 64$ feet per second.

3.6.40

a. The slope of the curve (which is the velocity) increases until about 5:30 p.m., so the car is speeding up over that time interval. From 5:30 p.m. until about 6:20 p.m. the velocity is decreasing. After that it is speeding up until 7:00 p.m.

b. The slope is the largest at about 5:30 p.m. and smallest at about 6:20 p.m.

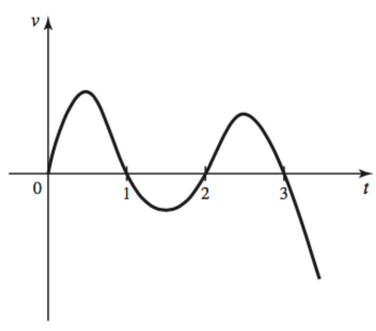
c. The maximum velocity is approximately 40 mph and the minimum is about 5 mph. These are estimates based on visually computing slopes of tangent lines. Your mileage may vary.

3.6.41

a. The velocity is zero at $t = 1, 2, \text{and } 3$.

b. The object is moving in the positive direction when the slope of s is positive, so from $t = 0$ to $t = 1$, and from $t = 2$ to $t = 3$. It is moving in the negative direction from $t = 1$ to $t = 2$, and for $t > 3$.

c.



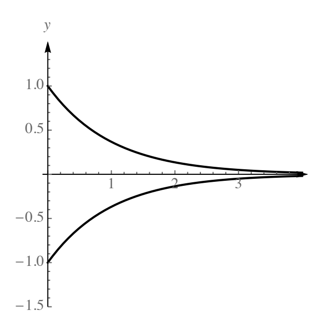
d. The speed is increasing on $(0, 1/2)$ as the velocity is positive and the acceleration is positive there. On $(1/2, 1)$ the speed is decreasing as the velocity is positive but the acceleration is negative. On $(1, 3/2)$ the speed is increasing as the velocity is negative and the acceleration is negative, but on $(3/2, 2)$ the speed is decreasing as the velocity is negative but the acceleration is positive. On approximately $(2, 2.6)$ the speed is increasing as the velocity is positive and the acceleration is positive, but on about $(2.6, 3)$ the velocity is positive but the acceleration is negative, so the speed is decreasing. On $(3, \infty)$ both the velocity and the acceleration are negative, so the object is speeding up.

3.6.42

a. $\dfrac{dL}{dt}$ represents the rate of change of the length of the species. The derivative is decreasing over time.

b. Over time, the species is getting bigger, but the rate of change is approaching zero.

c.



3.6.43

a. $P(x) = xp(x) - C(x) = 100x + 0.02x^2 - 50x - 100 = 0.02x^2 + 50x - 100.$

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = 0.02x + 50 - \dfrac{100}{x}$. The marginal profit is $P'(x) = .04x + 50$.

c. $\overline{P}(500) = 59.8$. $P'(500) = 70$.

d. The average profit for the first 500 items sold is \$59.80, while the profit on the 501st item is \$70.00.

3.6.44

a. $P(x) = xp(x) - C(x) = 100x - 0.1x^2 - (-0.02x^2 + 50x + 100) = 50x - 0.08x^2 - 100.$

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = -0.08x + 50 - \dfrac{100}{x}$. The marginal profit is $P'(x) = -0.16x + 50$.

c. $\overline{P}(500) = 9.8$. $P'(500) = -30$.

d. The average profit for the first 500 items sold is \$9.80, while the profit on the 501st item is −\$30.00.

3.6.45

a. $P(x) = xp(x) - C(x) = 100x + 0.04x^2 - 800.$

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = .04x + 100 - \dfrac{800}{x}$. The marginal profit is $P'(x) = .08x + 100$.

c. $\overline{P}(1000) = 139.2$. $P'(1000) = 180$.

d. The average profit for the first 1000 items sold is \$139.20, while the profit on the 1001st item is \$180.00.

3.6.46

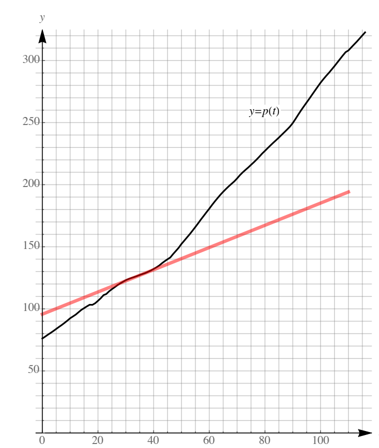
a. $P(x) = xp(x) - C(x) = 100x - 0.06x^2 - 800.$

b. The average profit is $\overline{P}(x) = \dfrac{P(x)}{x} = -0.06x + 100 - \dfrac{800}{x}$. The marginal profit is $P'(x) = -0.12x + 100$.

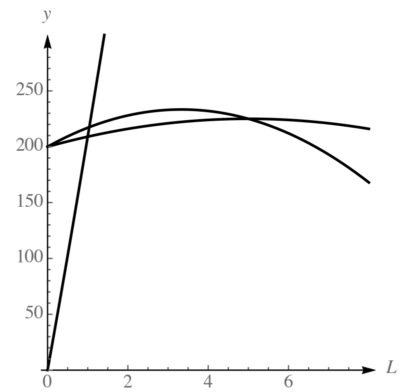
c. $\overline{P}(1000) = 39.2$. $P'(1000) = -20$.

d. The average profit for the first 1000 items sold is \$39.20, while the profit on the 1001st item is −\$20.00.

3.6.47 Since 1925, the population appears to have grown most slowly in about 1935. Using a straight edge and drawing the (approximate) tangent line, we obtain the following figure. The tangent line appears to pass through the points $(5, 100)$ and $(50, 140)$. Therefore the slope of the tangent line is approximately $\dfrac{(140 - 100)}{(50 - 5)} = 0.89$. So the population was growing at about 890,000 people per year in 1935 (answers will vary — between about 700,000 and 1,000,000 people per year are acceptable).

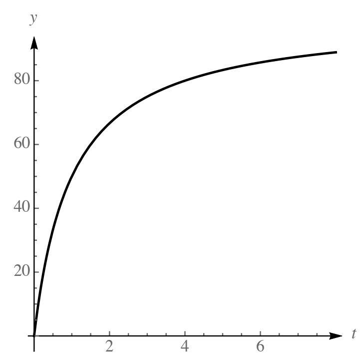


3.6.48

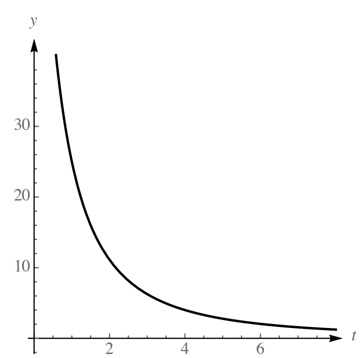
a. 

b. The peak of $A(L) = \dfrac{P(L)}{L} = -L^2 + 10L + 200$ occurs when the slope is zero. Note that $P'(L) = -3L^2 + 20L + 200$.
We seek L_0 so that $\dfrac{dA}{dL}(L_0) = 0$, which occurs when $\dfrac{L_0 \cdot P'(L_0) - P(L_0)}{L_0^2} = 0$, or when $P'(L_0) = \dfrac{P(L_0)}{L_0} = A(L_0)$. Thus if the peak of A occurs at L_0, we have $M(L_0) = A(L_0)$.

3.6.49

a. 

b. $v(t) = s'(t) = \dfrac{(t+1)100 - 100t \cdot 1}{(t+1)^2} = \dfrac{100}{(t+1)^2}$.

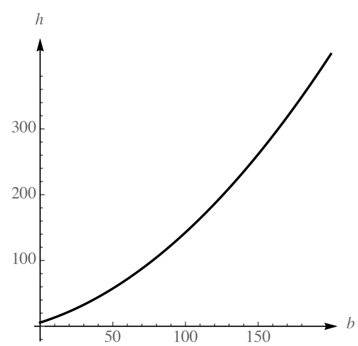
c. 

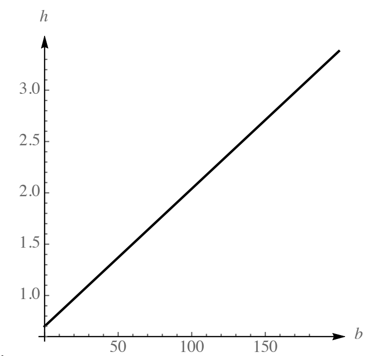
The velocity of the marble is decreasing.

d. $s(t) = 80$ when $\dfrac{100t}{t+1} = 80$, or $100t = 80t + 80$, which occurs when $t = 4$ seconds.

e. $v(t) = 50$ when $\dfrac{100}{(t+1)^2} = 50$, or $(t+1)^2 = 2$. This occurs for $t = \sqrt{2} - 1 \approx 0.414$ seconds.

3.6.50

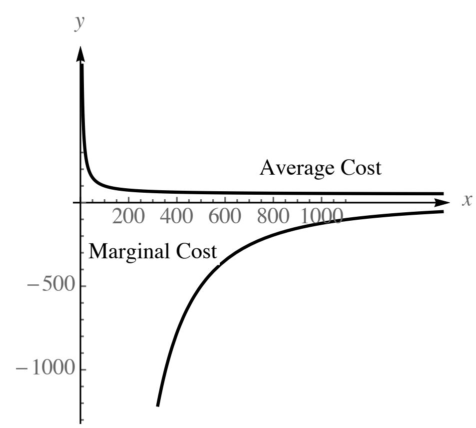
a. 

b. 

The function $\dfrac{dh}{db}$ shows the rate of increase in height (in meters) per cm increase in the base diameter of the tree.

3.6.51

a. The average cost function is $\overline{C}(x) = \dfrac{C(x)}{25000} = 50 + \dfrac{5000}{x} + 0.00006x$. The marginal cost function is $C'(x) = -\dfrac{125000000}{x^2} + 1.5$.
The average cost decreases to about 50 per unit as the batch size increases, while the marginal cost is negative but increases.

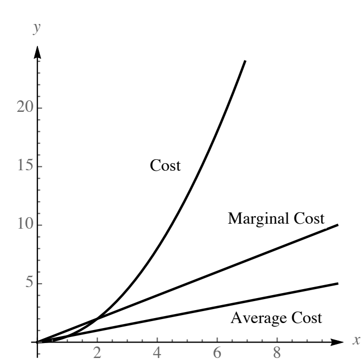


b. $\overline{C}(5000) = 51.3$, $C'(5000) = -3.5$.

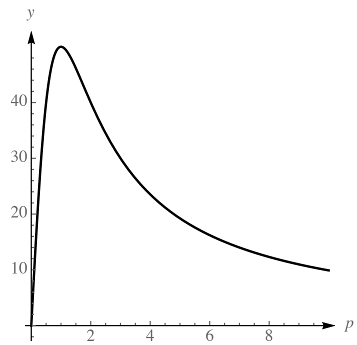
c. If the batch size is 5000, then the average cost of producing 25000 items is \$51.30 per item. If the batch size is increased from 5000 to 5001, then the cost of producing 25000 items would decrease by about \$3.50.

3.6.52

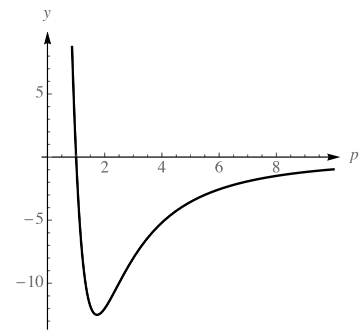
All these functions are increasing, so the cost per item increases as more items are produced. And we will have less revenue unless we charge more.



3.6.53

a. 

$$R(p) = \frac{100p}{p^2 + 1}$$

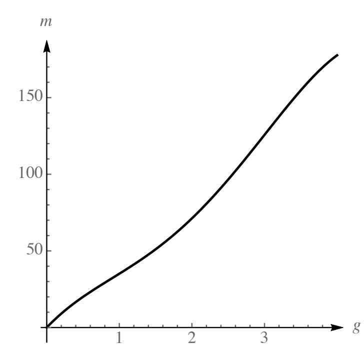
b. 

$$R'(p) = \frac{100(1 - p^2)}{(p^2 + 1)^2}$$

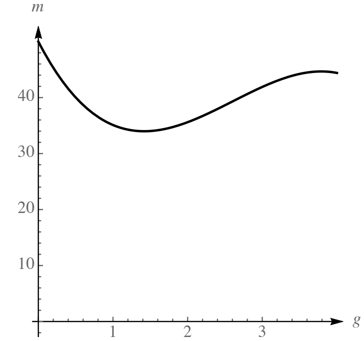
c. $R'(p)$ is zero at $p = 1$, and the maximum of $R(p)$ occurs at this same value of p, so that is the price to charge in order to maximize revenue. The revenue at this price is \$50.00.

3.6.54

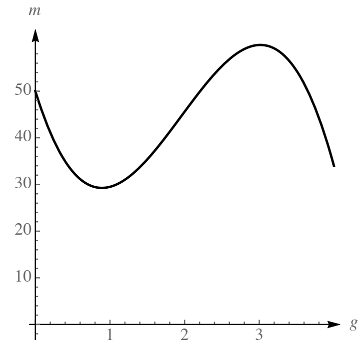
a. The number of miles increases with the number of gallons of gasoline.



b. The gas mileage is $m(g)/g$. The number of miles per gallon decreases during the first 1.5 gallons or so, then increases until it peaks again just short of 4 gallons.

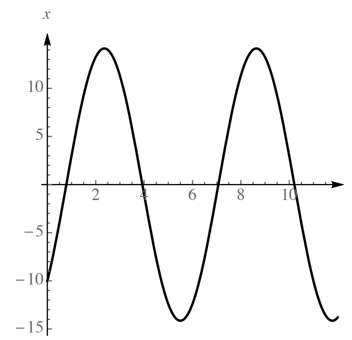


c. $\frac{dm}{dg}$ represents the instantaneous rate of change of the number of miles driven per unit of gasoline consumed.



3.6.55

a. The mass oscillates about the equilibrium point.

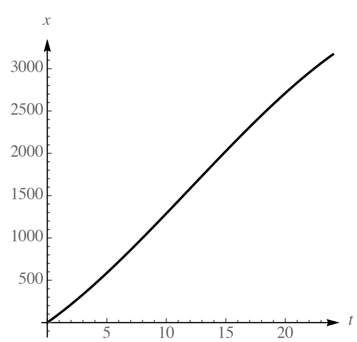


b. $\frac{dx}{dt} = 10\cos t + 10\sin t$ is the velocity of the mass at time t.

c. $\frac{dx}{dt} = 0$ when $\sin t = -\cos t$, which occurs when $t = \frac{4n+3}{4} \cdot \pi$ where n is any positive integer.

d. The model is unrealistic as it ignores the effects of friction and gravity. In reality, the amplitude would decrease as the mass oscillates.

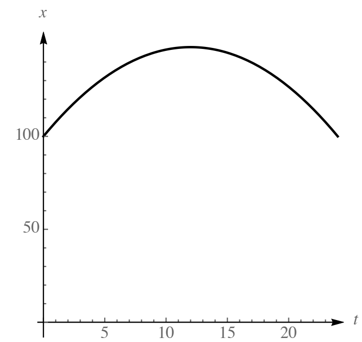
3.6.56

a. 

The energy function.

b. $P(t) = E'(t) = 100 + 8t - \frac{t^2}{3}$, $0 \le t \le 24$.

c. The power increases from midnight to noon, then decreases again until midnight. The units are kilowatts.



The power function.

3.6.57

a. Juan starts out faster, but slows toward the end, while Jean starts slower but increases her speed toward the end.

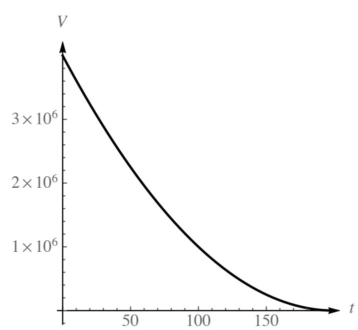
b. Because both start and finish at the same time, they finish with the same average angular velocity.

c. It is a tie.

d. Jean's velocity is given by $\theta'(t) = \dfrac{\pi t}{4}$. At $t = 2$, $\theta'(2) = \dfrac{\pi}{2}$ radians per minute. Her velocity is greatest at $t = 4$.

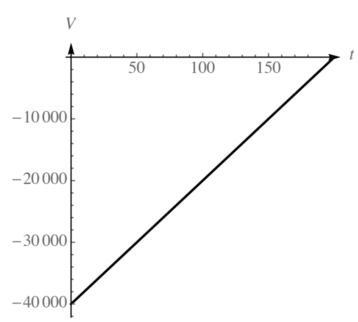
e. Juan's velocity is given by $\phi'(t) = \pi - \dfrac{\pi t}{4}$. At $t = 2$, $\phi'(2) = \dfrac{\pi}{2}$ radians per minute as well. His velocity is greatest at $t = 0$.

3.6.58

a. 

At the beginning the volume is 4,000,000 cubic meters.

b. The tank is empty when $V(t) = 100(200 - t)^2 = 0$, which occurs when $t = 200$.

c. 

Because $V(t)$ can be written as $V(t) = 4,000,000 - 40,000t + 100t^2$, the flow rate is $V'(t) = -40,000 + 200t$ cubic meters per minute.

d. The flow rate is largest (in absolute value) when $t = 0$ and smallest when $t = 200$.

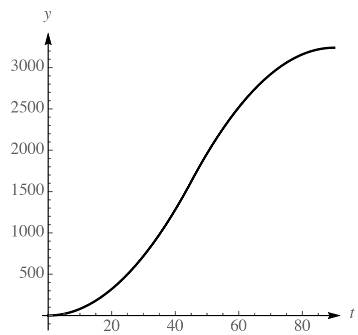
3.6.59

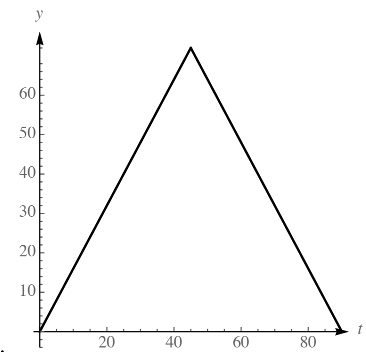
a. $v(t) = y'(t) = -15e^{-t}\cos t - 15e^{-t}\sin t$, so $v(1) \approx -7.625$ meters per second, and $v(3) \approx .63$ meters per second.

b. She is moving down for approximately 2.4 seconds, and then up until about 5.5 seconds, and then down again until about 8.6 seconds, and then up again.

c. The maximum velocity going up appears to be about about 0.65 meters per second.

3.6.60

a. 

b. 

$$V'(t) = \begin{cases} \dfrac{8}{5}t & \text{for } 0 \le t \le 45, \\ -\dfrac{8}{5}t + 144 & \text{for } 45 \le t \le 90. \end{cases}$$ This is in cubic feet per day.

c. The flow increases for the first 45 days, then decreases. The flow rate is at a maximum at 45 days.

3.6.61

a. $T'(t) = 160 - 80x$, so $T'(1) = 80$, so the heat flux at 1 is -80. At $x = 3$ we have $T'(3) = -80$, so the heat flux at 3 is 80.

b. The heat flux $-T'(x)$ is negative for $0 \le x < 2$ and positive for $2 < x \le 4$.

c. At any point other than the midpoint of the rod, heat flows toward the closest end of the rod, and "out the end."

3.7 The Chain Rule

3.7.1 If $y = f(x)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. Alternatively, we have $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$.

3.7.2 The inner function is $x^2 + 10$ and the outer function is u^{-5}, so with $y = f(u)$ and $u = g(x)$, we have $f(u) = u^{-5}$ and $g(x) = x^2 + 10$. Then $y = (x^2 + 10)^{-5}$.

3.7.3 The inner function is $u = x^3 + x + 1$ and the outer function is $y = u^4$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3(3x^2 + 1) = 4(x^3 + x + 1)^3(3x^2 + 1).$$

3.7.4 The inner function is $u = x^3 + 2x$ and the outer function is $y = e^u$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u(3x^2 + 2) = e^{x^3+2x}(3x^2 + 2).$$

3.7.5

a. $u = g(x) = \cos x$, $y = f(u) = u^3$. So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 3\cos^2 x \cdot (-\sin x) = -3\cos^2 x \sin x$.

b. $u = g(x) = x^3$, $y = f(u) = \cos u$. So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = -\sin x^3 \cdot 3x^2 = -3x^2 \sin x^3$.

3.7.6 We would need to know $f'(3)$. This is because $h'(1) = f'(g(1))g'(1) = f'(3) \cdot 5$, but we can't finish this calculation unless we know $f'(3)$.

3.7.7 The derivative of $f(g(x))$ equals f' evaluated at $g(x)$ multiplied by g' evaluated at x.

3.7.8 Let $g(x) = x^2 + 2x + 1$ and $f(x) = x^2$. Then $f(g(x)) = (x^2 + 2x + 1)^2$. We have $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x) = 2(g(x))(2x + 2) = 2(x^2 + 2x + 1)(2x + 2) = 4(x^2 + 2x + 1)(x + 1) = 4(x + 1)^3$.

3.7.9 Let $g(x) = 4x + 1$ and $f(x) = \sqrt{x}$ so that $f(g(x)) = \sqrt{4x + 1}$. Then $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x) = \dfrac{1}{2\sqrt{g(x)}} \cdot 4 = \dfrac{2}{\sqrt{4x + 1}}$.

3.7.10 Let $h(x) = x^2 + 1$, $g(u) = \cos u$, and $f(v) = v^4$. Then $f(g(h(x))) = f(g(x^2 + 1)) = f(\cos(x^2 + 1)) = (\cos(x^2 + 1))^4 = Q(x)$.

3.7.11 $h'(3) = f'(g(3))g'(3) = f'(4)g'(3) = 10 \cdot 5 = 50$. Note that the value of $f(4)$ isn't needed to calculate this.

3.7.12 $h'(4) = f'(g(4))g'(4) = f'(5)g'(4) = -\dfrac{1}{2} \cdot \dfrac{1}{4} = -\dfrac{1}{8}$.

3.7.13 $y' = e^{kx} \cdot k = ke^{kx}$.

3.7.14 $f'(x) = 15e^{3x} \cdot 3 = 45e^{3x}$.

3.7.15 With $u = 3x + 7$ and $y = u^{10}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{dx} \cdot \dfrac{du}{dx} = 10u^9 \cdot 3 = 30(3x + 7)^9$.

3.7.16 With $u = 5x^2 + 11x$ and $y = u^{4/3}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{4}{3} u^{1/3} \cdot (10x + 11) = \dfrac{4}{3} \sqrt[3]{5x^2 + 11x}(10x + 11)$.

3.7.17 With $u = \sin x$ and $y = u^5$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 5u^4 \cdot \cos x = 5\sin^4 x \cos x$.

3.7.18 With $u = x^5$ and $y = \sin u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u \cdot 5x^4 = 5x^4 \cos x^5$.

3.7.19 With $u = x^2 + 1$ and $y = \sqrt{u}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot (2x) = \dfrac{x}{\sqrt{x^2 + 1}}$.

3.7.20 With $u = 7x - 1$ and $y = \sqrt{u}$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{2\sqrt{u}} \cdot 7 = \dfrac{7}{2\sqrt{7x - 1}}$.

3.7.21 With $u = 4x^2 + 1$ and $y = e^u$, we have $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^u \cdot 8x = 8xe^{4x^2+1}$.

3.7.22 With $u = \sqrt{x}$ and $y = e^u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = e^u \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

3.7.23 With $u = 5x^2$ and $y = \tan u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \sec^2 u \cdot (10x) = 10x \sec^2 5x^2$.

3.7.24 With $u = x/4$ and $y = \sin u$ we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \cos u \cdot (1/4) = \dfrac{1}{4} \cdot \cos(x/4)$.

3.7.25

a. $h'(3) = f'(g(3))g'(3) = f'(1) \cdot 20 = 5 \cdot 20 = 100$.

b. $h'(2) = f'(g(2))g'(2) = f'(5) \cdot 10 = -10 \cdot 10 = -100$.

c. $p'(4) = g'(f(4))f'(4) = g'(1) \cdot (-8) = 2 \cdot (-8) = -16$.

d. $p'(2) = g'(f(2))f'(2) = g'(3) \cdot 2 = 20 \cdot 2 = 40$.

e. $h'(5) = f'(g(5))g'(5) = f'(2) \cdot 20 = 2 \cdot 20 = 40$.

3.7.26

a. $h'(1) = f'(g(1))g'(1) = f'(4) \cdot 9 = 7 \cdot 9 = 63$.

b. $h'(2) = f'(g(2))g'(2) = f'(1) \cdot 7 = (-6) \cdot 7 = -42$.

c. $h'(3) = f'(g(3))g'(3) = f'(5) \cdot 3 = 2 \cdot 3 = 6$.

d. $k'(3) = g'(g(3))g'(3) = g'(5) \cdot 3 = (-5) \cdot 3 = -15$.

e. $k'(1) = g'(g(1))g'(1) = g'(4) \cdot 9 = (-1) \cdot 9 = -9$.

f. $k'(5) = g'(g(5))g'(5) = g'(3) \cdot (-5) = 3 \cdot (-5) = -15$.

3.7.27 With $g(x) = 3x^2 + 7x$ and $f(u) = u^{10}$ we have $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 10(3x^2 + 7x)^9(6x + 7)$.

3.7.28 With $g(x) = x^2 + 2x + 7$ and $f(u) = u^8$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 8(x^2 + 2x + 7)^7(2x + 2) = 16(x^2 + 2x + 7)^7(x + 1).$$

3.7.29 With $g(x) = 10x + 1$ and $f(u) = \sqrt{u}$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{1}{2\sqrt{10x+1}} \cdot 10 = \frac{5}{\sqrt{10x+1}}.$$

3.7.30 With $g(x) = x^2 + 9$ and $f(u) = \sqrt[3]{u}$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{1}{3u^{2/3}} \cdot (2x) = \frac{2x}{3(x^2+9)^{2/3}}.$$

3.7.31 With $g(x) = 7x^3 + 1$ and $f(u) = 5u^{-3}$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = -15(7x^3 + 1)^{-4}(21x^2) = -315(7x^3 + 1)^{-4} \cdot x^2.$$

3.7.32 With $g(t) = 5t$ and $f(u) = \cos u$ we have $\dfrac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = -\sin 5t \cdot 5 = -5\sin 5t$.

3.7.33 With $g(x) = 3x + 1$ and $f(u) = \sec u$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \sec(3x+1)\tan(3x+1) \cdot 3 = 3\sec(3x+1)\tan(3x+1).$$

3.7.34 With $g(x) = e^x$ and $f(u) = \csc u$, we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = -(\csc e^x \cot e^x)e^x = -e^x \csc e^x \cot e^x.$$

3.7.35 With $g(x) = e^x$ and $f(u) = \tan u$ we have $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \sec^2 u \cdot e^x = e^x \sec^2 e^x$.

3.7.36 With $g(t) = \tan t$ and $f(u) = e^u$ we have $\dfrac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = e^{\tan t} \cdot \sec^2 t$.

3.7.37 With $g(x) = 4x^3 + 3x + 1$ and $f(u) = \sin u$ we have

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \cos u \cdot (12x^2 + 3) = (12x^2 + 3) \cdot \cos(4x^3 + 3x + 1).$$

3.7.38 With $g(t) = t^2 + t$ and $f(u) = \csc u$ we have

$$\frac{d}{dt}[f(g(t))] = f'(g(t))g'(t) = -(\csc u)(\cot u) \cdot (2t + 1) = -(2t + 1)\csc(t^2 + t)\cot(t^2 + t).$$

3.7.39 With $g(x) = 5x + 1$ and $f(u) = x^{2/3}$ we have

$$\frac{d}{dx}(f(g(x)) = f'(g(x)g'(x) = \frac{2}{3}(5x + 1)^{-\frac{1}{3}} \cdot 5 = \frac{10}{3(5x + 1)^{\frac{1}{3}}}.$$

3.7.40 We use the product rule and the chain rule when calculating the derivative of $y = (x + 1)^{1/3}$.

$$y' = 1 \cdot (x + 1)^{1/3} + x \cdot (1/3)(x + 1)^{-2/3} = \frac{3(x + 1)}{3(x + 1)^{2/3}} + \frac{x}{3(x + 1)^{2/3}} = \frac{4x + 3}{3(x + 1)^{2/3}}.$$

3.7.41 With $u = \dfrac{2x}{4x - 3}$ and $f(u) = u^{1/4}$ we have

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{2x}{4x - 3}\right)^{-\frac{3}{4}} \cdot \left(\frac{2(4x - 3) - 2x \cdot 4}{(4x - 3)^2}\right) = -\frac{3}{2}\left(\frac{4x - 3}{2x}\right)^{\frac{3}{4}} \cdot \frac{1}{(4x - 3)^2} = -\frac{3}{2^{7/4}x^{3/4}(4x - 3)^{5/4}}.$$

3.7.42 First note that $\dfrac{dy}{d\theta} = \dfrac{d}{d\theta}\left(\cos^4 \theta\right) + \dfrac{d}{d\theta}\left(\sin^4 \theta\right)$. To compute the first term, let $g_1(\theta) = \cos \theta$ and $f(u) = u^4$. Then $\dfrac{d}{d\theta} \cos^4 \theta = 4\cos^3(\theta)(-\sin \theta) = -4 \sin \theta \cos^3 \theta$.

Similarly, to compute the second term, let $g_2(\theta) = \sin \theta$ and $f(u) = u^4$. Then $\dfrac{d}{d\theta} \sin^4 \theta = 4 \sin^3 \theta \cos \theta = 4 \cos \theta \sin^3 \theta$. Thus, $\dfrac{dy}{d\theta} = -4 \sin \theta \cos^3 \theta + 4 \cos \theta \sin^3 \theta$. This can be further simplified to $4 \cos \theta \sin \theta(\sin^2 \theta - \cos^2 \theta) = 2 \sin 2\theta(-\cos 2\theta) = -\sin 4\theta$.

3.7.43 With $g(x) = \sec x + \tan x$ and $f(u) = u^5$ we have $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = 5u^4 \cdot (\sec x \tan x + \sec^2 x) = 5(\sec x + \tan x)^4(\sec x \tan x + \sec^2 x) = 5 \sec x(\sec x + \tan x)^5$.

3.7.44 With $g(z) = 4 \cos z$ and $f(u) = \sin u$ we have

$$\frac{dy}{dz} = f'(g(z))g'(z) = \cos(4 \cos z) \cdot (-4 \sin z) = -4 \sin z \cos(4 \cos z).$$

3.7.45 Take $g(x) = 2x^6 - 3x^3 + 3$, and $n = 25$. Then $y' = n(g(x))^{n-1}g'(x) = 25(2x^6 - 3x^3 + 3)^{24}(12x^5 - 9x^2)$.

3.7.46 Take $g(x) = \cos x + 2 \sin x$, and $n = 8$. Then $y' = n(g(x))^{n-1}g'(x) = 8(\cos x + 2 \sin x)^7(2 \cos x - \sin x)$.

3.7.47 Take $g(x) = 1 + 2 \tan u$, and $n = 4.5$. Then

$$y' = n(g(x))^{n-1}g'(x) = 4.5(1 + 2 \tan u)^{3.5}(2 \sec^2 u) = 9(1 + 2 \tan u)^{3.5} \sec^2 u.$$

3.7.48 Take $g(x) = 1 - e^x$, and $n = 4$. Then $y' = n(g(x))^{n-1}g'(x) = 4(1 - e^x)^3(-e^x) = -4e^x(1 - e^x)^3$.

3.7.49

$$\frac{d}{dx}\sqrt{1+\cot^2 x} = \frac{1}{2\sqrt{1+\cot^2 x}}\cdot\frac{d}{dx}(1+\cot^2 x) = \frac{1}{2\sqrt{1+\cot^2 x}}\cdot 2\cot x\cdot\frac{d}{dx}\cot x$$
$$= \frac{1}{2\sqrt{1+\cot^2 x}}\cdot 2\cot x\cdot(-\csc^2 x) = -\frac{\cot x\csc^2 x}{\sqrt{1+\cot^2 x}}.$$

3.7.50 $g'(x) = \dfrac{e^{3x}-x3e^{3x}}{e^{6x}} = \dfrac{e^{3x}(1-3x)}{e^{6x}} = \dfrac{1-3x}{e^{3x}}.$

3.7.51 Note that $\dfrac{d}{dx}e^{-x} = -e^{-x}$. Then we have $y' = \dfrac{2e^x-3e^{-x}}{3}$.

3.7.52 Using the product rule, we have $f'(x) = e^{7x} + xe^{7x}\cdot 7 = e^{7x}(1+7x)$.

3.7.53

$$\begin{aligned}\frac{d}{dx}\sin(\sin(e^x)) &= \cos(\sin(e^x))\frac{d}{dx}\sin(e^x)\\ &= \cos(\sin(e^x))\cdot\cos(e^x)\cdot e^x\end{aligned}$$

3.7.54

$$\begin{aligned}\frac{d}{dx}\sin^2(e^{3x+1}) &= 2\sin(e^{3x+1})\frac{d}{dx}\sin(e^{3x+1})\\ &= 2\sin(e^{3x+1})\cos(e^{3x+1})\frac{d}{dx}e^{3x+1}\\ &= 2\sin(e^{3x+1})\cos(e^{3x+1})e^{3x+1}\cdot 3\\ &= 3e^{3x+1}\sin(2e^{3x+1})\end{aligned}$$

3.7.55

$$\begin{aligned}\frac{d}{dx}\sin^5(\cos 3x) &= 5\sin^4(\cos 3x)\cdot\frac{d}{dx}(\sin(\cos 3x))\\ &= 5\sin^4(\cos 3x)\cdot\cos(\cos 3x)\cdot\frac{d}{dx}\cos 3x\\ &= 5\sin^4(\cos 3x)\cdot\cos(\cos 3x)\cdot(-\sin 3x)\cdot 3\\ &= -15\sin^4(\cos 3x)\cos(\cos 3x)\sin 3x.\end{aligned}$$

3.7.56

$$\frac{d}{dx}\cos^{7/4}(4x^3) = \frac{7}{4}\cos^{3/4}(4x^3)\cdot\frac{d}{dx}\cos(4x^3) = \frac{7}{4}\cos^{3/4}(4x^3)(-\sin(4x^3))\cdot\frac{d}{dx}(4x^3)$$
$$= \frac{7}{4}\cos^{3/4}(4x^3)(-\sin(4x^3))\cdot 12x^2 = -21x^2\cos^{3/4}(4x^3)\sin(4x^3).$$

3.7.57

$$\begin{aligned}\frac{d}{dt}\left(\frac{e^{2t}}{1+e^{2t}}\right) &= \frac{(1+e^{2t})2e^{2t}-e^{2t}\cdot 2e^{2t}}{(1+e^{2t})^2}\\ &= \frac{2e^{2t}(1+e^{2t}-e^{2t})}{(1+e^{2t})^2}\\ &= \frac{2e^{2t}}{(1+e^{2t})^2}.\end{aligned}$$

3.7.58

$$\begin{aligned}\frac{d}{dx}(1-e^{-0.05x})^{-1} &= -\frac{1}{(1-e^{-0.05x})^2}\cdot\frac{d}{dx}(1-e^{-0.05x})\\ &= -\frac{1}{(1-e^{-0.05x})^2}\cdot(0.05e^{-0.05x}) = -\frac{0.05e^{-0.05x}}{(1-e^{-0.05x})^2}.\end{aligned}$$

3.7.59 $\frac{d}{dx}\sqrt{x+\sqrt{x}} = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{d}{dx}(x+\sqrt{x}) = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1+\frac{1}{2\sqrt{x}}\right)$.

3.7.60

$$\begin{aligned}\frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x}}} &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \frac{d}{dx}(x+\sqrt{x+\sqrt{x}}) \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1+\frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1+\frac{1}{2\sqrt{x}}\right)\right).\end{aligned}$$

Note that on the last step, we used the result of the previous problem.

3.7.61 $\frac{d}{dx}f(g(x^2)) = f'(g(x^2)) \cdot \frac{d}{dx}(g(x^2)) = f'(g(x^2)) \cdot g'(x^2) \cdot 2x$.

3.7.62 $\frac{d}{dx}[f(g(x^m))]^n = n[f(g(x^m))]^{n-1}f'(g(x^m))g'(x^m)(mx^{m-1})$.

3.7.63 $y' = 5\left(\frac{x}{x+1}\right)^4 \cdot \frac{(x+1)(1)-x(1)}{(x+1)^2} = \frac{5x^4}{(x+1)^6}$.

3.7.64 $y' = 8\left(\frac{e^x}{x+1}\right)^7 \cdot \frac{(x+1)e^x-e^x}{(x+1)^2} = \frac{8xe^{8x}}{(x+1)^9}$.

3.7.65 $y' = e^{x^2+1}(2x)\sin x^3 + e^{x^2+1}(\cos x^3)3x^2 = xe^{x^2+1}(2\sin x^3 + 3x\cos x^3)$.

3.7.66 $y' = \sec^2(xe^x)((1)e^x + xe^x) = e^x(1+x)\sec^2(xe^x)$.

3.7.67 $\frac{dy}{d\theta} = 2\theta\sec 5\theta + \theta^2(5\sec 5\theta\tan 5\theta) = \theta\sec 5\theta(2+5\theta\tan 5\theta)$.

3.7.68 $5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{(4x+2)3-(3x)4}{(4x+2)^2} = 5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{6}{(4x+2)^2} = \frac{5(3x)^4}{(4x+2)^4} \cdot \frac{6}{(4x+2)^2} = \frac{2430x^4}{(4x+2)^6}$.

3.7.69 $y' = 4((x+2)(x^2+1))^3 \cdot ((1)(x^2+1)+(x+2)(2x)) = 4((x+2)(x^2+1))^3(3x^2+4x+1) =$ $4(x+2)^2(x^2+1)^3(3x+1)(x+1)$.

3.7.70

$$\begin{aligned}y' &= 2e^{2x}(2x-7)^5 + e^{2x}(5(2x-7)^4 2) = e^{2x}(2x-7)^4(2(2x-7)+10) \\ &= e^{2x}(2x-7)^4(4x-4) = 4e^{2x}(2x-7)^4(x-1).\end{aligned}$$

3.7.71 $y' = \frac{1}{5}(x^4+\cos 2x)^{-4/5}(4x^3-2\sin 2x) = \frac{4x^3-2\sin 2x}{5(x^4+\cos 2x)^{4/5}}$.

3.7.72 $y' = \frac{(t+1)(1\cdot e^t+te^t)-te^t\cdot 1}{(t+1)^2} = \frac{te^t+t^2e^t+e^t+te^t-te^t}{(t+1)^2} = \frac{e^t(t^2+t+1)}{(t+1)^2}$.

3.7.73

$$\begin{aligned}y' &= 2(p+3)^1\sin p^2 + (p+3)^2(\cos p^2)(2p) = (p+3)(2\sin p^2 + 2p^2\cos p^2 + 6p\cos p^2) \\ &= 2(p+3)(\sin p^2 + p^2\cos p^2 + 3p\cos p^2).\end{aligned}$$

3.7.74 $y' = 1.75(2z+5)^{0.75}(2)\tan z + (2z+5)^{1.75}\sec^2 z = (2z+5)^{0.75}(3.5\tan z + (2z+5)\sec^2 z)$.

3.7.75 $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{1}{2\sqrt{f(x)}} \cdot f'(x)$.

3.7.76 $\dfrac{d}{dx}\sqrt[5]{f(x)g(x)} = \dfrac{1}{5(f(x)g(x))^{4/5}} \cdot \dfrac{d}{dx}(f(x)g(x)) = \dfrac{f'(x)g(x)+f(x)g'(x)}{5(f(x)g(x))^{4/5}}$.

3.7.77

a. True. The product rule alone will suffice.

b. True. This function is the composition of e^x with $\sqrt{x+1}$.

c. True. The derivative of the composition $f(g(x))$ is the product of $f'(g(x))$ with $g'(x)$, so it is the product of two derivatives.

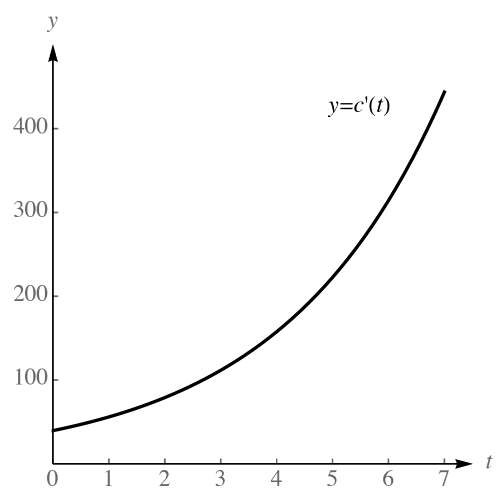
d. False. In fact, $\dfrac{d}{dx}P(Q(x)) = P'(Q(x))Q'(x)$.

3.7.78

a. The average growth rate between 2007 and 2009 was $\dfrac{c(2)-c(0)}{2-0} \approx 57.1$ million smartphones sold per year. Between 2012 and 2014 it was $\dfrac{c(7)-c(5)}{7-5} \approx 320.4$ million smartphones sold per year. Therefore the average growth rate was much larger between 2012 and 2014.

b. The growth rate in 2008 was $c'(1) \approx 56$ million smartphones sold per year and the growth rate in 2013 was $c'(6) \approx 314.1$ million smartphones sold per year. The growth rate was much larger in 2013.

c.



The growth rate is positive and increasing from 2007 to 2014.

3.7.79 Note that $a(70) = 13330$.

$$\frac{d}{dt}\,p(a(t))\big|_{t=70} = p'(a(70))a'(70) \approx \frac{738-765}{14330-13330}\cdot\frac{13440-13330}{80-70} = -0.297\,\text{hPa per minute.}$$

3.7.80

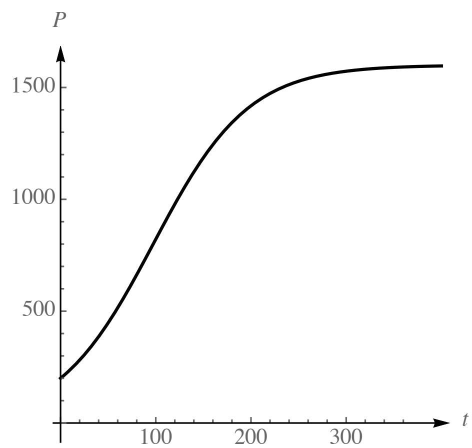
a. $\dfrac{dT}{dt} = \dfrac{dT}{dA}\dfrac{dA}{dt} \approx 6.5\cdot\dfrac{2.5-2.1}{2-1.5} = 6.5\cdot .8 = 5.2$ degrees per hour. The temperature is dropping at about 5.2 degrees per hour.

b. An increase in lapse rate would increase the answer to part (a).

c. No. The calculation depends on the lapse rate and on the rate at which the balloon is ascending, but not on the actual temperature.

3.7.81 $m'(t) = 64e^{0.004t} \cdot (0.004) = 0.256e^{0.004t}$. $m'(65) = 0.256e^{0.26} \approx 0.33$. 65 days after the diet switch, the mass of the tortoise is increasing at about 1/3 of a gram per day.

3.7.82

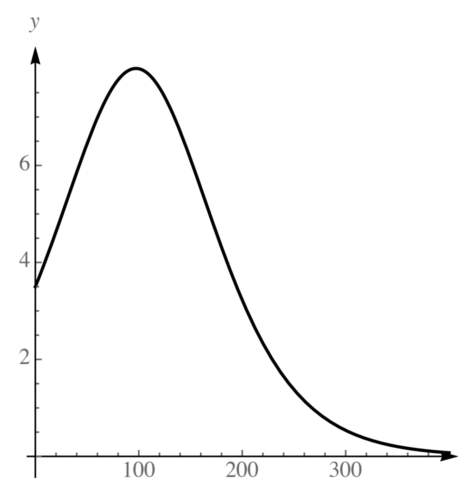
a. 

b. The average growth rate for the first ten days is $\dfrac{P(10)-P(0)}{10} \approx \dfrac{237.7-200}{10} = 3.77$ cells per day.

c. The maximum growth rate is where the curve $P(t)$ is the steepest, which appears to be at just shy of 100 days.

d. $P'(t) = \dfrac{0 - 1600 \cdot (-.14e^{-.02t})}{(1+7e^{-.02t})^2} = \dfrac{224e^{-.02t}}{(1+7e^{-.02t})^2}$.

e. At 100 days the populations is a little larger than 800. By doing a little bit of zooming, we can see that the maximum occurs at about $t = 97.3$ days with a population of 800.



3.7.83

a. After 10 years we will have $A(10) = 200e^{0.0398} \approx \$297.77$.

b. The growth rate is $A'(t) = 200 \cdot (0.0398e^{0.0398t}) = 7.96e^{0.0398t}$. After 10 years, the growth rate is $A'(10) = 7.96e^{0.0398} \approx 11.85$ dollars/year.

c. The tangent line is given by $y - 297.77 = 11.85(t-10)$, or $y = 11.85t + 179.27$.

3.7.84

a. $p(10) = 1000e^{-1} \approx 368$ mb, so the pressure on Mt. Everest is about 632 mb less than at sea level.

b. The average pressure change is $\dfrac{p(5) - p(0)}{5} = \dfrac{1000(e^{-0.5} - 1)}{5} \approx -78.7$ mb per km.

c. The rate of change in pressure is $p'(5) = -100e^{-5/10} \approx -60.7$ mb per km.

d. Because $p'(z) = -100e^{-z/10}$, it increases as z increases.

e. $\lim\limits_{z\to\infty} p(z) = 0$ means that if we go high enough, there is essentially no atmospheric pressure.

3.7.85

a. The slope is $f'(x) = e^{2x} + 2xe^{2x}$. This is zero when $e^{2x}(1 + 2x) = 0$, which occurs when $x = -\frac{1}{2}$.

b. The graph of f has a horizontal tangent line at $x = -1/2$.

3.7.86 $\dfrac{d^2}{dx^2}(x\cos(x^2)) = \dfrac{d}{dx}(\cos(x^2) - 2x^2\sin(x^2)) = -2x\sin(x^2) - 4x\sin(x^2) - 4x^3\cos(x^2) = -6x\sin(x^2) - 4x^3\cos(x^2)$.

3.7.87

$$\begin{aligned}\frac{d^2}{dx^2}\sin x^2 &= \frac{d}{dx}(2x\cos x^2) = 2(\cos x^2 - 2x^2\sin x^2)\\ &= 2\cos x^2 - 4x^2\sin x^2.\end{aligned}$$

Note that in the middle of this calculation we used a result from the middle of the previous problem – namely the derivative of $x\cos x^2$.

3.7.88

$$\begin{aligned}\frac{d^2}{dx^2}\sqrt{x^2+2} &= \frac{d}{dx}\left(\frac{1}{2}(x^2+2)^{-1/2}2x\right)\\ &= \frac{d}{dx}\left(\frac{x}{\sqrt{x^2+2}}\right)\\ &= \frac{\sqrt{x^2+2} - x\cdot\left(\frac{x}{\sqrt{x^2+2}}\right)}{x^2+2}\\ &= \frac{\frac{x^2+2-x^2}{\sqrt{x^2+2}}}{x^2+2}\\ &= \frac{2}{(x^2+2)^{3/2}}.\end{aligned}$$

3.7.89 $\dfrac{d^2}{dx^2}e^{-2x^2} = \dfrac{d}{dx}\left(-4xe^{-2x^2}\right) = -4e^{-2x^2} + 16x^2e^{-2x^2} = 4e^{-2x^2}(4x^2 - 1)$.

3.7.90

a. $\dfrac{d}{dx}(x^2+x)^2 = 2(x^2+x)\cdot\dfrac{d}{dx}(x^2+x) = 2(x^2+x)(2x+1) = 4x^3 + 6x^2 + 2x$.

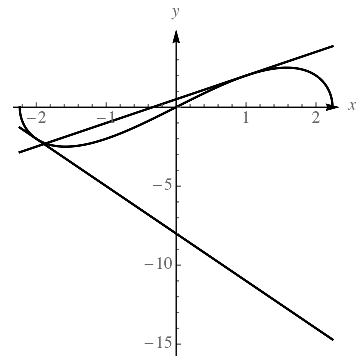
b. $\dfrac{d}{dx}(x^2+x)^2 = \dfrac{d}{dx}(x^4 + 2x^3 + x^2) = 4x^3 + 6x^2 + 2x$.

3.7.91 $y' = \dfrac{(x^3 - 6x - 1)(2)(x^2-1)(2x) - (x^2-1)^2(3x^2-6)}{(x^3-6x-1)^2}$,

so $y'(0) = \dfrac{(-1)(2)(-1)(0) - (-1)^2(-6)}{(-1)^2} = 6$. The equation of the tangent line is thus $y + 1 = 6(x - 0)$, or $y = 6x - 1$.

3.7.92

$y' = \sqrt{5-x^2} - \frac{x^2}{\sqrt{5-x^2}}$. Thus we have $y'(1) = 2-(1/2) = 3/2$, and $y'(-2) = 1-(4/1) = -3$. The tangents line we are seeking are $y-2 = (3/2)(x-1)$ and $y+2 = -3(x+2)$, or $y = \frac{3}{2}x + \frac{1}{2}$ and $y = -3x - 8$.



3.7.93

a. $g'(4) = 3$, $g(4) = 3 \cdot 4 - 5 = 7$. $f'(7) = -2$, $f(7) = -2 \cdot 7 + 23 = 9$. Thus, $h(4) = f(g(4)) = f(7) = 9$, and $h'(4) = f'(g(4))g'(4) = f'(7) \cdot 3 = -2 \cdot 3 = -6$.

b. The tangent line to h at $(4, 9)$ is given by $y - 9 = -6(x-4)$, or $y = -6x + 33$.

3.7.94

a. $g(1) = f(1^2) = f(1) = 4$.

b. $g'(x) = f'(x^2) \cdot 2x$.

c. Using the previous result, $g'(1) = f'(1) \cdot 2 = 3 \cdot 2 = 6$.

d. The tangent line is given by $y - 4 = 6(x-1)$, or $y = 6x - 2$.

3.7.95 $y'(x) = 2e^{2x}$, so $y'\left(\frac{\ln 3}{2}\right) = 2e^{\ln 3} = 6$. Also, $y\left(\frac{\ln 3}{2}\right) = e^{\ln 3} = 3$. The tangent line is therefore given by $y - 3 = 6\left(x - \frac{\ln 3}{2}\right)$, or $y = 6x + 3 - 3\ln 3$.

3.7.96 First, note that $g'(x) = f'(\sin x) \cdot \cos x$.

a. $g'(0) = f'(0) \cdot \cos 0 = 3 \cdot 1 = 3$.

b. $g'(\pi/2) = f'(1) \cdot \cos(\pi/2) = 5 \cdot 0 = 0$.

c. $g'(\pi) = f'(0) \cdot \cos \pi = 3 \cdot (-1) = -3$.

3.7.97 First, note that $g'(x) = \cos(\pi f(x)) \cdot \pi f'(x)$.

a. $g'(0) = \cos(\pi \cdot f(0)) \cdot \pi f'(0) = \cos(-3\pi) \cdot 3\pi = -3\pi$.

b. $g'(1) = \cos(\pi \cdot f(1)) \cdot \pi f'(1) = \cos(3\pi) \cdot 5\pi = -5\pi$.

3.7.98

a. $\dfrac{dy}{dt} = -y_0\sqrt{\dfrac{k}{m}} \sin\left(t\sqrt{\dfrac{k}{m}}\right)$.

b. The amplitude of the velocity (which is $y_0\sqrt{\frac{k}{m}}$) would decrease by a factor of 2, and the period would increase by a factor of 2.

c. The amplitude of the velocity would increase by a factor of 2, and the period would decrease by a factor of 2.

d. The units for $-y_0\sqrt{\frac{k}{m}}$ would be meters$\cdot\sqrt{\frac{\text{kg/sec}^2}{\text{kg}}} = \frac{\text{meters}}{\text{sec}}$. Inside the sine function the units for $t\cdot\sqrt{\frac{k}{m}}$ are sec $\cdot\,\frac{1}{\text{sec}} = 1$, so the factor involving the sine function is unit-less (as it should be).

3.7.99

a. $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(-y_0\sqrt{\dfrac{k}{m}}\sin\left(t\sqrt{\dfrac{k}{m}}\right)\right) = -y_0\cdot\dfrac{k}{m}\cdot\cos\left(t\sqrt{\dfrac{k}{m}}\right).$

b. $-\frac{k}{m}y = -\frac{k}{m}\left(y_0\cos\left(t\sqrt{\frac{k}{m}}\right)\right) = \frac{d^2y}{dt^2}.$

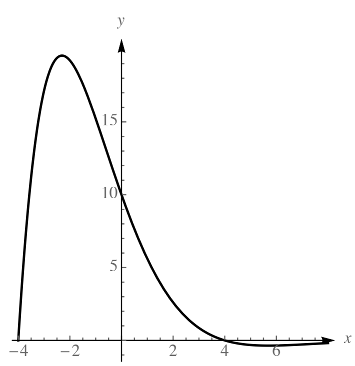
3.7.100

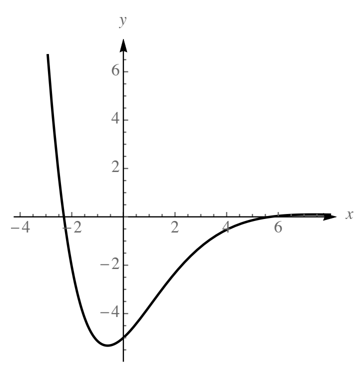
a. The period of $\cos x$ is 2π. The period of a function of the form $y = a\cos bx$ is $\dfrac{2\pi}{b}$. Thus, the period of y is $\dfrac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\dfrac{m}{k}}$.

b. $\dfrac{dT}{dm} = \dfrac{d}{dm}\left(2\pi\sqrt{\dfrac{m}{k}}\right) = \dfrac{2\pi}{\sqrt{k}}\cdot\dfrac{1}{2\sqrt{m}} = \dfrac{\pi}{\sqrt{mk}}.$

c. Because k and m are greater than 0, and π is greater than 0, this quotient is greater than 0. Physically this means that the period is increasing as mass increases: the oscillations get slower.

3.7.101

a. 

b. 

$$\frac{dy}{dt} = -5e^{-t/2}\cos\left(\frac{\pi t}{8}\right) - \frac{5\pi}{4}e^{-t/2}\sin\left(\frac{\pi t}{8}\right).$$

c. The velocity is zero at about -2.3 and at about 5.7, and the displacement has a maximum and a minimum at these points.

3.7.102 $\dfrac{dy}{dt} = -e^{-t}(\sin 2t - 2\cos 2t) + e^{-t}(2\cos 2t + 4\sin 2t) = e^{-t}(3\sin 3t + 4\cos 2t).$

$\dfrac{d^2y}{dt^2} = -e^{-t}(3\sin 2t + 4\cos 2t) + e^{-t}(6\cos 2t - 8\sin 2t) = e^{-t}(-11\sin 2t + 2\cos 2t).$

Then $y''(t) + 2y'(t) + 5y(t) = e^{-t}(-11\sin 2t + 2\cos 2t) + e^{-t}(6\sin 2t + 8\cos 2t) + e^{-t}(5\sin 2t - 10\cos 2t) = e^{-t}((-11+6+5)\sin 2t + (2+8-10)\cos 2t) = e^{-t}(0+0) = 0$, as desired.

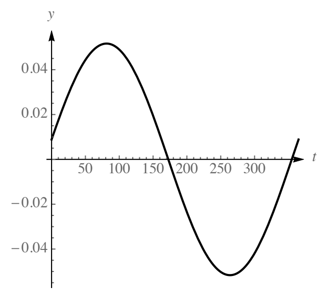
3.7.103

a. Assuming a non leap year, March 1st corresponds to $t = 59$. We have $D(59) = 12 - 3\cos\left(\frac{2\pi(69)}{365}\right) \approx$ 10.88 hours.

b. $\dfrac{d}{dt}D(t) = 3 \cdot \dfrac{2\pi}{365} \sin\left(\dfrac{2\pi(t+10)}{365}\right)$ hours per day.

c. $D'(59) \approx 0.048$ hours per day ≈ 2 minutes and 52 seconds per day. This means that on March 1st, the days are getting longer by just shy of 3 minutes per day.

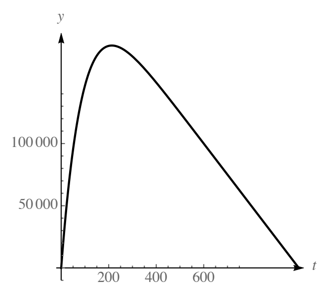
d.



e. The largest increase in the length of the days appears to be at about $t = 81$, and the largest decrease at about $t = 265$. These correspond to March 22nd and to September 22nd. The least rapid changes occur at about $t = 172$ and $t = 355$. These correspond to June 21st and December 21st.

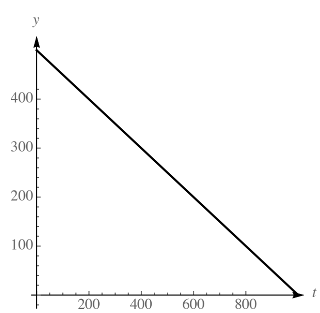
3.7.104

a.



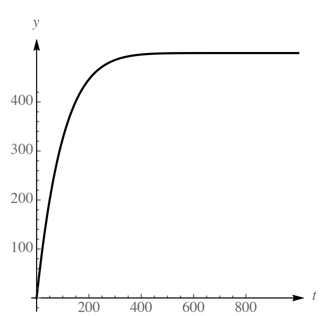
$M(0) = 250(1000)(1 - (10)^{-30} \cdot 10^{30}) = 250,000 \cdot (1-1) = 0$ grams.

b.



$V(1000) = 500 - (.5)(1000) = 500 - 500 = 0.$

c.



$C(0) = \frac{M(0)}{V(0)} = \frac{0}{500} = 0$. $C(1000)$ isn't defined because $V(1000) = 0$, but it appears that $\lim_{t\to1000} C(t) = 500$. The concentration of the salt in the tank increases with time, although it levels off as it nears 500 grams per liter.

d. $M'(t) = 250(1000-t)(10^{-30}(10(1000-t)^9)) - 250(1 - 10^{-30}(1000-t)^{10}) = \frac{250(1-t)^{10}}{10^{29}} - 250\left(1 - \frac{(1000-t)^{10}}{10^{30}}\right).$

e. It is convenient to rewrite $C(t)$ first. We can rewrite

$$C(t) = \frac{M(t)}{V(t)} = \frac{250(1000-t)}{\frac{1000-t}{2}} \cdot \left(1 - \frac{(1000-t)^{10}}{1000^{10}}\right) = 500 \cdot \left(1 - \left(1 - \frac{t}{1000}\right)^{10}\right).$$

Then $C'(t) = 500 \cdot \left(0 - 10\left(1 - \frac{t}{1000}\right)^9 \cdot -\frac{1}{1000}\right) = 5\left(1 - \frac{t}{1000}\right)^9.$

f. The derivative is positive for $0 \le t \le 1000$, so the concentration is increasing on this interval.

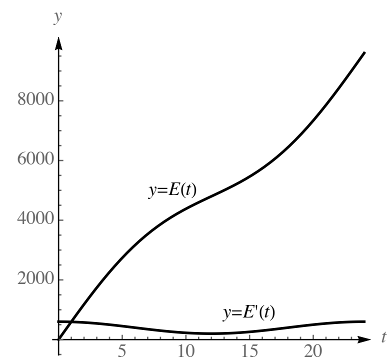
3.7.105

a. $E'(t) = 400 + 200\cos\left(\frac{\pi t}{12}\right)$ MW.

b. Because the maximum value of $\cos\theta$ is 1, the maximum value of $E'(t)$ will be 600 MW, where $\cos\left(\frac{\pi t}{12}\right) = 1$, which is where $t = 0$, which corresponds to noon.

c. Because the minimum value of $\cos\theta$ is -1, the minimum value of $E'(t)$ will be 200 MW, where $\cos\left(\frac{\pi t}{12}\right) = -1$, which is where $\frac{\pi t}{12} = \pi$, or $t = 12$, which corresponds to midnight.

d.



3.7.106

a. $\frac{d}{dt}\cos 2t = -2\sin 2t$, and $\frac{d}{dt}(\cos^2 t - \sin^2 t) = -2\sin t\cos t - 2\sin t\cos t = -4\sin t\cos t$. Thus, $-2\sin 2t = -4\sin t\cos t$, so $\sin 2t = 2\sin t\cos t$.

b. $\frac{d}{dt}(2\cos^2 t - 1) = -4\cos t\sin t$, so again, $\sin 2t = 2\sin t\cos t$.

c. $\frac{d}{dt}(\sin 2t) = 2\cos 2t$, and $\frac{d}{dt}(2\sin t\cos t) = 2\cos t\cos t + 2\sin t(-\sin t) = 2\cos^2 t - 2\sin^2 t$, so $\cos 2t = \cos^2 t - \sin^2 t$.

3.7.107

$$\begin{aligned}\frac{d}{dx}\left(f(x)(g(x))^{-1}\right) &= f'(x)(g(x))^{-1} + f(x)(-(g(x))^{-2}g'(x)) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.\end{aligned}$$

3.7.108

a.

$$\begin{aligned}\frac{d^2}{dx^2}[f(g(x))] &= \frac{d}{dx}[f'(g(x))g'(x)] \\ &= f''(g(x))g'(x)g'(x) + f'(g(x))g''(x) \\ &= f''(g(x))(g'(x))^2 + f'(g(x))g''(x).\end{aligned}$$

b. Let $g(x) = 3x^4 + 5x^2 + 2$. Then $g'(x) = 12x^3 + 10x$ and $g''(x) = 36x^2 + 10$. Let $f(u) = \sin u$. Then $f'(u) = \cos u$ and $f''(u) = -\sin u$.

We have $\dfrac{d^2}{dx^2}\sin(3x^4+5x^2+2) = -\sin(3x^4+5x^2+2)\cdot(12x^3+10x)^2 + \cos(3x^4+5x^2+2)\cdot(36x^2+10)$.

3.7.109

a. $h(x) = (x^2-3)^5$, $a = 2$.

b. $h'(x) = 5(x^2-3)^4(2x) = 10x(x^2-3)^4$, so the value of this limit is $h'(2) = 20$.

3.7.110

a. $h(x) = \sqrt{4+\sin x}$, $a = 0$.

b. $h'(x) = \dfrac{1}{2}\cdot(4+\sin x)^{-1/2}\cdot\cos x = \dfrac{\cos x}{2\sqrt{4+\sin x}}$, so the value of this limit is $h'(0) = \frac{1}{4}$.

3.7.111

a. $h(x) = \sin x^2$, $a = \frac{\pi}{2}$.

b. $h'(x) = (\cos x^2)(2x)$, so the value of this limit is $h'\left(\dfrac{\pi}{2}\right) = \pi\cdot\cos\left(\dfrac{\pi^2}{4}\right) \approx -2.45$.

3.7.112

a. $h(x) = \dfrac{1}{3(x^5+7)^{10}}$, $a = 1$.

b. $h'(x) = -\dfrac{50x^4}{3(x^5+7)^{11}}$, so the value of this limit is $h'(1) = -\dfrac{50}{3\cdot 8^{11}} \approx -1.94\times 10^{-9}$.

3.7.113 $\displaystyle\lim_{x\to 5}\frac{f(x^2)-f(25)}{x-5} = \frac{d}{dx}\left[f(x^2)\right]_{x=5} = 2\cdot 5\cdot f'(25) = 10f'(25)$.

3.7.114

a. First note that $\dfrac{d}{dx}f(-x) = -f'(-x)$.

If f is even then $f(-x) = f(x)$. Because the derivatives of both sides of this equation must be equal, we have $-f'(-x) = f'(x)$ or $f'(-x) = -f'(x)$, so f' must be an odd function.

b. If f is odd then $f(-x) = -f(x)$. Because the derivatives of both sides of this equation must be equal, we have $-f'(-x) = -f'(x)$ or $f'(-x) = f'(x)$, so f' must be an even function.

3.7.115

a. $\displaystyle\lim_{v\to u} H(v) = \lim_{v\to u}\left(\frac{f(v)-f(u)}{v-u} - f'(u)\right) = \lim_{v\to u}\left(\frac{f(v)-f(u)}{v-u}\right) - f'(u) = f'(u) - f'(u) = 0$.

b. Suppose $u = v$. Then clearly both sides of the given expression are 0, so they are equal. Suppose $u \neq v$. Then $H(v) = \frac{f(v)-f(u)}{v-u} - f'(u)$, so $H(v) + f'(u) = \frac{f(v)-f(u)}{v-u}$, so the result holds by multiplying both sides of this equation by $v - u$.

c.

$$\begin{aligned} h'(a) &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a}\frac{H(g(x))+f'(g(a))}{x-a}\cdot(g(x)-g(a)) \\ &= \lim_{x\to a}\left[(H(g(x))+f'(g(a)))\cdot\frac{g(x)-g(a)}{x-a}\right]. \end{aligned}$$

d. $\displaystyle h'(a) = \lim_{x\to a}\left[(H(g(x))+f'(g(a)))\cdot\frac{g(x)-g(a)}{x-a}\right] = (0+f'(g(a)))\cdot g'(a) = f'(g(a))g'(a)$.

3.8 Implicit Differentiation

3.8.1 Implicit differentiation gives a single unified derivative, whereas solving for y explicitly yields two different functions.

3.8.2 In implicit differentiation, the independent and dependent variables may both appear on the same side of an equation, so one must keep track of which is which.

3.8.3 The result of implicit differentiation is often an expression involving both the dependent and independent variables, so one would need to know both in order to calculate the value of the derivative.

3.8.4 The derivative of y^2 with respect to x is not $2y$ as stated in the argument. Instead, $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$. Remember to treat y as a function of x when differentiating terms involving y. With this correction, we have $2y\frac{dy}{dx} + 2\frac{dy}{dx} = 6x^2$, which implies that $\frac{dy}{dx} = \frac{3x^2}{y+1}$.

3.8.5 Differentiating both sides with respect to x gives $1 = 2y\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{2y}$.

3.8.6 Differentiating both sides with respect to x gives $3 + 12y^2\frac{dy}{dx} = 0$.Thus, $\frac{dy}{dx} = -\frac{3}{12y^2} = -\frac{1}{4y^2}$.

3.8.7 Differentiating both sides with respect to x gives $\cos y \cdot \frac{dy}{dx} + 0 = 1$, so $\frac{dy}{dx} = \frac{1}{\cos y}$.

3.8.8 Differentiating both sides with respect to x gives $e^y\frac{dy}{dx} - e^x = 0$, so $\frac{dy}{dx} = \frac{e^x}{e^y} = e^{x-y}$.

3.8.9

a. When $x = 0$ we have $-y + y^3 = 0$, so $y(y^2 - 1) = 0$, so y can be either 0 or ± 1. So the y-intercepts are $(0,0)$, $(0,1)$, and $(0,-1)$.

b. Differentiating both sides with respect to x gives $2 - \frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$. Thus $2 = \frac{dy}{dx} - 3y^2\frac{dy}{dx}$, so $2 = \frac{dy}{dx}\left(1 - 3y^2\right)$ and $\frac{dy}{dx} = \frac{2}{1-3y^2}$.

c. At $(0,0)$ we have $\frac{dy}{dx} = 2$, at $(0,1)$ and at $(0,-1)$ we have $\frac{dy}{dx} = \frac{2}{1-3} = -1$.

3.8.10 Differentiating both sides with respect to x gives $2x + 3y^2\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x}{3y^2}$. The slope at $(-1,1)$ is thus $\frac{2}{3}$ and at $(1,1)$ is $-\frac{2}{3}$.

3.8.11 Differentiating both sides with respect to x gives $1 = 3y^2\frac{dy}{dx}$. Thus $\frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3}y^{-2}$. Differentiating both sides again gives $\frac{d^2y}{dx^2} = -\frac{2}{3}y^{-3}\frac{dy}{dx} = -\frac{2}{3}y^{-3} \cdot \frac{1}{3}y^{-2} = -\frac{2}{9y^5}$.

3.8.12 Differentiating both sides with respect to x gives $1 = e^y\frac{dy}{dx}$. Thus $\frac{dy}{dx} = \frac{1}{e^y} = e^{-y}$. Differentiating both sides again gives $\frac{d^2y}{dx^2} = -e^{-y}\frac{dy}{dx} = -e^{-y} \cdot e^{-y} = -e^{-2y}$.

3.8.13

a. $4x^3 + 4y^3\frac{dy}{dx} = 0$. Thus $4y^3\frac{dy}{dx} = -4x^3$, so $\frac{dy}{dx} = -\frac{x^3}{y^3}$.

b. When $x = 1$ and $y = -1$, we have $\dfrac{dy}{dx} = \dfrac{-1}{-1} = 1$.

3.8.14

a. $1 = e^y \dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

b. When $x = 2$, $\dfrac{dy}{dx} = \dfrac{1}{x} = \dfrac{1}{2}$.

3.8.15

a. $2y\dfrac{dy}{dx} = 4$, so $\dfrac{dy}{dx} = \dfrac{2}{y}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,2)} = \dfrac{2}{2} = 1$.

3.8.16

a. $2y\dfrac{dy}{dx} + 3 = 0$, so $\dfrac{dy}{dx} = -\dfrac{3}{2y}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,\sqrt{5})} = -\dfrac{3}{2\sqrt{5}} = -\dfrac{3\sqrt{5}}{10}$.

3.8.17

a. $\dfrac{dy}{dx}\cos y = 20x^3$, so $\dfrac{dy}{dx} = \dfrac{20x^3}{\cos y}$.

b. $\left.\dfrac{dy}{dx}\right|_{(1,\pi)} = \dfrac{20}{\cos\pi} = -20$.

3.8.18

a. $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{y}}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{2\sqrt{x}}$.

b. When $x = 4$ and $y = 1$ we have $\dfrac{dy}{dx} = \dfrac{\sqrt{1}}{2\sqrt{4}} = \dfrac{1}{4}$.

3.8.19

a. $-\dfrac{dy}{dx}\sin y = 1$, so $\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\csc y$.

b. $\left.\dfrac{dy}{dx}\right|_{(0,\pi/2)} = -\csc(\pi/2) = -1$.

3.8.20

a. $(y + x\dfrac{dy}{dx})\sec^2(xy) = 1 + \dfrac{dy}{dx}$, so $x\dfrac{dy}{dx}\sec^2(xy) - \dfrac{dy}{dx} = 1 - y\sec^2(xy)$. Factoring out $\dfrac{dy}{dx}$ on the left-hand side gives
$\dfrac{dy}{dx}(x\sec^2(xy) - 1) = 1 - y\sec^2(xy)$, so $\dfrac{dy}{dx} = \dfrac{1 - y\sec^2(xy)}{x\sec^2(xy) - 1}$.

b. $\left.\dfrac{dy}{dx}\right|_{(0,0)} = \dfrac{1-0}{0-1} = -1$.

3.8.21

a. $1 \cdot y + x\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y}{x}$

b. $\left.\dfrac{dy}{dx}\right|_{(1,7)} = -\dfrac{7}{1} = -7$.

3.8.22

a. We can rewrite the equation as $x = y^2 + 1$. Then $1 = 2y\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = \dfrac{1}{2y}$.

b. $\left.\frac{dy}{dx}\right|_{(10,3)} = \frac{1}{6}$.

3.8.23

a. $\frac{1}{3}x^{-\frac{2}{3}} + \frac{4}{3}y^{\frac{1}{3}}\frac{dy}{dx} = 0$, so $\frac{4}{3}y^{1/3}\frac{dy}{dx} = -\frac{1}{3x^{2/3}}$, so $\frac{dy}{dx} = -\frac{1}{4x^{2/3}y^{1/3}}$.

b. $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{4}$.

3.8.24

a. $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$, so $y^{-1/3}\frac{dy}{dx} = -\frac{1}{x^{1/3}}$, so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$.

b. $\left.\frac{dy}{dx}\right|_{(1,1)} = -1$.

3.8.25

a. $y^{1/3} + \frac{1}{3}xy^{-2/3}\frac{dy}{dx} + \frac{dy}{dx} = 0$, so $\frac{1}{3}xy^{-2/3}\frac{dy}{dx} + \frac{dy}{dx} = -y^{1/3}$ and therefore $\frac{dy}{dx}\left(\frac{1}{3}xy^{-2/3} + 1\right) = -y^{1/3}$, so $\frac{dy}{dx} = -\frac{y^{1/3}}{\frac{1}{3}xy^{-2/3} + 1} = -\frac{3y}{x + 3y^{2/3}}$.

b. $\left.\frac{dy}{dx}\right|_{(1,8)} = -\frac{24}{13}$.

3.8.26

a. $\frac{2}{3}(x+y)^{-\frac{1}{3}}\left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}$, so $1 + \frac{dy}{dx} = \frac{3}{2}(x+y)^{1/3}\frac{dy}{dx}$, and therefore $1 = \frac{3}{2}(x+y)^{1/3}\frac{dy}{dx} - \frac{dy}{dx}$. Then $1 = \left(\frac{3}{2}(x+y)^{1/3} - 1\right)\frac{dy}{dx}$, and so $\frac{dy}{dx} = \frac{1}{\left(\frac{3}{2}(x+y)^{1/3} - 1\right)} = \frac{2}{3(x+y)^{1/3} - 2}$.

b. $\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{2}{4} = \frac{1}{2}$.

3.8.27 $\cos x + \cos y\frac{dy}{dx} = \frac{dy}{dx}$, so $\cos x = \frac{dy}{dx} - \cos y\frac{dy}{dx}$, and therefore $\cos x = \frac{dy}{dx}(1 - \cos y)$, and thus $\frac{dy}{dx} = \frac{\cos x}{1 - \cos y}$.

3.8.28 $\frac{dy}{dx} = 1 \cdot e^y + xe^y\frac{dy}{dx}$, so $\frac{dy}{dx} - xe^y\frac{dy}{dx} = e^y$, so $\frac{dy}{dx}(1 - xe^y) = e^y$, and therefore $\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$. Now by the original equation, $y = xe^y$, so this answer can be written $\frac{dy}{dx} = \frac{e^y}{1 - y}$.

3.8.29 $1 + \frac{dy}{dx} = -\sin y \cdot \frac{dy}{dx}$, so $\frac{dy}{dx} + (\sin y)\frac{dy}{dx} = -1$, and $\frac{dy}{dx} = -\frac{1}{1 + \sin y}$.

3.8.30 $1 + 2\frac{dy}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$, so $1 = \frac{1}{2\sqrt{y}}\frac{dy}{dx} - 2\frac{dy}{dx}$, and thus $1 = \frac{dy}{dx}\left(\frac{1}{2\sqrt{y}} - 2\right)$. Because the right-hand side of this equation can be written as $\frac{dy}{dx}\left(\frac{1 - 4\sqrt{y}}{2\sqrt{y}}\right)$, we have $\frac{dy}{dx} = \frac{2\sqrt{y}}{1 - 4\sqrt{y}}$.

3.8.31 $(y + x\frac{dy}{dx})\cos(xy) = 1 + \frac{dy}{dx}$, so $y\cos(xy) + x\frac{dy}{dx}\cos(xy) = 1 + \frac{dy}{dx}$. If we rearrange terms in order to have the terms with a factor of $\frac{dy}{dx}$ all on the same side, we obtain $y\cos(xy) - 1 = \frac{dy}{dx} - x\frac{dy}{dx}\cos(xy)$. Factoring out the $\frac{dy}{dx}$ factor gives $y\cos(xy) - 1 = \frac{dy}{dx}(1 - x\cos(xy))$, so $\frac{dy}{dx} = \frac{y\cos(xy) - 1}{1 - x\cos(xy)}$.

3.8.32 $(y + x\frac{dy}{dx})e^{xy} = 2\frac{dy}{dx}$, so $ye^{xy} + x\frac{dy}{dx}e^{xy} = 2\frac{dy}{dx}$. We can write this as $ye^{xy} = 2\frac{dy}{dx} - x\frac{dy}{dx}e^{xy}$, and factoring out the factor of $\frac{dy}{dx}$ on the right yields $ye^{xy} = \frac{dy}{dx}(2 - xe^{xy})$. Finally, we can divide to obtain $\frac{dy}{dx} = \frac{ye^{xy}}{2 - xe^{xy}}$.

3.8.33 $-2y\frac{dy}{dx}\sin y^2 + 1 = \frac{dy}{dx}e^y$, which we can write as $1 = \frac{dy}{dx}e^y + 2y\frac{dy}{dx}\sin y^2$, or $1 = \frac{dy}{dx}(e^y + 2y\sin y^2)$. Thus, $\frac{dy}{dx} = \frac{1}{e^y + 2y\sin y^2}$.

3.8.34 $\frac{dy}{dx} = \frac{(y-1) - (x+1)\frac{dy}{dx}}{(y-1)^2}$, which we can write as $(y-1)^2 \cdot \frac{dy}{dx} = y - 1 - (x+1)\frac{dy}{dx}$. If we rearrange terms in order to have terms with a factor of $\frac{dy}{dx}$ on the same side, we obtain $\frac{dy}{dx}(y-1)^2 + \frac{dy}{dx}(x+1) = y - 1$. Factoring out the common factor of $\frac{dy}{dx}$ yields $\frac{dy}{dx}((y-1)^2 + (x+1)) = y - 1$, so $\frac{dy}{dx} = \frac{y-1}{(y-1)^2 + x + 1}$.

3.8.35

$$\begin{aligned}
3x^2 &= \frac{(x-y)(1+\frac{dy}{dx}) - (x+y)(1-\frac{dy}{dx})}{(x-y)^2} \\
3x^2(x-y)^2 &= x + x\frac{dy}{dx} - y - y\frac{dy}{dx} - x + x\frac{dy}{dx} - y + y\frac{dy}{dx} \\
3x^2(x-y)^2 + 2y &= 2x\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{3x^2(x-y)^2 + 2y}{2x}
\end{aligned}$$

3.8.36

$$\begin{aligned}
3(y + x\frac{dy}{dx})(xy+1)^2 &= 1 - 2y\frac{dy}{dx} \\
3x\frac{dy}{dx}(xy+1)^2 + 2y\frac{dy}{dx} &= 1 - 3y(xy+1)^2 \\
\frac{dy}{dx}(3x(xy+1)^2 + 2y) &= 1 - 3y(xy+1)^2 \\
\frac{dy}{dx} &= \frac{1 - 3y(xy+1)^2}{3x(xy+1)^2 + 2y}.
\end{aligned}$$

3.8.37

$$\begin{aligned}
18x^2 + 21\frac{dy}{dx}y^2 &= 13(y + x\frac{dy}{dx}) \\
21\frac{dy}{dx}y^2 - 13x\frac{dy}{dx} &= 13y - 18x^2 \\
\frac{dy}{dx} &= \frac{13y - 18x^2}{21y^2 - 13x}.
\end{aligned}$$

3.8.38

$$\begin{aligned}
\cos x\cos y+\sin x(-\sin y)\frac{dy}{dx} &= \cos x+(-\sin y)\frac{dy}{dx}\\
\cos x\cos y-\cos x &= \sin x\sin y\frac{dy}{dx}-\sin y\frac{dy}{dx}\\
\cos x\cos y-\cos x &= (\sin x\sin y-\sin y)\frac{dy}{dx}\\
\frac{dy}{dx} &= \frac{\cos x\cos y-\cos x}{\sin x\sin y-\sin y}.
\end{aligned}$$

3.8.39

$$\begin{aligned}
\frac{4x^3+2y\dfrac{dy}{dx}}{2\sqrt{x^4+y^2}} &= 5+6y^2\frac{dy}{dx}\\
y\frac{dy}{dx}-6\frac{dy}{dx}y^2\sqrt{x^4+y^2} &= 5\sqrt{x^4+y^2}-2x^3\\
\frac{dy}{dx} &= \frac{5\sqrt{x^4+y^2}-2x^3}{y-6y^2\sqrt{x^4+y^2}}.
\end{aligned}$$

3.8.40

$$\begin{aligned}
\frac{1}{2}(x+y^2)^{-1/2}\left(1+2y\frac{dy}{dx}\right) &= (\cos y)\frac{dy}{dx}\\
\frac{1}{2\sqrt{x+y^2}} &= \cos y\frac{dy}{dx}-\frac{y\cdot dy/dx}{\sqrt{x+y^2}}\\
\frac{1}{2\sqrt{x+y^2}} &= \left(\cos y-\frac{y}{\sqrt{x+y^2}}\right)\frac{dy}{dx}\\
\frac{1}{2\sqrt{x+y^2}} &= \left(\frac{\cos y\sqrt{x+y^2}-y}{\sqrt{x+y^2}}\right)\frac{dy}{dx}\\
\frac{dy}{dx} &= \frac{1}{2(\cos y\sqrt{x+y^2}-y)}
\end{aligned}$$

3.8.41

a. $1280=40L^{1/3}K^{2/3}$, so $0=\frac{40}{3}L^{-2/3}K^{2/3}+\frac{80}{3}L^{1/3}K^{-1/3}\cdot\frac{dK}{dL}$. Multiplying both sides by $\frac{3}{40}L^{2/3}K^{1/3}$ yields
$0=K+2L\frac{dK}{dL}$, so $\frac{dK}{dL}=-\frac{1}{2}\frac{K}{L}$.

b. With $L=8$ and $K=64$, $\frac{dK}{dL}=-\frac{64}{16}=-4$.

3.8.42

a. $A=\pi r\sqrt{r^2+h^2}=1500\pi$. So $\pi r'\sqrt{r^2+h^2}+\pi r\frac{rr'+h}{\sqrt{r^2+h^2}}=0$. So $r'(r^2+h^2)+r^2r'+rh=0$, so $r'=-\frac{rh}{2r^2+h^2}$.

b. At $r=30$ and $h=40$, we have $r'=-\frac{1200}{1800+1600}=-\frac{6}{17}$.

3.8.43

a. $V = \dfrac{\pi h^2(3r-h)}{3} = \dfrac{5\pi}{3}$. So

$$\frac{1}{3}[2\pi h(3r-h) + \pi h^2(3r'-1)] = 0,$$
$$6rh - 2h^2 + 3h^2r' - h^2 = 0,$$

so $r' = 1 - \dfrac{2r}{h}$.

b. At $r = 2$ and $h = 1$, we have $r' = 1 - 4 = -3$.

3.8.44

a. When $V = \dfrac{\pi^2(b+a)(b-a)^2}{4} = 64\pi^2$, then we have

$$(b'+1)(b-a)^2 + (b+a)\cdot 2\cdot(b-a)(b'-1) = 0,$$
$$b'(b-a)^2 + 2(b^2-a^2)b' = 2(b^2-a^2) - (b-a)^2,$$
$$b'(b-a)(b-a+2b+2a) = (b-a)(2b+2a-(b-a)),$$
$$\frac{db}{da} = \frac{(b-a)(b+3a)}{(b-a)(3b+a)} = \frac{b+3a}{3b+a}.$$

b. When $a = 6$ and $b = 10$ we have $\dfrac{db}{da} = \dfrac{28}{36} = \dfrac{7}{9}$.

3.8.45

a. $\sin 0 + 5\cdot 0 = 0^2$, so the point $(0,0)$ does lie on the curve.

b. $y'\cos y + 5 = 2yy'$, so $5 = y'(2y - \cos y)$, so $y' = \dfrac{5}{2y-\cos y}$. At the given point we have $y' = -5$. The equation of the tangent line is therefore $y - 0 = -5(x-0)$, or $y = -5x$.

3.8.46

a. $1^3 + 1^3 = 2\cdot 1\cdot 1$, so the point $(1,1)$ does lie on the curve.

b. $3x^2 + 3y^2y' = 2(y + xy')$, which can be written $(3y^2 - 2x)y' = 2y - 3x^2$, so $y' = \dfrac{2y-3x^2}{3y^2-2x}$. At the given point we have $y' = \dfrac{2-3}{3-2} = -1$. The equation of the tangent line is therefore $y - 1 = -1(x-1)$, or $y = -x + 2$.

3.8.47

a. $2^2 + 2\cdot 1 + 1^2 = 7$, so the point $(2,1)$ does lie on the curve.

b. $2x + y + xy' + 2yy' = 0$, which can be written $(x+2y)y' = -2x - y$. Solving for y' yields $y' = \dfrac{-2x-y}{x+2y}$. Thus, at the point $(2,1)$ we have $y' = -\dfrac{5}{4}$. The equation of the tangent line is therefore $y - 1 = -\dfrac{5}{4}(x-2)$, or $y = -\dfrac{5}{4}x + \dfrac{7}{2}$.

3.8.48

a. $(-1)^4 - (-1)^2\cdot 1 + 1^4 = 1$, so the point $(-1,1)$ does lie on the curve.

b. $4x^3 - 2xy - x^2y' + 4y^3y' = 0$, which can be written $y'(4y^3 - x^2) = 2xy - 4x^3$. Thus, $y' = \dfrac{2xy - 4x^3}{4y^3 - x^2}$. Thus, at the point $(-1, 1)$ we have $y' = \dfrac{2}{3}$. The equation of the tangent line is therefore $y - 1 = \dfrac{2}{3}(x+1)$, or $y = \dfrac{2}{3}x + \dfrac{5}{3}$.

3.8.49

a. $\cos(\pi/2 - \pi/4) + \sin(\pi/4) = (\sqrt{2}/2) + (\sqrt{2}/2) = \sqrt{2}$, so the point $(\pi/2, \pi/4)$ does lie on the curve.

b. $(1 - y')(-\sin(x - y)) + y' \cos y = 0$, which can be written as $y'(\cos y + \sin(x - y)) = \sin(x - y)$, so $y' = \dfrac{\sin(x - y)}{\cos y + \sin(x - y)}$. At the given point we have $y' = 1/2$. The equation of the tangent line is therefore $y - (\pi/4) = (1/2)(x - \pi/2)$, or $y = \dfrac{1}{2}x$.

3.8.50

a. $(1 + 2^2)^2 = 25 = \frac{25}{4} \cdot 1 \cdot 2^2$, so the point $(1, 2)$ does lie on the curve.

b. $2(x^2 + y^2)(2x + 2yy') = \frac{25}{4} \cdot (y^2 + 2xyy')$, which can be written as $y'[4y(x^2 + y^2) - (25/2)xy] = (25/4)y^2 - 4x(x^2 + y^2)$, so $y' = \dfrac{25y^2/4 - 4x(x^2 + y^2)}{4y(x^2 + y^2) - (25xy/2)}$. At the given point we have $y' = \dfrac{25 - 20}{40 - 25} = \dfrac{1}{3}$. The equation of the tangent line is therefore $y - 2 = \dfrac{1}{3}(x - 1)$, or $y = \dfrac{1}{3}x + \dfrac{5}{3}$.

3.8.51 $1 + 2yy' = 0$, so $y' = -\dfrac{1}{2y}$. Differentiating again, we obtain

$$y'' = -\frac{1}{2} \cdot \frac{-y'}{y^2} = \frac{y'}{2y^2} = \left(-\frac{1}{2y}\right) \cdot \frac{1}{2y^2} = -\frac{1}{4y^3}.$$

3.8.52 $4x + 2yy' = 0$, so $y' = -\dfrac{2x}{y}$. Differentiating again, we obtain

$$y'' = \frac{-2y + 2xy'}{y^2} = \frac{-2y + \frac{-4x^2}{y}}{y^2} = \frac{-2y^2 - 4x^2}{y^3}.$$

3.8.53 $1 + \dfrac{dy}{dx} = (\cos y)\dfrac{dy}{dx}$, so $1 = \dfrac{dy}{dx}(\cos y - 1)$, and thus $\dfrac{dy}{dx} = \dfrac{1}{\cos y - 1}$. Thus

$$\frac{d^2y}{dx^2} = -\frac{1}{(\cos y - 1)^2} \cdot \left(-\sin y \frac{dy}{dx}\right) = \frac{\sin y}{(\cos y - 1)^2} \cdot \frac{1}{(\cos y - 1)} = \frac{\sin y}{(\cos y - 1)^3}.$$

3.8.54 $4x^3 + 4y'y^3 = 0$, so $y' = -\dfrac{x^3}{y^3}$. Differentiating again, we obtain

$$y'' = -\frac{3x^2y^3 - x^3 \cdot 3y^2y'}{y^6} = \frac{3x^3y' - 3x^2y}{y^4} = \frac{3x^3 \cdot \left(-\frac{x^3}{y^3}\right) - 3x^2y}{y^4} = \frac{-3x^6 - 3x^2y^4}{y^7}.$$

3.8.55 $2y'e^{2y} + 1 = y'$, so $y' = \dfrac{1}{1 - 2e^{2y}}$. Differentiating again, we obtain

$$y'' = -\left(1 - 2e^{2y}\right)^{-2}(-4e^{2y}y') = \frac{4e^{2y}}{(1 - 2e^{2y})^3}.$$

3.8.56 $\cos x + 2xy + x^2y' = 0$, so $y' = -\dfrac{2xy + \cos x}{x^2}$. Differentiating again, we obtain

$$y'' = -\frac{x^2(2y + 2xy' - \sin x) - 2x(2xy + \cos x)}{x^4} = \frac{x^2 \sin x + 2x \cos x + 2x^2 y - 2x^3 y'}{x^4}$$
$$= \frac{x \sin x + 2\cos x + 2xy + 2(2xy + \cos x)}{x^3} = \frac{x \sin x + 4\cos x + 6xy}{x^3}.$$

3.8.57

a. False. For example, the equation $y \cos(xy) = x$, cannot be solved explicitly for y in terms of x.

b. True. We have $2x + 2yy' = 0$, and the result follows by solving for y'.

c. False. The equation $x = 1$ doesn't represent any sort of function – it is either just a number, or perhaps a vertical line, but it doesn't represent a differentiable function.

d. False. $y + xy' = 0$, so $y' = -\frac{y}{x}$, $x \neq 0$.

3.8.58

a. $y^{5/2} + \dfrac{5}{2}xy^{3/2}y' + \dfrac{3}{2}x^{1/2}y + x^{3/2}y' = 0$. Then we have

$$\begin{aligned}
\frac{5}{2}xy^{3/2}y' + x^{3/2}y' &= -y^{5/2} - \frac{3}{2}x^{1/2}y \\
y'\left(\frac{5}{2}xy^{3/2} + x^{3/2}\right) &= -y^{5/2} - \frac{3}{2}x^{1/2}y \\
y' &= \frac{-y^{5/2} - \frac{3}{2}x^{1/2}y}{\frac{5}{2}xy^{3/2} + x^{3/2}} = -\frac{y(3\sqrt{x} + 2y^{3/2})}{x(2\sqrt{x} + 5y^{3/2})}.
\end{aligned}$$

b. At $(4, 1)$ we have $y' = -\dfrac{6+2}{4(4+5)} = -\dfrac{2}{9}$.

3.8.59

a. $y + xy' + \dfrac{3}{2}x^{1/2}y^{-1/2} - \dfrac{1}{2}x^{3/2}y^{-3/2}y' = 0$. Multiplying through by $2y^{3/2}$ gives

$$2y^{5/2} + 2xy^{3/2}y' + 3x^{1/2}y - x^{3/2}y' = 0.$$

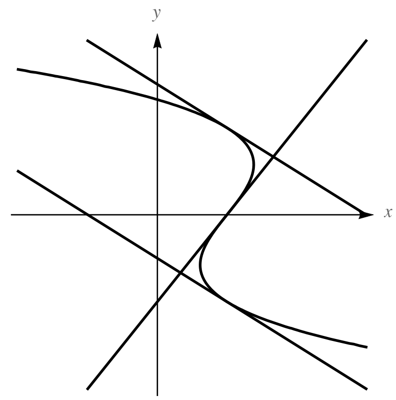
Then

$$\begin{aligned}
2xy^{3/2}y' - x^{3/2}y' &= -2y^{5/2} - 3x^{1/2}y \\
y'(2xy^{3/2} - x^{3/2}) &= -(2y^{5/2} + 3x^{1/2}y) \\
y' &= -\frac{2y^{5/2} + 3x^{1/2}y}{2xy^{3/2} - x^{3/2}} \\
&= \frac{y(2y^{3/2} + 3\sqrt{x})}{x(\sqrt{x} - 2y^{3/2})}.
\end{aligned}$$

b. At $(1, 1)$, $y' = -5$.

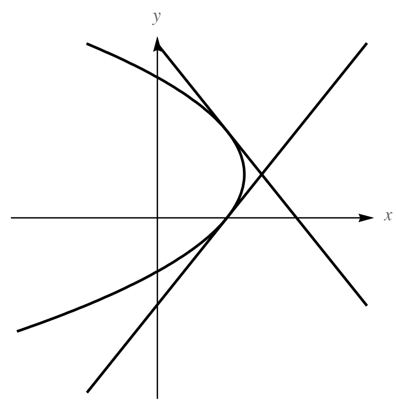
3.8.60

a. There are three points on the curve associated with $x = 1$. When $x = 1$, we have $1 + y^3 - y = 1$, so $y(y^2 - 1) = 0$. The three points are thus $(1, 0)$, $(1, 1)$ and $(1, -1)$. Differentiating yields $1 + 3y^2y' - y' = 0$, so $y' = \dfrac{1}{1 - 3y^2}$.
At $(1, 0)$, we have $y' = 1$, so the tangent line is given by $y = x - 1$.
At $(1, 1)$, we have $y' = -\dfrac{1}{2}$, so the tangent line is given by $y - 1 = -\dfrac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{3}{2}$.
At $(1, -1)$, we have $y' = -\dfrac{1}{2}$, so the tangent line is given by $y + 1 = -\frac{1}{2}(x - 1)$, or $y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

b. 

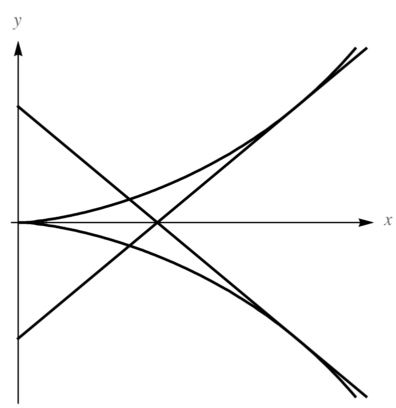
3.8.61

a. There are two points on the curve associated with $x = 1$. When $x = 1$, we have $1 + y^2 - y = 1$, so $y(y - 1) = 0$. The two points are thus $(1, 0)$ and $(1, 1)$. Differentiating yields $1 + 2yy' - y' = 0$, so $y' = \frac{1}{1-2y}$.
At $(1, 0)$, we have $y' = 1$, so the tangent line is given by $y = x - 1$.
At $(1, 1)$, we have $y' = -1$, so the tangent line is given by $y - 1 = -1(x - 1)$, or $y = -x + 2$.

b. 

3.8.62

a. There are two points on the curve associated with $x = 2$. When $x = 2$, we have $32 = 2y^2$, so $y^2 = 16$, so $y = \pm 4$. The two points are thus $(2, 4)$ and $(2, -4)$. Differentiating yields $12x^2 = 2yy'(4 - x) + -y^2$.
At $(2, -4)$, we have $48 = -8y'(2) - 16$, so $y' = -4$. Thus the tangent line is given by $y + 4 = -4(x - 2)$, or $y = -4x + 4$.
At $(2, 4)$, we have $y' = 4$, so the tangent line is given by $y - 4 = 4(x - 2)$, or $y = 4x - 4$.

b. 

3.8.63

a. $y(2x) + (x^2 + 4)y' = 0$, so $y' = -\dfrac{2xy}{x^2 + 4}$.

b. At $y = 1$ we have $x^2 + 4 = 8$, so $x = \pm 2$. At the point $(2, 1)$ we have $y' = -\dfrac{4}{8} = -\dfrac{1}{2}$. At the point $(-2, 1)$ we have $y' = \dfrac{4}{8} = \dfrac{1}{2}$. Thus, the equations of the tangent lines are given by $y - 1 = -\dfrac{1}{2}(x - 2)$ and $y - 1 = \dfrac{1}{2}(x + 2)$, or $y = -\dfrac{1}{2}x + 2$ and $y = \frac{1}{2}x + 2$.

c. $y = \dfrac{8}{x^2 + 4}$, so $y' = \dfrac{0 - 8 \cdot 2x}{(x^2 + 4)^2} = -\dfrac{16x}{(x^2 + 4)^2}$.

d. $y' = -\dfrac{16x}{(x^2 + 4)^2} = -\dfrac{2x}{x^2 + 4} \cdot \dfrac{8}{x^2 + 4} = -\dfrac{2x}{x^2 + 4} \cdot y = -\dfrac{2xy}{x^2 + 4}$.

3.8.64

a. From exercise 60 we have that $y' = \dfrac{1}{1 - 3y^2}$. A vertical tangent would occur at a point whose y value would make $1 - 3y^2$ equal to zero. So we are looking for where $3y^2 = 1$ or $y = \pm\dfrac{1}{\sqrt{3}}$.

If $y = \dfrac{1}{\sqrt{3}}$, then $x + \left(\dfrac{1}{\sqrt{3}}\right)^3 - \dfrac{1}{\sqrt{3}} = 1$, so $x = 1 + \dfrac{2\sqrt{3}}{9}$, and there is a vertical tangent at $\left(\dfrac{1}{\sqrt{3}}, 1 + \dfrac{2\sqrt{3}}{9}\right)$.

If $y = -\dfrac{1}{\sqrt{3}}$, then $x + \left(-\dfrac{1}{\sqrt{3}}\right)^3 - -\dfrac{1}{\sqrt{3}} = 1$, so $x = 1 - \dfrac{2\sqrt{3}}{9}$, and there is a vertical tangent at $\left(-\dfrac{1}{\sqrt{3}}, 1 - \dfrac{2\sqrt{3}}{9}\right)$.

b. Because y' is never zero, there are no horizontal tangent lines.

3.8.65

a. From exercise 61, we have that $y' = \dfrac{1}{1 - 2y}$. A vertical tangent would occur at a point whose y value would make $1 - 2y$ equal to zero. So we are looking for where $2y = 1$ or $y = \dfrac{1}{2}$.

If $y = \dfrac{1}{2}$, then $x + \dfrac{1}{4} - \dfrac{1}{2} = 1$, so $x = \dfrac{5}{4}$, and there is a vertical tangent at $\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$.

b. Because y' is never zero, there are no horizontal tangent lines.

3.8.66 Differentiating with respect to x gives $2x + 8y\dfrac{dy}{dx} + 2x\dfrac{dy}{dx} + 2y = 0$, so

$$\begin{aligned} 8y\frac{dy}{dx} + 2x\frac{dy}{dx} &= -2x - 2y \\ \frac{dy}{dx}(8y + 2x) &= -2x - 2y \\ \frac{dy}{dx} &= \frac{-x - y}{4y + x}. \end{aligned}$$

This quantity is zero when $y = -x$ (and not $x = y = 0$). Using the original equation, this means that $x^2 + 4x^2 - 2x^2 = 12$, or $3x^2 = 12$, so $x^2 = 4$ and we have $x = \pm 2$. Thus the points on the curve where the tangent line is horizontal are $(2, -2)$ and $(-2, 2)$.

The curve has vertical tangent lines where $x = -4y$ (and not $x = y = 0$). Using the original equation, this means that $16y^2 + 4y^2 - 8y^2 = 12$, or $12y^2 = 12$, so $y = \pm 1$. Thus the points on the curve where the tangent line is vertical are $(-4, 1)$ and $(4, -1)$.

3.8.67 Differentiating with respect to x gives $18x + 2y\dfrac{dy}{dx} - 36 + 6\dfrac{dy}{dx} = 0$, so

$$\begin{aligned} 2y\frac{dy}{dx} + 6\frac{dy}{dx} &= -18x + 36 \\ (2y+6)\frac{dy}{dx} &= -18x + 36 \\ \frac{dy}{dx} &= \frac{-9x+18}{y+3}. \end{aligned}$$

This is zero when $x = 2$. Using the original equation, we have $36 + y^2 - 72 + 6y + 36 = 0$, or $y^2 + 6y = 0$, or $y = 0$ and $y = -6$. Thus there are horizontal tangent lines at $(2, 0)$ and $(2, -6)$. So $y = 0$ and $y = -6$ are horizontal tangent lines.

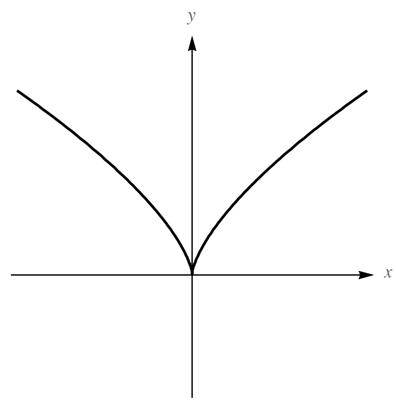
This curve has vertical tangent lines when $y = -3$. Using the original equation, we have $9x^2 + 9 - 36x - 18 + 36 = 0$ or $9x^2 - 36x + 27 = 0$, or $x^2 - 4x + 3 = 0$. This factors as $(x-3)(x-1) = 0$, so the corresponding values of x are 3 and 1. Thus the vertical tangent lines occur at $(3, -3)$ and $(1, -3)$.

3.8.68

a. $3y^2y' = 2ax$, so $y' = \frac{2ax}{3y^2}$.

b. $y = \sqrt[3]{ax^2}$.

c.



3.8.69

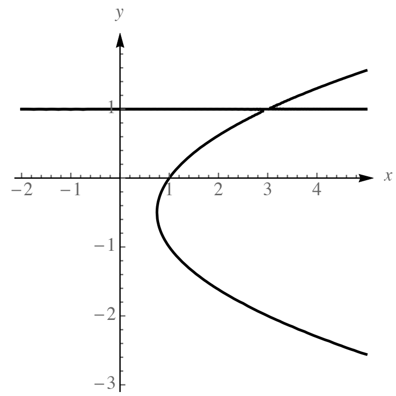
a. If we write $y^3 - 1 = xy - x$, we have $(y-1)(y^2+y+1) = x(y-1)$, so $y^2 + y + 1 = x$. Differentiating gives $2y\dfrac{dy}{dx} + \dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{1+2y}$. Note also that $y = 1$ satisfies the equation; $y' = 0$ on this branch.

b.

$$\begin{aligned} y^3 - 1 &= x(y-1) \\ (y-1)(y^2+y+1) &= x(y-1) \\ y^2+y+1 &= x \\ y^2 + y + (1-x) &= 0, \end{aligned}$$

so by the quadratic formula we have $y = \dfrac{-1 \pm \sqrt{4x-3}}{2}$. Note that this means that $\pm\sqrt{4x-3} = 2y+1$.

c.

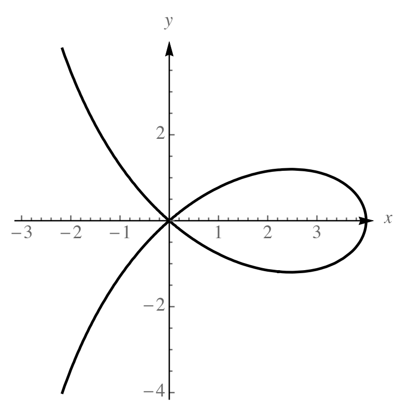


3.8.70

a. $2yy' = \dfrac{(2x(4-x)-x^2)(4+x)-x^2(4-x)}{(4+x)^2}$, so $2yy' = \dfrac{(8x-3x^2)(4+x)-4x^2+x^3}{(4+x)^2}$, and $2yy' = \dfrac{32x-8x^2-2x^3}{(4+x)^2}$. Thus $y' = \dfrac{16x-4x^2-x^3}{y(4+x)^2}$.

b. $y = \pm\sqrt{\dfrac{x^2(4-x)}{4+x}}$.

c.

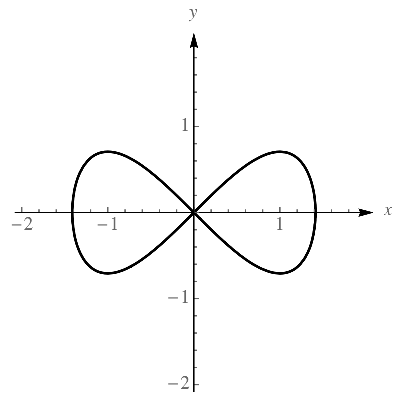


3.8.71

a. $4x^3 = 4x - 4yy'$, so $y' = \dfrac{x-x^3}{y}$.

b. $y = \pm\sqrt{x^2 - \dfrac{x^4}{2}}$.

c.

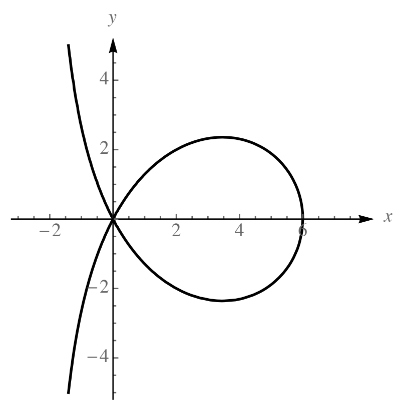


3.8.72

a. $2yy'(x+2)+y^2 = 12x-3x^2$. So $y' = \frac{12x-3x^2-y^2}{2y(x+2)}$.

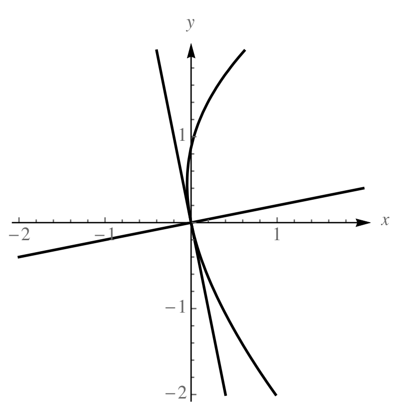
b. $y^2 = \dfrac{x^2(6-x)}{x+2}$, so $y = \pm\sqrt{\dfrac{x^2(6-x)}{x+2}}$.

c.



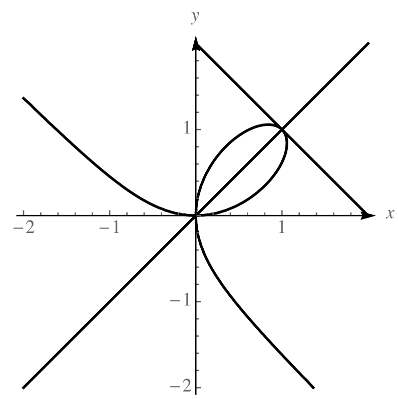
3.8.73

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 45: $y' = -5$, so the slope of the normal line is $\dfrac{1}{5}$. At the point (0,0), we have the normal line $y = \dfrac{1}{5}x$.



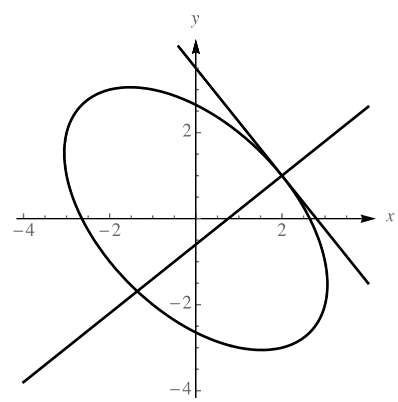
3.8.74

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 46: $y' = -1$, so the slope of the normal line is 1. At the point $(1,1)$ we have the line $y - 1 = x - 1$, or $y = x$.



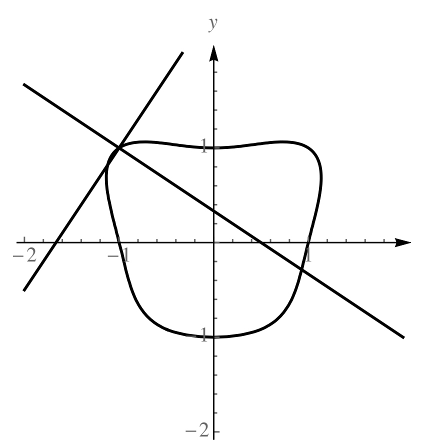
3.8.75

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 47: $y' = -\frac{5}{4}$, so the slope of the normal line is $\frac{4}{5}$. At the point $(2,1)$ we have the line $y - 1 = \frac{4}{5}(x-2)$, or $y = \frac{4}{5}x - \frac{3}{5}$.



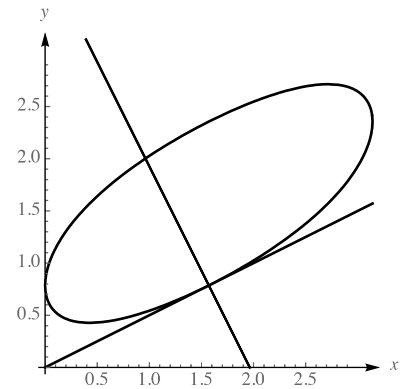
3.8.76

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 48: $y' = \frac{2}{3}$, so the slope of the normal line is $-\frac{3}{2}$. At the point $(-1,1)$ we have the line $y - 1 = -\frac{3}{2}(x+1)$, or $y = -\frac{3}{2}x - \frac{1}{2}$.



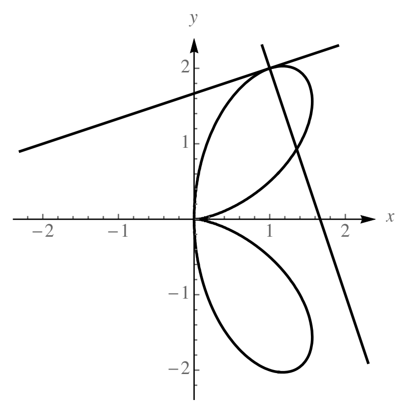
3.8.77

The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 49: $y' = \frac{1}{2}$, so the slope of the normal line is -2. At the point $(\frac{\pi}{2}, \frac{\pi}{4})$ we have the line $y - \frac{\pi}{4} = -2(x - \frac{\pi}{2})$, or $y = -2x + \frac{5\pi}{4}$.



3.8.78

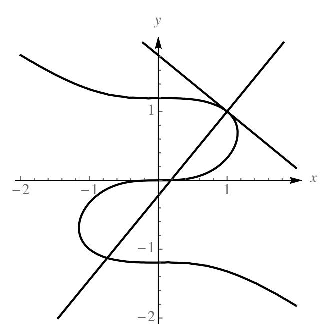
The slope of the normal line is the negative reciprocal of the slope of the tangent line. From 50: $y' = \frac{1}{3}$, so the slope of the normal line is -3. At the point $(1, 2)$ we have the line $y - 2 = -3(x - 1)$, or $y = -3x + 5$.



3.8.79

a. We have $9x^2 + 21y^2y' = 10y'$, so at the point $(1, 1)$ we have $9 + 21y' = 10y'$, so $y' = -\frac{9}{11}$. Thus, the tangent line is given by $y - 1 = -\frac{9}{11}(x-1)$, or $y = \frac{-9}{11}x + \frac{20}{11}$. The normal line is given by $y - 1 = \frac{11}{9}(x-1)$, or $y = \frac{11}{9}x - \frac{2}{9}$.

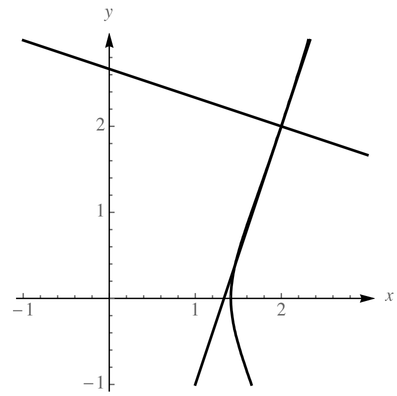
b.



3.8.80

a. We have $4x^3 = 4x + 4yy'$, so at the point $(2, 2)$ we have $32 = 8 + 8y'$, so $y' = 3$.

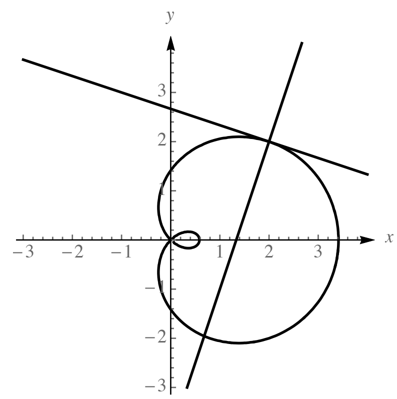
Thus, the tangent line is given by $y - 2 = 3(x - 2)$, or $y = 3x - 4$. The normal line is given by $y - 2 = -\frac{1}{3}(x - 2)$, or $y = -\frac{1}{3}x + \frac{8}{3}$.

b. 

3.8.81

a. We have $2(x^2 + y^2 - 2x)(2x + 2yy' - 2) = 4x + 4yy'$, so at the point $(2, 2)$ we have $2(4 + 4 - 4)(4 + 4y' - 2) = 8 + 8y'$, so $16 + 32y' = 8 + 8y'$, so $y' = -\frac{1}{3}$.

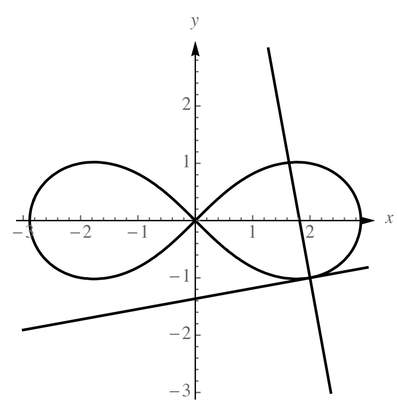
Thus, the tangent line is given by $y - 2 = -\frac{1}{3}(x - 2)$, or $y = \frac{-1}{3}x + \frac{8}{3}$. The normal line is given by $y - 2 = 3(x - 2)$, or $y = 3x - 4$.

b. 

3.8.82

a. We have $2(x^2 + y^2)(2x + 2yy') = \frac{25}{3}(2x - 2yy')$, so at the point $(2, -1)$ we have $2\cdot 5\cdot(4 - 2y') = \frac{25}{3}(4 + 2y')$, so $40 - 20y' = \frac{100}{3} + \frac{50}{3}y'$, so $120 - 60y' = 100 + 50y'$, and thus $y' = \frac{2}{11}$.

Thus, the tangent line is given by $y + 1 = \frac{2}{11}(x - 2)$, or $y = \frac{2}{11}x - \frac{15}{11}$. The normal line is given by $y + 1 = -\frac{11}{2}(x - 2)$, or $y = \frac{-11}{2}x + 10$.

b. 

3.8.83 Note for $y = mx$, $\dfrac{dy}{dx} = m = \dfrac{y}{x}$, and for $x^2 + y^2 = a^2$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$. So for any point (x, y), we have $\dfrac{y}{x}$ and $-\dfrac{x}{y}$ are negative reciprocals.

3.8.84 For $y = cx^2$ we have $y' = 2cx$ and for $x^2 + 2y^2 = k$, we have $y' = -\dfrac{x}{2y}$. Let (a, b) be a point on both curves. Then $b = ca^2$, so the point has the form (a, ca^2). A normal line to the ellipse $x^2 + 2y^2 = k$ would have slope $\dfrac{2y}{x} = \dfrac{2ca^2}{a} = 2ca$, which is the slope of the tangent line to the parabola $y = cx^2$ at the point in question. Thus the two curves are orthogonal at any points of intersection.

3.8.85 For $xy = a$ we have $xy' + y = 0$, so $y' = -\dfrac{y}{x}$. For $x^2 - y^2 = b$, we have $2x - 2yy' = 0$, so $y' = \dfrac{x}{y}$. Let (c, d) be a point on both curves. Then the slope of the normal line to the first curve is $\dfrac{c}{d}$, but that is the slope of the tangent line to the second curve. Thus the two curves are orthogonal at any points of intersection.

3.8.86

$\dfrac{5}{2\sqrt{x}} - \dfrac{5y'}{\sqrt{y}} = \cos x$, so $\dfrac{5y'}{\sqrt{y}} = \dfrac{5}{2\sqrt{x}} - \cos x$, and thus $y' = \dfrac{\sqrt{y}}{5} \cdot \left(\dfrac{5}{2\sqrt{x}} - \cos x\right) = \dfrac{\sqrt{y}}{5} \cdot \left(\dfrac{5 - 2\sqrt{x}\cos x}{2\sqrt{x}}\right)$

At the point $(4\pi, \pi)$ we have $y'(4\pi, \pi) = \dfrac{\sqrt{\pi}}{5} \cdot \left(\dfrac{5 - 2\sqrt{4\pi}(1)}{2\sqrt{4\pi}}\right) = \dfrac{5 - 4\sqrt{\pi}}{20}$.

3.8.87

$$\begin{aligned}
(2x + 2yy')(x^2 + y^2 + x) + (x^2 + y^2)(2x + 2yy' + 1) &= 8y^2 + 16xyy' \\
2yy'(x^2 + y^2 + x) + (x^2 + y^2)2yy' - 16xyy' &= 8y^2 - 2x(x^2 + y^2 + x) - (x^2 + y^2)(2x + 1) \\
y' &= \frac{8y^2 - 2x(x^2 + y^2 + x) - (x^2 + y^2)(2x + 1)}{2y(x^2 + y^2 + x) + 2y(x^2 + y^2) - 16xy} \\
&= \frac{8y^2 - 2x^3 - 2xy^2 - 2x^2 - 2x^3 - x^2 - 2xy^2 - y^2}{2y(x^2 + y^2 + x + x^2 + y^2 - 8x)} \\
&= \frac{7y^2 - 3x^2 - 4xy^2 - 4x^3}{2y(2x^2 + 2y^2 - 7x)}.
\end{aligned}$$

3.8.88

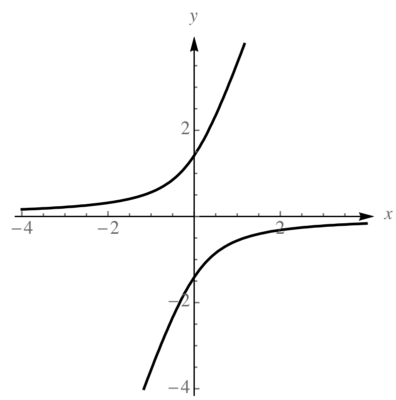
$$\begin{aligned}
\frac{21x^6 + 2yy'}{2\sqrt{3x^7 + y^2}} &= 2y' \sin y \cos y + 100(y + xy') \\
21x^6 + 2yy' &= 4y' \sin y \cos y\sqrt{3x^7 + y^2} + 200y\sqrt{3x^7 + y^2} + 200xy'\sqrt{3x^7 + y^2} \\
200y\sqrt{3x^7 + y^2} - 21x^6 &= 2yy' - 4y' \sin y \cos y\sqrt{3x^7 + y^2} - 200xy'\sqrt{3x^7 + y^2} \\
y' &= \frac{200y\sqrt{3x^7 + y^2} - 21x^6}{2y - 4\sin y \cos y\sqrt{3x^7 + y^2} - 200x\sqrt{3x^7 + y^2}}.
\end{aligned}$$

3.8.89 $\dfrac{y'}{2\sqrt{y}} + y + xy' = 0$, so $y' + 2x\sqrt{y}y' = -2y\sqrt{y}$, so $y' = \dfrac{-2y\sqrt{y}}{2x\sqrt{y} + 1} = -\dfrac{2y^{3/2}}{2x\sqrt{y} + 1}$.
Differentiating again we obtain

$$\begin{aligned}
y'' &= -\frac{(2x\sqrt{y} + 1)(3\sqrt{y}y') - 2y^{3/2}\left(2\sqrt{y} + \frac{xy'}{\sqrt{y}}\right)}{(2x\sqrt{y} + 1)^2} = \frac{-(2x\sqrt{y} + 1)(3\sqrt{y}y') + 4y^2 + 2xyy'}{(2x\sqrt{y} + 1)^2} \\
&= \left(\frac{-(2x\sqrt{y} + 1)(3\sqrt{y})\left(\frac{-2y\sqrt{y}}{2x\sqrt{y}+1}\right) + 4y^2 + 2xy\left(\frac{-2y\sqrt{y}}{1+2x\sqrt{y}}\right)}{(2x\sqrt{y} + 1)^2}\right) \cdot \frac{2x\sqrt{y} + 1}{2x\sqrt{y} + 1} \\
&= \frac{(2x\sqrt{y} + 1)(6y^2) + 4y^2(1 + 2x\sqrt{y}) - 4xy^{5/2}}{(2x\sqrt{y} + 1)^3} = \frac{10y^2 + 16xy^2\sqrt{y}}{(2x\sqrt{y} + 1)^3}.
\end{aligned}$$

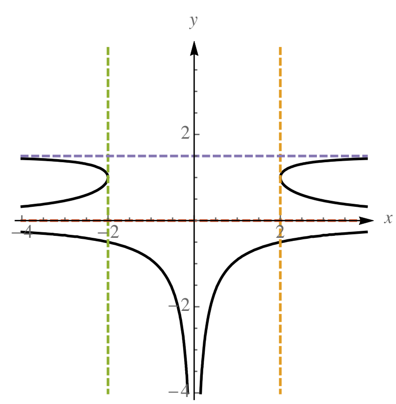
3.8.90 Differentiating with respect to x yields $2yy' - 3y - 3xy' = 0$; simplifying gives $y' = \frac{3y}{2y-3x}$. There could be a horizontal tangent line when $y = 0$, but using the original equation we see that there are no points on the curve where this occurs. (Letting $y = 0$ in the original equation gives the untrue equation $0 = 2$).

Considering where the tangent line might be vertical we consider points where $2y = 3x$. Using the original equation and replacing $3x$ by $2y$, we obtain $y^2 - 2y^2 = 2$, or $-y^2 = 2$, which is again impossible. So the tangent is never vertical nor horizontal.

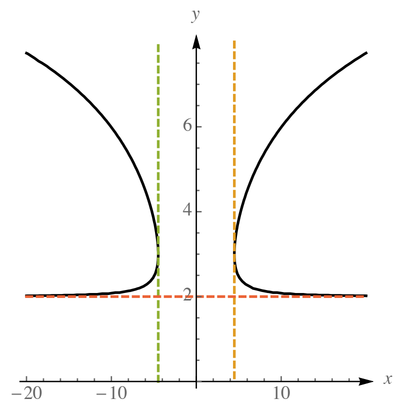


3.8.91 Differentiating with respect to x yields $2x(3y^2 - 2y^3) + x^2(6y - 6y^2)\dfrac{dy}{dx} = 0$. Solving for $\dfrac{dy}{dx}$ yields $\dfrac{dy}{dx} = \dfrac{-2x(3y^2 - 2y^3)}{6x^2(y - y^2)} = \dfrac{3y^2 - 2y^3}{3x(y^2 - y)} = \dfrac{3y - 2y^2}{3x(y - 1)}$. The numerator is zero when $y = 0$ or when $y = \frac{3}{2}$. Using the original equation, there are no points where $y = 0$, and there are also no points where $y = \frac{3}{2}$ because we obtain the equation $x^2(\frac{27}{4} - \frac{27}{4}) = 4$, which has no solutions. So there are no horizontal tangent lines.

There could be vertical tangent lines where $x = 0$ or $y = 1$; in fact, when $y = 1$ the original equation becomes $x^2(3 - 2) = 4$, so $x = \pm 2$. There are vertical tangents at $(2, 1)$ and $(-2, 1)$. Letting $x = 0$ doesn't yield any points in the original equation.

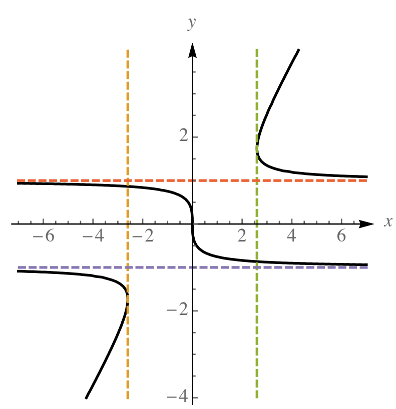


3.8.92 Differentiating with respect to x yields $2x(y-2)+x^2y'-e^yy' = 0$, and solving for y' yields $y' = \frac{2x(2-y)}{x^2-e^y}$. This quantity is zero for $x = 0$ or $y = 2$. Using the original equation we find no solutions for $x = 0$ or $y = 2$, so there are no horizontal tangent lines. There could be vertical tangent lines where $x^2 = e^y$. Using the original equation (and disallowing $x^2 = 0$) we have $x^2(y - 2) - x^2 = 0$, so $x^2(y - 2 - 1) = 0$, so $y = 3$. Using the original equation with $y = 3$ we have $x^2 - e^3 = 0$, so $x = \pm\sqrt{e^3}$. There are vertical tangent lines at $(\sqrt{e^3}, 3)$ and $-\sqrt{e^3}, 3)$.



3.8.93 Differentiating with respect to x yields $(1-y^2)+x(-2yy')+3y^2y'=0$. Solving for y' yields $y'=\dfrac{y^2-1}{3y^2-2xy}=\dfrac{y^2-1}{y(3y-2x)}$. There could be a horizontal tangent line where $y=\pm 1$, but the original equation with $y=\pm 1$ yields $\pm 1=0$, so there are no horizontal tangent lines.

There could be a vertical tangent line for $y=0$ or for $y=\dfrac{2}{3}x$. Using the original equation, letting $y=0$ yields $x=0$, and letting $y=\dfrac{2}{3}x$ yields $x\left(1-\dfrac{4x^2}{9}\right)+\dfrac{8x^3}{27}=0$. For $x\neq 0$, this yields $27-12x^2+8x^2=0$, or $x^2=\frac{27}{4}$, so $x=\pm\dfrac{\sqrt{27}}{2}=\pm\dfrac{3\sqrt{3}}{2}$. The corresponding y values are $y=\pm\dfrac{2}{3}x=\pm\sqrt{3}$. So there are vertical tangent lines at $(0,0)$ and at $(3\sqrt{3}/2,\sqrt{3})$ and $(-3\sqrt{3}/2,-\sqrt{3})$.

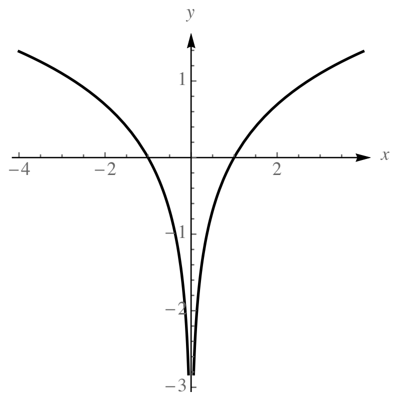


3.9 Derivatives of Logarithmic and Exponential Functions

3.9.1 $y=\ln x$ if and only if $x=e^y$. Differentiating implicitly yields $1=e^y\cdot y'$, so $y'=\frac{1}{e^y}=\frac{1}{e^{\ln x}}=\frac{1}{x}$ for $x>0$.

3.9.2

We have already established that if $y = \ln x$ for $x > 0$ then $y' = \frac{1}{x}$. By the symmetry about the y-axis, we know that for $x < 0$, the derivative of $y = \ln|x|$ should have the same absolute value but the opposite sign of the derivative for the corresponding positive x value. But this is the property that $\frac{1}{x}$ has for $x < 0$ – it is negative and has the right absolute value. So we see that for both $x > 0$ and $x < 0$, $\frac{d}{dx}\ln|x| = \frac{1}{x}$.



3.9.3 $\frac{d}{dx}\ln(kx) = \frac{1}{kx}\cdot k = \frac{1}{x}$. This is valid for $x > 0$ if $k > 0$ and $x < 0$ if $k < 0$. Also, we can write $\ln(kx) = \ln(k) + \ln(x)$, so its derivative is $0 + \frac{1}{x} = \frac{1}{x}$.

3.9.4 $\frac{d}{dx}b^x = b^x \ln b$, for $b > 0$ and all x. In the case $b = e$, the rule states that $\frac{d}{dx}e^x = e^x \ln e = e^x$ because $\ln e = 1$.

3.9.5 $\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$ for $b > 0$, $b \neq 1$ and $x > 0$. If $b = e$, we have $\frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x}$.

3.9.6 We use inverse property 3 from the text: $b^x = (e^{\ln b})^x = e^{(\ln b)\cdot x} = e^{x\ln b}$.

3.9.7 $e^{x\ln(x^2+1)} = e^{\ln(x^2+1)^x} = (x^2+1)^x$.

3.9.8 $\frac{d}{dx}\left(\ln\left(\frac{x}{x^2+1}\right)\right) = \frac{d}{dx}\left(\ln x - \ln(x^2+1)\right) = \frac{1}{x} - \frac{2x}{x^2+1} = \frac{x^2+1}{x(x^2+1)} - \frac{2x^2}{x(x^2+1)} = \frac{1-x^2}{x(x^2+1)}$.

3.9.9 $\frac{d}{dx}\left(\left(\ln\sqrt{x^2+1}\right)\right) = \frac{d}{dx}\left(\ln(x^2+1)^{1/2}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x^2+1)\right) = \frac{1}{2}\frac{2x}{(x^2+1)} = \frac{x}{x^2+1}$.

3.9.10 $\frac{d}{dx}(x^e + e^x) = ex^{e-1} + e^x$.

3.9.11 $f(x) = e^{\ln(g(x)^{h(x)})} = e^{h(x)\cdot\ln(g(x))}$.

3.9.12 $\frac{d}{dx}\left(\ln\sqrt{f(x)}\right) = \frac{d}{dx}(\ln(f(x)^{1/2})) = \frac{d}{dx}\left(\frac{1}{2}\ln(f(x))\right) = \frac{1}{2}\cdot\frac{d}{dx}(\ln(f(x))) = \frac{1}{2}\cdot\frac{f'(x)}{f(x)}$.

3.9.13 $\frac{d}{dx}(\ln(xe^x)) = \frac{d}{dx}(\ln x + \ln e^x) = \frac{d}{dx}(\ln x + x) = \frac{1}{x} + 1 = \frac{1+x}{x}$.

3.9.14 $\frac{d}{dx}(\ln x^{101}) = \frac{d}{dx}(101\ln x) = \frac{101}{x}$.

3.9.15 $\frac{d}{dx}(\ln(7x)) = \frac{d}{dx}(\ln 7 + \ln x) = 0 + \frac{1}{x} = \frac{1}{x}$.

3.9.16 $\frac{d}{dx}(x^2\ln x) = 2x\ln x + x^2\cdot\frac{1}{x} = 2x\ln x + x = x(2\ln x + 1)$.

3.9.17 $\dfrac{d}{dx}(\ln(x^2)) = \dfrac{d}{dx}(2\ln x) = \dfrac{2}{x}$.

3.9.18 $\dfrac{d}{dx}(\ln(2x^8)) = \dfrac{d}{dx}(8(\ln 2 + \ln x)) = 8(0 + \dfrac{1}{x}) = \dfrac{8}{x}$.

3.9.19 $\dfrac{d}{dx}(\ln|\sin x|) = \dfrac{1}{\sin x} \cdot (\cos x) = \cot x$.

3.9.20

$$\frac{d}{dx}\left(\frac{1+\ln x^3}{4x^3}\right) = \frac{d}{dx}\left(\frac{1+3\ln x}{4x^3}\right) = \frac{4x^3\left(\frac{3}{x}\right) - (1+3\ln x)12x^2}{16x^4}$$
$$= \frac{12x^2 - 12x^2 - 36x^2\ln x}{16x^4} = -\frac{9\ln x}{4x^4}.$$

3.9.21 $\dfrac{d}{dx}\ln(x^4+1) = \dfrac{1}{x^4+1} \cdot 4x^3 = \dfrac{4x^3}{x^4+1}$.

3.9.22 $\dfrac{d}{dx}\left(\ln\sqrt{x^4+x^2}\right) = \dfrac{d}{dx}\left(\dfrac{1}{2}\ln(x^4+x^2)\right) = \dfrac{4x^3+2x}{2(x^4+x^2)} = \dfrac{2x^3+x}{x^4+x^2} = \dfrac{2x^2+1}{x^3+x}$.

3.9.23 $\dfrac{d}{dx}\left(\ln\left(\dfrac{x+1}{x-1}\right)\right) = \dfrac{(x-1)}{(x+1)}\left(\dfrac{(x-1)-(x+1)}{(x-1)^2}\right) = -\dfrac{2}{(x+1)(x-1)} = \dfrac{2}{1-x^2}$.

3.9.24 $\dfrac{d}{dx}(x\ln x - x) = 1 \cdot \ln x + \dfrac{x}{x} - 1 = \ln x$.

3.9.25 $\dfrac{d}{dx}((x^2+1)\ln x) = 2x\ln x + \dfrac{x^2+1}{x}$.

3.9.26 $\dfrac{d}{dx}(\ln|x^2-1|) = \dfrac{1}{x^2-1} \cdot 2x = \dfrac{2x}{x^2-1}$.

3.9.27 $\dfrac{d}{dx}(x^2(1-\ln x^2)) = 2x(1-\ln x^2) + x^2\left(\dfrac{-1}{x^2}\right)2x = 2x - 2x\ln x^2 - 2x = -2x\ln x^2$.

3.9.28 $\dfrac{d}{dx}(3x^3\ln x - x^3) = 9x^2\ln x + 3x^3\left(\dfrac{1}{x}\right) - 3x^2 = 9x^2\ln x$.

3.9.29 $\dfrac{d}{dx}\ln(\ln x) = \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.

3.9.30 $\dfrac{d}{dx}(\ln(\cos^2 x)) = \dfrac{1}{\cos^2 x} \cdot (-2\sin x\cos x) = -2\tan x$.

3.9.31 $\dfrac{d}{dx}\left(\dfrac{\ln x}{\ln x+1}\right) = \dfrac{(\ln x+1)(1/x) - (\ln x)(1/x)}{(\ln x+1)^2} = \dfrac{1}{x(\ln x+1)^2}$.

3.9.32 $\dfrac{d}{dx}(\ln(e^x+e^{-x})) = \dfrac{1}{e^x+e^{-x}}(e^x-e^{-x}) = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

3.9.33 $y' = ex^{e-1}$.

3.9.34 $y' = e^x x^e + e^{x+1}x^{e-1} = e^x x^{e-1}(x+e)$.

3.9.35 $y' = \pi(2^x+1)^{\pi-1}2^x\ln 2$.

3.9.36 $y' = \dfrac{d}{dx}(\pi \cdot \ln(x^3+1)) = \pi \cdot \dfrac{3x^2}{x^3+1} = \dfrac{3\pi x^2}{x^3+1}$.

3.9.37 $y' = 8^x \ln 8$.

3.9.38 $y' = 5^{3t} \cdot \ln 5 \cdot 3 = 3(\ln 5)(5^{3t})$.

3.9.39 $y' = 5 \cdot \dfrac{d}{dx} 4^x = 5 \cdot \ln 4 \cdot 4^x$.

3.9.40 $y' = -\ln 4 \cdot 4^{-x} \cdot \sin x + 4^{-x} \cos x$.

3.9.41 $y' = 2^{3+\sin x}(\ln 2)\cos x$

3.9.42 $y' = 10^{\ln 2x}(\ln 10)\dfrac{1}{x}$.

3.9.43 $y' = 3x^2 3^x + x^3 3^x \ln 3 = 3^x x^2(3 + x\ln 3)$.

3.9.44 $\dfrac{dP}{dt} = \dfrac{0 - 40(-\ln 2 \cdot 2^{-t})}{(1+2^{-t})^2} = \dfrac{40\ln 2 \cdot 2^{-t}}{(1+2^{-t})^2}$.

3.9.45 $\dfrac{dA}{dt} = 250(1.045)^{4t} \cdot \ln(1.045) \cdot 4 = 1000 \cdot \ln 1.045 \cdot (1.045^{4t})$.

3.9.46 $\dfrac{d}{dx} 10^x(\ln 10^x - 1) = \dfrac{d}{dx}(10^x(x\ln 10 - 1)) = (10^x \ln 10)(x\ln 10 - 1) + 10^x \ln 10 = x10^x(\ln 10)^2$.

3.9.47 $f'(x) = \dfrac{(2^x+1)2^x \ln 2 - 2^x(2^x \ln 2)}{(2^x+1)^2} = \dfrac{2^x \ln 2}{(2^x+1)^2}$.

3.9.48 $s'(t) = -\sin(2^t) \cdot 2^t \ln 2$.

3.9.49 Let $y = x^{\cos x}$. Then $\ln y = \cos x \ln x$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = (-\sin x)\ln x + \cos x \cdot \frac{1}{x}.$$

Therefore,

$$y' = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

At $\pi/2$ we have $y'(\pi/2) = (\pi/2)^0(0/(\pi/2) - \ln(\pi/2)) = -\ln(\pi/2)$.

3.9.50 Let $y = x^{\ln x}$. Then $\ln y = \ln x \ln x = (\ln x)^2$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = 2\ln x \cdot \frac{1}{x}.$$

Therefore,

$$y' = x^{\ln x - 1} \cdot 2\ln x.$$

At e we have $y'(e) = e^{1-1} \cdot 2\ln e = 2$.

3.9.51 Let $y = x^{\sqrt{x}}$. Then $\ln y = \sqrt{x} \cdot \ln x$. Differentiating both sides gives

$$\frac{1}{y} \cdot y' = \frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x}.$$

Therefore,

$$y' = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right).$$

At 4 we have $y'(4) = 4^2\left(\frac{\ln 4 + 2}{4}\right) = 4\ln 4 + 8$.

3.9.52 Because $f(x) = (x^2+1)^x$, we have $\ln f(x) = x\ln(x^2+1)$. Thus,

$$\frac{1}{f(x)} f'(x) = (1)\ln(x^2+1) + x \cdot \frac{1}{x^2+1} \cdot 2x.$$

Therefore,

$$f'(x) = (x^2+1)^x \left(\ln(x^2+1) + \frac{2x^2}{x^2+1} \right).$$

We have $f'(1) = (1+1)^1(\ln(1+1) + \frac{2}{1+1}) = 2(\ln 2 + 1) = 2\ln 2 + 2$.

3.9.53 Because $f(x) = (\sin x)^{\ln x}$, we have $\ln f(x) = \ln x \ln \sin x$. Differentiating both sides gives

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} \cdot \ln \sin x + \ln x \cdot \frac{1}{\sin x} \cos x.$$

Therefore,

$$f'(x) = (\sin x)^{\ln x} \left(\frac{\ln \sin x + x \ln x \cot x}{x} \right).$$

We have $f'(\pi/2) = 0$ because $\cot \pi/2 = 0$ and $\ln \sin(\pi/2) = \ln 1 = 0$.

3.9.54 Because $f(x) = \tan^{x-1} x$, we have $\ln f(x) = (x-1)\ln \tan x$. Differentiating both sides gives

$$\frac{1}{f(x)} f'(x) = (1)\ln \tan x + (x-1)\frac{1}{\tan x} \sec^2 x.$$

Therefore,

$$f'(x) = (\tan^{x-1} x)(\ln \tan x + (x-1)\csc x \sec x).$$

We have $f'(\pi/4) = (1)^{\pi/4-1}(\ln 1 + (\pi/4 - 1)(\sqrt{2})(\sqrt{2})) = \pi/2 - 2$.

3.9.55 Because $f(x) = (4\sin x + 2)^{\cos x}$, we have $\ln f(x) = \cos x \ln(4\sin x + 2)$. Differentiating both sides gives

$$\frac{1}{f(x)} f'(x) = -\sin x \cdot \ln(4\sin x + 2) + \cos x \cdot \frac{4\cos x}{(4\sin x + 2)}.$$

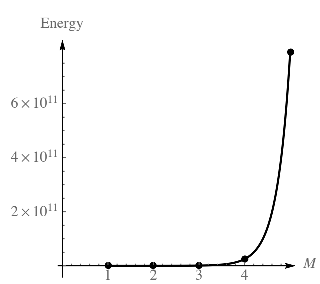
Therefore,

$$f'(x) = (4\sin x + 2)^{\cos x} \left(-\sin x \ln(4\sin x + 2) + \frac{2\cos^2 x}{(2\sin x + 1)} \right).$$

We have $f'(\pi) = 2^{-1}\left(0 + \frac{2}{1}\right) = \frac{1}{2} \cdot \frac{2}{1} = 1$.

3.9.56

a.



b. $\dfrac{dE}{dM} = 25000 \cdot 1.5 \cdot \ln 10 \cdot 10^{1.5M}$. At $M = 3$ we have $\dfrac{dE}{dM} = 25000 \cdot 1.5 \cdot \ln 10 \cdot 10^{9/2} \approx 2,730,530,025$ Joules per unit change in M. As the magnitude goes from 3 to 4, the energy goes up by this amount.

3.9.57

a. $T = 10 \cdot 2^{-0.274 \cdot 16}$ minutes ≈ 28.7 seconds.

b. $\dfrac{\Delta T}{\Delta a} = \dfrac{10 \cdot 2^{-0.274 \cdot 8} - 10 \cdot 2^{-0.274 \cdot 2}}{8-2} \approx -0.78$ minutes per 1000 feet, which is about -46.512 seconds per 1000 feet.

c. $\dfrac{dT}{da} = -2.74 \cdot 2^{-0.274 \cdot a} \cdot \ln 2$. At $a = 8$ we have $\dfrac{dT}{da} = -2.74 \cdot 2^{-0.274 \cdot 8} \cdot \ln 2 \approx -0.42$ minutes per 1000 feet. Every 1000 feet the airplane climbs, leaves about .42 minutes less time of consciousness, which corresponds to about 24.94 seconds.

3.9.58

a. At $Q = 10\mu$Ci we have $10 = 350 \cdot \left(\frac{1}{2}\right)^{t/13.1}$, so $\ln(1/35) = \frac{t}{13.1}\ln(1/2)$, so $t = 13.1 \cdot \frac{\ln 35}{\ln 2} \approx 67.19$ hours.

b. $\dfrac{dQ}{dt} = \dfrac{350}{13.1} \cdot \ln\left(\dfrac{1}{2}\right) \cdot \left(\dfrac{1}{2}\right)^{t/13.1}$. We have $Q'(12) \approx -9.81453$, $Q'(24) \approx -5.20136$, and $Q'(48) \approx -1.46087$. The rate at which the iodine decreases is decreasing in absolute value as time increases.

3.9.59 Let $y = x^{\sin x}$. Then $\ln y = \sin x \ln x$, so $\dfrac{1}{y} \cdot y' = \cos x \ln x + \dfrac{\sin x}{x}$. At the point $(1,1)$ we have $y' = \sin 1$, so the tangent line is given by $y - 1 = (\sin 1)(x-1)$, or $y = (\sin 1)x + 1 - \sin 1$.

3.9.60 Let $y = x^{\sqrt{x}}$. Then we have $\ln y = \sqrt{x} \cdot \ln x$, so $\dfrac{1}{y} \cdot y' = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}$, so $y' = x^{\sqrt{x}}\left(\dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}\right) = x^{\sqrt{x}}\left(\dfrac{\sqrt{x}\ln x + 2\sqrt{x}}{2x}\right)$. This expression is zero only when $\ln x + 2 = 0$, or $x = e^{-2}$.

3.9.61 Let $y = (x^2)^x = x^{2x}$. Then $\ln y = x \ln x^2$ and $\dfrac{1}{y} \cdot y' = \ln x^2 + 2$, so $y' = x^{2x}(\ln x^2 + 2)$. This quantity is zero when $\ln x^2 = -2$, or $x^2 = e^{-2}$. Thus there are horizontal tangents at $|x| = e^{-1}$, so for $x = \pm\frac{1}{e}$. The two tangent lines are given by $y = \dfrac{1}{e^{2/e}}$ (at $\left(\dfrac{1}{e}, \dfrac{1}{e^{2/e}}\right)$) and $y = e^{2/e}$ (at $\left(-\dfrac{1}{e}, e^{2/e}\right)$.)

3.9.62 Let $y = x^{\ln x}$. Then $\ln y = (\ln x)^2$. Thus $\dfrac{1}{y} \cdot y' = 2\ln x \cdot \dfrac{1}{x}$, so $y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$. This quantity is zero when $\ln x = 0$, which is at $x = 1$. The equation of the tangent line at $(1,1)$ is therefore $y = 1$.

3.9.63 $y' = 4 \cdot \dfrac{2x}{(x^2-1) \cdot \ln 3} = \dfrac{8x}{(x^2-1) \cdot \ln 3}$.

3.9.64 $y' = \dfrac{1}{x \ln 10}$.

3.9.65 $y' = -\sin x(\ln(\cos^2 x)) + \cos x \cdot \left(\dfrac{2\cos x(-\sin x)}{\cos^2 x}\right) = (-\sin x)(\ln(\cos^2 x) + 2)$.

3.9.66 $y' = \dfrac{1}{\ln 8 \tan x} \cdot \sec^2 x$.

3.9.67 $y' = \dfrac{d}{dx}(\log_4 x)^{-1} = -(\log_4 x)^{-2} \cdot \dfrac{1}{x \ln 4} = -\dfrac{1}{x(\ln 4)(\log_4 x)^2} = -\dfrac{\ln 4}{x \ln^2 x}$.

3.9.68 $y' = \dfrac{1}{(\ln 2)(\log_2 x)} \cdot \dfrac{1}{x \ln 2} = \dfrac{1}{(\ln 2)^2 \cdot x \cdot \log_2 x} = \dfrac{1}{x(\ln 2)\ln x}$.

3.9.69 $f'(x) = \dfrac{d}{dx}(4\ln(3x+1)) = \dfrac{4}{3x+1} \cdot 3 = \dfrac{12}{3x+1}$.

3.9.70 $f'(x) = \dfrac{d}{dx}(\ln 2x - 3\ln(x^2+1)) = \dfrac{1}{x} - \dfrac{6x}{x^2+1}$.

3.9.71 $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{2}\ln 10x\right) = \dfrac{d}{dx}\dfrac{1}{2}[\ln 10 + \ln x] = \dfrac{1}{2x}$.

3.9.72 $f'(x) = \dfrac{d}{dx}\left(\log_2 2^3 - \dfrac{1}{2}\log_2(x+1)\right) = 0 - \dfrac{1}{2}\cdot\dfrac{1}{(x+1)\ln 2} = -\dfrac{1}{(\ln 4)(x+1)}$.

3.9.73 $f'(x) = \dfrac{d}{dx}(\ln(2x-1) + 3\ln(x+2) - 2\ln(1-4x)) = \dfrac{2}{2x-1} + \dfrac{3}{x+2} + \dfrac{8}{1-4x}$.

3.9.74 $f'(x) = \dfrac{d}{dx}(4\ln(\sec x) + 2\ln(\tan x)) = \dfrac{4\sec x\tan x}{\sec x} + \dfrac{2\sec^2 x}{\tan x} = 4\tan x + 2\sec x\csc x$.

3.9.75 Let $y = x^{10x}$. Then $\ln y = 10x\ln x$, so $\dfrac{1}{y}\cdot y' = 10\ln x + 10$, and $y' = x^{10x}(10)(\ln x + 1)$.

3.9.76 Let $y = (2x)^{2x}$. Then $\ln y = 2x\ln(2x)$, and $\dfrac{1}{y}\cdot y' = 2\ln(2x) + 2$, so $y' = (2x)^{2x}(2)(\ln(2x)+1)$.

3.9.77 Let $y = \dfrac{(x+1)^{10}}{(2x-4)^8}$, so $\ln y = \ln\left(\dfrac{(x+1)^{10}}{(2x-4)^8}\right) = 10\ln(x+1) - 8\ln(2x-4)$. Then

$$\frac{1}{y}\cdot y' = \frac{10}{x+1} - \frac{8}{2x-4}\cdot 2,$$

$$y' = \frac{(x+1)^{10}}{(2x-4)^8}\cdot\left(\frac{10}{x+1} - \frac{8}{x-2}\right).$$

3.9.78 Let $y = x^2\cos x$. Then $\ln y = \ln(x^2\cos x) = 2\ln x + \ln(\cos x)$. So $\dfrac{1}{y}\cdot y' = \dfrac{2}{x} + \dfrac{1}{\cos x}\cdot(-\sin x)$, so

$$y' = x^2\cos x\cdot\left(\frac{2}{x} + \frac{1}{\cos x}\cdot(-\sin x)\right) = 2x\cos x - x^2\sin x.$$

3.9.79 Let $y = x^{\ln x}$. Then $\ln y = (\ln x)^2$. Thus $\dfrac{1}{y}\cdot y' = 2\ln x\cdot\dfrac{1}{x}$, so $y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$.

3.9.80 Let $y = \dfrac{\tan^{10} x}{(5x+3)^6}$. Then $\ln y = \ln\left(\dfrac{\tan^{10} x}{(5x+3)^6}\right) = 10\ln(\tan x) - 6\ln(5x+3)$. Then

$$\frac{1}{y}\cdot y' = \frac{10}{\tan x}\sec^2 x - \frac{6}{5x+3}\cdot 5,$$

$$y' = \frac{\tan^{10} x}{(5x+3)^6}\left(\frac{10\sec^2 x}{\tan x} - \frac{30}{5x+3}\right).$$

3.9.81 Let $y = \dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$. Then $\ln y = \ln\left(\dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\right) = \dfrac{3}{2}\ln(x+1) + \dfrac{5}{2}\ln(x-4) - \dfrac{2}{3}\ln(5x+3)$. Then

$$\frac{1}{y}\cdot y' = \frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)},$$

$$y' = \frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\cdot\left(\frac{3}{2(x+1)} + \frac{5}{2(x-4)} - \frac{10}{3(5x+3)}\right).$$

3.9.82 Let $y = \dfrac{x^8\cos^3 x}{\sqrt{x-1}}$. Then $\ln y = \ln\left(\dfrac{x^8\cos^3 x}{\sqrt{x-1}}\right) = 8\ln x + 3\ln\cos x - \dfrac{1}{2}\ln(x-1)$. Then

$$\frac{1}{y}\cdot y' = \frac{8}{x} - \frac{3\sin x}{\cos x} - \frac{1}{2x-2},$$

$$y' = \frac{x^8\cos^3 x}{\sqrt{x-1}}\left(\frac{8}{x} - 3\tan x - \frac{1}{2x-2}\right).$$

3.9.83 Let $y = (\sin x)^{\tan x}$, and assume $0 < x < \pi$, $x \neq \frac{\pi}{2}$. Then $\ln y = (\tan x)\ln(\sin x)$. Then

$$\frac{1}{y}\cdot y' = (\sec^2 x)\ln(\sin x) + \frac{\tan x\cos x}{\sin x},$$

$$y' = (\sin x)^{\tan x}\left((\sec^2 x)\ln(\sin x) + 1\right).$$

3.9.84 Let $y = (1+x^2)^{\sin x}$. Then $\ln y = \sin x\cdot\ln(1+x^2)$, so $\dfrac{1}{y}\cdot y' = \cos x\cdot\ln(1+x^2) + \sin x\cdot\dfrac{2x}{1+x^2}$. Therefore we have $y' = (1+x^2)^{\sin x}\left(\cos x\cdot\ln(1+x^2) + \dfrac{2x\sin x}{1+x^2}\right)$.

3.9.85 Let $y = \left(1+\dfrac{1}{x}\right)^x$. Then $\ln y = x\ln\left(1+\dfrac{1}{x}\right)$, so $\dfrac{1}{y}\cdot y' = \ln\left(1+\dfrac{1}{x}\right) + x\left(\dfrac{-1/x^2}{1+1/x}\right) = \ln\left(1+\dfrac{1}{x}\right) - \dfrac{1}{x+1}$. Therefore, $y' = \left(1+\dfrac{1}{x}\right)^x\left(\ln\left(1+\dfrac{1}{x}\right) - \dfrac{1}{x+1}\right)$.

3.9.86 Let $y = x^{x^{10}}$. Then $\ln y = x^{10}\ln x$, so $\dfrac{1}{y}\cdot y' = 10x^9\ln x + \dfrac{x^{10}}{x} = x^9(10\ln x + 1)$. Thus $y' = x^{x^{10}}\cdot x^9(10\ln x + 1)$.

3.9.87

a. False. $\log_2 9$ is a constant, so its derivative is 0.

b. False. If $x < -1$, then the right-hand side is defined while the left-hand side isn't.

c. False. The correct way to write that function would be $e^{(x+1)\ln 2}$.

d. False. $\dfrac{d}{dx}(\sqrt{2})^x = (\sqrt{2})^x\ln(\sqrt{2})$.

e. True. This follows from the generalized power rule.

f. True. Note that $\ln((4x+1)^{\ln x}) = (\ln x)\ln(4x+1) = \ln(4x+1)(\ln x) = \ln(x^{\ln(4x+1)})$. Then we have $\ln((4x+1)^{\ln x}) = \ln(x^{\ln(4x+1)})$, and exponentiating both sides gives the desired result.

3.9.88 $\dfrac{d^2}{dx^2}\left(\ln(x^2+1)\right) = \dfrac{d}{dx}\left(\dfrac{2x}{x^2+1}\right) = \dfrac{(x^2+1)\cdot 2 - 2x(2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

3.9.89 $\dfrac{d^2}{dx^2}(\log x) = \dfrac{d}{dx}\left(\dfrac{1}{x\ln 10}\right) = -\dfrac{1}{x^2\ln 10}$.

3.9.90 $\dfrac{d}{dx}(2^x) = (2^x)\ln 2$. $\dfrac{d^2}{dx^2}(2^x) = \dfrac{d}{dx}(2^x)\ln 2 = (2^x)(\ln 2)^2$. Clearly, each new derivative is the same as the old multiplied by a factor of $\ln 2$. So after n derivatives, the result is $\dfrac{d^n}{dx^n}(2^x) = (2^x)\cdot(\ln 2)^n$.

3.9.91 $\dfrac{d^3}{dx^3}(x^2\ln x) = \dfrac{d^2}{dx^2}(2x\ln x + x) = \dfrac{d}{dx}(2\ln x + 2 + 1) = \dfrac{2}{x}$.

3.9.92

i. $y' = \dfrac{d}{dx} e^{x\ln(x^2+1)} = e^{x\ln(x^2+1)}\left(\ln(x^2+1) + \dfrac{2x^2}{x^2+1}\right) = (x^2+1)^x\left(\ln(x^2+1) + \dfrac{2x^2}{x^2+1}\right).$

ii. Let $y = (x^2+1)^x$. Then $\ln y = x\ln(x^2+1)$, so $\dfrac{1}{y}\cdot y' = \ln(x^2+1) + \dfrac{2x^2}{x^2+1}$, and thus $y' = (x^2+1)^x\left(\ln(x^2+1) + \dfrac{2x^2}{x^2+1}\right)$.

3.9.93

i. $y' = \dfrac{d}{dx}\left(e^{x\ln 3}\right) = \left(e^{x\ln 3}\right)\cdot\ln 3 = 3^x\ln 3.$

ii. Let $y = 3^x$. Then $\ln y = x\ln 3$. So $\dfrac{1}{y}\cdot y' = \ln 3$, and $y' = 3^x\ln 3$.

3.9.94

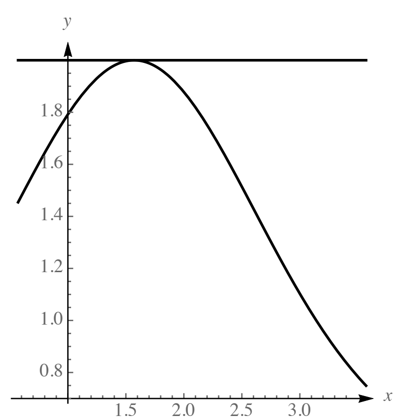
a.

$$\begin{aligned} y' &= \frac{d}{dx}\left(e^{\ln x\ln(4x+1)}\right) = \left(e^{\ln x\ln(4x+1)}\right)\cdot\left(\frac{\ln(4x+1)}{x} + \frac{4\ln x}{4x+1}\right) \\ &= (4x+1)^{\ln x}\left(\frac{(4x+1)\ln(4x+1) + 4x\ln x}{x(4x+1)}\right). \end{aligned}$$

b. Let $y = \ln(4x+1)^{\ln x}$ so that $\ln y = (\ln x)\ln(4x+1)$. Then $\dfrac{1}{y}\cdot y' = \dfrac{\ln(4x+1)}{x} + \dfrac{4\ln x}{4x+1}$, so $y' = (4x+1)^{\ln x}\left(\dfrac{(4x+1)\ln(4x+1) + 4x\ln x}{x(4x+1)}\right)$.

3.9.95

$y' = \dfrac{d}{dx} e^{\sin x\ln 2} = (\cos x)(\ln 2)2^{\sin x}$. At $x = \pi/2$ we have $y' = 0$, so the tangent line is given by $y = 2$.



3.9.96 We have

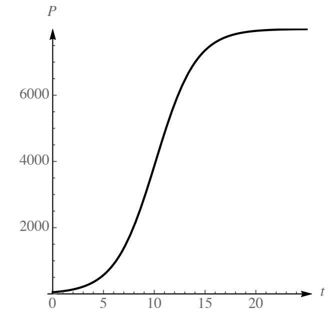
$$y' = -\sin x(\ln(\cos^2 x)) + \cos x\left(\frac{2(\cos x)(-\sin x)}{\cos^2 x}\right) = -\sin x(2 + \ln(\cos^2 x)).$$

This quantity is zero when $\sin x = 0$ or $2 + \ln(\cos^2 x) = 0$, and the latter occurs when $\cos^2 x = e^{-2}$, or $\cos x = \pm e^{-1}$.

$\sin x = 0$ for $x = 0, \pi, 2\pi$. $\cos x = e^{-1}$ for $x \approx 1.194$ and $x \approx 5.089$. Finally, $\cos x = -e^{-1}$ for $x \approx 1.948$ and $x \approx 4.336$. These seven numbers represent the locations of the horizontal tangent lines on $[0, 2\pi]$.

3.9.97

a. We used a graphing rectangle of $[0, 25] \times [0, 8000]$.



b. To find when $P(t)$ hits 5000, we solve $5000 = \frac{400000}{50+7950e^{-0.5t}}$, or $50 + 7950e^{-0.5t} = 80$. This leads to $7950e^{-0.5t} = 30$, or $-0.5t = \ln\left(\frac{30}{7950}\right)$. So we have $t = 2\ln(265) \approx 11.16$ years.

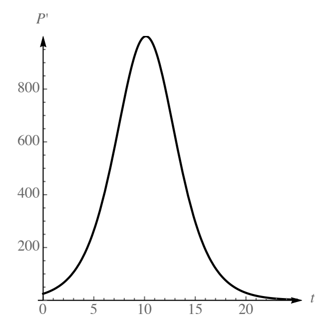
The carrying capacity is $\lim_{t\to\infty} P(t) = \dfrac{400,000}{50} = 8000$. Ninety percent of 8000 is 7200, so we seek the time when $P(t) = 7200$. We have $7200 = \frac{400000}{50+7950e^{-0.5t}}$, or $50 + 7950e^{-0.5t} = \frac{500}{9}$. This leads to $7950e^{-0.5t} = \frac{50}{9}$, or $-0.5t = \ln\left(\frac{50}{71550}\right)$. So we have $t = 2\ln\left(\frac{71550}{50}\right) \approx 14.53$ years.

c. $\dfrac{dP}{dt} = -\dfrac{400000}{(50 + 7950e^{-0.5t})^2} \cdot (7950)(-0.5)e^{-0.5t}$.

At $t = 0$ we have $P'(0) = \frac{400,000 \cdot 7950 \cdot .5}{8000^2} = \frac{1,590,000,000}{8000^2} \approx 25$ fish per year.

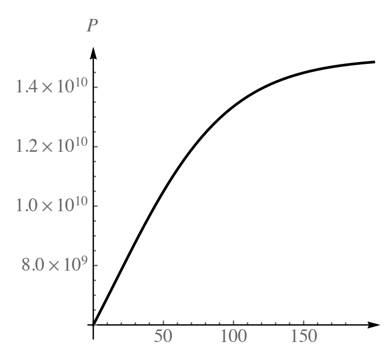
At $t = 5$ we have $P'(5) = \frac{1,590,000,000e^{-5/2}}{(50+7950e^{-5/2})^2} \approx 264$ fish per year.

d. The maximum is at about $t = 10$ years.



3.9.98

a. $P(t) = \frac{6\times10^9 \cdot 15\times10^9}{6\times10^9 + 9\times10^9 \cdot e^{-0.025t}} = \frac{3\times10^{10}}{2+3\cdot e^{-0.025t}}$.



b. $P(21) = \frac{3\times10^{10}}{2+3\cdot e^{-0.525}} \approx 7.95\times10^9$.

$P(t) = 12,000,000,000$ when $2+3e^{-0.025t} = \frac{5}{2}$, which is when $e^{-0.025t} = \frac{1}{6}$. This occurs for $t = 40\ln 6 \approx 71.67$ years.

3.9.99

a. $\ln(P(t)) = \ln(3\cdot10^{10}) - \ln(2+3e^{-0.025t})$. $\dfrac{d}{dt}\ln(P(t)) = \dfrac{P'(t)}{P(t)} = r(t) = \dfrac{0.075\cdot e^{-0.025t}}{2+3e^{-0.025t}}$. $r(0) = \dfrac{0.075}{5} = 0.015$, so the population is growing at 1.5% per year in 1999.

b. $r(11) = \dfrac{0.075e^{-0.275}}{2+3e^{-0.275}} \approx 0.0133$.

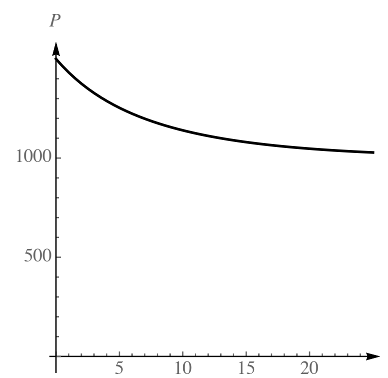
$r(21) = \dfrac{0.075e^{-0.525}}{2+3e^{-0.525}} \approx 0.0118$.

The relative growth rate decreases over time.

c. $\lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty} \dfrac{0.075}{3+2e^{0.025t}} = 0$, because the denominator increases without bound. The relative growth rate becomes smaller and smaller as the population nears the carrying capacity.

3.9.100

a. $P(t) = \frac{1500\cdot1000}{1500-500e^{-0.1t}}$. As $t\to\infty$, the population decreases and gets closer to the carrying capacity of 1000.



b. $P'(t) = -\dfrac{7.5\times10^7e^{-0.1t}}{(1500-500e^{-0.1t})^2}$. $P'(0) = -\dfrac{7.5\times10^7}{(1000)^2} = -75$ deer per year.

c. The population reaches 1200 deer when $1200 = \dfrac{1500\cdot1000}{1500-500e^{-0.1t}}$. This occurs when $-500e^{-0.1t} = \dfrac{1.5\times10^6}{1200} - 1500$, or when $-0.1t = \ln(0.5)$, or when $t = -10\ln(0.5) \approx 6.93$ years. It will take almost 7 years until the deer population reaches 1200.

3.9.101

a.

t	$A(t)$
5	\$17,442.50
15	\$72,704.68
25	\$173,248.49
35	\$356,177.57

Average growth on $[5,15]$ is $\dfrac{A(15)-A(5)}{10} \approx \$5526$ per year.

Average growth on $[15,25]$ is $\dfrac{A(25)-A(15)}{10} \approx \$10,054$ per year.

Average growth on $[25,35]$ is $\dfrac{A(35)-A(25)}{10} \approx \$18,293$ per year.

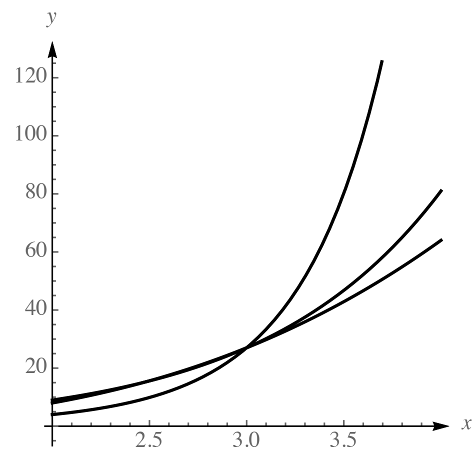
b. $A(40) \approx \$497,872.68$.

c. $A'(t) = 50,000 \cdot 12 \cdot (1.005)^{12t} \cdot \ln(1.005) \approx 2993 \cdot (1.005)^{12t}$. The rate of growth of the investment increases over time, so the earlier you start saving, the higher the rate of increase will be when you retire.

3.9.102 We search for a solution to $x^p = e^x$. If the two curves will have only one point of intersection, then they should be tangent at the point of intersection. So we need $px^{p-1} = e^x$, so we require $px^{p-1} = x^p$, so $x = p$. So $p^p = e^p$, and therefore we must have $p = e$. So we have $x^e = e^x$ intersecting at the point (e, e^e), and that is the only point of intersection.

3.9.103 We search for a solution to $x = p^x$. If the two curves will have only one point of intersection, then they should be tangent at the point of intersection. So we need $1 = p^x \ln p$, or $\dfrac{1}{\ln p} = p^x = x$. So $\ln p = \dfrac{1}{x}$ and $p = e^{1/x}$. Then we have $x = p^x = (e^{1/x})^x = e$. So the point of intersection is (e,e) and the value of p is $e^{1/e} \approx 1.44467$.

3.9.104 By inspection, we see that the point $(3, 27)$ is on all three curves



3.9.105 Let $f(x) = \ln x$ and $a = e$. Then $f'(e) = \lim\limits_{x\to e} \dfrac{\ln x - 1}{x - e} = \dfrac{1}{e}$.

3.9.106 Let $f(x) = \ln x$ and $a = e^8$. Then $f'(e^8) = \lim\limits_{h\to 0} \dfrac{\ln(e^8+h) - 8}{h} = \dfrac{1}{e^8}$.

3.9.107 Let $f(x) = x^x$ and $a = 3$. Then $f'(3) = \lim\limits_{h\to 0} \dfrac{(3+h)^{3+h} - 27}{h} = 3^3 \cdot (\ln 3 + 1) = 27(1 + \ln 3)$.

3.9.108 Let $f(x) = 5^x$ and $a = 2$. Then $f'(2) = \lim\limits_{x\to 2} \dfrac{5^x - 25}{x - 2} = 25 \ln 5$.

3.9.109 Let $y = u(x)^{v(x)}$. Then $\ln y = v(x) \ln u(x)$, so $\dfrac{1}{y} \cdot y' = v'(x) \ln u(x) + v(x) \cdot \dfrac{u'(x)}{u(x)}$. Thus we have $y' = u(x)^{v(x)} \cdot \left(v'(x) \ln u(x) + v(x) \cdot \dfrac{u'(x)}{u(x)}\right) = u(x)^{v(x)} \cdot \left(\dfrac{v(x)}{u(x)} \dfrac{du}{dx} + \ln u(x) \dfrac{dv}{dx}\right)$.

3.9.110 The slope of the tangent line to $y = b^x$ is $y' = b^x \ln b$. We are seeking the y-coordinate of the point (x, b^x) where $b^x \ln b = \dfrac{b^x}{x}$. we must have $x = \dfrac{1}{\ln b}$, in which case the y-coordinate is $b^{\frac{1}{\ln b}} = e$. To see that $b^{\frac{1}{\ln b}} = e$, let $z = b^{\frac{1}{\ln b}}$. Then $\ln z = \ln b^{\frac{1}{\ln b}} = \dfrac{\ln b}{\ln b} = 1$, so $z = e$.

3.10 Derivatives of Inverse Trigonometric Functions

3.10.1
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$.
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, $-\infty < x < \infty$.
$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

3.10.2 $y' = \frac{1}{\sqrt{1-x^2}}$. At $x = 0$ we have $y'(0) = 1$.

3.10.3 $y' = \frac{1}{1+x^2}$. At $x = -2$ we have $y'(-2) = \frac{1}{1+4} = \frac{1}{5}$.

3.10.4 $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = - - \frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}\cos^{-1} x$.

3.10.5 $(f^{-1})'(8) = \frac{1}{f'(2)} = \frac{1}{4}$.

3.10.6 $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$ where $f(x_0) = y_0$.

3.10.7

a. Note that $f(0) = 4$, so $f^{-1}(4) = 0$. $(f^{-1})'(4) = \frac{1}{f'(0)} = \frac{1}{2}$.

b. Note that $f(1) = 6$, so $f^{-1}(6) = 1$. $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{3/2} = \frac{2}{3}$.

c. Note that there is no given x so that $f(x) = 1$, so the desired derivative cannot be determined.

d. From the table directly, $f'(1) = \frac{3}{2}$.

3.10.8

a. $f'(f(0)) = f'(2) = 2$.

b. Note that $f(-4) = 0$, so $f^{-1}(0) = -4$. $(f^{-1})'(0) = \frac{1}{f'(-4)} = \frac{1}{5}$.

c. Note that $f(-2) = 1$, so $f^{-1}(1) = -2$. $(f^{-1})'(1) = \frac{1}{f'(-2)} = \frac{1}{4}$.

d. $(f^{-1})'(f(4)) = \frac{1}{f'(4)} = \frac{1}{1} = 1$.

3.10.9 Because $(f^{-1})'(8) = \frac{1}{f'(3)} = \frac{1}{7}$, the equation of the tangent line at $x = 8$ is $y - 3 = \frac{1}{7}(x - 8)$ or $y = \frac{1}{7}x + \frac{13}{7}$.

3.10.10 $f(3) = 16$ and $f'(3) = 5$, so $f^{-1}(16) = 3$ and $(f^{-1})'(16) = \frac{1}{5}$.

3.10.11 Let $f(x) = \sin x$. Then $f'(\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$, which implies that the slope of the curve $y = \sin^{-1} x$ at $(1/2, \pi/6)$ is $\frac{2}{\sqrt{3}}$.

3.10.12 Let $f(x) = \tan x$. Then $f'(\pi/4) = \sec^2(\pi/4) = 2$. So the slope of the curve $y = \tan^{-1} x$ at $(1, \pi/4)$ is $\frac{1}{2}$.

3.10.13 $\frac{d}{dx} \sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}$.

3.10.14 $\frac{d}{dx}(x \sin^{-1} x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$.

3.10.15 $\frac{d}{dw} \cos(\sin^{-1}(2w)) = (-\sin(\sin^{-1}(2w))) \cdot \frac{d}{dw}\left(\sin^{-1}(2w)\right) = -2w \cdot \frac{2}{\sqrt{1-4w^2}} = -\frac{4w}{\sqrt{1-4w^2}}$.

3.10.16 $\frac{d}{dx} \sin^{-1}(\ln x) = \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{d}{dx} \ln x = \frac{1}{x\sqrt{1-(\ln x)^2}}$.

3.10.17 $\frac{d}{dx} \sin^{-1}(e^{-2x}) = \frac{1}{\sqrt{1-e^{-4x}}} \cdot \frac{d}{dx} e^{-2x} = -\frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}$.

3.10.18 $\frac{d}{dx} \sin^{-1}(e^{\sin x}) = \frac{1}{\sqrt{1-e^{2\sin x}}} \cdot \frac{d}{dx} e^{\sin x} = \frac{(\cos x) \cdot (e^{\sin x})}{\sqrt{1-e^{2\sin x}}}$.

3.10.19 $f'(x) = \frac{1}{1+100x^2} \cdot 10 = \frac{10}{100x^2+1}$.

3.10.20 $f'(x) = 2\tan^{-1} x + \frac{2x}{1+x^2} - \frac{2x}{1+x^2} = 2\tan^{-1} x$.

3.10.21 $\frac{d}{dy} \tan^{-1}(2y^2-4) = \frac{1}{1+(2y^2-4)^2} \cdot \frac{d}{dy}(2y^2-4) = \frac{4y}{1+(2y^2-4)^2}$.

3.10.22 $\frac{d}{dz} \tan^{-1}(1/z) = \frac{1}{1+(1/z)^2} \cdot \frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2(1+(1/z)^2)} = -\frac{1}{z^2+1}$.

3.10.23 $\frac{d}{dz} \cot^{-1} \sqrt{z} = -\frac{1}{1+\sqrt{z}^2} \cdot \frac{d}{dz}\sqrt{z} = -\frac{1}{1+z} \cdot \frac{1}{2\sqrt{z}} = -\frac{1}{2\sqrt{z}(1+z)}$.

3.10.24 $\frac{d}{dx} \sec^{-1} \sqrt{x} = \frac{1}{\sqrt{x}\sqrt{x-1}} \cdot \frac{d}{dx}\sqrt{x} = \frac{1}{2x\sqrt{x-1}}$ for $x > 1$.

3.10.25

$$\begin{aligned}
f'(x) &= 2x + 6x^2 \cot^{-1} x - \frac{2x^3}{1+x^2} - \frac{2x}{1+x^2} = 2x + 6x^2 \cot^{-1} x - \frac{2x^3+2x}{1+x^2} \\
&= 2x + 6x^2 \cot^{-1} x - \frac{2x(x^2+1)}{1+x^2} = 6x^2 \cot^{-1} x.
\end{aligned}$$

3.10.26 $f'(x) = \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} - \frac{-2x}{2\sqrt{1-x^2}} = \cos^{-1} x$.

3.10.27 $f'(2) = 2w - \frac{2w}{1+w^4} = \frac{2w+2w^5}{1+w^4} - \frac{2w}{1+w^4} = \frac{2w^5}{1+w^4}$.

3.10.28 $f'(t) = \frac{1}{\sin^{-1} t^2} \cdot \frac{1}{\sqrt{1-(t^2)^2}} \cdot 2t = \frac{2t}{(\sin^{-1} t^2)\sqrt{1-t^4}}$.

3.10.29 $\frac{d}{dx} \cos^{-1} \frac{1}{x} = -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}} = \frac{|x|}{x^2\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$.

3.10.30 $\dfrac{d}{dt}(\cos^{-1} t)^2 = 2\cos^{-1} t \cdot \left(-\dfrac{1}{\sqrt{1-t^2}}\right) = -\dfrac{2\cos^{-1} t}{\sqrt{1-t^2}}.$

3.10.31 $\dfrac{d}{du}\csc^{-1}(2u+1) = -\dfrac{1}{|2u+1|\sqrt{(2u+1)^2-1}} \cdot 2 = -\dfrac{2}{|2u+1|\sqrt{(2u+1)^2-1}} = -\dfrac{1}{|2u+1|\sqrt{u^2+u}}.$

3.10.32 $\dfrac{d}{dt}\ln(\tan^{-1} t) = \dfrac{1}{\tan^{-1} t} \cdot \dfrac{1}{1+t^2}.$

3.10.33 $\dfrac{d}{dy}\cot^{-1}\left(\dfrac{1}{1+y^2}\right) = \left(-\dfrac{1}{1+\left(\dfrac{1}{1+y^2}\right)^2}\right) \cdot \left(-\dfrac{2y}{(1+y^2)^2}\right) = \dfrac{2y}{(1+y^2)^2+1}.$

3.10.34 $\dfrac{d}{dw}\sin[\sec^{-1} 2w] = \cos[\sec^{-1} 2w] \cdot \dfrac{2}{|2w|\sqrt{4w^2-1}} = \dfrac{1/w}{2|w|\sqrt{4w^2-1}} = \dfrac{1}{2w|w|\sqrt{4w^2-1}}.$

3.10.35 $\dfrac{d}{dx}\sec^{-1}(\ln x) = \dfrac{1}{|\ln x|\sqrt{(\ln x)^2-1}} \cdot \dfrac{1}{x} = \dfrac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}.$

3.10.36 $\dfrac{d}{dx}\tan^{-1}(e^{4x}) = \dfrac{1}{1+e^{8x}} \cdot 4e^{4x} = \dfrac{4e^{4x}}{1+e^{8x}}.$

3.10.37 $\dfrac{d}{dx}\csc^{-1}(\tan e^x) = -\dfrac{1}{|\tan e^x|\sqrt{(\tan e^x)^2-1}} \cdot (\sec^2 e^x) \cdot e^x.$

3.10.38 $\dfrac{d}{dx}\sin(\tan^{-1}(\ln x)) = \cos(\tan^{-1}(\ln x)) \cdot \dfrac{1}{1+(\ln x)^2} \cdot \dfrac{1}{x} = \dfrac{\cos(\tan^{-1}(\ln x))}{x(1+(\ln x)^2)}.$

3.10.39 $\dfrac{d}{ds}\cot^{-1}(e^s) = -\dfrac{1}{1+e^{2s}} \cdot e^s = -\dfrac{e^s}{1+e^{2s}}.$

3.10.40

$$\begin{aligned}\frac{d}{dx}\frac{1}{\tan^{-1}(x^2+4)} &= \frac{d}{dx}\left(\tan^{-1}(x^2+4)\right)^{-1} = -\left(\tan^{-1}(x^2+4)\right)^{-2} \cdot \frac{1}{1+(x^2+4)^2} \cdot 2x \\ &= -\frac{2x}{(1+(x^2+4)^2) \cdot (\tan^{-1}(x^2+4))^2}.\end{aligned}$$

3.10.41 $f'(x) = \dfrac{1}{1+4x^2} \cdot 2$, so $f'(1/2) = \dfrac{1}{1+1} \cdot 2 = 1$. Thus the equation of the tangent line is $y - \pi/4 = 1(x-1/2)$, or $y = x + \dfrac{\pi}{4} - \dfrac{1}{2}$.

3.10.42 $f'(x) = \dfrac{1}{\sqrt{1-x^2/16}} \cdot \dfrac{1}{4} = \dfrac{1}{\sqrt{16-x^2}}$, so $f'(2) = \dfrac{1}{\sqrt{12}}$. Thus the equation of the tangent line is $y - \pi/6 = \dfrac{1}{\sqrt{12}}(x-2)$, or $y = \dfrac{1}{2\sqrt{3}}x + \dfrac{\pi}{6} - \dfrac{1}{\sqrt{3}}$.

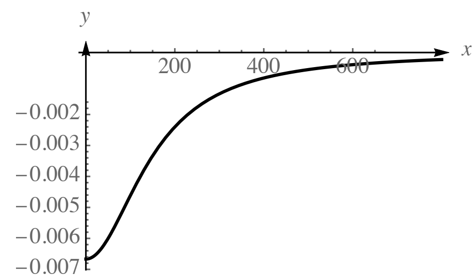
3.10.43 $f'(x) = -\dfrac{1}{\sqrt{1-x^4}} \cdot 2x = -\dfrac{2x}{\sqrt{1-x^4}}$, so $f'(1/\sqrt{2}) = -\dfrac{\sqrt{2}}{\sqrt{1-(1/4)}} = -\dfrac{2\sqrt{2}}{\sqrt{3}}$. Thus the equation of the tangent line is $y - \pi/3 = -\dfrac{2\sqrt{2}}{\sqrt{3}}\left(x - 1/\sqrt{2}\right)$, or $y = -\dfrac{2\sqrt{2}}{\sqrt{3}}x + \dfrac{\pi}{3} + \dfrac{2}{\sqrt{3}}$.

3.10.44 $f'(x) = \dfrac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x$, so $f'(\ln 2) = \dfrac{2}{2\sqrt{4-1}} = \dfrac{1}{\sqrt{3}}$. Thus the equation of the tangent line is $y - \pi/3 = \dfrac{1}{\sqrt{3}}(x - \ln 2)$, or $y = \dfrac{1}{\sqrt{3}}x + \dfrac{\pi}{3} - \dfrac{\ln 2}{\sqrt{3}}$.

3.10.45

a. $\dfrac{x}{150} = \cot\theta$, so $\theta = \cot^{-1}\left(\dfrac{x}{150}\right)$. Then $\dfrac{d\theta}{dx} = -\dfrac{1}{1+\left(\dfrac{x}{150}\right)^2}\cdot\dfrac{1}{150} = -\dfrac{150}{(150)^2+x^2}$. When $x = 500$, we have $\dfrac{d\theta}{dx} = -\dfrac{150}{150^2+500^2} \approx -0.00055$ radians per meter.

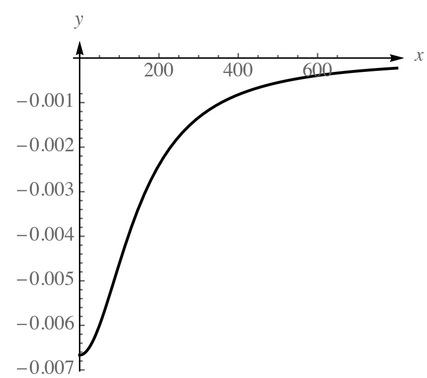
b. The most rapid change is at $x = 0$ where $\dfrac{d\theta}{dx} = -\dfrac{1}{150} \approx -0.0067$ radians per meter.



3.10.46

a. $\dfrac{x}{400} = \cot\theta$, so $\theta = \cot^{-1}\left(\dfrac{x}{400}\right)$. Then $\dfrac{d\theta}{dx} = -\dfrac{1}{1+\left(\dfrac{x}{400}\right)^2}\cdot\dfrac{1}{400} = -\dfrac{400}{(400)^2+x^2}$. When $x = 500$, we have $\dfrac{d\theta}{dx} = -\dfrac{400}{400^2+500^2} \approx -0.000976$ radians per meter.

b. The most rapid change is at $x = 0$, where the plane is directly over head.



3.10.47 $f(4) = 16$ so $(f^{-1})'(16) = \dfrac{1}{f'(4)} = \dfrac{1}{3}$.

3.10.48 $f(4) = 10$ so $(f^{-1})'(10) = \dfrac{1}{f'(4)} = \dfrac{1}{1/2} = 2$.

3.10.49 $f(0) = 1$ so $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{5/e} = \dfrac{e}{5}$.

3.10.50 $f(2) = 5$ so $(f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{4}$.

3.10.51 $f\left(\dfrac{\pi}{4}\right) = 1$ so $(f^{-1})'(1) = \dfrac{1}{f'\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{\sec^2\left(\dfrac{\pi}{4}\right)} = \dfrac{1}{2}$.

3.10.52 $f(-3) = 12$ so $(f^{-1})'(12) = \dfrac{1}{f'(-3)} = \dfrac{1}{-8} = -\dfrac{1}{8}$.

3.10.53 $f(4) = 2$ so $(f^{-1})'(2) = \dfrac{1}{f'(4)} = \dfrac{1}{(1/2\sqrt{4})} = 4$.

3.10.54 $f(0) = 4$ so $(f^{-1})'(4) = \dfrac{1}{f'(0)} = \dfrac{1}{40}$.

3.10.55 $f(4) = 36$ and $(f^{-1})'(36) = \dfrac{1}{f'(4)} = \dfrac{1}{2(4+2)} = \dfrac{1}{12}$.

3.10.56 $f(1/3) = 0$ so $(f^{-1})'(0) = \dfrac{1}{f'(1/3)} = \dfrac{1}{3/\ln 10} = \dfrac{\ln 10}{3}$.

3.10.57 Note that $f(1) = 3$. So $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

3.10.58 $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

3.10.59 $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{4}{5}$, so $f'(7) = \dfrac{5}{4}$.

3.10.60 $(f^{-1})'(7) = \dfrac{1}{f'(4)} = \dfrac{1}{1/5} = 5$.

3.10.61

a. True, because $\dfrac{d}{dx}\sin^{-1} x = -\dfrac{d}{dx}\cos^{-1} x$.

b. False. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$ for all x, and this doesn't equal $\sec^2 x$ anywhere except at the origin (one curve is always less than or equal to one, and the other is always greater than or equal to one).

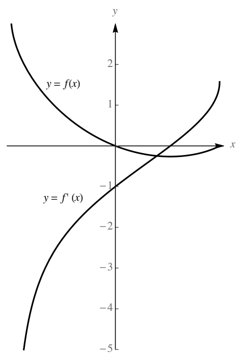
c. True. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$, and this is minimal when its denominator is as big as possible, which occurs when $x = 0$. So the smallest possible slope of a tangent line for this function on $(-1, 1)$ is $\dfrac{1}{\sqrt{1-0^2}} = 1$.

d. True. $\dfrac{d}{dx}\sin x = \cos x$ and $\cos x = 1$ for $x = 0$ and $-1 \le \cos x \le 1$ for all x. Thus 1 is the largest possible slope for a tangent line to the sine function.

e. True. This follows because the function $\dfrac{1}{x}$ is its own inverse. (Note that $f(f(x)) = \dfrac{1}{f(x)} = \dfrac{1}{1/x} = x$.) Thus, the derivative of the inverse of f is the derivative of f, which is $-\dfrac{1}{x^2}$.

3.10.62

a.

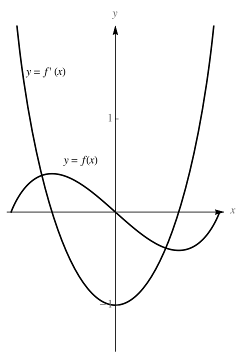


b. $f'(x) = \sin^{-1}(x) + \dfrac{x-1}{\sqrt{1-x^2}}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.53$.

3.10.63

a.

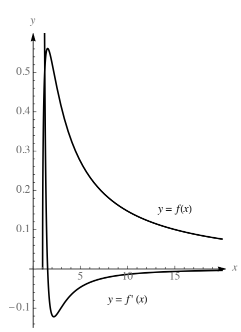


b. $f'(x) = 2x\sin^{-1}(x) + \dfrac{x^2-1}{\sqrt{1-x^2}}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = -0.61$ and at about $x = 0.61$.

3.10.64

a.

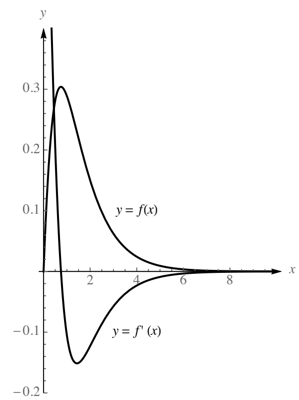


b. $f'(x) = \dfrac{x \cdot \dfrac{1}{|x|\sqrt{x^2-1}} - \sec^{-1} x}{x^2} = \dfrac{1}{x|x|\sqrt{x^2-1}} - \dfrac{\sec^{-1} x}{x^2}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 1.53$.

3.10.65

a. ,

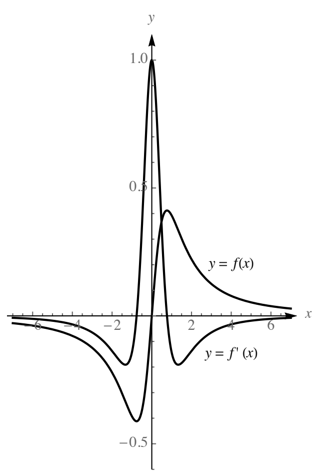


b. $f'(x) = -e^{-x}\tan^{-1}x + e^{-x}\dfrac{1}{1+x^2}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.75$.

3.10.66

a.



b. $f'(x) = \dfrac{(x^2+1)\dfrac{1}{x^2+1} - \tan^{-1}(x)(2x)}{(x^2+1)^2} = \dfrac{1-2x\tan^{-1}(x)}{(x^2+1)^2}$.

c. Note that f' is zero and f has a horizontal tangent line at about $x = 0.765$ and at about $x = -0.765$.

3.10.67 Let $f(y) = 3y - 4$. Then $f'(y) = 3$ for all y in the domain of f. Let $y = f^{-1}(x)$. $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{3}$.

3.10.68 Let $x = y^3 + 3$. Then $1 = 3y^2y'$, so $y' = \dfrac{1}{3y^2} = \dfrac{1}{3(x-3)^{2/3}}$, where $x \neq 3$.

3.10.69 Let $x = f(y) = y^2 - 4$ for $y > 0$. Note that this means that $y = \sqrt{x+4}$. Then $f'(y) = 2y$. So $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{2y} = \dfrac{1}{2\sqrt{x+4}}$.

3.10.70 Let $x = \dfrac{y}{y+5}$. Then $x(y+5) = y$, so $y - xy = 5x$. Thus, $y = f^{-1}(x) = \dfrac{5x}{1-x}$.
Therefore $(f^{-1})'(x) = \dfrac{(1-x)5 - 5x(-1)}{(1-x)^2} = \dfrac{5}{(1-x)^2}$.

3.10.71 $y = e^{3x+1}$, so $\ln y = 3x + 1$, and thus $x = \dfrac{\ln y - 1}{3}$. Then $f^{-1}(x) = \dfrac{\ln x - 1}{3}$, so $(f^{-1})'(x) = \dfrac{1}{3x}$.

3.10.72 $y = \ln(5x+4)$, so $e^y = 5x+4$, so $x = \dfrac{e^y-4}{5}$. Then $f^{-1}(x) = \dfrac{e^x-4}{5}$ and $(f^{-1})'(x) = \dfrac{e^x}{5}$.

3.10.73 $y = 10^{12x-6}$ so $\log_{10} y = 12x-6$, so $x = \dfrac{\log_{10} y + 6}{12}$. Then $f^{-1}(x) = \dfrac{\log_{10} x + 6}{12}$ and $(f^{-1})'(x) = \dfrac{1}{12x \ln 10}$.

3.10.74 $y = \log_{10} 2x+6$ so $10^y = 2x+6$, so $x = \dfrac{10^y-6}{2}$. Then $f^{-1}(x) = \dfrac{10^x-6}{2}$ and $(f^{-1})'(x) = \dfrac{10^x \ln 10}{2}$.

3.10.75 For $y \ge -2$, let $x = \sqrt{y+2}$. Note that it then follows that $x \ge 0$. Then $1 = \dfrac{y'}{2\sqrt{y+2}}$, and $x^2 = y+2$, so $y = x^2-2$. Thus we have $(f^{-1})'(x) = y' = 2\sqrt{x^2-2+2} = 2|x| = 2x$, because $x \ge 0$.

3.10.76 For $y > 0$, let $x = y^{2/3}$. Then $1 = \dfrac{2}{3}y^{-1/3}y'$. So $y' = \dfrac{3}{2}y^{1/3} = \dfrac{3}{2}(x^{3/2})^{1/3} = \dfrac{3}{2}x^{1/2}$ where $x > 0$.

3.10.77 For $y > 0$, let $x = y^{-1/2}$. Then $1 = -\dfrac{1}{2}y^{-3/2}y'$, so $y' = -2y^{3/2} = -2(x^{-2})^{3/2} = -2x^{-3}$ where $x > 0$.

3.10.78 Let $f(y) = |y+2|$ for $y \le -2$. Then $f(y) = -(y+2)$, and $f'(y) = -1$. Thus $(f^{-1})'(x) = \dfrac{1}{f'(y)} = \dfrac{1}{-1} = -1$.

3.10.79

a. Because $\dfrac{l}{10} = \csc(\theta)$, $\theta = \csc^{-1}\left(\dfrac{l}{10}\right)$, and $\dfrac{d\theta}{dl} = -\dfrac{1}{(l/10)\sqrt{(l/10)^2-1}} \cdot \dfrac{1}{10} = -\dfrac{10}{l\sqrt{l^2-100}}$.

b. $\left.\dfrac{d\theta}{dl}\right|_{l=50} = -\dfrac{10}{50\sqrt{2500-100}} \approx -0.0041$ radians per foot.

$\left.\dfrac{d\theta}{dl}\right|_{l=20} = -\dfrac{10}{20\sqrt{400-100}} \approx -0.029$ radians per foot.

$\left.\dfrac{d\theta}{dl}\right|_{l=11} = -\dfrac{10}{11\sqrt{121-100}} \approx -0.198$ radians per foot.

c. $\displaystyle\lim_{l\to 10^+} -\dfrac{10}{l\sqrt{l^2-100}} = -\infty$. The angle changes very quickly as we approach the dock.

d. $\dfrac{d\theta}{dl}$ is negative because this measures the change in θ as l increases – but when the boat is approaching the dock, l is decreasing.

3.10.80

a. Because the triangle from the top of the cliff to the falcon is isosceles and has a base of $80-h$, we get that the falcon is also $80-h$ feet from the cliff. So $\tan\theta = \dfrac{h}{80-h}$, or $\theta = \tan^{-1}\left(\dfrac{h}{80-h}\right)$.

b. $\dfrac{d\theta}{dh} = \dfrac{d}{dh}\tan^{-1}\left(\dfrac{h}{80-h}\right) = \dfrac{1}{1+\left(\dfrac{h}{80-h}\right)^2} \cdot \dfrac{80-h+h}{(80-h)^2} = \dfrac{80}{(80-h)^2+h^2}$.

$\left.\dfrac{d\theta}{dh}\right|_{h=60} = \dfrac{80}{20^2+60^2} = \dfrac{1}{50}$ radians per foot.

3.10.81

a. $\sin\theta = \dfrac{c}{D}$, so $\theta = \sin^{-1}\left(\dfrac{c}{D}\right)$. Thus $\dfrac{d\theta}{dc} = \dfrac{1/D}{\sqrt{1-\left(\dfrac{c}{D}\right)^2}} = \dfrac{1}{\sqrt{D^2-c^2}}$.

b. $\left.\dfrac{d\theta}{dc}\right|_{c=0} = \dfrac{1}{\sqrt{D^2}} = \dfrac{1}{D}$.

3.10.82

a. $\cos\theta = \dfrac{c}{D}$, so $\theta = \cos^{-1}\left(\dfrac{c}{D}\right)$. Thus $\dfrac{d\theta}{dc} = -\dfrac{1/D}{\sqrt{1-\left(\dfrac{c}{D}\right)^2}} = -\dfrac{1}{\sqrt{D^2-c^2}}$.

b. $\left.\dfrac{d\theta}{dc}\right|_{c=0} = -\dfrac{1}{D}$. This is the opposite result of exercise 69, as θ now increases with decreasing c.

3.10.83 $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$ where $y_0 = f(x_0)$.
$\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\cos(\sin^{-1}x)} = \dfrac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}} = \dfrac{1}{\sqrt{1-x^2}}$.

3.10.84

a. $\dfrac{d}{dx}\cos^{-1}x = \dfrac{1}{-\sin(\cos^{-1}x)} = -\dfrac{1}{\sqrt{1-\cos^2(\cos^{-1}x)}} = -\dfrac{1}{\sqrt{1-x^2}}$.

b. $\dfrac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \dfrac{d}{dx}\dfrac{\pi}{2} = 0$, so $\dfrac{d}{dx}\sin^{-1}x = -\dfrac{d}{dx}\cos^{-1}x$. But $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, so $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}}$.

3.10.85 Using the identity $\cot^{-1}x+\tan^{-1}x = \dfrac{\pi}{2}$, we have the $\dfrac{d}{dx}\cot^{-1}x+\dfrac{d}{dx}\tan^{-1}x = 0$, so $\dfrac{d}{dx}\cot^{-1}x = -\dfrac{d}{dx}\tan^{-1}x$. Likewise, because $\csc^{-1}x+\sec^{-1}x = \dfrac{\pi}{2}$, we have $\dfrac{d}{dx}\csc^{-1}x+\dfrac{d}{dx}\sec^{-1}x = 0$, so $\dfrac{d}{dx}\csc^{-1}x = -\dfrac{d}{dx}\sec^{-1}x$.

3.10.86

a. $y_0 = f(x_0)$, so $y_0 = ax_0 + b$ and $b = y_0 - ax_0$.

b. $x_0 = f^{-1}(y_0)$, so $x_0 = cy_0 + d$ and $c = \dfrac{x_0-d}{y_0}$. Also, because $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$and $f'(x_0) = a$, $(f^{-1})'(y_0) = c$, we have that $c = \dfrac{1}{a}$. So $d = x_0 - \dfrac{y_0}{a}$.

c. We show that $L(M(x)) = x$.
$L(M(x)) = a(cx+d)+b = acx+ad+b = x+ad+b = x+a(x_0-cy_0)+(y_0-ax_0) = x+ax_0-acy_0+y_0-ax_0 = x-y_0+y_0 = x$.

3.10.87 $\cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$ for $-1 \le x \le 1$.

3.10.88 $\cos(2\sin^{-1}(x)) = \cos^2(\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = 1-\sin^2(\sin^{-1}(x)) - \sin^2(\sin^{-1}(x)) = 1-2x^2$ for $-1 \le x \le 1$.

3.10.89 $\tan(2\tan^{-1}(x)) = \dfrac{2\tan(\tan^{-1}(x))}{1-\tan^2(\tan^{-1}(x))} = \dfrac{2x}{1-x^2}$ for $-1 < x < 1$.

3.10.90 $\sin(2\sin^{-1}(x)) = 2\sin(\sin^{-1}(x))\cos(\sin^{-1}(x)) = 2x\sqrt{1-x^2}$ for $-1 \le x \le 1$.

3.11 Related Rates

3.11.1 The area of a circle of radius r is $A(r) = \pi r^2$. If the radius $r = r(t)$ changes with time, then the area of the circle is a function of r and r is a function of t, so ultimately A is a function of t. If the radius changes at rate $\frac{dr}{dt}$, then the area changes at rate $2\pi r \frac{dr}{dt}$.

3.11.2 $\frac{dV}{dt} = k\frac{dT}{dt}$.

3.11.3 Because area is width times height, if one increases, the other must decrease in order for the area to remain constant.

3.11.4

a $\frac{dF}{dt} = \frac{9}{5}\frac{dC}{dt}$.

b. $\frac{dF}{dt} = \frac{9}{5}(10) = \frac{90}{5} = 18$ degrees F per minute.

3.11.5

a $V = 200h$, so $\frac{dV}{dt} = 200\frac{dh}{dt}$.

b. $\frac{dV}{dt} = 200 \cdot \frac{1}{4} = 50\,\text{ft}^3/\text{min}$.

c. $\frac{dh}{dt} = \frac{dV/dt}{200} = \frac{10}{200} = \frac{1}{20}$ ft/min.

3.11.6

a $A = lw = 2w \cdot w = 2w^2$.

b. $\frac{dA}{dt} = 4w\frac{dw}{dt}$.

3.11.7

a. $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$.

b. $\frac{dV}{dt} = 4\pi(4)^2(2) = 128\,\text{in}^3/\text{min}$.

c. $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{10}{4\pi 5^2} = \frac{1}{10\pi}$ in/min.

3.11.8 $A = \frac{3}{2}x^2$, so $\frac{dA}{dt} = 3x\frac{dx}{dt}$.

3.11.9 $\frac{dz}{dt} = \frac{dx}{dt} + 3y^2\frac{dy}{dt}$, so at the aforementioned time, $\frac{dz}{dt} = -1 + 3(2)^2(5) = 59$.

3.11.10 $\frac{dw}{dt} = \left(2x\frac{dx}{dt}\right)y^4 + x^2\left(4y^3\frac{dy}{dt}\right)$, so when $x = 3$, $\frac{dx}{dt} = 2$, $y = 1$, and $\frac{dy}{dt} = 4$, we have $\frac{dw}{dt} = 2(3)(2)(1)^4 + (3)^2(4(1)^3(4)) = 156$.

3.11.11

$A(x) = x^2$, $\frac{dx}{dt} = 2$ meters per second.

a. $\frac{dA}{dt} = 2x\frac{dx}{dt}$, so at $x = 10$ we have $\frac{dA}{dt} = 2 \cdot 10\text{m} \cdot 2\text{m/s} = 40\text{m}^2/\text{s}$.

b. At $x = 20\text{m}$ we have $\dfrac{dA}{dt} = 2 \cdot 20\text{m} \cdot 2\text{m/s} = 80\text{m}^2/\text{s}$.

3.11.12

a. Let x be the length of a side of the square. Then $\dfrac{dx}{dt} = -1$ meters per second. Because $A = x^2$, we have $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x \cdot (-1) = -2x$ square meters per second. Thus $A'(5) = -10$, and so the area of the square is decreasing at 10 square meters per second when $x = 5$.

b. If l is the length of a diagonal of a square with side length x, then $x^2 + x^2 = l^2$ by the Pythagorean Theorem, so $l(x) = \sqrt{2}x$. Thus $\dfrac{dl}{dt} = \dfrac{dl}{dx}\dfrac{dx}{dt} = \sqrt{2} \cdot (-1) = -\sqrt{2}$. The diagonals are decreasing at a rate of $\sqrt{2}$ meters per second.

3.11.13

a. Let x be the length of a leg of an isosceles right triangle. Then $\dfrac{dx}{dt} = 2$ meters per second. The area is given by $A(x) = \frac{1}{2}x^2$. Thus, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = x \cdot 2 = 2x$ square meters per second. When $x = 2$, we have $\dfrac{dA}{dx} = 4$, so the area is increasing at 4 square meters per second.

b. When the hypotenuse is 1 meter long, the legs are $1/\sqrt{2}$ meters long. So $A'(1/\sqrt{2}) = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$, so the area is increasing at $\sqrt{2}$ square meters per second.

c. If h is the length of the hypotenuse, then $x^2 + x^2 = h^2$, so $h = \sqrt{2}x$. So $\dfrac{dh}{dt} = \dfrac{dh}{dx}\dfrac{dx}{dt} = \sqrt{2}\dfrac{dx}{dt} = \sqrt{2} \cdot 2 = 2\sqrt{2}$ meters per second.

3.11.14

a. Let x be the length of a leg, and h the length of the hypotenuse. Then $x^2 + x^2 = h^2$, so $h = \sqrt{2}x$. Thus $\dfrac{dh}{dt} = \sqrt{2}\dfrac{dx}{dt}$, and because we are given that $\dfrac{dh}{dt} = -4$, we must have $\dfrac{dx}{dt} = -\dfrac{4}{\sqrt{2}} = -2\sqrt{2}$ meters per second.

Therefore, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x \cdot -2\sqrt{2} = -4\sqrt{2}x$. When $x = 5$, we have $\dfrac{dA}{dt} = -20\sqrt{2}$ square meters per second. The area is decreasing at a rate of $20\sqrt{2}$ square meters per second.

b. As mentioned above, $\dfrac{dx}{dt} = -2\sqrt{2}$, so the legs are decreasing at a rate of $2\sqrt{2}$ meters per second.

c. When the triangle has area 4 square meters, the legs have length $x = 2\sqrt{2}$ meters. At that time, $\dfrac{dA}{dt} = \dfrac{dA}{dx}\dfrac{dx}{dt} = 2x(-2\sqrt{2}) = -4\sqrt{2}x = -4\sqrt{2}(2\sqrt{2}) = -16$. The area is decreasing at 16 square meters per second.

3.11.15

a. Let r be the radius of the circle and A the area, and note that we are given $\dfrac{dA}{dt} = 1$ square cm per second. Because $A = \pi r^2$, we have $\dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt}$, so

$$1 = 2\pi r \frac{dr}{dt},$$

and thus

$$\frac{dr}{dt} = \frac{1}{2\pi r}.$$

When $r = 2$, we have $\dfrac{dr}{dt} = \dfrac{1}{4\pi}$ cm per second.

b. When $c = 2\pi r = 2$, we have $r = 1/\pi$. At this time, $\dfrac{dr}{dt} = \dfrac{1}{2\pi r} = \dfrac{1}{2\pi(1/\pi)} = \dfrac{1}{2}$ cm per second.

3.11.16 $V(x) = x^3$, so $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. At $x = 50\,\text{cm}$ and $\dfrac{dx}{dt} = 2\,\text{cm/s}$ we have $\dfrac{dV}{dt} = 3\cdot(50)^2\cdot 2 = 15000\,\text{cm}^3/\text{s}$.

3.11.17 $A(x) = \pi x^2$, so $\dfrac{dA}{dt} = 2\pi x\dfrac{dx}{dt}$. At $x = 10\,\text{ft}$ and $\dfrac{dx}{dt} = -2\,\text{ft/min}$ we have $\dfrac{dA}{dt} = 2\pi\cdot 10\cdot(-2) = -40\pi\,\text{ft}^2/\text{min}$.

3.11.18 $V(x) = x^3$, so $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt} = -0.5\,\text{ft}^3/\text{min}$. When $x = 12\,\text{ft}$ we have $3(144)\text{ft}^2\frac{dx}{dt} = -0.5\,\text{ft}^3/\text{min}$, so $\dfrac{dx}{dt} = -\dfrac{1}{864}\,\text{ft/min} \approx -0.0012\,\text{ft/min}$.

3.11.19 $V(r) = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 15\,\text{in}^3/\text{min}$. At $r = 10$ inches we have $4\pi(10\,\text{in})^2\frac{dr}{dt} = 15\,\text{in}^3/\text{min}$. Thus, $\dfrac{dr}{dt} = \dfrac{3}{80\pi}\,\text{in/min} \approx 0.012\,\text{in/min}$.

3.11.20 Let x be the excess of the short side of the rectangle over the original 2 cm, so that the at time t the rectangle has dimensions $2+x$ by $4+x$. Then the area $A(x)$ is given by $A(x) = (2+x)(4+x) = 8+6x+x^2$. So $\dfrac{dA}{dt} = (6+2x)\dfrac{dx}{dt}$. With $\dfrac{dx}{dt} = 1\,\text{cm/s}$, and at $t = 20\,\text{s}$ we have $x = 20\,\text{cm}$, so $\dfrac{dA}{dt} = (6+2\cdot 20)\,\text{cm}\cdot 1\,\text{cm/s} = 46\,\text{cm}^2/\text{s}$.

3.11.21 $V(r) = \dfrac{4}{3}\pi r^3$, and $S(r) = 4\pi r^2$. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2$, so $\dfrac{dr}{dt} = k$, the constant of proportionality.

3.11.22

a. Let $x(t)$ be the distance that the westbound boat has traveled at time t and $y(t)$ the distance the southbound boat has traveled at time t. Then $x(0.5) = 10$ mi and $y(0.5) = 7.5$ mi.

b. The distance s between them is given by $s = \sqrt{x^2+y^2}$. Also $\dfrac{dx}{dt} = 20$ and $\dfrac{dy}{dt} = 15$. We have

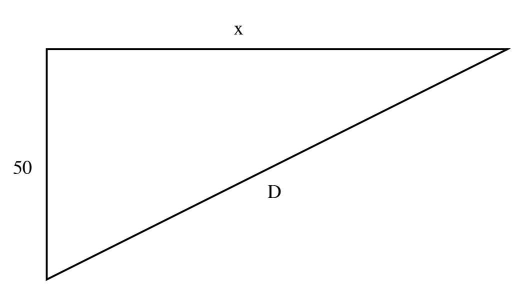
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\cdot\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{20x+15y}{\sqrt{x^2+y^2}}.$$

After 30 minutes $\sqrt{x^2+y^2} = \sqrt{10^2+(7.5)^2} = 12.5$. So $\dfrac{ds}{dt} = \dfrac{200+112.5}{12.5} = 25$ miles per hour.

3.11.23 Let x be the distance the westbound airliner has traveled between noon and t hours after 1:00, and let y be the distance the northbound airliner has traveled t hours after 1:00, and let D be the distance between the planes. We have $D^2 = x^2+y^2$, so $2D\dfrac{dD}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. We are given that $\dfrac{dx}{dt} = 500$ mph and $\dfrac{dy}{dt} = 550$ mph. At 2:30, we have that $x = 500 + 500\cdot 1.5 = 1250$, and $y = 550\cdot 1.5 = 825$ miles. $D = \sqrt{2243125} \approx 1497.7$ miles. Thus $\dfrac{dD}{dt} \approx \dfrac{1250\cdot 500 + 825\cdot 550}{1497.7} \approx 720.27$ miles per hour.

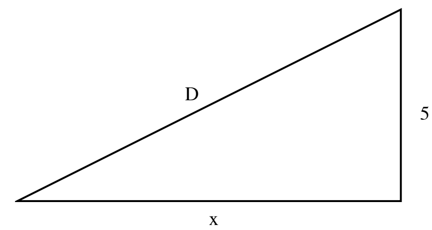
3.11.24

Let x be the horizontal distance of the kite, and let D be the length of the string. Then $D^2 = x^2 + 2500$, so $2D\dfrac{dD}{dt} = 2x\dfrac{dx}{dt}$, so $\dfrac{dD}{dt} = \dfrac{x}{D}\dfrac{dx}{dt}$. When $D = 120$ feet, then $x = \sqrt{11900} \approx 109$ feet. Therefore, $\dfrac{dD}{dt} \approx \dfrac{109}{120}\cdot 5 \approx 4.55$ feet per second.



3.11.25

Let D be the length of the rope from the boat to the capstan, and let x be the horizontal distance from the boat to the dock. By the Pythagorean Theorem, $x^2 + 25 = D^2$, so $2x\dfrac{dx}{dt} = 2D\dfrac{dD}{dt}$, so $\dfrac{dx}{dt} = \dfrac{D}{x}\dfrac{dD}{dt}$. We are given that $\dfrac{dD}{dt} = -3$ feet per second, so when $x = 10$, we have $\dfrac{dx}{dt} = \dfrac{\sqrt{125}}{10} \cdot (-3) = -\dfrac{3\sqrt{5}}{2}$ feet per second. The boat is approaching the dock at $\dfrac{3\sqrt{5}}{2}$ feet per second.



3.11.26

a. Let s be the distance from the origin to the bug's position $P(x, x^2)$ on the parabola. By the Pythagorean Theorem, $s^2 = x^2 + x^4$. Differentiating with respect to t, we have

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 4x^3\frac{dx}{dt} = 2x(1 + 2x^2)\frac{dx}{dt}.$$

Dividing through by 2 and solving for $\dfrac{dx}{dt}$ gives

$$\frac{dx}{dt} = \frac{s(ds/dt)}{x(1 + 2x^2)}.$$

When the bug is at $(2, 4)$ we have $s = \sqrt{20} = 2\sqrt{5}$, so

$$\frac{dx}{dt} = \frac{2\sqrt{5}(1)}{2(1 + 2(4))} = \frac{\sqrt{5}}{9}\text{ cm/min.}$$

b. Differentiating both sides of $y = x^2$ with respect to t gives

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

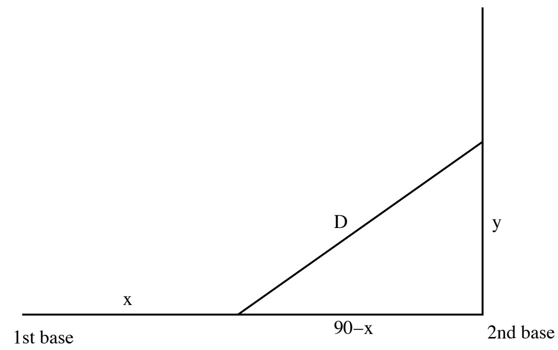
c. Using the previous parts of this problem, we have

$$\frac{dy}{dt} = 2(2)\left(\frac{\sqrt{5}}{9}\right) = \frac{4\sqrt{5}}{9}\text{ cm/min.}$$

3.11.27 Let x be the distance the motorcycle has traveled since the instant it went under the balloon, and let y be the height of the balloon above the ground t seconds after the motorcycle went under it. We have $x^2 + y^2 = D^2$ where D is the distance between the motorcycle and the balloon. Thus, $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2D\dfrac{dD}{dt}$, and we are given that $\dfrac{dy}{dt} = 10$ feet per second, and $\dfrac{dx}{dt} = 40\,\text{mph} = \dfrac{176}{3}$ ft/s. After 10 seconds have passed, we have that $y = 150 + 100 = 250\,\text{ft}$, $x = \dfrac{1760}{3}$ ft and $D = \sqrt{250^2 + \left(\frac{1760}{3}\right)^2} \approx 638\,\text{ft}$. Thus, $\dfrac{dD}{dt} \approx \dfrac{1}{638}\left(\dfrac{1760}{3} \cdot \dfrac{176}{3} + 2500\right) \approx 57.86$ feet per second.

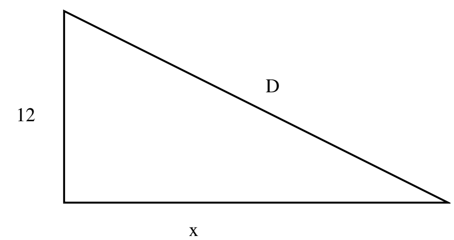
3.11.28

Let D, x and y be as pictured. By the Pythagorean theorem, we know that $D^2 = (90 - x)^2 + y^2$. We are given that $\frac{dx}{dt} = 18$ feet per second, and $\frac{dy}{dt} = 20$ feet per second. Differentiating, we obtain $2D\frac{dD}{dt} = -2(90-x)\frac{dx}{dt} + 2y\frac{dy}{dt}$. After 1 second, we have that $x = 18$ and $y = 20$, and $D = 4\sqrt{349}$ feet. So $\frac{dD}{dt} = \frac{1}{4\sqrt{349}}(-72\cdot 18 + 20\cdot 20) \approx -11.99$ feet per second. So the distance between the runners is decreasing at a rate of about 11.99 feet per second.



3.11.29

Let x be the distance between the fish and the fisherman's feet, and let D be the distance between the fish and the tip of the pole. Then $D^2 = x^2 + 144$, so $2D(dD/dt) = 2x(dx/dt)$. Note that $dD/dt = -1/3\,\text{ft/sec}$, so when $x = 20\,\text{ft}$, we have $dx/dt = \sqrt{400+144}/20\cdot(-1/3) \approx -0.3887\,\text{ft/sec} \approx -4.66\,\text{in/sec}$. The fish is moving toward the fisherman at about 4.66 in/sec.



3.11.30 $y = 50x - x^2$, so $\frac{dy}{dt} = 50\frac{dx}{dt} - 2x\frac{dx}{dt}$. We are given that $\frac{dx}{dt} = 30$ feet per second. For $x = 10$, we have $\frac{dy}{dt} = 1500 - 600 = 900$ feet per second. For $x = 40$, $\frac{dy}{dt} = 1500 - 2400 = -900$ feet per second.

3.11.31 Let $h(t)$ be the height of the water in the tank at time t. Then the volume of the water in the tank at time t is given by $V = \pi r^2 h = \pi h$. We are seeking $\frac{dV}{dt}$ when $\frac{dh}{dt} = -1/2$ foot per minute. Because $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = -\frac{1}{2}\pi$, the volume of the water in the tank is decreasing at $\pi/2$ cubic feet per minute, so the water is draining out at $\pi/2$ cubic feet per minute.

3.11.32 We have that $V = \pi r^2 h$, and r is a constant 2 inches, so $V = 4\pi h$, and $\frac{dV}{dt} = 4\pi\frac{dh}{dt}$. Because we are given that $\frac{dh}{dt} = -0.25$ inches per second, we have that $\frac{dV}{dt} = 4\pi(-0.25) = -\pi\,\text{in}^3/\text{s}$. Thus, the soda is being sucked out at a rate of π cubic inches per second.

3.11.33 Let x be the distance from the bottom of the cylinder to the position of the piston. Let $V(x)$ be the volume of the cylinder when the piston is at position x. $V(x) = 25\pi x$, so $\frac{dV}{dt} = 25\pi\frac{dx}{dt}$. Because $\frac{dx}{dt} = -3\,\text{cm/s}$ we have $\frac{dV}{dt} = 25\pi(-3)\,\text{cm}^3/\text{s} = -75\pi\,\text{cm}^3/\text{s}$.

3.11.34 For the small pool, $V_s = 25\pi h_s$, so $\frac{dV_s}{dt} = 25\pi\frac{dh_s}{dt}$, and we are given that $\frac{dh_s}{dt} = .5$ meters per minutes, so $\frac{dV_s}{dt} = 12.5\pi\,\text{m}^3/\text{min}$. Because the pools are being filled at the same rate, this number is also $\frac{dV_L}{dt}$ for the large pool. We have $V_L = 64\pi h_L$, so $\frac{dV_L}{dt} = 12.5\pi = 64\pi\frac{dh_L}{dt}$, so $\frac{dh_L}{dt} = \frac{25}{128}$ meters per minute.

3.11.35 $V = \frac{1}{3}\pi r^2 h$ where $r = 3h$, so $V = 3\pi h^3$. We have that $\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$, and we given that $\frac{dh}{dt} = 2$ at the moment when $h = 12$, so at that time, $\frac{dV}{dt} = 9\pi \cdot 144\,\mathrm{cm}^2 \cdot 2\,\mathrm{cm/sec} = 2592\pi\,\mathrm{cm}^3/\mathrm{s}$. This is the rate at which the volume of the sandpile is increasing, so it must also be the rate at which the sand is leaving the bin, because there is no other sand involved.

3.11.36 Let h be the depth of the water in the tank at time t, and let r be the radius of the cone-shaped water at time t. By similar triangles, we have that $\frac{h}{r} = \frac{12}{6}$, so $h = 2r$. The volume of the water in the tank is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{h^2}{4} \cdot h = \frac{\pi h^3}{12}$. Thus, $\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$, and so when $h = 3$ we have $-2\,\mathrm{ft}^3/\mathrm{s} = \frac{9\pi\,\mathrm{ft}^2}{4}\frac{dh}{dt}$, so $\frac{dh}{dt} = -\frac{8}{9\pi}$ ft/s. So the depth of the water is decreasing at a rate of $8/(9\pi)$ feet per second.

3.11.37 Let h be the depth of the water in the tank at time t, and let r be the radius of the cone-shaped water at time t. By similar triangles, we have that $\frac{h}{r} = \frac{12}{6}$, so $h = 2r$. The volume of the water in the tank is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\frac{h^2}{4} \cdot h = \frac{\pi h^3}{12}$. Thus, $\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$. When $\frac{dh}{dt} = -1$, we have $\frac{dV}{dt} = -\frac{\pi h^2}{4}$. when $h = 6$, we have $\frac{dV}{dt} = -9\pi$, so the water is draining from the tank at 9π cubic feet per minute.

3.11.38 The volume of the upper tank is $V_u = \frac{1}{3}\pi r^2 h$ with $\frac{h}{r} = \frac{5}{4}$, so $V_u = \frac{\pi}{3}\frac{16}{25}h^3$. We have $\frac{dV_u}{dt} = \frac{16\pi}{25}h^2\frac{dh}{dt}$, and we are given that $\frac{dh}{dt} = -0.5$ meters per minute. If $h = 3$, we have $\frac{dV_u}{dt} = -\frac{144\pi}{50}$ meters per minute.

The volume of the lower tank is given by $V_l = 16\pi h_l$, so $\frac{dV_l}{dt} = 16\pi\frac{dh_l}{dt} = \frac{144\pi}{50}$, so $\frac{dh_l}{dt} = \frac{9}{50}$ meters per minute.

Now suppose that $h = 1$. Then $\frac{dV_u}{dt} = \frac{16\pi}{50}$ meters per minute. Then $\frac{dV_l}{dt} = 16\pi\frac{dh_l}{dt} = \frac{16\pi}{50}$, so $\frac{dh_l}{dt} = \frac{1}{50}$ meters per minute.

3.11.39 The volume of a segment of water of height h within a hemisphere of radius 10 is given by $V = \frac{1}{3}\pi h^2(30 - h) = 10\pi h^2 - \frac{1}{3}\pi h^3$. We have that $\frac{dV}{dt} = 20\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$. We are given that $\frac{dV}{dt} = 3\,\mathrm{m}^3/\mathrm{min}$, so when $h = 5$ we have $3 = (100\pi - 25\pi)\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{3}{75\pi} = \frac{1}{25\pi}$ meters per minute.

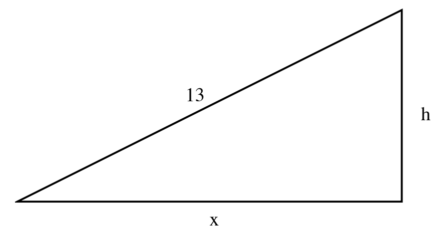
3.11.40 Let r be the radius of the exposed surface of the water of height h at time t. Consider the right triangle with legs of length r and $10 - h$ (from the center of the sphere measured down to where the water level is). The hypotenuse is given by the radius of the sphere, which is 10. By the Pythagorean theorem, we have

$$r^2 + (10 - h)^2 = 10^2,$$

which can be written as $r^2 + 100 - 20h + h^2 = 100$, so $r^2 + h^2 = 20h$. So $20\frac{dh}{dt} = 2h\frac{dh}{dt} + 2r\frac{dr}{dt}$. When $h = 5$, we have $10 \cdot \frac{1}{25\pi} = 5 \cdot \frac{1}{25\pi} + 5\sqrt{3}\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{\sqrt{3}}{75\pi}$. The surface area is given by $S = \pi r^2$, so $\frac{dS}{dt} = 2\pi r\frac{dr}{dt}$, so at this moment it is given by $\frac{dS}{dt} = 2\pi \cdot 5\sqrt{3} \cdot \frac{\sqrt{3}}{75\pi} = \frac{2}{5}$ square meters per minute.

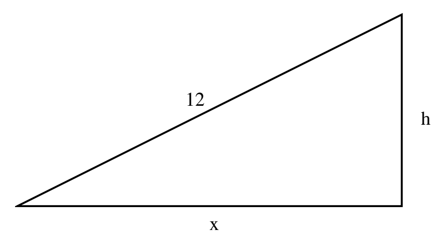
3.11.41

Let h be the vertical distance from the ground to the top of the ladder, and let x be the horizontal distance from the wall to the bottom of the ladder. By the Pythagorean Theorem, we have that $x^2 + h^2 = 169$. Thus, $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$, so $\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$, and we are given that $\frac{dx}{dt} = 0.5$ feet per second. At $x = 5$ we have $h = \sqrt{169 - 25} = 12$ feet. Thus, $\frac{dh}{dt} = -\frac{5}{12} \cdot \frac{1}{2} = -\frac{5}{24}$ feet per second. So the top of the ladder slides down the wall at $\frac{5}{24}$ feet per second.



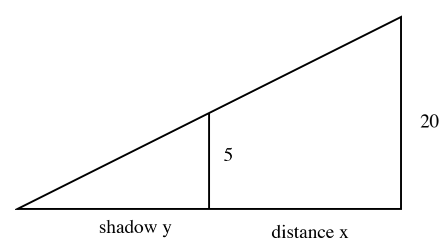
3.11.42

Let h be the vertical distance from the ground to the top of the ladder, and let x be the horizontal distance from the wall to the bottom of the ladder. By the Pythagorean Theorem, we have that $x^2 + h^2 = 144$. Thus, $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$. We are given that $\frac{dx}{dt} = 0.2$ feet per second. We are seeking the configuration when $\frac{dh}{dt} = -0.2$ feet per second. This occurs when $0.2x - 0.2h = 0$, or $x = h$. At this point in time, the triangle is forming a 45-45-90 triangle with $x = h = 6\sqrt{2}$.



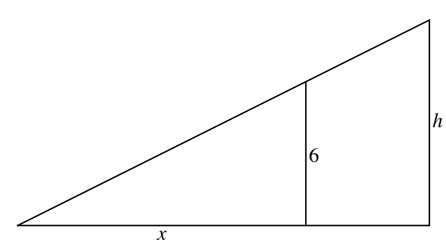
3.11.43

By similar triangles, $\frac{x+y}{20} = \frac{y}{5}$, so $x + y = 4y$, so $x = 3y$, and $\frac{dx}{dt} = 3\frac{dy}{dt}$. Because we are given that $\frac{dx}{dt} = -8$, we have $\frac{dy}{dt} = -\frac{8}{3}$ feet per second. The tip of her shadow is therefore moving at $-8 - \frac{8}{3} = -\frac{32}{3}$ feet per second.



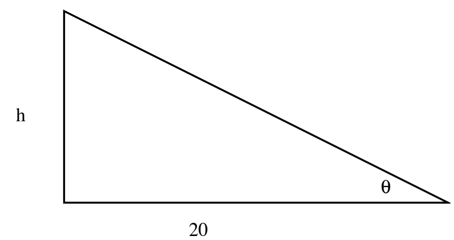
3.11.44

Let h be the height of the shadow of the 6-ft tall man who is x feet from the light. Then we have $\frac{x}{6} = \frac{15}{h}$ or $x = \frac{90}{h}$. So $\frac{dx}{dt} = -\frac{90}{h^2}\frac{dh}{dt}$. When he is 9 feet from the light, the height of his shadow is $h = \frac{90}{9} = 10$ ft and his walking speed is $\frac{dx}{dt} = -\frac{90}{10^2}(-2) = 1.8$ ft/s.



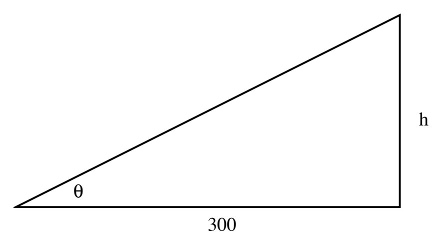
3.11.45

Let h be the vertical distance between the point on the elevator shaft positioned directly opposite the observer and the point on the elevator shaft that the observer is observing. So $h > 0$ corresponds to $\theta > 0$ and $h < 0$ corresponds to $\theta < 0$. We have $\frac{h}{20} = \tan\theta$, so $\frac{1}{20}\frac{dh}{dt} = \sec^2\theta\frac{d\theta}{dt}$. We are given that $\frac{dh}{dt} = 5\,\text{m/s}$. At $h = -10$, we have $\tan\theta = -.5$, so $\sec^2\theta = 1 + \tan^2\theta = 1 + (.5)^2 = 1.25$. So $\frac{d\theta}{dt} = \frac{1}{20 \cdot 1.25} \cdot 5 = \frac{1}{5}$ radian per second.
When $h = 20$, we have that $\tan\theta = 1$, so $\sec^2\theta = 1 + 1^2 = 2$, and thus $\frac{d\theta}{dt} = \frac{1}{20 \cdot 2} \cdot 5 = \frac{1}{8}$ radian per second.



3.11.46

Let h be the height of the balloon at time t. We have $\tan\theta = \frac{h}{300}$. Then $(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{300}\frac{dh}{dt}$, so $\frac{d\theta}{dt} = \frac{\cos^2\theta}{300} \cdot \frac{d\theta}{dt}$. When the balloon is 400 feet off the ground, $\frac{d\theta}{dt} = \frac{1}{300}\left(\frac{3}{5}\right)^2 \cdot 20 = 0.024$ radians/s. (Note that when $h = 400$ the hypotenuse of the pictured triangle is 500, so $\cos\theta = \frac{3}{5}$).



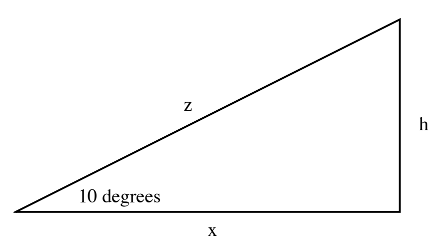
3.11.47 Let α be the angle between the line of sight to the bottom of the screen and the line of sight to the point 3 feet below where the floor and the wall meet. Note that $\cot\alpha = \frac{x}{3}$ and $\cot(\alpha + \theta) = \frac{x}{10}$, so $\alpha = \cot^{-1}(\frac{x}{3})$ and $\alpha + \theta = \cot^{-1}\left(\frac{x}{10}\right)$. Thus, $\theta = \cot^{-1}\left(\frac{x}{10}\right) - \cot^{-1}\left(\frac{x}{3}\right)$. So

$$\frac{d\theta}{dt} = -\frac{10x'}{100 + x^2} + \frac{3x'}{9 + x^2},$$

and at $x = 30$ feet, and with $\frac{dx}{dt} = 3$ feet per second, we have $\frac{d\theta}{dt} = -\frac{30}{1000} + \frac{9}{909} \approx -0.0201$ radians per second.

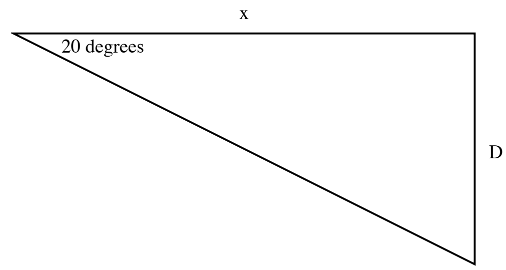
3.11.48

Let x be the distance the shadow has traveled, h the altitude of the jet, and z the line of flight of the jet. We have that $\frac{dz}{dt} = 550\,\text{mi/hr}$ and $h = z \cdot \sin(10°) \approx 0.174z$, so $\frac{dh}{dt} = 0.174\frac{dz}{dt} = 95.51\,\text{mi/hr}$. Also, $x = z \cdot \cos(10°) \approx 0.985z$, so $\frac{dx}{dt} = 0.985\frac{dz}{dt} = 541.64\,\text{mi/hr}$. So the shadow is moving at about 541.64 miles per hour.



3.11.49

Let x be the distance the surface ship has traveled and D the depth of the submarine. We have $\frac{dx}{dt} = 10\,\text{km/hr}$. Note that $\frac{D}{x} = \tan 20°$, so $D = x \cdot \tan 20° \approx 0.364x$. We have $\frac{dD}{dt} = 0.364\frac{dx}{dt} = 3.64\,\text{km/hr}$. The depth of the submarine is increasing at a rate of 3.64 km/hr.



3.11.50 Let θ be the angle RLP where L represents the lighthouse and R represents the point on the land where the light is currently hitting. Let s be the distance from the point P to the point R. We are given that $\frac{d\theta}{dt} = \frac{2\pi}{15}$ radians per second. Note that $\tan\theta = \frac{s}{500}$, so $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{ds}{dt}$. When the light is at point Q, $\tan\theta = \frac{2}{5}$, so $\sec^2\theta = 1 + \frac{4}{25} = \frac{29}{25}$. Then

$$\frac{ds}{dt} = 500 \cdot \frac{2\pi}{15} \cdot \frac{29}{25} = \frac{232\pi}{3}\ \text{m/s}.$$

The beam moves more slowly when R is near P, and more quickly when it is further away from P.

3.11.51

a. Let A be the point where the dragster started, let B be the point where camera 1 is located and let $C = y(t)$ be the position of the car at time t. Let θ be angle ABC. Note that $\tan\theta = \frac{y}{50}$, so $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$. At time $t = 2$, we have that $\tan^2\theta = 4$, so $\sec^2\theta = \tan^2\theta + 1 = 5$. So $\frac{dy}{dt} = 5 \cdot 50 \cdot .75 = 187.5$ feet per second.

b. Let D be the point where camera 2 is located, and let ϕ be angle ADC. The $\phi = \tan^{-1}\left(\frac{y}{100}\right)$, so $\frac{d\phi}{dt} = \frac{1}{100(1+\left(\frac{y}{100}\right)^2} \cdot \frac{dy}{dt}$. After 2 seconds, we know that $y = 100$ and $\frac{dy}{dt} = 187.5$. Thus $\frac{d\phi}{dt} = \frac{100}{20,000} \cdot 187.5 = .9375$ radians per second.

3.11.52

a. We have that the radius of the reel is 2 inches, so if L is the length and R is the number of revolutions, that $L = 4\pi R$.

b. Because $L = 4\pi R$, we have $\frac{dL}{dt} = 4\pi\frac{dR}{dt}$, so $\frac{dL}{dt} = 4\pi \cdot 1.5 = 6\pi$ inches per second.

3.11.53

a. $P = \frac{dE}{dt} = \frac{1}{2}v^2\frac{dm}{dt}$.

b. $P = \frac{1}{2}v^2\frac{dm}{dt} = \frac{1}{2}v^2(\rho A v) = \frac{1}{2}\rho A v^3$.

c. $P = \frac{1}{2}\rho A v^3 = \frac{1}{2}(1.23)(\pi)(3^2)(10^3) \approx 388.7$ W

d. $17388.7(0.25) \approx 4347.2$ W.

3.11.54 Because $V = 0.5$ when $P = 50$, we must have $0.5 = \frac{k}{50}$, so $k = 50(0.5) = 25$. Then $\frac{dV}{dt} = -\frac{25}{P^2}\frac{dP}{dt}$, so

$$\frac{dP}{dt} = -\frac{P^2}{25}\frac{dV}{dt} = -\frac{50^2}{25}(0.15) = -15\,\mathrm{kPa/min}.$$

3.11.55 Let θ be the angle between the hands of the clock, and D the distance between the tips of the hands. By the Law of Cosines, $D^2 = 2.5^2 + 3^2 - 15\cos\theta$. So $2D\frac{dD}{dt} = 15\sin\theta\frac{d\theta}{dt}$. At 9:00 AM, we have $D^2 = 6.25 + 9$, so $D = \sqrt{15.25}$. Also, $\theta = \pi/2$ so $\sin\theta = 1$. Thus, $\frac{dD}{dt} = \frac{15}{2\sqrt{15.25}}\frac{d\theta}{dt}$. Now $\frac{d\theta}{dt} = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}$ where $\frac{d\theta_1}{dt}$ is the angular change of the minute hand and $\frac{d\theta_2}{dt}$ is the angular change of the hour hand. We have $\frac{d\theta_1}{dt} = \frac{\pi}{30}$ radians per minute and $\frac{d\theta_2}{dt} = \frac{\pi}{360}$ radians per minute, so $\frac{d\theta}{dt} = \frac{11\pi}{360}$ radians per minute. Thus $\frac{dD}{dt} = \frac{15}{2\sqrt{15.25}} \cdot \frac{11\pi}{360} \approx 0.18436$ meters per minute, or about 11.06 meters per hour.

3.11.56 At time t, the boat traveling west has gone $20t$ miles while the boat traveling southwest has gone $15t$ miles. Let D be the distance between the boats; the line between the boats forms the third side of a triangle. Then by the Law of Cosines,

$$D^2 = (20t)^2 + (15t)^2 - 2\cdot 20t\cdot 15t\cos(\pi/4) = 625t^2 - 600t^2\cdot\frac{\sqrt{2}}{2} = (625 - 300\sqrt{2})t^2.$$

Thus,

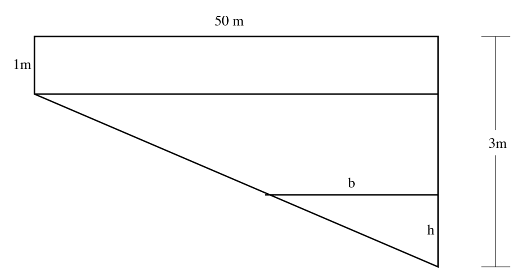
$$D = \left(625 - 30\sqrt{2}\right)^{1/2} t,$$

so that

$$\frac{dD}{dt} = \left(625 - 30\sqrt{2}\right)^{1/2} \approx 14.168\,\mathrm{mph}.$$

3.11.57

By similar triangles, $\frac{2}{50} = \frac{h}{b}$, so $b = 25h$. Also, $A = \frac{1}{2}bh = 12.5h^2$, so the volume for $0 \le h \le 2$ is $V(h) = 12.5\cdot h^2\cdot 20 = 250h^2$. For $2 < h \le 3$, $V(h) = 250\cdot 2^2 + 50\cdot 20\cdot (h-2) = 1000h - 1000$. When $t = 250$ minutes, then $V = 250\,\mathrm{min}\cdot 1\,\mathrm{m}^3/\mathrm{min} = 250\,\mathrm{m}^3$. So $V(h) = 250h^2 = 250$, so $h = 1\,\mathrm{m}$. At that time $\frac{dV}{dt} = 500h\frac{dh}{dt} = 500\cdot 1\cdot\frac{dh}{dt} = 1\,\mathrm{m}^3/\mathrm{min}$. So $\frac{dh}{dt} = \frac{1}{500}\,\mathrm{m/min} = 0.002\,\mathrm{m/min} = 2\,\mathrm{mm/min}$.



Fill time: The volume of the entire swimming pool is 2000 cubic meters, so at 1 cubic meter per minute, it will take 2000 minutes.

3.11.58 Let l be the length of a side of triangle, and let x be the line segment from a vertex to the midpoint of the opposite side. Then $\sin(\pi/3) = \frac{x}{l}$, so $l = \frac{2}{\sqrt{3}}x$. Now $A = \frac{xl}{2} = \frac{x^2}{\sqrt{3}}$. Thus $\frac{dA}{dt} = \frac{2x}{\sqrt{3}}\frac{dx}{dt}$, and when $x = 0$, this quantity is zero.

3.11.59 By the Law of Sines, $\frac{\sin\theta}{s} = \frac{\sin\left(\frac{3\pi}{4} - \theta\right)}{2}$, so

$$2\sin\theta = s\sin\left(\frac{3\pi}{4} - \theta\right) = s\left(\sin\left(\frac{3\pi}{4}\right)\cos\theta - \cos\left(\frac{3\pi}{4}\right)\sin\theta\right).$$

We have

$$\begin{aligned}
2\sin\theta &= \frac{\sqrt{2}}{2}s(\sin\theta+\cos\theta)\\
2\tan\theta &= \frac{\sqrt{2}}{2}s(\tan\theta+1)\\
\tan\theta &= \frac{(\sqrt{2}/2)\cdot s}{2-(\sqrt{2}/2)s} = \frac{\sqrt{2}s}{4-\sqrt{2}s}\\
\theta &= \tan^{-1}\left(\frac{\sqrt{2}s}{4-\sqrt{2}s}\right).
\end{aligned}$$

Thus, $\frac{d\theta}{dt} = \frac{\sqrt{2}\cdot\frac{ds}{dt}}{4-2\sqrt{2}s+s^2}$. When $\frac{ds}{dt}=15$ and $s=7.5$ we arrive at $\frac{d\theta}{dt}=0.54$ radians per hour.

3.11.60 Let s be the distance the ship has traveled. By the Law of Sines, $\frac{\sin\theta}{s} = \frac{\sin\left(\frac{3\pi}{4}-\theta\right)}{1.5}$, so $1.5\sin\theta = s\cdot(\sin\left(\frac{3\pi}{4}\right)\cos\theta - \cos\left(\frac{3\pi}{4}\right)\sin\theta)$.We have

$$\begin{aligned}
\sin\theta &= \frac{\sqrt{2}}{3}s(\sin\theta+\cos\theta)\\
\tan\theta &= \frac{\sqrt{2}}{3}s(\tan\theta+1)\\
\tan\theta &= \frac{(\sqrt{2}/3)\cdot s}{1-(\sqrt{2}/3)s} = \frac{\sqrt{2}s}{3-\sqrt{2}s}\\
\theta &= \tan^{-1}\left(\frac{\sqrt{2}s}{3-\sqrt{2}s}\right).
\end{aligned}$$

Thus, $\frac{d\theta}{dt} = \frac{3\sqrt{2}\cdot\frac{ds}{dt}}{9-6\sqrt{2}s+4s^2}$. At noon, $2s^2 = 1.5^2$, so $s=\frac{3}{\sqrt{8}}$. At 1:30 pm $s = 18+\frac{3}{\sqrt{8}} \approx 19.06\,\text{mi}$, and $\frac{ds}{dt} = 12\,\text{mi/hr}$, so $\frac{d\theta}{dt} \approx 0.04$ radians per hour.

3.11.61 Let x be the distance the eastbound boat has traveled at time t and let s be the distance the northeastbound boat has traveled. Note the diagram shown. By the Law of Sines, $\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{s} = \frac{\sin\left(\frac{\pi}{4}+\theta\right)}{x}$. Thus,

$$\begin{aligned}
x\left(\sin\left(\frac{\pi}{2}\right)\cos\theta - \cos\left(\frac{\pi}{2}\right)\sin\theta\right) &=\\
s\left(\sin\left(\frac{\pi}{4}\right)\cos\theta + \cos\left(\frac{\pi}{4}\right)\sin\theta\right)
\end{aligned}$$

So

$$\begin{aligned}
x\cos\theta &= \frac{\sqrt{2}}{2}\cdot s\cdot\cos\theta + \frac{\sqrt{2}}{2}\cdot s\cdot\sin\theta,\\
x &= \frac{\sqrt{2}}{2}\cdot s + \frac{\sqrt{2}}{2}\cdot s\cdot\tan\theta,
\end{aligned}$$

and thus $\tan\theta = \frac{x-\frac{\sqrt{2}}{2}s}{\frac{\sqrt{2}}{2}s} = \frac{\sqrt{2}x-s}{s}$, and therefore
$\theta = \tan^{-1}\left(\frac{\sqrt{2}x-s}{s}\right)$.

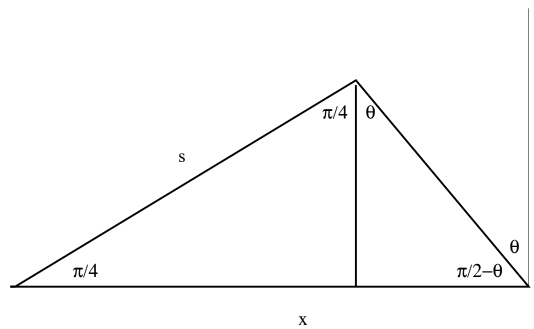
We have

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{\sqrt{2}x-s}{s}\right)^2} \cdot \frac{\left(\sqrt{2}\left(\frac{dx}{dt}\right) - \left(\frac{ds}{dt}\right)\right)\cdot s - \left(\sqrt{2}x - s\right)\cdot \frac{ds}{dt}}{s^2} = \frac{\sqrt{2}\left(s\frac{dx}{dt} - x\frac{ds}{dt}\right)}{s^2 + (\sqrt{2}x - s)^2}.$$

At time t, we have $s(t) = 15t$ and $x(t) = 12t$. Note that

$$s\frac{dx}{dt} - x\frac{ds}{dt} = 15t \cdot 12 - 12t \cdot 15 = 0.$$

Thus $\theta' = 0$ for every value of t, so that the angle is constant.



3.11.62

Let D be the distance from the bottom center of the Ferris wheel to the cart. Note that $\tan\theta = \dfrac{D}{20}$, so $\theta = \tan^{-1}\left(\dfrac{D}{20}\right)$, and

$$\frac{d\theta}{dt} = \frac{20 \cdot \frac{dD}{dt}}{400 + D^2} = \frac{20}{400+D^2} \cdot \frac{dD}{dt}.$$

Let α be the angle pictured. By the Law of Cosines,

$$D^2 = 5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cdot \cos\alpha = 50 - 50\cos\alpha.$$

So $2D\dfrac{dD}{dt} = 50\dfrac{d\alpha}{dt} \cdot \sin\alpha$, and solving for $\dfrac{dD}{dt}$ gives

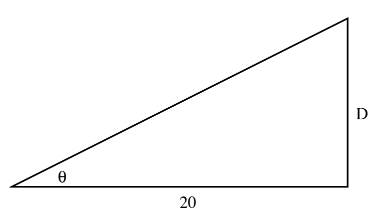
$$\frac{dD}{dt} = \frac{25\sin\alpha}{D} \cdot \frac{d\alpha}{dt}.$$

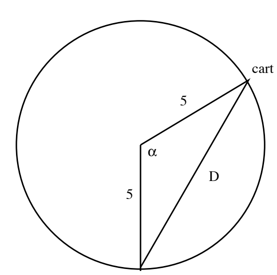
At $t = 40$ seconds (which is $\dfrac{2}{3}$ minutes), we have $\alpha = \dfrac{2\pi}{3}$, so that $D^2 = 50 - 50\cos(2\pi/3) = 75$, and thus $D = 5\sqrt{3}$. Also $\sin\alpha = \dfrac{\sqrt{3}}{2}$. Finally, we are given that $\dfrac{d\alpha}{dt} = \pi$ radians per minute. Then we have

$$\frac{dD}{dt} = \frac{25 \cdot \frac{\sqrt{3}}{2}}{5\sqrt{3}} \cdot \pi = \frac{5}{2}\pi.$$

Finally, we have

$$\frac{d\theta}{dt} = \frac{20}{400+75} \cdot \frac{5}{2}\pi = \frac{2\pi}{19} \approx 0.331\,\text{radians per second.}$$





3.11.63

a. The volume of the water in the tank (as a function of h – the depth of the water in the tank) is given by 5 times the area of the segment of water in a cross-sectional circle. For a tank of radius 1, the formula for such a segment is $\cos^{-1}(1-h) - (1-h)\sqrt{2h-h^2}$. Thus the volume of the water in the tank is given by $V = 5(\cos^{-1}(1-h) - (1-h)\sqrt{2h-h^2})$. We have

$$\begin{aligned}
\frac{dV}{dt} &= 5 \cdot \left(-\frac{1}{\sqrt{1-(1-h)^2}} \cdot \left(-\frac{dh}{dt}\right) + \frac{dh}{dt}\sqrt{2h-h^2} - \frac{(1-h)^2}{\sqrt{2h-h^2}}\frac{dh}{dt} \right) \\
&= 5\left(\sqrt{2h-h^2} + \frac{1-(1-h)^2}{\sqrt{2h-h^2}}\right)\frac{dh}{dt} \\
&= 5\left(\frac{2h-h^2+1-1+2h-h^2}{\sqrt{2h-h^2}}\right)\frac{dh}{dt} \\
&= 5\left(\frac{2(2h-h^2)}{\sqrt{2h-h^2}}\right)\frac{dh}{dt} \\
&= 10\sqrt{2h-h^2} \cdot \frac{dh}{dt}
\end{aligned}$$

When $h = 0.5$, we have $-\frac{3}{2} = \frac{dV}{dt} = 5\sqrt{3}\frac{dh}{dt}$, so $\frac{dh}{dt} = -\frac{\sqrt{3}}{10}$ meters per hr.

b. The surface area of the water is given by $S = 5 \cdot 2\sqrt{2h-h^2}$. So $\frac{dS}{dt} = 10 \cdot \frac{2-2h}{2\sqrt{2h-h^2}} \cdot \frac{dh}{dt}$, so at $h = .5$, we have $\frac{5}{\sqrt{3/4}} \cdot -\frac{\sqrt{3}}{10} = -1$ square meter per hr.

3.11.64 Let r be the distance from the point on the highway perpendicular to the searchlight to the right-hand edge of the beam, and let l be the distance from that point to the left-hand edge of the beam. Then $w = l - r$. We have that

$$r = 100\tan\left(\theta - \frac{\pi}{32}\right) \quad \text{and} \quad l = 100\tan\left(\theta + \frac{\pi}{32}\right).$$

Thus

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dl}{dt} - \frac{dr}{dt} \\
&= 100\left(\sec^2\left(\theta + \frac{\pi}{32}\right) - \sec^2\left(\theta - \frac{\pi}{32}\right)\right) \cdot \frac{d\theta}{dt} \\
&= 100\left(\tan^2\left(\theta + \frac{\pi}{32}\right) - 1 - \left(\tan^2\left(\theta - \frac{\pi}{32}\right) - 1\right)\right) \cdot \frac{d\theta}{dt} \\
&= 100\left(\tan^2\left(\theta + \frac{\pi}{32}\right) - \tan^2\left(\theta - \frac{\pi}{32}\right)\right) \cdot \frac{d\theta}{dt}.
\end{aligned}$$

With $\theta' = \frac{\pi}{6}$ radians per second and $\theta = \frac{\pi}{3}$, we have

$$\frac{dw}{dt} = 100\left(\tan^2\left(\frac{35\pi}{96}\right) - \tan^2\left(\frac{29\pi}{96}\right)\right) \cdot \frac{\pi}{6} \approx 153.081 \text{ meters per second.}$$

Chapter Three Review

1

a. False. This function is not differentiable at $x = -\frac{1}{2}$. It is possible for a function to be continuous at a point and not differentiable at that point.

b. False. For example, $f(x) = x^2 + 3$ and $g(x) = x^2 + 100$ have the same derivative, but aren't the same function.

c. False. For example, $\dfrac{d}{dx}|e^{-x}| = \dfrac{d}{dx}e^{-x} = -e^{-x} \neq |-e^{-x}|$.

d. False. For example, the function $f(x) = |x|$ has no derivative at 0, but there is no vertical tangent there.

e. True. For example, a ball dropping from a high tower has acceleration due to gravity which is negative, but it is speeding up as it falls because the velocity (which is negative also) is in the same direction as the acceleration.

2

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) + 9 - (x^2+2x+9)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h + 9 - x^2 - 2x - 9}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(2x+h+2)}{h} \\
&= \lim_{h\to 0}(2x+h+2) = 2x+2.
\end{aligned}$$

3

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+5} - \frac{1}{x^2+5}}{h} = \\
&= \lim_{h\to 0} \frac{x^2+5-(x^2+2xh+h^2+5)}{h((x+h)^2+5)(x^2+5)} \\
&= \lim_{h\to 0} \frac{-h(2x+h)}{h((x+h)^2+5)(x^2+5)} \\
&= \lim_{h\to 0} \frac{-(2x+h)}{((x+h)^2+5)(x^2+5)} \\
&= -\frac{2x}{(x^2+5)^2}.
\end{aligned}$$

4

$$\begin{aligned}
h'(t) &= \lim_{x\to t} \frac{h(x)-h(t)}{x-t} = \lim_{x\to t} \frac{\sqrt{3x+5} - \sqrt{3t+5}}{x-t} \\
&= \lim_{x\to t} \frac{(\sqrt{3x+5} - \sqrt{3t+5})}{(x-t)} \cdot \frac{(\sqrt{3x+5} + \sqrt{3t+5})}{(\sqrt{3x+5} + \sqrt{3t+5})} \\
&= \lim_{x\to t} \frac{3x+5-(3t+5)}{(x-t)(\sqrt{3x+5} + \sqrt{3t+5}} \\
&= \lim_{x\to t} \frac{3(x-t)}{(x-t)(\sqrt{3x+5} + \sqrt{3t+5})} \\
&= \lim_{x\to t} \frac{3}{\sqrt{3x+5} + \sqrt{3t+5}} = \frac{3}{2\sqrt{3t+5}}.
\end{aligned}$$

5 $\dfrac{d}{dx}(\tan^3 - 3\tan x + 3x) = 3\tan^2 x \sec^2 x - 3\sec^2 x + 3 = 3\sec^2 x(\tan^2 x - 1) + 3 = 3(\tan^2 x + 1)(\tan^2 x - 1) + 3 = 3\tan^4 x - 3 + 3 = 3\tan^4 x$.

6

$$\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{\sqrt{1-x^2} - x\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2} = \frac{\frac{1-x^2}{\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{\frac{1}{\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)^{3/2}}.$$

7

$$\frac{d}{dx}(x^4 - \ln(x^4+1)) = 4x^3 - \frac{4x^3}{x^4+1} = \frac{4x^3(x^4+1)}{x^4+1} - \frac{4x^3}{x^4+1} = \frac{4x^7+4x^3-4x^3}{x^4+1} = \frac{4x^7}{x^4+1}.$$

8

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right) &= \frac{d}{dx}\left(\ln(1-\sqrt{x}) - \ln(1+\sqrt{x})\right)\\ &= \frac{-1}{2\sqrt{x}(1-\sqrt{x})} - \frac{1}{2\sqrt{x}(1+\sqrt{x})}\\ &= \frac{-1}{2\sqrt{x}}\left(\frac{1}{1-\sqrt{x}} + \frac{1}{1+\sqrt{x}}\right)\\ &= \frac{-1}{2\sqrt{x}}\left(\frac{1+\sqrt{x}+1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)\\ &= \frac{-1}{\sqrt{x}}\left(\frac{1}{1-x}\right) = \frac{1}{\sqrt{x}(x-1)}.\end{aligned}$$

9 $y' = 2x^2 + 2\pi x + 7.$

10 $y' = 16x^3 \ln x + \dfrac{4x^4}{x} - 4x^3 = 16x^3\ln x + 4x^3 - 4x^3 = 16x^3 \ln x.$

11 $y' = 2^x(\ln 2).$

12 $y' = 2\sqrt{2}x^{\sqrt{2}-1}.$

13 $y' = e^{2\theta}\cdot 2 = 2e^{2\theta}.$

14 $y' = 2x^{3/2} + \frac{3}{2}(2x-3)x^{1/2} = 5x^{3/2} - \frac{9}{2}x^{1/2} = \sqrt{x}\left(5x - \frac{9}{2}\right).$

15 $y' = \dfrac{3}{2}\sqrt{1+x^4}(4x^3) = 6x^3\sqrt{1+x^4}.$

16 $y' = 2\sqrt{x^2-2x+2} + 2x\cdot\dfrac{1}{2\sqrt{x^2-2x+2}}(2x-2) = 2\left(\sqrt{x^2-2x+2} + \dfrac{x^2-x}{\sqrt{x^2-2x+2}}\right) = \dfrac{4x^2-6x+4}{\sqrt{x^2-2x+2}}.$

17 $y' = 10t\sin t + 5t^2\cos t.$

18 $y' = 5 + 3\sin^2 x\cos x + 3x^2\cos x^3.$

19 $y' = -e^{-x}(x^2+2x+2) + e^{-x}(2x+2) = -e^{-x}(x^2+2x+2-2x-2) = -x^2e^{-x}.$

20 $y' = \dfrac{(\ln x + a)\frac{1}{x} - (\ln x)\frac{1}{x}}{(\ln x + a)^2} = \dfrac{a}{x(\ln x + a)^2}.$

21 $y' = \dfrac{((\sec 2w + 1)2\sec 2w\tan 2w - 2\sec 2w\sec 2w\tan 2w)}{(\sec 2w+1)^2} = \dfrac{2\sec 2w\tan 2w}{(\sec 2w+1)^2}.$

22

$$\begin{aligned}y' &= \frac{1}{3}\left(\frac{\sin x}{\cos x+1}\right)^{-2/3}\left(\frac{(\cos x+1)\cos x - (\sin x)(-\sin x)}{(\cos x+1)^2}\right)\\ &= \frac{1}{3}\left(\frac{\sin x}{\cos x+1}\right)^{-2/3}\left(\frac{\cos^2 x + \cos x + \sin^2 x}{(\cos x+1)^2}\right)\\ &= \frac{1}{3}\left(\frac{(\cos x+1)^{2/3}}{\sin^{2/3} x}\right)\left(\frac{\cos x+1}{(\cos x+1)^2}\right)\\ &= \frac{1}{3\sin^{2/3} x(1+\cos x)^{1/3}}.\end{aligned}$$

23 $y' = \dfrac{3}{\sec 3x} \sec 3x \tan 3x = 3\tan 3x.$

24 $y' = \dfrac{-7\csc 7x \cot 7x - 7\csc^2 7x}{\csc 7x + \cot 7x} = \dfrac{-7\csc 7x(\cot 7x + \csc 7x)}{\csc 7x + \cot 7x} = -7\csc 7x.$

25 $y' = 100(5t^2+10)^{99}(10t) = 1000t(5t^2+10)^{99}.$

26 $y' = e^{\sin x + 2x + 1} \cdot (\cos x + 2).$

27 $y' = \dfrac{1}{\sin x^3} \cdot \cos x^3 \cdot 3x^2 = \dfrac{3x^2 \cos x^3}{\sin x^3} = 3x^2 \cot x^3.$

28 $y' = e^{\tan x}\sec^2 x(\tan x - 1) + e^{\tan x}(\sec^2 x) = e^{\tan x}\sec^2 x(\tan x - 1 + 1) = e^{\tan x}\sec^2 x \tan x.$

29 $y' = \dfrac{1}{(\sqrt{t^2-1})^2+1} \cdot \dfrac{1}{2}(t^2-1)^{-1/2} \cdot 2t = \dfrac{2t}{2t^2\sqrt{t^2-1}} = \dfrac{1}{t\sqrt{t^2-1}}.$

30 Let $y = x^{\sqrt{x+1}}$. Then $\ln y = \sqrt{x+1}\ln x$. Then

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x+1}} \cdot \ln x + \frac{\sqrt{x+1}}{x} \\
&= \frac{\ln x}{2\sqrt{x+1}} + \frac{\sqrt{x+1}}{x} \\
&= \frac{x\ln x}{2x\sqrt{x+1}} + \frac{2(x+1)}{2x\sqrt{x+1}} \\
&= \frac{x\ln x + 2x + 2}{2x\sqrt{x+1}}.
\end{aligned}$$

Thus

$$\frac{dy}{dx} = y \cdot \frac{x\ln x + 2x + 2}{2x\sqrt{x+1}} = x^{\sqrt{x+1}}\left(\frac{x\ln x + 2x + 2}{2x\sqrt{x+1}}\right).$$

31 $y' = (4\sec^2(\theta^2+3\theta+2)) \cdot (2\theta+3) = (8\theta+12)\sec^2(\theta^2+3\theta+2).$

32 $y' = 5\csc^4 3x \cdot (-\csc 3x \cot 3x) \cdot 3 = -15\csc^5 3x \cot 3x.$

33 $y' = \dfrac{w^5 \cdot \frac{1}{w} - \ln w \cdot 5w^4}{w^{10}} = \dfrac{w^4(1-5\ln w)}{w^{10}} = \dfrac{1-5\ln w}{w^6}.$

34 $y' = \dfrac{e^{as} \cdot 1 - s \cdot e^{as} \cdot a}{e^{2as}} = \dfrac{e^{as}(1-as)}{e^{2as}} = \dfrac{1-as}{e^{as}}.$

35 $y' = \dfrac{(8u+1)(8u+1) - (4u^2+u)(8)}{(8u+1)^2} = \dfrac{64u^2+16u+1-32u^2-8u}{(8u+1)^2} = \dfrac{32u^2+8u+1}{(8u+1)^2}.$

36 $y' = -3\left(\dfrac{3t^2-1}{3t^2+1}\right)^{-4} \cdot \dfrac{(3t^2+1)(6t)-(3t^2-1)(6t)}{(3t^2+1)^2} = -36 \cdot \dfrac{(3t^2+1)^2 \cdot t}{(3t^2-1)^4}.$

37 $y' = \sec^2(\sin\theta) \cdot \cos\theta.$

38 $y' = \dfrac{4}{3}\left(\dfrac{v}{v+1}\right)^{1/3}\left(\dfrac{(v+1)-v}{(v+1)^2}\right) = \dfrac{4}{3}\left(\dfrac{v}{v+1}\right)^{1/3}\left(\dfrac{1}{(v+1)^2}\right) = \dfrac{4v^{1/3}}{3(v+1)^{7/3}}.$

39 $y' = \cos\sqrt{\cos^2 x + 1} \cdot \dfrac{1}{2\sqrt{\cos^2 x+1}} \cdot -2\cos x \sin x = -\dfrac{\cos\sqrt{\cos^2 x+1}\cos x \sin x}{\sqrt{\cos^2 x + 1}}.$

40 $y' = e^{\sin(\cos x)} \cdot \cos(\cos x) \cdot (-\sin x) = -\sin x \cos(\cos x) e^{\sin(\cos x)}$.

41 $y' = \frac{d}{dt} \ln \sqrt{e^t+1} = \frac{d}{dt}\left(\frac{1}{2}\ln(e^t+1)\right) = \frac{1}{2} \cdot \frac{1}{e^t+1} \cdot e^t = \frac{e^t}{2(e^t+1)}$.

42 $y' = 1 \cdot e^{-10x} + x(-10e^{-10x}) = e^{-10x}(1-10x)$.

43 $y' = 2x + 2\tan^{-1}(\cot x) + 2x \cdot \frac{1}{1+\cot^2 x} \cdot (-\csc^2 x) = 2x + 2\tan^{-1}(\cot x) + \frac{2x}{\csc^2 x} \cdot (-\csc^2 x) = 2\tan^{-1}(\cot x)$.

44 $y' = \frac{1}{2}(1-x^4)^{-1/2}(-4x^3) + 2x\sin^{-1} x^2 + x^2 \frac{1}{\sqrt{1-x^4}} \cdot 2x = \frac{-2x^3}{\sqrt{1-x^4}} + 2x\sin^{-1} x^2 + \frac{2x^3}{\sqrt{1-x^4}} = 2x\sin^{-1} x^2$.

45 $y' = \ln^2 x + x \cdot 2\ln x \cdot \left(\frac{1}{x}\right) = \ln x \cdot (\ln x + 2)$.

46 $y' = 6e^{6x}\sin x + e^{6x}\cos x = e^{6x}(6\sin x + \cos x)$.

47 $y' = 2^{x^2-x} \cdot \ln 2 \cdot (2x-1)$.

48 $y' = 10^{\sin x} \ln 10 \cdot \cos x + 10\sin^9 x \cos x = \cos x(10^{\sin x} \ln 10 + 10\sin^9 x)$.

49 Let $y = (x^2+1)^{\ln x}$. Then $\ln y = \ln x \ln(x^2+1)$. Then

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}\ln(x^2+1) + (\ln x)\frac{1}{x^2+1} \cdot 2x \\ &= \frac{\ln(x^2+1)}{x} + \frac{2x\ln x}{x^2+1},\end{aligned}$$

so

$$\frac{dy}{dx} = (x^2+1)^{\ln x}\left(\frac{\ln(x^2+1)}{x} + \frac{2x\ln x}{x^2+1}\right).$$

50

$$\begin{aligned}\frac{d}{dx}x^{\cos 2x} = \frac{d}{dx}e^{(\cos 2x)\ln x} &= e^{(\cos 2x)\ln x}\left((-2\sin 2x)\ln x + (\cos 2x) \cdot \frac{1}{x}\right) \\ &= x^{\cos 2x}\left((-2\sin 2x)\ln x + \frac{\cos 2x}{x}\right).\end{aligned}$$

51 $y' = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot -\frac{1}{x^2} = -\frac{1}{|x|\sqrt{x^2-1}}$.

52 $y' = \frac{1}{(x+8)\ln 3}$.

53 $y' = 6\cot^{-1} 3x - \frac{18x}{9x^2+1} + \frac{18x}{9x^2+1} = 6\cot^{-1} 3x$.

54 $y' = 4x\cos^{-1} x + 2x^2 \cdot \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 4x\cos^{-1} x + \frac{1-2x^2}{\sqrt{1-x^2}}$.

55 Differentiate implicitly. $1 = -\sin(x-y)(1-y')$, so $y' - 1 = \frac{1}{\sin(x-y)}$, so $y' = \frac{1}{\sin(x-y)} + 1 = \csc(x-y) + 1$.

56 Differentiate implicitly. $1 \cdot y^4 + x \cdot 4y^3y' + 4x^3y + x^4y' = 0$, so $(4xy^3 + x^4)y' = -y^4 - 4x^3y$. Then $y' = \frac{-y(y^3+4x^3)}{x(4y^3+x^3)}$.

57 Because
$$y' = \frac{(1+\sin x)y'e^y - e^y\cos x}{(1+\sin x)^2},$$
collecting terms gives
$$y'\left(1 - \frac{e^y}{1+\sin x}\right) = -\frac{e^y\cos x}{(1+\sin x)^2},$$
so
$$y'(1-y) = -\frac{\cos x}{1+\sin x}\cdot y.$$
Thus $y' = -\dfrac{y\cos x}{(1-y)(1+\sin x)}$. This can also be written as $y' = \dfrac{y\cos x}{e^y - 1 - \sin x}$.

58 $\cos x\cos(y-1) - (\sin x)y'\sin(y-1) = 0$, so $y' = \cot x\cot(y-1)$.

59 $y'\sqrt{x^2+y^2} + y\cdot\dfrac{x+yy'}{\sqrt{x^2+y^2}} = 0$, and thus $y'\left(\sqrt{x^2+y^2} + \dfrac{y^2}{\sqrt{x^2+y^2}}\right) = -\dfrac{xy}{\sqrt{x^2+y^2}}$. This can be written as $y'\left(\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}\right) = -\dfrac{xy}{\sqrt{x^2+y^2}}$, so $y' = -\dfrac{xy}{x^2+2y^2}$.

60 Use logarithmic differentiation. $\ln y = 3\ln(x^2+1) - 8\ln(x^4+7) - 7\ln(2x+1)$. Then
$$\frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2+1} - \frac{32x^3}{x^4+7} - \frac{14}{2x+1}.$$
So
$$\frac{dy}{dx} = \frac{(x^2+1)^3}{(x^4+7)^8(2x+1)^7}\left(\frac{6x}{x^2+1} - \frac{32x^3}{x^4+7} - \frac{14}{2x+1}\right).$$

61 Use logarithmic differentiation. $\ln y = 10\ln(3x+5) + \frac{1}{2}\ln(x^2+5) - 50\ln(x^3+3)$. Then
$$\frac{1}{y}\frac{dy}{dx} = \frac{30}{3x+5} + \frac{x}{x^2+5} - \frac{150x^2}{x^3-3}.$$
So
$$\frac{dy}{dx} = \frac{(3x+5)^{10}\sqrt{x^2+5}}{(x^3+1)^{50}}\left(\frac{30}{3x+5} + \frac{x}{x^2+5} - \frac{150x^2}{x^3-3}\right).$$

62 $f'(x) = \dfrac{1}{1+(4x^2)^2}\cdot 8x = \dfrac{8x}{1+16x^4}$. So $f'(1) = \dfrac{8}{17}$.

63 $f'(x) = \sec^{-1}x + \dfrac{1}{\sqrt{x^2-1}}$. So $f'(2/\sqrt{3}) = \dfrac{\pi}{6} + \sqrt{3}$.

64 $f'(x) = \dfrac{1}{1+e^{-2x}}\cdot\left(-e^{-x}\right) = -\dfrac{1}{e^{-x}+e^x}$. So $f'(0) = -\dfrac{1}{2}$.

65 $\dfrac{d}{dx}x^{1/x} = \dfrac{d}{dx}e^{\frac{\ln x}{x}} = e^{\frac{\ln x}{x}}\cdot\left(\dfrac{1-\ln x}{x^2}\right) = x^{1/x}\left(\dfrac{1-\ln x}{x^2}\right)$. So $\dfrac{d}{dx}x^{1/x}\Big|_{x=1} = 1\cdot\dfrac{1-0}{1^2} = 1$.

66 $y' = 2xe^{x^2+1}$, so $y'' = 2e^{x^2+1} + 2x(2xe^{x^2+1}) = (4x^2+2)e^{x^2+1}$.

67 $y' = 2^x x\ln 2 + 2^x = 2^x(x\ln 2+1)$, so $y'' = 2^x\ln 2(x\ln 2+1) + 2^x\ln 2 = 2^x\ln 2(x\ln 2+2)$.

68 $y' = \dfrac{(x+1)3-(3x-1)}{(x+1)^2} = \dfrac{4}{(x+1)^2}$. Thinking of y' as $4(x+1)^{-2}$, we have
$$y'' = -8(x+1)^{-3} = -\frac{8}{(x+1)^3}.$$

69 $y' = \dfrac{x^2 \cdot \frac{1}{x} - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3}$. Then

$$y'' = \frac{x^3(\frac{-2}{x}) - (1-2\ln x)3x^2}{x^6} = \frac{-2x^2 - 3x^2 + 6x^2\ln x}{x^6} = \frac{6\ln x - 5}{x^4}.$$

70 Differentiating implicitly: $1 + y'\cos y = y'$, so $y'(\cos y - 1) = -1$, so $y' = \dfrac{1}{1-\cos y}$. Then $y'' = -(1-\cos y)^{-2}(\sin y)y' = -\dfrac{1}{(1-\cos y)^2}\cdot \sin y \cdot \dfrac{1}{1-\cos y} = -\dfrac{\sin y}{(1-\cos y)^3} = \dfrac{\sin y}{(\cos y - 1)^3}$.

71 Differentiating implicitly: $y + xy' + 2yy' = 0$, so $y'(x+2y) = -y$, so $y' = -\dfrac{y}{x+2y}$. Then $y'' = -\dfrac{(x+2y)y' - y(1+2y')}{(x+2y)^2} = \dfrac{y - xy'}{(x+2y)^2} = \dfrac{y + x\left(\frac{y}{x+2y}\right)}{(x+2y)^2} = \dfrac{y(x+2y)+xy}{(x+2y)^3} = \dfrac{2xy+2y^2}{(x+2y)^3} = \dfrac{2}{(x+2y)^3}$.

Note that for the last equality we are using the fact (from the original equation) that $xy + y^2 = 1$.

72 $y' = 2^{3x-6}3\ln 2$ so $y'(2) = 3\ln 2$. The equation of the tangent line is $y - 1 = 3\ln 2(x-2)$, or $y = (3\ln 2)x - 6\ln 2 + 1$.

73 $y' = 9x^2 + \cos x$. At $x = 0$, $y' = 1$. So the tangent line is given by $y - 0 = 1(x-0)$, or $y = x$.

74 $y' = \dfrac{4(x^2+3) - 8x^2}{(x^2+3)^2}$, so $y'(1) = \dfrac{1}{2}$. The tangent line is given by $y - 1 = \dfrac{1}{2}(x-1)$, or $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

75 $y' + \dfrac{y + xy'}{2\sqrt{xy}} = 0$. At the point $(1,4)$, we have $y' + \dfrac{4+y'}{4} = 0$, so $y' = -\dfrac{4}{5}$. The tangent line is given by $y - 4 = -\dfrac{4}{5}(x-1)$, or $y = -\dfrac{4}{5}x + \dfrac{24}{5}$.

76 $2xy + x^2y' + 3y^2y' = 0$. At the point $(2,1)$ we have $4 + 4y' + 3y' = 0$, so $y' = -\dfrac{4}{7}$. The tangent line is given by $y - 1 = -\dfrac{4}{7}(x-2)$, or $y = -\dfrac{4}{7}x + \dfrac{15}{7}$.

77 $\dfrac{d}{dx}[x^2 f(x)] = 2xf(x) + x^2 f'(x)$.

78 $\dfrac{d}{dx}\sqrt{\dfrac{f(x)}{g(x)}} = \dfrac{1}{2\sqrt{\frac{f(x)}{g(x)}}} \cdot \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

79 $\dfrac{d}{dx}\left(\dfrac{xf(x)}{g(x)}\right) = \dfrac{(f(x) + xf'(x))g(x) - xf(x)g'(x)}{g(x)^2}$.

80 $\dfrac{d}{dx}f\left(\sqrt{g(x)}\right) = f'\left(\sqrt{g(x)}\right) \cdot \dfrac{1}{2\sqrt{g(x)}} \cdot g'(x)$.

81

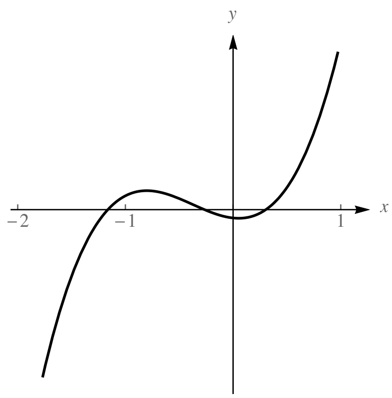
a. This has (D) as its derivative. Note that it consists of two pieces each of which are linear with the same slope. So its derivative is constant – but at $x = 2$ the derivative doesn't exist. We can easily know that this is true because the function isn't continuous at $x = 2$, so it can't be differentiable there.

b. This has (C) as its derivative. The slope of the tangent line is positive for $x < 2$ and negative for $x > 2$ and doesn't exist at $x = 2$. Also, near $x = 2$ the slope is near zero.

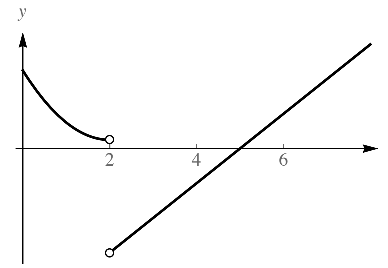
c. This has (B) as its derivative. Note that the slope of the tangent line is always positive, and gets infinitely steep at $x = 2$.

d. This has (A) as its derivative. Note that the slope of the tangent line is positive for $x < 2$, negative for $x > 2$, and is infinitely steep at $x = 2$ where the cusp occurs.

82



83



84

a. $5f'(1) + 3g'(1) = 5(3) + 3\left(\dfrac{1}{2}\right) = \dfrac{33}{2}$.

b. $f'(1)g(1) + f(1)g'(1) = 3(1) + 2\left(\frac{1}{2}\right) = 4$.

c. $\dfrac{g(3)f'(3) - f(3)g'(3)}{g(3)^2} = \dfrac{2(-2) - 3\left(\frac{1}{2}\right)}{2^2} = -\dfrac{11}{8}$.

d. $f'(f(4))f'(4) = f'(1)f'(4) = 3(-2) = -6$.

e. $g'(f(1))f'(1) = g'(2)f'(1) = \dfrac{1}{2}(3) = \dfrac{3}{2}$.

85

a. $\dfrac{d}{dx}\left[f(x) + 2g(x)\right]_{x=3} = f'(3) + 2g'(3) = 9 + 2 \cdot 9 = 27$.

b. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]_{x=1} = \dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{9(7) - 3(5)}{9^2} = \dfrac{48}{81} = \dfrac{16}{27}$.

c. $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=3} = f'(3)g(3) + f(3)g'(3) = 9(7) + 1(9) = 72$.

d. $\dfrac{d}{dx}\,(f(x))^3\Big|_{x=5} = 3f(5)^2f'(5) = 3(9)^2 \cdot 5 = 1215$.

e. $(g^{-1})'(7) = \dfrac{1}{g'(3)} = \dfrac{1}{9}$.

86 $\left(f^{-1}(x)\right)'\Big|_{x=f(0)} = \dfrac{1}{f'(0)} = -\dfrac{1}{(0+1)^2} = -1$.

87 Note that for $x = 2$, we have $y = \sqrt{8+2-1} = 3$. $\left(f^{-1}(x)\right)'\Big|_{x=f(2)} = \dfrac{1}{f'(2)} = \dfrac{1}{\frac{3(2^2)+1}{2\sqrt{2^3+2-1}}} = \dfrac{6}{13}$.

88 $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{12}$.

89 If $f(x) = x^{-1/3}$, then $f^{-1}(x) = x^{-3}$. So $\left(f^{-1}\right)'(x) = -3x^{-4}$ for $x \neq 0$.

90

a. $\frac{d}{dx}\left(xf(x)\right)\big|_{x=2} = \left(f(x)+xf'(x)\right)\big|_{x=2} = f(2)+2f'(2) = 5+2\cdot 3 = 11.$

b. $\frac{d}{dx}\left(f(x^2)\right)\Big|_{x=1} = \left(2xf'(x^2)\right)\Big|_{x=1} = 2f'(1) = 2.$

c. $\frac{d}{dx}\left(f(f(x))\right)\big|_{x=1} = f'(f(1))\cdot f'(1) = f'(3)\cdot 1 = 4.$

91

a. Because $f^{-1}(7) = 3$, we have $\left(f^{-1}\right)'(7) = \frac{1}{f'(3)} = \frac{1}{4}$.

b. Because $f^{-1}(3) = 1$, we have $\left(f^{-1}\right)'(3) = \frac{1}{f'(1)} = 1$.

c. $\left(f^{-1}\right)'(f(2)) = \frac{1}{f'(2)} = \frac{1}{3}$.

92 The value of $f(3)$ is the same as the value of the tangent line to f at $x = 3$, so it is $4(3) - 10 = 2$. The value of $f'(3)$ is the slope of the tangent line to $y = f(x)$ at $x = 3$, so it is 4. The value of $g(5)$ is the same as the value of the tangent line to g at $x = 5$, so it is $30 - 27 = 3$. The value of $g'(5)$ is the slope of the tangent line to $y = g(x)$ at $x = 5$, so it is 6.

93 $\frac{d}{dx}(f(g(x)))\Bigg|_{x=5} = f'(g(5))g'(5) = f'(3)g'(5) = 4(6) = 24$. Also, $f(g(5)) = f(3) = 2$. So the tangent line is given by $y - 2 = 24(x - 5)$, or $y = 24x - 118$.

94

a. $\frac{ds}{dt} = v(t) = 27 - 3t^2$. This is 0 for $t > 0$ only for $t = 3$. On the interval $(0, 3)$ $v(t) > 0$ and on $(3, \infty)$ $v(t) < 0$. So the object is stationary at $t = 3$, is moving to the right for $0 < t < 3$, and is moving to the left for $t > 3$.

b. $v(2) = 27 - 12 = 15$ ft/s. $a(t) = \frac{dv}{dt} = -6t$, so $a(2) = -12\,\text{ft/s}^2$.

c. $a(3) = -6(3) = -18\,\text{ft/s}^2$.

d. Note that for $t > 0$ the acceleration is always negative. So the speed is decreasing when the object is moving to the right, which occurs for $0 < t < 3$.

95

a. $\frac{ds}{dt} = v(t) = 96 - 12t$, so $v(1) = 84$ ft/s.

b. $v(t) = 12$ when $96 - 12t = 12$, or $12t = 84$, so $t = 7$ s.

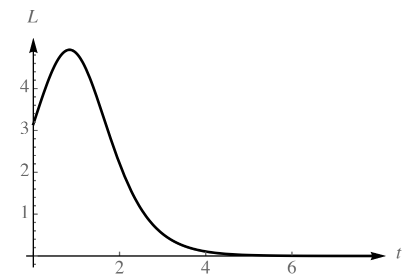
c. $v(t) = 0$ when $96 - 12t = 0$, or $t = 8$ s. The height at $t = 8$ is $s(8) = 96(8) - 6(64) = 384$ ft.

d. $s(t) = 0$ when $96t - 6t^2 = 0$, or $t(16 - t) = 0$, so $t = 0$ and $t = 16$. The velocity at time $t = 16$ is $v(16) = -96$, so it strikes the ground with a speed of 96 ft/s.

96

a. We can write $L(t) = 11.94(1 + 4e^{-1.65t})^{-1}$, so $L'(t) = -11.94(1 + 4e^{-1.65t})^{-2}((-1.65)4e^{-1.65t}) = \frac{78.804e^{-1.65t}}{(1 + 4e^{-1.65t})^2}$. Then $L'(1) \approx 4.8$. The culmen of a one-week-old Indian Spotted Owlet is growing at almost 5 mm/week.

b. The culmen grows quickly at first, reaching a maximum growth rate just before the first week of life, and then the growth rate gradually decreases over the next 4 weeks. By the fifth week, the growth rate is approximately 0 which means it has reached its adult size.



97

a. $A(20) = 40,000(1.0075^{240} - 1) \approx \$200,366$.
$A'(t) = 40,000(1.0075^{12t})\ln(1.0075) \cdot 12 \approx 3586.57(1.0075^{12t})$. $A'(20) \approx \$21,552$ dollars/year.

b. $A(t) = 100,000$ when $40,000(1.0075^{12t} - 1) = 100,000$, which occurs when

$$1.0075^{12t} - 1 = \frac{100,000}{40,000} = 2.5,$$

so when

$$1.0075^{12t} = 3.5.$$

This occurs when $12t\ln(1.0075) = \ln 3.5$, or $t = \dfrac{\ln 3.5}{12\ln(1.0075)} \approx 13.97 \approx 14$ years. At that time we have $A'(13.97) \approx \$12,551$ per year.

98

a. The instantaneous rate of change is $Q'(t) = -1.386e^{-0.0693t}$ mg/hr.

b. At $t = 0$ hours, we have $Q'(0) = -1.386$, so the amount of antibiotic is decreasing at a rate of 1.386 mg/hr. At $t = 2$ hours, we have $Q'(2) = -1.386e^{-0.1386} \approx -1.207$, so the amount of antibiotic is decreasing at a rate of about 1.207 mg/hr.

c. $\lim_{t\to\infty} Q(t) = 20 \lim_{t\to\infty} e^{-0.0693t} = 0$. In the long run, the antibiotic is all used up. $\lim_{t\to\infty} Q'(t) = -1.386 \lim_{t\to\infty} e^{-0.0693t} = 0$. The rate of change of the amount of antibiotic in the bloodstream also goes to zero as $t \to \infty$.

99

a. Average growth is $\dfrac{p(60) - p(50)}{10} = 2.7$ million people per year.

b. The curve is pretty straight between $t = 50$ and $t = 60$, so the secant line between these two points is approximately as steep as the tangent line at a point in between.

c. A reasonable estimate to the instantaneous grow rate at 1985 would be the slope of the secant line between $t = 80$ and $t = 90$. This is $\dfrac{p(90) - p(80)}{10} = 2.217$ million people per year.

100

a. The graph has the steepest slope at about $t = 18$. At this point the rate is about $\dfrac{N(20) - N(16)}{4} = \dfrac{3500 - 1900}{4} = 400$ bacteria per hour.

b. It is smallest at $t = 0$ or $t = 36$, where it is about $\dfrac{N(36) - N(32)}{4} \approx \dfrac{4900 - 4800}{4} = 25$ bacteria per hour.

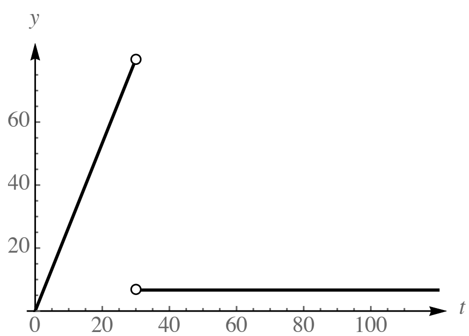
c. The average growth rate over $[0, 36]$ is $\dfrac{N(36) - N(0)}{36} \approx \dfrac{4900 - 400}{36} = \dfrac{4500}{36} = 125$ bacteria per hour.

101

a. $v(15) \approx \frac{400-200}{5} = 40$ meters per second.

b. Because the graph is a straight line for $t \geq 30$, $v(70) = \frac{D(90)-D(60)}{30} = \frac{1600-1400}{30} = \frac{20}{3}$ meters per second. The points at 60 and 90 were chosen because it is easier to detect the function values at those points using the given grid.

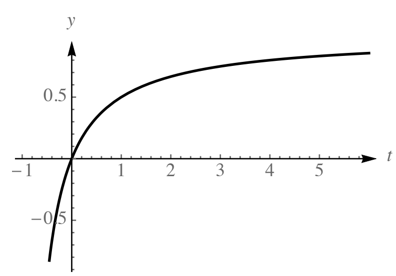
c. The average velocity is $\dfrac{D(90) - D(20)}{70} \approx \dfrac{1600 - 550}{70} = 15$ meters per second.

d. 

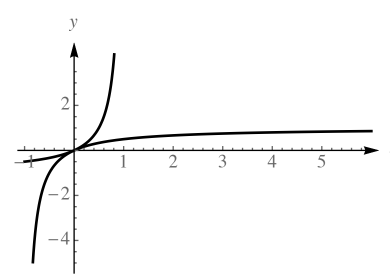
e. The parachute was deployed.

102

a.



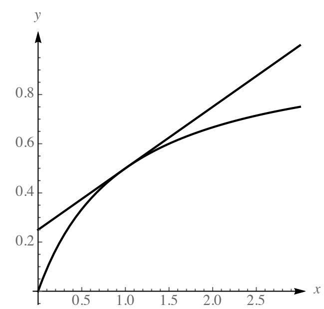
b.



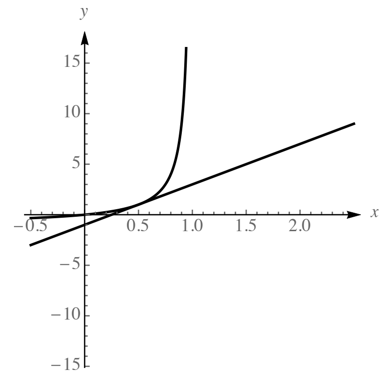
If $y = \dfrac{x}{x+1}$, then $yx + y = x$, so $y = x - yx$, and $y = x(1 - y)$, so $x = \dfrac{y}{1 - y}$. The inverse function is given by $f^{-1}(x) = \dfrac{x}{1 - x}$.

c. $\left(f^{-1}(x)\right)' = \dfrac{1 - x + x}{(1 - x)^2} = \dfrac{1}{(1 - x)^2}$. So $\left(f^{-1}\right)\left(\frac{1}{2}\right) = 4$.

d1.



d2.



103 We are looking for values of x so that $y'(x) = 0$. We have $y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$, and this quantity is zero when $2(6-x) - x = 0$, or $12 - 3x = 0$, so when $x = 4$. So at the point $(4, 4\sqrt{2})$ there is a horizontal tangent line. There is a vertical tangent line at $x = 6$, because $\lim_{x\to 6^-} y'(x) = -\infty$.

104 With $a = \dfrac{\pi}{4}$, $f(x) = \sin^2(x)$ we have

$$f'\left(\frac{\pi}{4}\right) = \lim_{h\to 0} \frac{f(\pi/4+h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin^2(\pi/4+h) - (1/2)}{h}$$
$$= 2\sin(\pi/4)\cos(\pi/4) = 2(\sqrt{2}/2)(\sqrt{2}/2) = 1.$$

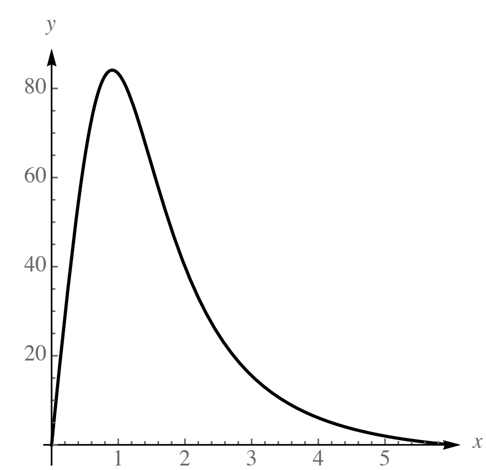
Note that we used the fact that $\dfrac{d}{dx}\sin^2(x) = 2\sin x \cos x$ in the middle of this derivation.

105 Let $a = 5$ and $f(x) = \tan(\pi\sqrt{3x-11})$. Note that $f'(5) = \dfrac{3\pi}{2}\dfrac{\sec^2(2\pi)}{2} = \dfrac{3\pi}{4}$.

So $\displaystyle\lim_{x\to 5} \frac{f(x) - f(5)}{x-5} = \lim_{x\to 5} \frac{\tan(\pi\sqrt{3x-11}) - 0}{x-5} = f'(5) = \frac{3\pi}{4}$.

106

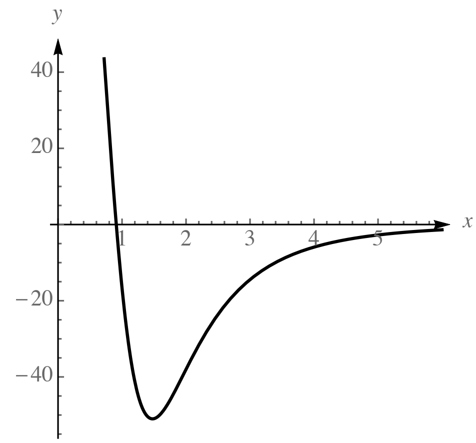
a. The probe climbs quickly, and achieves a maximum height of about 84.1 at about $t = 0.91$.



b.

$$v(t) = s'(t) = \frac{(t^3+2)(300-100t)-(300t-50t^2)(3t^2)}{(t^3+2)^2}$$
$$= \frac{300t^3-100t^4+600-200t-900t^3+150t^4}{(t^3+2)^2}$$
$$= \frac{50t^4-600t^3-200t+600}{(t^3+2)^2}.$$

c.



The maximum velocity is attained at $t = 0$.

107

a. The average cost is $\frac{C(3000)}{3000} = \frac{1025000}{3000} \approx \$341.67$. The marginal cost is $C'(3000) = -0.04(3000) + 400 = \$280$.

b. The average cost of producing 3000 lawnmowers is \$341.67 per mower. The cost of producing the 3001st lawnmower is approximately \$280.

108

a. The marginal cost is given by $C'(x) = -0.0003x^2 + 0.1x + 60$, so $C'(400) = \$52$. The average cost of producing 400 fly rods is $\frac{C(400)}{400} = \$66$.

b. The average cost of producing 400 fly rods is \$66 per fly rod. The cost of producing the 401st fly rod is approximately \$52.

109

a. The average growth rate is $\frac{p(50)-p(0)}{50} = \frac{407500-80000}{50} = 6550$ people per year.

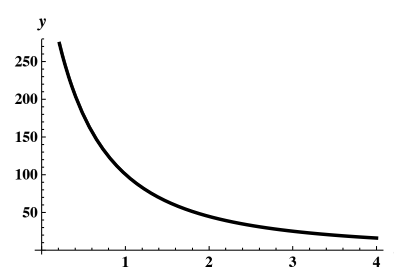
b. The growth rate in 1990 is $p'(40) = -5.1(40^2) + 144 \cdot 40 + 7200 = 4800$ people per year.

110

a. $v(t) = \pi \cdot 4^2 \cdot \frac{8t}{t+1} = \frac{128\pi t}{t+1}$ cubic cm.

b. $v'(t) = 128\pi \cdot \frac{(t+1)-t}{(t+1)2} = \frac{128\pi}{(t+1)^2}$.

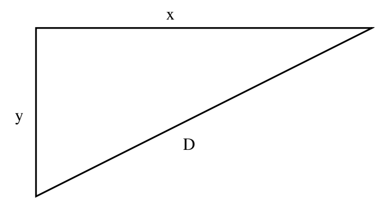
c.



Because the rate of change of volume is strictly positive, the volume function must be increasing for $t > 0$.

111

Let x be the distance the eastbound boat has traveled, and y the distance the southbound boat has traveled. By the Pythagorean Theorem, $D^2 = x^2 + y^2$, so $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $\frac{dD}{dt} = \frac{x \cdot x' + y \cdot y'}{D}$. We are given that $x' = 40$, $y' = 30$, and at $t = .5$ hours, we have $x = 20$, $y = 15$, and $D = 25$. Thus, $\frac{dD}{dt} = \frac{20 \cdot 40 + 30 \cdot 15}{25} = 50$ mph.



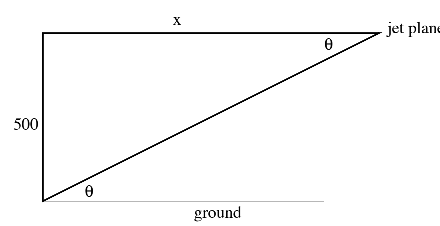
112 $V = \frac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$, so $V' = \frac{\pi d^2 d'}{2}$. With $V' = 10\,\text{cm}^3/\text{min}$ and $d = 5$ cm, we have $d' = \frac{20}{25\pi} = \frac{4}{5\pi}$ cm/min.

113 Let h be the elevation of the balloon, and s the length of the rope. We have $h = s\sin(65°)$, so $h' = s'\sin(65°) = -5 \cdot \sin(65°) \approx -4.53$ feet per second.

114 $\frac{r}{h} = \frac{2}{3}$, so $r = \frac{2}{3}h$. $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. So $\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$. When $h = 2$, $\frac{dV}{dt} = 2$, so $\frac{dh}{dt} = \frac{2}{\frac{4}{9}\pi \cdot 4} = \frac{9}{8\pi}$ feet per minute.

115

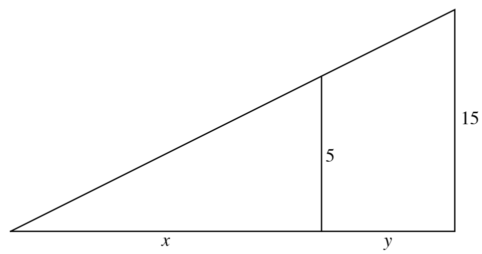
Let x be the distance the jet has flown since it went over the spectator. Let θ be the angle of elevation between the ground and the line from the spectator to the jet. Note that θ is also the angle pictured, and that $\cot\theta = \frac{x}{500}$. Thus, $\theta = \cot^{-1}\left(\frac{x}{500}\right)$. We are given that $x' = 450\,\text{mph} = 660\,\text{ft/sec}$.
$\theta' = -\frac{x'}{500 \cdot \left(1 + \left(\frac{x}{500}\right)^2\right)} = -\frac{500x'}{250,000 + x^2}$. After 2 seconds, $x = 1320$ feet, so at this time $\theta' = -\frac{500 \cdot 660}{250,000 + (1320)^2} \approx -0.166$ radians per second.



116 Let D be the distance the man is from the billboard, and let α be the angle between his eye level and the line of sight to the bottom of the billboard, and let θ be the angle between his line of sight to the bottom of the billboard and his line of sight to the top of the billboard. We have that $\cot\alpha = \frac{D}{4}$, so $\alpha = \cot^{-1}\left(\frac{D}{4}\right)$. Also, $\cot(\alpha + \theta) = \frac{D}{19}$, so $\theta = \cot^{-1}\left(\frac{D}{19}\right) - \alpha = \cot^{-1}\left(\frac{D}{19}\right) - \cot^{-1}\left(\frac{D}{4}\right)$.

So $\theta' = -\frac{19D'}{361 + D^2} + \frac{4D'}{16 + D^2}$. We are given that $D' = -2$ feet per second, so at $D = 30$ we have $\theta' \approx 0.03 - 0.009 = 0.021$ radians per second.

117 Let x equal the length of her shadow and let y equal her distance from the pole. Using similar triangles, we have $\frac{x}{5} = \frac{x+y}{15}$, which implies that $3x = x + y$ or $x = \frac{y}{2}$. It follows that the rate at which her shadow is increasing is $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}(3) = 1.5$ ft/s.



118

a. $f'(a) = \lim_{x\to a} \frac{x^2 - a^2}{x - a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x\to a}(x+a) = 2a.$

b. $f'(a) = \lim_{h\to 0} \frac{b(a+h)^2 + c(a+h) + d - ba^2 - ca - d}{h} = \lim_{h\to 0} \frac{2bah + bh^2 + ch}{h} = \lim_{h\to 0}(2ba + bh + c) = 2ab + c.$

119

a. $\left(f^{-1}\right)'\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}.$

b. $\frac{d}{dx}\sin^{-1}(x)\Big|_{x=1/\sqrt{2}} = \frac{1}{\sqrt{1-(1/2)}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}.$

120

a. Note that $f'(x) = 2x$, so $f'\left(\frac{x+y}{2}\right) = 2\cdot\frac{x+y}{2} = x+y$. The quantity $\frac{f(x)-f(y)}{x-y}$ can be written as $\frac{x^2-y^2}{x-y} = \frac{(x-y)(x+y)}{x-y} = x+y$, so these quantities are equal for $x \neq y$.

b. Yes. Note that $f'(x) = 2ax$, so $f'\left(\frac{x+y}{2}\right) = 2a\cdot\frac{x+y}{2} = a(x+y)$. The quantity $\frac{f(x)-f(y)}{x-y}$ can be written as $\frac{ax^2-ay^2}{x-y} = a\cdot\frac{(x-y)(x+y)}{x-y} = a(x+y)$, so these quantities are equal for $x \neq y$.

c. The line through $(x, f(x))$ and $(y, f(y))$ is parallel to the tangent line at the midpoint between x and y.

d. No. For example, consider $a = 1$, $x = 0$, and $y = 1$. Note that $f'(x) = 3x^2$. Then $f'\left(\frac{x+y}{2}\right) = f'(1/2) = 3/4$. On the other hand, $\frac{f(x)-f(y)}{x-y} = \frac{1-0}{1-0} = 1$.

Chapter 4

Applications of the Derivative

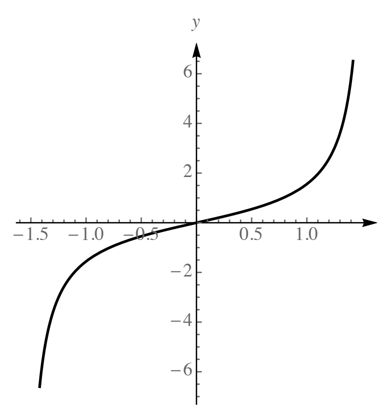
4.1 Maxima and Minima

4.1.1 A number $M = f(c)$ where $c \in [a, b]$ with the property that $f(x) \leq M$ for all $x \in [a, b]$ is an absolute maximum for f on $[a, b]$, and a number $m = f(d)$ where $d \in [a, b]$ with the property that $f(x) \geq m$ for all $x \in [a, b]$ is an absolute minimum for f on $[a, b]$.

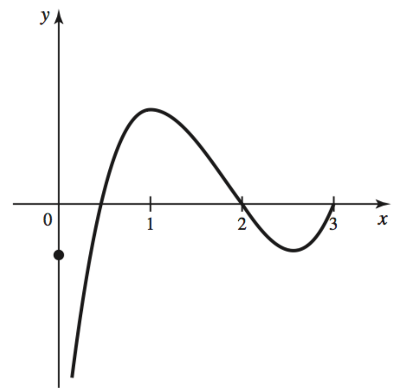
4.1.2 A number $M = f(c)$ is a local maximum for f if there is an interval (r, s) containing c so that $f(x) \leq M$ for all $x \in (r, s)$. A number $m = f(d)$ is a local minimum for f if there is an interval (r, s) containing d so that $f(x) \geq m$ for all $x \in (r, s)$.

4.1.3 The function must be a continuous function defined on a closed interval.

4.1.4 The tangent function on the interval $(-\pi/2, \pi/2)$ is continuous but has no maximum or minimum.

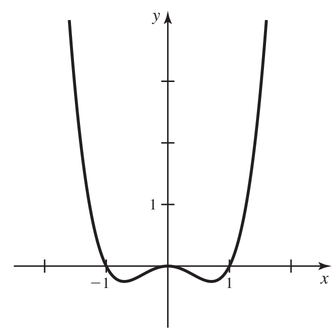


4.1.5 The function shown has no absolute minimum on $[0,3]$ because $\lim_{x\to 0^+} f(x) = -\infty$. It has an absolute maximum near $x = 1$ and a local minimum near $x = 2.5$.

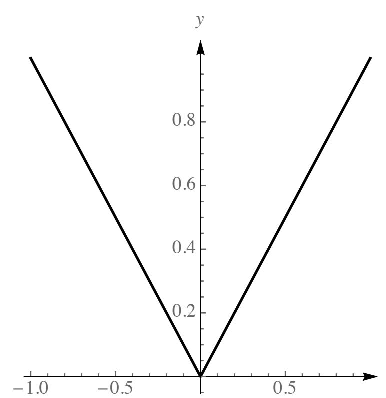


4.1.6 An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ doesn't exist is a critical point of f.

4.1.7 Note the existence of a horizontal tangent line at $x = 0$ where the maximum occurs.



4.1.8 Note the minimum at $x = 0$ where $f'(x)$ does not exist. Note that $\lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ is different for $h \to 0^-$ and $h \to 0^+$.



4.1.9 First find all the critical points by seeking all points x in the domain of f so that $f'(x) = 0$ or $f'(x)$ doesn't exist. Now compare the y-values of all of these points, together with the y-values of the endpoints. The largest y-value from among these is the maximum, and the smallest is the minimum.

4.1.10 If a is an endpoint of the given interval, and $f(a) \le f(x)$ for all x in the interval, then $f(a)$ is the absolute minimum. This happens, for example, for a line of positive slope defined on an interval $[a, b]$ – the y-value at the left endpoint is the smallest y-value over the interval.

4.1.11 $y = h(x)$ has an absolute maximum at $x = b$ and an absolute minimum at $x = c_2$.

4.1.12 $y = f(x)$ has an absolute maximum at $x = c$ and no absolute minimum.

4.1.13 $y = g(x)$ has no absolute maximum, but has an absolute minimum at $x = a$.

4.1.14 $y = g(x)$ has an absolute maximum at $x = a$ and an absolute minimum at $x = c$.

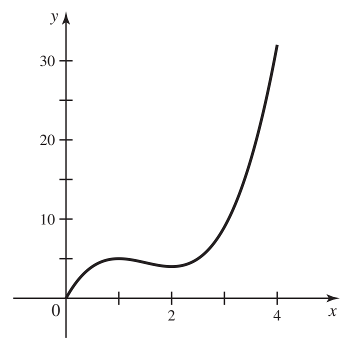
4.1.15 $y = f(x)$ has an absolute maximum at $x = b$ and an absolute minimum at $x = a$. It has local maxima at $x = p$ and $x = r$, and local minima at $x = q$ and $x = s$.

4.1.16 $y = f(x)$ has an absolute maximum at $x = p$, and an absolute minimum at $x = a$. It has local minima at $x = q$ and $x = s$, and local maxima at $x = r$ and $x = p$.

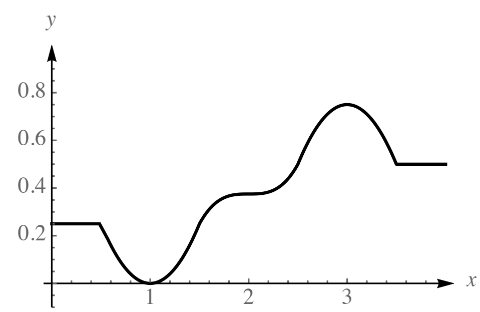
4.1.17 $y = g(x)$ has an absolute minimum at $x = b$ and an absolute maximum at $x = p$. It has local maxima at $x = p$ and $x = r$. It has a local minimum at $x = q$.

4.1.18 $y = h(x)$ has an absolute maximum at $x = p$ and an absolute minimum at $x = u$. It has local maxima at $x = p$, $x = r$ and $x = t$. It has local minima at $x = q$, $x = s$, and $x = u$.

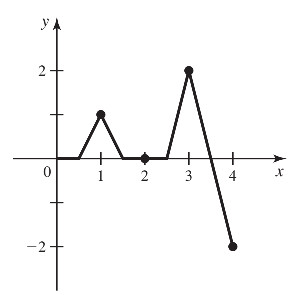
4.1.19 Note the horizontal tangent lines at 1 and 2, and the minimum at 0 and the maximum at 4.



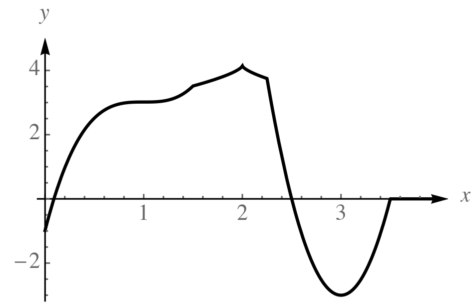
4.1.20 Note the minimum at $x = 1$, the maximum at $x = 3$, and the horizontal tangent lines at 1, 2, and 3.



4.1.21 Note the horizontal tangent line at $x = 2$, and the "corners" at $x = 1$ and $x = 3$. Also note the absolute maximum at $x = 3$ and the absolute minimum at $x = 4$.



4.1.22 Note the maximum at 2, and the minimum at 3. Note also the horizontal tangent lines at $x = 1$ and $x = 3$, and the sharp "corner" at $x = 2$.



4.1.23 $f'(x) = 6x - 4$, which is zero when $x = \frac{2}{3}$. So that is the only critical point.

4.1.24 $f'(x) = \frac{3}{8}x^2 - \frac{1}{2}$, which is zero when $3x^2 - 4 = 0$, which occurs for $x = \pm\frac{2}{\sqrt{3}}$.

4.1.25 $f'(x) = x^2 - 9$, which is zero for $x = \pm 3$. So the critical points are $x = \pm 3$.

4.1.26 $f'(x) = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x-3)(x+2)$, which is zero for $x = 0, 3$, and -2. So the critical points are $x = 0, 3$, and -2.

4.1.27 $f'(x) = 9x^2 + 3x - 2 = (3x+2)(3x-1)$, which is zero for $x = -\frac{2}{3}$ and $x = \frac{1}{3}$. So the critical points are $x = -\frac{2}{3}$ and $x = \frac{1}{3}$.

4.1.28 $f'(x) = 4x^4 - 9x^2 = x^2(4x^2 - 9) = x^2(2x+3)(2x-3)$, which is zero for $x = 0$ and $x = \pm\frac{3}{2}$. So the critical points are $x = \pm\frac{3}{2}$.

4.1.29 $f'(x) = 3x^2 - 4a^2$, which is zero for $3x^2 = 4a^2$, or $x^2 = \frac{4a^2}{3}$. So the critical points are $x = \pm\frac{2a}{\sqrt{3}}$.

4.1.30 $f'(x) = 1 - \frac{5}{1+x^2}$, which is zero for $1 + x^2 = 5$, so $x = \pm 2$. So those are the critical points.

4.1.31 $f'(t) = \frac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$. This quantity is zero exactly when $1 - t^2 = 0$, so $t = 1$ and $t = -1$ are the critical points.

4.1.32 $f'(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1)$, which is zero on the given interval when $x = \pm 1$ and when $x = 0$, so those are the critical points.

4.1.33 $f'(x) = \frac{e^x - e^{-x}}{2}$, which is zero when $e^x = e^{-x}$ or $x = -x$, so only for $x = 0$.

4.1.34 $f'(x) = \cos x \cos x - \sin x \sin x$, which is zero when $\sin^2 x = \cos^2 x$, so when $\sin x = \cos x$ or $\sin x = -\cos x$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$ where k is an integer.

4.1.35 $f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}$. There is a critical point at $x = 1$.

4.1.36 $f'(t) = 2t - \frac{4t}{t^2+1} = \frac{2t^3 - 2t}{t^2+1} = \frac{2t(t+1)(t-1)}{t^2+1}$. This is zero for $t = 0, \pm 1$.

4.1.37 $f'(x) = 2x\sqrt{x+5} + x^2 \cdot \dfrac{1}{2\sqrt{x+5}} = \dfrac{4x(x+5)}{2\sqrt{x+5}} + \dfrac{x^2}{2\sqrt{x+5}} = \dfrac{5x^2+20x}{2\sqrt{x+5}}$. This is zero when $5x^2+20x = 5x(x+4)$ is zero, which occurs for $x = 0$ and $x = -4$. The critical points are $x = 0$ and $x = -4$.

4.1.38 $f'(x) = \dfrac{1}{\sqrt{1-x^2}}\cos^{-1}(x) + \sin^{-1}(x) \cdot \left(-\dfrac{1}{\sqrt{1-x^2}}\right) = \dfrac{\cos^{-1}(x) - \sin^{-1}(x)}{\sqrt{1-x^2}}$. This is zero when $\sin^{-1}(x) = \cos^{-1}(x)$, which occurs for $x = \dfrac{\sqrt{2}}{2}$.

4.1.39 $f'(x) = \sqrt{x-a} + \dfrac{x}{2\sqrt{x-a}} = \dfrac{2x-2a+x}{2\sqrt{x-a}} = \dfrac{3x-2a}{2\sqrt{x-a}}$. This expression is zero when $x = \dfrac{2a}{3}$; however, that number is not in the domain of f if $a > 0$. However, if $a < 0$, then $\dfrac{2a}{3}$ is in the domain, and thus gives a critical point.

4.1.40 $f'(x) = \dfrac{\sqrt{x-a} - \frac{x}{2\sqrt{x-a}}}{x-a} = \dfrac{\sqrt{x-a} - \frac{x}{2\sqrt{x-a}}}{x-a} \cdot \dfrac{2\sqrt{x-a}}{2\sqrt{x-a}} = \dfrac{2x-2a-x}{2(x-a)^{3/2}} = \dfrac{x-2a}{2(x-a)^{3/2}}$. This is zero when $x = 2a$, so there is a critical point at $x = 2a$ for $a > 0$.

4.1.41 $f'(t) = t^4 - a^4$, which is zero when $t^4 = a^4$, or $|t| = a$. So there are critical points at $t = a$ and at $t = -a$.

4.1.42 $f'(x) = 3x^2 - 6ax + 3a^2 = 3(x-a)^2$. So the point $x = a$ is a critical point.

4.1.43 $f'(x) = 2x$, which is zero for $x = 0$. We have that $f(-2) = -6$, $f(0) = -10$, and $f(3) = -1$, so the maximum value of f on this interval is -1 and the minimum is -10.

4.1.44 $f'(x) = \dfrac{4}{3}(x+1)^{1/3}$, which is zero for $x = -1$. So $(-1, 0)$ is the only critical point. Checking endpoints and critical points we have that $f(-9) = (-8)^{4/3} = 16$, $f(-1) = 0$, and $f(7) = 8^{4/3} = 16$, so the maximum value of f on this interval is 16 and the minimum is 0.

4.1.45 $f'(x) = 3x^2 - 6x = 3x(x-2)$. This is zero for $x = 0$ and $x = 2$. Checking endpoints and critical points we have $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, and $f(3) = 0$. So there is an absolute maximum of 0 at $x = 0$ and $x = 3$, and an absolute minimum of -4 at $x = -1$ and $x = 2$.

4.1.46 $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$, which is zero for $x = 0$, $x = 1$, and $x = 2$. Checking endpoints and critical points we have $f(-1) = 9$, $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, and $f(3) = 9$. There is an absolute maximum of 9 at $x = 3$ and $x = -1$, and an absolute minimum of 0 at $x = 0$ and $x = 2$.

4.1.47 $f'(x) = 15x^4 - 75x^2 + 60 = 15(x^4 - 5x^2 + 4) = 15(x^2-4)(x^2-1) = 15(x-2)(x+2)(x-1)(x+1)$. So the critical points are $x = 1, -1, 2, -2$. Checking the endpoints and critical points we have $f(-2) = -16$, $f(-1) = -38$, $f(0) = 0$, $f(1) = 38$, $f(2) = 16$, and $f(3) = 234$. There is an absolute minimum of -38 at $x = -1$, and an absolute maximum of 234 at $x = 3$.

4.1.48 $f'(x) = 2e - 2x$, which is zero for $x = e$. Checking endpoints and critical points we have $f(0) = 0$, $f(e) = e^2$, and $f(2e) = 0$, so there is an absolute maximum of e^2 at $x = e$ and an absolute minimum of 0 at $x = 0$ and $x = 2e$.

4.1.49 $f'(x) = -2\cos x \sin x$, which is zero for $x = 0$, $x = \pi/2$, and $x = \pi$. Because there are endpoints at $x = 0$ and $x = \pi$, only $(\pi/2, 0)$ is a critical point. We have that $f(0) = 1$, $f(\pi/2) = 0$, and $f(\pi) = 1$, so the maximum value of f on this interval is 1 and the minimum is 0.

4.1.50 $f'(x) = \dfrac{(x^2+3)^2 - x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4} = \dfrac{(x^2+3) - 4x^2}{(x^2+3)^3} = \dfrac{3-3x^2}{(x^2+3)^3}$, which is zero for $x = \pm 1$. We have that $f(-2) = -\dfrac{2}{49} \approx -0.041$, $f(2) = \dfrac{2}{49} \approx 0.041$, $f(\pm 1) = \pm\dfrac{1}{16} = \pm 0.0625$. The absolute maximum of f on the given interval is 0.0625 at $x = 1$ and the absolute minimum is -0.0625 at $x = -1$.

4.1.51 $f'(x) = 3\cos 3x$, which is zero when

$$3x = \ldots, -\pi/2,\, \pi/2,\, 3\pi/2, \ldots,$$

so when $x = \ldots, -\pi/6, \pi/6, \pi/2, \ldots$. The only such values on the given interval are $x = -\pi/6$ and $x = \pi/6$. We have

$$f(-\pi/4) = -\sqrt{2}/2 \approx -0.707,$$

$f(-\pi/6) = -1$, $f(\pi/6) = 1$, and $f(\pi/3) = 0$, so the absolute maximum of f is 1 at $x = \pi/6$ and the absolute minimum is -1 at $x = -\pi/6$.

4.1.52 $f'(x) = 3 \cdot \dfrac{2}{3}x^{-1/3} - 1 = \dfrac{2}{\sqrt[3]{x}} - 1 = \dfrac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$, which is zero for $x = 8$. Also, f' doesn't exist for $x = 0$ (even though 0 is in the domain of f). We have $f(8) = 4$, and $f(0) = 0$ and $f(27) = 0$. So the absolute maximum of f on this interval is 4 at $x = 8$ and the absolute minimum is 0 at $x = 0$ and $x = 27$.

4.1.53 Let $y = (2x)^x$, so that $\ln y = x\ln(2x)$. Then $\dfrac{1}{y}y' = \ln(2x) + \dfrac{x}{2x} \cdot 2 = 1 + \ln(2x)$. Thus $y' = (2x)^x(1 + \ln(2x))$. This quantity is zero when $1 + \ln(2x) = 0$, which occurs when $\ln(2x) = -1$, or $x = \dfrac{1}{2e} \approx 0.184$. We have $f(0.1) \approx 0.851$, $f\left(\dfrac{1}{2e}\right) = e^{-(1/2e)} \approx 0.832$, and $f(1) = 2$. So the absolute minimum is $e^{-(1/2e)}$ and the absolute maximum is 2.

4.1.54 $f'(x) = e^{1-x/2} + xe^{1-x/2} \cdot \left(-\dfrac{1}{2}\right) = e^{1-x/2}\left(\dfrac{2-x}{2}\right)$. Because the exponential function is never zero, this expression is zero only when $x = 2$. So $x = 2$ is the only critical point. We have $f(0) = 0$ and $f(2) = 2$, and $f(5) \approx 1.12$. So the absolute maximum of f on this interval is 2 and the absolute minimum is 0.

4.1.55 $f'(x) = 2x - \dfrac{1}{\sqrt{1-x^2}} = \dfrac{2x\sqrt{1-x^2} - 1}{\sqrt{1-x^2}}$. This is zero on $(-1, 1)$ when the numerator is zero, which is when $2x\sqrt{1-x^2} = 1$, so when $(4x^2)(1-x^2) = 1$, or $4x^4 - 4x^2 + 1 = 0$. This factors as $(2x^2-1)(2x^2-1) = 0$, so we have solutions for $x = \pm\sqrt{1/2}$. We have $f(-1) = 1 + \pi$, $f(-1/\sqrt{2}) = \dfrac{1}{2} + \dfrac{3\pi}{4}$, $f(1/\sqrt{2}) = \dfrac{1}{2} + \dfrac{\pi}{4}$, and $f(1) = 1$. So the maximum for f is $1 + \pi$ and the minimum is 1.

4.1.56 $f'(x) = \sqrt{2-x^2} + x\left(-\dfrac{2x}{2\sqrt{2-x^2}}\right) = \dfrac{2-2x^2}{\sqrt{2-x^2}}$. This is zero on $(-\sqrt{2}, \sqrt{2})$ when $2 - 2x^2 = 0$, which occurs for $x = \pm 1$. $f(-\sqrt{2}) = 0 = f(\sqrt{2})$, $f(-1) = -1$, and $f(1) = 1$, so the absolute maximum of f is 1 and the absolute minimum is -1.

4.1.57 $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-4)(x-1)$. This is zero at $x = 4$ and $x = 1$. $f(1) = 11$ and $f(4) = -16$. At the endpoints we have $f(0) = 0$ and $f(5) = -5$. The absolute maximum is 11 and the absolute minimum is -16.

4.1.58 $f'(x) = e^x - 2$, which is 0 for $e^x = 2$, or $x = \ln 2$. Checking endpoints and critical points, we have $f(0) = 1$, $f(\ln 2) = 2 - 2\ln 2 \approx 0.61$, and $f(2) = e^2 - 4 \approx 3.39$. So the absolute maximum is $e^2 - 4$ at $x = 2$ and the absolute minimum is $2 - 2\ln 2$ at $x = \ln 2$.

4.1.59 $f'(x) = 4x^2 + 10x - 6 = 2(2x^2 + 5x - 3) = 2(x+3)(2x-1)$. This is zero when $x = -3$ and when $x = 1/2$. $f(-3) = 27$ and $f(1/2) = -19/12$. At the endpoints we have $f(-4) = 56/3 \approx 18.7$, and $f(1) = 1/3$. The absolute maximum is 27 and the absolute minimum is $-19/12$.

4.1.60 $f'(x) = 12x^5 - 60x^3 + 48x = 12x(x^4 - 5x^2 + 4) = 12x(x^2-4)(x^2-1) = 12x(x+2)(x-2)(x+1)(x-1)$. This is zero for $x = 0$, $x = \pm 2$, and $x = \pm 1$. The critical points occur at 0 and ± 1, since the endpoints are ± 2. $f(\pm 2) = -16$, $f(\pm 1) = 11$, and $f(0) = 0$. The absolute maximum is 11 and the absolute minimum is -16.

4.1.61 $f'(x) = \dfrac{(x^2+9)^5 - x\cdot 5(x^2+9)^4(2x)}{(x^2+9)^{10}} = \dfrac{(x^2+9)^4(x^2+9-10x^2)}{(x^2+9)^{10}} = \dfrac{9-9x^2}{(x^2+9)^6} = \dfrac{9(1-x)(1+x)}{(x^2+9)^2}$. This expression is zero for $x=\pm 1$. Checking the endpoints and critical points gives $f(-2) = -\dfrac{2}{13^5}$, $f(-1) = -\dfrac{1}{10^5}$, $f(1) = \dfrac{1}{10^5}$ and $f(2) = \dfrac{2}{13^5}$. The absolute maximum is $\dfrac{1}{100000}$ at $x=1$ and the absolute minimum is $-\dfrac{1}{100000}$ at $x=-1$.

4.1.62 $f'(x) = \dfrac{1}{2\sqrt{x}}\cdot(x^2/5-4) + \sqrt{x}\left(\dfrac{2x}{5}\right) = \dfrac{x^2-20}{10\sqrt{x}} + \dfrac{4x^2}{10\sqrt{x}} = \dfrac{x^2-4}{2\sqrt{x}}$. On the given domain, this expression is zero only for $x=2$. $f(0)=0$, $f(2) = -\dfrac{16\sqrt{2}}{5} \approx -4.5255$, and $f(4) = -\dfrac{8}{5}$. So the absolute maximum is 0 and the absolute minimum is $-\dfrac{16\sqrt{2}}{5} \approx -4.5255$.

4.1.63 $f'(x) = \sec x \tan x$ which is zero when $\tan x = 0$ (since $\sec x$ is never zero.) So we are looking for where $\dfrac{\sin x}{\cos x} = 0$, which is when $\sin x = 0$, which is at $x=0$. $f(-\pi/4) = \sqrt{2} = f(\pi/4)$ and $f(0)=1$. So the absolute maximum for f is $\sqrt{2}$ and the absolute minimum is 1.

4.1.64 $f'(x) = \dfrac{1}{3}\cdot x^{-2/3}\cdot(x+4) + x^{1/3} = \dfrac{x+4}{3x^{2/3}} + \dfrac{3x}{3x^{2/3}} = \dfrac{4x+4}{3x^{2/3}}$. This expression is zero when $x=-1$, and is undefined when $x=0$ (although 0 is in the domain of f.) So $(0,0)$ and $(-1,-3)$ are the critical points. $f(-27)=69$, $f(-1)=-3$, $f(0)=0$, and $f(27)=93$. So the absolute maximum is 93 and the absolute minimum is -3.

4.1.65 $f'(x) = 3x^2e^{-x} + x^3\cdot(-e^{-x}) = e^{-x}\cdot\left(3x^2-x^3\right) = e^{-x}\cdot x^2\cdot(3-x)$. This expression is zero when $x=0$ and when $x=3$, so $(0,0)$ and $(3,(27/e^3))$ are the critical points. $f(-1)=-e$, $f(0)=0$, $f(3)=\dfrac{27}{e^3}\approx 1.344$, and $f(5)\approx 0.8422$. So the absolute maximum of f on the given interval is about 1.344, and the absolute minimum is $-e\approx -2.718$.

4.1.66 $f'(x) = \ln(x/5) + x\cdot\dfrac{5}{x}\cdot\dfrac{1}{5} = \ln(x/5)+1$. This expression is zero when $\ln(x/5) = -1$, or $x = 5e^{-1} = \dfrac{5}{e} \approx 1.8394$. $f(0.1)\approx -0.391$, $f\left(\dfrac{5}{e}\right) = -\dfrac{5}{e} \approx -1.8394$, and $f(5)=0$. So the absolute maximum of f is 0, and the absolute minimum is $-\dfrac{5}{e}\approx -1.8394$.

4.1.67 $f'(x) = x^{2/3}(-2x) + (4-x^2)\cdot\dfrac{2}{3\sqrt[3]{x}} = \dfrac{-6x^2+8-2x^2}{3\sqrt[3]{x}} = \dfrac{8-8x^2}{3\sqrt[3]{x}}$. This quantity is zero when $x=\pm 1$ and it doesn't exist at $x=0$. So there are critical points at $(0,0)$, $(-1,3)$, and $(1,3)$. Checking the endpoints, we have $f(\pm 2)=0$. So the absolute minimum is 0 at $x=\pm 2$ and $x=0$, and the absolute maximum is 3 at $x=\pm 1$.

4.1.68 $f'(t) = \dfrac{(t^2+1)\cdot 3 - 3t(2t)}{(t^2+1)^2} = \dfrac{3-3t^2}{(t^2+1)^2}$. This quantity is zero when $t=\pm 1$. So there are critical points at $t=\pm 1$. Checking endpoints and critical points we see that $f(-2) = -\dfrac{6}{5}$ and $f(2)=\dfrac{6}{5}$, $f(1)=\dfrac{3}{2}$ and $f(-1) = -\dfrac{3}{2}$. There is an absolute maximum of $\dfrac{3}{2}$ at $x=1$ and an absolute minimum of $-\dfrac{3}{2}$ at $x=-1$.

4.1.69

a. The wind either passes through the blade without slowing down or the turbine blades slow down the wind so that $0\le v_2\le v_1$, and because $v_1>0$, $0\le\dfrac{v_2}{v_1}\le 1$.

b. $R(1) = \frac{1}{2}(1+1)(1-1) = 0$. Given that $r = 1$, it is the case that $v_1 = v_2$ which means that the wind does not slow down after passing through the blades and therefore no power is captured by the turbine blades. It stands to reason that the blades should slow down the wind, at least a little bit, and therefore it seems unlikely or impossible for $r = 1$.

c. $R(0) = \frac{1}{2}$. A value of $r = 0$ means that $v_2 = 0$, so that the wind turbine stops the wind stream, which is unlikely or impossible to occur.

d. Finding the derivative of R and setting it equal to zero, we have

$$R'(r) = \frac{1}{2}((1-r^2) + (1+r)(-2r)) = -\frac{1}{2}(3r^2 + 2r - 1) = -\frac{1}{2}(3r-1)(r+1) = 0.$$

Given that $R(0) = \frac{1}{2}$, $R\left(\frac{1}{3}\right) = \frac{1}{2}\left(\frac{4}{3}\right)\left(\frac{8}{9}\right) = \frac{16}{27} \approx 0.593$, R is maximized on $[0,1]$ when $r = \frac{1}{3}$. At best, a wind turbine can extract about 59.3% of the available power from the wind stream.

4.1.70

a. Equating the two power equations, we have $v^2\rho A(v_1 - v_2) = \frac{1}{2}\rho v A(v_1^2 - v_2^2)$, or

$$v(v_1 - v_2) = \frac{1}{2}(v_1^2 - v_2^2) = \frac{1}{2}(v_1 - v_2)(v_1 + v_2).$$

Dividing through by $v_1 - v_2$ gives

$$v = \frac{1}{2}(v_1 + v_2).$$

b. By replacing v by $\frac{v_1 + v_2}{2}$ in the power equation $P = \frac{1}{2}\rho v A(v_1^2 - v_2^2)$, we have

$$P = \frac{1}{2}\rho\left(\frac{v_1 + v_2}{2}\right)A(v_1^2 - v_2^2) = \frac{\rho A}{4}(v_1 + v_2)(v_1^2 - v_2^2).$$

c.

$$\begin{aligned}
\frac{P}{P_0} &= \frac{\frac{\rho A}{4}(v_1 + v_2)(v_1^2 - v_2^2)}{\frac{\rho A v_1^3}{2}} \\
&= \frac{(v_1 + v_2)(v_1^2 - v_2^2)}{2v_1^3} = \frac{1}{2} \cdot \frac{v_1 + v_2}{v_1} \cdot \frac{v_1^2 - v_2^2}{v_1^2} \\
&= \frac{1}{2}\left(1 + \frac{v_2}{v_1}\right)\left(1 - \left(\frac{v_2}{v_1}\right)^2\right) = \frac{1}{2}(1+r)(1-r^2) = R(r).
\end{aligned}$$

4.1.71 $s'(t) = 32 - 4t^3$ which is 0 when $t^3 = 8$, so $t = 2$. Because $s(0) = 0$, $s(2) = 48$, and $s(3) = 15$, the object is furthest to the right at time $t = 2$.

4.1.72 $S'(x) = 4x - \frac{200}{x^2} = \frac{4x^3 - 200}{x^2}$, which is zero when $x^3 = 50$, so for $x = \sqrt[3]{50} \approx 3.684$. This critical point does indeed yield a minimum (which can be determined via a graphing calculator, or by techniques in an upcoming section). So the minimum surface area is given by $S(\sqrt[3]{50}) \approx 81.433$, when the box has dimensions $\sqrt[3]{50} \times \sqrt[3]{50} \times \sqrt[3]{50}$.

4.1.73 The stone will reach its maximum height when its velocity is zero, which occurs at the only critical point for this inverted parabola. We have that $v(t) = s'(t) = -32t + 64$, which is zero when $t = 2$. The height at this time is $s(2) = 256$, the maximum height.

4.1.74

a. $R'(x) = -120x + 300$, which is zero when $x = 2.5$. This is the only critical number.

b. The maximum must occur at either an endpoint or a critical point. Note that $R(0) = 0$, $R(2.5) = 375$, and $R(5) = 0$, so the maximum revenue is \$375, which occurs when the price is \$2.50.

4.1.75

a. Note that $P(n) = 50n - .5n^2 - 100$, so $P'(n) = 50 - n$, which is zero when $n = 50$. It is clear that this is a maximum, since the graph of P is an inverted parabola.

b. Given a domain of $[0, 45]$, since the only critical point is not in the domain, the maximum must occur at an endpoint. Because $P(0) = -100$, and $P(45) = \$1137.50$, he should take 45 people on the tour.

4.1.76 $P(x) = 2x + \dfrac{128}{x}$, $x > 0$, so $P'(x) = 2 - \frac{128}{x^2}$, which is zero when $x^2 = 64$, or when $x = 8$. So $(8, 32)$ is the only critical point. This does turn out to be a minimum, so the dimensions of the rectangle with minimal perimeter are 8×8.

4.1.77

a. False. The derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$ is never zero, and the function has no critical points.

b. False. For example, the function $f(x) = \begin{cases} \sin x & \text{if } -5 \le x \le 0, \\ -8 & \text{if } 0 < x \le 5 \end{cases}$ is not continuous on $[-5, 5]$, but has an absolute maximum of 1.

c. False. For example, the function $f(x) = (x-2)^3$ satisfies $f'(2) = 0$, but it has neither a maximum nor a minimum at $x = 2$.

d. True. This follows from the theorems in this section.

4.1.78

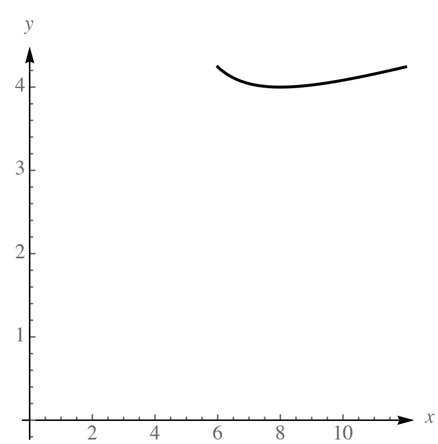
a.

$$f'(x) = \frac{\sqrt{x-4}\cdot 1 - x \cdot \frac{1}{2\sqrt{x-4}} \cdot 1}{x-4}$$
$$= \frac{\sqrt{x-4}\cdot 1 - x \cdot \frac{1}{2\sqrt{x-4}} \cdot 1}{x-4} \cdot \frac{2\sqrt{x-4}}{2\sqrt{x-4}}$$
$$= \frac{2x - 8 - x}{2(x-4)^{3/2}} = \frac{x-8}{2(x-4)^{3/2}}.$$

This expression is 0 when $x = 8$. So $(8, 4)$ is the only critical point.

b. $f(6) = 3\sqrt{2} = f(12)$, and $f(8) = 4$. Note that $3\sqrt{2} \approx 4.24 > 4$. So the absolute maximum of f on this interval is $3\sqrt{2}$, and the absolute minimum is 4.

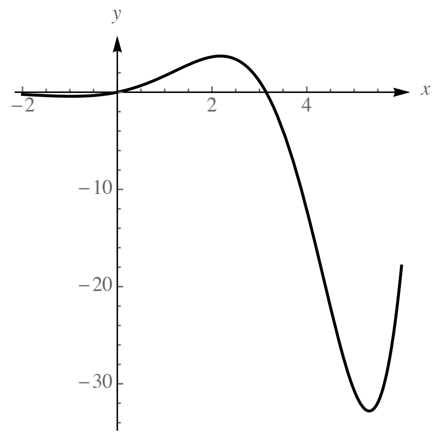
c.



4.1.79

a. $f'(x) = 2^x \cdot \ln 2 \cdot \sin x + 2^x \cos x = 2^x((\ln 2) \cdot \sin x + \cos x)$. Because 2^x is never zero, this expression is zero only when $(\ln 2) \cdot \sin x + \cos x = 0$, or $(\ln 2) \cdot \tan x = -1$, or $\tan x = \left(-\dfrac{1}{\ln 2}\right)$. So one solution is $x = \tan^{-1}\left(-\dfrac{1}{\ln 2}\right) \approx -0.9647$. And since the tangent function is periodic with period π, we also have solutions at approximately $-0.9647 + \pi \approx 2.1769$, and $-0.9647 + 2\pi \approx 5.3185$. These are the only solutions on the given interval.

b. $f(-2) \approx -0.2273$, $f(-0.9647) \approx -0.4211$, $f(2.1769) \approx 3.7164$, $f(5.3185) \approx -32.7968$, and $f(6) \approx -17.8826$. Thus the absolute maximum is about 3.7164 and the absolute minimum is about -32.7968.

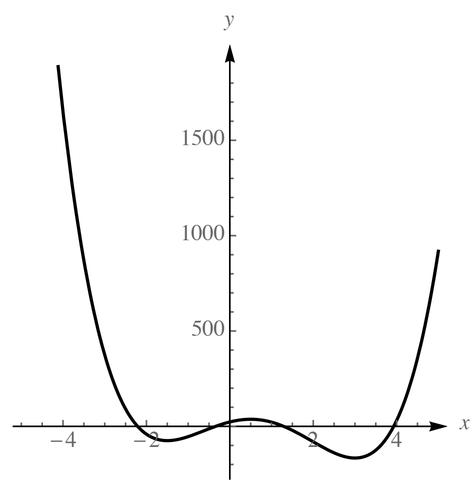
c. 

4.1.80

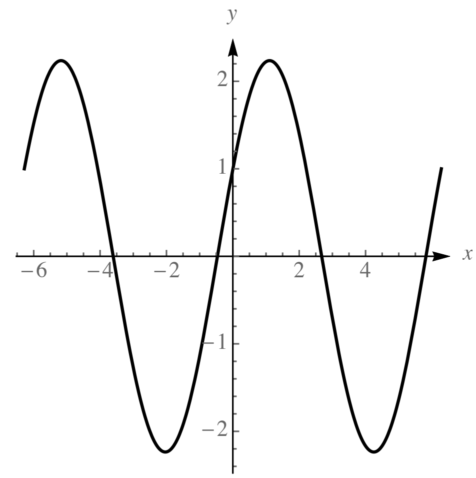
a. $f'(x) = 24x^3 - 48x^2 - 90x + 54 = 6(4x^3 - 8x^2 - 15x + 9)$, which can be written as $6(2x+3)(2x-1)(x-3)$. Note: to get this factorization, we used the rational root theorem to establish candidates for roots, then used trial-and-error to find that $x = 3$ was one of the roots, which means that $(x-3)$ is one of the factors of f'. Then we used long division to determine that $f'(x) = 6(x-3)(4x^2 + 4x - 3)$, then factored the quadratic. After factoring, it is clear that the roots of f' are 3, $-\dfrac{3}{2}$, and $\dfrac{1}{2}$, so these are the locations of the critical points.

b. From the graph, it appears that there is a local minimum at $x = -\dfrac{3}{2}$, a local maximum at $x = \dfrac{1}{2}$, and a local minimum at $x = 3$.

c. The local minimum at $x = 3$ is also an absolute minimum. The value of the absolute minimum is -166 and the value of the absolute maximum (which occurs at the left endpoint $x = -5$) is 4378.



4.1.81

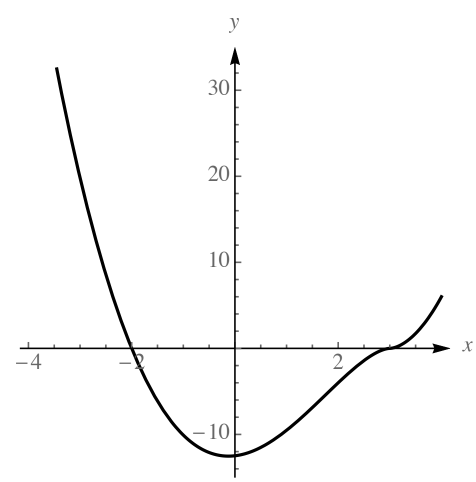


a. $f'(\theta) = 2\cos\theta - \sin\theta$, which is zero when $\tan\theta = 2$. So one critical point occurs at $\theta = \tan^{-1}(2) \approx 1.107$. And since the tangent function is periodic with period π, there are will also be solutions at this number plus or minus integer multiples of π. On the given interval, these are located at approximately $1.107 - 2\pi \approx -5.176$, at $1.107 - \pi \approx -2.034$, and at $1.107 + \pi \approx 4.249$.

b. From the graph, it appears that there is a local minimum at about $\theta = -2.034$ and at $\theta = 4.249$, and there is a local maximum at about $\theta = -5.176$, and at about $\theta = 1.107$.

c. From the graph, it appears that the local minimum at about $\theta = -2.034$ is also an absolute minimum, as is the one at $\theta = 4.249$. The local maximum at about $\theta = -5.176$, and at about $\theta = 1.107$ are also absolute maxima. The value of the absolute maximum appears to be about 2.24 and the value of the absolute minimum appears to be about −2.24.

4.1.82



a. $f'(x) = (x-3)^{5/3} + (x+2)\cdot\dfrac{5}{3}(x-3)^{2/3} = \dfrac{(x-3)^{2/3}}{3}(3x - 9 + 5x + 10) = \dfrac{(x-3)^{2/3}}{3}(8x + 1)$.This is zero when $x = 3$ and when $x = -\dfrac{1}{8}$. There are critical points at $x = -\dfrac{1}{8}$ and at $x = 3$.

b. From the graph, it appears that there is a local minimum of about −12.52 at $x = -\dfrac{1}{8}$.

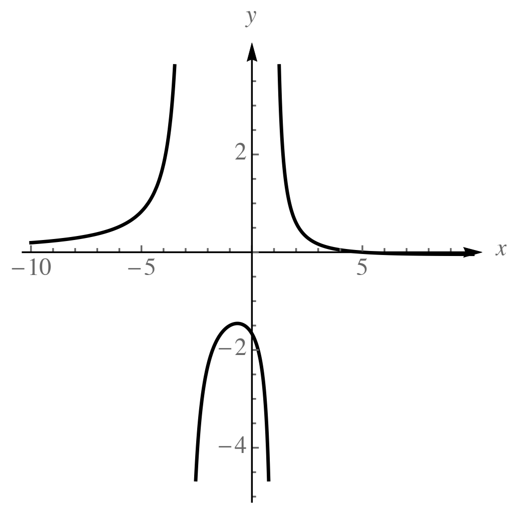
c. The local minimum mentioned above is also an absolute minimum. The absolute maximum occurs at the left endpoint $x = -4$, where the value of f is about 51.23.

4.1.83

a. $h'(x) = \dfrac{(x^2+2x-3)(-1)-(5-x)(2x+2)}{(x^2+2x-3)^2} = \dfrac{x^2-10x-7}{(x^2+2x-3)^2}$. By the quadratic formula, the numerator is zero (making the quotient zero) when $x = \dfrac{10 \pm \sqrt{100-4(-7)}}{2} = 5 \pm \dfrac{1}{2}\sqrt{128} = 5 \pm 4\sqrt{2}$. Note that $5+4\sqrt{2}$ isn't in the domain, so the only critical point is at $x = 5-4\sqrt{2}$.

b. From the graph, it appears that the one critical point mentioned above yields a local maximum.

c. The function has no absolute maximum and no absolute minimum on the given interval.

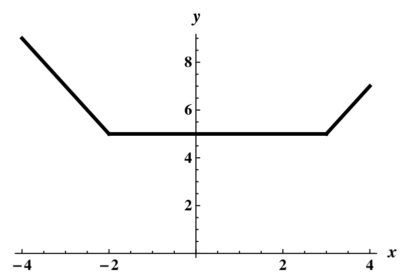


4.1.84

Note that

$$f(x) = \begin{cases} 3-x-x-2 = 1-2x & \text{if } -4 \le x \le -2, \\ 3-x+x+2 = 5 & \text{if } -2 \le x \le 3, \\ x-3+x+2 = 2x-1 & \text{if } 3 \le x \le 4. \end{cases}$$

There is an absolute maximum of 9 and an absolute minimum of 5. The absolute maximum occurs at $x = -4$, and the absolute minimum occurs at all of the values of x between -2 and 3.

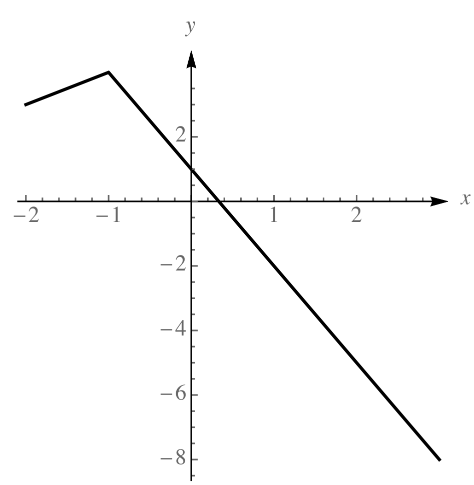


4.1.85

Note that

$$g(x) = \begin{cases} 3-x+2x+2 = x+5 & \text{if } -2 \le x \le -1, \\ 3-x-2x-2 = 1-3x & \text{if } -1 \le x \le 3. \end{cases}$$

There is an absolute maximum of 4 and an absolute minimum of -8. The absolute maximum occurs at $x = -1$, and the absolute minimum occurs at a $x = 3$.



4.1.86

a. $f'(x) = 2x$, so $f'(a) = 2a$ is the slope at $x = a$. The slope of the line perpendicular to the tangent line is $-\dfrac{1}{2a}$.

b. We are looking for a line through (a, a^2) with slope $-\dfrac{1}{2a}$, so the equation is given by $y = -\dfrac{1}{2a}(x-a) + a^2$.

c. To find B's position, we find where the parabola and the line from the last part of this problem intersect. So we seek the solution to $x^2 = -\dfrac{1}{2a}x + \dfrac{1}{2} + a^2$, or $x^2 + \dfrac{1}{2a}x + (-a^2 - \dfrac{1}{2}) = 0$. By the quadratic formula, we find that $x = \dfrac{-2a^2-1}{2a}$ is the desired point. The other root is $x = a$ which corresponds to point A.

d. $F(a) = \left(a - \left(\dfrac{-2a^2-1}{2a}\right)\right)^2 + \left(a^2 - \dfrac{(2a^2+1)^2}{4a^2}\right)^2$
$= \dfrac{(4a^2+1)^2}{4a^2} + \left(\dfrac{4a^4 - 4a^4 - 4a^2 - 1}{4a^2}\right)^2 = \dfrac{64a^6 + 48a^4 + 12a^2 + 1}{16a^4} = \dfrac{(4a^2+1)^3}{16a^4}$.

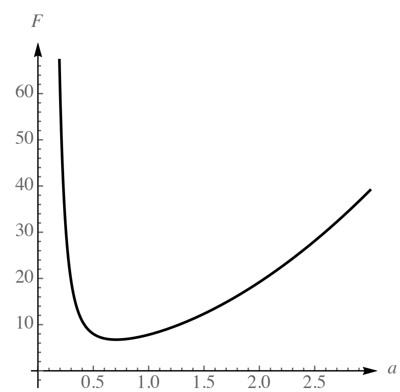
e.

$$\begin{aligned} F'(a) &= \frac{16a^4 \cdot 3(4a^2+1)^2 \cdot 8a - (4a^2+1)^3 \cdot 64a^3}{256a^8} \\ &= \frac{64a^3 \cdot (4a^2+1)^2 \left(6a^2 - (4a^2+1)\right)}{256a^8} \\ &= \frac{(4a^2+1)^2 \cdot (2a^2-1)}{4a^5}. \end{aligned}$$

The critical point of F for $a > 0$ occurs at $a = \sqrt{0.5}$.

f. The value of F at the critical point is $F(\sqrt{0.5}) = \dfrac{27}{4}$. The points are at $A = (\sqrt{0.5}, 0.5)$ and $B = (-\sqrt{2}, 2)$.

g.



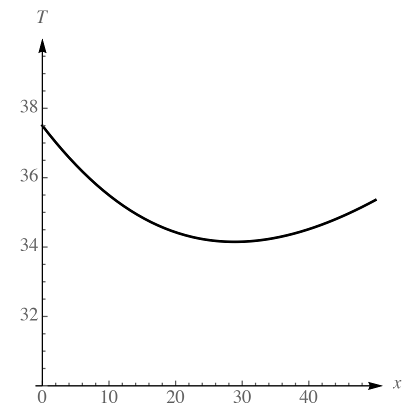
4.1.87

a. Because distance is rate times time, the time will be distance over rate. The swim distance is given by $\sqrt{2500 + x^2}$ meters, so the time for swimming is $\dfrac{\sqrt{2500+x^2}}{2}$. For running, the distance is $50 - x$, so the time is $\dfrac{50-x}{4}$. Thus we have $T(x) = \dfrac{\sqrt{2500+x^2}}{2} + \dfrac{50-x}{4}$.

b. $T'(x) = \frac{1}{2} \cdot \frac{1}{2} \left(x^2 + 2500\right)^{-1/2} \cdot 2x - \frac{1}{4} = \frac{x}{2\sqrt{x^2+2500}} - \frac{1}{4}$. This expression is zero when $\frac{x^2}{x^2+2500} = \frac{1}{4}$, so when $4x^2 = x^2 + 2500$, which occurs when $x^2 = \frac{2500}{3}$. So $x = \sqrt{\frac{2500}{3}} \approx 28.868$.

c. $T(0) = 37.5$, $T(28.868) \approx 34.151$, and $T(50) = 25\sqrt{2} \approx 35.355$. The absolute minimum occurs at the only critical point. The minimal crossing time is approximately 34.151 seconds.

d.



4.1.88 Because a parabola either opens up and has a minimum at its vertex, or open down and has a maximum at its vertex, it will always have exactly one extreme value. We have $f'(x) = 2ax + b$, which is zero when $x = -\frac{b}{2a}$, so that one critical point is the location of the vertex which gives the extreme point.

4.1.89

a. Note that since there is a local extreme value at 2 for f and since f is differentiable everywhere, we must have $f'(2) = 0$.

$g(2) = 2f(2) + 1 = 1$.

$h(2) = 2f(2) + 2 + 1 = 3$.

$g'(2) = 2 \cdot f'(2) + f(2) = 0$.

$h'(2) = 2f'(2) + f(2) + 1 = 1$.

b. h doesn't, since its derivative isn't zero at $x = 2$. However g might: for example, if $f(x) = (x-2)^2$ then $g(x) = x(x-2)^2 + 1$ has a local minimum at $x = 2$.

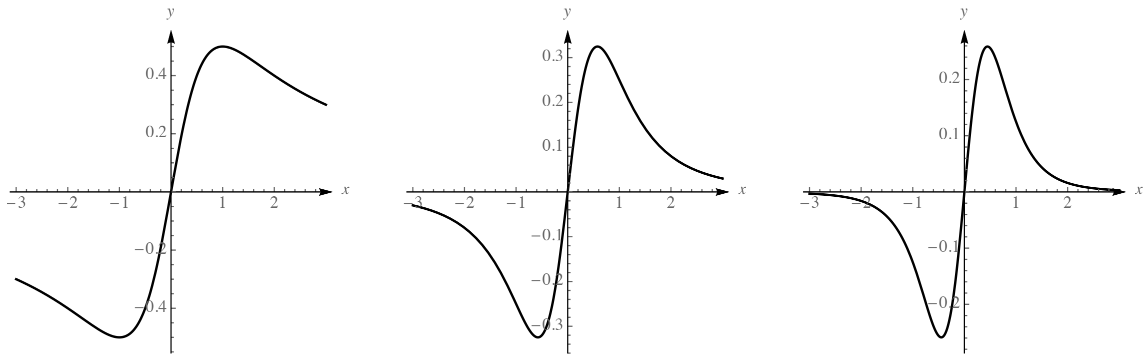
4.1.90

a. $f(-x) = -\frac{x}{((-x)^2+1)^n} = -\frac{x}{(x^2+1)^n} = -\frac{x}{(x^2+1)^n} = -f(x)$.

b. $f'(x) = \frac{(x^2+1)^n - x \cdot n(x^2+1)^{n-1} \cdot 2x}{(x^2+1)^{2n}} = \frac{(x^2+1)^{n-1}\left[x^2+1-2x^2n\right]}{(x^2+1)^{2n}} = \frac{1-(2n-1)x^2}{(x^2+1)^{n+1}}$. This quantity is zero when $x^2 = \frac{1}{2n-1}$, so $x = \pm\sqrt{\frac{1}{2n-1}}$ is a critical point.

c. The maximum value occurs at the positive critical number. The value of the maximum is given by $\frac{\sqrt{\frac{1}{2n-1}}}{\left(\left(\sqrt{\frac{1}{2n-1}}\right)^2+1\right)^n} = \frac{1}{\sqrt{2n-1}} \cdot \left(1 - \frac{1}{2n}\right)^n$. As $n \to \infty$, this quantity has limit 0.

d. Here are the graphs for $n = 1$, $n = 2$, and $n = 3$.



4.1.91

a. If $f(c)$ is a local maximum, then when x is near c but not equal to c, $f(c) \geq f(x)$, so $f(x) - f(c) \leq 0$.

b. When x is near to c but a little bigger than c, $x - c > 0$. So in this case, $\dfrac{f(x) - f(c)}{x - c} \leq 0$, since the numerator is negative (or 0) and the denominator is positive.

Thus, $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'(c) \leq 0$

c. When x is near to c but a little smaller than c, $x - c < 0$. So in this case, $\dfrac{f(x) - f(c)}{x - c} \geq 0$, since the numerator is negative (or 0) and the denominator is negative, making the quotient positive (or 0).

Thus, $\displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) \geq 0$.

d. From the above, we have that $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$.

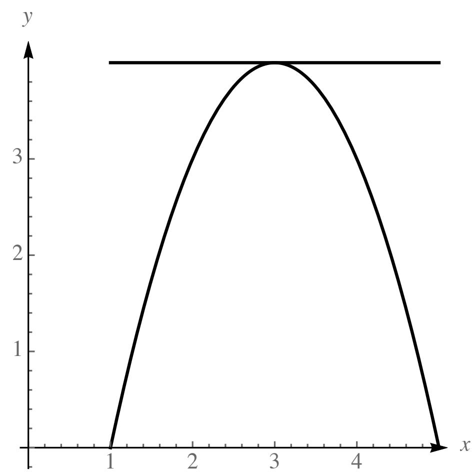
4.1.92

a. Because of the symmetry about the y-axis for an even function, a minimum at $x = c$ will correspond to a minimum at $x = -c$ as well.

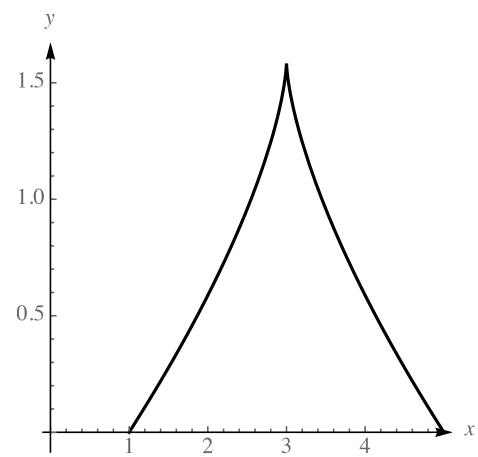
b. Because of the symmetry about the origin, a minimum at $x = c$ will correspond to a maximum at $x = -c$. It is helpful to think about the symmetry about the origin as being the result of flipping about the y-axis and then flipping about the x-axis.

4.2 Mean Value Theorem

4.2.1 If f is a continuous function on the closed interval $[a, b]$ and is differentiable on (a, b) and the slope of the secant line that joins $(a, f(a))$ and $(b, f(b))$ is zero, then there is at least one value c in (a, b) at which the slope of the line tangent to f at $(c, f(c))$ is also zero.

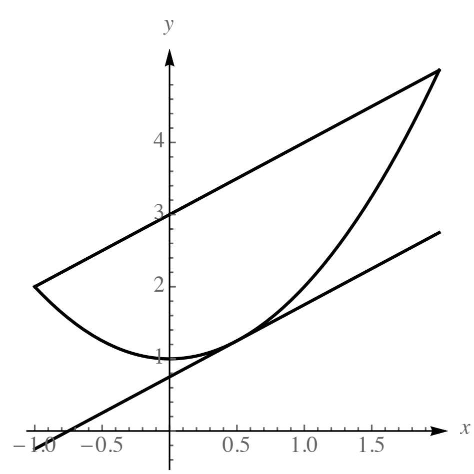


4.2.2 We seek a continuous function over an interval for which it isn't true that there is a horizontal tangent line to the function.



4.2.3 The function $f(x) = |x|$ is not differentiable at 0.

4.2.4 If f is a continuous function on the closed interval $[a, b]$ and is differentiable on (a, b), then there is at least one value c in (a, b) at which the slope of the line tangent to f at $(c, f(c))$ is equal to the slope of the secant line that joins $(a, f(a))$ and $(b, f(b))$.



4.2.5

a. It appears that at $c = 1$ the tangent line is parallel to the secant line shown.

b. $f'(x) = \frac{x}{2}$. The slope of the secant line is $\frac{f(4) - f(-2)}{4 - (-2)} = \frac{5 - 2}{6} = \frac{1}{2}$. We have $\frac{x}{2} = \frac{1}{2}$ at $x = 1$, as conjectured.

4.2.6

a. It appears that at $c = 1$ the tangent line is parallel to the secant line shown.

b. $f'(x) = \frac{1}{\sqrt{x}}$. The slope of the secant line is $\frac{f(4) - f(0)}{4 - 0} = \frac{4 - 0}{4 - 0} = 1$. We have $\frac{1}{\sqrt{x}} = 1$ at $x = 1$, as conjectured.

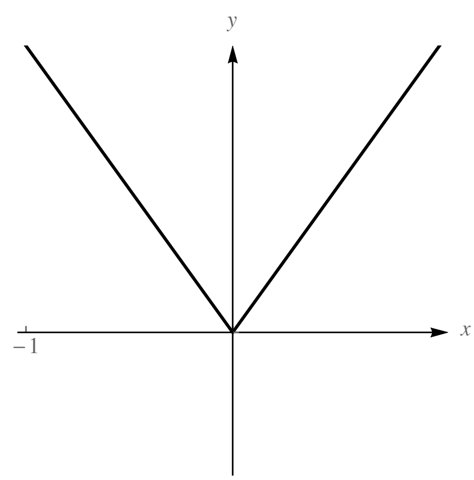
4.2.7

a. It appears that at a point a little bigger than 1 (but less than 1.5) the tangent line is parallel to the secant line shown, and then again at a point between -1 and -1.5.

b. $f'(x) = \frac{5x^4}{16}$. The slope of the secant line is $\frac{f(2) - f(-2)}{2 - (-2)} = \frac{2 - (-2)}{2 - (-2)} = 1$. We have $\frac{5x^4}{16} = 1$ when $x^4 = \frac{16}{5}$, or $c = \pm\frac{2}{\sqrt[4]{5}} \approx \pm 1.34$.

4.2.8 The average rate of change of f on the interval $[-10, 10]$ is $\frac{f(10) - f(-10)}{10 - (-10)} = \frac{10^3 - (-10)^3}{20} = 100$. We wish to find a point x in $(-10, 10)$ such that $f'(x) = 100$, or equivalently $3x^2 = 100$, which gives $x = \pm 10/\sqrt{3}$.

4.2.9 We seek a function over an interval for which it isn't true that there is a tangent line parallel to the secant line between the endpoints.



4.2.10 $f'(x) = \frac{2}{3\sqrt[3]{x}}$, while $\frac{f(8) - f(-1)}{8 - (-1)} = \frac{4 - 1}{9} = \frac{1}{3}$. In order for $f'(x) = \frac{1}{3}$, we would need $\frac{2}{3\sqrt[3]{x}} = \frac{1}{3}$. This occurs only when $\sqrt[3]{x} = 2$, or $x = 8$. However, $x = 8$ is not in the interval $(-1, 8)$. This does violate the MVT because f is not differentiable at $x = 0$.

4.2.11 The function f is differentiable on $[0, 1]$ and $f(0) = f(1) = 0$, so Rolle's theorem applies. We wish to find a point x in $(0, 1)$ such that $f'(x) = 0$; we have $f'(x) = (x - 1)^2 + 2x(x - 1) = (x - 1)(3x - 1)$, so $x = 1/3$ satisfies the conclusion of Rolle's theorem.

4.2.12 The function f is differentiable on $[0, \pi/2]$ and $f(0) = f(\pi/2) = 0$, so Rolle's theorem applies. We wish to find a point x in $(0, \pi/2)$ such that $f'(x) = 0$; we have $f'(x) = 2\cos 2x$, so $x = \pi/4$ satisfies the conclusion of Rolle's theorem.

4.2.13 The function f is differentiable on $[\pi/8, 3\pi/8]$ and $f(\pi/8) = f(3\pi/8) = 0$, so Rolle's theorem applies. We wish to find a point x in $(\pi/8, 3\pi/8)$ such that $f'(x) = 0$; we have $f'(x) = -4\sin 4x$, so $x = \pi/4$ satisfies the conclusion of Rolle's theorem.

4.2.14 The function f is not differentiable at $x = 0$, so Rolle's theorem does not apply.

4.2.15 The function f is not differentiable at $x = 0$, so Rolle's theorem does not apply.

4.2.16 The function f is continuous on $[-2, 4]$ and differentiable on $(-2, 4)$, and $f(-2) = f(4) = 0$, so Rolle's theorem does apply. $f'(x) = 3x^2 - 4x - 8$, which is zero at $x = \frac{4 \pm \sqrt{16+96}}{6} = \frac{4 \pm \sqrt{112}}{6} = 2\left(\frac{1 \pm \sqrt{7}}{3}\right)$. Both of these values lie between -2 and 4, so both satisfy the conclusion of Rolle's theorem.

4.2.17 g is continuous on $[-1, 3]$ and differentiable on $(-1, 3)$, and $g(-1) = 0 = g(3)$, so Rolle's theorem does apply. $g'(x) = 3x^2 - 2x - 5 = (x+1)(3x-5)$. This is zero for $x = -1$ (which is not on $(-1, 3)$) and for $x = 5/3$ (which is on $(-1, 3)$.) So $x = 5/3$ satisfies the conclusion of Rolle's theorem.

4.2.18 h is continuous on $[-a, a]$ and differentiable on $(-a, a)$, and $h(-a) = e^{-a^2} = h(a)$, so Rolle's theorem does apply. $h'(x) = -2xe^{-x^2}$, which is zero at $x = 0$. So $x = 0$ satisfies the conclusion of Rolle's theorem.

4.2.19 The average rate of change of the temperature from 3.2 km to 6.1 km is $\dfrac{-10.3 - 8.0}{6.1 - 3.2} \approx -6.3°/\text{km}$. Based on this, we cannot conclude that the lapse rate exceeds the critical value of 7°/ km.

4.2.20 The average acceleration over the 4.45 seconds is $\dfrac{330 - 0}{4.45 - 0} \approx 74.2\text{mi/hr/s}$, so at some point during the race, the maximum acceleration of the drag racer is at least 74 mi/hr/s.

4.2.21

a. The function f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. so the Mean Value Theorem applies.

b. The average rate of change of f on $[-1, 2]$ is $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{3 - 6}{3} = -1$. We wish to find a point c in $(-1, 2)$ such that $f'(c) = -1$, or equivalently $-2c = -1$ which gives $c = 1/2$.

4.2.22

a. The function f is differentiable on $(0, 1)$ and continuous on $[0, 1]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1]$ is $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{-1 - 0}{1} = -1$.

   We wish to find a point c in $(0, 1)$ such that $f'(c) = -1$, or equivalently $3c^2 - 4c = -1$. This can be written as $0 = 3c^2 - 4c + 1 = (3c - 1)(c - 1)$, which has solutions $c = \dfrac{1}{3}$ and $c = 1$. Only $c = \dfrac{1}{3}$ is in the interval $(0, 1)$.

4.2.23

a. The MVT does not apply because f is not differentiable at $x = 0$. To see this note that $\displaystyle\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1$ and $\displaystyle\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -2$.

4.2.24

a. The MVT does not apply because f is not continuous at $x = 1$ (because 1 isn't even in the domain of f).

4.2.25

a. The function f is differentiable on $(0, 1)$ and continuous on $[0, 1]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1]$ is $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{e - 1}{1} = e - 1$. We wish to find a point c in $(0, 1)$ such that $f'(c) = e - 1$, or equivalently $e^c = e - 1$ which gives $c = \ln(e - 1) \approx 0.541$.

4.2.26

a. The function f is differentiable on $(1, e)$ and continuous on $(1, e)$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[1, e]$ is $\dfrac{f(e) - f(1)}{e - 1} = \dfrac{\ln e + \ln 2 - \ln 2}{e - 1} = \dfrac{1}{e - 1}$. We wish to find a point c in $(1, e)$ such that $f'(c) = 1/(e - 1)$, or equivalently $1/c = 1/(e - 1)$ which gives $c = e - 1 \approx 1.71828$.

4.2.27

a. f is continuous on $[-\pi/2, \pi/2]$ and differentiable on $(-\pi/2, \pi/2)$, so the Mean Value Theorem applies.

b. The average rate of change of f on $[-\pi/2, \pi/2]$ is $\dfrac{f(\pi/2) - f(-\pi/2)}{\pi/2 - (-\pi/2)} = \dfrac{1 - (-1)}{\pi} = \dfrac{2}{\pi}$. We wish to find a point c in $(-\pi/2, \pi/2)$ such that $f'(c) = \dfrac{2}{\pi}$, or equivalently $\cos c = \dfrac{2}{\pi}$ which gives $c \approx \pm 0.881$ (using a root finder).

4.2.28

a. f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$, so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, \pi/4]$ is $\dfrac{f(\pi/4) - f(0)}{\pi/4 - 0} = \dfrac{1 - 0}{\pi/4} = \dfrac{4}{\pi}$. We wish to find a point c in $(0, \pi/4)$ such that $f'(c) = \dfrac{4}{\pi}$, or equivalently $\sec^2 c = \dfrac{4}{\pi}$ which gives $c \approx 0.482$ (using a root finder).

4.2.29

a. The function f is differentiable on $(0, 1/2)$ and continuous on $[0, 1/2]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[0, 1/2]$ is $\dfrac{f(1/2) - f(0)}{\frac{1}{2} - 0} = \dfrac{\frac{\pi}{6} - 0}{\frac{1}{2}} = \dfrac{\pi}{3}$.

   We wish to find a point c in $(0, 1/2)$ such that $f'(c) = \pi/3$, or equivalently $\dfrac{1}{\sqrt{1 - c^2}} = \dfrac{\pi}{3}$, so $c = \sqrt{1 - \dfrac{9}{\pi^2}}$.

4.2.30

a. The function f is differentiable on $(1, 3)$ and continuous on $[1, 3]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[1, 3]$ is $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{\frac{10}{3} - 2}{2} = \dfrac{2}{3}$. We wish to find a point c in $(1, 3)$ such that $f'(c) = 2/3$, or equivalently $1 - \dfrac{1}{c^2} = \dfrac{2}{3}$, so $c = \sqrt{3}$.

4.2.31

a. The Mean Value Theorem does not apply because the function f is not differentiable at $x = 0$.

b. Even though the Mean Value Theorem doesn't apply, it still happens to be the case that there are numbers c between -8 and 8 where the tangent line has slope $\dfrac{f(8) - f(-8)}{8 - (-8)} = \dfrac{1}{2}$. This occurs where $\dfrac{2}{3}c^{-2/3} = 1/2$, which gives $c = \pm\dfrac{8}{9} \cdot \sqrt{3}$.

4.2.32

a. The function f is differentiable on $(-1, 2)$ and continuous on $[-1, 2]$ so the Mean Value Theorem applies.

b. The average rate of change of f on $[-1, 2]$ is $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{\frac{1}{2} - (-1)}{3} = \dfrac{1}{2}$. We wish to find a point c in $(-1, 2)$ such that $f'(c) = 1/2$, or equivalently $\dfrac{2}{(c+2)^2} = \dfrac{1}{2}$, so $c = 0$.

4.2.33

a. False. The function f is not differentiable at $x = 0$.

b. True. If $f(x) - g(x) = c$ is constant, then $f'(x) - g'(x) = 0$.

c. False. If $f'(x) = 0$ then we can conclude that $f(x) = c$ for some constant.

4.2.34

a. Let $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$. Then $f'(x) = \dfrac{1}{1+x^2} + \dfrac{1}{1+(1/x)^2}\left(-\dfrac{1}{x^2}\right) = \dfrac{1}{1+x^2} - \dfrac{1}{1+x^2} = 0$ for all $x \neq 0$. By Theorem 4.5, f is constant on $(-\infty, 0)$ and $(0, \infty)$.

b. Because $f(1) = \tan^{-1} 1 + \tan^{-1} 1 = \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$, and f is constant, we must have that $f(x) = \dfrac{\pi}{2}$ on $(0, \infty)$.

c. Similarly, $f(-1) = \tan^{-1}(-1) + \tan^{-1}(-1) = -\dfrac{\pi}{4} - \dfrac{\pi}{4} = -\dfrac{\pi}{2}$. Because f is constant on $(-\infty, 0)$, we must have $f(x) = -\dfrac{\pi}{2}$ there. Combining this result with the result of part b allows us to conclude that

$$f(x) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0. \end{cases}$$

4.2.35

a.

$$\frac{d}{dx}\tan^{-1}(2/x^2) = \frac{1}{1+4/x^4} \cdot -\frac{4}{x^3} = \frac{1}{1+4/x^4} \cdot -\frac{4}{x^3} \cdot \frac{x}{x} = -\frac{4x}{x^4+4}.$$

$$\begin{aligned}
\frac{d}{dx}\left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right) &= \frac{1}{1+(x+1)^2} - \frac{1}{1+(x-1)^2} \\
&= \frac{x^2 - 2x + 2 - (x^2 + 2x + 2)}{1 + (x+1)^2 + (x-1)^2 + (x^2-1)^2} \\
&= -\frac{4x}{1 + x^2 + 2x + 1 + x^2 - 2x + 1 + x^4 - 2x^2 + 1} \\
&= -\frac{4x}{x^4+4}.
\end{aligned}$$

Because these two functions have the same derivative, they differ by a constant. So for any x, $\tan^{-1}(2/x^2) - \left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right)$ is a constant.

b. Because the two function in part (a) differ by a constant, we can compute the constant by evaluating for a specific number x. Choosing $x = 1$, we have $\tan^{-1} 2 - \left(\tan^{-1} 2 - \tan^{-1} 0\right) = 0$, so the constant is 0, and we have
$$\tan^{-1}(2/x^2) = \left(\tan^{-1}(x+1) - \tan^{-1}(x-1)\right).$$

4.2.36 Observe that $\ln 2x = \ln 2 + \ln x$ and $\ln 10x^2 = \ln 10 + \ln x^2$, so the pairs $f(x), g(x)$ and $h(x), p(x)$ have the same derivative.

4.2.37 The functions $h(x)$ and $p(x)$ have the same derivative as $f(x)$ because they differ from $f(x)$ by a constant.

4.2.38 One example of a function f with $f'(x) = x+1$ is $f(x) = \dfrac{x^2}{2} + x$; therefore the most general function with derivative $x + 1$ is $f(x) = \dfrac{x^2}{2} + x + C$ where C is a constant.

4.2.39 The secant line between the endpoints has slope $\dfrac{f(4) - f(-4)}{4 - (-4)} = \dfrac{4-1}{8} = \dfrac{3}{8}$. The slope of the tangent line to the graph appears to have this value at approximately -2.5 and at about 2.6. These are eyeballed estimates, so your personal estimate may differ.

4.2.40 Because $f(3) = f(1) \approx 2$, the average rate of change of f on $[1, 3]$ is $\dfrac{f(3) - f(1)}{3 - 1} = 0$. However, there is no point in $(1, 3)$ such that the slope of the tangent to f at that point is zero. This does not contradict the Mean Value Theorem because f is not differentiable everywhere on $(1, 3)$; in particular, it is not differentiable at $x = 2$.

4.2.41 Because $f(1) \approx 2$ and $f(3) \approx 2$, the average rate of change of f on $[1, 3]$ is $\dfrac{f(3) - f(1)}{3 - 1} = 0$. However, the tangent to f between $x = 1$ and $x = 2$ is the graph of f itself, which has slope 2, and the tangent to f between $x = 2$ and $x = 3$ is also the graph of f, which has slope 1. So there is no point in $(1, 3)$ where the tangent line has slope 0. This does not contradict the Mean Value Theorem because f is not continuous everywhere on $[1, 3]$, nor differentiable everywhere on $(1, 3)$ both of these hypotheses fail at x = 2.

4.2.42

a. The average temperature gradient from $h = 0$ to $h = 1.1$ m is $\dfrac{-2 - (-16)}{1.1 - 0} \approx 12.7°/\text{m}$, so by the Mean Value Theorem the temperature gradient must equal $12.7°/\text{m}$ somewhere in the snowpack, and the formation of a weak layer is likely.

b. The average temperature gradient from $h = 0$ to $h = 1.4$ m is $\dfrac{-1 - (-12)}{1.4 - 0} \approx 7.86°/\text{m}$. While it is still possible that the temperature gradient exceeds $10°/\text{m}$ somewhere in the snowpack, one may suspect that the formation of a weak layer is not likely in this case.

c. If the surface temperature and temperature at the bottom of the snowpack are both roughly constant, then the temperature gradient will be larger in areas where the snowpack is less deep.

d. If all layers of the snowbank are the same temperature, then the temperature gradient is 0 and a weak layer is not likely to form.

4.2.43 The average speed of the car over the 28 minute period ($= 28/60$ hr) is $\dfrac{30 - 0}{28/60} \approx 64.3\text{mi/hr}$, so the officer can conclude by the Mean Value Theorem that at some point the car exceeded the speed limit.

4.2.44 The average speed of the car over the 30 minute period ($= 1/2$ hr) is exactly 60 mi/hr. But because the car started from rest, the average speed for the first few seconds of the trip is less than 60 mi/hr, and therefore the average speed for the remainder of the trip must exceed 60 mi/hr, and the officer can conclude that the driver exceeded the speed limit.

4.2.45 The runner's average speed is $6.2/(32/60) \approx 11.6$ mi /hr. By the Mean Value Theorem, the runner's speed was 11.6 mi/hr at least once. By the intermediate value theorem, all speeds between 0 and 11.6 mi/hr were reached. Because the initial and final speed was 0 mi/hr, the speed of 11 mi/hr was reached at least twice.

4.2.46 For linear functions $f(x)$ the average rate of change of f on any interval $[a,b]$ is the same as the slope $f'(c)$ for *any* point c in (a,b).

4.2.47 Observe that
$$\frac{f(b)-f(a)}{b-a} = \frac{A(b^2-a^2)+B(b-a)}{b-a} = A(a+b)+B$$
and $f'(c) = 2Ac+B$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = (a+b)/2$.

4.2.48

a. Observe that $\dfrac{f(b)-f(a)}{b-a} = \dfrac{b^2-a^2}{b-a} = (a+b)$ and $f'(c) = 2c$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = (a+b)/2$.

b. Observe that
$$\frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)} = -\frac{1}{ab}$$
and $f'(c) = -1/c^2$, so the point c that satisfies the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

4.2.49 Note that $f'(x) = 2\tan x \sec^2 x$ and $g'(x) = 2\sec x \sec x \tan x = 2\tan x \sec^2 x$, so $f'(x) = g'(x)$. This implies that $f-g$ is a constant, which also follows from the trigonometric identity $\sec^2 x = \tan^2 x + 1$.

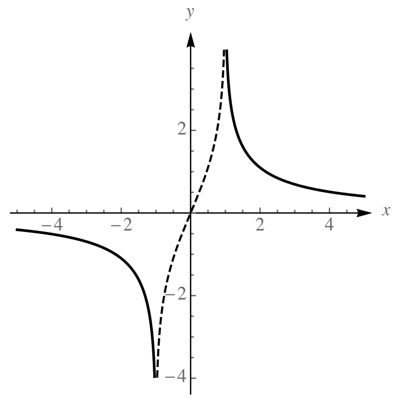
4.2.50 Bolt's average speed during the race was $\dfrac{100}{9.58}$ m/s $= \dfrac{100}{9.58}\cdot\dfrac{3600}{1000}$ km/hr ≈ 37.58 km/hr, so by the Mean Value Theorem he must have exceeded 37 km/hr during the race.

4.2.51 Let $f(x) = \sec^{-1} x$ and $g(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$. Then
$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}}$$
and
$$g'(x) = -\frac{1}{\sqrt{1-(1/x)^2}}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{|x|^2\sqrt{1-(1/x)^2}} = \frac{1}{|x|\sqrt{x^2}\sqrt{1-(1/x)^2}} = \frac{1}{|x|\sqrt{x^2-1}}.$$
Because $f'(x) = g'(x)$ for $x > 0$, we must have that $f(x) - g(x) = C$ for $x > 0$. Let $x = 1$. Then $C = f(1) - g(1) = 0 - 0 = 0$, so $f(x) - g(x) = 0$, so $f(x) = g(x)$ for $x > 0$. Similarly, $f(x) - g(x) = C$ for $x < 0$, and letting $x = -1$ gives $f(x) - g(x) = 0$ so $f(x) = g(x)$ for $x < 0$. Thus $\sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right)$ for all $x \neq 0$.

4.2.52

a.
$$f'(x) = \frac{(x-1)}{(x+1)}\left(\frac{(x-1)-(x+1)}{(x-1)^2}\right) = -\frac{2}{(x+1)(x-1)} = -\frac{2}{x^2-1} \text{ for all } x \text{ such that } |x| > 1.$$
Also,
$$\begin{aligned} g'(x) &= \frac{(1-x)}{(x+1)}\left(\frac{(1-x)-(x+1)(-1)}{(1-x)^2}\right) = \frac{2}{(x+1)(1-x)} = -\frac{2}{(x+1)(x-1)} \\ &= -\frac{2}{x^2-1} \text{ for all } x \text{ such that } |x| > 1. \end{aligned}$$

b.



The solid curve is $f(x)$ and the dashed curve is $g(x)$.

c. f' and g' do not equal each other on a common interval I, so Theorem 4.6 is not violated.

4.2.53 By the Mean Value Theorem, there is a number c in $(2,4)$ so that $\dfrac{f(4)-f(2)}{4-2} = f'(c)$, or $f(4)-7 = 2f'(c)$. Because $f'(c) < 2$, we must have $f(4)-7 < 4$, which implies that $f(4) < 11$.

4.2.54 Suppose $x > 0$. By the Mean Value Theorem, there is a number c in $(0,x)$ so that $\dfrac{f(x)-f(0)}{x-0} = f'(c) > 1$. This implies that $\dfrac{f(x)}{x} > 1$, so $f(x) > x$.

4.2.55 Let $a > 0$ and let $f(x) = \sqrt{1+x}$ on $[0,a]$. By the Mean Value Theorem, there is a number c in $(0,a)$ so that $\dfrac{f(a)-f(0)}{a-0} = f'(c)$, which implies that

$$\frac{\sqrt{1+a}-1}{a} = \frac{1}{2\sqrt{1+c}}.$$

Because $\sqrt{1+c} > 1$ for $c > 0$, $\dfrac{1}{2\sqrt{1+c}} < \dfrac{1}{2}$. So

$$\frac{\sqrt{1+a}-1}{a} = \frac{1}{2\sqrt{1+c}} < \frac{1}{2}$$

so

$$\frac{\sqrt{1+a}-1}{a} < \frac{1}{2}.$$

Rewriting this inequality gives

$$1 + \frac{a}{2} > \sqrt{1+a}$$

for $a > 0$.

4.2.56

a. Let $f(x) = \sin x$. If $a = b$, then the inequality $|\sin a - \sin b| \le |a-b|$ holds because both sides of the inequality are 0. Now assume $a < b$. By the Mean Value Theorem on (a,b), we know that there exists a number c in (a,b) so that

$$\frac{f(b)-f(a)}{b-a} = f'(c) = \cos c.$$

Because $|\cos c| \le 1$, we have

$$|f(b)-f(a) \le |b-a|$$

so

$$|\sin b - \sin a| \leq |b - a|.$$

For the case $a > b$, a similar argument using the Mean Value Theorem on the interval (b, a) shows that $|\sin a - \sin b| \leq |a - b|$, which can be written $|\sin b - \sin a| \leq |b - a|$. Thus this inequality holds for all a and b.

b. Using the above inequalty, let $b = 0$. Then

$$|\sin b - \sin a| = |\sin a| \leq |b - a| = |a|.$$

So $|\sin a| \leq |a|$ for all real numbers a.

4.2.57

a. If $g(x) = x$ then $g'(x) = 1$ and hence $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f'(c)}{g'(c)} = f'(c)$.

b. We have $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{0 - (-1)}{6 - 2} = \dfrac{1}{4}$; $\dfrac{f'(c)}{g'(c)} = \dfrac{2c}{4} = \dfrac{c}{2}$; so $c = 1/2$.

4.2.58 Observe that f' is positive and decreasing for $x > a$ (because $f'' < 0$ for $x > a$). Fix some $b > a$ and let $a < x < b$. Then by the Mean Value Theorem $\dfrac{f(x) - f(a)}{x - a} = f'(c)$ for some c in (a, x). We have $f'(c) > f'(b) > 0$, so therefore if $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, we must have $f'(a) \geq f'(b) > 0$. On the other hand f' is negative for $x < a$ so the Mean Value Theorem implies $\dfrac{f(x) - f(a)}{x - a} < 0$ for $x < a$, and so if $f'(a)$ exists we must have $f'(a) \leq 0$. This gives a contradiction, so we conclude that $f'(a)$ does not exist.

More generally, if f' and f'' both change signs at some point a, then one of the functions $f(x)$, $-f(x)$,$f(-x)$ or $-f(-x)$ satisfies the hypotheses above, and so $f'(a)$ does not exist.

4.3 What Derivatives Tell Us

4.3.1 If f' is positive on an interval, f is increasing on that interval. If f' is negative on an interval, f is decreasing on that interval.

4.3.2 The First Derivative Test can be used to tell whether or not a critical point is a local maximum or minimum, as follows: If $(c, f(c))$ is a critical point, we investigate the sign of f' for points that are just to the left and just to the right of c. If the sign of f' changes from positive to negative, then f is changing from increasing to decreasing at c, so there is a local maximum at c. If the signs of f' are changing from negative to positive, then f is changing from decreasing to increasing at c, so there is a local minimum at c. If the signs of f' are the same on either side of c, then there is neither kind of local extremum at $x = c$.

Note that if we find *all* of the critical points of f, and if the domain of f is an interval or union of intervals, then the critical points naturally divide up the domain into intervals on which we can check the sign of f' and look for places where the sign changes.

4.3.3

a. $x - 3 = 0$ for $x = 3$, so $x = 3$ is a critical point of f.

b. $f'(x) > 0$ for $x > 3$, so f is increasing on $(3, \infty)$. $f' < 0$ on $(-\infty, 3)$, so f is decreasing on $(-\infty, 3)$.

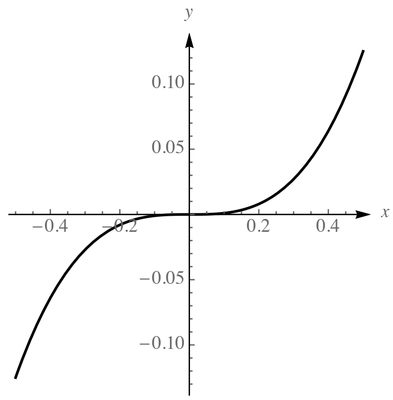
4.3.4

a. $(x - 1)(x - 2) = 0$ for $x = 1$ and $x = 2$, so these are the critical points.

b. $f' > 0$ on $(-\infty, 1)$ and on $(2, \infty)$, so f is increasing there. $f' < 0$ on $(1, 2)$ so f is decreasing there.

4.3.5

One such example is $f(x) = x^3$ at $x = 0$.



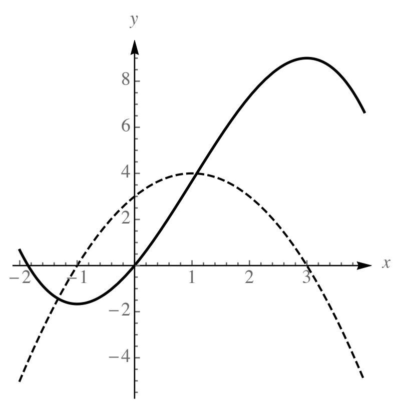
4.3.6

a. $g'(x) = 0$ for $x = 1$ and $x = 2$, so those are the critical points.

b. $g' > 0$ on $(1, 2)$, so g is increasing there. $g' < 0$ on $(0, 1)$ and on $(2, 3)$, so g is decreasing there.

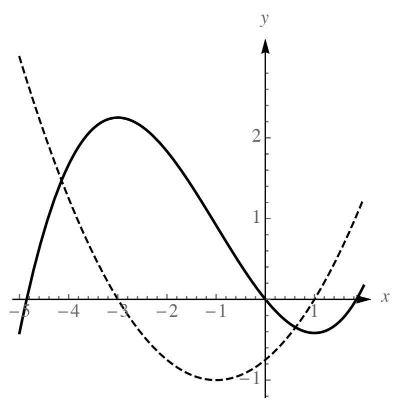
c. By the First Derivative Test, g has a local maximum at $x = 2$ and a local minimum at $x = 1$.

4.3.7



f' is dotted and possible f is solid.

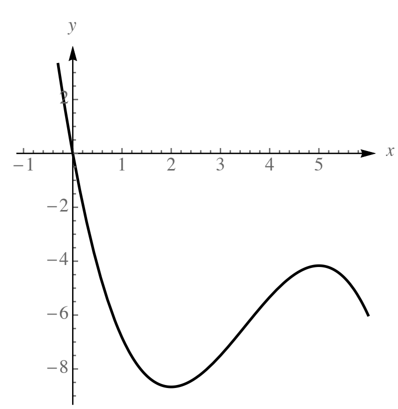
4.3.8



f' is dotted and possible f is solid.

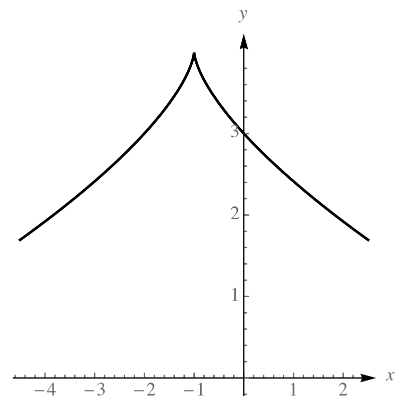
4.3.9

Such a function would be decreasing until $x = 2$, then increasing until $x = 5$, and then decreasing again after that.



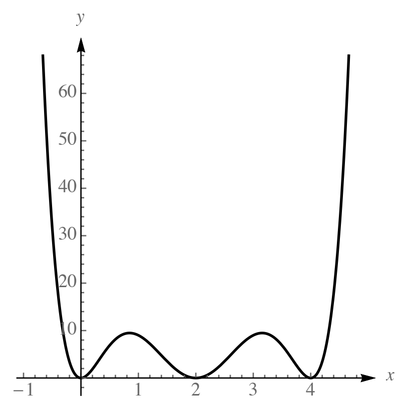
4.3.10

Such a function would be increasing on $(-\infty, -1)$, and decreasing on $(-1, \infty)$. It should have a point of non-differentiability at $x = -1$.



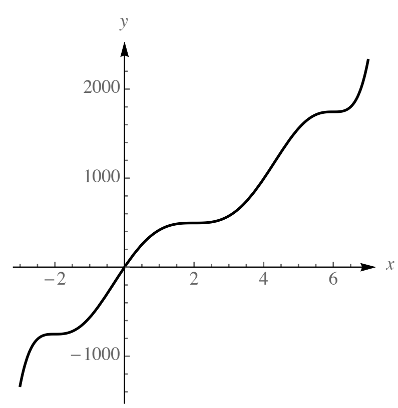
4.3.11

Such a function has extrema (minima) at 0 and 4, where the y value is zero. The function should never go below the x axis.



4.3.12

Such a function is never decreasing, but is flat at -2, 2, and 6.



4.3.13

a. $g'' > 0$ on $(-\infty, 2)$ so g is concave up there; $g'' < 0$ on $(2, \infty)$, so g is concave down there.

b. There is an inflection point at $x = 2$.

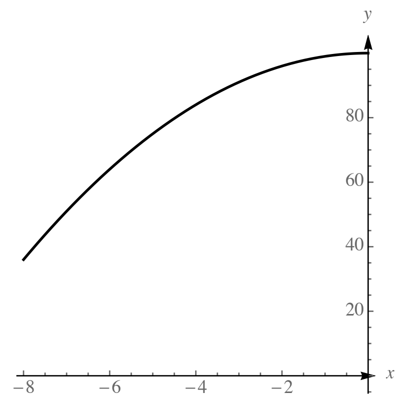
4.3.14

a. $g'' > 0$ on $(-3, 0)$ and $(2, 3)$, so g is concave up on those intervals. $g'' < 0$ on $(-4, 3)$ and $(0, 2)$, so g is concave down on those intervals.

b. There are inflection points at $x = -3$, $x = 0$, and $x = 2$.

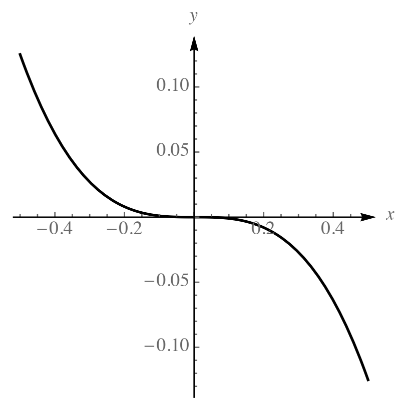
4.3.15

Yes, for example, consider $f(x) = 100 - x^2$ on the interval $(-8, 0)$. It is above the x axis, increasing, and concave down on that interval.



4.3.16

The second derivative is positive to the left of the inflection point, and negative to the right.



4.3.17 $f(x) = x^4$ has this property at 0. Note that $f''(x) = 12x^2$, which is 0 at $x = 0$, but the function doesn't have an inflection point there.

4.3.18 Because the Second Derivative Test is inconclusive, the First Derivative Test should be used in this case.

4.3.19 $f'(x) = -2x$, which is zero exactly when $x = 0$. On $(-\infty, 0)$ we note that $f' > 0$, so that f is increasing on this interval. On $(0, \infty)$, we note that $f' < 0$, so f is decreasing on this interval.

4.3.20 $f'(x) = 2x$, which is zero exactly when $x = 0$. On $(-\infty, 0)$ we note that $f' < 0$, so that f is decreasing on this interval. On $(0, \infty)$, we note that $f' > 0$, so f is increasing on this interval.

4.3.21 $f'(x) = 2(x - 1)$, which is zero exactly when $x = 1$. On $(-\infty, 1)$ we note that $f' < 0$, so that f is decreasing on this interval. On $(1, \infty)$, we note that $f' > 0$, so f is increasing on this interval.

4.3.22 $f'(x) = 3x^2 + 4$, which is always positive, because it is always 4 or greater. So f is increasing on $(-\infty, \infty)$.

4.3.23 $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, which is 0 for $x = 1$ and $x = 4$. On $(-\infty, 1)$ $f' > 0$ so f is increasing; on $(1, 4)$ $f' < 0$ so f is decreasing; and on $(4, \infty)$ $f' > 0$ so f is increasing.

4.3.24 $f'(x) = -x^2 + x + 2 = -(x^2 - x - 2) = -(x-2)(x+1)$, which is 0 for $x = 2$ and $x = -1$. On $(-\infty, -1)$ $f' < 0$ so f is decreasing; on $(-1, 2)$ $f' > 0$ so f is increasing; on $(2, \infty)$ $f' < 0$ so f is decreasing.

4.3.25 $f'(x) = 1 - 2x$, which is 0 when $x = 1/2$. On $(-\infty, 1/2)$ $f' > 0$ so f is increasing on this interval, while on $(1/2, \infty)$ $f' < 0$, so f is deceasing on this interval.

4.3.26 $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-2)(x-1)$, which is 0 when x is 0, 1, or 2. On $(-\infty, 0)$ $f' < 0$ so f is decreasing. On $(0, 1)$, $f' > 0$ so f is increasing, on $(1, 2)$ $f' < 0$ so f is decreasing, and on $(2, \infty)$ $f' > 0$ so f is increasing.

4.3.27 $f'(x) = -x^3 + 3x^2 - 2x = -x(x^2 - 3x + 2) = -x(x-1)(x-2)$. This is zero when $x = 0$, $x = 1$, and $x = 2$. Note that $f'(-1) > 0$, and $f'(1.5) > 0$, while $f'(.5) < 0$, and $f'(3) < 0$. So f is increasing on $(-\infty, 0)$ and on $(1, 2)$, while it is decreasing on $(0, 1)$ and on $(2, \infty)$.

4.3.28 $f'(x) = 10x^4 - 15x^3 + 5x^2 = 5x^2(2x^2 - 3x + 1) = 5x^2(x-1)(2x-1)$. This is zero when $x = 0$, $x = 1$ and $x = 1/2$. Note that $f'(-1) > 0$, and $f'(1/4) > 0$. Because the given function is continuous, we can combine the intervals and conclude that f is increasing on $(-\infty, 1/2)$. Also, $f'(3/4) < 0$, so f is decreasing on $(1/2, 1)$, and $f'(2) > 0$, so f is increasing on $(1, \infty)$.

4.3.29 $f'(x) = 2x \ln x^2 + x^2 \cdot \dfrac{1}{x^2} \cdot 2x = 2x(\ln x^2 + 1)$. This is undefined for $x = 0$, and when $\ln x^2 + 1 = 0$. For $x > 0$, this occurs when $2 \ln x + 1 = 0$, which occurs when $\ln x = -\dfrac{1}{2}$, or $x = \dfrac{1}{\sqrt{e}}$. By symmetry, we also have that $f'(x)$ is zero for $x = -\dfrac{1}{\sqrt{e}}$. Note that $\dfrac{1}{\sqrt{e}} \approx 0.6$, and that $f'(-1) < 0$, $f'(-1/2) > 0$, $f'(1/2) < 0$, and $f'(1) > 0$. Thus, f is decreasing on $(-\infty, -1/\sqrt{e})$ and on $(0, 1/\sqrt{e})$, and is increasing on $(-1/\sqrt{e}, 0)$ and on $(1/\sqrt{e}, \infty)$.

4.3.30 $f'(x) = \dfrac{(e^{2x}+1)e^x - e^x(2e^{2x})}{(e^{2x}+1)^2} = \dfrac{e^x(1 - e^{2x})}{(e^{2x}+1)^2}$. This is zero when $e^{2x} = 1$, which occurs when $x = 0$. Note that $f'(-1) > 0$ and $f'(1) < 0$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

4.3.31 $f'(x) = 2 \sin x - 1$ which is 0 for $\sin x = 1/2$, or (on the given interval) $x = \pi/6, 5\pi/6$. On $(0, \pi/6)$ $f' < 0$ so f is decreasing; on $(\pi/6, 5\pi/6)$ $f' > 0$ so f is increasing; on $(5\pi/6, 2\pi)$ $f' < 0$ so f is decreasing.

4.3.32 $f'(x) = \sqrt{2} \cos x - 1$, which is 0 for $\cos x = 1/\sqrt{2}$, or (on the given interval) $x = \pi/4, 7\pi/4$. On $(0, \pi/4)$ $f' > 0$ so f is increasing; on $(\pi/4, 7\pi/4)$ $f' < 0$ so f is decreasing; on $(7\pi/4, 2\pi)$ $f' > 0$ so f is increasing.

4.3.33 $f'(x) = -9 \sin 3x$, which is 0 for $3x = -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \text{and } 3\pi$, which corresponds to $x = -\pi, -2\pi/3, -\pi/3, 0, \pi/3, 2\pi/3$ and π. Note that $f'(-5\pi/6) = 9 > 0$, $f'(-\pi/2) = -9 < 0$, $f'(-\pi/6) = 9 > 0$, $f'(\pi/6) = -9 < 0$, $f'(\pi/2) = 9 > 0$, and $f'(5\pi/6) = -9 < 0$. Thus f is increasing on $(-\pi, -2\pi/3)$, on $(-\pi/3, 0)$, and on $(\pi/3, 2\pi/3)$, while f is decreasing on $(-2\pi/3, -\pi/3)$, on $(0, \pi/3)$, and on $(2\pi/3, \pi)$.

4.3.34 $f'(x) = 2(\cos x)(-\sin x) = -\sin 2x$. This is 0 for $2x = -2\pi, -\pi, 0, \pi$, and 2π, which corresponds to $x = -\pi, -\pi/2, 0, \pi/2$, and π. Note that $f'(-3\pi/4) = -1 < 0$, $f'(-\pi/4) = 1 > 0$, $f'(\pi/4) = -1 < 0$, and $f'(3\pi/4) = 1 > 0$. So f is decreasing on $(-\pi, -\pi/2)$ and on $(0, \pi/2)$, and is increasing on $(-\pi/2, 0)$ and on $(\pi/2, \pi)$.

4.3.35 $f'(x) = (2/3)x^{-1/3}(x^2 - 4) + x^{2/3} \cdot 2x = \dfrac{2(x^2-4)}{3x^{1/3}} + \dfrac{2x^{5/3}}{1} \cdot \dfrac{3x^{1/3}}{3x^{1/3}} = \dfrac{8x^2 - 8}{3x^{1/3}} = \dfrac{8(x+1)(x-1)}{3x^{1/3}}$. This is zero for $x = \pm 1$ and is undefined for $x = 0$. Note that $f'(-8) = -84 < 0$, $f'(-1/8) = \frac{21}{4} > 0$, $f'(1/8) = -\frac{21}{4} < 0$, and $f''(8) = 84 > 0$. Thus f is decreasing on $(-\infty, -1)$ and on $(0, 1)$, while f is increasing on $(-1, 0)$ and on $(1, \infty)$.

4.3.36 $f'(x) = 2x\sqrt{9-x^2} + x^2 \cdot \dfrac{1}{2\sqrt{9-x^2}} \cdot (-2x) = \dfrac{2x(9-x^2)}{\sqrt{9-x^2}} + -\dfrac{x^3}{\sqrt{9-x^2}} = \dfrac{18x-3x^3}{\sqrt{9-x^2}} = \dfrac{3x(6-x^2)}{\sqrt{9-x^2}}$. This is zero when $x = 0$, and when $x = \pm\sqrt{6}$. Note that $\sqrt{6} \approx 2.4$. Also note that $f'(-2.7) > 0$, $f'(-1) < 0$, $f'(1) > 0$ and $f'(2.7) < 0$. Thus, f is increasing on $(-3, -\sqrt{6})$ and on $(0, \sqrt{6})$, and is decreasing on $(-\sqrt{6}, 0)$ and on $(\sqrt{6}, 3)$.

4.3.37 $f'(x) = \dfrac{1}{2\sqrt{9-x^2}} \cdot (-2x) + \dfrac{1}{3\sqrt{1-\frac{x^2}{9}}} = \dfrac{-x}{\sqrt{9-x^2}} + \dfrac{1}{\sqrt{9-x^2}} = \dfrac{1-x}{\sqrt{9-x^2}}$. This is 0 for $x = 1$ and is undefined at the endpoints of the domain which are 3 and -3. On $(-3, 1)$ $f' > 0$ so f is increasing; on $(1, 3)$ $f' < 0$ so f is decreasing.

4.3.38 $f'(x) = \ln x + \dfrac{x}{x} - 2 = \ln x - 1$. This is 0 for $\ln x = 1$, or $x = e$. On $(0, e)$ $f' < 0$ so f is decreasing; on (e, ∞) $f' > 0$ so f is increasing.

4.3.39 $f'(x) = -60x^4+300x^3-240x^2 = -60x^2(x^2-5x+4) = -60x^2(x-4)(x-1)$. This is 0 for $x = 0$, $x = 1$, and $x = 4$. Note that $f'(-1) = -600 < 0$, $f'(1/2) = -26.25 < 0$, $f'(2) = 480 > 0$, and $f'(5) = -6000 < 0$. Thus f is increasing on $(1, 4)$ and is decreasing on $(-\infty, 1)$ and on $(4, \infty)$.

4.3.40 $f'(x) = 2x - \dfrac{2}{x} = \dfrac{2(x^2-1)}{x}$ which is 0 for $x = 1$. Note that the domain of f is $(0, \infty)$ and that $f'(1/2) = -3 < 0$ and $f'(2) = 3 > 0$, so f is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

4.3.41 $f'(x) = -8x^3 + 2x = -2x(4x^2 - 1) = -2x(2x+1)(2x-1)$. This is zero for $x = 0$ and $x = \pm 1/2$. Note that $f'(-1) > 0$, $f'(-1/4) < 0$, $f'(1/4) > 0$, and $f'(1) < 0$, so f is increasing on $(-\infty, -1/2)$ and on $(0, 1/2)$, while it is decreasing on $(-1/2, 0)$ and on $(1/2, \infty)$.

4.3.42 $f'(x) = x^3 - 8x^2 + 15x = x(x^2 - 8x + 15) = x(x-5)(x-3)$. This is zero when $x = 0$, $x = 3$, and $x = 5$. Note that $f'(-1) < 0$, $f'(1) > 0$, $f'(4) < 0$, and $f'(6) > 0$. Thus f is increasing on $(0, 3)$ and on $(5, \infty)$, while it is decreasing on $(-\infty, 0)$ and on $(3, 5)$.

4.3.43 We have $f'(x) = e^{-x^2/2} + xe^{-x^2/2} \cdot (-x) = (1-x^2)e^{-x^2/2}$. This is zero only when $x = \pm 1$. Note that $f'(-2) = -3e^{-2} < 0$, $f'(0) = 1 > 0$, and $f'(2) = -3e^{-2} < 0$. Thus f is decreasing on $(-\infty, -1)$ and on $(1, \infty)$, and is increasing on $(-1, 1)$.

4.3.44 $f'(x) = \dfrac{1}{1+\left(\frac{x}{x^2+2}\right)^2} \cdot \dfrac{(x^2+2) - x(2x)}{(x^2+2)^2} = \dfrac{1}{1+\left(\frac{x}{x^2+2}\right)^2} \cdot \dfrac{2-x^2}{(x^2+2)^2}$. Note that the first factor is always positive, so the expression is zero exactly when $2 - x^2 = 0$, so only at $\pm\sqrt{2}$. Note that $f'(0) > 0$ while $f'(\pm 2) < 0$, so f is increasing on $(-\sqrt{2}, \sqrt{2})$ and decreasing on $(-\infty, -\sqrt{2})$ and on $(\sqrt{2}, \infty)$.

4.3.45

a. $f'(x) = 2x$, so $x = 0$ is the only critical point.

b. Note that $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, so f has a local minimum of $f(0) = 3$ at $x = 0$.

c. Note that $f(-3) = 12$, $f(0) = 3$ and $f(2) = 7$, so the absolute maximum is 12 and the absolute minimum is 3.

4.3.46

a. $f'(x) = -2x - 1$, which exists everywhere and is zero only for $x = -1/2$, so that is the only critical point.

b. Note that $f'(-2) = 3 > 0$ and $f'(0) = -1 < 0$, so f has a local maximum of $f(-1/2) = 9/4$ at $x = -1/2$.

c. Note that $f(-4) = -10$ and $f(4) = -18$, so the absolute maximum is $9/4$ at $x = -1/2$ and the absolute minimum is -18 at $x = 4$.

4.3.47

a. $f'(x) = x \cdot \frac{1}{2}(4-x^2)^{-1/2} \cdot (-2x) + \sqrt{4-x^2} \cdot 1 = \frac{4-2x^2}{\sqrt{4-x^2}}$, which exists everywhere on $(-2, 2)$ and is zero only for $x = \pm\sqrt{2}$, so those are the only critical points.

b. Note that $f'(-1.5) < 0$, $f'(0) > 0$ and $f'(1.5) < 0$, so f has a local minimum of $f(-\sqrt{2}) = -2$ and a local maximum of $f(\sqrt{2}) = 2$.

c. Note that $f(-2) = 0 = f(2)$. So the absolute maximum is 2 at $x = \sqrt{2}$ and the absolute minimum is -2 at $x = -\sqrt{2}$.

4.3.48

a. $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$, which exists everywhere and is 0 at $x = -2$ (an endpoint of the given domain) and $x = 1$.

b. Note that $f'(-1.5) < 0$ and $f'(2) > 0$, so f has a local minimum at $x = 1$ of $f(1) = -6$.

c. Note that $f(-2) = 21$ and $f[4] = 129$, so the absolute maximum of f on $[-2, 4]$ is 129 and the absolute minimum is -6.

4.3.49

a. $f'(x) = -3x^2 + 9$, which is zero when $9 = 3x^2$, or $x^2 = 3$. So the critical points are at $x = \pm\sqrt{3}$.

b. Note that $f'(-2) < 0$, $f'(0) > 0$, and $f'(2) < 0$, so there is a local minimum of $f(-\sqrt{3}) = -6\sqrt{3}$ and a local maximum of $f(\sqrt{3}) = 6\sqrt{3}$.

c. There is an absolute maximum of 28 at $x = -4$ and an absolute minimum of $-6\sqrt{3}$ at $x = -\sqrt{3}$.

4.3.50

a. $f'(x) = 10x^4 - 20x^3 - 30x^2 = 10x^2(x^2 - 2x - 3) = 10x^2(x+1)(x-3)$. This is zero for $x = 0$, $x = -1$, and $x = 3$.

b. Note that $f'(-2) > 0$, $f'(-.5) < 0$, $f'(1) < 0$, and $f'(4) > 0$. So there is a local maximum of $f(-1) = 7$ and a local minimum of $f(3) = -185$. There isn't any sort of extremum at $x = 0$.

c. The local minimum value of -185 is an absolute minimum, and the absolute maximum is $f(4) = 132$.

4.3.51

a. $f'(x) = x^{2/3} + (x-5) \cdot \frac{2}{3}x^{-1/3} = \frac{5x-10}{3x^{1/3}}$, which is undefined at $x = 0$ and is 0 at $x = 2$. So these are the two critical points.

b. Note that $f'(-1) > 0$ and $f'(1) < 0$, and $f'(3) > 0$ so f has a local maximum at $x = 0$ of $f(0) = 0$ and a local minimum at $x = 2$ of $-3\sqrt[3]{4} \approx -4.762$.

c. Note that $f(-5) = -10\sqrt[3]{25} \approx -29.24$, $f(0) = 0$, and $f(5) = 0$, so the absolute maximum of f on $[-5, 5]$ is 0 and the absolute minimum is $-10\sqrt[3]{25}$.

4.3.52 First note that even though the interval given is $[-4, 4]$, the function isn't defined at $x = \pm 1$, so we will assume that the given domain is $[-4, -1) \cup (-1, 1) \cup (1, 4]$.

a. $f'(x) = \frac{(x^2-1) \cdot 2x - (x^2)(2x)}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}$, which is 0 only at $x = 0$.

b. Note that $f'(-2) > 0$ and $f'(-1/2) > 0$, and $f'(1/2) < 0$ and $f'(2) < 0$, so f is increasing on $(-4, -1)$ and on $(-1, 0)$, while it is decreasing on $(0, 1)$ and on $(1, 4)$. There is a local maximum of 0 at $x = 0$.

c. Because f becomes arbitrarily large as x approaches 1 from the left, and arbitrarily large in the negative sense as x approaches 1 from the right, it has no absolute extrema.

4.3.53

a. $f'(x) = \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$. This is defined everywhere on $(0,\infty)$ and is 0 only at $x = e^{-2}$.

b. Note that $f' < 0$ on $\left(0, \frac{1}{e^2}\right)$ and $f' > 0$ on $\left(\frac{1}{e^2}, \infty\right)$, so there is a local minimum at $x = \frac{1}{e^2}$.

c. Because there is only one critical point, the local minimum at $x = \frac{1}{e^2}$ yields an absolute minimum of $f(1/e^2) = -\frac{2}{e} \approx -0.736$. There is no absolute maximum because f increases without bound as $x \to \infty$.

4.3.54

a. $f'(x) = 1 - \frac{2}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{2}{1+x^2} = \frac{x^2-1}{x^2+1} = \frac{(x-1)(x+1)}{x^2+1}$. This is zero for $x = \pm 1$.

b. $f' > 0$ on $(-\sqrt{3}, -1)$ and on $(1, \sqrt{3})$ while $f' < 0$ on $(-1, 1)$. So there is a local maximum value of $f(-1) = -1 + \frac{\pi}{2} \approx 0.57$ at $x = -1$ and a local minimum value of $f(1) = 1 - \frac{\pi}{2} \approx -0.57$ at $x = 1$.

c. Checking endpoints, we have $f(-\sqrt{3}) = -\sqrt{3} + \frac{2\pi}{3} \approx 0.36$ and $f(\sqrt{3}) = \sqrt{3} - \frac{2\pi}{3} \approx -0.36$. So the absolute maximum of f is $f(-1) = -1 + \frac{\pi}{2} \approx 0.57$ at $x = -1$ and the absolute minimum value of f is $f(1) = 1 - \frac{\pi}{2} \approx -0.57$ at $x = 1$.

4.3.55 $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$, which is 0 only for $x = 1$. f is continuous on $(-\infty, \infty)$ and contains only one critical point. Note that $f' > 0$ for $x < 1$ and $f' < 0$ for $x > 1$. So there is a local maximum of $f(1) = 1/e$ at $x = 1$. The local maximum of $1/e$ at $x = 1$ is an absolute maximum. There is no absolute minimum, because the function is unbounded in the negative direction as $x \to -\infty$.

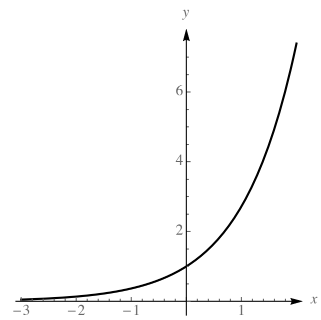
4.3.56 f is continuous on $(0,\infty)$. $f'(x) = 4 - \frac{1}{2x^{3/2}} = \frac{8x^{3/2}-1}{2x^{3/2}}$, which exists for $x > 0$. This quantity is 0 for $x = \frac{1}{4}$. So f has only one critical point on $(0,\infty)$. Also, $f' < 0$ on $(0, 1/4)$ and $f' > 0$ on $(1/4, \infty)$. So there is a local minimum of $f(1/4) = 3$ at $x = 1/4$. The local minimum of 3 at $x = 1/4$ is an absolute minimum. There is no absolute maximum, because the function is unbounded as $x \to \infty$.

4.3.57 A is continuous on $(0,\infty)$. $A'(r) = -\frac{24}{r^2} + 4\pi r = \frac{4\pi r^3 - 24}{r^2}$, which is 0 for $r = \sqrt[3]{6/\pi}$, so there is only one critical point on the stated interval. Note that $A' < 0$ on $(0, \sqrt[3]{6/\pi})$ and $A' > 0$ on $(\sqrt[3]{6/\pi}, \infty)$, so there is a local minimum of $A(\sqrt[3]{6/\pi}) = 36\sqrt[3]{\pi/6}$. The local minimum mentioned above is an absolute minimum. There is no absolute maximum, because A is unbounded as $r \to \infty$.

4.3.58 f is continuous on $(-\infty, 3)$. $f'(x) = -\frac{x}{2\sqrt{3-x}} + \sqrt{3-x} = \frac{-3x+6}{2\sqrt{3-x}}$, which is 0 only for $x = 2$, so there is only one critical point on the stated interval. Also, $f' > 0$ for $x < 2$ and $f' < 0$ on $(2, 3)$. Thus there is a local maximum of $f(2) = 2$ which is also an absolute maximum. There is no absolute minimum, because the function is unbounded in the negative direction as $x \to -\infty$.

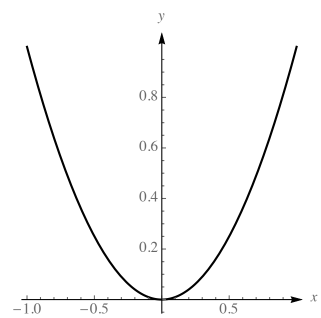
4.3.59

The function sketched should be increasing and concave up everywhere.



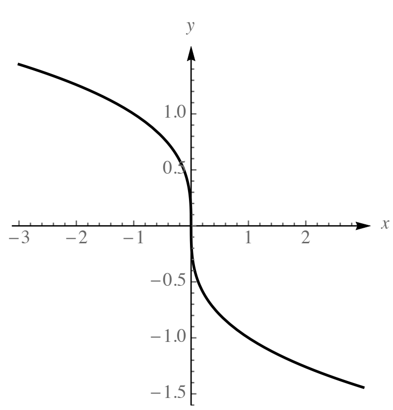
4.3.60

The function sketched should be concave up everywhere, decreasing for $x < 0$ and increasing for $x > 0$.



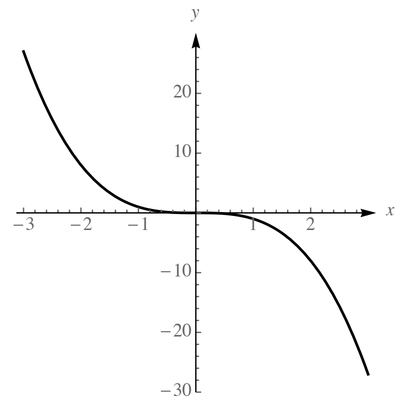
4.3.61

The function sketched should be decreasing everywhere, concave down for $x < 0$, and concave up for $x > 0$.



4.3.62

The function sketched should be decreasing everywhere, concave up for $x < 0$, and concave down for $x > 0$.



4.3.63 $f'(x) = 4x^3 - 6x^2$, so $f''(x) = 12x^2 - 12x = 12x(x-1)$. Note that f'' is zero when $x = 0$ and $x = 1$, so these are potential inflection points. Also note that $f''(-1) > 0$, $f''(0.5) < 0$, and $f''(2) > 0$, so f is concave up on $(-\infty, 0)$ and on $(1, \infty)$, and is concave down on $(0, 1)$. There are inflection points at $(0, 1)$ and $(1, 0)$.

4.3.64 $f'(x) = -4x^3 - 6x^2 + 24x$, so $f''(x) = -12x^2 - 12x + 24 = -12(x^2 + x - 2) = -12(x+2)(x-1)$. Note the f'' is zero for $x = -2$ and $x = 1$, so these are potential inflection points. Now note that $f''(-3) < 0$, $f''(0) > 0$, and $f''(2) < 0$. Thus f is concave up on $(-2, 1)$ and is concave down on $(-\infty, -2)$ and on $(1, \infty)$. There are inflection points at $(-2, 48)$ and $(1, 9)$.

4.3.65 $f'(x) = 20x^3 - 60x^2$, and $f''(x) = 60x^2 - 120x = 60x(x-2)$. This is 0 for $x = 0$ and for $x = 2$. Note that $f''(-1) > 0$, $f''(1) < 0$, and $f''(3) > 0$. So f is concave up on $(-\infty, 0)$, concave down on $(0, 2)$, and concave up on $(2, \infty)$. There are inflection points at $x = 0$ and $x = 2$.

4.3.66 $f'(x) = -1(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$. $f''(x) = \dfrac{(-2)(1+x^2)^2 - (-2x) \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4} = \dfrac{6x^2 - 2}{(1+x^2)^3}$. Note that f'' is 0 for $x = \pm\sqrt{1/3}$. Also, $f''(-1) > 0$, $f''(0) < 0$, and $f''(1) > 0$, so f is concave up on $(-\infty, \sqrt{1/3})$, concave down on $(-\sqrt{1/3}, \sqrt{1/3})$, and concave up on $(\sqrt{1/3}, \infty)$. There are inflection points at $x = \pm\sqrt{1/3}$.

4.3.67 $f'(x) = e^x(x-3) + e^x = e^x(x-3+1) = e^x(x-2)$. $f''(x) = e^x(x-2) + e^x = e^x(x-2+1) = e^x(x-1)$. Note that f'' is zero only at $x = 1$. Also note that $f''(0) < 0$ and $f''(2) > 0$, so f is concave down on $(-\infty, 1)$ and is concave up on $(1, \infty)$. The point $(1, -2e)$ is an inflection point.

4.3.68 $f'(x) = 4x \ln x + 2x^2 \cdot \dfrac{1}{x} - 10x = 4x \ln x - 8x$. $f''(x) = 4 \ln x + 4x \cdot \frac{1}{x} - 8 = 4 \ln x - 4$. Note that f'' is zero when $\ln x = 1$, which occurs for $x = e$. Also note that $f''(1) < 0$ and $f''(4) > 0$, so f is concave down on $(0, e)$ and is concave up on (e, ∞). There is an inflection point at $(e, -3e^2)$.

4.3.69 $g'(t) = \dfrac{6t}{3t^2 + 1}$, and $g''(t) = \dfrac{(3t^2 + 1) \cdot 6 - 6t(6t)}{(3t^2 + 1)^2} = \dfrac{6 - 18t^2}{(3t^2 + 1)^2}$. Note that g'' is 0 for $t = \pm\sqrt{1/3}$. Also, $g''(-1) < 0$, $g''(0) > 0$, and $g''(1) < 0$, so g is concave down on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$, and is concave up on $(-\sqrt{1/3}, \sqrt{1/3})$. There are inflection points at $t = \pm\sqrt{1/3}$.

4.3.70 $g'(x) = \dfrac{1}{3\sqrt[3]{(x-4)^2}}$, and $g''(x) = -\dfrac{2}{9\sqrt[3]{(x-4)^5}}$. Note that g'' is never zero, but is undefined at $x = 4$. On $(-\infty, 4)$ we have $g'' > 0$ so g is concave up, and on $(4, \infty)$ we have $g'' < 0$, so g is concave down. There is an inflection point at $(4, 0)$.

4.3.71 $f'(x) = -xe^{-x^2/2}$, and $f''(x) = (-x)(-xe^{-x^2/2}) + e^{-x^2/2} \cdot -1 = e^{-x^2/2}(x^2-1)$. $f''(x)$ is 0 for $x = \pm 1$. Also $f'' > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so f is concave up there, while on $(-1, 1)$ f is concave down because $f'' < 0$ on that interval. The inflection points are at $(\pm 1, e^{-1/2})$.

4.3.72 $p'(x) = 4x^3e^x + x^4e^x + 1$ and $f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x = x^2e^x(12 + 4x + 4x + x^2) = x^2e^x(x^2 + 8x + 12) = x^2e^x(x+6)(x+2)$. So $p'' = 0$ for $x = -6, -2, 0$. Checking the sign of p'' shows that $p'' > 0$ on $(-\infty, -6)$ and on $(-2, 0)$, and on $(0, \infty)$, so p is concave up on these intervals, while $p'' < 0$ on $(-6, -2)$ so p is concave down there. The inflection points occur at $x = -6$ and $x = -2$.

4.3.73 $f'(x) = \sqrt{x}/x + (\ln x)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{2 + \ln x}{2\sqrt{x}}$. $f''(x) = \dfrac{2\sqrt{x}/x - (2 + \ln x)/\sqrt{x}}{(2\sqrt{x})^2} = -\dfrac{\ln x}{4\sqrt{x^3}}$. Note that f'' is 0 only at $x = 1$. On $(0, 1)$ we note that $f'' > 0$ so f is concave up, and on $(1, \infty)$ we note that $f'' < 0$ so f is concave down. There is an inflection point at $(1, 0)$.

4.3.74 $h'(t) = -2\sin 2t$ and $h''(t) = -4\cos 2t$, which on the stated domain is 0 when $2t = \pi/2$, and $3\pi/2$, which means for $t = \pi/4$, and $3\pi/4$. $h'' < 0$ on $(0, \pi/4)$ and on $(3\pi/4, \pi)$, so h is concave down on those intervals, while $h'' > 0$ on $(\pi/4, 3\pi/4)$, so h is concave up on that interval. There are inflection points at $t = \pi/4$, and $t = 3\pi/4$.

4.3.75 $g'(t) = 15t^4 - 120t^3 + 240t^2$, and $g''(t) = 60t^3 - 360t^2 + 480t = 60t(t-2)(t-4)$. Note that g'' is 0 for $t = 0, 2$, and 4. Note also that $g'' < 0$ on $(-\infty, 0)$ and on $(2, 4)$, so g is concave down on those intervals, while $g'' > 0$ on $(0, 2)$ and on $(4, \infty)$, so g is concave up there. There are inflection points at $t = 0, 2$, and 4.

4.3.76 $f'(x) = 8x^3 + 24x^2 + 24x - 1$, and $f''(x) = 24x^2 + 48x + 24 = 24(x+1)^2$. Note that this quantity is always greater than 0 for $x \neq -1$, and is 0 only at $x = -1$. Thus f is concave up on $(-\infty, -1)$ and on $(-1, \infty)$, and because f and f' are continuous at -1, we can say that f is concave up on $(-\infty, \infty)$.

4.3.77 $f'(x) = 3x^2 - 6x = 3x(x-2)$. This is zero when $x = 0$ and when $x = 2$, and these are the critical points. $f''(x) = 6x - 6$. Note that $f''(0) < 0$ and $f''(2) > 0$. Thus by the Second Derivative Test, there is a local maximum at $x = 0$ and a local minimum at $x = 2$.

4.3.78 $f'(x) = 12x - 3x^2 = 3x(4-x)$. This is zero when $x = 0$ and when $x = 4$, and these are the critical points. $f''(x) = 12 - 6x$. Note that $f''(0) > 0$ and $f''(4) < 0$, so there is a local minimum at 0 and a local maximum at 4.

4.3.79 $f'(x) = -2x$, so $x = 0$ is a critical point. $f''(x) = -2$, so $f''(0) = -2$ and the critical point yields a local maximum.

4.3.80 $f'(x) = 3x^2 - 3x - 36 = 3(x^2 - x - 12) = 3(x-4)(x+3)$, which is zero for $x = 4$ and $x = -3$. $f''(x) = 6x - 3$. We have $f''(4) = 21 > 0$, so there is a local minimum at $x = 4$, while $f''(-3) = -21 < 0$, so there is a local maximum at $x = -3$.

4.3.81 $f'(x) = e^x(x-7) + e^x = e^x(x-6)$. This is zero when $x = 6$, and this is a critical point. $f''(x) = e^x(x-6) + e^x = e^x(x-5)$. Note that $f''(6) > 0$, so there is a local minimum at $x = 6$.

4.3.82 $f'(x) = e^x(x-2)^2 + e^x \cdot 2(x-2) = e^x(x-2)(x-2+2) = xe^x(x-2)$. This is zero for $x = 0$ and $x = 2$. $f''(x) = e^x(x-2) + xe^x(x-2) + xe^x = e^x(x - 2 + x^2 - 2x + x) = e^x(x^2 - 2)$. We have $f''(0) = -2 < 0$ so there is a local maximum at $x = 0$, while $f''(2) = 2e^2 > 0$ so there is a local minimum at $x = 2$.

4.3.83 $f'(x) = 6x^2 - 6x = 6x(x-1)$, so $x = 0$ and $x = 1$ are critical points. $f''(x) = 12x - 6$, so $f''(1) = 6 > 0$, so the critical point at $x = 1$ yields a local minimum. Also, $f''(0) = -6 < 0$, so the critical point at 0 yields a local maximum.

4.3.84 $f'(x) = \dfrac{(x+1)e^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$. This is zero for $x = 0$, so that is the only critical point. $f''(x) = \dfrac{(x+1)^2(e^x + xe^x) - xe^x \cdot 2(x+1)}{(x+1)^4} = \dfrac{(x+1)e^x((x+1)^2 - 2x)}{(x+1)^4} = \dfrac{e^x(x^2+1)}{(x+1)^3}$. Because $f''(0) = 1 > 0$, there is a local minimum at $x = 0$.

4.3.85 $f'(x) = x^2 \cdot (-e^{-x}) + e^{-x} \cdot 2x = e^{-x}(2x - x^2)$, which is zero for $x = 0$ and $x = 2$, so these are the critical points. $f''(x) = e^{-x}(2 - 2x) + (2x - x^2)(-e^{-x}) = e^{-x}(2 - 4x + x^2)$. Note that $f''(0) = 2 > 0$, so there is a local minimum at $x = 0$. Also, $f''(2) = -2e^{-2} < 0$, so there is a local maximum at $x = 2$.

4.3.86

$$g'(x) = \frac{(2 - 12x^2)(4x^3) - x^4(-24x)}{(2 - 12x^2)^2} = \frac{-24x^5 + 8x^3}{(2 - 12x^2)^2} = 2\left(\frac{-3x^5 + x^3}{(1 - 6x^2)^2}\right).$$

g' is zero at 0 and at $\pm\frac{1}{\sqrt{3}}$. We have

$$\begin{aligned} g''(x) &= 2\left(\frac{(1 - 6x^2)^2(-15x^4 + 3x^2) - (-3x^5 + x^3)(2(1 - 6x^2) \cdot (-12x))}{(1 - 6x^2)^4}\right) \\ &= 2\left(\frac{(1 - 6x^2)((1 - 6x^2)(-15x^4 + 3x^2) - (72x^6 - 24x^4))}{(1 - 6x^2)^4}\right) \\ &= 2\left(\frac{-15x^4 + 3x^2 + 90x^6 - 18x^4 - 72x^6 + 24x^4}{(1 - 6x^2)^3}\right) \\ &= 2\left(\frac{3x^2 - 9x^4 + 18x^6}{(1 - 6x^2)^3}\right) \\ &= 6\left(\frac{x^2 - 3x^4 + 6x^6}{(1 - 6x^2)^3}\right). \end{aligned}$$

Then $g''\left(\pm\frac{1}{\sqrt{3}}\right) = -\frac{4}{3}$, so these critical points give local maxima by the Second Derivative Test. However, $g''(0) = 0$, so the Second Derivative Test is inconclusive at $x = 0$. Using the First Derivative Test, note that $0.5 < \frac{1}{\sqrt{3}}$, and $g'(0.5) = 0.25 > 0$, and using the fact that g' is an odd function, $g'(-0.5) = -0.25 < 0$, so there is a local minimum for g at $x = 0$.

4.3.87 $f'(x) = 4x \ln x + 2x^2 \cdot \frac{1}{x} - 22x = 4x \ln x - 20x = 4x(\ln x - 5)$. This is zero for $x = e^5$, so that is the critical point. $f''(x) = 4 \ln x + 4x \cdot \dfrac{1}{x} - 20 = 4 \ln x - 16$. Note that $f''(e^5) > 0$, so there is a local minimum at e^5.

4.3.88 Note that $f(x)$ can be written as $f(x) = \dfrac{12}{7}x^{7/2} - 4x^{5/2}$. Thus $f'(x) = 6x^{5/2} - 10x^{3/2} = 2x^{3/2}(3x - 5)$. This is zero on the given interval only at $x = 5/3$, so that is the critical point. $f''(x) = 15x^{3/2} - 15x^{1/2}$, and $f''(5/3) > 0$, so there is a local minimum at $x = 5/3$.

4.3.89 $p'(t) = 6t^2 + 6t - 36 = 6(t + 3)(t - 2)$, which is 0 at $t = -3$ and $t = 2$. Note that $p''(t) = 12t + 6$, so $p''(-3) = -30 < 0$ and $p''(2) = 30 > 0$, so there is a local maximum at $t = -3$ and a local minimum at $t = 2$.

4.3.90 $f'(x) = x^3 - 5x^2 - 8x + 48 = (x - 4)^2(x + 3)$. (This can be obtained by using trial-and-error to determine that $x = 4$ is a root, and then using long division of polynomials to see that $f'(x) = (x - 4)(x^2 - x - 12)$.) Note that $f''(x) = 3x^2 - 10x - 8$, so $f''(-3) = 49 > 0$ and $f''(4) = 0$. So there is a local minimum at $x = -3$, but the test is inconclusive for $x = 4$. The first derivative test shows that there is neither a maximum nor a minimum at $x = 4$.

4.3.91 $f'(x) = 4(x + a)^3$, which is 0 for $x = -a$. Note that $f''(x) = 12(x + a)^2$, which is 0 at $x = -a$, so the test is inconclusive. The first derivative test shows that there is a local minimum at $x = -a$.

4.3.92 $f'(x) = 3x^2 - 26x - 9 = (3x + 1)(x - 9)$. This is zero for $x = 9$ and $x = -\dfrac{1}{3}$, so those are the critical points. $f''(x) = 6x - 26$. $f''(9) > 0$ so there is a local minimum at $x = 9$, while $f''(-1/3) < 0$, so there is a local maximum at $x = -\dfrac{1}{3}$.

4.3.93 $f'(x) = 24x^3 \ln x^2 + 6x^4\left(\frac{2}{x}\right) - 28x^3 = 24x^3 \ln x^2 + 12x^3 - 28x^3 = 24x^3 \ln x^2 - 16x^3 = 8x^3(3\ln x^2 - 2)$.
Note that 0 is not in the domain of f, so the only values of x that makes f' 0 are those for which $\ln x^2 = \frac{2}{3}$, or $x^2 = e^{2/3}$, so $x = \pm e^{1/3}$.

$$f''(x) = 24x^2(3\ln x^2 - 2) + 8x^3\left(\frac{6}{x}\right) = 24x^2(3\ln x^2 - 2 + 2) = 72x^2 \ln x^2.$$

A check of f'' evaluated at either $\pm e^{1/3}$ shows a positive value, so there are local minimums at both $x = e^{1/3}$ and $x = -e^{1/3}$.

4.3.94 $f'(x) = -6x^{-4} + 2x^{-3} = \frac{-6}{x^4} + \frac{2x}{x^4} = \frac{2(x-3)}{x^4}$. $x = 3$ is the only critical point.

$$f''(x) = 24x^{-5} - 6x^{-4} = \frac{24}{x^5} - \frac{6x}{x^5} = \frac{6(4-x)}{x^5}.$$

We have $f''(3) > 0$, so there is a local minimum at $x = 3$.

4.3.95

a. True. $f'(x) > 0$ implies that f is increasing, and $f''(x) < 0$ implies that f' is decreasing. So f is increasing, but at a decreasing rate.

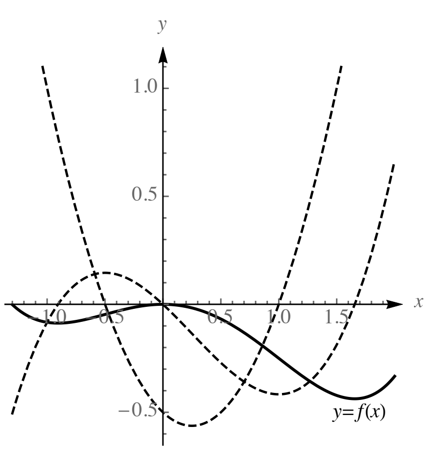
b. False. In fact, if $f'(c)$ exists and isn't zero, then there isn't any kind of local extrema at $x = c$.

c. True. In fact, if two functions differ by a constant, then all of their derivatives are the same.

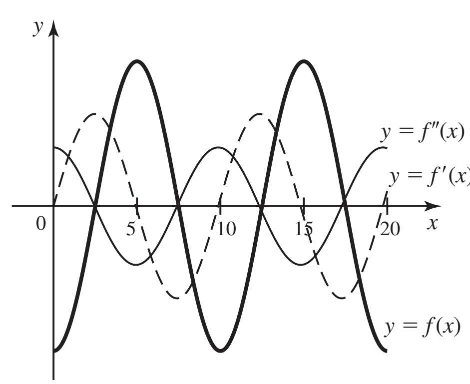
d. False. For example, consider $f(x) = x$ and $g(x) = x - 10$. Both are increasing, but $f(x)g(x) = x^2 - 10x$ is decreasing on $(-\infty, 5)$.

e. False. A continuous function with two local maxima must have a local minimum in between.

4.3.96

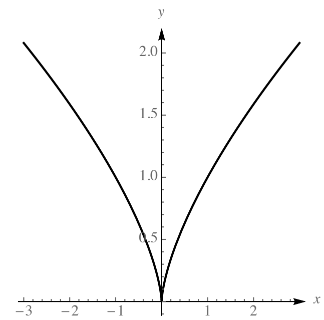


4.3.97

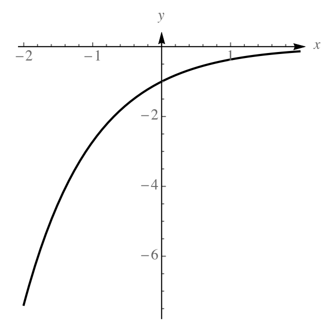


4.3.98

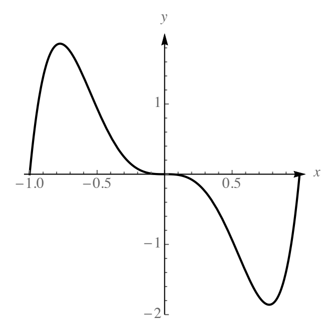
a. Not Possible. The closest thing would be a function like $f(x) = x^{2/3}$ which is positive and is concave down on $(-\infty, 0)$ and on $(0, \infty)$.



b. This is possible, for example $y = -e^{-x}$ has this property.



c. This is possible. See graphic pictured.



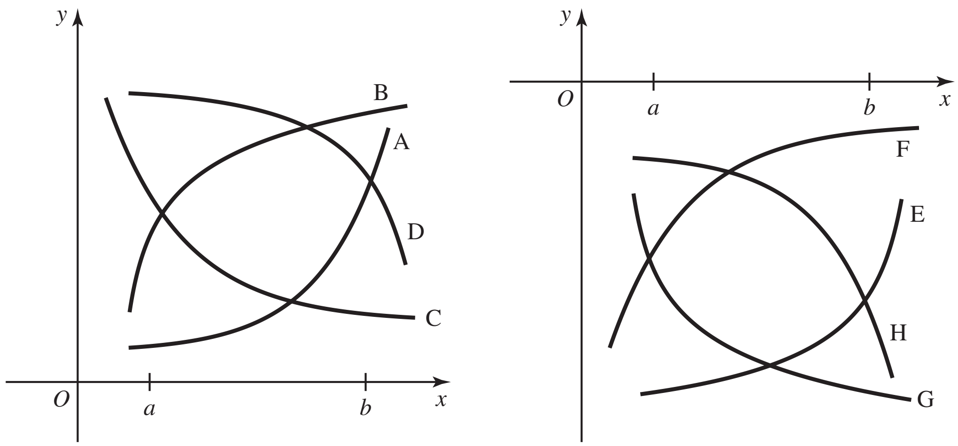
d. This is not possible. Between every pair of zeros, there must be a maximum or a minimum, so a continuous function with four zeros must have at least three local extrema.

4.3.99 The graphs match as follows: (a) – (f) – (g); (b) – (e) –(i); (c) – (d) –(h). Note that (a) is always increasing, so its derivative must be always positive, and (f) switches from decreasing to increasing at 0, so its derivative must be negative for $x < 0$ and positive for $x > 0$.

Note that (b) has three extrema where there are horizontal tangent lines, so its derivative must cross the x-axis three times, and (e) has two extrema, so its derivative must cross the x-axis two times.

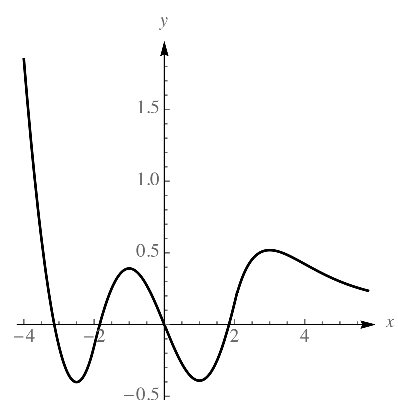
4.3.100 Note that C is increasing where B is positive, and is decreasing where B is negative, so it seems reasonable to assert that B is the derivative of C. Also, B is increasing where A is positive and decreasing where A is negative, so it is reasonable to assert that A is the derivative of B. So it appears that $C = f(x)$, $B = f'(x)$, and $A = f''(x)$.

4.3.101



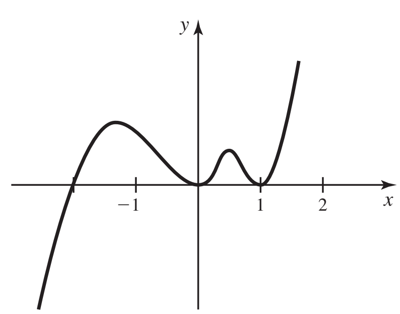
4.3.102

The graph sketched must be concave up on $(-\infty, -2)$ and on $(4, \infty)$, and must have a flat tangent line at $x = -1$, $x = 1$, and $x = 3$. A convenient way to ensure that $f''(-2) = f''(2) = 0$ is to have inflection points occur there. The example to the right is only one possible such graph.



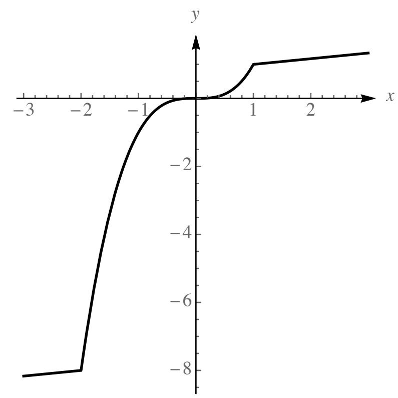
4.3.103

The graph sketched must have a flat tangent line at $x = -3/2$, $x = 0$, and $x = 1$, and must contain the points $(-2, 0)$, $(0, 0)$, and $(1, 0)$. The example to the right is only one possible such graph.



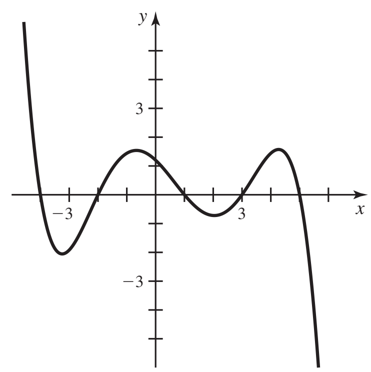
4.3.104

The graph sketched must be increasing everywhere and differentiable everywhere except at $x = -2$ and $x = 1$. We also have $f''(0) = 0$. The example to the right is only one possible such graph.

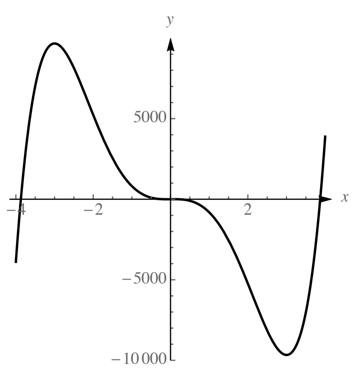


4.3.105

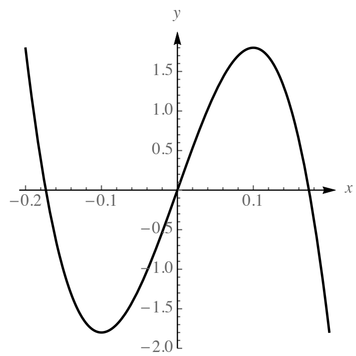
The graph sketched must be concave up on $(-\infty, -2)$ and on $(1, 3)$, and concave down on $(-2, 1)$ and on $(3, \infty)$. The example to the right is only one possible such graph.



4.3.106



It appears that there are 2 extreme values near $x = \pm 3$ and perhaps a flat tangent line corresponding to an inflection point at $(0, 0)$. However, note that $f'(x) = 300x^4 - 2703x^2 + 27 = 3(100x^4 - 901x^2 + 9) = 3(x^2 - 9)(100x^2 - 1) = 3(x - 3)(x + 3)(10x - 1)(10x + 1)$. It turns out that there are 4 critical points at $x = \pm 3$ and $x = \pm \frac{1}{10}$ (and there isn't one at $x = 0$). Below is a graph on the range $[-.2, .2]$, which clearly shows a maximum at $x = 0.1$ and a minimum at $x = -0.1$.



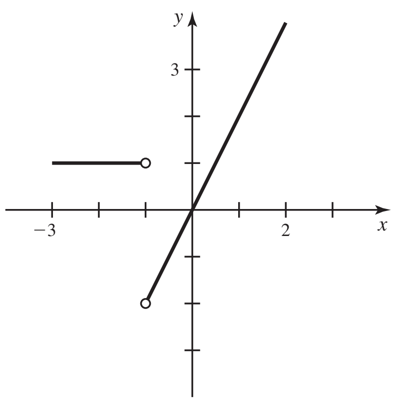
4.3.107

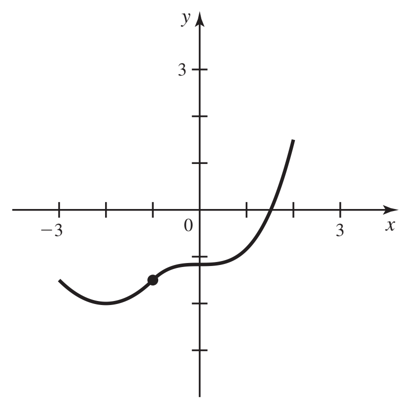
a. f is increasing on $(-2, 2)$. It is decreasing on $(-3, -2)$.

b. There are critical points of f at $x = -2$ and at $x = 0$. There is a local minimum at $x = -2$ and no extremum at $x = 0$.

c. There are inflection points of f at $x = -1$ and at $x = 0$.

d. f is concave up on $(-3, -1)$ and on $(0, 2)$, while it is concave down on $(-1, 0)$.

e. 

f. 

4.3.108

$$r'(t) = \frac{(37.98e^{-2.2t}+1)^2 \cdot (-2.2) \cdot 10147.9e^{-2.2t} - 10147.9e^{-2.2t} 2(37.98e^{-2.2t}+1) \cdot (-2.2)(37.98)e^{-2.2t}}{(37.98e^{-2.2t}+1)^4}.$$

After some algebra, this can be written as

$$\frac{e^{-2.2t}(847{,}998.7358e^{-2.2t} - 22{,}325.39)}{(37.98e^{-2.2t}+1)^3}.$$

The numerator is 0 when

$$e^{-2.2t} = \frac{22{,}325.39}{847{,}993.7358}$$

or at about 1.7 weeks. Using the First Derivative Test it can be verfied that this value maximizes r. The owlet is growing at its fastest rate when it is 1.7 weeks old.

4.3.109

a. $E = \frac{dD}{dp} \cdot \frac{p}{D} = -10\frac{p}{500 - 10p} = \frac{p}{p - 50}$.

b. $E = \dfrac{12}{12-50} \cdot .045 = -1.42\%$.

c. If $D(p) = a - bp$, then $E(p) = -b \cdot \dfrac{p}{a-bp} = \dfrac{bp}{bp-a}$. So $E'(p) = \dfrac{(bp-a)b - bpb)}{(bp-a)^2} = -\dfrac{ab}{(bp-a)^2}$, which is less than 0 for $a, b > 0$ and $p \neq a/b$.

d. If $D(p) = \dfrac{a}{p^b}$, then $E(p) = -\dfrac{ab}{p^{b+1}} \cdot \dfrac{p}{a/p^b} = -b$.

4.3.110 The growth rate is given by the slope of the tangent line to the curve. Up to the inflection point, the curve is concave up, meaning that the slopes are increasing. After the inflection point, the curve is concave down, meaning that the slopes are decreasing. So the maximum slope occurs at the inflection point.

4.3.111

a. $\displaystyle\lim_{t\to\infty} \frac{300t^2}{t^2+30} \cdot \frac{1/t^2}{1/t^2} = \lim_{t\to\infty} \frac{300}{1+(30/t^2)} = 300.$

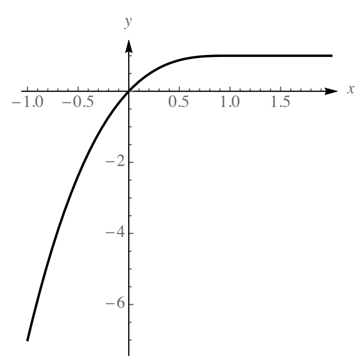
b. Note that $P'(t) = \frac{(t^2+30)(600t)-300t^2(2t)}{(t^2+30)^2} = \frac{18000t}{(t^2+30)^2}$. We want to maximize this, so we compute its derivative $P''(t) = \frac{(t^2+30)^2\cdot 18000 - 18000t\cdot 2(t^2+30)\cdot 2t}{(t^2+30)^4} = \frac{54000(10-t^2)}{(t^2+30)^3}$. This is 0 for $t = \sqrt{10}$, and an analysis of $P''(t)$ reveals that $P''(t) > 0$ for $t < \sqrt{10}$ and $P''(t) < 0$ for $t > \sqrt{10}$ so there is a local maximum for $P'(t)$ at $t = \sqrt{10}$.

c. Following the outline from the previous problem, we see that $P'(t) = \frac{2bKt}{(t^2+b)^2}$, and $P''(t) = \frac{2bK(b-3t^2)}{(t^2+b)^3}$. $P''(t)$ is 0 for $t = \sqrt{b/3}$, and the first derivative test reveals that this is a local maximum.

4.3.112 If f is concave up at $x = c$, then $f''(c) > 0$. This means that f' is increasing in a neighborhood near c. So for $x < c$, the slope of the function is less than the slope of the tangent line at c, and for $x > c$ it is greater than the slope of the tangent line. This means that the curve is "bending upward," away from its tangent line, so the tangent line is below the curve in a neighborhood near c.

4.3.113 $f'(x) = 4x^3 + 3ax^2 + 2bx + c$, and $f''(x) = 12x^2 + 6ax + 2b = 2(6x^2 + 3ax + b)$. Note that $f''(x) = 0$ exactly when $x = \frac{-3a\pm\sqrt{9a^2-24b}}{12}$. This represents no real solutions when $9a^2 - 24b < 0$, which occurs when $b > 3a^2/8$. When $b = 3a^2/8$, there is one root, but in this case the sign of f'' doesn't change at the double root $x = -a/4$, so there are no inflection points for f. In the case $b < 3a^2/8$, there are two roots of f'', both of which yield inflection points of f, as can be seen by the change in sign of f'' at its two roots.

4.3.114 One possible such function is $f(x) = \begin{cases} (x-1)^3 + 1 & \text{if } x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$



Note that the derivative of f to the left of 1 is positive, and to the right it is 0 (so it is neither positive nor negative). This example highlights the fact that the statement of the First Derivative Test does not include all possible cases for what might be happening on either side of the critical point. The missing cases include:

- f' changes from negative to 0 as x increases through c (in which case f has a local minimum at c).
- f' changes from positive to 0 as x increases through c (in which case f has a local maximum at c).
- f' changes from 0 to positive as x increases through c (in which case f has a local maximum at c).
- f' changes from 0 to negative as x increases through c (in which case f has a local minimum at c).
- f' remains 0 as x increases through c (in which case f has both a local minimum and local maximum at c – f is constant near c.

4.3.115

a. $f'(x) = 3x^2 + 2ax + b$, which is 0 when $x = \dfrac{-2a \pm \sqrt{4a^2 - 12b}}{6} = \dfrac{-a \pm \sqrt{a^2 - 3b}}{3}$. These solutions represent distinct real numbers when $a^2 > 3b$. Let the two distinct roots be $r_1 < r_2$. Note that f' is negative on the interval (r_1, r_2) and positive on $(-\infty, r_1)$ and on (r_2, ∞) so there is a maximum at r_1 and a minimum at r_2.

b. If $a^2 < 3b$, then there are no real critical points, so there are no extreme values.

4.3.116 $f'(x) = 2ax + b$, and $f''(x) = 2a$. Note that $f''(x)$ is positive for $a > 0$ and negative for $a < 0$. So f is concave up for $a > 0$ and concave down for $a < 0$.

4.4 Graphing Functions

4.4.1 Because the intervals of increase and decrease and the intervals of concavity must be subsets of the domain, it is helpful to know what the domain is at the outset.

4.4.2 If a function is symmetric, then only one half the function needs to be graphed, and the information about the other half will follow immediately. Also, knowledge about symmetry can help catch mistakes.

4.4.3 No. Polynomials are continuous everywhere, so they have no vertical asymptotes. Also, polynomials in x always tend to $\pm\infty$ as $x \to \pm\infty$.

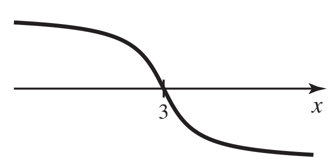
4.4.4 If a rational function is in simplified form (with no common factors in the numerator and denominator), then there is a vertical asymptote wherever the denominator is zero.

4.4.5 The maximum and minimum must occur at either an endpoint or a critical point. So to find the absolute maximum and minimum, it suffices to find all the critical points, and then compare the values of the function at those points and at the endpoints. The largest such value is the maximum and the smallest is the minimum.

4.4.6 For every polynomial $p(x)$, $\lim\limits_{x\to\pm\infty} p(x) = \pm\infty$.

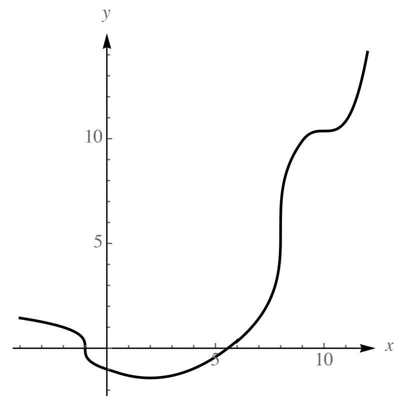
4.4.7

The function sketched should be decreasing and concave down for $x < 3$ and decreasing and concave up for $x > 3$.

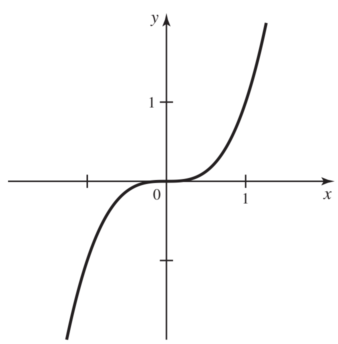


4.4.8

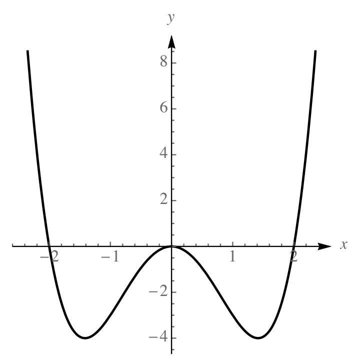
The function sketched should be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. It should be concave down on $(-\infty, -1)$ and on $(8, 10)$. It should be concave up on $(-1, 8)$ and on $(10, \infty)$.



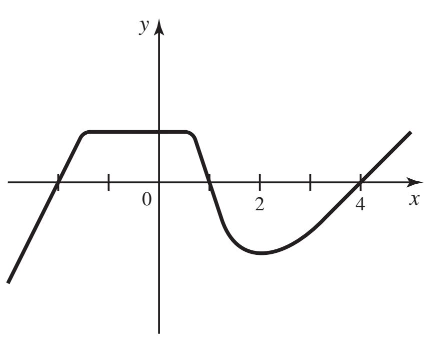
4.4.9



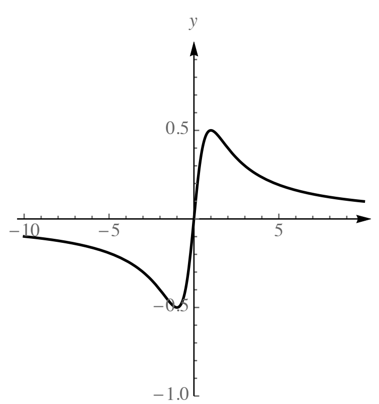
4.4.10



4.4.11



4.4.12



4.4.13

a. $f'(x) = 1 \cdot (x+3)^2 + (x-3)(2)(x+3) = (x+3)(x+3+2x-6) = (x+3)(3x-3) = 3(x-1)(x+3)$. Also, $f''(x) = 3(x+3) + 3(x-1) = 3x+9+3x-3 = 6x+6 = 6(x+1)$.

b. $f'(x) = 0$ for $x = 1$ and $x = -3$ (so those are the critical points), while $f''(x) = 0$ for $x = -1$, so that is the location of a potential inflection point.

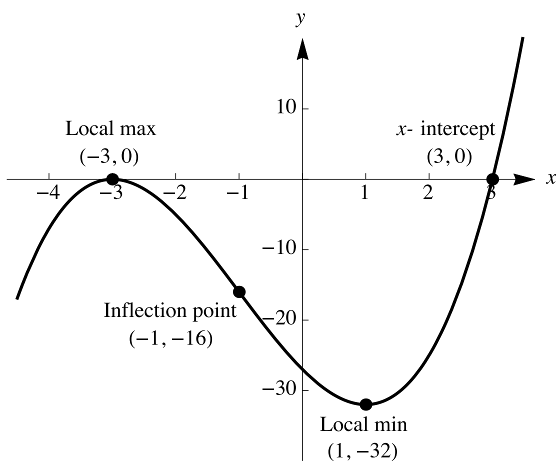
c. $f' > 0$ on $(-\infty, -3)$ and $(1, \infty)$, so f is increasing there. $f' < 0$ on $(-3, 1)$ so f is decreasing there.

d. $f'' > 0$ on $(-1, \infty)$ so f is concave up there; $f'' < 0$ on $(-\infty, -1)$, so f is concave down there.

e. f has a local max of $f(-3) = 0$ at $x = -3$ and a local minimum of $f(1) = -32$ at $x = 1$. The point $(-1, -16)$ is an inflection point.

f. The y-intercept is $f(0) = -27$. The x-intercepts are $x = 3, -3$.

g.



4.4.14

a. $f'(x) = 0$ for $x = 0$ (so that is the only critical point), while $f''(x = 0$ for $x = -1$, and $x = 1$, so those are the locations of potential inflection points.

b. $f' > 0$ on $(-\infty, 0)$, so f is increasing there. $f' < 0$ on $(0, \infty)$, so f is decreasing there.

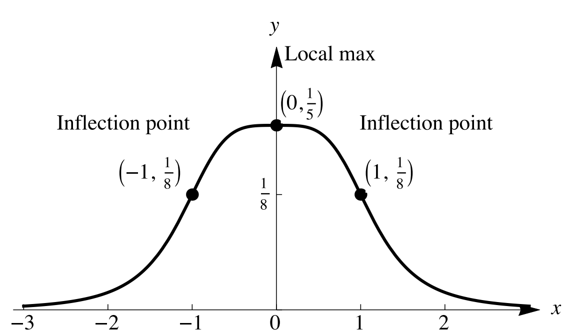
c. $f'' > 0$ on $(-\infty, -1)$ and $(1, \infty)$, so f is concave up there; $f'' < 0$ on $(-1, 1)$, so f is concave down there.

d. f has a local max of $f(0) = 1/5$ at $x = 0$. The points $(-1, 1/8)$ and $(1, 1/8)$ are inflection points.

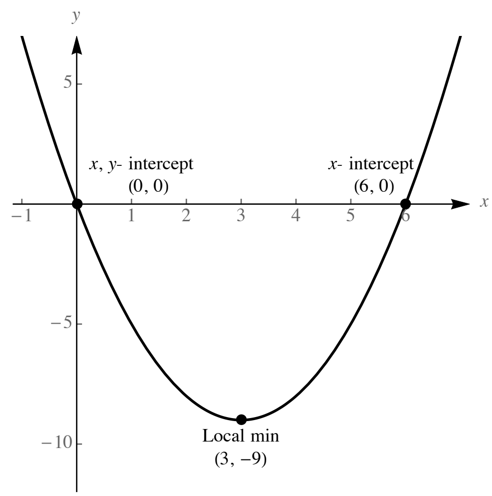
e. The y-intercept is $f(0) = 1/5$. There are no x-intercepts.

f. There are no vertical asymptotes. There s a horizontal asymptote of $y = 0$ because $\lim_{x \to \pm\infty} \dfrac{1}{3x^4 + 5} = 0$.

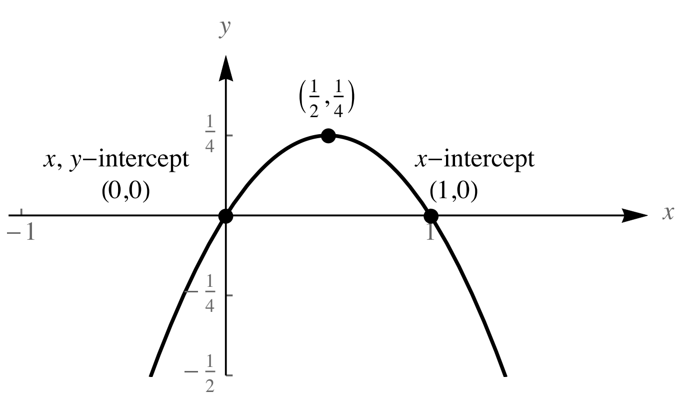
g.



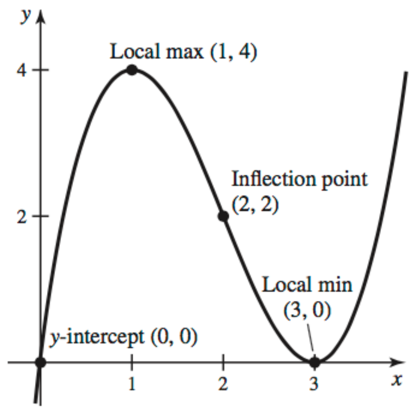
4.4.15 Because f is a polynomial, its domain is $(-\infty, \infty)$. There is neither even nor odd symmetry because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. $f'(x) = 2x - 6 = 2(x - 3)$ which is zero for $x = 3$. We have $f' > 0$ on $(3, \infty)$ so f is increasing there, while $f' < 0$ on $(-\infty, 3)$, so f is decreasing there, and there is a local minimum at $x = 3$. $f''(x) = 2$ which is always positive, so f is concave up on $(-\infty, \infty)$ and there are no inflection points. The local minimum value is $f(3) = -9$, the y-intercept is $(0, 0)$, which is also an x-intercept, and $(6, 0)$ is also an x-intercept.



4.4.16 Because f is a polynomial, its domain is $(-\infty, \infty)$. There is neither even nor odd symmetry because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. $f'(x) = 1 - 2x$ which is zero for $x = 1/2$. We have $f' > 0$ on $(-\infty, 1/2)$ so f is increasing there, while $f' < 0$ on $(1/2, -\infty)$, so f is decreasing there, and there is a local maximum at $x = 1/2$. $f''(x) = -2$ which is always negative, so f is concave down on $(-\infty, \infty)$ and there are no inflection points. The local maximum value is $f(1/2) = 1/4$, the y-intercept is $(0, 0)$, which is also an x-intercept, and $(1, 0)$ is also an x-intercept.

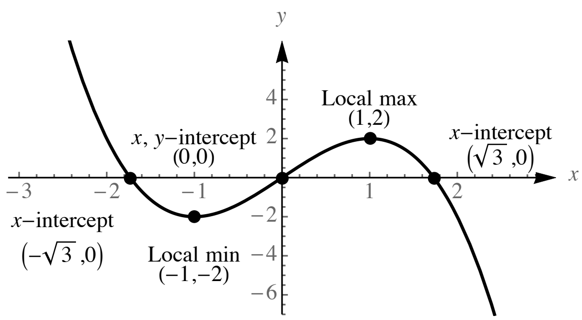


4.4.17 The domain of f is $(-\infty, \infty)$, and there is no symmetry. The y intercept is $f(0) = 0$, and the x-intercepts are 0 and 3 because $f(x) = x(x-3)^2$. $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$. This is zero when $x = 1$ and $x = 3$. Note that $f'(0) > 0$, $f'(2) < 0$ and $f'(4) > 0$, so f is increasing on $(-\infty, 1)$ and on $(3, \infty)$. It is decreasing on $(1, 3)$. Note that $f''(x) = 6x - 12$ which is zero at $x = 2$. Because $f''(1) < 0$ and $f''(3) > 0$, we conclude that f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. There is an inflection point at $(2, 2)$, a local maximum at $(1, 4)$ and a local minimum at $(3, 0)$.

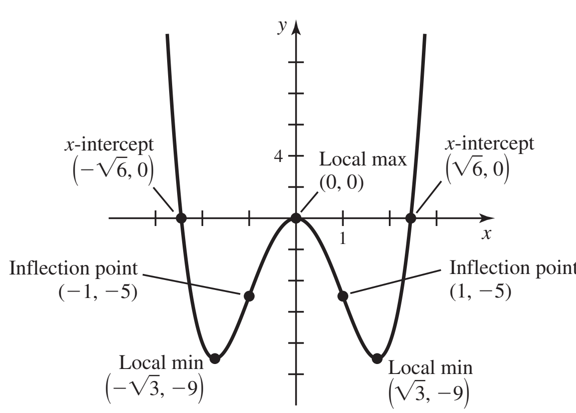


4.4.18 The domain of f is $(-\infty, \infty)$, and there is odd symmetry, because $f(-x) = 3(-x) - (-x)^3 = -(3x - x^3) = -f(x)$. The y intercept is $f(0) = 0$, and the x-intercepts are 0 and $\pm\sqrt{3}$ because $f(x) = x(3 - x^2)$. $f'(x) = 3 - 3x^2 = 3(1 - x^2)$, which is zero for $x = \pm 1$. Note that $f'(-2) < 0$, $f'(0) > 0$, and $f'(2) < 0$, so f is decreasing on $(-\infty, -1)$ and on $(1, \infty)$, and is increasing on $(-1, 1)$. There is a local minimum at $(-1, -2)$ and a local maximum at $(1, 2)$.

$f''(x) = -6x$, which is zero at $x = 0$. Note that $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$, so f is concave up on $(-\infty, 0)$ and is concave down on $(0, \infty)$. There is an inflection point at $(0, 0)$.

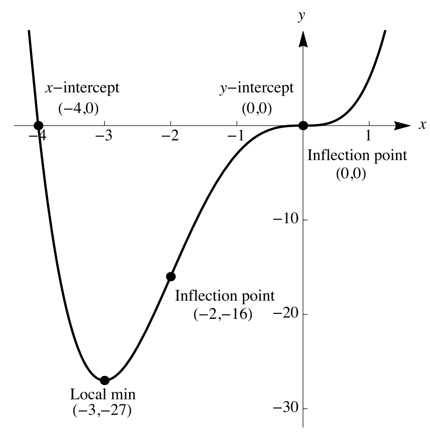


4.4.19 The domain of f is $(-\infty, \infty)$, and there is even symmetry, because $f(-x) = f(x)$. $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$. This is 0 when $x = \pm\sqrt{3}$ and when $x = 0$. $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$, which is 0 when $x = \pm 1$. Note that $f'(-2) < 0$, $f'(-1) > 0$, $f'(1) < 0$, and $f'(2) > 0$. So f is decreasing on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$. It is increasing on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$. There is a local maximum of 0 at $x = 0$ and local minima of -9 at $x = \pm\sqrt{3}$. Note also that $f''(x) > 0$ for $x < -1$ and for $x > 1$ and $f''(x) < 0$ for $-1 < x < 1$, so there are inflection points at $x = \pm 1$. Also, f is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and on $(1, \infty)$. There is a y-intercept at $f(0) = 0$ and x-intercepts where $f(x) = x^4 - 6x^2 = x^2(x^2 - 6) = 0$, which is at $x = \pm\sqrt{6}$ and $x = 0$.



4.4.20 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$. This is zero for $x = 0$ and $x = -3$. $f' > 0$ on $(-3, 0)$ and on $(0, \infty)$ so f is increasing there, while $f' < 0$ on $(-\infty, -3)$, so f is decreasing there. There is a local minimum at $x = -3$ with y-value $f(-3) = -27$.

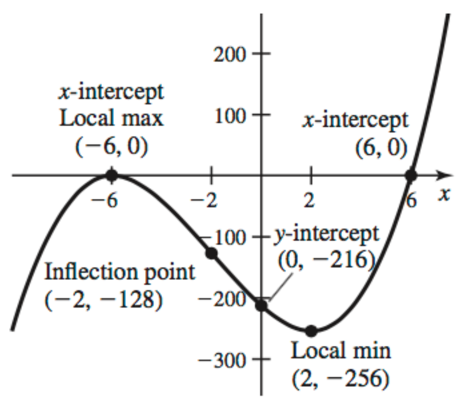
$f''(x) = 12x^2 + 24x = 12x(x+2)$ which is zero for $x = 0$ and $x = -2$. $f'' > 0$ on $(-\infty, -2)$ and on $(0, \infty)$, so f is concave up there, while $f'' < 0$ on $(-2, 0)$, so f is concave down there. There are inflection points at $(-2, -16)$ and $(0, 0)$. In addition to the x- and y-intercept at $(0, 0)$ there is an x-intercept at $x = -4$.



4.4.21 The domain of f is $(-\infty, \infty)$, and there is no symmetry. The y- intercept is $f(0) = -216$. The x-intercepts are 6 and -6.

$f'(x) = (x+6)^2 + (x-6)2(x+6) = (x+6)(x+6+2x-12) = (x+6)(3x-6) = 3(x+6)(x-2)$. The critical numbers are -6 and 2. Note that $f'(-7) > 0$, $f'(-2) < 0$, and $f'(3) > 0$, so f is increasing on $(-\infty, -6)$ and on $(2, \infty)$. It is decreasing on $(-6, 2)$. There is a local maximum of 0 at -6 and a local minimum of -256 at $x = 2$.

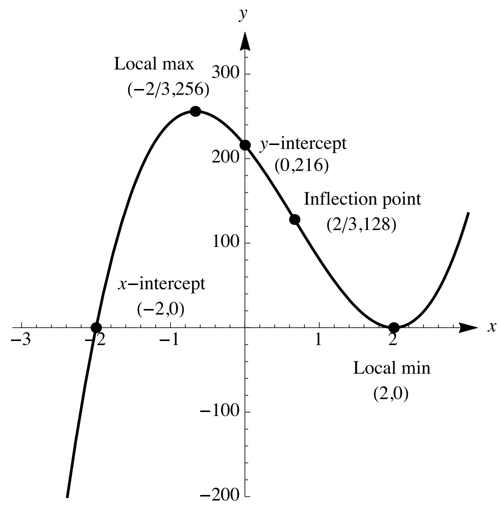
$f''(x) = 3(x-2) + 3(x+6) = 3(x-2+x+6) = 3(2x+4) = 6(x+2)$, which is zero for $x = -2$. Note that $f''(x) < 0$ for $x < -2$ and $f''(x) > 0$ for $x > -2$, so f is concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$. The point $(-2, -128)$ is an inflection point.



4.4.22 The domain of f is $(-\infty, \infty)$ and there is no symmetry. The y- intercept is $f(0) = 216$ and the x-intercepts are ± 2.

$f'(x) = 27 \cdot 2(x-2)(x+2) + 27(x-2)^2 = 27(x-2)(2x+4+x-2) = 27(x-2)(3x+2)$. This is zero for $x = 2$ and $x = -2/3$. Note that $f'(-1) > 0$, $f'(0) < 0$, and $f'(3) > 0$. Thus f is increasing on $(-\infty, -2/3)$ and on $(2, \infty)$, and is decreasing on $(-2/3, 2)$. There is a local maximum of 256 at $x = -2/3$ and a local minimum of 0 at $x = 2$.

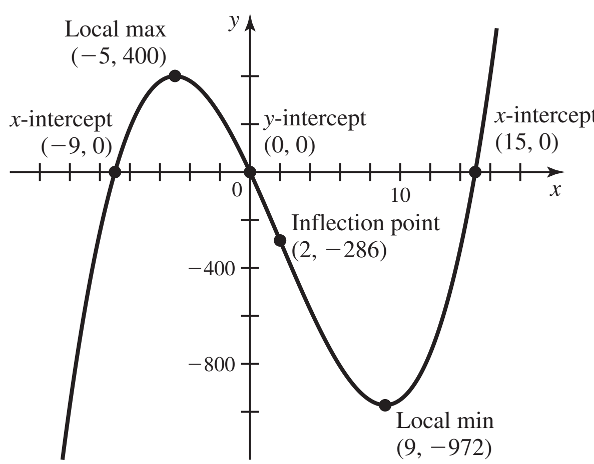
$f''(x) = 27(3x+2) + 27(x-2) \cdot 3 = 27(3x+2+3x-6) = 27(6x-4) = 54(3x-2)$. This is zero when $x = 2/3$. Note that $f''(x) < 0$ for $x < 2/3$ and $f''(x) > 0$ for $x > 2/3$, so f is concave down on $(-\infty, 2/3)$ and concave up on $(2/3, \infty)$. There is an inflection point at $(2/3, 128)$.



4.4.23 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 3x^2 - 12x - 135 = 3(x-9)(x+5)$, which is 0 for $x = 9$ and $x = -5$. $f'(x) > 0$ on $(-\infty, -5)$ and on $(9, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-5, 9)$, so f is decreasing on that interval. There is a local maximum at $x = -5$ and a local minimum at $x = 9$.

$f''(x) = 6x - 12$, which is 0 for $x = 2$. $f''(x) > 0$ on $(2, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty, 2)$, so f is concave down on that interval. There is a point of inflection at $x = 2$. The y-intercept is 0 and the x-intercepts are -9 and 15.



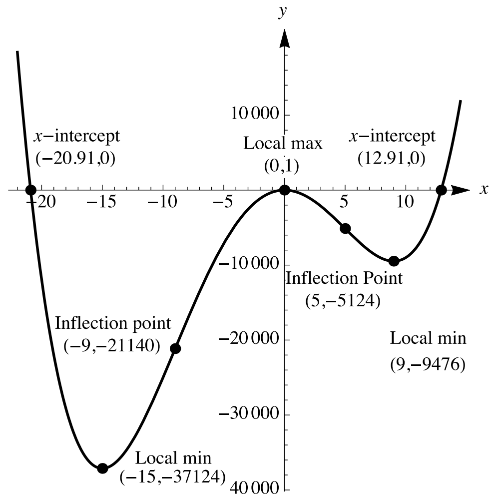
4.4.24 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 4x^3 + 24x^2 - 540x = 4x(x-9)(x+15)$, which is 0 for $x = 9$, $x = -15$, and $x = 0$.

$f'(x) > 0$ on $(-15, 0)$ and on $(9, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-\infty, -15)$ and on $(0, 9)$, so f is decreasing on those intervals. There is a local maximum at $x = 0$ and local minima at $x = -15$ and $x = 9$.

$f''(x) = 12x^2 + 48x - 540 = 12(x-5)(x+9)$, which is 0 for $x = 5$ and $x = -9$.

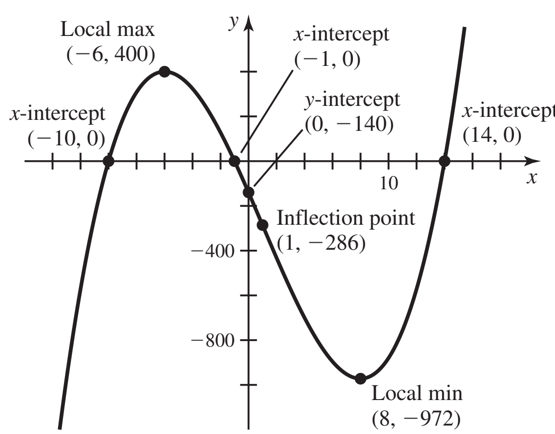
$f''(x) > 0$ on $(-\infty, -9)$ and on $(5, \infty)$, so f is concave up on those intervals. $f''(x) < 0$ on $(-9, 5)$ so f is concave down on that interval. There are points of inflection at $x = -9$ and $x = 5$. The y-intercept is 1 and the x-intercepts are ≈ -20.912 and ≈ 12.911. There are also x-intercepts at approximately ± 0.061.



4.4.25 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

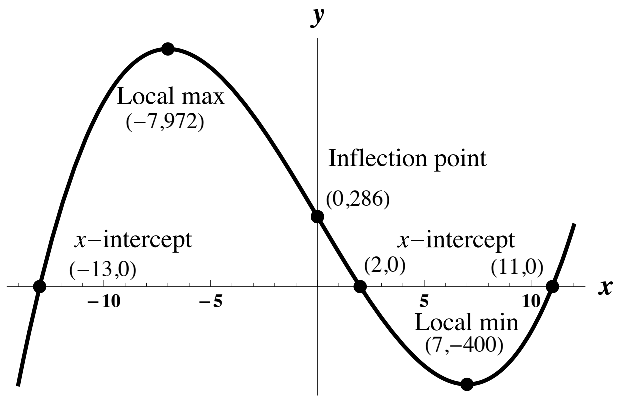
$f'(x) = 3x^2 - 6x - 144 = 3(x+6)(x-8)$, which is 0 for $x = -6$ and $x = 8$. $f'(x) > 0$ on $(-\infty, -6)$ and on $(8, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-6, 8)$, so f is decreasing on that interval. There is a local maximum at $x = -6$ and a local minimum at $x = 8$.

$f''(x) = 6x - 6$, which is 0 for $x = 1$. $f''(x) > 0$ on $(1, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty, 1)$, so f is concave down on that interval. There is a point of inflection at $x = 1$. The y-intercept is -140 and the x-intercepts are at -10, -1, and 14.



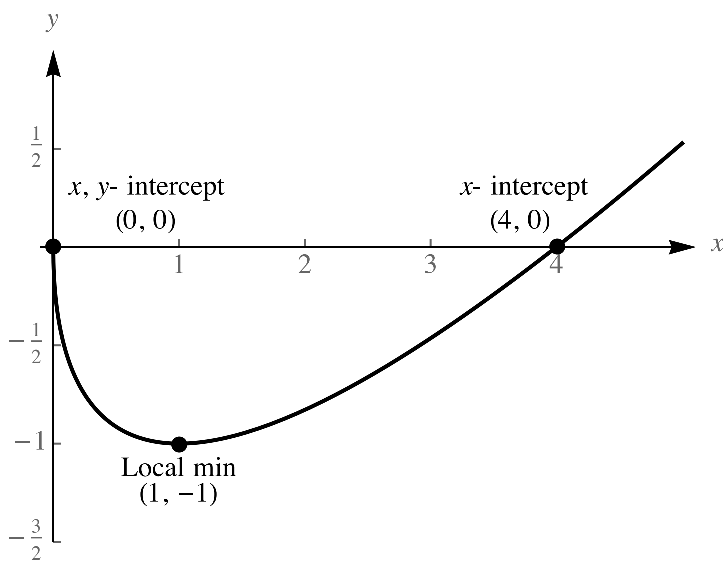
4.4.26 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial. $f'(x) = 3x^2 - 147 = 3(x+7)(x-7)$, which is 0 for $x = 7$ and $x = -7$. $f'(x) > 0$ on $(-\infty, -7)$ and on $(7, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-7, 7)$, so f is decreasing on that interval. There is a local maximum at $x = -7$ and a local minimum at $x = 7$.

$f''(x) = 6x$, which is 0 for $x = 0$. $f''(x) > 0$ on $(0, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(-\infty, 0)$, so f is concave down on that interval. There is a point of inflection at $x = 0$. The y-intercept is 286 and the x-intercepts are at -13, 2, and 11.



4.4.27 The domain of f is $[0, \infty)$ and there is no symmetry. $f'(x) = 1 - \dfrac{1}{\sqrt{x}} = \dfrac{\sqrt{x}-1}{\sqrt{x}}$ which is zero for $x = 1$. $f' > 0$ on $(1, \infty)$, so f is increasing there, while $f' < 0$ on $(0, 1)$, so f is decreasing there. There is a local minimum of -1 at $x = 1$.

If we write $f'(x)$ as $f'(x) = 1 - x^{-1/2}$, then we see that $f''(x) = \dfrac{1}{2}x^{-3/2} = \dfrac{1}{2x^{3/2}}$. This is never zero, and is always positive. So f is concave up and has no inflection points. The x- and y-intercept is $(0, 0)$ and there is another x-intercept at $(4, 0)$.

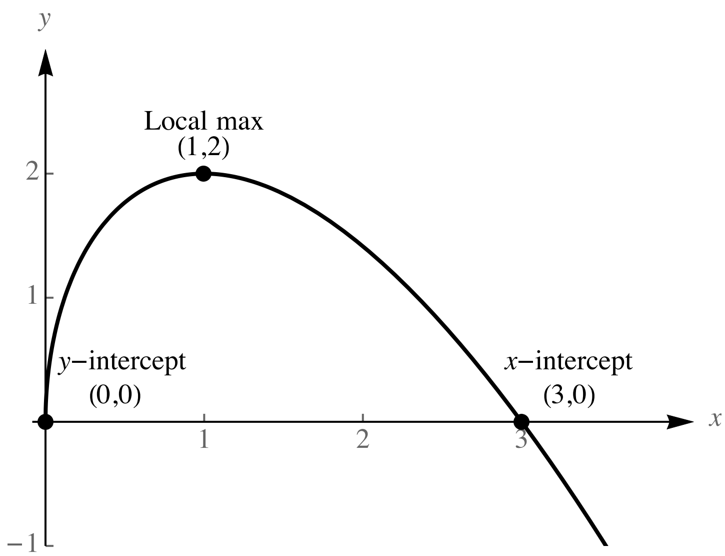


4.4.28 The domain of f is $[0, \infty)$ and there is no symmetry. $f'(x) = \dfrac{3}{2\sqrt{x}} - \dfrac{3\sqrt{x}}{2} = \dfrac{3(1-x)}{2\sqrt{x}}$ which is zero for $x = 1$. $f' < 0$ on $(1, \infty)$, so f is decreasing there, while $f' > 0$ on $(0, 1)$, so f is increasing there. There is a local maximum of 2 at $x = 1$.

If we write $f'(x)$ as $f'(x) = \dfrac{3}{2}x^{-1/2} - \dfrac{3}{2}x^{1/2}$, then we see that

$$f''(x) = -\frac{3}{4}x^{-3/2} - \frac{3}{4}x^{-1/2} = -\frac{3}{4}\left(\frac{1}{x^{3/2}} + \frac{1}{x^{1/2}}\right) = -\frac{3}{4}\left(\frac{x+1}{x^{3/2}}\right),$$

which is never zero on the domain of f, and is always negative. So f is always concave down and has no inflection points. The x- and y- intercept is $(0, 0)$, and another x-intercept is $(3, 0)$.



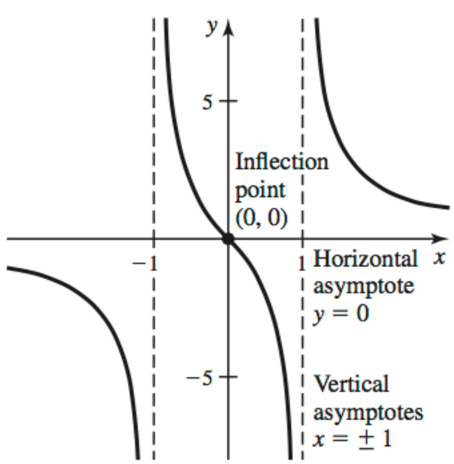
4.4.29 The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, and there is odd symmetry, because $f(-x) = \dfrac{3(-x)}{(-x)^2-1} = -\dfrac{3x}{x^2-1} = -f(x)$. The only intercept is $(0,0)$ which is both the y- and x-intercept.

Note that $\lim_{x\to-1^+} f(x) = \infty$ and $\lim_{x\to-1^-} f(x) = -\infty$, so there is a vertical asymptote at $x=-1$. Also, $\lim_{x\to1^+} f(x) = \infty$ and $\lim_{x\to1^-} f(x) = -\infty$, so there is a vertical asymptote at $x = 1$

Note that $\lim_{x\to\pm\infty} \dfrac{3x}{x^2-1}\cdot\dfrac{1/x^2}{1/x^2} = \lim_{x\to\pm\infty}\dfrac{3/x}{1-(1/x^2)} = 0$, so $y=0$ is a horizontal asymptote.

$f'(x) = \dfrac{(x^2-1)\cdot 3 - 3x\cdot 2x}{(x^2-1)^2} = \dfrac{-3x^2-3}{(x^2-1)^2} = -\dfrac{3(x^2+1)}{(x^2-1)^2}$. This is never 0, and is in fact negative wherever it is defined. Thus, f is decreasing on $(-\infty,-1)$, on $(-1,1)$, and on $(1,\infty)$. There are no extrema.

$f''(x) = \dfrac{(x^2-1)^2(-6x)+3(x^2+1)\cdot 2\cdot(x^2-1)\cdot 2x}{(x^2-1)^4} = \dfrac{-6x^3+6x+12x^3+12x}{(x^2-1)^3} = \dfrac{6x(x^2+3)}{(x^2-1)^3}$. This is 0 for $x=0$. The point $(0,0)$ is an point of inflection, because it is an interior point on the domain, and the second derivative changes from positive to negative there. The other concavity changes take place at the asymptotes. Note that f is concave down on $(-\infty,-1)$ and on $(0,1)$, and concave up on $(-1,0)$ and on $(1,\infty)$.

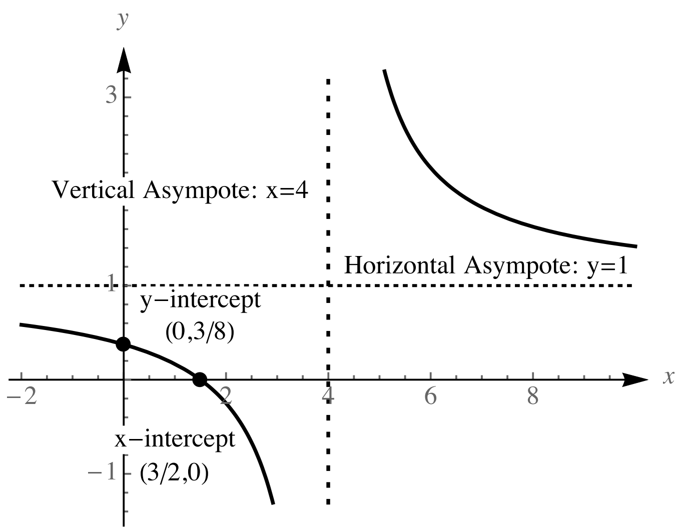


4.4.30 The domain of f is $(-\infty,4)\cup(4,\infty)$, and there is no symmetry.

Because $\lim_{x\to\pm\infty} \frac{2x-3}{2x-8} \cdot \frac{1/x}{1/x} = \lim_{x\to\pm\infty} \frac{2-(3/x)}{2-(8/x)} = \frac{2}{2} = 1$, there is a horizontal asymptote of $y = 1$. Also, because $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$, there is a vertical asymptote at $x = 4$. $f'(x) = \frac{(2x-8)\cdot 2 - (2x-3)\cdot 2}{(2x-8)^2} = -\frac{10}{(2x-8)^2}$. This is never 0. $f''(x) = \frac{40}{(2x-8)^3}$ which is also never 0.

Note that $f'(x) < 0$ on $(-\infty, 4)$ and on $(4, \infty)$. So f is decreasing on $(-\infty, 4)$ and on $(4, \infty)$. There are no extrema.

Note also that $f''(x) < 0$ for $x < 4$, and $f''(x) > 0$ for $x > 4$, so f is concave down on $(-\infty, 4)$ and is concave up on $(4, \infty)$. There are no inflection points because the only change in concavity occurs at the vertical asymptote. The x-intercept is $x = 3/2$ and the y-intercept is $f(0) = 3/8$.



4.4.31 The domain of f is $(-\infty, 2) \cup (2, \infty)$, and there is no symmetry. Note that $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, so there is a vertical asymptote at $x = 2$. There isn't a horizontal asymptote, because $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

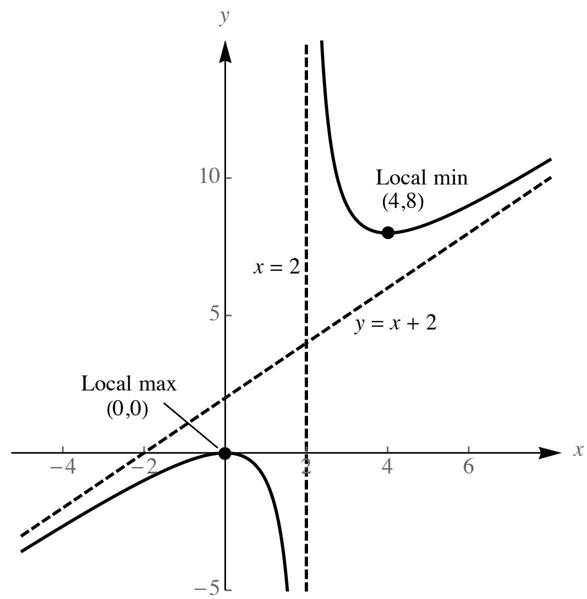
$f'(x) = \frac{(x-2)\cdot 2x - x^2}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$. This is 0 when $x = 4$ and when $x = 0$.

$f''(x) = \frac{(x-2)^2(2x-4) - (x^2-4x)\cdot 2\cdot (x-2)}{(x-2)^4} = \frac{8}{(x-2)^3}$. This is never 0.

Note that $f'(-1) > 0$, $f'(1) < 0$, $f'(3) < 0$ and $f'(5) > 0$. So f is decreasing on $(0, 2)$ and on $(2, 4)$. It is increasing on $(-\infty, 0)$ and on $(4, \infty)$. There is a local maximum of 0 at $x = 0$ and a local minimum of 8 at $x = 4$.

Note that $f''(x) > 0$ for $x > 2$ and $f''(x) < 0$ for $x < 2$, So f is concave up on $(2, \infty)$ and concave down on $(-\infty, 4)$. There are no inflection points, because the only change in concavity occurs at a vertical asymptote. The only intercept is $(0, 0)$.

By long division, we can write f as $x + 2 + \frac{4}{x-2}$. Therefore the line $y = x + 2$ is a slant asymptote.



4.4.32 The domain of f is $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$, and there is even symmetry because $f(-x) = \dfrac{(-x)^2}{(-x)^2-4} = \dfrac{x^2}{x^2-4} = f(x)$.

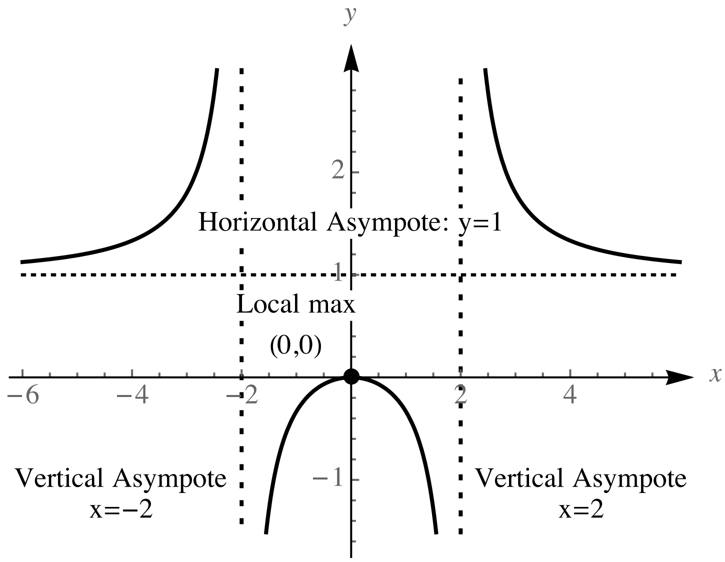
Because $\displaystyle\lim_{x\to\pm\infty} \frac{x^2}{x^2-4}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\pm\infty}\frac{1}{1-(4/x^2)} = 1$, there is a horizontal asymptote at $y=1$. Also, because $\lim_{x\to-2^-} f(x)=\infty$, $\lim_{x\to-2^+} f(x) = -\infty$, $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x)=\infty$, there are vertical asymptotes at $x=-2$ and $x=2$.

$f'(x) = \dfrac{(x^2-4)\cdot 2x - x^2\cdot 2x}{(x^2-4)^2} = -\dfrac{8x}{(x^2-4)^2}$. This is 0 when $x=0$.

$f''(x) = \dfrac{(x^2-4)^2(-8) - (-8x)\cdot 2\cdot(x^2-4)\cdot 2x}{(x^2-4)^4} = \dfrac{8(3x^2+4)}{(x^2-4)^3}$, which is never 0.

Note that $f'(x) > 0$ on $(-\infty,-2)$ and on $(-2,0)$, while $f'(x) < 0$ on $(0,2)$ and on $(2,\infty)$. So f is increasing on $(-\infty,-2)$ and on $(0,2)$, and is decreasing on $(0,2)$ and on $(2,\infty)$. There is a local maximum of 0 at $x=0$.

Note also that $f''(x) > 0$ for $x < -2$, and $f''(x) > 0$ for $x > 2$, while $f''(x) < 0$ for $-2 < x < 2$. So f is concave up on $(-\infty,-2)$ and on $(2,\infty)$, while it is concave down on $(-2,2)$. There are no inflection points because the only changes in concavity occur at asymptotes.



4.4.33 The domain of f is $(-\infty,-1/2)\cup(-1/2,\infty)$, and there is no symmetry.

Because $\displaystyle\lim_{x\to\pm\infty}\frac{x^2+12}{2x+1}\cdot\frac{1/x}{1/x}=\lim_{x\to\pm\infty}\frac{x+(12/x)}{2+(1/x)}=\pm\infty$, there is no horizontal asymptote. However, there is a slant asymptote of $y=\dfrac{x}{2}-\dfrac{1}{4}$, because we can write f as $f(x)=\dfrac{x}{2}-\dfrac{1}{4}+\dfrac{49/4}{2x+1}$ by long division.

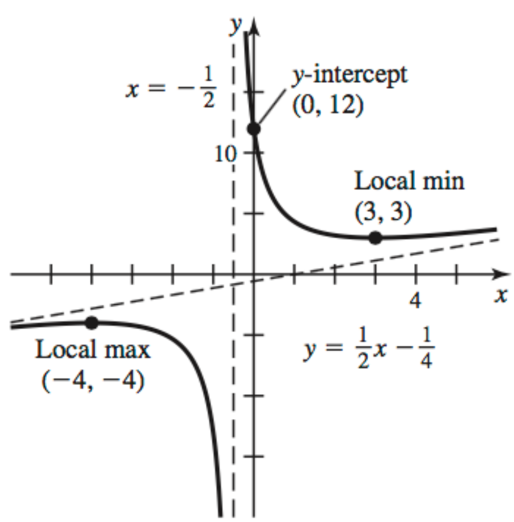
Also, because $\displaystyle\lim_{x\to(-1/2)^-}f(x)=-\infty$ and $\displaystyle\lim_{x\to(-1/2)^+}f(x)=\infty$, there is a vertical asymptote at $x=-1/2$.

$f'(x)=\dfrac{(2x+1)\cdot 2x-(x^2+12)\cdot 2}{(2x+1)^2}=\dfrac{2x^2+2x-24}{(2x+1)^2}=\dfrac{2(x+4)(x-3)}{(2x+1)^2}$. This is 0 for $x=-4$ and $x=3$.

$f''(x)=\dfrac{(2x+1)^2(4x+2)-(2x^2+2x-24)\cdot 2(2x+1)\cdot 2}{(2x+1)^4}=\dfrac{98}{(2x+1)^3}$, which is never 0.

Note that $f'(x)>0$ on $(-\infty,-4)$ and on $(3,\infty)$. So f is increasing on $(-\infty,-4)$ and on $(3,\infty)$. Also, $f'(x)<0$ on $(-4,-1/2)$ and on $(-1/2,3)$. So f is decreasing on those intervals. There is a local maximum of -4 at $x=-4$ and a local minimum of 3 at $x=3$.

Note also that $f''(x)<0$ for $x<-1/2$, and $f''(x)>0$ for $x>-1/2$, so f is concave down on $(-\infty,-1/2)$ and is concave up on $(-1/2,\infty)$. There are no inflection points because the only change in concavity occurs at the vertical asymptote. There are no x-intercepts because $x^2+12>0$ for all x, and the y-intercept is $f(0)=12$.



4.4.34 The domain of f is $(-\infty, \infty)$. There is odd symmetry because

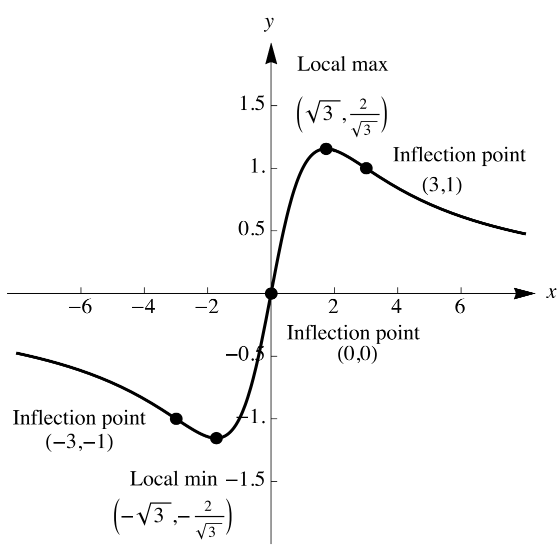
$$f(-x) = \frac{-4x}{(-x)^2+3} = -\frac{4x}{x^2+3} = -f(x).$$

The only intercept is $(0,0)$. We have $\lim_{x\to\infty} \frac{4x}{x^2+3} = \lim_{x\to\infty} \frac{4/x}{1+3/x^2} = 0$, so $y = 0$ is a horizontal asymptote.

$f'(x) = \frac{(x^2+3)4 - 4x(2x)}{(x^2+3)^2} = \frac{4(3-x^2)}{(x^2+3)^2}$. This is zero for $x = \pm\sqrt{3}$. $f' > 0$ on $(-\sqrt{3}, \sqrt{3})$ so f is increasing there, while $f' < 0$ on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$, so f is decreasing there. There is therefore a local maximum at $x = \sqrt{3}$ and a local minimum at $x = -\sqrt{3}$. The y value of the local maximum is $\frac{2}{\sqrt{3}}$ and for the minimum it is $-\frac{2}{\sqrt{3}}$

$$\begin{aligned} f''(x) &= \frac{(x^2+3)^2(-8x) - 4(3-x^2)(2)(x^2+3)(2x)}{(x^2+3)^4} \\ &= \frac{-8x(x^2+3)(x^2+3+2(3-x^2))}{(x^2+3)^4} \\ &= \frac{-8x(9-x^2)}{(x^2+3)^3} \\ &= \frac{8x(x-3)(x+3)}{(x^2+3)^3}. \end{aligned}$$

This is zero for $x = -3, 0, 3$, $f'' < 0$ on $(-\infty, -3)$ and on $(0, 3)$ so f is concave down there, while $f' > 0$ on $(-3, 0)$ and on $(3, \infty)$, so f is concave up there. There are inflection points at $(-3, -1)$, $(0, 0)$, and $(3, 1)$.



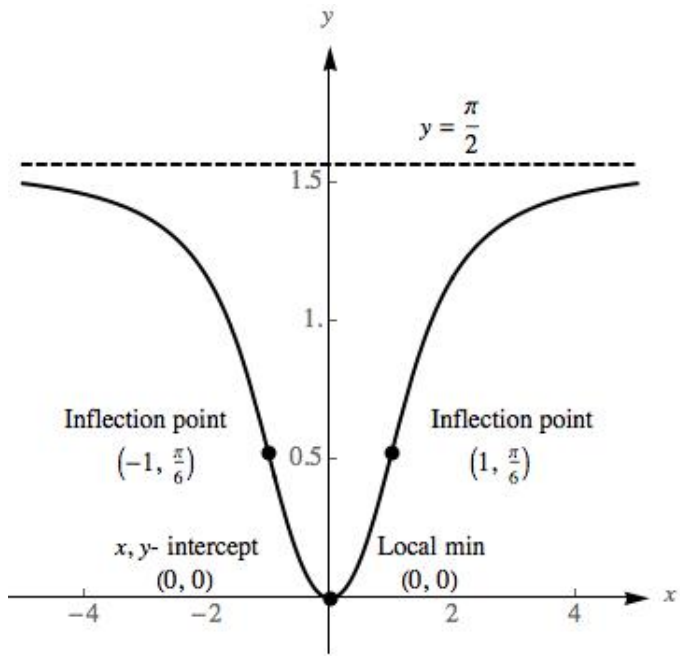
4.4.35 The domain of f is $(-\infty, \infty)$. There is even symmetry because

$$f(-x) = \tan^{-1}\left(\frac{(-x)^2}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right) = f(x).$$

The only intercept is $(0,0)$. $f'(x) = \frac{1}{x^4/3+1} \cdot \frac{2x}{\sqrt{3}} = \frac{2\sqrt{3}x}{x^4+3}$. This is zero for $x = 0$, and $f' > 0$ on $(0, \infty)$ so f is increasing there, while $f' < 0$ on $(-\infty, 0)$ so f is deceasing there. There is a local minimum of 0 at $x = 0$.

$f''(x) = \dfrac{(x^4+3)(2\sqrt{3}) - 2\sqrt{3}x(4x^3)}{(x^4+3)^2} = \dfrac{2\sqrt{3}(3-3x^4)}{(x^4+3)^2} = \dfrac{6\sqrt{3}(1-x)(1+x)(1+x^2)}{(x^4+3)^2}$. This is zero for $x = 1$ and $x = -1$. $f'' < 0$ on $(-\infty, -1)$ and on $(1, \infty)$ so f is concave down there, while $f'' > 0$ on $(-1, 1)$, so f is concave up there. There are inflection points at $(1, \pi/6)$ and $(-1, \pi/6)$.

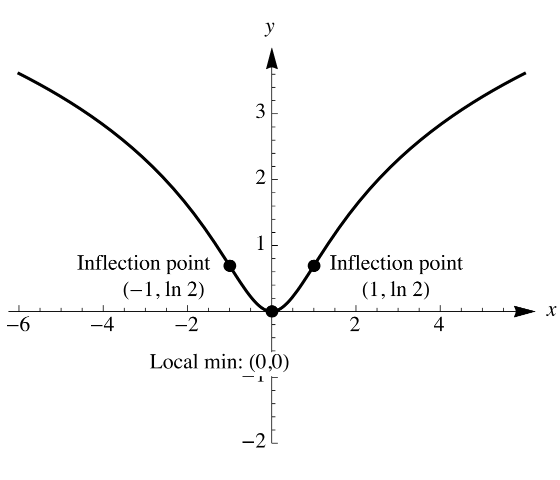
$\lim_{x\to\pm\infty} f(x) = \dfrac{\pi}{2}$, so $y = \dfrac{\pi}{2}$ is a horizontal asymptote.



4.4.36 The domain of f is $(-\infty, \infty)$, and f is symmetric about the y-axis, because $f(-x) = \ln((-x)^2+1) = f(x)$. The only intercept is $(0,0)$. Note that $\lim_{x\to\pm\infty} \ln(x^2+1) = \infty$, so there are no horizontal asymptotes.

We have $f'(x) = \dfrac{2x}{1+x^2}$, which is zero for $x = 0$. Note that f' is negative when x is negative and positive when x is positive, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, and there is a local (in fact, absolute) minimum at $(0,0)$.

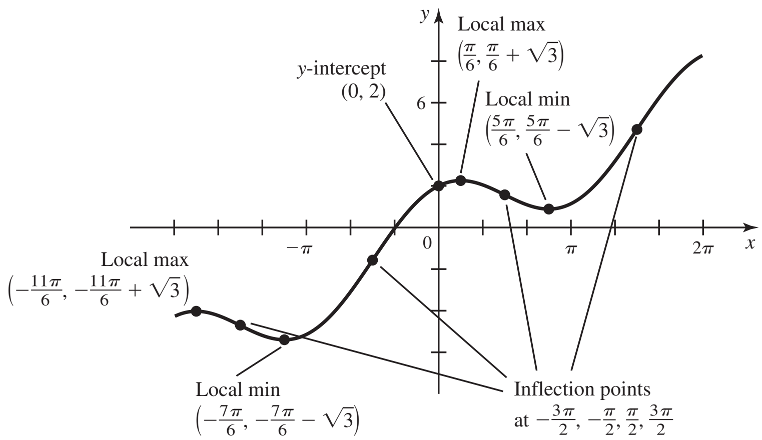
$f''(x) = \dfrac{(1+x^2)\cdot 2 - 2x \cdot 2x}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2}$. This is zero for $x = \pm 1$. Also note that $f''(-2) < 0$, $f''(0) > 0$, and $f''(2) < 0$, so f is concave down on $(-\infty, -1)$ and on $(1, \infty)$, and is concave up on $(-1, 1)$. There are inflection points at $(-1, \ln 2)$ and $(1, \ln 2)$.



4.4.37 The domain of f is given to be $[-2\pi, 2\pi]$, and there is no symmetry, and no vertical asymptotes. There are no horizontal asymptotes to consider on this restricted domain.

$f'(x) = 1 - 2\sin x$. This is 0 when $\sin x = 1/2$, which occurs on the given interval for $x = -11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. $f''(x) = -2\cos x$, which is 0 for $x = -3\pi/2, -\pi/2, \pi/2$, and $3\pi/2$.

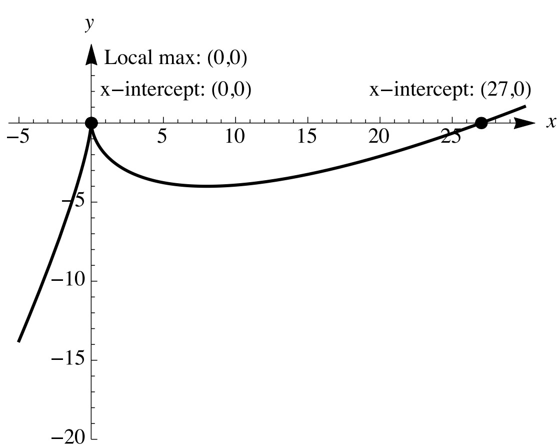
Note that $f'(x) > 0$ on $(-2\pi, -11\pi/6)$, and on $(-7\pi/6, \pi/6)$, and on $(5\pi/6, 2\pi)$. So f is increasing on those intervals, while $f'(x) < 0$ on $(-11\pi/6, -7\pi/6))$ and on $(\pi/6, 5\pi/6)$, so f is decreasing there. f has local maxima at $x = -11\pi/6$ and at $x = \pi/6$ and local minima at $x = -7\pi/6$ and at $x = 5\pi/6$. Note also that $f''(x) < 0$ on $(-2\pi, -3\pi/2)$ and on $(-\pi/2, \pi/2)$ and on $(3\pi/2, 2\pi)$, so f is concave down on those intervals, while $f''(x) > 0$ on $(-3\pi/2, -\pi/2)$ and on $(\pi/2, 3\pi/2)$, so f is concave up there and there are inflection points at $x = \pm 3\pi/2$ and $x = \pm\pi/2$. The y-intercept is $f(0) = 2$ and the x-intercept is at approximately -1.030.



4.4.38 The domain of f is $(-\infty, \infty)$. There are no asymptotes. There are x-intercepts at $(0,0)$ and $(27,0)$.

$f'(x) = 1 - \frac{2}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}-2}{\sqrt[3]{x}}$. This is undefined at $x = 0$, and is equal to zero at $x = 8$. Note that $f'(-1) > 0$, $f'(1) < 0$, and $f'(27) > 0$, so f is increasing on $(-\infty, 0)$, decreasing on $(0, 8)$, and increasing on $(8, \infty)$. There is a local maximum at $(0,0)$ and a local minimum at $(8, -4)$.

$f''(x) = \frac{2}{3\sqrt[3]{x^4}}$, which is never zero, but is undefined at $x = 0$. Because this is always positive, f is concave up on $(-\infty, 0)$ and on $(0, \infty)$. There are no inflection points.

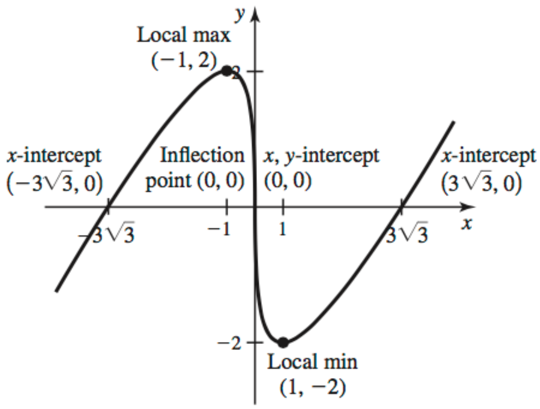


4.4.39 The domain of f is $(-\infty, \infty)$. There are no asymptotes. There are x-intercepts at $(0,0)$ and $(\pm 3\sqrt{3}, 0)$. f does have odd symmetry, because $f(-x) = -x - 3((-x)^{1/3}) = -(x - 3x^{1/3}) = -f(x)$.

$f'(x) = 1 - \dfrac{1}{\sqrt[3]{x^2}} = \dfrac{\sqrt[3]{x^2} - 1}{\sqrt[3]{x^2}}$. This is undefined at $x = 0$, and is equal to zero at ± 1. Note that $f'(-2) > 0$, $f'(-1/2) < 0$, $f'(1/2) < 0$, $f'(2) > 0$. Thus, f is increasing on $(-\infty, -1)$ and on $(1, \infty)$. Because f is continuous at 0 (even though f' doesn't exist there), we can combine the intervals $(-1, 0)$ and

$(0,1)$ and state that f is decreasing on $(-1,1)$. There is a local maximum at $(-1,2)$ and a local minimum at $(1,-2)$.

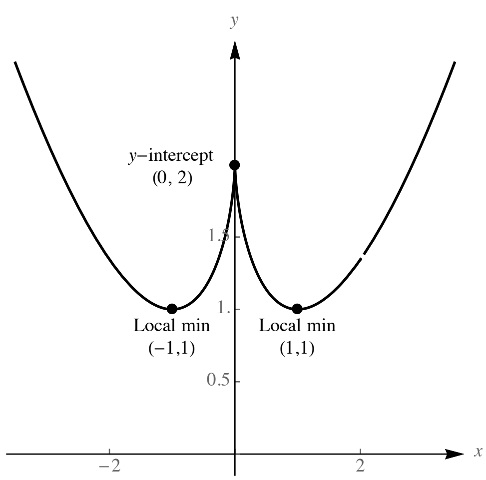
$f''(x) = \dfrac{2}{3\sqrt[3]{x^5}}$, which is never zero, but is undefined at $x=0$. Note that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is concave down on $(-\infty, 0)$ and is concave up on $(0,\infty)$. There is an inflection point at $(0,0)$.



4.4.40 The domain of f is $(-\infty,\infty)$. There are no asymptotes, nor x-intercepts. f does have even symmetry, because $f(-x) = 2 - 2(-x)^{2/3} + (-x)^{4/3} = 2 - 2x^{2/3} + x^{4/3} = f(x)$.

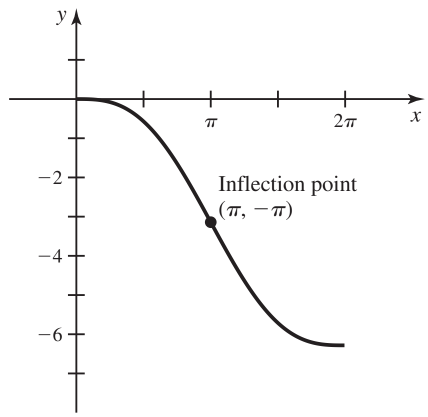
$f'(x) = -\dfrac{4}{3}x^{-1/3} + \dfrac{4}{3}x^{1/3} = -\dfrac{4}{3}\left(\dfrac{1}{x^{1/3}} - \dfrac{x^{1/3}}{1}\right) = -\dfrac{4}{3}\left(\dfrac{1-x^{2/3}}{x^{1/3}}\right)$. The numerator is zero for $x = \pm 1$, so these are critical points, and the denominator is zero at 0, and 0 is in the domain of the original function, so 0 is also a critical point. $f' > 0$ on $(-1,0)$ and on $(1,\infty)$ so f is increasing there, while $f' < 0$ on $(-\infty,-1)$ and on $(0,1)$, so f is decreasing there. There is a local max of 2 at $x=0$ and local minimums of 1 at $x = \pm 1$.

Working from $f'(x) = -\dfrac{4}{3}x^{-1/3} + \dfrac{4}{3}x^{1/3}$, we have $f''(x) = \dfrac{4}{9}x^{-4/3} + \dfrac{4}{9}x^{-2/3} = \dfrac{4(x^{2/3}+1)}{9x^{4/3}}$. $f'' > 0$ on $(-\infty,0)$ and on $(0,\infty)$, so there are no inflection points.



4.4.41 The domain of f is given to be $[0, 2\pi]$, so questions about symmetry and horizontal asymptotes aren't relevant. There are no vertical asymptotes.

$f'(x) = \cos x - 1$. This is never 0 on $(0, 2\pi)$. $f''(x) = -\sin x$, which is 0 on the given interval only for $x = \pi$. Note that $f'(x) < 0$ on $(0, 2\pi)$, so f is decreasing on the given interval and there are no relative extrema. Note also that $f''(x) < 0$ on $(0,\pi)$ and $f''(x) > 0$ on $(\pi, 2\pi)$, so f is concave down on $(0,\pi)$ and is concave up on $(\pi, 2\pi)$, and there is an inflection point at $x = \pi$. The only intercept is the origin $(0,0)$.



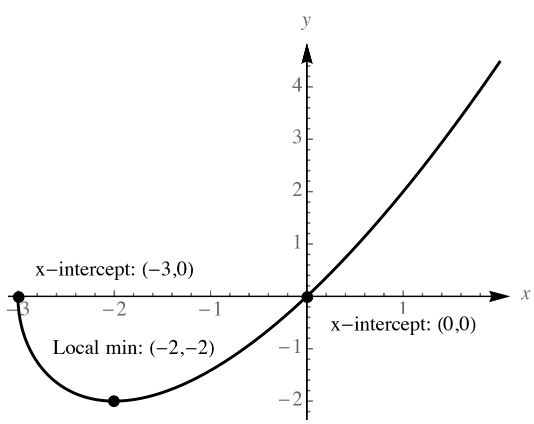
4.4.42 The domain of f is $[-3, \infty)$, and there is no symmetry, and there are no vertical asymptotes. Note that $\lim_{x \to \infty} f(x) = \infty$, so there are no horizontal asymptotes.

$f'(x) = x \cdot (1/2) \cdot (x+3)^{-1/2} + (x+3)^{1/2} = \dfrac{3x+6}{2\sqrt{x+3}}$. This is 0 for $x = -2$.

$f''(x) = \dfrac{2(x+3)^{1/2} \cdot 3 - (3x+6)(x+3)^{-1/2}}{4(x+3)} = \dfrac{3x+12}{4(x+3)^{3/2}}$. The numerator is zero for $x = -4$, but that number isn't in the domain of f.

Note that $f'(x) < 0$ on $(-3, -2)$ and $f'(x) > 0$ on $(-2, \infty)$, so f is decreasing on $(-3, -2)$ and increasing on $(-2, \infty)$, and there is a local (and absolute) minimum of -2 at $x = -2$.

Note also that $f''(x) > 0$ for all x in the domain of f, so f is concave up on its domain, and there are no inflection points. The function is equal to zero at the x-intercepts $(-3, 0)$ and $(0, 0)$, and the latter is also the y-intercept.

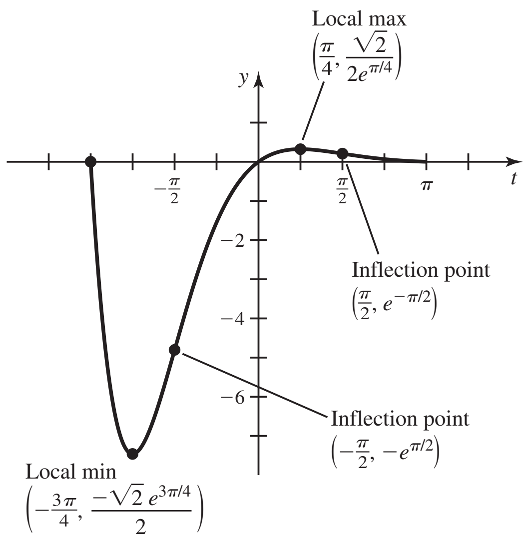


4.4.43 The domain of g is given to be $[-\pi, \pi]$, and there is neither symmetry nor vertical asymptotes. Because the domain is finite, questions about horizontal asymptotes are not relevant.

$g'(t) = e^{-t} \cos t + \sin t \cdot (-e^{-t}) = e^{-t} (\cos t - \sin t)$. This is 0 on the given interval for $t = -3\pi/4$ and $t = \pi/4$. $g''(t) = e^{-t}(-\sin t - \cos t) + (\cos t - \sin t)(-e^{-t}) = -2e^{-t} \cos t$, which is 0 for $t = -\pi/2$ and $t = \pi/2$.

Note that $g'(t) < 0$ on $(-\pi, -3\pi/4)$ and on $(\pi/4, \pi)$, so g is decreasing on those intervals. On $(-3\pi/4, \pi/4)$ we have $g'(t) > 0$ and so g is increasing. There is a local minimum of about -7.460 at $t = -3\pi/4$ and a local maximum of about 0.322 at $t = \pi/4$.

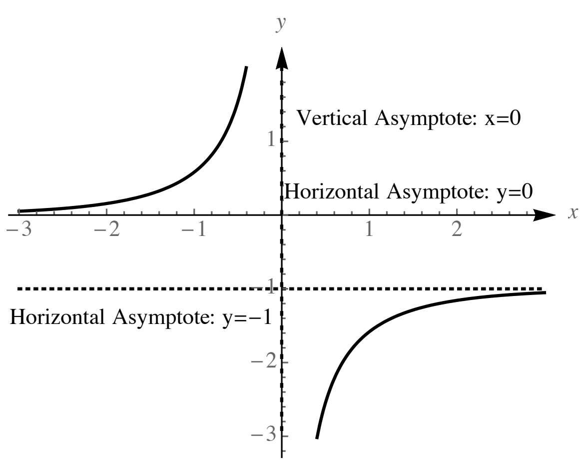
Note also that $g''(t) > 0$ on $(-\pi, -\pi/2)$ and on $(\pi/2, \pi)$, while $g''(t) < 0$ on $(-\pi/2, \pi/2i)$, so g is concave down on $(-\pi/2, \pi/2)$ and is concave up on $(\pi/2, \pi)$ and on $(-\pi, -\pi/2)$. There are inflection points at $t = \pm\pi/2$. The origin is both the $y-$intercept and an x-intercept. The endpoints are x-intercepts as well.



4.4.44 The domain of f is $(-\infty, 0) \cup (0, \infty)$, and f has no symmetry. Note that $\lim_{x \to \infty} \frac{1}{e^{-x} - 1} = \frac{1}{0-1} = -1$, so $y = -1$ is a horizontal asymptote as $x \to \infty$. Also, $\lim_{x \to -\infty} \frac{1}{e^{-x} - 1} = 0$, so $y = 0$ is a horizontal asymptote as $x \to -\infty$.

$f'(x) = -(e^{-x} - 1)^{-2}(-e^{-x}) = \frac{e^{-x}}{(e^{-x} - 1)^2}$, which is never 0, and is positive on $(-\infty, 0)$ and on $(0, \infty)$, so f is increasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. Thus f has no extrema.

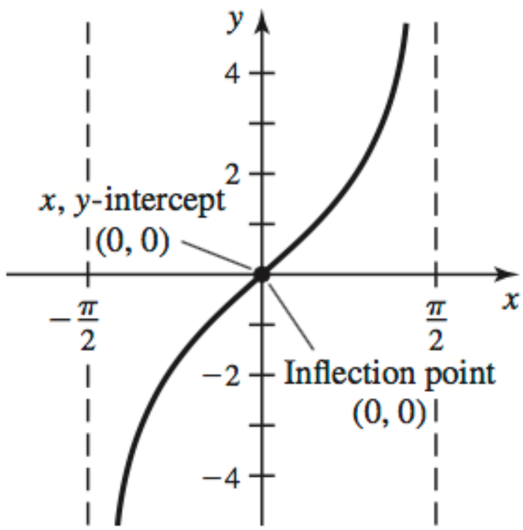
$f''(x) = \frac{(e^{-x} - 1)^2(-e^{-x}) - e^{-x} \cdot 2 \cdot (e^{-x} - 1)(-e^{-x})}{(e^{-x} - 1)^4} = \frac{e^{-x}(e^{-x} + 1)}{(e^{-x} - 1)^3}$, which is never 0. It is positive on $(-\infty, 0)$ and negative on $(0, \infty)$, so there are no inflection points as the only change in concavity occurs at the vertical asymptote, where the concavity changes from concave up (for $x < 0$) to concave down (for $x > 0$).



4.4.45 Note that $f(-x) = -x + \tan(-x) = -x - \tan x = -(x + \tan x) = -f(x)$, so f has odd symmetry. f has vertical asymptotes at $x = \pm\pi/2$, because the tangent function increases or decreases without bound as x approaches these values.

$f'(x) = 1 + \sec^2 x$ which is always greater than 0. Thus f is increasing on each interval on which it is defined, and it has no extrema. $f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$. This is 0 at $x = 0$.

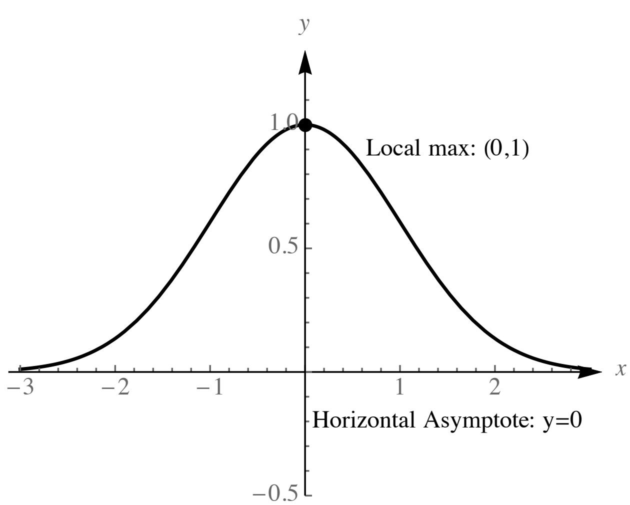
Note that $f''(x)$ is positive on $(0, \pi/2)$, so f is concave up there. Also, $f''(x)$ is negative on $(-\pi/2, 0)$, so f is concave down there. There is a point of inflection at $(0, 0)$.



4.4.46 The domain of f is $(-\infty, \infty)$. Note that $f(-x) = e^{-(-x)^2/2} = e^{-x^2/2} = f(x)$, so f has even symmetry. There are no vertical asymptotes, but $\lim_{x\to\pm\infty} \frac{1}{e^{x^2}} = 0$, so the x-axis is a horizontal asymptote.

$f'(x) = -xe^{-x^2/2}$, which is 0 only for $x = 0$, and is positive on $(-\infty, 0)$ and is negative on $(0, \infty)$, so f is increasing on $(-\infty, 0)$ and is decreasing on $(0, \infty)$, so there is a local maximum which is actually an absolute maximum of $f(0) = 1$ at $x = 0$.

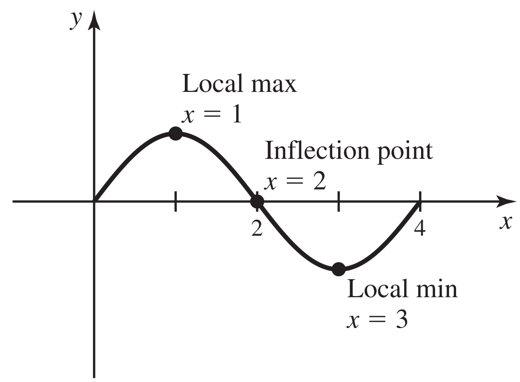
$f''(x) = -x \cdot (-xe^{-x^2/2}) + e^{-x^2/2}(-1) = e^{-x^2/2}(x^2 - 1)$ which is 0 only for $x = \pm 1$. Note that $f''(x) < 0$ on $(-1, 1)$ (so f is concave down there) and $f''(x) > 0$ on $(-\infty, -1)$ and on $(1, \infty)$, where f is concave up. There are inflection points at $x = \pm 1$, and there are no x-intercepts. There is a y-intercept at $(0, 1)$.



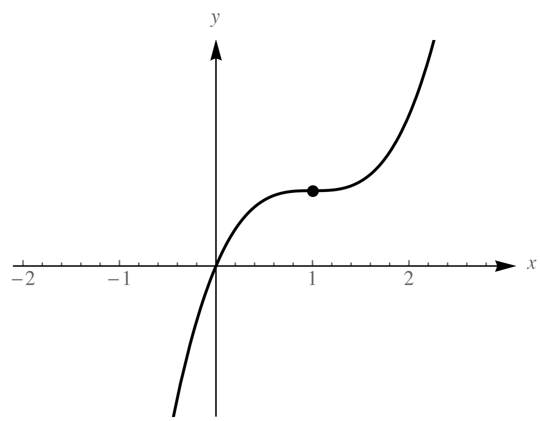
4.4.47 $f'(x)$ is 0 at $x = 1$ and $x = 3$.

$f'(x) > 0$ on $(0, 1)$ and on $(3, 4)$, so f is increasing on those intervals. $f'(x) < 0$ on $(1, 3)$, so f is decreasing on that interval. There is a local maximum at $x = 1$ and a local minimum at $x = 3$.

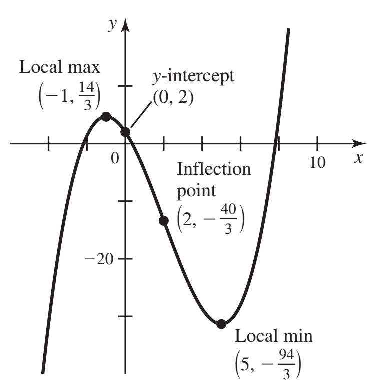
$f''(x)$ changes sign at $x = 2$ from negative to positive, so $x = 2$ is an inflection point where the concavity of f changes from down to up. An example of such a function is sketched.



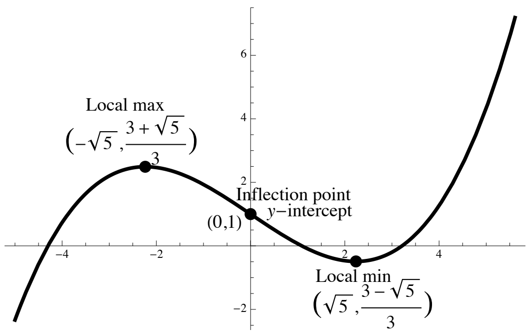
4.4.48 $f'(x) > 0$ on $(-\infty, 1)$ and on $(1, \infty)$, so f should be increasing on both of those intervals. There should be an inflection point at $x = 1$, because the 2nd derivative changes from negative to positive there, so f should change from concave down to concave up at that point.



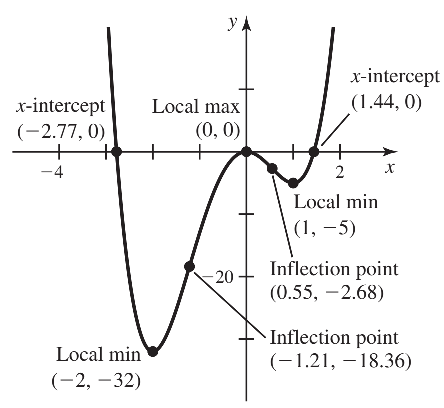
4.4.49 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = x^2 - 4x - 5 = (x-5)(x+1)$. This is 0 when $x = -1, 5$. $f''(x) = 2x - 4$, which is 0 when $x = 2$. Note that $f'(-2) > 0$, $f'(0) < 0$, and $f'(6) > 0$. So f is increasing on $(-\infty, -1)$ and on $(5, \infty)$. It is decreasing on $(-1, 5)$. There is a local maximum of $14/3$ at $x = -1$ and a local minimum of $-94/3$ at $x = 5$. Note also that $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$, so there is an inflection point at $(2, -40/3)$, and f is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$. The y intercept is $f(0) = 2$.



4.4.50 The domain of f is $(-\infty, \infty)$, and there is no symmetry. $f'(x) = \frac{1}{5}x^2 - 1$. This is 0 when $x = \pm\sqrt{5}$. $f''(x) = \frac{2}{5}x$, which is 0 when $x = 0$. Note that $f'(-3) > 0$, $f'(0) < 0$, and $f'(3) > 0$. So f is increasing on $(-\infty, -\sqrt{5})$ and on $(\sqrt{5}, \infty)$. It is decreasing on $(-\sqrt{5}, \sqrt{5})$. There is a local maximum of $\frac{3+2\sqrt{5}}{3}$ at $x = -\sqrt{5}$ and a local minimum of $\frac{3-2\sqrt{5}}{3}$ at $x = \sqrt{5}$. Note also that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so there is an inflection point at the y-intercept $(0,1)$, and f is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

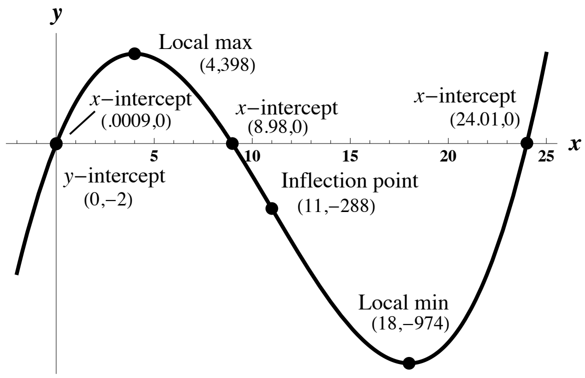


4.4.51 The domain of f is $(-\infty,\infty)$, and there is no symmetry. $f'(x) = 12x^3+12x^2-24x = 12x(x+2)(x-1)$. This is 0 when $x=-2$, when $x=1$, and when $x=0$. $f''(x) = 36x^2+24x-24 = 12(3x^2+2x-2)$, which is 0 when $x = \frac{-1\pm\sqrt{7}}{3}$. These values are at approximately -1.215 and 0.549. Note that $f'(-3)<0$, $f'(-1)>0$, $f'(.5)<0$, and $f'(2)>0$. So f is decreasing on $(-\infty,-2)$ and on $(0,1)$. It is increasing on $(-2,0)$ and on $(1,\infty)$. There is a local maximum of 0 at $x=0$ and a local minimum of -32 at $x=-2$ and a local minimum of -5 at $x=1$. Let $r_1<r_2$ be the two roots of $f''(x)$ mentioned above. Note that $f''(x)>0$ for $x<r_1$ and for $x>r_2$ and $f''(x)<0$ for $r_1<x<r_2$, so there are inflection points at $x=r_1$ and at $x=r_2$. Also, f is concave down on (r_1,r_2) and concave up on $(-\infty,r_1)$ and on (r_2,∞). There is a y-intercept at $f(0)=0$ and x-intercepts where $f(x) = 3x^4+4x^3-12x^2 = x^2(3x^2+4x-12) = 0$, which is at $x = \frac{-2\pm2\sqrt{10}}{3}$ and $x=0$.



4.4.52 The domain of f is $(-\infty,\infty)$, and there is no symmetry. $f'(x) = 3x^2-66x+216 = 3(x-4)(x-18)$. This is 0 when $x=4$ and when $x=18$. $f''(x) = 6x-66 = 6(x-11)$, which is 0 when $x=11$. Note that $f'(0)>0$, $f'(10)<0$, and $f'(20)>0$. Thus f is decreasing on $(4,18)$, while it is increasing on $(-\infty,4)$ and on $(18,\infty)$.

There is a local maximum of 398 at $x=4$ and a local minimum of -974 at $x=18$. Note also that $f''(x)<0$ for $x<11$, and $f''(x)>0$ for $x>11$, so there is an inflection point at $x=11$, and f is concave up on $(11,\infty)$ and is concave down on $(-\infty,11)$.



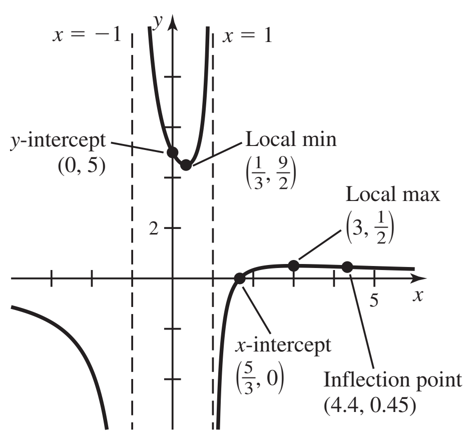
4.4.53 The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, and there is no symmetry. Note that $\lim_{x\to-1^+} f(x)=\infty$ and $\lim_{x\to-1^-} f(x)=-\infty$, so there is a vertical asymptote at $x=-1$. Also, $\lim_{x\to1^+} f(x)=-\infty$ and $\lim_{x\to1^-} f(x)=\infty$, so there is a vertical asymptote at $x=1$

Note that $\lim_{x\to\pm\infty} \frac{3x-5}{x^2-1}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\pm\infty}\frac{3/x}{1-(1/x^2)}=0$, so $y=0$ is a horizontal asymptote.

$f'(x)=\frac{(x^2-1)\cdot3-(3x-5)\cdot2x}{(x^2-1)^2}=\frac{-3x^2+10x-3}{(x^2-1)^2}=\frac{(-3x+1)(x-3)}{(x^2-1)^2}$. This is 0 when $x=3$ and when $x=1/3$. $f''(x)=\frac{(x^2-1)^2(-6x+10)-(-3x^2+10x-3)(2)(x^2-1)\cdot2x}{(x^2-1)^4}=\frac{2(3x^3-15x^2+9x-5)}{(x^2-1)^3}$. This is 0 for $x\approx4.405$. Let r be this root of $f''(x)$.

Note that $f'(-2)<0$, $f'(-1/2)<0$, $f'(1/2)>0$, $f'(2)>0$ and $f'(4)<0$. So f is decreasing on $(-\infty,-1)$, on $(-1,1/3)$ and on $(3,\infty)$. It is increasing on $(1/3,1)$ and on $(1,3)$. There is a local maximum of $1/2$ at $x=3$ and a local minimum of $9/2$ at $x=1/3$.

Note that $f''(x)<0$ for $x<-1$ and $f''(x)<0$ for $1<x<r$, while $f''(x)>0$ for $-1<x<1$, and for $x>r$. Thus f is concave up on $(-1,1)$ and on (r,∞) and concave down on $(-\infty,-1)$ and on $(1,r)$. There is an inflection point at r. There is a y-intercept at $f(0)=5$ and an x-intercept at $(5/3,0)$.



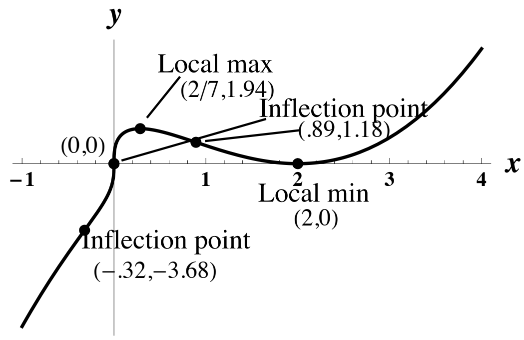
4.4.54 The domain of f is $(-\infty,\infty)$, and there is no symmetry, and no vertical asymptotes. Note that $\lim_{x\to\pm\infty} f(x)=\pm\infty$, so there are no horizontal asymptotes.

$f'(x)=x^{1/3}\cdot2\cdot(x-2)+(x-2)^2(1/3)x^{-2/3}=\frac{(x-2)(7x-2)}{3x^{2/3}}$. This is 0 for $x=2$ and $x=2/7$, and does not exist for $x=0$.

$f''(x)=\frac{3x^{2/3}(14x-16)-(7x^2-16x+4)(2x^{-1/3})}{9x^{4/3}}=\frac{4(7x^2-4x-2)}{9x^{5/3}}$. The numerator of this last expression has two roots, which we will call r_1 and r_2. Note that $r_1\approx-0.320$ and $r_2\approx0.892$. Note also that f'' doesn't exist for $x=0$.

$f'(x) > 0$ on $(-\infty, 0)$ and $(0, 2/7)$ and $(2, \infty)$, so f is increasing on those intervals, while $f'(x) < 0$ on $(2/7, 2)$, so f is decreasing there. f has a local maximum at $x = 2/7$ and a local minimum of 0 at $x = 2$.

Note also that $f''(x) < 0$ for $x < r_1$ and for $0 < x < r_2$, so f is concave down on $(-\infty, r_1)$ and on $(0, r_2)$. However, $f''(x) > 0$ for $r_1 < x < 0$ and for $x > r_2$, so f is concave up on $(r_1, 0)$ and on (r_2, ∞). There are inflection points at each of r_1, r_2, and 0. The inflection point $(0, 0)$ serves also as the x- and y- intercept.



4.4.55

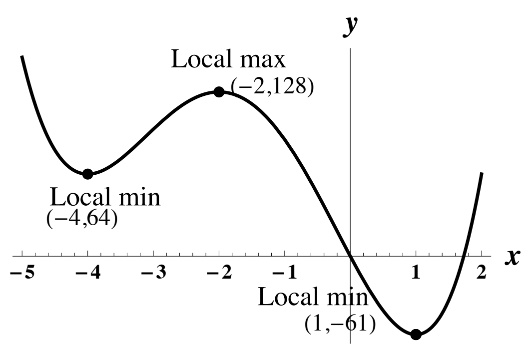
a. False. Maxima and minima can also occur at points where $f'(x)$ doesn't exist. Also, it is possible to have a zero of f' which doesn't correspond to an extreme point.

b. False. Inflection points can also occur at points where $f''(x)$ doesn't exist, and a zero of f'' might not correspond to an inflection point.

c. False. For example, $f(x) = \dfrac{(x^2-9)(x^2-16)}{(x+3)(x-4)}$ doesn't have a vertical asymptote at $x = -3$ or $x = 4$.

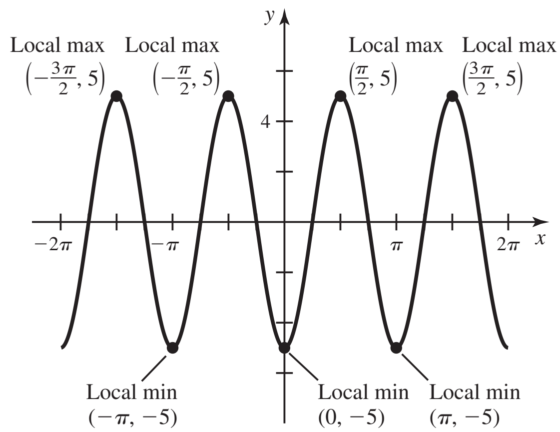
d. True. The limit of a rational function as $x \to \infty$ is a finite number when the degree of the denominator is greater than or equal to that of the numerator. If they both have the same degree, the limit is the ratio of the leading coefficients, and this is also true of the limit as $x \to -\infty$. In the case where the denominator has greater degree than the numerator, the limit is 0 as $x \to -\infty$ and as $x \to \infty$.

4.4.56 $f'(x)$ is 0 at $x = -4$, $x = -2$, and $x = 1$. $f'(x) > 0$ on $(-4, -2)$ and on $(1, \infty)$, so f is increasing there, while $f'(x) < 0$ on $(-\infty, -4)$ and on $(-2, 1)$, so f is decreasing on those intervals. There must be a local maximum at $x = -2$ and local minimums at $x = -4$ and $x = 1$. An example of such a function is sketched.

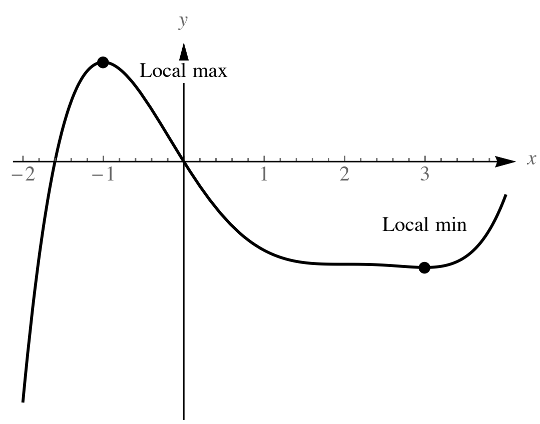


4.4.57 $f'(x)$ is 0 on the interior of the given interval at $x = \pm 3\pi/2$, $x = \pm\pi$, $x = \pm\pi/2$, and at $x = 0$.

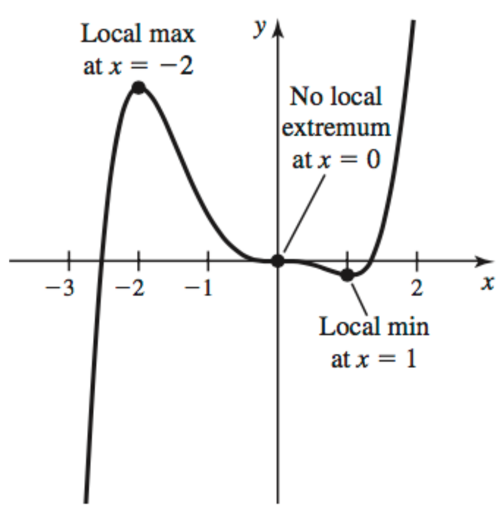
$f'(x) > 0$ on $(-2\pi, -3\pi/2)$, $(-\pi, -\pi/2)$, $(0, \pi/2)$, and on $(\pi, 3\pi/2)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-3\pi/2, -\pi)$, $(-\pi/2, 0)$, $(\pi/2, \pi)$, and on $(3\pi/2, 2\pi)$, so f is decreasing on those intervals. There are local maxima at $x = \pm 3\pi/2$ and $x = \pm\pi/2$, and local minima at $x = 0$ and at $x = \pm\pi$. An example of such a function is sketched.



4.4.58 $f'(x)$ is 0 at -1, 2, and 3. On the interval $(-\infty, -1)$, f' is positive (so f is increasing), and likewise on the interval $(3, \infty)$. On the interval $(-1, 2)$ and on $(2, 3)$, f' is negative (so f is decreasing). There is a local maximum at $x = -1$ and a local minimum at $x = 3$.



4.4.59 $f'(x)$ is 0 at $x = 0$, $x = -2$, and $x = 1$. $f'(x) > 0$ on $(-\infty, -2)$ and on $(1, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-2, 0)$ and on $(0, 1)$, so f is decreasing on those intervals. There is a local maximum at $x = -2$ and a local minimum at $x = 1$. There isn't an extremum at $x = 0$.

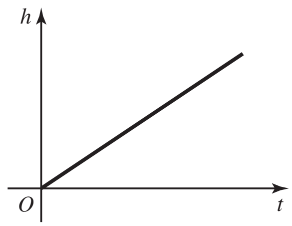


4.4.60 Let $f(x) = \dfrac{\ln x}{x}$. $f'(x) = \dfrac{1 - \ln x}{x^2}$, which is 0 at $x = e$. Note that $f'(x) > 0$ on $(0, e)$ and $f'(x) < 0$ on (e, ∞), so $f(x)$ has its maximal value at $x = e$. Thus, $f(\pi) < f(e)$, so $\dfrac{\ln \pi}{\pi} < \dfrac{1}{e}$, so $\ln \pi < \pi/e$, so $e \ln \pi < \pi$. Thus $\ln \pi^e < \pi$, and so $\pi^e < e^\pi$.

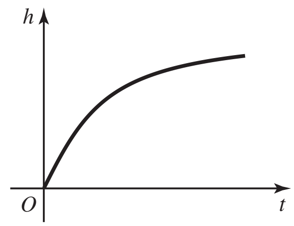
4.4.61

a.

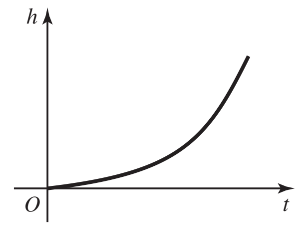
(a)



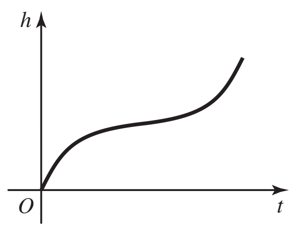
(b)



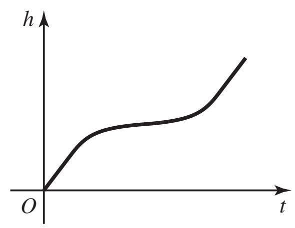
(c)



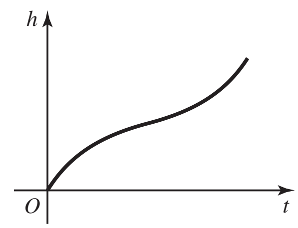
(d)



(e)



(f)



b. The water is being poured in at a constant rate, so the depth is always increasing, so $y = h(t)$ is an increasing function.

c. (a) No concavity

(b) Always concave down.

(c) Always concave up.

(d) Concave down for for the first half and concave up for the second half.

(e) At the beginning, in the middle, and at the end, there is no concavity. In the lower middle it is concave down and in the upper middle it will be concave up.

(f) This is concave down for the first half, and concave up for the second half.

d. (a) $h'(t)$ is constant, so there is no local max/min.

(b) $h'(t)$ is maximal at $t = 0$.

(c) $h'(t)$ is maximal at $t = 10$.

(d) $h'(t)$ is maximal at $t = 0$ and $t = 10$.

(e) $h'(t)$ is maximal on the first and last straight parts of $h(t)$.

(f) $h'(t)$ is maximal at $t = 0$ and $t = 10$.

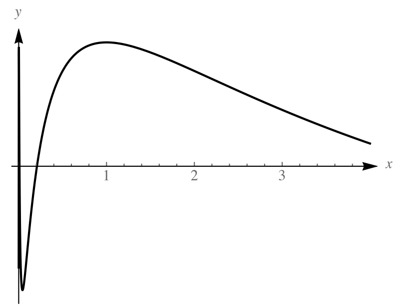
4.4.62

a. $f'(x) = -\sin(\ln x) \cdot \dfrac{1}{x} = -\dfrac{\sin(\ln x)}{x}$, which is 0 when $\ln x = k\pi$ for an integer k, which occurs for $x = e^{k\pi}$. On the given interval, this occurs for $x = 1$, $x = e^{-\pi}$, $x = e^{-2\pi}$, $\ldots$.

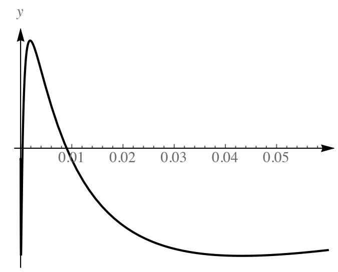
b. $f''(x) = \dfrac{x(-\cos(\ln x) \cdot (1/x)) - (-\sin(\ln x))}{x^2} = \dfrac{\sin(\ln x) - \cos(\ln x)}{x^2}$. This is 0 when $\ln x = \dfrac{4k+1}{4}\pi$ where k is an integer. For our domain, this occurs for $x = e^{\pi/4}$, $x = e^{-3\pi/4}$, $x = e^{-7\pi/4}$, $\ldots$.

c. Using a computer algebra system, the three smallest zeroes on $(0.1, \infty)$ are at ≈ 0.208, ≈ 4.81, and ≈ 111.318.

d.

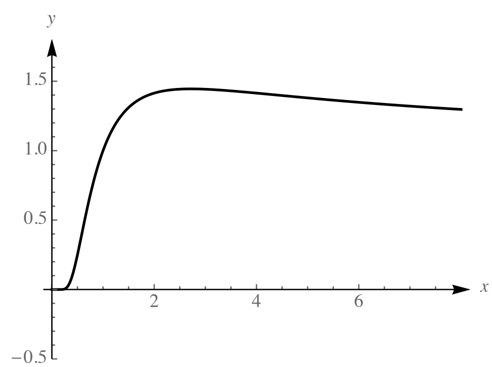


Here it is shown graphed on $(0, 4)$.

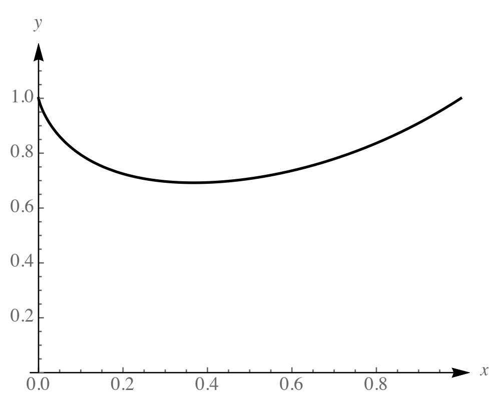


Here it is shown graphed on $(0, .06)$.

4.4.63 f can be written as $f(x) = e^{(\ln x)/x}$. $f'(x) = e^{(\ln x)/x}\left(\dfrac{1 - \ln x}{x^2}\right)$. This is 0 for $x = e$, and is positive on $(0, e)$ and negative on (e, ∞). There is a local maximum at $x = e$ of $e^{1/e}$.

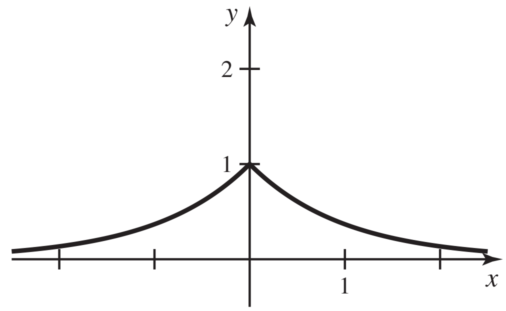


4.4.64 f can be written as $f(x) = e^{x \ln x}$. $f'(x) = e^{x \ln x}(1 + \ln x)$. This is 0 for $x = 1/e$, and is positive on $(1/e, \infty)$ and negative on $(0, 1/e)$. There is a local minimum at $x = 1/e$ of $1/(e^{1/e})$.



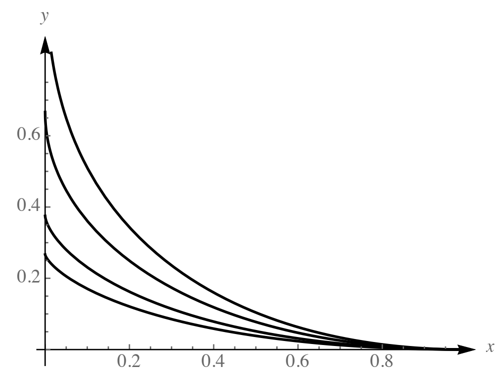
4.4.65

If $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, \infty)$, then $f'(x)$ is increasing on both of those intervals. But if there is a local max at 0, the function f must be switching from increasing to decreasing there. This means that f' must be switching from positive to negative. But if f' is switching from positive to negative, but increasing, there must be a cusp at $x = 0$, so $f'(0)$ does not exist.



4.4.66

As s increases, the man reaches the dog faster.



The curves shown (from top to bottom) are for $s = 1.5, 2, 3, 4$.

4.4.67 The domain of f is given to be $[-2\pi, 2\pi]$. There are no vertical asymptotes. Note that $f(-x) = \dfrac{(-x)(\sin(-x))}{((-x)^2+1)} = \dfrac{x\sin x}{x^2+1} = f(x)$, f has even symmetry. Questions about horizontal asymptotes aren't relevant because the given domain is an interval with finite length.

$f'(x) = \dfrac{(x^2+1)(x\cos x + \sin x) - x\sin x \cdot (2x)}{(x^2+1)^2}$, which can be simplified to

$$\frac{x(x^2+1)\cos x + (1-x^2)\sin x}{(x^2+1)^2},$$

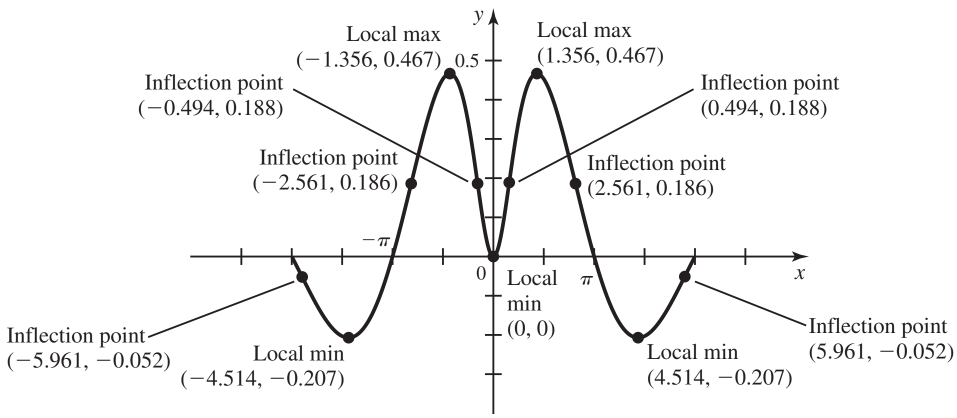
and with the aid of a computer algebra system, the roots of this expression can be found to be approximately ± 4.514 and ± 1.356, as well as $x = 0$. We will call the non-zero roots $\pm r_1$ and $\pm r_2$ where $0 < r_1 < r_2$. Note that $f'(x) < 0$ on $(-2\pi, -r_2)$ and on $(-r_1, 0)$ and (r_1, r_2), so f is decreasing there, while $f'(x) > 0$ on $(-r_2, -r_1)$, on $(0, r_1)$, and on $(r_2, 2\pi)$, so f is increasing on these intervals. There are local maxima at $x = \pm r_1$ and local minima at $x = 0$ and at $x = \pm r_2$.

$f''(x)$ has numerator $(x^2+1)^2((x^3+x)(-\sin x) + \cos x(3x^2+1) + (1-x^2)(\cos x) + \sin x(-2x)) - ((x^3+x)\cos x + (1-x^2)\sin x)(4x)(x^2+1)$ and denominator $(x^2+1)^4$. This simplifies to

$$f''(x) = \frac{(-x^5 - 7x)\sin x + (-2x^4+2)\cos x}{(x^2+1)^3},$$

which is 0 at approximately ± 5.961 and ± 2.561 and ± 0.494. We will call these 6 roots $\pm r_3$, $\pm r_4$ and $\pm r_5$ where $0 < r_3 < r_4 < r_5$. Note that $f''(x) < 0$ on $(-2\pi, -r_5)$ and on $(-r_4, -r_3)$, and on (r_3, r_4), and on $(r_5, 2\pi)$, so f is concave down on these intervals, while $f''(x) > 0$ on $(-r_5, -r_4)$, and on $(-r_3, r_3)$, and on (r_4, r_5), so f is concave up on these intervals. There are points of inflection at each of $\pm r_3$, $\pm r_4$, and $\pm r_5$.

There is an x-intercept at $(0, 0)$, which is also the y-intercept, as well as x-intercepts at $\pm 2\pi$.



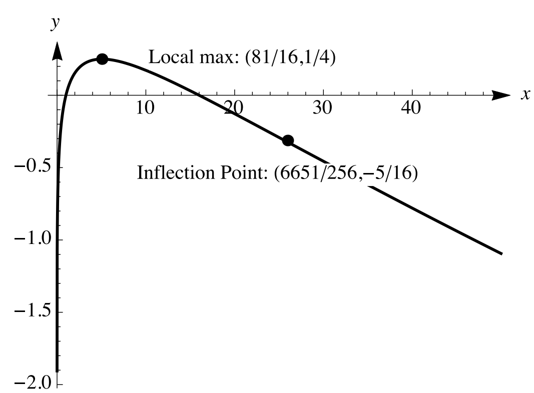
4.4.68 The domain of f is $[0, \infty)$ and there is no symmetry. There are no asymptotes.

$f'(x) = (3/4)x^{-3/4} - (1/2)x^{-1/2} = \frac{3-2x^{1/4}}{4x^{3/4}}$, which is 0 for $x = \frac{81}{16}$.

$f'(x) > 0$ on $(0, 81/16)$, so f is increasing on that interval. $f'(x) < 0$ on $(81/16, \infty)$, so f is decreasing on that interval. There is a local maximum at 81/16 (which also gives an absolute maximum.)

$f''(x) = (-9/6)x^{-7/4} + (1/4)x^{-3/2} = \frac{4x^{1/4}-9}{16x^{7/4}}$, which is 0 at $x = 6561/256$.

$f''(x) > 0$ on $(6561/256, \infty)$, so f is concave up on that interval. $f''(x) < 0$ on $(0, 6561/256)$, so f is concave down on that interval. There is a point of inflection at $x = 6561/256$.

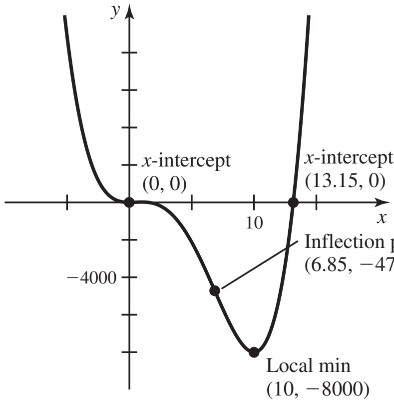


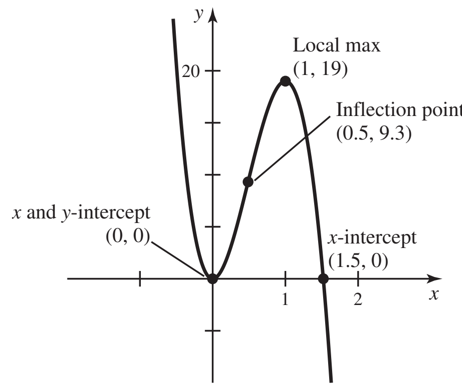
4.4.69 The domain of f is $(-\infty, \infty)$ and there is no symmetry. There are no asymptotes because f is a polynomial.

$f'(x) = 12x^3 - 132x^2 + 120x = 12x(x - 10)(x - 1)$, which is 0 for $x = 0$, $x = 1$, and $x = 10$.

$f'(x) > 0$ on $(0, 1)$ and on $(10, \infty)$, so f is increasing on those intervals. $f'(x) < 0$ on $(-\infty, 0)$ and on $(1, 10)$, so f is decreasing on those intervals. There is a local maximum at $x = 1$, and local minima at $x = 0$ and at $x = 10$.

$f''(x) = 36x^2 - 264x + 120 = 12(3x^2 - 22x + 10)$. This is 0 at approximately $x = .487$ and $x = 6.846$. Let these two roots be r_1 and r_2 with $r_1 < r_2$. $f''(x) > 0$ on $(-\infty, r_1)$ and on (r_2, ∞), so f is concave up on those intervals. $f''(x) < 0$ on (r_1, r_2), so f is concave down on that interval. There are points of inflection at $x = r_1$ and $x = r_2$.





4.4.70 The domain of f is $(0,3/2)\cup(3/2,2)$ and there is no symmetry. There is a vertical asymptote at $x=3/2$, because $\lim_{x\to 3/2} f(x)=-\infty$.

$$f'(x)=\frac{(1+\sin\pi x)\cos(\pi x)\cdot\pi-\sin(\pi x)\cos(\pi x)\cdot\pi}{(1+\sin(\pi x))^2}=\frac{\pi\cos\pi x}{(1+\sin\pi x)^2},$$

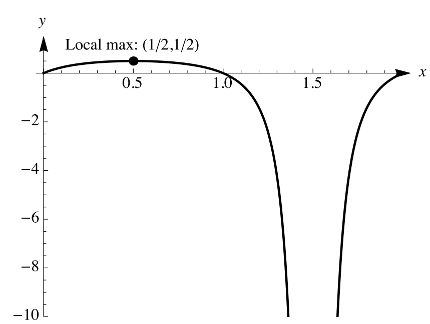
which is 0 for $x=1/2$.

$f'(x)>0$ on $(0,1/2)$ and on $(3/2,2)$, so f is increasing on those intervals. $f'(x)<0$ on $(1/2,3/2)$, so f is decreasing on that interval. There is a local maximum at $x=1/2$.

$$f''(x)=\frac{(1+\sin\pi x)^2(-\pi^2)(\sin\pi x)-2\pi^2\cos^2\pi x(1+\sin\pi x)}{(1+\sin\pi x)^4}=\frac{-\pi^2(2-\sin\pi x)(1+\sin\pi x)}{(1+\sin\pi x)^3}.$$

This expression isn't 0 on the given interval.

$f''(x)<0$ on both $(0,3/2)$ and on $(3/2,2)$, so f is concave down on those intervals. There are no points of inflection.



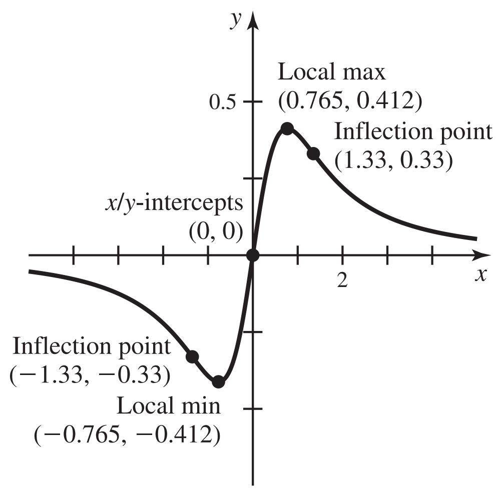
4.4.71 The domain of f is $(-\infty,\infty)$. There are no vertical asymptotes. Note that $f(-x)=\dfrac{\tan^{-1}(-x)}{(-x)^2+1}=-\dfrac{\tan^{-1}(x)}{x^2+1}=-f(x)$, so f has odd symmetry. Because $\lim\limits_{x\to\pm\infty}\tan^{-1}(x)=\pm\pi/2$, $\lim\limits_{x\to\pm\infty} f(x)=0$, so $y=0$ is a horizontal asymptote.

$f'(x)=\dfrac{(x^2+1)(1/(x^2+1))-\tan^{-1}x\cdot 2x}{(x^2+1)^2}=\dfrac{1-2x\tan^{-1}x}{(x^2+1)^2}$. Using a computer algebra system shows that the numerator has two roots at approximately ± 0.765. Let the roots be $\pm r_1$ where $r_1>0$. Note that $f'(x)<0$ on $(-\infty,-r_1)$ and on (r_1,∞), so f is decreasing there, while $f'(x)>0$ on $(-r_1,r_1)$, so f is increasing on that interval. There is a local minimum at $-r_1$ and a local maximum at r_1.

$$\begin{aligned}f''(x)&=\frac{(x^2+1)^2\left[(-2x)(1/(x^2+1))-\tan^{-1}x\cdot 2\right]-(1-2x\tan^{-1}x)\cdot 2(x^2+1)(2x)}{(x^2+1)^4}\\&=\frac{(6x^2-2)\tan^{-1}x-6x}{(x^2+1)^3}.\end{aligned}$$

Again, using a computer algebra system reveals roots at approximately ± 1.330 in addition to the root at 0. Let the non-zero roots of the numerator be $\pm r_2$ where $r_2 > 0$. We see that $f''(x) < 0$ on $(-\infty, -r_2)$, and on $(0, r_2)$, so f is concave down on those intervals, while $f''(x) > 0$ on $(-r_2, 0)$ and on (r_2, ∞), so f is concave up on those intervals, and there are points of inflection at $-r_2$, 0, and r_2.

There is an x-intercept at $(0, 0)$, which is also the y-intercept.



4.4.72 The domain of f is $(-\infty, \infty)$. Note that $f(-x) = \frac{\sqrt{4(-x)^2+1}}{(-x)^2+1} = \frac{\sqrt{4x^2+1}}{x^2+1} = f(x)$, so f has even symmetry.

Note that $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+1}}{x^2+1} \cdot \dfrac{\sqrt{(1/x^4)}}{1/x^2} = \lim\limits_{x\to\infty} \dfrac{\sqrt{(4/x^2)+(1/x^4)}}{1+(1/x^2)} = 0$, so $y = 0$ is a horizontal asymptote as $x \to \infty$, and by symmetry it is a horizontal asymptote as $x \to -\infty$ as well.

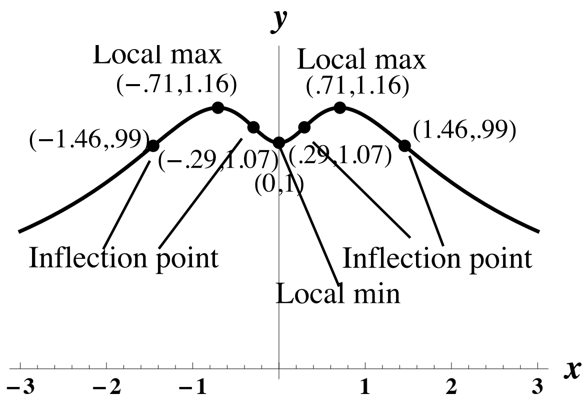
$$\begin{aligned} f'(x) &= \frac{(x^2+1)(4x)(4x^2+1)^{-1/2} - \sqrt{4x^2+1}(2x)}{(x^2+1)^2} \\ &= \frac{4x^3+4x-8x^3-2x}{(x^2+1)^2 \cdot \sqrt{4x^2+1}} \\ &= \frac{2x-4x^3}{(x^2+1)^2 \cdot \sqrt{4x^2+1}} \\ &= \frac{2x(1-2x^2)}{(x^2+1)^2 \cdot \sqrt{4x^2+1}}. \end{aligned}$$

This expression is 0 for $x = 0$ and $x = \pm\sqrt{1/2}$. Note that $f'(x) > 0$ on the interval $(-\infty, -\sqrt{1/2})$ and on $(0, \sqrt{1/2})$, so f is increasing on those intervals, while $f'(x) < 0$ on $(-\sqrt{1/2}, 0)$ and on $(\sqrt{1/2}, \infty)$, so f is decreasing on those intervals. There is a local minimum at $(0, 1)$ and local maxima at approximately $(\pm\sqrt{1/2}, 1.16)$.

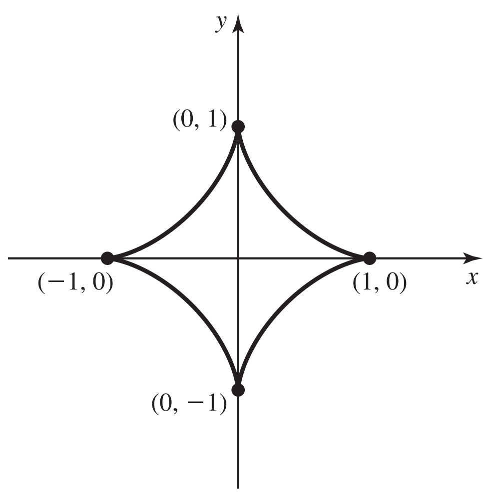
$f''(x)$ has numerator $(4x^2+1)^{1/2}(x^2+1)^2(2-12x^2)$
$-(2x-4x^3)((4x^2+1)^{1/2}(2)(x^2+1)(2x) + (x^2+1)^2(4x)(4x^2+1)^{-1/2}$ and denominator $(4x^2+1)(x^2+1)^4$. When simplified, this yields

$$f''(x) = \frac{2(16x^6 - 30x^4 - 9x^2 + 1)}{(4x^2+1)^{3/2}(x^2+1)^3}.$$

Using a computer algebra system, the roots of the polynomial are determined to be approximately ± 1.458 and $\pm .295$. We will refer to these roots as $\pm r_1$ and $\pm r_2$ where $0 < r_1 < r_2$. Note that $f''(x) < 0$ on $(-r_2, -r_1)$ and on (r_1, r_2), so f is concave down there, while $f''(x) > 0$ on $(-\infty, -r_2)$, and on $(-r_1, r_1)$, and on (r_2, ∞), so f is concave up on these intervals, and there are inflection points at each of $\pm r_1$ and $\pm r_2$. The y-intercept is $(0, 1)$.



4.4.73 The equation is valid only for $|x| \le 1$ and $|y| \le 1$. Using implicit differentiation, we have $(2/3)x^{-1/3} + (2/3)y^{-1/3}y' = 0$, so $y' = \frac{-y^{1/3}}{x^{1/3}}$. This is 0 for $y = 0$ (in which case $x = \pm 1$) and doesn't exist for $x = 0$ (in which case $x = \pm 1$.) In the first quadrant the curve is decreasing, in the 2nd it is increasing, in the 3rd it is decreasing, and in the 4th it is increasing. Differentiating y' yields $y'' = \frac{x^{1/3}(-1/3)y^{-2/3}y' + y^{1/3}(1/3)(x^{-2/3})}{x^{2/3}} = \frac{y^{2/3} + x^{2/3}}{3x^{4/3}y^{1/3}}$, which is positive when y is positive and negative when y is negative, so the curve is concave up in the first and 2nd quadrants, and concave down in the 3rd and 4th.

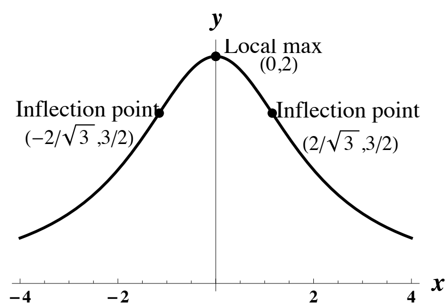


4.4.74 The domain of f is $(-\infty, \infty)$ and f has even symmetry, because $f(-x) = f(x)$.
Note that $\lim_{x\to\infty} \frac{8}{x^2+4} = 0$, so $y = 0$ is a horizontal asymptote. $f'(x) = -\frac{16x}{(x^2+4)^2}$, which is negative for $x > 0$ and positive for $x < 0$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, and there is a local maximum of 2 at $x = 0$.

$$f''(x) = \frac{(x^2+4)^2(-16) - (-16x)(2)(x^2+4)(2x)}{(x^2+4)^4}$$

$$= -\frac{16(4-3x^2)}{(x^2+4)^3},$$

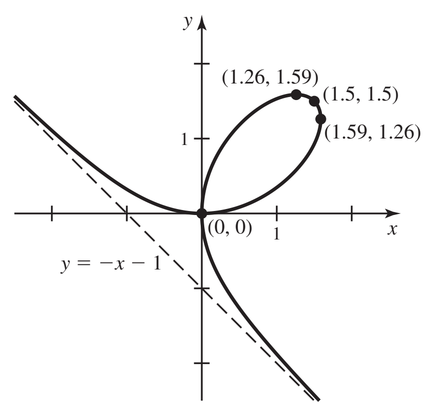
which is 0 for $x = \pm\sqrt{4/3}$. Note that $f'' < 0$ on $(-2/\sqrt{3}, 2/\sqrt{3})$, and is positive elsewhere, so f is concave up on $(-\infty, -2/\sqrt{3})$ and on $(2/\sqrt{3}, \infty)$, and is concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$. There are inflection points at $x = \pm 2/\sqrt{3}$.



4.4.75 First note that the expression is symmetric when x and y are switched, so the curve should be symmetric about the line $y = x$. Also, if $y = x$, then $2x^3 = 3x^2$, so either $x = 0$ or $x = 3/2$, so this is where the curve intersects the line $y = x$.

Differentiating implicitly yields $3x^2 + 3y^2y' = 3xy' + 3y$, so $y' = \dfrac{y - x^2}{y^2 - x}$. This is 0 when $y = x^2$, but this occurs on the curve when $x^3 + x^6 = 3x^3$, which yields $x = 0$ (and $y = 0$), or $x^3 = 2$, so $x = \sqrt[3]{2} \approx 1.260$. Note also that the derivative doesn't exist when $x = y^2$, which again yields $(0, 0)$ and $y^6 + y^3 = 3y^3$, or $y = \sqrt[3]{2}$. So there should be a flat tangent line at approximately $(1.260, 1.587)$ and a vertical tangent line at about $(1.587, 1.260)$.

Differentiating again and solving for y'' yields $y''(x) = \dfrac{2xy\left(x^3 - 3xy + y^3 + 1\right)}{(x - y^2)^3} = \dfrac{2xy}{(x - y^2)^3}$. In the first quadrant, when $x > y^2$, the curve is concave up, when $x < y^2$, the curve is concave down. In both the 2nd and 4th quadrants, the curve is concave up.



4.4.76 Note that the curve requires $0 \le x < 1$.

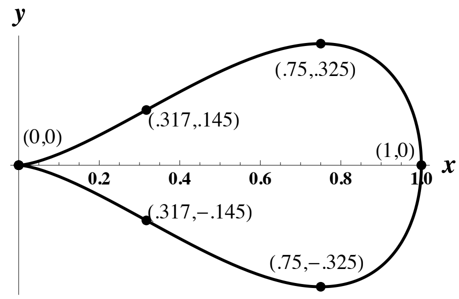
Note also that the curve is symmetric about the x-axis, so we can just consider the first quadrant, and obtain the rest by reflection.

Differentiating implicitly and solving for y' yields $y'(x) = \dfrac{3x^2 - 4x^3}{2y}$. This quantity is positive on the first quadrant for $0 < x < 3/4$ and negative for $3/4 < x < 1$, so f is increasing in the first quadrant for $0 < x < 3/4$ and decreasing for $3/4 < x < 1$. There is a maximum at $x = 3/4$.

Differentiating again and solving for y'' yields

$$y''(x) = \frac{-x\left((3 - 4x)^2x^3 + 12(2x - 1)y^2\right)}{4y^3}.$$

Rewriting and simplifying yields $y''(x) = \dfrac{x^4(8x^2 - 12x + 3)}{4y^3}$ which is positive in the first quadrant for $0 < x < r_1$ where $r_1 \approx .317$, and negative for $r_1 < x < 1$. So the function in the first quadrant is concave up for $0 < x < r_1$ and concave down for $r_1 < x < 1$, and there is a point of inflection at r_1.



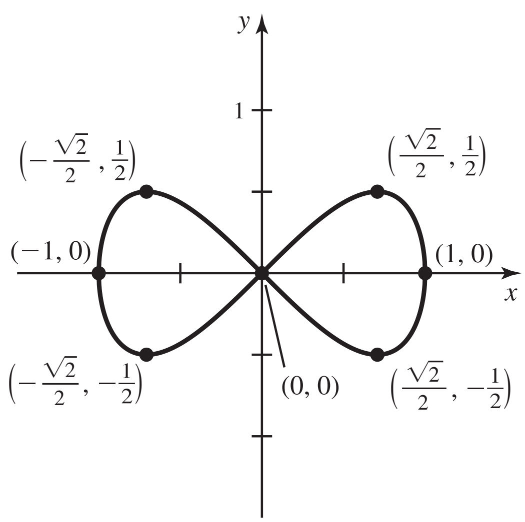
4.4.77 Note that the curve requires $-1 \le x < 1$.

Note also that the curve is symmetric about both the x-axis and the y-axis, so we can just consider the first quadrant, and obtain the rest by reflection.

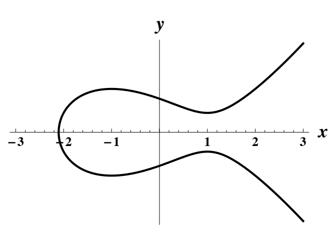
Differentiating implicitly yields $4x^3 - 2x + 2yy' = 0$, so $y' = \dfrac{x - 2x^3}{y}$. This is 0 in the first quadrant for $x = \sqrt{2}/2$. Note also that there is a vertical tangent line at the point $(1, 0)$. The derivative is positive on $(0, \sqrt{2}/2)$ and negative on $(\sqrt{2}/2, 1)$, so in the first quadrant the curve is increasing on that first interval and decreasing on the second.

Differentiating again and solving for y'' (and rewriting) yields $y''(x) = \dfrac{x^4(2x^2 - 3)}{y}$, which is negative in the first quadrant for $0 < x < 1$, so this curve is concave down in the first quadrant.

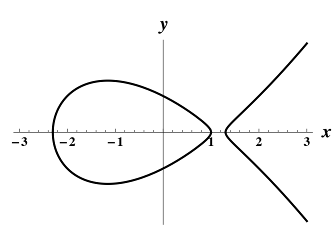
The rest of the curve can be found by reflection.



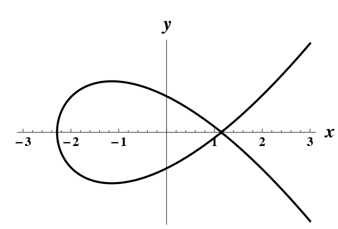
4.4.78



a.

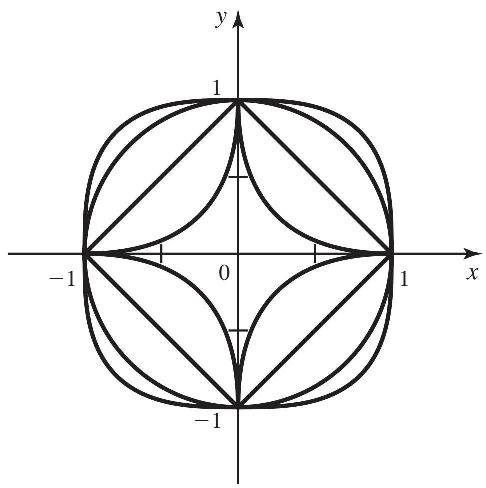


b.



c. This occurs for $a \approx 3.931$.

4.4.79 As n increases, the curves retain their symmetry, but move "outward." That is, the curves enclose a greater area. It appears that the figures approach the 2×2 square centered at the origin with sides parallel to the coordinate axes.



4.5 Optimization Problems

4.5.1 . . . objective. . . constraints . . . .

4.5.2 The constraints are used to express all but one of the variables in terms of one independent variable.

4.5.3 The constraint is $x + y = 10$, so we can express $y = 10 - x$ or $x = 10 - y$. Therefore the objective function can be expressed $Q = x^2(10 - x)$ or $Q = (10 - y)^2 y$.

4.5.4 The minimum will occur at one of the endpoints of the closed interval.

4.5.5

a. From the constraint, we have $y = 100 - 10x$. Substituting this into the objective function, we have $P(x) = x(100 - 10x) = 100x - 10x^2$.

b. $P'(x) = 100 - 20x$, which is zero for $x = 5$. Because $P''(x) = -20$, P is concave down everywhere, so the critical point $x = 5$ yields an absolute maximum. The value of $P(5)$ is $100(5) - 10(25) = 250$.

4.5.6

a. Because $y = \dfrac{50}{x}$, we can write $S = x + 2\left(\dfrac{50}{x}\right) = x + \dfrac{100}{x}$.

b. $S'(x) = 1 - \dfrac{100}{x^2}$, which is zero for $x = 10$. $S''(x) = \dfrac{200}{x^3}$, so $S''(10) > 0$ so there is a local minimum at $x = 10$. Because $x = 10$ is the only critical point, this is actually an absolute minimum. The value of $S(10) = 20$.

4.5.7 Let x and y be the two non-negative numbers. The constraint is $x + y = 23$, which gives $y = 23 - x$. The objective function to be maximized is the product of the numbers, $P = xy$. Using $y = 23 - x$, we have $P = xy = x(23 - x) = 23x - x^2$. Now x must be at least 0, and cannot exceed 23 (otherwise $y < 0$). Therefore we need to maximize $P(x) = 23x - x^2$ for $0 \le x \le 23$. The critical points of the objective function satisfy $P'(x) = 23 - 2x = 0$, which has the solution $x = 23/2$. To find the absolute maximum of P, we check the endpoints of $[0, 23]$ and the critical point $x = 23/2$. Because $P(0) = P(23) = 0$ and $P(23/2) = (23/2)^2$, the absolute maximum occurs when $x = y = 23/2$.

4.5.8 Let a and b be the two non-negative numbers. The constraint is $a + b = 23$, which gives $b = 23 - a$. The objective function to be maximized/minimized is the quantity $Q = a^2 + b^2$. Using $b = 23 - a$, we have $Q = a^2 + b^2 = a^2 + (23 - a)^2 = 2a^2 - 46a + 529$. Now a must be at least 0, and cannot exceed 23 (otherwise $b < 0$). Therefore we need to maximize $Q(a) = 2a^2 - 46a + 529$ for $0 \le a \le 23$. The critical points of the objective function satisfy $Q'(a) = 4a - 46 = 0$, which has the solution $a = 23/2$. To find the absolute maximum/minimum of Q, we check the endpoints of $[0, 23]$ and the critical point $a = 23/2$. Observe that $Q(0) = Q(23) = 529$ and $Q(23/2) = 529/2$, so the absolute maximum occurs when $a, b = 0, 23$ or $23, 0$ and the absolute minimum occurs when $a = b = 23/2$.

4.5.9 Let x and y be the two positive numbers. The constraint is $xy = 50$, which gives $y = 50/x$. The objective function to be minimized is the sum of the numbers, $S = x + y$. Using $y = 50/x$, we have $S = x + y = x + \frac{50}{x}$. Now x can be any positive number, so we need to maximize $S(x) = x + 50/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $S'(x) = 1 - \frac{50}{x^2} = 0$, which has the solution $x = \sqrt{50} = 5\sqrt{2}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the numbers with minimum sum are $x = 5\sqrt{2}$ and $y = \frac{50}{5\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$, so $x = y = 5\sqrt{2}$.

4.5.10 We seek to maximize $P = xy$ subject to the constraint $y = 12 - 3x$. Substituting gives $P = x(12 - 3x) = 12x - 3x^2$. Then $P'(x) = 12 - 6x$, which is zero for $x = 2$. Because $P'(x) > 0$ for $0 < x < 2$ and $P'(x) < 0$ for $x > 2$, we have a maximum at $x = 2$. When $x = 2$, we have $y = 12 - 3x = 6$. So the two numbers are 2 and 6.

4.5.11 Let x and y be the dimensions of the rectangle. The perimeter is $2x + 2y$, so the constraint is $2x + 2y = 10$, which gives $y = 5 - x$. The objective function to be maximized is the area of the rectangle, $A = xy$. Thus we have $A = xy = x(5 - x) = 5x - x^2$. We have $x, y \geq 0$, which also implies $x \leq 5$ (otherwise $y < 0$). Therefore we need to maximize $A(x) = 5x - x^2$ for $0 \leq x \leq 5$. The critical points of the objective function satisfy $A'(x) = 5 - 2x = 0$, which has the solution $x = 5/2$. To find the absolute maximum of A, we check the endpoints of $[0, 5]$ and the critical point $x = 5/2$. Because $A(0) = A(5) = 0$ and $A(5/2) = 25/4$, the absolute maximum occurs when $x = y = 5/2$, so width = length = 5/2 m.

4.5.12 Let x and y be the dimensions of the rectangle. The perimeter is $2x + 2y$, so the constraint is $2x + 2y = P$, which gives $y = P/2 - x$. The objective function to be maximized is the area of the rectangle, $A = xy$. Thus we have $A = xy = x(P/2 - x) = (P/2)x - x^2$. We have $x, y \geq 0$, which also implies $x \leq P/2$ (otherwise $y < 0$). Therefore we need to maximize $A(x) = (P/2)x - x^2$ for $0 \leq x \leq P/2$. The critical points of the objective function satisfy $A'(x) = P/2 - 2x = 0$, which has the solution $x = P/4$. To find the absolute maximum of A, we check the endpoints of $[0, P/2]$ and the critical point $x = P/4$. Because $A(0) = A(P/2) = 0$ and $A(P/4) = P^2/8 - P^2/16 = P^2/16$, the absolute maximum occurs when $x = y = P/4$, so width = length = $P/4$.

4.5.13 Let x and y be be the dimensions of the rectangle. The area is $xy = 100$, so the constraint is $y = 100/x$. The objective function to be minimized is the perimeter of the rectangle, $P = 2x + 2y$. Using $y = 100/x$, we have $P = 2x + 2y = 2x + \dfrac{200}{x}$. Because $xy = 100 > 0$ we must have $x > 0$, so we need to minimize $P(x) = 2x + 200/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $P'(x) = 2 - \dfrac{200}{x^2} = 0$, which has the solution $x = 10$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions of the rectangle with minimum perimeter are $x = 10$ and $y = \frac{100}{10} = 10$, so width = length = 10.

4.5.14 Let x and y be be the dimensions of the rectangle. The area is $xy = A$ and A is fixed, so the constraint is $xy = A$, which gives $y = A/x$. The objective function to be minimized is the perimeter of the rectangle, $P = 2x + 2y$. Using $y = A/x$, we have $P = 2x + 2y = 2x + \dfrac{2A}{x}$. Because $xy = A > 0$ we must have $x > 0$, so we need to minimize $P(x) = 2x + 2A/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $P'(x) = 2 - \dfrac{2A}{x^2} = 0$, which has the solution $x = \sqrt{A}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions of the rectangle with minimum perimeter are $x = \sqrt{A}$ and $y = A/\sqrt{A} = \sqrt{A}$, so width = length = $\sqrt{A}$.

4.5.15 We seek to minimize $S = 2x+y$ subject to the constraint $y = 12/x$. Substituting gives $S = 2x+12/x$, so $S'(x) = 2 - 12/x^2$. This is zero when $x^2 = 6$, or $x = \sqrt{6}$. Note that for $0 < x < \sqrt{6}$ we have $S'(x) < 0$, and for $x > \sqrt{6}$ we have $S'(x) > 0$, so we have a minimum at $x = \sqrt{6}$. Note that when $x = \sqrt{6}$, we have $y = 12/x = 12/\sqrt{6} = 2\sqrt{6}$.

4.5.16

a. Let x and y be the lengths of the sides of the pen, with y the side parallel to the barn. Then the constraint is $2x + y = 200$, which gives $y = 200 - 2x$. The objective function to be maximized is the area of the pen, $A = xy$. Using $y = 200 - 2x$, we have $A = xy = x(200 - 2x) = 200x - 2x^2$. The length x must be at least 0, and cannot exceed 100 (otherwise $y < 0$). Therefore we need to maximize $A(x) = 200x - 2x^2$ for $0 \le x \le 100$. The critical points of the objective function satisfy $A'(x) = 200 - 4x = 0$, which has the solution $x = 50$. To find the absolute maximum of A, we check the endpoints of $[0, 100]$ and the critical point $x = 50$. Because $A(0) = A(100) = 0$ and $A(50) = 5000$, the absolute maximum occurs when $x = 50$ m and $y = 200 - 2 \cdot 50 = 100$ m.

b. Let x and y be the lengths of the sides of each individual rectangular pen, with y the side parallel to the barn. Then the constraint is $xy = 100$, which gives $y = 100/x$. The objective function to be minimized is the total amount of fencing required, which is $Q = 5x + 4y$. Using the constraint, we have $Q = 5x + 4y = 5x + \dfrac{400}{x}$. Now x can be any positive number, so so we need to minimize $Q(x) = 5x + 400/x$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $Q'(x) = 5 - \dfrac{400}{x^2} = 0$, which has the solution $x = \sqrt{80} = 4\sqrt{5}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the dimensions which require the least fencing are $x = 4\sqrt{5}$ m and $y = 100/4\sqrt{5} = 5\sqrt{5}$ m.

4.5.17 Let the coordinates of the base of the rectangle be $(x, 0)$ and $(-x, 0)$ where $0 \le x \le 5$. Then the width of the rectangle is $2x$ and the height is $\sqrt{25 - x^2}$, so the area A is given by $A(x) = 2x\sqrt{25 - x^2}$. The critical points of this function satisfy $A'(x) = 2\sqrt{25 - x^2} + \dfrac{2x \cdot (-x)}{\sqrt{25 - x^2}} = \dfrac{2(25 - 2x^2)}{\sqrt{25 - x^2}} = 0$, which has unique solution $x = 5/\sqrt{2}$ in $(0, 5)$. We have $A(0) = A(5) = 0$, so the rectangle of maximum area has width $2x = 10/\sqrt{2}$ cm, height $y = \sqrt{25 - (25/2)} = 5/\sqrt{2}$ cm.

4.5.18 Let the coordinates of the base of the rectangle be $(x, 0)$ and $(-x, 0)$ where $0 \le x \le \sqrt{48}$. Then the width of the rectangle is $2x$ and the height is $48 - x^2$, so the area A is given by $A(x) = 2x(48 - x^2) = 2(48x - x^3)$. The critical points of this function satisfy $A'(x) = 2(48 - 3x^2) = 0$, which has unique solution $x = 4$ in $(0, \sqrt{48})$. Note that $A''(x) = 2(-6x) = -12x$ which is always negative on the domain interval, so $x = 4$ gives a maximum. The biggest rectangle has dimensions 8×32 and has area 256.

4.5.19 Let x be the length of the sides of the base of the box and y be the height of the box. The volume is $x \cdot x \cdot y = 8$, so the constraint is $x^2 y = 8$, which gives $y = \dfrac{8}{x^2}$. The objective function to be minimized is the surface area S of the box, which consists of $2x^2$ (for the top and base) + $4xy$ (for the 4 sides); therefore $S = 2x^2 + 4xy$. Using $y = \dfrac{8}{x^2}$, we have $S = 2x^2 + 4xy = 2x^2 + 4x \cdot \dfrac{8}{x^2} = 2x^2 + \dfrac{32}{x}$. The base side length can be any $x > 0$, so we need to maximize $S(x) = 2x^2 + \dfrac{32}{x}$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $S'(x) = 4x - \dfrac{32}{x^2} = 0$; clearing denominators gives $4x^3 = 32$ so $x^3 = 8$, so $x = 2$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. When $x = 2$ we have $y = \dfrac{8}{4} = 2$, so the box with the smallest surface area is $2 \times 2 \times 2$.

4.5.20 Let x be the length of the sides of the base of the box and y be the height of the box. The constraint is $2x+y = 108$, which gives $y = 108-2x$. The objective function to be maximized is the volume V of the box, which is given by $V = x \cdot x \cdot y = x^2y$. Using $y = 108 - 2x$, we have $V = x^2y = x^2(108 - 2x) = 108x^2 - 2x^3$. The length x must be at least 0, and cannot exceed $108/2 = 54$ (otherwise $y < 0$). Therefore we need to maximize $V(x) = 108x^2 - 2x^3$ for $0 \le x \le 54$. The critical points of the objective function satisfy $V'(x) = 216x - 6x^2 = 0$, which has solutions $x = 0$ and $x = 216/6 = 36$. To find the absolute maximum of V, we check the endpoints of $[0, 54]$ and the critical point $x = 36$. Because $V(0) = V(54) = 0$ and $V(36) = 36^3$, the absolute maximum occurs when $x = 36$ in and $y = 108 - 2 \cdot 36 = 36$ in.

4.5.21 Let x be the length of the sides of the base of the box and y be the height of the box. The volume of the box is $x \cdot x \cdot y = x^2y$, so the constraint is $x^2y = 16$, which gives $y = 16/x^2$. Let c be the cost per square foot of the material used to make the sides. Then the cost to make the base is $2cx^2$, the cost to make the 4 sides is $4cxy$, and the cost to make the top is $\frac{1}{2}cx^2$. The objective function to be minimized is the total cost, which is $C = 2cx^2 + 4cxy + \dfrac{1}{2}cx^2 = \dfrac{5}{2}cx^2 + 4cx \cdot \dfrac{16}{x^2} = c\left(\dfrac{5x^2}{2} + \dfrac{64}{x}\right)$. The base side length can be any $x > 0$, so we need to maximize $C(x) = c(5x^2/2 + 64/x)$ on the interval $(0, \infty)$. The critical points of the objective function satisfy $5x - \dfrac{64}{x^2} = 0$, which gives $x^3 = 64/5$ or $x = 4/\sqrt[3]{5}$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. Therefore the box with minimum cost has base $4/\sqrt[3]{5}$ ft by $4/\sqrt[3]{5}$ ft and height $y = 16/(4/\sqrt[3]{5})^2 = 5^{2/3}$ ft.

4.5.22 A point on the line $y = 3x + 4$ has the form $(x, 3x + 4)$, which has distance L to the origin given by $L^2 = x^2 + (3x + 4)^2 = 10x^2 + 24x + 16$. Because L is positive, it suffices to minimize L^2. The quadratic function $10x^2 + 24x + 16$ takes its minimum at $x = -24/20 = -6/5$, and the corresponding value of $y = 2/5$. Therefore the point closest to the origin on this line is $(-6/5, 2/5)$.

4.5.23 A point on the parabola $y = 1 - x^2$ has the form $(x, 1 - x^2)$, which has distance L to the point (1,1) given by $L^2 = (x - 1)^2 + \left(1 - (1 - x^2)\right)^2 = x^4 + x^2 - 2x + 1$. Because L is positive, it suffices to minimize L^2. The critical points of L^2 satisfy $\dfrac{dL^2}{dx} = 4x^3 + 2x - 2 = 2(2x^3 + x - 1) = 0$ This cubic equation has a unique root $x \approx 0.590$, so the point closest to $(1, 1)$ on this parabola is approximately $(0.590, 0.652)$.

4.5.24 The distance between (x, x^2) and $(18, 0)$ is $d(x) = \sqrt{(x^2 - 0)^2 + (x - 18)^2}$. Instead of working with the distance, we can instead work with the square of the distance, because these two functions have minima which occur at the same place. So consider

$$d(x)^2 = D(x) = (x^2)^2 + (x - 18)^2 = x^4 + x^2 - 36x + 324$$

$\dfrac{dD}{dx} = 4x^3 + 2x - 36 = 2(2x^3 + x - 18) = 2(x - 2)\left(2x^2 + 4x + 9\right)$, which has only one real root at $x = 2$. This critical point gives a minimum, so the closest point is $(2, 4)$. The distance at this point is $d(4) = \sqrt{16 + 256} = \sqrt{272} = 4\sqrt{17}$.

4.5.25 The distance between $(x, 3x)$ and $(50, 0)$ is $d(x) = \sqrt{(3x - 0)^2 + (x - 50)^2}$. Instead of working with the distance, we can instead work with the square of the distance, because these two functions have minima which occur at the same place. So consider

$$(d(x))^2 = D(x) = (3x)^2 + (x - 50)^2 = 9x^2 + x^2 - 100x + 2500 = 10(x^2 - 10x + 250).$$

$\dfrac{dD}{dx} = 10(2x - 10)$, which is zero for $x = 5$. Because $\dfrac{d^2D}{dx^2} = 20 > 0$, we see that the critical point at $x = 5$ is a minimum. So the minimum of D (and d) occurs at $x = 5$. The value of d at the point $(5, 15)$, is $d(5) = \sqrt{15^2 + (-45)^2} = 15\sqrt{10} \approx 47.4$.

4.5.26

a. Label the starting point, finishing point and transition point P as in Figure 4.62 in the text. In terms of the angle θ, the swimming distance is $2\sin(\theta/2)$ and the walking distance is $\pi - \theta$, as derived in

Example 3. So the time for the swimming leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{2\sin(\theta/2)}{2} = \sin\dfrac{\theta}{2}$ and the time for the walking leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{\pi-\theta}{4}$. The total travel time for the trip is the objective function $T(\theta) = \sin\dfrac{\theta}{2} + \dfrac{\pi-\theta}{4}$, $\quad 0 \le \theta \le \pi$. The critical points of T satisfy $\dfrac{dT}{d\theta} = \dfrac{1}{2}\cos\dfrac{\theta}{2} - \dfrac{1}{4} = 0$ or $\cos\dfrac{\theta}{2} = \dfrac{1}{2}$. Because $0 \le \theta/2 \le \pi/2$, the only solution is given by $\theta/2 = \pi/3$ or $\theta = 2\pi/3$. Evaluating the objective function at the critical point and the endpoints, we find that $T(2\pi/3) = \sqrt{3}/2 + \pi/12 \approx 1.128$ hr, $T(0) = \pi/4 \approx 0.785$ hr and $T(\pi) = 1$ hr. Therefore the minimum travel time is $T(0) \approx 0.785$ hr when the entire trip is done walking. The maximum travel time, corresponding to $\theta = 120^o$, is $T \approx 1.128$ hr.

b. In this case the time for the walking leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{\pi-\theta}{1.5}$, so the total travel time for the trip is now given by $T(\theta) = \sin\dfrac{\theta}{2} + \dfrac{\pi-\theta}{1.5}$, $\quad 0 \le \theta \le \pi$. The critical points of T satisfy $\dfrac{dT}{d\theta} = \dfrac{1}{2}\cos\dfrac{\theta}{2} - \dfrac{1}{1.5} = 0$ or $\cos\dfrac{\theta}{2} = \dfrac{4}{3}$. Therefore in this case there are no critical points. Evaluating the objective function at the endpoints, we find that $T(0) = \pi/1.5 \approx 2.09$ hr and $T(\pi) = 1$ hr. Therefore the minimum travel time is $T(\pi) = 1$ hr when the entire trip is done swimming, and the maximum travel time is $T(0) \approx 2.09$ hr when the entire trip is done walking.

c. Denote the walking speed by $v > 0$. Then the time for the walking leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{\pi-\theta}{v}$, so the total travel time for the trip is now given by $T(\theta) = \sin\dfrac{\theta}{2} + \dfrac{\pi-\theta}{v}$, $\quad 0 \le \theta \le \pi$. Observe that $\dfrac{d^2T}{d\theta^2} = -\dfrac{1}{4}\sin\dfrac{\theta}{2} < 0$ on the interval $(0,\pi)$. Therefore by the Second Derivative Test, the function T cannot have a local minimum in $(0,\pi)$, and so the minimum travel time must occur either at $\theta = 0$ (all walking) or $\theta = \pi$ (all swimming) in all cases. Evaluating the objective function at the endpoints, we find that $T(0) = \pi/v$ and $T(\pi) = 1$. In the case $v > \pi$ we have $T(0) < 1$ and the minimum corresponds to all walking; when $v < \pi$ we have $T(0) > 1$ and the minimum corresponds to all swimming (and when $v = \pi$ the travel time is 1 hr for both all walking and all swimming). Hence the minimum walking speed for which it is quickest to walk the entire distance is $v = \pi$ m/hr.

4.5.27

a. Let x be the distance from the point on the shoreline nearest to the boat to the point where the woman lands on shore; then the remaining distance she must travel on shore is $6 - x$. By the Pythagorean Theorem, the distance the woman must row is $\sqrt{x^2+16}$. So the time for the rowing leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{\sqrt{x^2+16}}{2}$ and the time for the walking leg is $\dfrac{\text{distance}}{\text{rate}} = \dfrac{6-x}{3}$. The total travel time for the trip is the objective function $T(x) = \dfrac{\sqrt{x^2+16}}{2} + \dfrac{6-x}{3}$. We wish to minimize this function for $0 \le x \le 6$. The critical points of the objective function satisfy $T'(x) = \dfrac{x}{2\sqrt{x^2+16}} - \dfrac{1}{3} = 0$, which when simplified gives $5x^2 = 64$, so $x = 8/\sqrt{5}$ is the only critical point in $(0,6)$. From the First Derivative Test we see that T has a local minimum at this point, so $x = 8/\sqrt{5}$ must give the minimum value of T on $[0,6]$.

b. Let $v > 0$ be the woman's rowing speed. Then the total travel time is now given by $T(x) = \dfrac{\sqrt{x^2+16}}{v} + \dfrac{6-x}{3}$. The derivative of the objective function is $T'(x) = \dfrac{x}{v\sqrt{x^2+16}} - \dfrac{1}{3}$. If we try to solve the equation $T'(x) = 0$ as in part (a) above, we see that there is at most one solution $x > 0$. Therefore there can be at most one critical point of T in the interval $(0,6)$. Observe also that $T'(0) = -1/3 < 0$ so the absolute minimum of T on $[0,6]$ cannot occur at $x = 0$. So one of two things must happen: there is a unique critical point for T in $(0,6)$ which is the absolute minimum for T on $[0,6]$, and then $T'(6) > 0$;

or, T is decreasing on $[0,6]$, and then $T'(6) \le 0$ (the quickest way to the restaurant is to row directly in this case). The condition $T'(6) \le 0$ is equivalent to $\dfrac{6}{\sqrt{6^2+16}} \le \dfrac{v}{3}$ which gives $v \ge 9/\sqrt{13}$ mi/hr.

4.5.28 Let x be the distance from the point on shore nearest the island to the point where the underwater cable meets the shore, and let y be the be the length of the underwater cable. By the Pythagorean Theorem, $y = \sqrt{x^2+3.5^2}$. The objective function to be minimized is the cost given by $C(x) = 2400\sqrt{x^2+3.5^2} + 1200\cdot(8-x) = 2400\sqrt{x^2+3.5^2} - 1200x + 9600$. We wish to minimize this function for $0 \le x \le 8$. The critical points of $C(x)$ satisfy $C'(x) = \dfrac{2400x}{\sqrt{x^2+3.5^2}} - 1200 = 1200\left(\dfrac{2x}{\sqrt{x^2+3.5^2}} - 1\right) = 0$, which we solve to obtain $x = 7\sqrt{3}/6$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,8]$. Therefore the optimal point on shore has distance $x = 7\sqrt{3}/6$ mi from the point on shore nearest the island, in the direction of the power station.

4.5.29 Let x be the distance from the point on shore nearest the island to the point where the underwater cable meets the shore, and let y be the be the length of the underwater cable. In terms of the angle θ in the figure, $\tan\theta = 3.5/x$ so $x = 3.5\cot\theta$, and $\sin\theta = 3.5/y$ so $y = 3.5\csc\theta$. The objective function to be minimized is the cost given by $C(\theta) = 2400\cdot 3.5\csc\theta + 1200\cdot(8-3.5\cot\theta) = 8400\csc\theta - 4200\cot\theta + 9600$. The angle θ must be between $\tan^{-1}(3.5/8)$ (≈ 0.412) and $\pi/2$. The critical points of $C(\theta)$ satisfy $C'(\theta) = 8400(-\csc\theta\cot\theta) - 4200(-\csc^2\theta) = 4200\csc^2\theta(1-2\cos\theta)$, which has unique solution $\theta = \pi/3$ in the interval under consideration. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[\tan^{-1}(3.5/8), \pi/2]$. Therefore the optimal point on shore has distance $x = 3.5\cot(\pi/3) = 7\sqrt{3}/6$ mi from the point on shore nearest the island, in the direction of the power station.

4.5.30 Let L be the ladder length and x be the distance between the foot of the ladder and the fence. The Pythagorean Theorem gives the relationship $L^2 = (x+4)^2 + b^2$, where b is the height of the top of the ladder. We see that $b/(x+4) = 10/x$ by similar triangles, which gives $b = 10(x+4)/x$. Substituting in the expression for L^2 above gives $L^2 = (x+4)^2 + 100\dfrac{(x+4)^2}{x^2} = (x+4)^2\left(1+\dfrac{100}{x^2}\right)$. It suffices to minimize L^2 for $x > 0$ because L and L^2 have the same local extrema (L is positive). We have $\dfrac{d}{dx}L^2 = (x+4)^2\left(-\dfrac{200}{x^3}\right) + 2(x+4)\left(1+\dfrac{100}{x^2}\right) = \dfrac{2(x+4)(x^3-400)}{x^3}$. Because $x > 0$, the only critical point is $x = \sqrt[3]{400} \approx 7.368$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0,\infty)$. Substituting $x \approx 7.368$ in the expression for L^2 we find the length of the shortest ladder $L \approx 19.16$ ft.

4.5.31 Let L be the ladder length and x be the distance between the foot of the ladder and the fence. The Pythagorean Theorem gives the relationship $L^2 = (x+5)^2 + b^2$, where b is the height of the top of the ladder. We see that $b/(x+5) = 8/x$ by similar triangles, which gives $b = 8(x+5)/x$. Substituting in the expression for L^2 above gives $L^2 = (x+5)^2 + 64\dfrac{(x+5)^2}{x^2} = (x+5)^2\left(1+\dfrac{64}{x^2}\right)$. It suffices to minimize L^2 instead of L. However in this case x and b must satisfy $x, b \le 20$. Solving $20 = 8(x+5)/x$ for x gives $x = 10/3$, so the condition $b \le 20$ corresponds to $x \ge 10/3$, and we see that we must minimize L^2 for $10/3 \le x \le 20$. We have $\dfrac{d}{dx}L^2 = (x+5)^2\left(-\dfrac{128}{x^3}\right) + 2(x+5)\left(1+\dfrac{64}{x^2}\right) = \dfrac{2(x+5)(x^3-320)}{x^3}$. Because $x > 0$, the only critical point is $x = \sqrt[3]{320} \approx 6.840$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[10/3, 20]$. Substituting $x \approx 6.840$ in the expression for L^2 we find the length of the shortest ladder $L \approx 18.220$ ft.

4.5.32 Let x be the length of the piece of wire used to make the circle; then $60-x$ is the length of the piece used to make the square. Let r be the radius of the circle and s the side length of the square. The circle has circumference $2\pi r$ so we have $x = 2\pi r$ or $r = x/2\pi$; the square has perimeter $4s$ so $60 - x = 4s$

which gives $s = (60 - x)/4$. The objective function to be maximized/minimized is the combined area of the circle and square given by $A = \pi r^2 + s^2 = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{60-x}{4}\right)^2 = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{15}{2}x + 225$. The critical points of this function satisfy $A'(x) = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{15}{2} = \left(\frac{4+\pi}{8\pi}\right)x - \frac{15}{2} = 0$, which has unique solution $x = 60\pi/(4+\pi) \approx 26.39$. By the First (or Second) Derivative test, this critical point gives a local minimum, which by Theorem 4.5 must be the absolute minimum of A over the interval $[0, 60]$. So the area is minimized by using 26.394 cm of wire for the circle and 33.606 cm of wire for the square. The maximum area must therefore occur at the endpoints, and because $A(0) = 225$, $A(60) \approx 286.479$, the maximum area occurs when all 60 cm of wire is used to make the circle.

4.5.33 If we remove a sector of angle θ from a circle of radius 20, the remaining circumference is $2\pi\cdot 20 - \theta\cdot 20 = 20(2\pi - \theta)$, so the base of the cone formed has radius $r = \frac{20(2\pi-\theta)}{2\pi} = \frac{10(2\pi-\theta)}{\pi}$. As θ varies from 0 to 2π, the radius ranges from 0 to 20, but all possible cones formed have side length 20. The height h of the cone is given by the Pythagorean Theorem: $h^2 + r^2 = 20^2$, so $h = \sqrt{400 - r^2}$. The volume of the cone given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}r^2\sqrt{400 - r^2}$.

Thus

$$\begin{aligned} V'(r) &= \frac{2\pi}{3}r\sqrt{400 - r^2} + \frac{\pi}{3}r^2(400 - r^2)^{-1/2}\cdot\frac{1}{2}\cdot(-2r) \\ &= \frac{\pi}{3}\cdot\frac{2r(400 - r^2) - r^3}{\sqrt{400 - r^2}} \\ &= \frac{\pi}{3}\cdot\frac{800r - 3r^3}{\sqrt{400 - r^2}}. \end{aligned}$$

The only positive critical point occurs where $r = \sqrt{\frac{800}{3}} = 20\sqrt{\frac{2}{3}}$. An application of the First Derivative Test shows that this is a maximum. So

$$h = \sqrt{400 - \left(20\sqrt{\frac{2}{3}}\right)^2} = \sqrt{400 - 400\cdot\frac{2}{3}} = 20\sqrt{\frac{1}{3}}.$$

4.5.34 Let r and h be the radius and height of the cone; then we have the constraint $r^2 + h^2 = 3^2 = 9$, which gives $r^2 = 9 - h^2$. The objective function to be maximized is the volume of the cone, given by $V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}(9 - h^2)h = \frac{\pi}{3}(9h - h^3)$. Because $r, h \geq 0$ we must have $0 \leq h \leq 3$. Therefore we need to maximize $V(h)$ over $[0, 3]$. The critical points of $V(h)$ satisfy $V'(h) = \frac{\pi}{3}(9 - 3h^2) = \pi(3 - h^2) = 0$, so $h = \sqrt{3}$ is the only critical point in $[0, 3]$. Because $V(0) = V(3) = 0$, the cone of maximum volume has height $h = \sqrt{3}$ and radius $r = \sqrt{6}$.

4.5.35

a. Let r and h be the radius and height of the can. The volume of the can is $V = \pi r^2 h$, which gives the constraint $\pi r^2 h = 354$ or $h = 354/(\pi r^2)$. The objective function to be minimized is the surface area, which consists of $2\pi r^2$ (for the top and bottom of the can) and $2\pi rh$ (for the side of the can). Therefore the objective function to be minimized is $A = 2\pi r^2 + 2\pi rh = 2\pi\left(r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 2\pi\left(r^2 + \frac{354}{\pi r}\right)$. We need to minimize $A(r)$ for $r > 0$. The critical points of $A(r)$ satisfy $A'(r) = 2\pi\left(2r - \frac{354}{\pi r^2}\right) = 0$, which gives $r = \sqrt[3]{(177/\pi)} \approx 3.834$ cm. The corresponding value of h is $h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = 2r \cdot \frac{177}{\pi r^3} = 2r$, so $h = 2\sqrt[3]{(177/\pi)} \approx 7.667$ cm. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$.

b. We modify the objective function in part (a) above to account for the fact that the top and bottom of the can have double thickness: $A = 4\pi r^2 + 2\pi rh = 2\pi\left(2r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 4\pi\left(r^2 + \frac{177}{\pi r}\right)$. We need to minimize $A(r)$ for $r > 0$. The critical points of $A(r)$ satisfy $A'(r) = 4\pi\left(2r - \frac{177}{\pi r^2}\right) = 0$, which gives $r = \sqrt[3]{(177/2\pi)} \approx 3.043$ cm. The corresponding value of h is $h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = 4r \cdot \frac{177}{2\pi r^3} = 4r$, so $h = 4\sqrt[3]{(177/2\pi)} \approx 12.171$ cm. These dimensions are closer to those of a real soda can.

4.5.36 If we fill the pot with just enough water to cover the marble, the water in the pot will have height $2r$. Because the pot has radius 4, the water and marble together have volume $\pi \cdot 4^2 \cdot 2r = 32\pi r$. The marble has volume $4\pi r^3/3$, so the volume of water needed to cover the marble is $V(r) = 32\pi r - \frac{4}{3}\pi r^3$. The critical points of this function satisfy $V'(r) = 32\pi - 4\pi r^2 = 0$, which has unique solution $r = \sqrt{8} = 2\sqrt{2}$ cm. By the First (or Second) Derivative test, this critical point gives a local maximum, which by Theorem 4.5 must be the absolute maximum of A over the interval $[0, 4]$.

4.5.37 Let x and y be the dimensions of the flower garden; the area of the flower garden is 30, so we have the constraint $xy = 30$ which gives $y = 30/x$. The dimensions of the garden and borders are $x + 4$ and $y + 2$, so the objective function to be minimized for $x > 0$ is $A = (x+4)(y+2) = (x+4)\left(\frac{30}{x} + 2\right) = 2x + \frac{120}{x} + 38$. The critical points of $A(x)$ satisfy $A'(x) = 2 - \frac{120}{x^2} = 0$, which has unique solution $x = \sqrt{60} = 2\sqrt{15}$. By the First (or Second) Derivative test, this critical point gives a local minimum, which by Theorem 4.5 must be the absolute minimum of A over $(0, \infty)$. The corresponding value of y is $30/2\sqrt{15} = \sqrt{15}$, so the dimensions are $\sqrt{15}$ by $2\sqrt{15}$ m.

4.5.38

a. Suppose the side on the x-axis extends to the point $(a, 0)$ and the side on the y-axis to $(0, b)$. Then $b = 10 - 2a$ and the rectangle has area $A = ab = a(10 - 2a) = 10a - 2a^2$. We must have $0 \le a \le 5$ to ensure that both $a, b \ge 0$. The critical points of $A(a)$ satisfy $A'(a) = 10 - 4a = 0$, which has unique solution $a = 5/2$. Because $A(0) = A(5) = 0$, $a = 5/2$ gives the maximum area. The corresponding b value is 5, and the maximum area is $25/2$.

b. Let $(a, 0)$ be the vertex on the x-axis and $(0, b)$ the vertex on the y-axis, and label the two vertices on the line $y = 10 - 2x$ as P and Q. The line joining $(a, 0)$ and $(0, b)$ must be parallel to the line $y = 10 - 2x$, which has slope -2. This gives the constraint $\frac{0 - b}{a - 0} = -\frac{b}{a} = -2$, so $b = 2a$. The segment joining $(a, 0)$ and $(0, b)$ has length $\sqrt{a^2 + b^2} = \sqrt{5}a$. The other side length of the rectangle can be found by observing that the triangle with vertices (0,0), $(a, 0)$, $(0, b)$ is similar to the triangle with vertices P, $(5, 0)$, $(a, 0)$

in that order, so the remaining side of the rectangle has length l satisfying $\frac{l}{5-a} = \frac{2a}{\sqrt{5}a}$; hence $l = \frac{2}{\sqrt{5}}(5-a)$. Therefore the area of the rectangle is given by $A(a) = \sqrt{5}a\frac{2}{\sqrt{5}}(5-a) = 10a - 2a^2$, which is the same function as in part (a) above. The maximum again occurs when $a = 5/2$, and the dimensions of the rectangle of maximum area are $5\sqrt{5}/2$ and $\sqrt{5}$.

4.5.39 Because the length of the box (in inches) is $18 - 2x$, the width is $9 - x$, and the height is x, the volume of the box is $V(x) = x(9-x)(18-2x) = 2x^3 - 36x^2 + 162x$, where $0 \le x \le 9$. Taking the derivative of V, we have $V'(x) = 6x^2 - 72x + 162 = 6(x^2 - 12x + 27) = 6(x-3)(x-9)$. Setting the derivative equal to 0 and solving for x, we find that $x = 3$ and $x = 9$. Because $V(0) = V(9) = 0$ and $V(3) = 216$, the box of maximum volume is $12 \times 6 \times 3$ and has a volume of 216 in^3.

4.5.40

a. The dimensions of the box are $5-2x$, $8-2x$, and x, so the volume is given by $V(x) = x(5-2x)(8-2x)$. The dimensions cannot be negative, so we must have $0 \le x \le 2.5$. The critical points of $V(x)$ satisfy

$$\begin{aligned} V'(x) &= (5-2x)(8-2x) + x(-2)(8-2x) + x(5-2x)(-2) \\ &= 40 - 10x - 16x + 4x^2 - 16x + 4x^2 - 10x + 4x^2 \\ &= 12x^2 - 52x + 40 = 4(3x-10)(x-1) = 0. \end{aligned}$$

Thus the critical points are $x = 0$ and $x = 1$ (the other root $x = 10/3$ isn't in the domain). We have $V(0) = V(2.5) = 0$, and $V(1) = 18$, so the maximum is 18 cubic feet.

b. In this case the dimensions of the box are $l - 2x$, $l - 2x$ and x, so the volume is given by $V(x) = x(l-2x)^2 = 4x^3 - 4lx^2 + l^2x$. The dimensions cannot be negative, so we must have $0 \le x \le l/2$. The critical points of $V(x)$ satisfy $V'(x) = 12x^2 - 8lx + l^2 = (6x - l)(2x - l) = 0$. This quadratic equation has roots $x = l/6$ and $l/2$, so the only critical point in $(0, l/2)$ is $x = l/6$. We have $V(0) = V(l/2) = 0$, so the maximum volume is $V(l/6) = 2l^3/27$.

4.5.41 Let x and y equal the width and height, respectively, of the rectangular part of the window. Then the perimeter is $x + 2y + \frac{\pi x}{2} = 20$ and the area of the window is $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{\pi x^2}{8}$. Solving the perimeter equation for y, we have $y = \frac{1}{2}\left(20 - x - \frac{\pi x}{2}\right) = 10 - \frac{x}{2} - \frac{\pi x}{4}$. By substitution, $A(x) = x\left(10 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = \left(\frac{\pi}{8} - \frac{1}{2} - \frac{\pi}{4}\right)x^2 + 10x = \left(-\frac{1}{2} - \frac{\pi}{8}\right)x^2 + 10x$. Taking the derivative of A, we have $A'(x) = \left(-1 - \frac{2\pi}{8}\right)x + 10$, which is 0 for $x = \frac{10}{1 + 2\pi/8} = \frac{40}{4+\pi} \approx 5.6$ ft. At this x value, $y \approx 2.8$ ft. Because $A''\left(\frac{40}{4+\pi}\right) < 0$, the Second Derivative Test implies that the area is maximized when the dimensions of the rectangular pane are approximately 5.6 ft wide by 2.8 ft high.

4.5.42 Suppose that the base of the rectangle has dimension x and the side of the rectangle has dimension y. The semicircular pane has radius $x/2$, so the perimeter of the window is $P = 2y + x + \frac{1}{2} \cdot 2\pi \cdot \frac{x}{2} = 2y + \left(1 + \frac{\pi}{2}\right)x$ which gives the constraint $y = \frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)x$. The rectangular pane has area xy and the semicircular pane has area $(1/2)\pi(x/2)^2$, so the amount of light transmitted through the window is proportional to $L = 2xy + \frac{\pi x^2}{8} = 2x\left(\frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)x\right) + \frac{\pi x^2}{8} = Px - \left(1 + \frac{3\pi}{8}\right)x^2$. Because $x, y \ge 0$ we must have $0 \le x \le P/(1 + (\pi/2)) \approx 0.389P$. The quadratic function $L(x)$ has maximum at $x = \frac{P}{2 + \frac{3\pi}{4}} = \frac{4P}{8+3\pi} \approx 0.230P$, which is in the interval under consideration. The corresponding value of y is

$$y = \frac{P}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)\frac{4P}{8+3\pi} = \frac{4+\pi}{16+6\pi}P \approx 0.205P.$$

4.5.43 The radius r and height h of the barrel satisfy the constraint $r^2 + h^2 = d^2$, which we can rewrite as $r^2 = d^2 - h^2$. The volume of the barrel is given by $V = \pi r^2 h = \pi(d^2 - h^2)h = \pi(d^2h - h^3)$. The height h must satisfy $0 \le h \le d$, so we need to maximize $V(h)$ on the interval $[0, d]$. The critical points of V satisfy $V'(h) = \pi(d^2 - 3h^2) = 0$. The only critical point in $(0, d)$ is $h = d/\sqrt{3}$, which gives the maximum volume because at the endpoints $V(0) = V(d) = 0$. The corresponding r value satisfies $r^2 = d^2 - d^2/3 = 2d^2/3$, so $r = \sqrt{2}d/\sqrt{3}$ and we see that the ratio r/h that maximizes the volume is $\sqrt{2}$.

4.5.44 The objective function to be minimized is the average number of tests required, given by $A(x) = N\left(1 - q^x + \dfrac{1}{x}\right)$ where x is the group size, $N = 10{,}000$ and $q = 0.95$. We may assume that $1 \le x \le 10{,}000$. The critical points of this function satisfy $A'(x) = N\left(-(\ln q)q^x - \dfrac{1}{x^2}\right) = 0$, which is equivalent to the equation $(0.95)^{-x} + \ln(0.95)x^2 = 0$. Using a numerical solver, we find that this equation has one root between 5 and 6, and one between 132 and 133. By the First Derivative Test, we see that the smaller of these roots gives a local minimum and the larger a local maximum. Therefore the minimum value of $A(x)$ for $1 \le x \le 10,000$ occurs either at the smaller root or at the endpoint 10,000. The group size x must be an integer, so the possible optimal choices are $x = 5, 6$ and 10,000; comparing the value of $A(x)$ at these points shows that $x = 5$ is the optimal group size.

4.5.45 Let r and h be the radius and height of the cylinder. The distance d from the centroid of the cylinder (the midpoint of the cylinder's axis of rotation) to any point on the top or bottom edge satisfies $d^2 = r^2 + \left(\dfrac{h}{2}\right)^2$ so the constraint is $r^2 + (h/2)^2 = R^2$. The volume of the cylinder is given by $V = \pi r^2 h = \pi\left(R^2 - \left(\dfrac{h}{2}\right)^2\right)h = \pi\left(R^2h - \dfrac{h^3}{4}\right)$. Because $r, h \ge 0$ we must have $0 \le h \le 2R$. We wish to maximize $V(h)$ on this interval. The critical points of $V(h)$ satisfy $V'(h) = \pi\left(R^2 - \dfrac{3h^2}{4}\right) = 0$ which gives $h = 2R/\sqrt{3}$, and from the constraint we obtain $r = \sqrt{2}R/\sqrt{3}$. The volume $V(h) = 0$ at the endpoints $h = 0$ and $h = 2R$, so the maximum volume must occur at this critical point.

4.5.46 Let x be the number of tickets sold. The cost per ticket is $30 - 0.25x$, and the fixed expenses are 200, so the profit is $P(x) = x(30 - 0.25x) - 200 = -0.25x^2 + 30x - 200$. We wish to maximize this function for $20 \le x \le 70$. The critical points of $P(x)$ satisfy $P'(x) = -0.5x + 30 = 0$ so $x = 60$ is the only critical point. By the First (or Second) Derivative Test, this critical point corresponds to a local maximum, and by Theorem 4.5, this solitary local maximum is also the absolute maximum on the interval $[20, 70]$. Therefore the profit is maximized by selling 60 tickets.

4.5.47 Let R and H be the radius and height of the larger cone and let r and h be the radius and height of the smaller inscribed cone. The region that lies above the smaller cone inside the larger cone is a cone with radius r and height $H - h$; by similar triangles we have $\dfrac{H-h}{r} = \dfrac{H}{R}$ so $h = \dfrac{H}{R}(R - r)$. The volume of the smaller cone is $V = \dfrac{\pi}{3}r^2h = \dfrac{\pi H}{3R}\left(Rr^2 - r^3\right)$, which we must maximize over $0 \le r \le R$. The critical points of $V(r)$ satisfy $V'(r) = \dfrac{\pi H}{3R}\left(2Rr - 3r^2\right) = 0$ which has unique solution $r = 2R/3$ in $(0, R)$. Because $V(r) = 0$ at the endpoints $r = 0$ and $r = R$, the smaller cone with maximum volume has radius $r = 2R/3$ and height $h = H/3$, so the optimal ratio of the heights is 3:1.

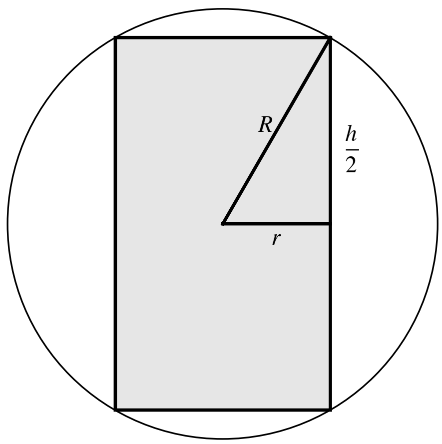
4.5.48 The objective function is the volume of the cylinder, which is $V = \pi r^2 h$. By using the right triangle in the diagram below, we see that $r^2 = R^2 - \dfrac{h^2}{4}$. Thus we can write $V = \pi\left(R^2 - \dfrac{h^2}{4}\right)h = \pi\left(R^2h - \dfrac{h^3}{4}\right)$, which is valid for $0 < h < 2R$. Differentiating leads to $V' = \pi\left(R^2 - \dfrac{3h^2}{4}\right)$. This is zero for $h^2 = \dfrac{4R^2}{3}$, or $h = \dfrac{2R}{\sqrt{3}}$. Using the Second Derivative Test, it can be verified that this critical point yields a maximum for

V. At this value of h, we have

$$r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{4R^2}{12} = \frac{8R^2}{12} = \frac{2R^2}{3},$$

so $r = \frac{\sqrt{2}R}{\sqrt{3}}$. The corresponding volume is

$$V = \pi r^2 h = \frac{\pi(2R^2)(2R)}{3\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}.$$



4.5.49

a. We find $g(0) = 0$, $g(40) = 30$ and $g(60) = 25$ miles per gallon. The value at $v = 0$ is reasonable because when a car first starts moving it needs a lot of power from its engine, so the gas mileage is very low. The decline from 30 to 25 mi/gal as v increases from 40 mi/hr to 60 mi/hr reflects the fact that gas mileage tends to decrease at speeds over 55 mi/hr.

b. The quadratic function $g(v) = (85v - v^2)/60$ takes its maximum value at $v = 85/2 = 42.5$ mi/hr.

c. At speed v the amount of gas needed to drive one mile is $1/g(v)$ and the time it takes is $1/v$. Hence the cost of gas for one mile is $p/g(v)$ and the cost for the driver is w/v, and so the cost for L miles is $C(v) = Lp/g(v) + Lw/v$.

d. We have $C(v) = 400\left(\frac{4}{g(v)} + \frac{20}{v}\right) = 1600\left(\frac{1}{g(v)} + \frac{5}{v}\right)$. The critical points of $C(v)$ satisfy $\frac{g'(v)}{g(v)^2} + \frac{5}{v^2} = 0$, which simplifies to $v^2 g'(v) + 5g(v)^2 = 0$. Substituting the formula for $g(v)$ above and using $g'(v) = (85 - 2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2 - 194v + 8245 = 0$, which has roots $v \approx 62.883, 131.117$. The First (or Second) Derivative Test shows that $C(v)$ has a local minimum at $v \approx 62.9$, which is the unique critical point for $0 \le v \le 131$. Therefore the cost is minimized at this value of v.

e. Because L is a constant factor in the cost function $C(v)$, changing L will not change the critical points of $C(v)$.

f. The critical points of $C(v)$ now satisfy the equation $\frac{4.2g'(v)}{g(v)^2} + \frac{20}{v^2} = 0$, which simplifies to $4.2v^2 g'(v) + 20g(v)^2 = 0$. As above, substituting the formula for $g(v)$ above and using $g'(v) = (85 - 2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2 - 195.2v + 8296 = 0$, which has roots $v \approx 62.532, 132.668$. As in part (d), the minimum cost occurs for $v \approx 62.532$, slightly less than the speed in part (d).

g. The critical points of $C(v)$ now satisfy the equation $\dfrac{4g'(v)}{g(v)^2} + \dfrac{15}{v^2} = 0$, which simplifies to $4v^2g'(v) + 15g(v)^2 = 0$. As above, substituting the formula for $g(v)$ above and using $g'(v) = (85 - 2v)/60$, we can factor out v^2 and reduce to the quadratic equation $v^2 - 202v + 8585 = 0$, which has roots $v \approx 60.800, 141.200$. As in part (d), the minimum cost occurs for $v \approx 60.8$, less than the speed in part (d).

4.5.50

a. The dog runs distance $z - y$ and swims distance $\sqrt{x^2 + y^2}$. Using the fact that time is distance/speed, we see that the total time it takes the dog to get to the tennis ball is $T(y) = \dfrac{z-y}{r} + \dfrac{\sqrt{x^2+y^2}}{s}$.

b. The critical points of $T(y)$ satisfy $T'(y) = -\dfrac{1}{r} + \dfrac{y}{s\sqrt{x^2+y^2}} = 0$, which simplifies to $\dfrac{\sqrt{x^2+y^2}}{y} = \dfrac{r}{s}$ so $\left(\dfrac{r^2}{s^2} - 1\right)y^2 = x^2$, so $y = \dfrac{x}{\sqrt{r/s+1}\sqrt{r/s-1}}$. The First (or Second) Derivative Test shows that a local minimum occurs for this value of y, which by Theorem 4.5 must give the absolute minimum of $T(y)$ for $y > 0$. If $y \le z$ then $T(y)$ is minimized for this value of y; otherwise, the minimum occurs at $y = z$ (all swimming).

c. The ratio is $\dfrac{y}{x} = \dfrac{1}{\sqrt{8+1}\sqrt{8-1}} = \dfrac{1}{\sqrt{63}}$.

d. Elvis's optimal ratio is $\dfrac{y}{x} \approx \dfrac{1}{\sqrt{7.033+1}\sqrt{7.033-1}} \approx 0.144$, so Elvis appears to know calculus!

4.5.51 The viewing angle θ is given by $\theta = \cot^{-1}\left(\dfrac{x}{10}\right) - \cot^{-1}\left(\dfrac{x}{3}\right)$, and we wish to maximize this function for $x > 0$. The critical points satisfy $\theta'(x) = -\dfrac{1}{1+\left(\frac{x}{10}\right)^2}\cdot\dfrac{1}{10} - (-)\dfrac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\dfrac{1}{3} = \dfrac{3}{x^2+3^2} - \dfrac{10}{x^2+10^2} = 0$ which simplifies to $3(x^2+100) = 10(x^2+9)$ or $x^2 = 30$. Therefore $x = \sqrt{30} \approx 5.477$ ft is the only critical point in $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local maximum, and by Theorem 4.5, this solitary local maximum must be the absolute maximum on the interval $(0, \infty)$.

4.5.52 The two cables joined to the ceiling each have length $\sqrt{x^2+1}$ by the Pythagorean Theorem, and the vertical cable has length $6 - x$. The objective function to be minimized is the total length of the three cables, given by $L(x) = 2\sqrt{x^2+1} + 6 - x$. Because the lengths cannot be negative, we must have $0 \le x \le 6$. The critical points of $L(x)$ satisfy $L'(x) = \dfrac{2x}{\sqrt{x^2+1}} - 1 = 0$, which occurs when $3x^2 = 1$, so $x = 1/\sqrt{3} = \sqrt{3}/3$ is the unique critical point in $(0, 6)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, 6]$. Therefore the cables should be joined at distance $x = \sqrt{3}/3$ m below the ceiling.

4.5.53 Let x be the distance between the point and the weaker light source; then $12 - x$ is the distance to the stronger light source. The intensity is proportional to $I(x) = \dfrac{1}{x^2} + \dfrac{2}{(12-x)^2}$, so we can take this as our objective function to be minimized for $0 < x < 12$. The critical points of $I(x)$ satisfy $I'(x) = -\dfrac{2}{x^3} + \dfrac{4}{(12-x)^3} = 0$ which gives $\left(\dfrac{12-x}{x}\right)^3 = 2$, or $\dfrac{12-x}{x} = \sqrt[3]{2}$, or $x = \dfrac{12}{\sqrt[3]{2}+1} \approx 5.310$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, 12)$. Therefore the intensity is weakest at the point $12/(\sqrt[3]{2}+1) \approx 5.310$ m from the weaker source.

4.5.54

a. Setting $h(18) = 10$ gives $\dfrac{-32(18^2)}{v^2}+18+8 = 10$, so $\dfrac{32(18)^2}{v^2} = 16$, so $v^2 = 2(18)^2$, and $v = 18\sqrt{2} \approx 25.46$ ft/s.

b. Using the result above, we have $h(x) = -\dfrac{4x^2}{81} + x + 8$. We know that the center of the hoop is located at the point (18,10) and because the radius of the hoop is 0.75 feet, the front of the hoop is located at the point (17.25,10).

   In this case, the distance formula is

$$s = \sqrt{(x-17.25)^2 + \left(-\frac{4x^2}{81} + x + 8 - 10\right)^2} = \sqrt{(x-17.25)^2 + \left(-\frac{4x^2}{81} + x - 2\right)^2}.$$

c. A women's basketball has a radius of slightly less than 4.62 inches. To determine the value of x at which the ball reaches its closest distance from the rim, we calculate the derivative of the function underneath the square root, set it equal to 0 and solve for x:

$$2(x-17.25) + 2\left(-\frac{4x^2}{81} + x - 2\right)\left(-\frac{8}{81}x + 1\right) = 0$$

   . Using a root-finder, we find that $x \approx 17.52$, which means that the center of the ball is at its closest point to the front of the rim when $x \approx 17.52$ ft. Plugging this value of x into the formula for s, we find that $s \approx 0.45$ ft or about 5.42 inches. So at its closest point, the center of the ball is 5.42 inches from the front of the rim and because this distance is greater than the radius of the basketball, the ball will not hit the front of the rim.

d. A men's basketball has a radius of about 4.75 inches, which is less than the value of s found in part (c). So the men's ball will not hit the front of the rim either.

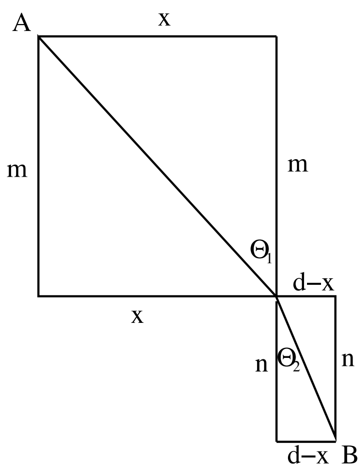
4.5.55

a. Let x, $d-x$ be the distances from the point where the rope meets the ground to the poles of height m, n respectively. Then the rope has length $L(x) = \sqrt{x^2+m^2} + \sqrt{(d-x)^2+n^2}$. We wish to minimize this function for $0 \le x \le d$. The critical points of $L(x)$ satisfy $L'(x) = \dfrac{x}{\sqrt{x^2+m^2}} - \dfrac{d-x}{\sqrt{(d-x)^2+n^2}} = 0$, which is equivalent to $\dfrac{x}{\sqrt{x^2+m^2}} = \dfrac{d-x}{\sqrt{(d-x)^2+n^2}}$, or in terms of the angles θ_1 and θ_2 in the figure, $\sec\theta_1 = \sec\theta_2$ and therefore $\theta_1 = \theta_2$. Observe that $L'(0) < 0$ and $L'(d) > 0$, so the minimum value of $L(x)$ must occur at some $x \in (0, d)$. There must be exactly one critical point, because as x ranges from 0 to d, θ_1 decreases and θ_2 increases, and so $\theta_1 = \theta_2$ can occur for at most one value of x.

b. Because the speed of light is constant, travel time is minimized when distance is minimized, which we saw in part (a) occurs when $\theta_1 = \theta_2$.

4.5.56

Let x, d, m, n be the distances labeled in the figure below. Then using time = distance/speed, we see that the time for light to travel from A to B is $T(x) = \dfrac{\sqrt{x^2+m^2}}{v_1} + \dfrac{\sqrt{(d-x)^2+n^2}}{v_2}$. We wish to minimize this function for $0 \le x \le d$. The critical points of $T(x)$ satisfy $T'(x) = \dfrac{x}{v_1\sqrt{x^2+m^2}} - \dfrac{d-x}{v_2\sqrt{(d-x)^2+n^2}} = 0$, which is equivalent to $\dfrac{x}{v_1\sqrt{x^2+m^2}} = \dfrac{d-x}{v_2\sqrt{(d-x)^2+n^2}}$, or in terms of the angles θ_1 and θ_2 in the figure, $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$ (Snell's Law). Observe that $T'(0) < 0$ and $T'(d) > 0$, so the minimum value of $T(x)$ must occur at some $x \in (0, d)$. There must be exactly one critical point, because as x ranges from 0 to d, $\sin\theta_1$ decreases and $\sin\theta_2$ increases, so Snell's Law can hold for at most one value of x.



4.5.57 Let h be the height of the cylindrical tower and r the radius of the dome. The cylinder has volume $\pi r^2 h$, and the hemispherical dome has volume $2\pi r^3/3$ (half the volume of a sphere of radius r). The total volume is 750, so we have the constraint $\pi r^2 h + \dfrac{2\pi r^3}{3} = 750$ which gives $h = \dfrac{750}{\pi r^2} - \dfrac{2r}{3}$. We must have $h \geq 0$, which is equivalent to $r \leq \sqrt[3]{1125/\pi}$. The objective function to be maximized is the cost of the metal to make the silo, which is proportional to the surface area of the cylinder ($= 2\pi rh$) plus 1.5 times the surface area of the hemisphere ($= 2\pi r^2$). So we can take as objective function

$$C = 2\pi rh + 1.5 \cdot 2\pi r^2 = 2\pi r\left(\frac{750}{\pi r^2} - \frac{2r}{3}\right) + 3\pi r^2 = \frac{1500}{r} + \frac{5}{3}\pi r^2.$$

The critical points of $C(r)$ satisfy $C'(r) = -\dfrac{1500}{r^2} + \dfrac{10}{3}\pi r = 0$, which gives $\pi r^3 = 450$ and hence $r = \sqrt[3]{450/\pi}$. The corresponding value of h is

$$h = \frac{750}{\pi r^2} - \frac{2r}{3} = \frac{750r}{\pi r^3} - \frac{2r}{3} = \left(\frac{750}{450} - \frac{2}{3}\right) r = r.$$

By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, \sqrt[3]{1125/\pi}]$. Therefore the dimensions that minimize the cost are $r = h = \sqrt[3]{450/\pi}$ m.

4.5.58

a. Using the figure, observe that $\sin\theta = \dfrac{l_3}{l_2}$ and $\cot\theta = \dfrac{l_4 - l_1}{l_3}$. It follows that $l_2 = l_3 \csc\theta$ and $l_1 = l_4 - l_3 \cot\theta$. So by substitution, we have

$$T = k\left(\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right) = k\left(\frac{l_4 - l_3\cot\theta}{r_1^4} + \frac{l_3\csc\theta}{r_2^4}\right).$$

b. Differentiating gives

$$\frac{dT}{d\theta} = k\left(\frac{l_3}{r_1^4}\csc^2\theta - \frac{l_3}{r_2^4}\csc\theta\cot\theta\right) = k\left(\frac{l_3}{r_1^4}\csc^2\theta - \frac{l_3}{r_2^4}\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\right) = kl_3\csc^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right).$$

For $0 < \theta < \pi/2$, this can only be zero for $\cos\theta = \dfrac{r_2^4}{r_1^4} = \left(\dfrac{r_2}{r_1}\right)^4$. To verify that this corresponds to a minimum using the First Derivative Test, we rewrite the derivative in a different form:

$$\frac{dT}{d\theta} = kl_3\csc^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right) = \frac{kl_3\csc^2\theta}{r_2^4}\cdot\left(\frac{r_2^4}{r_1^4} - \cos\theta\right).$$

Suppose that $\theta^* < \pi/2$ is the solution to $\cos\theta = \left(\frac{r_2}{r_1}\right)^4$. If $\theta < \theta^*$, then $\cos\theta > \cos\theta^* = \left(\frac{r_2}{r_1}\right)^4$, which implies that $\frac{dT}{d\theta} < 0$. Similarly, if $\theta > \theta^*$, then $\frac{dT}{d\theta} > 0$. So by the First Derivative Test, T has a local minimum at $\cos\theta = \left(\frac{r_2}{r_1}\right)^4$. Because the local minimum corresponds to the only critical point in $(0, \pi/2)$, it is an absolute minimum.

c. We have $r_2 = 0.85r_1$, which implies that $\cos\theta = 0.85^4$. Using a calculator, we find that $\theta \approx 1.02$ rad which is about $58.5°$.

4.5.59

a. $f'(x) = 2(x - a_1) + 2(x - a_2) = 4x - 2(a_1 + a_2)$. This is zero for $x = \frac{a_1 + a_2}{2}$. Because $f'(x) < 0$ for $x < \frac{a_1 + a_2}{2}$ and $f'(x) > 0$ for $x > \frac{a_1 + a_2}{2}$, we have a minimum at $x = \frac{a_1 + a_2}{2}$.

b. $f'(x) = 2(x - a_1) + 2(x - a_2) + 2(x - a_3) = 6x - 2(a_1 + a_2 + a_3)$. This is zero for $x = \frac{a_1 + a_2 + a_3}{3}$. Because $f'(x) < 0$ for $x < \frac{a_1 + a_2 + a_3}{3}$ and $f'(x) > 0$ for $x > \frac{a_1 + a_2 + a_3}{3}$, we have a minimum at $x = \frac{a_1 + a_2 + a_3}{3}$.

c. $f'(x) = 2\sum_{k=1}^{n}(x - a_k) = 2nx - 2\sum_{k=1}^{n} a_k$. This is zero when $x = \frac{\sum_{k=1}^{n} a_k}{n}$. An application of the First Derivative Test shows that this value of x yields a minimum.

4.5.60 We have $x = 100\tan\theta$, so the rate at which the beam sweeps along the highway is

$$\frac{dx}{dt} = 100\sec^2\theta\frac{d\theta}{dt} = 100\sec^2\theta \cdot \frac{\pi}{6} = \frac{50\pi}{3}\sec^2\theta.$$

The beam meets the highway provided that the angle θ satisfies $-\pi/2 < \theta < \pi/2$. The function $\sec^2\theta$ is unbounded on this interval, and so has no maximum. The minimum value occurs at $\theta = 0$, because everywhere else $\sec^2\theta > 1$. Therefore the minimum rate is $50\pi/3 \approx 52.360$ m/s, and there is no maximum rate.

4.5.61 The cross-section is a trapezoid with height $3\sin\theta$; the larger of the parallel sides has length $3 + 2 \cdot 3\cos\theta = 3 + 6\cos\theta$ and the smaller parallel side has length 3. The area of this trapezoid is given by

$$A(\theta) = \frac{1}{2}(3 + (3 + 6\cos\theta)) \cdot 3\sin\theta = 9(1 + \cos\theta)\sin\theta = 9\left(\sin\theta + \frac{\sin 2\theta}{2}\right),$$

using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. We wish to maximize this function for $0 \le \theta \le \pi/2$. The critical points of $A(\theta)$ satisfy $\cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = 0$, using the identity $\cos 2\theta = 2\cos^2\theta - 1$. Therefore $x = \cos\theta$ satisfies the quadratic equation $2x^2 + x - 1 = 0$, which has roots $x = 1/2$ and -1. So the only critical point in $(0, \pi/2)$ is $\theta = \cos^{-1}(1/2) = \pi/3$, which by the First (or Second) Derivative Test and Theorem 4.5 gives the maximum area.

4.5.62

a. Gliding is more efficient if $S > 0$; substituting $m = 200$ in the equation for $S(m, \theta)$ gives

$$S > 0 \iff 8.46 \cdot 200^{2/3} - 1.36 \cdot 200\tan\theta > 0 \iff \tan\theta < 1.064$$

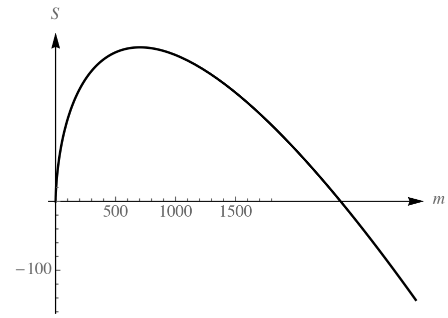
so $\theta < \tan^{-1} 1.064 \approx 46.768°$.

b. We solve $S(m, \theta) = 0$ for θ, which gives $\tan\theta \approx 6.22m^{-1/3}$, so $\theta = g(m) \approx \tan^{-1}(6.22m^{-1/3})$. This is a decreasing function of body mass.

c. Because $\theta = g(m)$ is a decreasing function of body mass, larger gliders have a smaller selection of glide angles for which gliding is more efficient than walking.

d.

Gliding is more efficient when $S(m, 25°) > 0$, which is true if and only if $8.46m^{2/3} > 1.36m \tan 25°$, which is true if and only if $m^{1/3} < \dfrac{8.46}{1.36\tan 25°} \approx 13.34$, which again is true if and only if $m < 2374$ g.



e. We have $\frac{d}{dm}S(m,\theta) = 5.64m^{-1/3} - 1.36\tan\theta$, which is 0 for $m^* \approx 71.32\cot^3\theta$. For $\theta = 25°$ this gives $m^* \approx 703$ g. The First Derivative Test shows that m^* is a local maximum, which by Theorem 4.5 is the absolute maximum for $m \ge 0$.

f. Because $\cot\theta$ is a decreasing function on $0 < \theta < 90°$, we see that m^* decreases with increasing θ.

g. From part (b) we have $g(10^6) \approx \tan^{-1}(6.22m^{-1/3}) \approx 3.56°$, so any angle $\theta < 3.56°$.

4.5.63 Let the radius of the Ferris wheel have length r, and let α be the angle the specific seat on the Ferris wheel makes with the center of the wheel (see the figure in the text). This point has coordinates $(r\cos\alpha, r + r\sin\alpha)$ so the distance from the seat to the base of the wheel is

$$d = \sqrt{r^2\cos^2\alpha + r^2(1+\sin\alpha)^2} = \sqrt{2}r\sqrt{1+\sin\alpha}.$$

Therefore the observer's angle satisfies $\tan\theta = \dfrac{r\sqrt{2}}{20}\sqrt{1+\sin\alpha}$. Think of θ and α as functions of time t and differentiate: $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{r\sqrt{2}}{20}\cdot\dfrac{\cos\alpha}{2\sqrt{1+\sin\alpha}}\dfrac{d\alpha}{dt} = \dfrac{\pi r\sqrt{2}}{40}\cdot\dfrac{\cos\alpha}{\sqrt{1+\sin\alpha}}$. Therefore

$$\frac{d\theta}{dt} = \frac{\pi r\sqrt{2}}{40}\frac{\cos^2\theta\cos\alpha}{\sqrt{1+\sin\alpha}}.$$

Observe that $\left|\dfrac{d\theta}{dt}\right| = \dfrac{\pi r\sqrt{2}}{40}\dfrac{\cos^2\theta|\cos\alpha|}{\sqrt{1+\sin\alpha}}\dfrac{\sqrt{1-\sin\alpha}}{\sqrt{1-\sin\alpha}}$, which can be written as $\dfrac{\pi r\sqrt{2}}{40}\cos^2\theta\sqrt{1-\sin\alpha}$. When the seat on the Ferris wheel is at its lowest point we have $\theta = 0$ and $\alpha = -\pi/2$, which gives $\cos^2\theta = 1$ and $\sqrt{1-\sin\alpha} = \sqrt{2}$. At any other point on the wheel we have $\cos^2\theta \le 1$ and $\sqrt{1-\sin\alpha} < \sqrt{2}$, so θ is changing most rapidly when the seat is at its lowest point.

4.5.64 Let x and y be the base and height of the triangle that is folded over, and z the height of point P above the base (see figure in the text). The Pythagorean Theorem gives $z^2 = x^2 - (a-x)^2 = 2ax - a^2$, so $z = \sqrt{2ax - a^2}$. The Pythagorean Theorem also gives $(y - \sqrt{2ax-a^2})^2 + a^2 = y^2$, which can be simplified to $2y\sqrt{2ax-a^2} = 2ax$, so $y = ax/\sqrt{2ax-a^2}$. The length L of the crease satisfies $L^2 = x^2 + y^2 = x^2 + \dfrac{a^2x^2}{2ax-a^2} = x^2\left(1 + \dfrac{a^2}{2ax-a^2}\right) = \dfrac{x^3}{x - \frac{a}{2}}$. Because L is positive, it suffices to minimize the function L^2 over $a/2 < x \le a$. We have $\dfrac{dL^2}{dx} = \dfrac{3x^2}{x-\frac{a}{2}} - \dfrac{x^3}{(x-\frac{a}{2})^2} = \dfrac{x^2}{x-\frac{a}{2}}\left(3 - \dfrac{x}{x-\frac{a}{2}}\right)$, and we solve $3(x - a/2) = x$

to obtain $x = 3a/4$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(a/2, a]$. Substituting in the equation for L^2 above, we find that the shortest crease has length $L = 3\sqrt{3}a/4$, and the height of the point P is $z = a/\sqrt{2}$. (The corresponding value of $y = 3\sqrt{2}a/4 \approx 1.061a$; so the height b of the rectangle must satisfy $b > 1.061a$ to be able to form the minimal crease.)

4.5.65 The critical points of the function $a(\theta)$ satisfy

$$a'(\theta) = \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{L}\right) = -\omega^2 r\sin\theta\left(1 + \frac{4r\cos\theta}{L}\right) = 0,$$

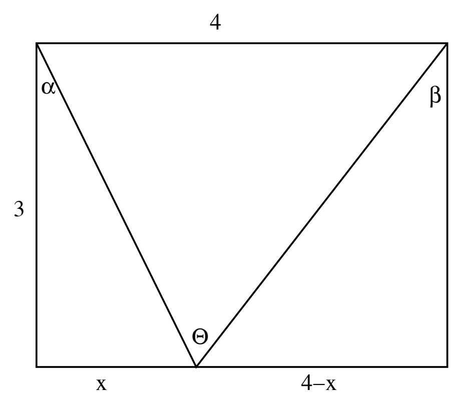
using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. There are two cases to consider separately: (a) $0 < L < 4r$ and (b) $L \geq 4r$. In case (a) the critical points in $[0, 2\pi]$ are $\theta = 0, \pi, 2\pi$ and also $\theta = \cos^{-1}(-L/(4r))$ and $2\pi - \cos^{-1}(-L/(4r))$. Comparing the values of $a(\theta)$ at these points shows that the maximum acceleration occurs at $\theta = 0$ and 2π and the minimum occurs at $\theta = \cos^{-1}(-L/(4r))$ and $2\pi - \cos^{-1}(-L/(4r))$. (There is a local maximum at $\theta = \pi$.) In case (b) the only critical points are $\theta = 0, \pi$ and 2π, and comparing the values of $a(\theta)$ at these points shows that the maximum acceleration occurs at $\theta = 0$ and 2π as in case (a), whereas the minimum occurs at $\theta = \pi$ in this case.

4.5.66 Let α and β be the angles labeled below (see figure). Then $\left(\frac{\pi}{2} - \alpha\right) + \theta + \left(\frac{\pi}{2} - \beta\right) = \pi$ so $\theta = \alpha + \beta$. We have $\tan\alpha = x/3$ and $\tan\beta = (4 - x)/3$, so we can express θ in terms of x as $\theta(x) = \tan^{-1}\left(\frac{x}{3}\right) + \tan^{-1}\left(\frac{4-x}{3}\right)$. We wish to maximize this function for $0 \leq x \leq 4$. The critical points of $\theta(x)$ satisfy

$$\theta'(x) = \frac{1}{1 + \left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} + \frac{1}{1 + \left(\frac{4-x}{3}\right)^2} \cdot \left(-\frac{1}{3}\right)$$

which can be written as $\dfrac{3}{x^2+9} - \dfrac{3}{(4-x)^2+9}$.

This is equal to zero for $x^2 = (4 - x)^2$ and because $x, 4 - x \geq 0$ we must have $x = 4 - x$, so $x = 2$ is the only critical point. We compare $\theta(x)$ at $x = 2$ and the endpoints $x = 0, 4$: $\theta(2) = 2\tan^{-1}\left(\frac{2}{3}\right) \approx 1.176$, $\quad \theta(0) = \theta(4) = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.937$. Therefore the maximum angle occurs when $x = 2$.



4.5.67

a. Using the Pythagorean Theorem, we find that the height of this triangle is 2. Let x be the distance from the point P to the base of the triangle; then the distance from P to the top vertex is $2 - x$ and the distance to each of the base vertices is $\sqrt{x^2+4}$, again by the Pythagorean Theorem. Therefore the sum of the distances to the three vertices is given by $S(x) = 2\sqrt{x^2+4} + 2 - x$. We wish to minimize this function for $0 \le x \le 2$. The critical points of $S(x)$ satisfy $S'(x) = \dfrac{2x}{\sqrt{x^2+4}} - 1 = 0$, which has unique solution $x = 2/\sqrt{3}$ in $(0,2)$. By the First Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,2]$. Therefore the optimal location for P is $2/\sqrt{3}$ units above the base.

b. In this case the objective function to be minimized is $S(x) = 2\sqrt{x^2+4} + h - x$ where $0 \le x \le h$. Exactly as above, we find that the only critical point $x > 0$ is $x = 2/\sqrt{3}$. This will give the absolute minimum on $[0,h]$ as long as $h \ge 2/\sqrt{3}$. When $h < 2/\sqrt{3}$, $S(x)$ is decreasing on $[0,h]$ and the minimum occurs at the endpoint $x = h$.

4.5.68 Let r and h be the radius and height of both the cylinder and cones. The surface area of each cone is $\pi r\sqrt{r^2+h^2}$ and the surface area of the cylinder is $2\pi rh$, so we have the constraint $2\pi r\sqrt{r^2+h^2} + 2\pi rh = A$, which we rewrite as $h + \sqrt{r^2+h^2} = \dfrac{A}{2\pi r}$. Square to obtain $h^2 + 2h\sqrt{r^2+h^2} + r^2 + h^2 = \left(\dfrac{A}{2\pi r}\right)^2$, and substitute $\sqrt{r^2+h^2} = A/(2\pi r) - h$ in this equation to obtain $h^2 + 2h\left(\dfrac{A}{2\pi r} - h\right) + r^2 + h^2 = \left(\dfrac{A}{2\pi r}\right)^2$. Solving for h yields $h = \dfrac{\pi r}{A}\left(\dfrac{A^2}{4\pi^2 r^2} - r^2\right) = \dfrac{A}{4\pi r} - \dfrac{\pi r^3}{A}$.

We must have $h \ge 0$, which is equivalent to the condition $r \le \sqrt{A}/\sqrt{2\pi}$. So the possible r under consideration satisfy $0 \le r \le \sqrt{A}/\sqrt{2\pi}$. The objective function to be maximized is the combined volume of the cylinder and cones, which is given by

$$V = \pi r^2 h + 2\cdot\frac{\pi}{3}r^2 h = \frac{5\pi}{3}r^2 h = \frac{5\pi}{3}r^2\left(\frac{A}{4\pi r} - \frac{\pi r^3}{A}\right) = \frac{5A}{12}r - \frac{5\pi^2}{3A}r^5.$$

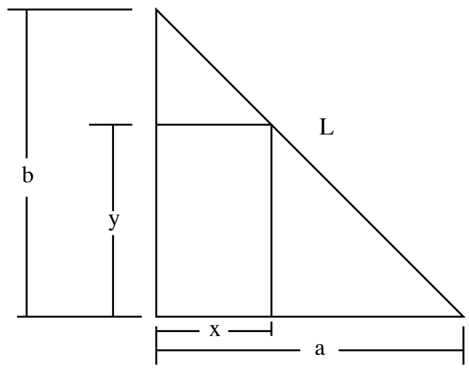
The critical points of $V(r)$ satisfy $V'(r) = \dfrac{5A}{12} - \dfrac{25\pi^2}{3A}r^4 = 0$, which has unique positive solution $r = \sqrt{A}/(\sqrt[4]{20}\sqrt{\pi})$. To find the corresponding value of h, observe that $\pi r^3/A = A/(20\pi r)$, so $h = \dfrac{A}{4\pi r} - \dfrac{\pi r^3}{A} = \dfrac{A}{4\pi r} - \dfrac{A}{20\pi r} = \dfrac{A}{5\pi r}$ which gives $h = \sqrt{A}\sqrt[4]{20}/(5\sqrt{\pi})$. Note that $V(r) = 0$ at the endpoints of the interval $[0, \sqrt{A}/\sqrt{2\pi}]$, so the maximum volume must occur at the values of r and h given above.

4.5.69 Let x be the distance between the point on the track nearest your initial position to the point where you catch the train. If you just catch the back of the train, then the train will have travelled $x + 1/3$ miles, which will require time $T = \dfrac{\text{distance}}{\text{rate}} = \dfrac{x+\frac{1}{3}}{20}$. The distance you must run is $\sqrt{x^2 + 1/(16)}$, so your running speed must be $v = \dfrac{\text{distance}}{\text{time}} = \dfrac{20\sqrt{x^2+\frac{1}{16}}}{x+\frac{1}{3}}$. We wish to minimize this function for $x \ge 0$. The derivative of $v(x)$ can be written $v'(x) = \left(\dfrac{x}{x^2+\frac{1}{16}} - \dfrac{1}{x+\frac{1}{3}}\right)v(x)$, so the critical points of $v(x)$ satisfy $\dfrac{x}{x^2+\frac{1}{16}} = \dfrac{1}{x+\frac{1}{3}}$ so $x\left(x+\dfrac{1}{3}\right) = x^2 + \dfrac{1}{16}$ which gives $x = 3/16$ mi. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0,\infty)$. The minimum running speed is $v\left(\dfrac{3}{16}\right) = \dfrac{20\sqrt{\left(\frac{3}{16}\right)^2 + \frac{1}{16}}}{\frac{3}{16}+\frac{1}{3}} = \dfrac{60\sqrt{9+16}}{9+16} = 12$ mph.

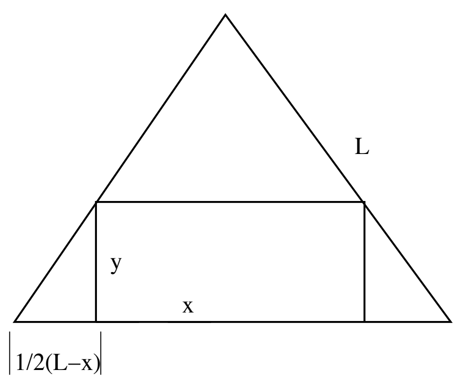
4.5.70

a. Let a and b be the side lengths of a particular right triangle with hypotenuse L, and let x and y be the side lengths of an inscribed rectangle (see figure). Then using similar triangles we see that

$y/(a-x) = b/a$, so $y = (b/a)(a-x)$. The area to be maximized is $A(x) = \frac{b}{a}x(a-x)$ over $0 \le x \le a$. This function has unique critical point $x = a/2$ and is 0 at the endpoints $x = 0$ and $x = a$; hence the maximum occurs when $x = a/2$. The other side of the rectangle has length $y = b/2$, so the maximum area is $A(a/2) = ab/4$. Note: We could also consider inscribing the rectangle so that one side rests on the hypotenuse. The maximum area is also $ab/4$ using this configuration. Now consider all possible right triangles with hypotenuse L. The side lengths a and b must satisfy $a, b \ge 0$ and $a^2 + b^2 = L^2$, which gives $b = \sqrt{L^2 - a^2}$. For each triangle, the largest area of an inscribed rectangle is $A = \dfrac{ab}{4} = \dfrac{a}{4}\sqrt{L^2 - a^2}$, which now we must maximize over $0 \le a \le L$. This function has unique critical point $a = L/\sqrt{2}$, and is 0 at the endpoints $a = 0$ and $a = L$. The constraint gives $b = a$, so the optimal triangle is an isosceles right triangle, and the largest inscribed rectangle is a square with side length $L/(2\sqrt{2})$ and area $L^2/8$.

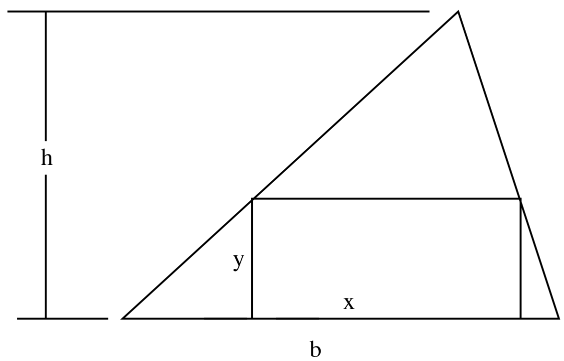


b. Let x and y be the dimensions of a rectangle inscribed in an equilateral triangle with side length L (see figure). Then $\tan\dfrac{\pi}{3} = \dfrac{y}{\frac{1}{2}(L-x)} = \sqrt{3}$ so $y = \dfrac{\sqrt{3}}{2}(L-x)$. The area to be maximized is $A = xy = \dfrac{\sqrt{3}}{2}x(L-x)$ over $0 \le x \le L$. This function has unique critical point $x = L/2$, and is 0 at the endpoints $x = 0$ and $x = L$; hence the maximum occurs when $x = L/2$. The other side of the rectangle has length $y = L\sqrt{3}/4$, so the maximum area is $A = L^2\sqrt{3}/8$.



c. Let a and b be the (non-hypotenuse) side lengths of a right triangle; then as shown in part (a) above, the inscribed rectangle of maximum area has side lengths $a/2$ and $b/2$ and area $ab/4 = A/2$.

d. Let b and h be the base and height of the triangle, assume the angles to the base are both less than or equal to $90°$, and let x and y be the side lengths of an inscribed rectangle (see figure). Then by similar triangles $\dfrac{h-y}{h} = \dfrac{x}{b}$ so $y = h\left(1 - \dfrac{x}{b}\right)$. The rectangle has area $xy = (h/b)x(b-x)$, and the maximum value of this function over $0 \le x \le b$ occurs at $x = b/2$, which gives $y = h/2$ and area $= bh/4 = A/2$ if the triangle has area A. (Note that if a rectangle is inscribed in a triangle, then two of the vertices of the rectangle must lie on the same side of the triangle, so the rectangle must rest on one of the sides of the triangle, and therefore the angles to that side must both be less than or equal to 90°. So the maximum area of an inscribed rectangle is $A/2$ in all cases.)



4.5.71

a. Let r and h be the radius and height of the inscribed cylinder. The region that lies above the cylinder inside the cone is a cone with radius r and height $H - h$; by similar triangles we have $\frac{H-h}{r} = \frac{H}{R}$ so $h = \frac{H}{R}(R - r)$. The volume of the cylinder is $V = \pi r^2 h = \frac{\pi H}{R}\left(Rr^2 - r^3\right)$, which we must maximize over $0 \le r \le R$. The critical points of $V(r)$ satisfy $V'(r) = \frac{\pi H}{R}\left(2Rr - 3r^2\right) = 0$, which has unique solution $r = 2R/3$ in $(0, R)$. Because $V(r) = 0$ at the endpoints $r = 0$ and $r = R$, the cylinder with maximum volume has radius $r = 2R/3$, height $h = H/3$ and volume $V = \pi r^2 h = \frac{4\pi}{27}R^2 H = \frac{4}{9} \cdot \frac{\pi}{3} R^2 H$; i.e. 4/9 the volume of the cone.

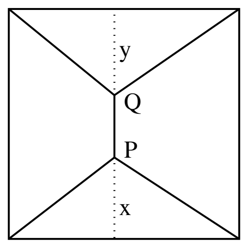
b. The lateral surface area of the cylinder is $A = 2\pi r h = 2\pi r \cdot \frac{H}{R}(R - r) = \frac{2\pi H}{R} r(R - r)$. This function takes its maximum over $0 \le r \le R$ at $r = R/2$, so the cylinder with maximum lateral surface area has dimensions $r = R/2$ and $h = H/2$.

4.5.72

a. Referring to the diagram in the text, note that if we drop a perpendicular from the 150° angle, the trapezoid is divided into a 30–60–90 triangle and an $x \times y$ rectangle. The slanted side has length $y \sec 60^\circ = 2y$, and the base of the trapezoid has length $x + 2y\cos 30^\circ = x + \sqrt{3}y$. The perimeter of the trapezoid is 1000, so we get the constraint $P = 2x + (3 + \sqrt{3})y = 1000$, which gives $x = 500 - (3 + \sqrt{3})y/2$. The objective function to be maximized is the area of the trapezoid, which is $A = \frac{1}{2}\left(x + (x + \sqrt{3}y)\right) y = (x + \frac{\sqrt{3}}{2}y)y = (500 - \frac{3}{2}y)y = 500y - \frac{3}{2}y^2$. Because we need both $x, y \ge 0$, we also must have $y \le 1000/(3 + \sqrt{3}) \approx 211.325$. The maximum value of the quadratic function $A(y)$ occurs at $y = 500/3 \approx 166.667$ ft, which is in the interval under consideration; the corresponding value of x is $250(3 - \sqrt{3}/3) \approx 105.662$ ft.

b. In this case we do not use fencing for the slanted side with length $2y$, so we modify the constraint to be $2x + (1 + \sqrt{3})y = 1000$, which gives $x = 500 - (1 + \sqrt{3})y/2$. The area of the trapezoid is $A = \frac{1}{2}\left(x + (x + \sqrt{3}y)\right) y = (x + \frac{\sqrt{3}}{2}y)y = (500 - \frac{1}{2}y)y = 500y - \frac{1}{2}y^2$. Because we need both $x, y \ge 0$ we also must have $y \le 1000/(1 + \sqrt{3}) \approx 366.025$. But the maximum value of the quadratic function $A(y)$ occurs at $y = 500$, which is outside the interval under consideration. Hence $A(y)$ is increasing over the interval $[0, 1000/(1+\sqrt{3})]$ and the maximum area occurs when $y = 1000/(1+\sqrt{3}) \approx 366.025$ ft and $x = 0$.

4.5.73 Following the hint, place two points P and Q above the midpoint of the base of the square, at distances x and y to the sides (see figure), where $0 \le x, y \le 1/2$. Then join the bottom vertices of the square to P, the upper vertices to Q and join P to Q. This road system has total length $L = 2\sqrt{x^2 + \frac{1}{4}} + 2\sqrt{y^2 + \frac{1}{4}} + (1 - x - y) = 1 + \left(\sqrt{4x^2+1} - x\right) + \left(\sqrt{4y^2+1} - y\right)$. We can minimize the contributions from x and y separately; the critical points of the function $f(x) = \sqrt{4x^2+1} - x$ satisfy $f'(x) = \dfrac{4x}{\sqrt{4x^2+1}} - 1 = 0$ which gives $\sqrt{4x^2+1} = 4x$, so $12x^2 = 1$ and $x = 1/(2\sqrt{3})$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $[0, 1/2]$. The minimum value of $f(x)$ on this interval is $f(1/(2\sqrt{3})) = \sqrt{3}/2$, so the shortest road system has length $L = 1 + 2 \cdot \dfrac{\sqrt{3}}{2} = 1 + \sqrt{3} \approx 2.732$.



4.5.74

a. Let x be the diameter of the smaller semicircle joining points A and B; then the other smaller semicircle has diameter $1 - x$. The area of a semicircle with diameter d is $\pi d^2/8$, so the area of the arbelos is given by
$$A(x) = \frac{\pi}{8}\left(1 - x^2 - (1-x)^2\right) = \frac{\pi}{8}\left(2x - 2x^2\right) = \frac{\pi}{4}x(1-x).$$
The quadratic function $x(1-x)$ takes its maximum at $x = 1/2$, so the largest area is obtained when we position point B at the center of the larger semicircle.

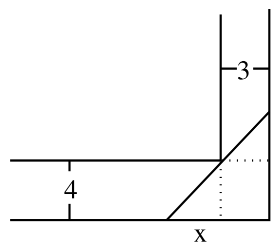
b. Point B has distance $|x - 1/2|$ to the center of the larger semicircle, so the length l of the segment BD can be found using the Pythagorean Theorem: $l^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 = x(1-x)$, so $l = \sqrt{x(1-x)}$ and a circle with diameter l has area $\pi\left(\frac{l}{2}\right)^2 = \frac{\pi}{4}x(1-x)$, which is the area of the arbelos.

4.5.75 A point on the curve $y = \sqrt{x}$ has the form $(x, \sqrt{x})$, which has distance L to the point $(p, 0)$ given by $L^2 = (x-p)^2 + \sqrt{x}^2 = x^2 + (1-2p)x + p^2$. Because L is positive, it suffices to minimize L^2 for $x \ge 0$. This quadratic function takes its minimum at $x = -(1-2p)/2 = p - 1/2$, so in case (i) the minimum occurs at the point $(p - 1/2, \sqrt{p - 1/2})$ and in case (ii) there are no critical points for $x > 0$, the function L^2 is increasing on $[0, \infty)$ so the minimum occurs at $(0, 0)$.

4.5.76

a. The length of the longest pole that can be carried around the corner is equal to the shortest length of a segment that joins the outer walls of the corridor while touching the inner corner (see figure). Using similar triangles, we can express this length in terms of the length x in the figure:
$$L(x) = \sqrt{x^2 + 4^2} + \frac{3}{x}\sqrt{x^2+4^2} = \left(1 + \frac{3}{x}\right)\sqrt{x^2+16}.$$
We wish to minimize this function for $x > 0$. The critical points of $L(x)$ satisfy
$$L'(x) = \left(1 + \frac{3}{x}\right)\frac{x}{\sqrt{x^2+16}} - \frac{3}{x^2}\sqrt{x^2+16} = 0$$

which simplifies to $(x^2+3x)x = 3(x^2+16)$ or $x^3 = 48$. This gives a unique critical point $x = \sqrt[3]{48}$ in the interval $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. The length of the longest pole is therefore $L(\sqrt[3]{48}) \approx 9.866$ ft.



b. For this case we replace 4 with a and 3 with b in the objective function from part (a) above, so we need to minimize the function $L(x) = \left(1 + \frac{b}{x}\right)\sqrt{x^2+a^2}$ for $x > 0$. The critical points of $L(x)$ satisfy

$$L'(x) = \left(1 + \frac{b}{x}\right)\frac{x}{\sqrt{x^2+a^2}} - \frac{b}{x^2}\sqrt{x^2+a^2} = 0$$

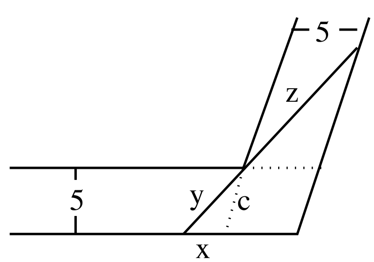
which simplifies to $(x^2+bx)x = b(x^2+a^2)$ or $x^3 = ba^2$. As above, this gives a unique critical point $x = \sqrt[3]{ba^2}$ in the interval $(0, \infty)$ which minimizes $L(x)$. The length of the longest pole is therefore

$$L(a^{\frac{2}{3}}b^{\frac{1}{3}}) = \left(1 + b^{\frac{2}{3}}a^{-\frac{2}{3}}\right)\sqrt{a^{\frac{4}{3}}b^{\frac{2}{3}} + a^2} = \left(1 + b^{\frac{2}{3}}a^{-\frac{2}{3}}\right)a^{\frac{2}{3}}\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

c. As above, we need to minimize the length $L = y + z$ as shown in the figure for $x > 0$. Note that $c = 5\csc 60^\circ = \frac{10}{\sqrt{3}}$ and by the law of cosines, $y^2 = x^2 + c^2 - 2cx\cos 120^\circ = x^2 + cx + c^2$. By similar triangles $z/y = c/x$, so we have $L(x) = y + z = \left(1 + \frac{c}{x}\right)\sqrt{x^2+cx+c^2}$. The critical points of $L(x)$ satisfy

$$L'(x) = \left(1 + \frac{c}{x}\right)\frac{2x+c}{2\sqrt{x^2+cx+c^2}} - \frac{c}{x^2}\sqrt{x^2+cx+c^2} = 0,$$

which simplifies to $(x^2+cx)(2x+c) = 2c(x^2+cx+c^2)$ or $2x^3+cx^2-c^2x-2c^3 = (x-c)(2x^2+3cx+2c^2) = 0$. The quadratic equation $2x^2+3cx+2c^2 = 0$ has discriminant $-7c^2 < 0$, so $x = c$ is the only critical point in $(0, \infty)$. By the First (or Second) Derivative Test, this critical point corresponds to a local minimum, and by Theorem 4.5, this solitary local minimum is also the absolute minimum on the interval $(0, \infty)$. The length of the longest pole is therefore $L(c) = 2\sqrt{3}c = 20$ ft.



d. Imagine a pole in any position in this corridor, and form a right triangle by dropping a segment perpendicular to the floor from the highest point on the pole down to the height of the lowest point on the pole. The base of this triangle can be no longer than the maximum length for the two-dimensional corridor in part (b), and the height is at most 8, so the maximum length L is given by combining the result from part (b) with the Pythagorean Theorem:

$$L = \sqrt{64 + \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^3}.$$

4.5.77 Let the angle of the cuts with the horizontal be ϕ_1 and ϕ_2, where $\phi_1 + \phi_2 = \theta$. The volume of the notch is proportional to $\tan\phi_1 + \tan\phi_2 = \tan\phi_1 + \tan(\theta - \phi_1)$, so it suffices to minimize the objective function $V(\phi_1) = \tan\phi_1 + \tan(\theta - \phi_1)$ for $0 \le \phi_1 \le \theta$. The critical points of $V(\phi_1)$ satisfy $\sec^2\phi_1 - \sec^2(\theta - \phi_1) = 0$, which is equivalent to the condition $\cos\phi_1 = \cos(\theta - \phi_1)$. This is satisfied if $\phi_1 = \theta - \phi_1$ which gives $\phi_1 = \theta/2$. There are no other solutions in $(0, \theta)$, because $\cos\phi_1$ is decreasing and $\cos(\theta - \phi_1)$ is increasing on $(0, \theta)$ and therefore can intersect at most once. So the only critical point occurs when $\phi_1 = \phi_2 = \theta/2$, and the First Derivative Test shows that this critical point is a local minimum; by Theorem 4.5, this must be the absolute minimum on $[0, \theta]$.

4.5.78 The radius r of a circle inscribed in a triangle is given by the formula $r = 2A/P$, where A is the area and P the perimeter of the triangle. Let x be the length of the base of the isosceles triangle. The height is then $\sqrt{1 - (x/2)^2}$ by the Pythagorean Theorem, and therefore the area is given by

$$A = \frac{1}{2}x\sqrt{1 - \frac{x^2}{4}} = \frac{1}{4}x\sqrt{4 - x^2}.$$

The perimeter is $x + 2$, so the radius of the inscribed triangle is

$$r = \frac{2A}{P} = \frac{1}{2} \cdot \frac{x\sqrt{4 - x^2}}{x + 2}.$$

The possible x values here satisfy $0 \le x \le 2$, so we need to maximize the function $r(x)$ on $[0, 2]$. We have

$$r'(x) = \frac{1}{2}\left(\frac{\sqrt{4 - x^2}}{x + 2} + \frac{x}{x + 2}\left(\frac{-x}{\sqrt{4 - x^2}}\right) - \frac{x\sqrt{4 - x^2}}{(x + 2)^2}\right),$$

which simplifies to

$$r'(x) = \frac{4 - 2x - x^2}{2(x + 2)\sqrt{4 - x^2}}.$$

Thus the critical points satisfy $x^2 + 2x - 4 = 0$. This equation has roots $-1 \pm \sqrt{5}$, so the only critical point in $(0, 2)$ is $x = \sqrt{5} - 1$. Because $r(0) = r(2) = 0$, the maximum radius must occur at $x = \sqrt{5} - 1$. For this value of x we have

$$r = \frac{\sqrt{2}}{8}(\sqrt{5} - 1)^{5/2} \approx 0.300, \quad \pi r^2 = \frac{\pi}{32}(\sqrt{5} - 1)^5 \approx 0.283.$$

4.5.79 Let x and y be the lengths of the sides of the pen, with y the side parallel to the barn. The diagonal has length $\sqrt{x^2 + y^2}$, by the Pythagorean Theorem. Therefore the constraint is $2x + y + \sqrt{x^2 + y^2} = 200$, which we rewrite as $2x + y = 200 - \sqrt{x^2 + y^2}$. Square both sides to obtain $4x^2 + 4xy + y^2 = 40{,}000 - 400\sqrt{x^2 + y^2} + x^2 + y^2$, which simplifies to $3x^2 + 4xy = 40{,}000 - 400\sqrt{x^2 + y^2}$. Now substitute $\sqrt{x^2 + y^2} = 200 - 2x - y$ in this equation and simplify to obtain $(3x - 200)(x - 200) = 4(100 - x)y$ so $y = \frac{(3x-200)(x-200)}{4(100-x)}$. The objective function to be maximized is the area of the pen, $A = xy$. Using the expression above for y in terms of x, we have

$$A = xy = \frac{x(3x - 200)(x - 200)}{4(100 - x)} = -\frac{1}{4} \cdot \frac{x(3x - 200)(x - 200)}{(x - 100)}.$$

The length x must be at least 0, and because the diagonal is at least as long as x, we must have $3x \le 200$; so x cannot exceed $200/3$. Therefore we need to maximize the function $A(x)$ defined above for $0 \le x \le 200/3$. We have

$$\begin{aligned} A'(x) &= -\frac{1}{4} \cdot \left(\frac{(3x - 200)(x - 200)}{(x - 100)} + \frac{x \cdot 3(x - 200)}{(x - 100)} + \frac{x(3x - 200)}{(x - 100)} - \frac{x(3x - 200)(x - 200)}{(x - 100)^2}\right) \\ &= \left(\frac{1}{x} + \frac{3}{3x - 200} + \frac{1}{x - 200} - \frac{1}{x - 100}\right)A(x). \end{aligned}$$

Because $A(x) > 0$ for $0 < x < 200/3$, the critical points of the objective function satisfy $\dfrac{1}{x} + \dfrac{3}{3x - 200} + \dfrac{1}{x - 200} = \dfrac{1}{x - 100}$ which when simplified gives the equation $6x^3 - 1700x^2 + 160{,}000x - 4{,}000{,}000 = 0$. Using a numerical solver, we find that this equation has exactly one solution in the interval $(0, 200/3)$, which is $x \approx 38.81$. To find the absolute maximum of A, we check the endpoints of $[0, 200/3]$ and the critical point $x \approx 38.814$. We have $A(0) = A(200/3) = 0$, so the absolute maximum occurs when $x \approx 38.814$ m; using the formula for y in terms of x above gives $y \approx 55.030$ m.

4.5.80 The dimensions of the box are $l - 2x$, $L - 2x$, and x, so the volume is given by

$$V(x) = x(l - 2x)(L - 2x) = 4x^3 - 2(l + L)x^2 + lLx.$$

The dimensions cannot be negative, so we must have $0 \le x \le l/2$ (because we are letting $L \to \infty$, we may assume that $l \le L$). The critical points of $V(x)$ satisfy

$$V'(x) = 12x^2 - 4(l + L)x + lL = 0,$$

and this quadratic equation has roots

$$x = \frac{L + l \pm \sqrt{L^2 - lL + l^2}}{6}.$$

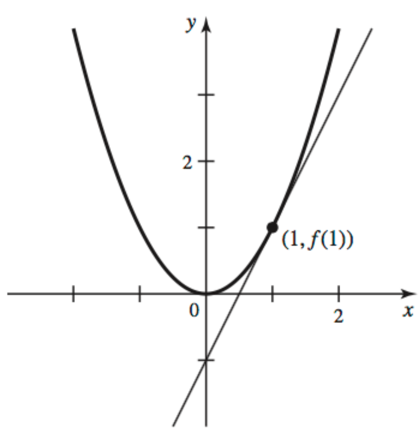
Now $V(x)$ is a cubic polynomial with roots $x = 0, l/2, L/2$, and so has exactly one critical point between 0 and $l/2$, which gives the maximum of $V(x)$ on the interval $[0, l/2]$. Because $V'(0) > 0$, this critical point is given by the smaller root of the quadratic above:

$$\begin{aligned} x &= \frac{L + l - \sqrt{L^2 - lL + l^2}}{6} \\ &= \frac{(L + l - \sqrt{L^2 - lL + l^2})}{6} \frac{(L + l + \sqrt{L^2 - lL + l^2})}{(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{(L + l)^2 - (L^2 - lL + l^2)}{6(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{3lL}{6(L + l + \sqrt{L^2 - lL + l^2})} \\ &= \frac{l}{2\left(1 + \frac{l}{L} + \sqrt{1 - \frac{l}{L} + \frac{l^2}{L^2}}\right)} \end{aligned}$$

(for the last step, divide all terms by L). As $L \to \infty$ with l fixed, $l/L \to 0$ so the size x of the corner squares that maximizes the volume has limit $l/4$ as $L \to \infty$.

4.6 Linear Approximation and Differentials

4.6.1



4.6.2 The derivative of a function is 0 at a local maximum, so the linear approximation is a horizontal line.

4.6.3 If f is differentiable at the point, then near that point, f is approximately linear, so the function nearly coincides with the tangent line at that point.

4.6.4 The change in $y = f(x)$ may be approximated by the formula $\Delta y \approx f'(x)\Delta x$.

4.6.5 $L(x) = f(1) + f'(1)(x-1) = 2 + 3(x-1) = 3x - 1$, so $f(1.1) \approx L(1.1) = 3(1.1) - 1 = 2.3$.

4.6.6 $L(x) = 5x - 3$, so $f(2.01) \approx 5(2.01) - 3 = 7.05$.

4.6.7 $L(x) = f(4) + f'(4)(x-4) = 3 + 2(x-4)$. So $f(3.85) \approx L(3.85) = 3 + 2(3.85 - 4) = 3 - .3 = 2.7$.

4.6.8 $L(x) = f(5) + f'(5)(x-5) = 10 - 2(x-5)$. So $f(5.1) \approx L(5.1) = 10 - 2(.1) = 9.8$.

4.6.9 The relationship is given by $dy = f'(x)dx$, which is the linear approximation of the change Δy in $y = f(x)$ corresponding to a change dx in x.

4.6.10 The differential dy is precisely the change in the linear approximation to f, which is an approximation of the change in f for small changes dx in x.

4.6.11 $\Delta y \approx f'(a)\Delta x$, so $f'(a) \approx \dfrac{\Delta y}{\Delta x}$. Therefore $f'(5) \approx \dfrac{f(5.01) - f(5)}{5.01 - 5} = \dfrac{0.25}{0.01} = 25$.

4.6.12 $\Delta y \approx f'(a)\Delta x$, so $f'(a) \approx \dfrac{\Delta y}{\Delta x}$. Therefore $f'(6) \approx \dfrac{f(5.99) - f(6)}{5.99 - 6} = \dfrac{-0.002}{-0.01} = 0.2$.

4.6.13 The approximate average speed is $L(-1) = 60 - (-1) = 61$ miles per hour. The exact speed is $\dfrac{3600}{59}$ miles per hour which is about 61.02 miles per hour.

4.6.14 The approximate average speed is $L(3) = 60 - (3) = 57$ miles per hour. The exact speed is $\dfrac{3600}{63}$ miles per hour which is about 57.14 miles per hour.

4.6.15 Let $T(x) = \dfrac{60D}{60 + x}$. Then $T'(x) = -\dfrac{60D}{(60+x)^2}$, so $T'(0) = -\dfrac{D}{60}$. The linear approximation is given by $L(x) = T(0) - \dfrac{D}{60}(x - 0)$, or $L(x) = D\left(1 - \dfrac{x}{60}\right)$.

4.6.16 With $D = 45$, we have $T(2) \approx L(2) = 45(1 - \dfrac{2}{60}) = 45 - (3/2) = 43.5$ minutes. The exact time required is $T(2) = \dfrac{60 \cdot 45}{60 + 2} \approx 43.55$ minutes.

4.6.17 With $D = 80$, we have $T(-3) \approx L(-3) = 80(1 - \dfrac{-3}{60}) = 80 + 4 = 84$ minutes. The exact time required is $T(-3) = \dfrac{60 \cdot 80}{60 - 3} \approx 84.211$ minutes.

4.6.18 With $D = 93$, we have $T(3) \approx L(3) = 93(1 - \dfrac{3}{60}) = 93 - \dfrac{93}{20} = 88.35$ minutes. The exact time required is $T(3) = \dfrac{60 \cdot 93}{60 + 3} \approx 88.57$ minutes.

4.6.19 $f'(x) = 8x + 1$, so $f'(1) = 9$.

$$L(x) = f(1) + f'(1)(x-1) = 5 + 9(x-1) = 9x - 4.$$

4.6.20 $f'(x) = 3x^2 - 5$, so $f'(2) = 7$.

$$L(x) = f(2) + f'(2)(x-2) = 1 + 7(x-2) = 7x - 13.$$

4.6.21 $g'(t) = \dfrac{2}{2\sqrt{2t+9}} = \dfrac{1}{\sqrt{2t+9}}$, so $g'(-4) = 1$.

$$L(t) = g(-4) + g'(-4)(t+4) = 1 + 1(t+4) = t+5.$$

4.6.22 $h'(w) = \dfrac{5}{2\sqrt{5w-1}}$, so $h'(1) = \dfrac{5}{4}$.

$$L(w) = h(1) + h'(1)(w-1) = 2 + \frac{5}{4}(w-1) = \frac{5}{4}w + \frac{3}{4}.$$

4.6.23 $f'(x) = 3e^{3x-6}$, so $f'(2) = 3$.

$$L(x) = f(2) + f'(2)(x-2) = 1 + 3(x-2) = 3x-5.$$

4.6.24 $f'(x) = 9(2/3)(4x+11)^{-1/3}(4) = \dfrac{24}{(4x+11)^{1/3}}$, so $f'(4) = \dfrac{24}{3} = 8$.

$$L(x) = f(4) + f'(4)(x-4) = 81 + 8(x-4) = 8x+49.$$

4.6.25

a. Note that $f(a) = f(2) = 8$ and $f'(a) = -2a = -4$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 8 + (-4)(x-2) = -4x+16.$$

b. We have $f(2.1) \approx L(2.1) = 7.6$.

c. The percentage error is $100 \cdot \dfrac{|7.6-7.59|}{7.59} \approx 0.13\%$.

4.6.26

a. Note that $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$ and $f'(a) = \cos a = \sqrt{2}/2$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = \sin\frac{\pi}{4} + \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\left(x+1-\frac{\pi}{4}\right)$.

b. We have $f(0.75) \approx L(0.75) \approx 0.68$.

c. The percentage error is $100 \cdot \dfrac{\left|\frac{\sqrt{2}}{2}\left(1.75-\frac{\pi}{4}\right) - \sin 0.75\right|}{\sin 0.75} \approx 0.064\%$.

4.6.27

a. Note that $f(a) = f(0) = \ln 1 = 0$ and $f'(a) = 1/(1+a) = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = x.$$

b. We have $f(0.9) \approx L(0.9) = 0.9$.

c. The percentage error is $100 \cdot \dfrac{|0.9-\ln 1.9|}{|\ln 1.9|} \approx 40\%$.

4.6.28

a. Note that $f(a) = \dfrac{a}{a+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$ and $f'(a) = \dfrac{1}{(1+a)^2} = \dfrac{1}{4}$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = \dfrac{1}{2} + \dfrac{1}{4}(x-1) = \dfrac{1}{4}(x+1)$.

b. We have $f(1.1) \approx L(1.1) = 0.525$.

c. The percentage error is $100 \cdot \dfrac{|0.525 - (1.1/2.1)|}{1.1/2.1} \approx 0.23\%$.

4.6.29

a. Note that $f(a) = f(0) = \cos 0 = 1$ and $f'(a) = -\sin a = 0$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 1.$$

b. We have $f(-0.01) \approx L(-0.01) = 1$.

c. The percentage error is $100 \cdot \dfrac{|1 - \cos(-0.01)|}{\cos(-0.01)} \approx 0.005\%$.

4.6.30

a. Note that $f(a) = e^a = e^0 = 1$ and $f'(a) = e^a = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 1 + x.$$

b. We have $f(0.05) \approx L(0.05) = 1.05$.

c. The percentage error is $100 \cdot \dfrac{|1.05 - e^{0.05}|}{e^{0.05}} \approx 0.12\%$.

4.6.31

a. Note that $f(a) = 8^{-1/3} = 1/2$ and $f'(a) = (-1/3)(8)^{-4/3} = -1/48$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = \frac{1}{2} + \frac{-1}{48}x.$$

b. We have $f(-0.1) \approx L(-0.1) \approx 0.50208333$

c. The percentage error is $100 \cdot \dfrac{|7.9^{-1/3} - .050208333|}{(7.9)^{-1/3}} \approx 0.003\%$.

4.6.32

a. Note that $f(a) = \sqrt[4]{81} = 3$ and $f'(a) = \frac{1}{4}(81)^{-3/4} = \frac{1}{108}$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 3 + \frac{1}{108}(x - 81).$$

b. We have $f(85) \approx L(85) = 3 + \frac{4}{108} = 3 + \frac{1}{27} \approx 3.04$.

c. The percentage error is $100 \cdot \dfrac{|3.04 - \sqrt[4]{85}|}{\sqrt[4]{85}} \approx 0.12\%$.

4.6.33

a. Note that $f(a) = f(0) = 1$ and $f'(a) = -1/(1+a)^2 = -1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 1 - x.$$

b. The linear approximation to $1/1.1$ is $\dfrac{1}{1.1} \approx L(0.1) = 0.9$.

c. The percentage error is $100 \cdot \dfrac{|0.9 - \frac{1}{1.1}|}{\frac{1}{1.1}} = 1\%$.

4.6.34

a. Note that $f(a) = \cos(\pi/4) = \sqrt{2}/2$ and $f'(a) = -\sin(\pi/4) = -\sqrt{2}/2$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4} - x\right).$$

b. The linear approximation to $\cos(0.8)$ is $\cos(0.8) \approx L(0.8) = \dfrac{\sqrt{2}}{2}\left(0.2 + \dfrac{\pi}{4}\right) \approx 0.697$.

c. The percentage error is $100 \cdot \dfrac{\left|\frac{\sqrt{2}}{2}\left(0.2 + \frac{\pi}{4}\right) - \cos(0.8)\right|}{\cos(0.8)} \approx 0.011\%$.

4.6.35

a. Note that $f(a) = f(0) = 1$ and $f'(a) = -e^{-a} = -1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 1 - x.$$

b. The linear approximation to $e^{-0.03}$ is $e^{-0.03} \approx L(0.03) = 0.97$.

c. The percentage error is $100 \cdot \dfrac{|0.97 - e^{-0.03}|}{e^{-0.03}} \approx 0.046\%$.

4.6.36

a. Note that $f(a) = \tan 0 = 0$ and $f'(a) = \sec^2 0 = 1$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = x.$$

b. The linear approximation to $\tan 3^\circ$ is $\tan 3^\circ = \tan \dfrac{\pi}{60} \approx L\left(\dfrac{\pi}{60}\right) = \dfrac{\pi}{60} \approx 0.052$ Note that we must convert 3° to radians before applying the linear approximation formula.

c. The percentage error is $100 \cdot \dfrac{\left|\dfrac{\pi}{60} - \tan\dfrac{\pi}{60}\right|}{\dfrac{\pi}{60}} \approx 0.091\%$.

4.6.37 Let $f(x) = 1/x$, $a = 200$. Then $f(a) = 0.005$ and $f'(a) = -1/a^2 = -0.000025$, so the linear approximation to f near $a = 200$ is

$$L(x) = f(a) + f'(a)(x-a) = 0.005 - 0.000025(x - 200).$$

Therefore

$$\frac{1}{203} = f(203) \approx L(203) = 0.004925.$$

4.6.38 Let $f(x) = \tan x$, $a = 0$. Then $f(a) = 0$ and $f'(a) = \sec^2 a = 1$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = x.$$

Therefore

$$\tan(-2^\circ) = \tan\left(\frac{-\pi}{90}\right) = f\left(\frac{-\pi}{90}\right) \approx L\left(\frac{-\pi}{90}\right) \approx -0.0349.$$

Note that we must convert -2° to radians before applying the linear approximation formula.

4.6.39 Let $f(x) = \sqrt{x}$, $a = 144$. Then $f(a) = 12$ and $f'(a) = 1/(2\sqrt{a}) = 1/24$, so the linear approximation to f near $a = 144$ is

$$L(x) = f(a) + f'(a)(x-a) = 12 + \frac{1}{24}(x - 144).$$

Therefore

$$\sqrt{146} = f(146) \approx L(146) = \frac{145}{12}.$$

4.6.40 Let $f(x) = x^{1/3}$, $a = 64$. Then $f(a) = 4$ and $f'(a) = (1/3)a^{-2/3} = 1/48$, so the linear approximation to f near $a = 144$ is

$$L(x) = f(a) + f'(a)(x-a) = 4 + \frac{1}{48}(x - 64).$$

Therefore

$$\sqrt[3]{65} = f(65) \approx L(65) = \frac{193}{48}.$$

4.6.41 Let $f(x) = \ln x$, $a = 1$. Then $f(a) = 0$ and $f'(a) = 1/a = 1$, so the linear approximation to f near $a = 1$ is

$$L(x) = f(a) + f'(a)(x-a) = x - 1.$$

Therefore

$$\ln(1.05) = f(1.05) \approx L(1.05) = 0.05.$$

4.6.42 Let $f(x) = \sqrt{x}$, $a = 0.16$ (note that $5/29 \approx 0.17$). Then $f(a) = 0.4$ and $f'(a) = 1/(2\sqrt{a}) = 1.25$, so the linear approximation to f near $a = 0.16$ is

$$L(x) = f(a) + f'(a)(x-a) = 0.4 + 1.25(x - 0.16).$$

Therefore

$$\sqrt{5/29} = f(5/29) \approx L(5/29) \approx 0.416.$$

4.6.43 Let $f(x) = e^x$, $a = 0$. Then $f(a) = 1$ and $f'(a) = e^a = 1$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = 1 + x.$$

Therefore

$$e^{0.06} = f(0.06) \approx L(0.06) \approx 1.060.$$

4.6.44 Let $f(x) = 1/\sqrt{x}$, $a = 121$. Then $f(a) = 1/11$ and $f'(a) = -1/(2a^{3/2}) = -1/2662$, so the linear approximation to f near $a = 121$ is

$$L(x) = f(a) + f'(a)(x-a) = \frac{1}{11} - \frac{1}{2662}(x - 121).$$

Therefore

$$\frac{1}{\sqrt{119}} = f(119) \approx L(119) \approx 0.0917.$$

4.6.45 Let $f(x) = 1/\sqrt[3]{x}$, $a = 512$. Then $f(a) = 1/8$ and $f'(a) = -1/(3a^{4/3}) = -1/12{,}288$, so the linear approximation to f near $a = 512$ is

$$L(x) = f(a) + f'(a)(x-a) = \frac{1}{8} - \frac{1}{12{,}288}(x - 512).$$

Therefore

$$\frac{1}{\sqrt[3]{510}} = f(510) \approx L(510) = \frac{769}{6144} \approx 0.1252.$$

4.6.46 Let $f(x) = \cos x$, $a = \pi/6$ $(= 30°)$. Then $f(a) = \sqrt{3}/2$ and $f'(a) = -\sin a = -1/2$, so the linear approximation to f near $a = 0$ is

$$L(x) = f(a) + f'(a)(x-a) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right).$$

Therefore

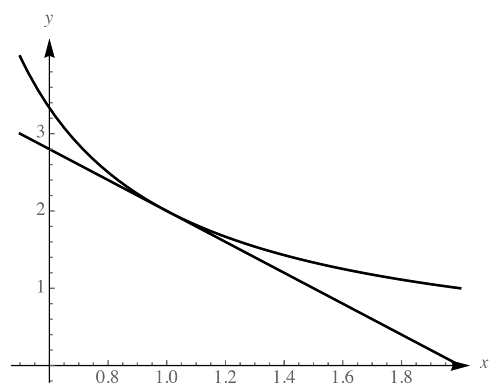
$$\cos 31° = \cos\frac{31\pi}{180} = f\left(\frac{31\pi}{180}\right) \approx L\left(\frac{31\pi}{180}\right) \approx 0.857.$$

Note that we must convert 31° to radians before applying the linear approximation formula.

4.6.47

a. With $f(x) = \dfrac{2}{x}$ and $a = 1$, we have $f(a) = 2$ and $f'(a) = -\frac{2}{a^2} = -2$. Thus the linear approximation to $f(x)$ at $x = 1$ is $L(x) = f(1) + f'(1)(x-1) = 2 + -2(x-1) = -2x + 4$.

b. A plot of f with L:



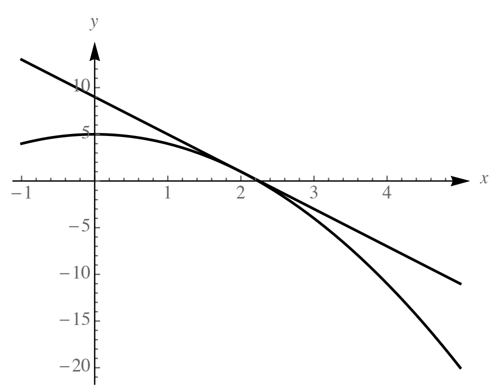
c. The linear approximation in part (b) appears to be an underestimate everywhere, because it lies below the graph of f.

d. Because $f'(x) = -\dfrac{2}{x^2}$, we have $f''(x) = \dfrac{4}{x^3}$, so that $f''(1) > 0$, and f is concave up at $x = 1$. This is consistent with L being an underestimate near $x = 1$.

4.6.48

a. With $f(x) = 5 - x^2$ and $a = 2$, we have $f(a) = 1$ and $f'(a) = -2a = -4$. Thus the linear approximation to f at $x = 2$ is $L(x) = f(2) + f'(2)(x-2) = 1 + -4(x-2) = -4x + 9$.

b. A plot of f with L:



c. The linear approximation in part (b) appears to be an overestimate everywhere, because it lies above the graph of f

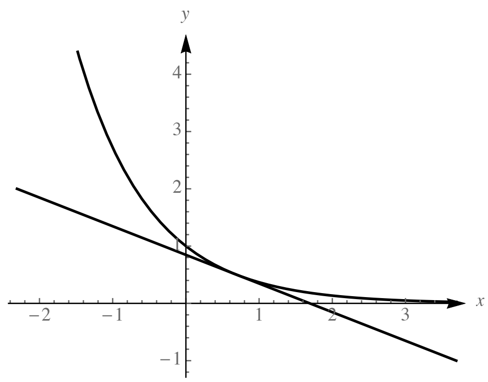
d. Because $f'(x) = -2x$, we have $f''(x) = -2$, so that $f''(x) < 0$ for all values of x and f is therefore concave down everywhere. This is consistent with L being an overestimate.

4.6.49

a. With $f(x) = e^{-x}$ and $a = \ln 2$, we have $f(a) = \frac{1}{2}$ and $f'(a) = -e^{-a} = -\frac{1}{2}$. Thus the linear approximation to f at $x = \ln 2$ is

$$L(x) = f(\ln 2) + f'(\ln 2)(x - \ln 2) = \frac{1}{2} + -\frac{1}{2}(x - \ln 2) = -\frac{1}{2}x + \frac{1}{2}(1 + \ln 2).$$

b. A plot of f with L:



c. The linear approximation in part (b) appears to be an underestimate everywhere, because it lies below the graph of f

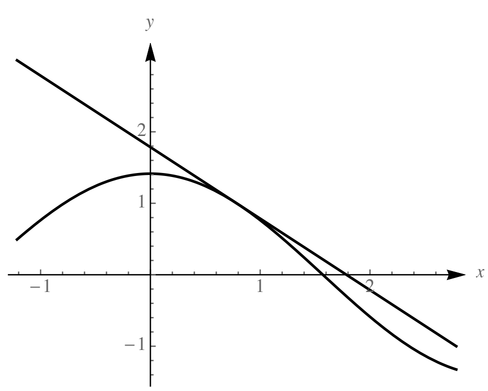
d. Because $f'(x) = -e^{-x}$, we have $f''(x) = e^{-x}$, so that $f''(x) > 0$ for all values of x and f is therefore concave up everywhere. This is consistent with L being an underestimate.

4.6.50

a. With $f(x) = \sqrt{2}\cos x$ and $a = \frac{\pi}{4}$, we have $f(a) = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$ and $f'(a) = -\sqrt{2}\sin(\pi/4) = -1$. Thus the linear approximation to f at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a) = 1 + -1\left(x - \frac{\pi}{4}\right) = -x + 1 + \frac{\pi}{4}.$$

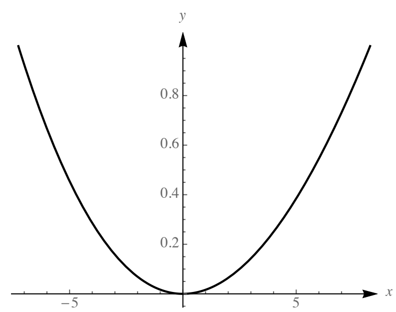
b. A plot of f with L:



c. The linear approximation in part (b) appears to be an overestimate everywhere, because it lies above the graph of f

d. Because $f'(x) = -\sqrt{2}\sin x$, we have $f''(x) = -\sqrt{2}\cos x$, so that $f''(a) < 0$ for x near $a = \frac{\pi}{4}$ and f is therefore concave down near $x = a$. This is consistent with L being an overestimate.

4.6.51 $E(x) = |L(x) - s(x)| = \left|60 - x - \dfrac{3600}{60 + x}\right|$. A graph is shown below, for x from -7.26 to 8.26.



$E(x) \le 1$ when $-7.26 \le x \le 8.26$, which corresponds to driving times for 1 mi from about 53 s to 68 s. Therefore, $L(x)$ gives approximations to $s(x)$ that are within 1 mi/hr of the true value when you drive 1 mile in t seconds, where $53 < t < 68$.

4.6.52

a. We have $P = T/V$ with V held constant, which is a linear function of T. Hence $\Delta P = \frac{\Delta T}{V} = \frac{0.05}{V}$. Because this is greater than 0, the pressure increases.

b. If T is held constant then $dP = -(T/V^2)dV$, and the approximate change in pressure is $\Delta P \approx dP = -0.1\dfrac{T}{V^2} < 0$, so the pressure decreases.

c. We have $T = PV$ with P held constant, which is a linear function of V. Hence $\Delta T = P\Delta V = 0.1P > 0$, so the temperature increases.

4.6.53

a. True. Note that $f(0) = 0$ and $f'(0) = 0$, so the linear approximation at 0 is in fact $L(x) = 0$.

b. False. The function $f(x) = |x|$ is not differentiable at $x = 0$, so there is no good linear approximation at 0.

c. True. For linear functions, the linear approximation at any point and the function are equal.

d. True. Note that $f'(x) = \frac{1}{x}$, so that $f''(x) = \frac{-1}{x^2}$, and $f''(e) < 0$, so f is concave down near $x = e$. Thus L is an overestimate of f.

4.6.54 Because $T = \dfrac{D}{\text{speed}}$, we have $T = \dfrac{D}{60 + x}$ hours which is $\dfrac{60D}{60 + x}$ minutes.

4.6.55 Note that $V'(r) = 4\pi r^2$, so $\Delta V \approx V'(a)\Delta r = 4\pi a^2 \Delta r$. Substituting $a = 5$ and $\Delta r = 0.1$ gives $\Delta V \approx 4\pi \cdot 25 \cdot 0.1 = 10\pi \approx 31.416$ ft^3.

4.6.56 Note that $P'(z) = -100e^{-z/10}$, so $\Delta P \approx P'(a)\Delta z = -100e^{-a/10}\Delta z$. Substituting $a = 2$ and $\Delta z = 0.01$ gives $\Delta P \approx -100e^{-0.2} \cdot 0.01 = -0.819$.

4.6.57 Note that V is a linear function of h with $V'(h) = \pi r^2 = 400\pi$, so $\Delta V = V'(a)\Delta r = 400\pi\Delta r$. Substituting $\Delta r = -0.1$ gives $\Delta V = -40\pi \approx -125.664$ cm^3.

4.6.58 Note that $V'(r) = 2\pi rh/3 = 8\pi r/3$, so $\Delta V \approx V'(a)\Delta h = \dfrac{8\pi a}{3}\Delta h$. Substituting $a = 3$ and $\Delta h = 0.05$ gives $\Delta V \approx 8\pi \cdot (0.05) = 0.4\pi \approx 1.257$ cm^3.

4.6.59 Note that $S'(r) = \pi\sqrt{r^2+h^2} + \pi r \cdot \dfrac{r}{\sqrt{r^2+h^2}} = \pi\dfrac{2r^2+h^2}{\sqrt{r^2+h^2}}$, so $\Delta S \approx S'(a)\Delta r = \pi\dfrac{2a^2+h^2}{\sqrt{a^2+h^2}}\Delta r$. Substituting $h = 6$, $a = 10$ and $\Delta r = -0.1$ gives $\Delta S \approx \pi\dfrac{236}{\sqrt{136}}(-0.1) = \dfrac{-59\pi}{5\sqrt{34}} \approx -6.358$ m^2.

4.6.60 Note that $F'(r) = -0.02r^{-3}$, so $\Delta F \approx F'(a)\Delta r = -0.02a^{-3}\Delta r$. Substituting $a = 20$ and $\Delta r = 1$ gives $\Delta F \approx -0.02 \cdot 20^{-3} \cdot 1 = -2.5 \cdot 10^{-6}$.

4.6.61 We have $f'(x) = 2$, so $dy = 2\,dx$.

4.6.62 We have $f'(x) = 2\sin x\cos x$, so $dy = 2\sin x\cos x\,dx$.

4.6.63 We have $f'(x) = -3/x^4$, so $dy = -\dfrac{3}{x^4}dx$.

4.6.64 We have $f'(x) = 2e^{2x}$, so $dy = 2e^{2x}\,dx$.

4.6.65 We have $f'(x) = a\sin x$, so $dy = a\sin x\,dx$.

4.6.66 We have $f'(x) = \dfrac{(4-x)\cdot 1 - (4+x)\cdot(-1)}{(4-x)^2} = \dfrac{8}{(x-4)^2}$, so $dy = \dfrac{8}{(x-4)^2}\,dx$.

4.6.67 We have $f'(x) = 9x^2 - 4$, so $dy = (9x^2-4)\,dx$.

4.6.68 We have $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, so $dy = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

4.6.69 We have $f'(x) = \sec^2 x$, so $dy = \sec^2 x\,dx$.

4.6.70 We have $f'(x) = -1/(1-x) = 1/(x-1)$, so $dy = \dfrac{1}{x-1}\,dx$.

4.6.71 Note that $f(a) = f(8) = 2$ and $f'(a) = (1/3)a^{-2/3} = 1/12$, so the linear approximation has equation

$$y = L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{12}(x-8) = \frac{x}{12} + \frac{4}{3}.$$

x	Linear approx	Exact value	Percent error
8.1	$2.008\overline{3}$	2.00829885	1.717×10^{-3}
8.01	$2.0008\overline{3}$	2.000832986	1.734×10^{-5}
8.001	$2.00008\overline{3}$	2.00008333	1.736×10^{-7}
8.0001	$2.000008\overline{3}$	2.00000833	1.736×10^{-9}
7.9999	$1.999991\overline{6}$	1.999991667	1.736×10^{-9}
7.999	$1.99991\overline{6}$	1.999916663	1.736×10^{-7}
7.99	$1.9991\overline{6}$	1.999166319	1.738×10^{-5}
7.9	$1.991\overline{6}$	1.991631701	1.756×10^{-3}

The percentage errors become extremely small as x approaches 8 In fact, each time we decrease Δx by a factor of 10, the percentage error decreases by a factor of 100.

4.6.72 Note that $f(a) = f(0) = 1$ and $f'(a) = -1(1+a)^2 = -1$, so the linear approximation has equation $y = L(x) = f(a) + f'(a)(x-a) = 1 - x$.

x	Linear approx	Exact value	Percent error
0.1	0.9	$0.\overline{90}$	1
0.01	0.99	$0.\overline{9900}$	1×10^{-2}
0.001	0.999	$0.\overline{999000}$	1×10^{-4}
0.0001	0.9999	$0.\overline{99990000}$	1×10^{-6}
-0.0001	1.0001	$1.\overline{0001}$	1×10^{-6}
-0.001	1.001	$1.\overline{001}$	1×10^{-4}
-0.01	1.01	$1.\overline{01}$	1×10^{-2}
$-.1$	1.1	$1.\overline{1}$	1

The percentage errors become extremely small as x approaches 0. In fact, each time we decrease Δx by a factor of 10, the percentage error decreases by a factor of 100.

4.6.73

a. The linear approximation near $x = 1$ is more accurate for f because the rate at which f' is changing at 1 is smaller than the rate at which g' is changing at 1. The graph of f bends away from the linear function more slowly than the graph of g.

b. The larger the value of $|f''(a)|$, the greater the deviation of the curve $y = f(x)$ from the tangent line at points near $x = a$.

4.7 L'Hôpital's Rule

4.7.1 If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{0}{0}$.

4.7.2 In general, limits with the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can have *any* value, and so cannot be evaluated by direct substitution.

4.7.3 Take the limit of the quotient of the derivatives of the numerator and denominator.

4.7.4

a. Let $f(x) = 3\sin x$ and $g(x) = x$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{3\sin x}{x} = 3$.

b. Let $f(x) = 4\sin x$ and $g(x) = x$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{4\sin x}{x} = 4$.

4.7.5

a. Let $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$, but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} x^2 \cdot \frac{1}{x^2} = \lim_{x\to 0} 1 = 1$.

b. Let $f(x) = 2x^2$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = \infty$, but $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} 2x^2 \cdot \frac{1}{x^2} = \lim_{x\to 0} 2 = 2$.

4.7.6

a. L'Hôpital's Rule is not needed. This is not an indeterminate form. $\lim_{x\to 0} \frac{\sin x}{x^2 + 2x + 1} = \frac{0}{1} = 0$.

b. L'Hôpital's Rule is needed. $\lim_{x\to 0} \frac{\sin x}{x^3+2x} = \lim_{x\to 0} \frac{\cos x}{3x^2+2} = \frac{1}{2}$.

4.7.7 If $\lim_{x\to a} f(x)g(x)$ has the form $0\cdot\infty$, then $\lim_{x\to a} \frac{f(x)}{1/g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

4.7.8 A simple example is $\lim_{x\to 0} \frac{1/x^2}{1/x^2}$.

4.7.9 $\lim_{x\to 0}(\tan^{-1} x)\left(\frac{1}{5x}\right) = \lim_{x\to 0} \frac{\tan^{-1} x}{5x} = \lim_{x\to 0} \frac{1/(1+x^2)}{5} = \frac{1}{5}$.

4.7.10 By l'Hôpital's rule :

$$\lim_{x\to 2} \frac{x^3-3x^2+2x}{x-2} = \lim_{x\to 2} \frac{3x^2-6x+2}{1} = 2.$$

By a method from a previous chapter:

$$\lim_{x\to 2} \frac{x^3-3x^2+2x}{x-2} = \lim_{x\to 2} \frac{x(x^2-3x+2)}{x-2} = \lim_{x\to 2} \frac{x(x-2)(x-1)}{x-2} = \lim_{x\to 2} x(x-1) = 2.$$

4.7.11 If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then $f(x)^{g(x)} \to 1^\infty$ as $x \to a$, which is meaningless; so direct substitution does not work.

4.7.12 First, evaluate $L = \lim_{x\to a} g(x)\ln f(x)$, which can usually be handled by L'Hôpital's rule. Then we have $\lim_{x\to a} f(x)^{g(x)} = e^L$.

4.7.13 This means $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

4.7.14 This means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$ where $0 < M < \infty$.

4.7.15 By Theorem 4.14, we have $\ln x, x^3, 2^x, x^x$ in order of increasing growth rates.

4.7.16 By Theorem 4.14, we have $\ln x^{10}, x^{100}, 10^x, x^x$ in order of increasing growth rates.

4.7.17 L'Hôpital's rule gives $\lim_{x\to 2} \frac{x^2-2x}{8-6x+x^2} = \lim_{x\to 2} \frac{2x-2}{-6+2x} = \frac{2}{-2} = -1$.

4.7.18 L'Hôpital's rule gives $\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} = \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} = -4+3+2 = 1$.

4.7.19 $\lim_{x\to 1} \frac{x^2+2x}{x+3} = \frac{1^2+2}{1+3} = \frac{3}{4}$. Note that this is not an indeterminate form.

4.7.20 $\lim_{x\to 0} \frac{e^x-1}{2x+5} = \frac{1-1}{5} = \frac{0}{5} = 0$. Note that this is not an indeterminate form.

4.7.21 L'Hôpital's rule gives $\lim_{x\to 1} \frac{\ln x}{4x-x^2-3} = \lim_{x\to 1} \frac{1/x}{4-2x} = \frac{1}{2}$.

4.7.22 L'Hôpital's rule gives $\lim_{x\to 0} \frac{e^x-1}{x^2+3x} = \lim_{x\to 0} \frac{e^x}{2x+3} = \frac{1}{3}$.

4.7.23 Apply L'Hôpital's rule three times:

$$\lim_{x\to\infty} \frac{3x^4-x^2}{6x^4+12} = \lim_{x\to\infty} \frac{12x^3-2x}{24x^3} = \lim_{x\to\infty} \frac{36x^2-2}{72x^2} = \lim_{x\to\infty} \frac{72x}{144x} = \frac{1}{2}.$$

4.7.24 Apply L'Hôpital's rule three times:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \lim_{x\to\infty} \frac{12x^2 - 4x}{3\pi x^2} = \lim_{x\to\infty} \frac{24x - 4}{6\pi x} = \lim_{x\to\infty} \frac{24}{6\pi} = \frac{4}{\pi}.$$

4.7.25 L'Hôpital's rule gives $\lim_{x\to e} \frac{\ln x - 1}{x - e} = \lim_{x\to e} \frac{1/x}{1} = \frac{1}{e}$.

4.7.26 L'Hôpital's rule gives $\lim_{x\to 1} \frac{4\tan^{-1} x - \pi}{x - 1} = \lim_{x\to 1} \frac{4/(1+x^2)}{1} = 2$.

4.7.27 Apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{1 - \ln x}{1 + \ln x} = \lim_{x\to 0^+} \frac{-1/x}{1/x} = \lim_{x\to 0^+} (-1) = -1.$$

4.7.28 Apply L'Hôpital's rule once, then simplify:

$$\lim_{x\to 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}} = \lim_{x\to 0^+} \frac{1 - 3/(2\sqrt{x})}{1 - 1/\sqrt{x}} = \lim_{x\to 0^+} \frac{(1 - 3/(2\sqrt{x}))}{(1 - 1/\sqrt{x})} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} = \lim_{x\to 0^+} \frac{2\sqrt{x} - 3}{2\sqrt{x} - 2} = \frac{3}{2}.$$

4.7.29 L'Hôpital's rule gives $\lim_{x\to 0} \frac{3\sin 4x}{5x} = \lim_{x\to 0} \frac{12\cos 4x}{5} = \frac{12}{5}$.

4.7.30 L'Hôpital's rule gives $\lim_{x\to 2\pi} \frac{x\sin x + x^2 - 4\pi^2}{x - 2\pi} = \lim_{x\to 2\pi} \frac{x\cos x + \sin x + 2x}{1} = 2\pi + 0 + 4\pi = 6\pi$.

4.7.31 L'Hôpital's rule gives $\lim_{u\to \pi/4} \frac{\tan u - \cot u}{u - \pi/4} = \lim_{u\to \pi/4} \frac{\sec^2 u + \csc^2 u}{1} = 2 + 2 = 4$.

4.7.32 L'Hôpital's rule gives $\lim_{z\to 0} \frac{\tan 4z}{\tan 7z} = \lim_{z\to 0} \frac{4\sec^2 4z}{7\sec^2 7z} = \frac{4}{7}$.

4.7.33 Apply L'Hôpital's rule twice: $\lim_{x\to 0} \frac{1 - \cos 3x}{8x^2} = \lim_{x\to 0} \frac{3\sin 3x}{16x} = \lim_{x\to 0} \frac{9\cos 3x}{16} = \frac{9}{16}$.

4.7.34 Observe that $\lim_{x\to 0} \frac{\sin^2 3x}{x^2} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)^2$, and apply L'Hôpital's rule to obtain $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3\cos 3x}{1} = 3$. Therefore $\lim_{x\to 0} \frac{\sin^2 3x}{x^2} = 9$.

4.7.35 Apply L'Hôpital's rule twice:

$$\lim_{x\to \pi} \frac{\cos x + 1}{(x - \pi)^2} = \lim_{x\to \pi} \frac{-\sin x}{2(x - \pi)} = \lim_{x\to \pi} \frac{-\cos x}{2} = \frac{1}{2}.$$

4.7.36 Apply L'Hôpital's rule twice:

$$\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} = \lim_{x\to 0} \frac{e^x - 1}{10x} = \lim_{x\to 0} \frac{e^x}{10} = \frac{1}{10}.$$

4.7.37 Apply L'Hôpital's rule three times:

$$\lim_{x\to \pi/2^-} \frac{\tan x}{3/(2x - \pi)} = \lim_{x\to \pi/2^-} \frac{\sec^2 x}{-6/(2x - \pi)^2} = -(1/6) \lim_{x\to \pi/2^-} \frac{(2x - \pi)^2}{\cos^2 x} = (-1/6) \lim_{x\to \pi/2^-} \frac{4(2x - \pi)}{2\cos x(-\sin x)} =$$

$$(-1/6) \lim_{x\to \pi/2^-} \frac{8x - 4\pi}{-\sin(2x)} = (-1/6) \lim_{x\to \pi/2^-} \frac{8}{-2\cos(2x)} = -\frac{2}{3}.$$

4.7.38 Applying L'Hôpital's rule gives:

$$\lim_{x\to\infty}\frac{e^{3x}}{3e^{3x}+5}=\lim_{x\to\infty}\frac{3e^{3x}}{9e^{3x}}=\frac{3}{9}=\frac{1}{3}.$$

4.7.39 Apply L'Hôpital's rule twice:

$$\lim_{x\to0}\frac{e^x-\sin x-1}{x^4+8x^3+12x^2}=\lim_{x\to0}\frac{e^x-\cos x}{4x^3+24x^2+24x}=\lim_{x\to0}\frac{e^x+\sin x}{12x^2+48x+24}=\frac{1}{24}.$$

4.7.40 Apply L'Hôpital's rule three times:

$$\lim_{x\to0}\frac{\sin x-x}{7x^3}=\lim_{x\to0}\frac{\cos x-1}{21x^2}=\lim_{x\to0}\frac{-\sin x}{42x}=\lim_{x\to0}\frac{-\cos x}{42}=-\frac{1}{42}.$$

4.7.41 L'Hôpital's rule gives:

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{1/x}=\lim_{x\to\infty}\frac{e^{1/x}(-1/x^2)}{(-1/x^2)}=\lim_{x\to\infty}e^{1/x}=1.$$

4.7.42 Apply L'Hôpital's rule twice:

$$\lim_{x\to\infty}\frac{\tan^{-1}x-\pi/2}{1/x}=\lim_{x\to\infty}\frac{1/(1+x^2)}{(-1/x^2)}=\lim_{x\to\infty}\frac{-x^2}{x^2+1}=\lim_{x\to\infty}\frac{-2x}{2x}=-1.$$

4.7.43 Apply L'Hôpital's rule twice:

$$\lim_{x\to-1}\frac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}=\lim_{x\to-1}\frac{3x^2-2x-5}{4x^3+6x^2-2x-4}=\lim_{x\to-1}\frac{6x-2}{12x^2+12x-2}=4.$$

4.7.44 L'Hôpital's rule gives $\displaystyle\lim_{x\to1}\frac{x^n-1}{x-1}=\lim_{x\to1}\frac{nx^{n-1}}{1}=n.$

4.7.45 Applying L'Hôpital's rule twice gives:

$$\lim_{x\to\infty}\frac{\ln(3x+5)}{\ln(7x+3)+1}=\lim_{x\to\infty}\frac{3/(3x+5)}{7/(7x+3)}=\frac{3}{7}\lim_{x\to\infty}\frac{7x+3}{3x+5}=\frac{3}{7}\lim_{x\to\infty}\frac{7}{3}=1.$$

4.7.46 Applying L'Hôpital's rule numerous times gives:

$$\lim_{x\to\infty}\frac{\ln(3x+5e^x)}{\ln(7x+3e^{2x})}=\lim_{x\to\infty}\frac{(3+5e^x)/(3x+5e^x)}{(7+6e^{2x})/(7x+3e^{2x})}=\lim_{x\to\infty}\frac{3+5e^x}{3x+5e^x}\cdot\lim_{x\to\infty}\frac{7x+3e^{2x}}{7+6e^{2x}}=$$

$$\lim_{x\to\infty}\frac{5e^x}{3+5e^x}\cdot\lim_{x\to\infty}\frac{7+6e^{2x}}{12e^{2x}}=\lim_{x\to\infty}\frac{5e^x}{5e^x}\cdot\lim_{x\to\infty}\frac{12e^{2x}}{24e^{2x}}=\frac{1}{2}.$$

4.7.47 L'Hôpital's rule gives $\displaystyle\lim_{v\to3}\frac{v-1-\sqrt{v^2-5}}{v-3}=\lim_{v\to3}\frac{1-\dfrac{v}{\sqrt{v^2-5}}}{1}=-\frac{1}{2}.$

4.7.48 L'Hôpital's rule gives $\displaystyle\lim_{y\to2}\frac{y^2+y-6}{\sqrt{8-y^2}-y}=\lim_{y\to2}\frac{2y+1}{-\dfrac{y}{\sqrt{8-y^2}}-1}=-\frac{5}{2}.$

4.7.49 Apply L'Hôpital's rule twice:

$$\lim_{x\to2}\frac{x^2-4x+4}{\sin^2\pi x}=\lim_{x\to2}\frac{2x-4}{2\pi(\sin\pi x)(\cos\pi x)}=\lim_{x\to2}\frac{2x-4}{\pi\sin2\pi x}=\lim_{x\to2}\frac{2}{2\pi^2\cos2\pi x}=\frac{2}{2\pi^2}=\frac{1}{\pi^2}.$$

4.7.50 L'Hôpital's rule gives $\lim_{x\to 2} \frac{(3x+2)^{1/3}-2}{x-2} = \lim_{x\to 2} \frac{(3x+2)^{-2/3}}{1} = 8^{-2/3} = \frac{1}{4}$. (Notice that this limit is the derivative of $(3x+2)^{1/3}$ at $x=2$.)

4.7.51 Applying L'Hôpital's rule twice gives:

$$\lim_{x\to\infty} \frac{x^2 - \ln(2/x)}{3x^2+2x} = \lim_{x\to\infty} \frac{2x+(1/x)}{6x+2} = \lim_{x\to\infty} \frac{2-1/x^2}{6} = \frac{2}{6} = \frac{1}{3}.$$

4.7.52 Observe that $\lim_{x\to 1+} \left(\frac{1}{x-1} - \frac{1}{\sqrt{x-1}}\right) = \lim_{x\to 1+} \frac{1-\sqrt{x-1}}{x-1} = \infty.$

4.7.53 Observe that $\lim_{x\to 0} x\csc x = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$, by L'Hôpital's rule.

4.7.54 Observe that $\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^-} \frac{1-x}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1^-} \frac{-1}{-\left(\frac{\pi}{2}\right)\csc^2\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi}$ by L'Hôpital's rule.

4.7.55 By L'Hôpital's rule: $\lim_{x\to 0} \frac{\sin 7x}{\sin 6x} = \lim_{x\to 0} \frac{7\cos 7x}{6\cos 6x} = \frac{7}{6}$.

4.7.56 By L'Hôpital's rule: $\lim_{x\to\infty} \frac{e^{1/x}-1}{\sin(1/x)} = \lim_{x\to\infty} \frac{e^{1/x}\cdot(-1/x^2)}{\cos(1/x)(-1/x^2)} = \frac{1}{1} = 1.$

4.7.57 Observe that $\lim_{x\to(\pi/2)^-} \left(\frac{\pi}{2} - x\right)\sec x = \lim_{x\to(\pi/2)^-} \frac{\pi/2 - x}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-1}{-\sin x} = 1$ by L'Hôpital's rule.

4.7.58 Observe that $\lim_{x\to 0^+} \sin x\sqrt{\frac{1-x}{x}} = \lim_{x\to 0^+} \sin x\sqrt{\frac{x(1-x)}{x^2}} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \sqrt{x(1-x)} = 1\cdot 0 = 0,$ where we use L'Hôpital's rule for $\lim_{x\to 0^+} \sin x/x = 1$.

4.7.59

$$\begin{aligned}
\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \\
&= \lim_{x\to 1^+} \frac{1/x - 1}{\ln x + (x-1)(1/x)} \\
&= \lim_{x\to 1^+} \frac{1/x-1}{\ln x + 1 - 1/x} = \lim_{x\to 1^+} \frac{-1/x^2}{1/x + 1/x^2} = -\frac{1}{2},
\end{aligned}$$

where L'Hôpital's rule was used twice.

4.7.60

$$\begin{aligned}
\lim_{x\to 1^-} \left(\frac{x}{x-1} - \frac{x}{\ln x}\right) &= \lim_{x\to 1^-} \frac{x\ln x - x^2 + x}{(x-1)\ln x} \\
&= \lim_{x\to 1^-} \frac{\ln x + 1 - 2x + 1}{\ln x + \frac{x-1}{x}} \\
&= \lim_{x\to 1^-} \frac{\ln x - 2x + 2}{\ln x + 1 - 1/x} = \lim_{x\to 1^-} \frac{1/x - 2}{1/x + 1/x^2} = -\frac{1}{2},
\end{aligned}$$

where L'Hôpital's rule was used twice.

4.7.61 Observe that $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x}$. Apply L'Hôpital's rule twice:

$$\begin{aligned}\lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} &= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\ &= -\lim_{x\to 0}\frac{x\sin x}{\sin x + x\cos x} = -\lim_{x\to 0}\frac{\sin x + x\cos x}{\cos x + \cos x - x\sin x} = -\frac{0}{2} = 0.\end{aligned}$$

4.7.62 Observe that

$$\lim_{\theta\to(\pi/2)^-}(\tan\theta - \sec\theta) = \lim_{\theta\to(\pi/2)^-}\left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim_{\theta\to(\pi/2)^-}\frac{\sin\theta - 1}{\cos\theta}.$$

By L'Hôpital's rule

$$\lim_{\theta\to(\pi/2)^-}\frac{\sin\theta - 1}{\cos\theta} = \lim_{\theta\to(\pi/2)^-}\frac{\cos\theta}{-\sin\theta} = \frac{0}{-1} = 0.$$

4.7.63 Make the substitution $x = \frac{1}{t}$. Then

$$\begin{aligned}\lim_{x\to\infty}(x^2 - \sqrt{x^4 + 16x^2}) &= \lim_{t\to 0^+}\left(\frac{1}{t^2} - \sqrt{\frac{1}{t^4} + \frac{16}{t^2}}\right)\\ &= \lim_{t\to 0^+}\frac{1 - \sqrt{1 + 16t^2}}{t^2} = \lim_{t\to 0^+}\frac{\frac{-32t}{2\sqrt{1+16t^2}}}{2t}\\ &= \lim_{t\to 0^+}\frac{-8}{\sqrt{1 + 16t^2}} = -8.\end{aligned}$$

4.7.64 Observe that

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + 4x}\right) = \lim_{x\to\infty} x\left(1 - \sqrt{1 + 4/x}\right).$$

Make the change of variables $t = 1/x$:

$$\lim_{x\to\infty} x\left(1 - \sqrt{1 + 4/x}\right) = \lim_{t\to 0+}\frac{1 - \sqrt{1 + 4t}}{t} = \lim_{t\to 0+}\frac{-\dfrac{4}{2\sqrt{1 + 4t}}}{1} = -2.$$

4.7.65 Observe that

$$(\sqrt{x-2} - \sqrt{x-4})\cdot\frac{\sqrt{x-2} + \sqrt{x-4}}{\sqrt{x-2} + \sqrt{x-4}} = \frac{x - 2 - (x-4)}{\sqrt{x-2} + \sqrt{x-4}} = \frac{2}{\sqrt{x-2} + \sqrt{x-4}},$$

so

$$\lim_{x\to\infty}\sqrt{x-2} - \sqrt{x-4} = \lim_{x\to\infty}\frac{2}{\sqrt{x-2} + \sqrt{x-4}} = 0.$$

4.7.66 Let $t = \frac{1}{x}$. Then

$$\lim_{x\to\infty} x^2\ln\left(\cos\frac{1}{x}\right) = \lim_{t\to 0^+}\frac{\ln(\cos t)}{t^2} = \lim_{t\to 0^+}\frac{-\tan t}{2t} = \lim_{t\to 0^+}\frac{-\sec^2 t}{2} = -\frac{1}{2}.$$

4.7.67 Use the identity $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$; then

$$\lim_{n\to\infty}\frac{1 + 2 + \cdots + n}{n^2} = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

4.7.68 Let $y = x^{1/\ln x}$. Then $\ln y = \ln x^{1/\ln x} = 1$, so $\lim_{x\to 0^+}\ln y = 1$, so $\lim_{x\to 0^+} y = e^1 = e$.

4.7.69

$$\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}} = \frac{\ln 3}{\ln 2}.$$

4.7.70 Note that $\log_2 x = \ln x/\ln 2$ and $\log_3 x = \ln x/\ln 3$; therefore

$$\lim_{x\to\infty} (\log_2 x - \log_3 x) = \lim_{x\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) \ln x = \infty.$$

4.7.71 By L'Hôpital's rule, $\lim_{x\to 6} \frac{(5x+2)^{1/5} - 2}{x^{-1} - 6^{-1}} = \lim_{x\to 6} \frac{(5x+2)^{-4/5}}{-x^{-2}} = -\frac{9}{4}$.

4.7.72

$$\lim_{x\to\pi/2} (\pi - 2x)\tan x = \lim_{x\to\pi/2} \frac{\pi - 2x}{\cot x} = \lim_{x\to\pi/2} \frac{-2}{-\csc^2 x} = \lim_{x\to\pi/2} 2\sin^2 x = 2.$$

4.7.73 Make the substitution $t = 1/x$; then

$$\lim_{x\to\infty} x^3 \left(\frac{1}{x} - \sin\frac{1}{x}\right) = \lim_{t\to 0+} \frac{t - \sin t}{t^3} = \lim_{t\to 0+} \frac{1 - \cos t}{3t^2} = \lim_{t\to 0+} \frac{\sin t}{6t} = \lim_{t\to 0+} \frac{\cos t}{6} = \frac{1}{6},$$

using L'Hôpital's rule.

4.7.74 Make the substitution $t = 1/x$; then

$$\lim_{x\to\infty} (x^2 e^{1/x} - x^2 - x) = \lim_{t\to 0+} \frac{e^t - 1 - t}{t^2} = \lim_{t\to 0+} \frac{e^t - 1}{2t} = \lim_{t\to 0+} \frac{e^t}{2} = \frac{1}{2},$$

using L'Hôpital's rule.

4.7.75 Note that $\ln x^{2x} = 2x \ln x$, so we evaluate

$$L = \lim_{x\to 0^+} 2x \ln x = 2 \lim_{x\to 0^+} \frac{\ln x}{1/x} = 2 \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 2 \lim_{x\to 0^+} (-x) = 0$$

by L'Hôpital's rule. Therefore $\lim_{x\to 0^+} x^{2x} = e^L = 1$.

4.7.76 Note that $\ln(1+4x)^{3/x} = \frac{3\ln(1+4x)}{x}$, so we evaluate

$$L = \lim_{x\to 0} \frac{3\ln(1+4x)}{x} = 3 \lim_{x\to 0} \frac{4/(1+4x)}{1} = 12$$

by L'Hôpital's rule. Therefore $\lim_{x\to 0} (1+4x)^{3/x} = e^L = e^{12}$.

4.7.77 Note that $\ln(\tan\theta)^{\cos\theta} = \cos\theta \ln\tan\theta$, so we evaluate

$$L = \lim_{\theta\to\pi/2-} \cos\theta \ln\tan\theta = \lim_{\theta\to\pi/2-} \frac{\ln\tan\theta}{\sec\theta}.$$

L'Hôpital's rule gives

$$\lim_{\theta\to\pi/2-} \frac{\ln\tan\theta}{\sec\theta} = \lim_{\theta\to\pi/2-} \frac{\sec^2\theta/\tan\theta}{\sec\theta\tan\theta} = \lim_{\theta\to\pi/2-} \frac{\sec\theta}{\tan^2\theta} = \lim_{\theta\to\pi/2-} \frac{\cos\theta}{\sin^2\theta} = 0,$$

so $\lim_{\theta\to\pi/2-} (\tan\theta)^{\cos\theta} = e^L = 1$.

4.7.78 Note that $\ln(\sin\theta)^{\tan\theta} = \tan\theta \ln\sin\theta$, so we evaluate

$$L = \lim_{\theta\to 0+} \tan\theta \ln\sin\theta = \lim_{\theta\to 0+} \frac{\ln\sin\theta}{\cot\theta}.$$

L'Hôpital's rule gives

$$\lim_{\theta\to 0+} \frac{\ln\sin\theta}{\cot\theta} = \lim_{\theta\to 0+} \frac{\cos\theta/\sin\theta}{-\csc^2\theta} = -\lim_{\theta\to 0+} \cos\theta\sin\theta = 0,$$

so $\displaystyle\lim_{\theta\to 0+} (\tan\theta)^{\cos\theta} = e^L = 1.$

4.7.79 Note that $\ln(1+x)^{\cot x} = \cot x \ln(1+x)$, so we evaluate

$$L = \lim_{x\to 0+} \cot x \ln(1+x) = \lim_{x\to 0+} \frac{\ln(1+x)}{\tan x} = \lim_{x\to 0+} \frac{1/(1+x)}{\sec^2 x} = \lim_{x\to 0+} \frac{\cos^2 x}{1+x} = 1$$

by L'Hôpital's rule. Therefore $\displaystyle\lim_{x\to 0^+} (1+x)^{\cot x} = e^L = e.$

4.7.80 Note that $\ln(1+a/x)^x = x\ln(1+a/x)$, so we evaluate

$$L = \lim_{x\to\infty} x\ln(1+a/x) = \lim_{x\to\infty} \frac{\ln(1+a/x)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+a/x}\cdot\frac{-a}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{a}{1+a/x} = a$$

by L'Hôpital's rule. Therefore $\displaystyle\lim_{x\to\infty} (1+a/x)^x = e^L = e^a.$

4.7.81 Note that $\ln(e^{ax}+x)^{1/x} = \frac{1}{x}\ln(e^{ax}+x)$, so we evaluate

$$L = \lim_{x\to 0} \frac{\ln(e^{ax}+x)}{x} = \lim_{x\to 0} \frac{(ae^{ax}+1)/(e^{ax}+x)}{1} = \frac{(a+1)/1}{1} = a+1.$$

Therefore, $\displaystyle\lim_{x\to 0} (e^{ax}+x)^{1/x} = e^{a+1}.$

4.7.82 Note that $\ln(1+10/z^2)^{z^2} = z^2\ln(1+10/z^2)$, so we evaluate

$$L = \lim_{z\to\infty} z^2 \ln\left(1+\frac{10}{z^2}\right) = \lim_{z\to\infty} \frac{\ln\left(1+\frac{10}{z^2}\right)}{\frac{1}{z^2}} = \lim_{z\to\infty} \frac{\frac{1}{1+\frac{10}{z^2}}\cdot\frac{-20}{z^3}}{-\frac{2}{z^3}} = \lim_{z\to\infty} \frac{10z^2}{z^2+10} = 10$$

by L'Hôpital's rule. Therefore $\displaystyle\lim_{z\to\infty} \left(1+\frac{10}{z^2}\right)^{z^2} = e^L = e^{10}.$

4.7.83 Note that $\ln(x+\cos x)^{1/x} = (\ln(x+\cos x))/x$, so we evaluate

$$L = \lim_{x\to 0} \frac{\ln(x+\cos x)}{x} = \lim_{x\to 0} \frac{(x+\cos x)^{-1}(1-\sin x)}{1} = 1$$

by L'Hôpital's rule. Therefore $\displaystyle\lim_{x\to 0} (x+\cos x)^{1/x} = e^L = e.$

4.7.84 Observe that

$$\lim_{\delta\to 2m\pi} \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} = \left(\lim_{\delta\to 2m\pi} \frac{\sin(N\delta/2)}{\sin(\delta/2)}\right)^2,$$

and

$$\lim_{\delta\to 2m\pi} \frac{\sin(N\delta/2)}{\sin(\delta/2)} = \lim_{\delta\to 2m\pi} \frac{(N/2)\cos(N\delta/2)}{(1/2)\cos(\delta/2)} = \pm N,$$

so $\displaystyle\lim_{\delta\to 2m\pi} \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} = N^2.$

4.7.85

a. After each year the balance increases by the factor $1+r$; therefore the balance after t years is $B(t) = P(1+r)^t$.

b. Let $L = \lim_{m\to\infty} mt\ln\left(1+\frac{r}{m}\right) = t\lim_{m\to\infty}\frac{\ln\left(1+\frac{r}{m}\right)}{1/m}$. By L'Hôpital's rule we have

$$L = t\lim_{m\to\infty}\frac{\frac{1}{1+\frac{r}{m}}\left(\frac{-r}{m^2}\right)}{-1/m^2} = t\lim_{m\to\infty}\frac{r}{1+\frac{r}{m}} = rt,$$

so $\lim_{m\to\infty} B(t) = Pe^L = Pe^{rt}$.

4.7.86 Observe that $\lim_{x\to\infty}\frac{2x^3-x^2+1}{5x^3+2x} = \frac{2}{5}$ by Theorem 2.7. We can also use L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{2x^3-x^2+1}{5x^3+2x} = \lim_{x\to\infty}\frac{6x^2-2x}{15x^2+2} = \lim_{x\to\infty}\frac{12x-2}{30x} = \lim_{x\to\infty}\frac{12}{30} = \frac{2}{5}.$$

4.7.87 By factoring, we have

$$\lim_{x\to 0}\frac{(e^x-1)(e^x+5)}{(e^x+1)(e^x-1)} = \lim_{x\to 0}\frac{e^x+5}{e^x+1} = \frac{6}{2} = 3.$$

Or, using L'Hôpital's rule, we have

$$\lim_{x\to 0}\frac{2e^{2x}+4e^x}{2e^{2x}} = \frac{2+4}{2} = 3.$$

4.7.88 Note that

$$\ln\left(\frac{\sin x}{x}\right)^{1/x^2} = \frac{\ln\sin x - \ln x}{x^2},$$

so we evaluate

$$L = \lim_{x\to 0}\frac{\ln\sin x - \ln x}{x^2} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x} = \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\sin x}$$

by L'Hôpital's rule. Next, observe that

$$\lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\sin x} = \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2\cos x} = \lim_{x\to 0}\frac{-\frac{\sin x}{x}}{\frac{4\sin x}{x} + 2\cos x} = -\frac{1}{6}$$

using L'Hôpital's rule and $\lim_{x\to 0}\sin x/x = 1$. Therefore $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^L = e^{-1/6}$.

4.7.89

$$\lim_{x\to 1}\frac{x\ln x - x + 1}{x\ln^2 x} = \lim_{x\to 1}\frac{\ln x + x\cdot\frac{1}{x} - 1}{\ln^2 x + 2x\ln x\cdot\frac{1}{x}} = \lim_{x\to 1}\frac{\ln x}{\ln^2 x + 2\ln x} = \lim_{x\to 1}\frac{1}{\ln x + 2} = \frac{1}{2},$$

where the first equality follows from L'Hôpital's rule and the penultimate follows by dividing the numerator and denominator by $\ln x$.

4.7.90

$$\begin{aligned}
\lim_{x\to 1}\frac{x\ln x+\ln x-2x+2}{x^2\ln^3 x} &= \lim_{x\to 1}\frac{\ln x+x\cdot\frac{1}{x}+\frac{1}{x}-2}{2x\ln^3 x+x^2\cdot 3\ln^2 x\cdot\frac{1}{x}}\\
&= \lim_{x\to 1}\frac{\ln x+\frac{1}{x}-1}{2x\ln^3 x+3x\ln^2 x}\\
&= \lim_{x\to 1}\frac{\frac{1}{x}-\frac{1}{x^2}}{2\ln^3 x+6x\ln^2 x\cdot\frac{1}{x}+3\ln^2 x+6x\ln x\cdot\frac{1}{x}}\\
&= \lim_{x\to 1}\frac{\frac{-1}{x^2}+\frac{2}{x^3}}{6\ln x\cdot\frac{1}{x}+12\ln x\cdot\frac{1}{x}+6\ln x\cdot\frac{1}{x}+\frac{6}{x}}\\
&= \frac{1}{6},
\end{aligned}$$

where each equality except the second follows from L'Hôpital's rule.

4.7.91 Let $z=\ln x^{\frac{1}{1+\ln x}}$. Then $z=\frac{\ln x}{1+\ln x}$, and $\lim_{x\to 0^+} z=\lim_{x\to 0^+}\frac{\ln x}{1+\ln x}=\lim_{x\to 0^+}\frac{1/x}{1/x}=\lim_{x\to 0^+} 1=1$. Then $\lim_{x\to 0^+} e^z=\lim_{x\to 0^+} x^{\frac{1}{1+\ln x}}=e^1=e$.

4.7.92 The given limit can be written as $\lim_{n\to\infty}\frac{\ln(n\sin(1/n))}{\frac{1}{n^2}}$, and by L'Hôpital's rule this is equal to

$$\lim_{n\to\infty}\frac{1}{n\sin(1/n)}\cdot\frac{\sin(1/n)+n\cos(1/n)\left(\frac{-1}{n^2}\right)}{\frac{-2}{n^3}}=\lim_{n\to\infty}\left(-\frac{n^2}{2}+\frac{n\cot(1/n)}{2}\right).$$

This is exactly $\frac{1}{2}$ times $\lim_{n\to\infty}(n\cot(1/n)-n^2)$.

Let $z=\frac{1}{n}$. Then as $n\to\infty$ we have $z\to 0^+$. Then

$$\begin{aligned}
\frac{1}{2}\lim_{n\to\infty}(n\cot(1/n)-n^2) &= \frac{1}{2}\lim_{z\to 0^+}\frac{\cot z}{z}-\frac{1}{z^2}\\
&= \frac{1}{2}\lim_{z\to 0^+}\frac{z\cos z-\sin z}{z^2\sin z}\\
&= \frac{1}{2}\lim_{z\to 0^+}\frac{\cos z-z\sin z-\cos z}{2z\sin z+z^2\cos z}\\
&= \frac{1}{2}\lim_{z\to 0^+}\frac{-\sin z}{2\sin z+z\cos z}\\
&= \frac{1}{2}\lim_{z\to 0^+}\frac{-\cos z}{2\cos z+\cos z-z\sin z}\\
&= \frac{1}{2}\left(-\frac{1}{2+1-0}\right)=\frac{1}{2}\left(-\frac{1}{3}\right)=-\frac{1}{6},
\end{aligned}$$

where the third and fifth equalities follow from L'Hôpital's rule.

4.7.93 Apply L'Hôpital's rule: $\lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{(\ln a)a^x-(\ln b)b^x}{1}=\ln a-\ln b$.

4.7.94 Note that $\ln(1+ax)^{b/x}=b\ln(1+ax)/x$, so we evaluate

$$L=\lim_{x\to 0}\frac{b\ln(1+ax)}{x}=b\lim_{x\to 0}\frac{\frac{a}{1+ax}}{1}=ab$$

by L'Hôpital's rule. Therefore $\lim_{x\to 0}(1+ax)^{b/x}=e^L=e^{ab}$.

4.7.95 By Theorem 4.14, $e^{0.01x}$ grows faster than x^{10} as $x \to \infty$.

4.7.96 Observe that $\lim_{x\to\infty} \frac{x^2 \ln x}{(\ln x)^2} = \lim_{x\to\infty} \frac{x^2}{\ln x} = \infty$, so $x^2 \ln x$ grows faster than $(\ln x)^2$ as $x \to \infty$.

4.7.97 Note that $\ln x^{20} = 20 \ln x$, so $\ln x^{20}$ and $\ln x$ have comparable growth rates as $x \to \infty$.

4.7.98 Make the substitution $y = \ln x$; then $y \to \infty$ if $x \to \infty$ and $\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{y\to\infty} \frac{\ln y}{y} = 0$. Therefore $\ln x$ grows faster than $\ln(\ln x)$ as $x \to \infty$.

4.7.99 By Theorem 4.14, x^x grows faster than 100^x as $x \to \infty$.

4.7.100 Observe that $\lim_{x\to\infty} \frac{x^2 \ln x}{x^3} = \lim_{x\to\infty} \frac{\ln x}{x} = 0$, so x^3 grows faster than $x^2 \ln x$ as $x \to \infty$.

4.7.101 By Theorem 4.14, 1.00001^x grows faster than x^{20} as $x \to \infty$.

4.7.102 Observe that $\ln \sqrt{x} = (\ln x)/2$. Note that $\lim_{x\to\infty} \frac{\ln^2 x}{(\ln x)/2} = \lim_{x\to\infty} 2 \ln x = \infty$, so $\ln^2 x$ grows faster than $\ln \sqrt{x}$ as $x \to \infty$.

4.7.103 Note that $\lim_{x\to\infty} \frac{e^{x^2}}{e^{10x}} = \lim_{x\to\infty} e^{x^2-10x} = \infty$, so e^{x^2} grows faster than e^{10x} as $x \to \infty$.

4.7.104 Observe that $\lim_{x\to\infty} \frac{x^{x/10}}{e^{x^2}} = \lim_{x\to\infty} \left(\frac{x^{1/10}}{e^x}\right)^x = 0$ by Theorem 4.14, so e^{x^2} grows faster than $x^{x/10}$ as $x \to \infty$.

4.7.105

a. False. $\lim_{x\to 2} x^2 - 1 = 3$, so L'Hôpital's rule does not apply. In fact, $\lim_{x\to 2} \frac{x-2}{x^2-1} = \frac{0}{3} = 0$.

b. False. L'Hôpital's rule does not say $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f'(x) \lim_{x\to a} g'(x)$. In fact, $\lim_{x\to 0} x \sin x = 0 \cdot 0 = 0$.

c. False. This limit has the form $0^\infty = 0$.

d. False. This limit has the indeterminate form 1^∞ which is not always 1.

e. True. $\ln x^{100} = 100 \ln x$.

f. True. Note that $\lim_{x\to\infty} \frac{e^x}{2^x} = \lim_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty$ because $e/2 > 1$.

4.7.106

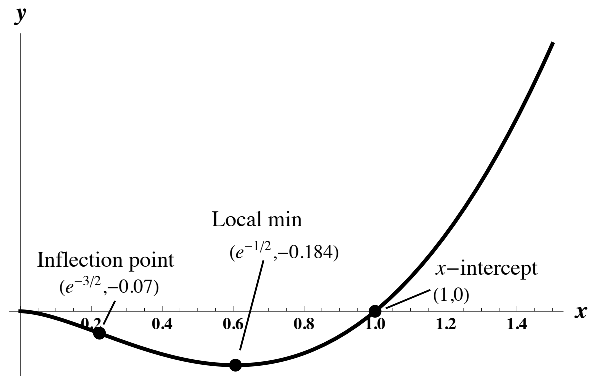
The domain of g is $(0, \infty)$, and there is no symmetry, and there are no vertical asymptotes. Note that $\lim_{x\to\infty} g(x) = \infty$, so there are no horizontal asymptotes. To see the behavior as $x \to 0^+$, we compute

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

$g'(x) = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x = x(1 + 2\ln x)$. This is 0 for $x = e^{-1/2}$.
$g''(x) = x(2/x) + (1 + 2\ln x) = 3 + 2\ln x$. This is 0 for $x = e^{-3/2}$.

Note that $g'(x) < 0$ on $(0, e^{-1/2})$ and $g'(x) > 0$ on $(e^{-1/2}, \infty)$, so g is decreasing on $(0, e^{-1/2})$ and increasing on $(e^{-1/2}, \infty)$, and there is a local (and absolute) minimum at $x = e^{-1/2}$.

Note also that $g''(x) < 0$ for $0 < x < e^{-3/2}$ and $g''(x) > 0$ for $x > e^{-3/2}$, so g is concave down on $(0, e^{-3/2})$ and concave up on $(e^{-3/2}, \infty)$ and there is an inflection point at $x = e^{-3/2}$. The only x-intercept is $(1, 0)$.



4.7.107

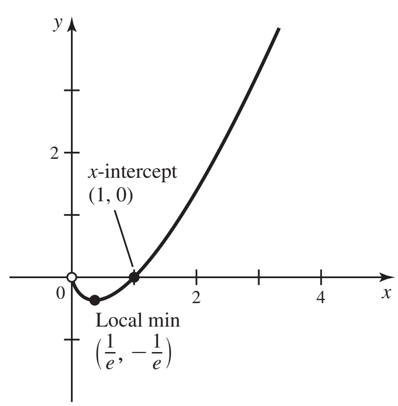
The domain of f is $(0, \infty)$, so questions about symmetry aren't relevant. There are no asymptotes. To see the behavior as $x \to 0^+$ we compute

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

$f'(x) = x \cdot 1/x + \ln x = 1 + \ln x$. This is 0 for $x = 1/e$. Note that $f'(x) < 0$ for $0 < x < 1/e$ and $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e, \infty)$ and there is a local minimum (which is also an absolute minimum) at $x = 1/e$.

$f''(x) = 1/x$, which is always positive on the domain, so f is concave up on its domain and there are no inflection points.

There is an x-intercept at $x = 1$.



4.7.108

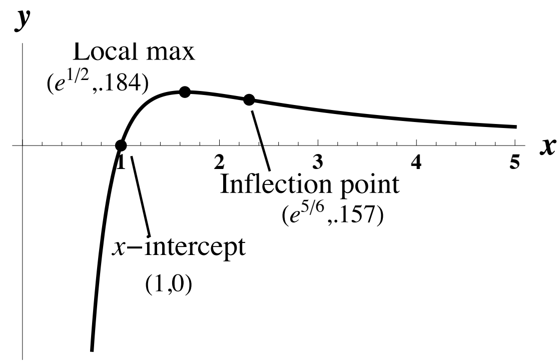
The domain of f is $(0, \infty)$. Note that $\lim_{x\to 0^+} \frac{\ln x}{x^2} = -\infty$, so $x = 0$ is a vertical asymptote. Questions about symmetry aren't relevant. We have

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0,$$

so $y = 0$ is a horizontal asymptote.

$f'(x) = \dfrac{x^2 \cdot (1/x) - \ln x \cdot 2x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$, which is 0 for $x = e^{1/2}$. Note that $f'(x) > 0$ for $0 < x < e^{1/2}$ and $f'(x) < 0$ for $x > e^{1/2}$, so f is increasing on $(0, e^{1/2})$ and decreasing on $(e^{1/2}, \infty)$, and there is a local maximum (which is actually an absolute maximum) at $x = e^{1/2}$ of about .184.

$f''(x) = \dfrac{x^3(-2/x) - (1 - 2\ln x) \cdot 3x^2}{x^6} = \dfrac{6\ln x - 5}{x^4}$, which is 0 for $x = e^{5/6}$. Note that $f''(x) < 0$ for $0 < x < e^{5/6}$ and $f''(x) > 0$ for $x > e^{5/6}$, so there is a point of inflection at $e^{5/6}$ where the concavity changes from down (to the left) to up (to the right).



4.7.109

The domain of p is $(-\infty, \infty)$. There are no vertical asymptotes. Note that $p(-x) = -xe^{-(-x)^2/2} = -(xe^{-x^2/2}) = -p(x)$, so p has odd symmetry. Note that

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} \frac{x}{e^{x^2/2}} = \lim_{x\to\pm\infty} \frac{1}{xe^{x^2/2}} = 0,$$

so $y = 0$ is a horizontal asymptote.

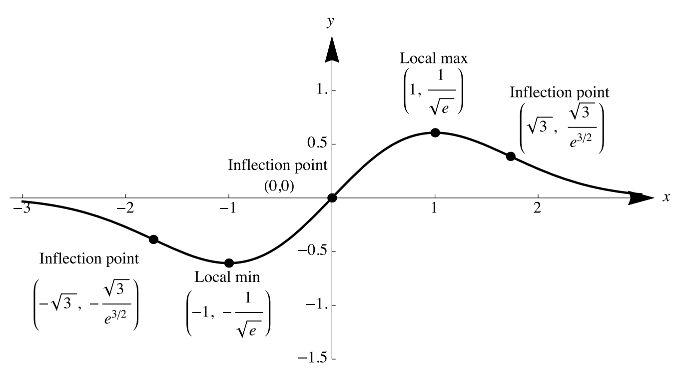
$$p'(x) = x \cdot (-xe^{-x^2/2}) + e^{-x^2/2} \cdot 1 = e^{-x^2/2}(1 - x^2) = e^{-x^2/2}(1 - x)(1 + x).$$

This is 0 for $x = \pm 1$. Note that $p'(x) < 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so p is decreasing on those intervals, and $p'(x) > 0$ on $(-1, 1)$, so p is increasing on that interval. There is a local maximum at $x = 1$ and a local minimum at $x = -1$.

$$p''(x) = e^{-x^2/2}(-2x) + (1 - x^2) \cdot (-x)e^{-x^2/2} = e^{-x^2/2}x(x^2 - 3),$$

which is 0 at $x = 0$ and at $x = \pm\sqrt{3}$. Note that $p''(x) > 0$ on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$, so p is concave up on those intervals, while $p''(x) < 0$ on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$, so p is concave down on those intervals. There are inflection points at each of $x = \pm\sqrt{3}$ and at $x = 0$.

There is an x-intercept at $(0, 0)$, which is also the y-intercept.



4.7.110 Note that

$$\sqrt{n}\log_2 n << n\log_2 n << n(\log_2 n)^2 << n^{3/2}.$$

For the last relation, we note that $n^{3/2}/n(\log_2 n)^2 = n^{1/2}/(\log_2 n)^2 \to \infty$ as $n \to \infty$. Therefore the ranking in order of least to most efficient is A, C, B, D.

4.7.111 L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{\sqrt{ax+b}}{\sqrt{cx+d}} = \lim_{x\to\infty} \frac{a}{\sqrt{ax+b}} \cdot \frac{\sqrt{cx+d}}{c} = \frac{a}{c} \lim_{x\to\infty} \frac{\sqrt{cx+d}}{\sqrt{ax+b}},$$

which is the same form as the original limit, so L'Hôpital's rule fails in this case. We can evaluate this limit as follows: first observe that

$$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \frac{a}{c}$$

by L'Hôpital's rule; therefore

$$\lim_{x\to\infty} \frac{\sqrt{ax+b}}{\sqrt{cx+d}} = \lim_{x\to\infty} \sqrt{\frac{ax+b}{cx+d}} = \sqrt{\frac{a}{c}}.$$

4.7.112 Observe that

$$(ax - \sqrt{a^2x^2 - bx}) \cdot \frac{(ax + \sqrt{a^2x^2 - bx})}{(ax + \sqrt{a^2x^2 - bx})} = \frac{bx}{(ax + \sqrt{a^2x^2 - bx})} = \frac{b}{(a + \sqrt{a^2 - b/x})},$$

so

$$\lim_{x\to\infty} (ax - \sqrt{a^2x^2 - bx}) = \frac{b}{2a}.$$

4.7.113 Let $t = b^x$, then $x = \ln t/\ln b$ and we have

$$\lim_{x\to\infty} \frac{x^p}{b^x} = \lim_{t\to\infty} \frac{\ln^p t}{t \ln^p b} = 0,$$

by Theorem 4.14.

4.7.114 Observe that $\lim\limits_{x\to\infty} \dfrac{a^x}{b^x} = \lim\limits_{x\to\infty} \left(\dfrac{a}{b}\right)^x = \infty$, because $a/b > 1$.

4.7.115 Note that $\log_a x = \ln x/\ln a$, so

$$\frac{\log_a x}{\log_b x} = \frac{\ln b}{\ln a},$$

and therefore $\log_a x$ and $\log_b x$ grow at a comparable rate as $x \to \infty$.

4.7.116 We have

$$b^n << n! << n^n$$

as $n \to \infty$ for any $b > 1$. To see this, observe that

$$\lim_{n\to\infty} \frac{n!}{b^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\, n^n}{(be)^n} = \infty$$

and

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} = 0,$$

by Theorem 4.14.

4.7.117 The triangle ABP has base $1 - \cos\theta$ and height $\sin\theta$, so its area is

$$f(\theta) = \frac{1}{2}\sin\theta(1 - \cos\theta).$$

The sector OBP has area $\theta/2$, and the triangle OBP has base 1 and height $\sin\theta$; therefore

$$g(\theta) = \frac{1}{2}(\theta - \sin\theta).$$

We have

$$\lim_{\theta\to 0} \frac{g(\theta)}{f(\theta)} = \lim_{\theta\to 0} \frac{\theta - \sin\theta}{\sin\theta(1 - \cos\theta)} = \lim_{\theta\to 0} \frac{\theta - \sin\theta}{\sin\theta - (1/2)\sin 2\theta}.$$

Three applications of L'Hôpital's rule gives

$$\lim_{\theta\to 0} \frac{g(\theta)}{f(\theta)} = \lim_{\theta\to 0} \frac{1 - \cos\theta}{\cos\theta - \cos 2\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{2\sin 2\theta - \sin\theta} = \lim_{\theta\to 0} \frac{\cos\theta}{4\cos 2\theta - \cos\theta} = \frac{1}{3}.$$

4.7.118 The function isn't defined for $a = 0$ Assume $a > 0$. Then by L'Hôpital's rule,

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \lim_{x\to a} \frac{\frac{2a^3-4x^3}{2\sqrt{2a^3x-x^4}} - \frac{1}{3}a^{5/3}x^{-2/3}}{-\frac{3}{4}a^{1/4}x^{-1/4}} = \frac{-a - \frac{a}{3}}{-\frac{3}{4}} = \frac{16}{9}a.$$

If $a < 0$, then the limit of the denominator of the given function satisfies $\lim\limits_{x\to a} a - \sqrt[4]{ax^3} = \lim\limits_{x\to a} a - \sqrt[4]{a^4} = \lim\limits_{x\to a} a - |a| = 2a$. So this is not an indeterminate form. The numerator has limit $\lim\limits_{x\to a} \sqrt{a^4} - a^2 = 0$. So the given limit is $\begin{cases} \dfrac{16a}{9} & \text{if } a > 0 \\ 0 & \text{if } a < 0. \end{cases}$

4.7.119 Note that

$$\ln\left(1 + \frac{a}{x}\right)^x = \frac{\ln(1 + a/x)}{1/x}$$

so we evaluate

$$L = \lim_{x\to\infty} \frac{\ln(1 + a/x)}{1/x} = \lim_{x\to\infty} \frac{1}{1 + a/x} \cdot -\frac{a}{x^2} \cdot \frac{1}{-1/x^2} = \lim_{x\to\infty} \frac{a}{1 + a/x} = a$$

by L'Hôpital's rule. Therefore

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^L = e^a.$$

4.7.120 Because $x \to \infty$, eventually $x > 2b$. Then

$$\frac{x^x}{b^x} > \frac{(2b)^x}{b^x} = 2^x \to \infty$$

as $x \to \infty$, so x^x grows faster than b^x as $x \to \infty$.

4.7.121

a. Observe that $\lim\limits_{x\to\infty} \dfrac{b^x}{e^x} = \lim\limits_{x\to\infty} \left(\dfrac{b}{e}\right)^x$. This limit is ∞ exactly when $b > e$.

b. Observe that $\lim\limits_{x\to\infty} \dfrac{e^{ax}}{e^x} = \lim\limits_{x\to\infty} e^{(a-1)x}$. This limit is ∞ exactly when $a > 1$.

4.8 Newton's Method

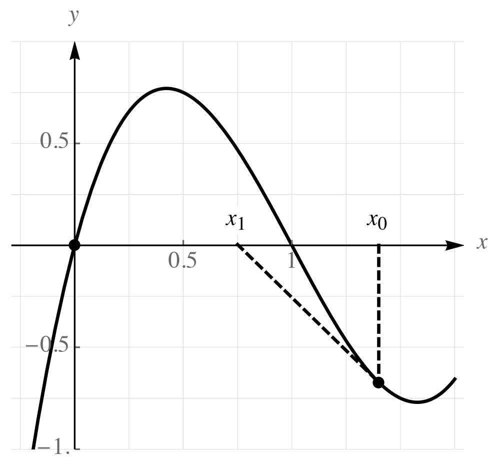
4.8.1 Newton's method generates a sequence of x-intercepts of lines tangent to the graph of f to approximate the roots of f.

4.8.2 To get the x_{n+1} iterate from x_n, one puts x_n into the recursive formula and computes x_{n+1}. Thus, starting with x_0, a sequence $x_0, x_1, x_2, x_3, \ldots$ is generated.

4.8.3 Starting at $x_0 = 3$ on the x-axis, travel with your eye downward until you hit the point on the curve at $(3, -3)$. Then follow the tangent line at that point until you hit the x-axis, and you will be at the point $x_1 = 2$. Repeating, we start at the point $(2, 0)$ and go down to the point on the curve which is $(2, -1)$. Then following the tangent line until it hits the x-axis again, we find ourselves at $x_2 = 1$. Repeating, we go down to the point on the curve at about $(1, -1/3)$, and then follow the tangent line until hitting the x-axis at $x_3 = 0$.

4.8.4 Starting at $x_0 = 3$ on the x-axis, travel with your eye downward until you hit the point on the curve at $(3, -5)$. Then follow the tangent line at that point until you hit the x-axis, and you will be at the point $x_1 = -3$. Repeating, we start at the point $(-3, 0)$ and go up to the point on the curve which is $(-3, 3)$. Then following the tangent line until it hits the x-axis again, we find ourselves at $x_2 = 2$. Repeating, we go down to the point on the curve at about $(2, -3.9)$, and then follow the tangent line until hitting the x-axis at $x_3 = -1$.

4.8.5 It is hard to find these exactly by hand, but you should find (approximately) $x_1 = 0.75$.



4.8.6

a. The line tangent to the graph of f at $x = 2$ does not intersect the x-axis because it is horizontal.

b. The expression $x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is undefined because $f'(x_0) = 0$.

4.8.7 Generally, if two successive Newton approximations agree in their first p digits, then those approximations have p digits of accuracy. The method is terminated when the desired accuracy is reached.

4.8.8 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{2x_n^2 - x_n^2 + 5}{2x_n} = \frac{x_n^2 + 5}{2x_n}.$$

4.8.9 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{2x_n^2 - x_n^2 + 6}{2x_n} = \frac{x_n^2 + 6}{2x_n}.$$

$x_1 = 2.5$ and $x_2 = 2.45$.

4.8.10 Because $f'(x_n) = 2x_n - 2$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 2x_n - 3}{2x_n - 2} = \frac{2x_n^2 - 2x_n - (x_n^2 - 2x_n - 3)}{2x_n - 2} = \frac{x_n^2 + 3}{2x_n - 2}.$$

$x_1 = 3.5$ and $x_2 = 3.05$.

4.8.11 Because $f'(x_n) = -e^{-x_n} - 1$, we have

$$x_{n+1} = x_n - \frac{e^{-x_n} - x_n}{-e^{-x_n} - 1} = \frac{(-e^{-x_n} - 1)x_n - (e^{-x_n} - x_n)}{-e^{-x_n} - 1} = \frac{-e^{-x_n}x_n - e^{-x_n}}{-e^{-x_n} - 1} = \frac{x_n + 1}{e^{x_n} + 1}.$$

$x_1 = 0.564382$ and $x_2 = 0.567142$.

4.8.12 Because $f'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{3x_n^3 - (x_n^3 - 2)}{3x_n^2} = \frac{2x_n^3 + 2}{3x_n^2}.$$

$x_1 = 1.5$ and $x_2 = 1.2963$.

4.8.13 Because $f'(x_n) = 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} = \frac{2x_n^2 - (x_n^2 - 10)}{2x_n} = \frac{x_n^2 + 10}{2x_n}.$$

n	x_n
0	3.0
1	3.16667
2	3.16228
3	3.16228

4.8.14 Because $f'(x_n) = 3x_n^2 + 2x_n$, we have

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + 1}{3x_n^2 + 2x_n}$$
$$= \frac{x_n(3x_n^2 + 2x_n) - (x_n^3 + x_n^2 + 1)}{3x_n^2 + 2x_n}$$
$$= \frac{2x_n^3 + x_n^2 - 1}{3x_n^2 + 2x_n}.$$

n	x_n
0	−1.5
1	−1.46667
2	−1.46557
3	−1.46557

4.8.15 Because $f'(x_n) = \cos(x_n) + 1$, we have

$$x_{n+1} = x_n - \frac{\sin x_n + x_n - 1}{\cos x_n + 1}.$$

n	x_n
0	0.5
1	0.51096
2	0.51097
3	0.51097

4.8.16 Because $f'(x_n) = e^{x_n} + 1$, we have

$$x_{n+1} = x_n - \frac{e^{x_n} + x - 5}{e^{x_n} + 1}.$$

n	x_n
0	1.6
1	1.33912
2	1.30697
3	1.30656
4	1.30656

4.8.17 Because $f'(x_n) = \sec^2(x_n) - 2$, we have

$$x_{n+1} = x_n - \frac{\tan x_n - 2x_n}{\sec^2(x_n) - 2}.$$

n	x_n
0	1.2
1	1.16934
2	1.16561
3	1.16556
4	1.16556

4.8.18 Because $f'(x_n) = \ln(x_n+1) + \frac{x_n}{x_n+1}$, we have

$$x_{n+1} = x_n - \frac{x_n \ln(x_n+1) - 1}{\ln(x_n+1) + \frac{x_n}{x_n+1}}$$

n	x_n
0	1.7
1	1.27574
2	1.24027
3	1.23998
4	1.23998

4.8.19 Because $f'(x_n) = \frac{-1}{\sqrt{1-x_n^2}} - 1$, we have

$$x_{n+1} = x_n - \frac{\cos^{-1}(x_n) - x_n}{\frac{-1}{\sqrt{1-x_n^2}} - 1}$$

n	x_n
0	0.75
1	0.73915
2	0.73909
3	0.73909

4.8.20 A preliminary sketch of f indicates that there are two roots on $[0, 2\pi]$, near $x = 1.5.$ and $x = 5.5$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\cos x_n - (x_n/7)}{-\sin x_n - (1/7)}.$$

We have:

n	x_n
0	5.5
1	5.63692
2	5.65202
3	5.65222
4	5.65222

n	x_n
0	1.5
1	1.37412
2	1.37333
3	1.37333

The roots are approximately 5.65222 and 1.37333.

4.8.21 A preliminary sketch of f indicates that there are two roots, near $x = -0.4.$ and $x = 1.3$ The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\cos(2x_n) - x_n^2 + 2x_n}{-2\sin(2x_n) - 2x_n + 2}.$$

We have:

n	x_n
0	−0.4
1	−0.337825
2	−0.335412
3	−0.335408
4	−0.335408

n	x_n
0	1.3
1	1.33256
2	1.33306
3	1.33306

The roots are approximately −0.335408 and 1.33306.

4.8.22 A preliminary sketch of f indicates that there are two roots on $[0, 8]$, near $x = 6.1$ and $x = 6.9$
The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n/6 - \sec x_n}{1/6 - \sec x_n \tan x_n}.$$

We have:

n	x_n
0	6.1
1	6.10098
2	6.10099
3	6.10099

n	x_n
0	6.9
1	6.79197
2	6.76435
3	6.7627
4	6.7627

The roots are approximately 6.10099 and 6.7627.

4.8.23 A preliminary sketch of f indicates that there is one root, near $x = 0.2$
The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{e^{-x_n} - (x_n + 4)/5}{-e^{-x_n} - 1/5}.$$

We have:

n	x_n
0	0.2
1	0.179122
2	0.179295
3	0.179295

The root is approximately 0.179295.

4.8.24 A preliminary sketch of f indicates that there are three roots, near $x = -1$. $x = -0.7$ and $x = 1.2$. In fact, -1 is a root because $f(-1) = (-1/5) - (-1/4) - (1/20) = 0$.
The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n^5/5 - x_n^3/4 - 1/20}{x_n^4 - 3x_n^2/4}.$$

We have:

n	x_n
0	-0.7
1	-0.683234
2	-0.683551
3	-0.683551

n	x_n
0	1.2
1	1.18424
2	1.18355
3	1.18355

The roots are -1 and approximately -0.683551 and 1.18355.

4.8.25 A preliminary sketch of f indicates that there are two roots, near $x = .5$ and near $x = 3$.
The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{\ln x_n - x_n^2 + 3x_n - 1}{(1/x_n) - 2x_n + 3}.$$

We have:

n	x_n
0	0.5
1	0.610787
2	0.620655
3	0.620723
4	0.620723

n	x_n
0	3
1	3.03698
2	3.03645
3	3.03645

The roots are approximately 0.620723 and 3.03645.

4.8.26 A preliminary sketch of f indicates that there are three roots; two near $x = 0$ and one near $x = 100$. To avoid division by 0, we use initial estimates of $x_0 = \pm 0.1$ and $x_0 = 100$.

The Newton's method recursion for f is given by

$$x_{n+1} = x_n - \frac{x_n^3 - 100x_n^2 + 1}{3x_n^2 - 200x_n}.$$

We have:

n	x_n
0	-0.1
1	-0.0999501
2	-0.0999501

n	x_n
0	0.1
1	0.10005
2	0.10005

n	x_n
0	100
1	99.9999
2	99.9999

The roots are approximately -0.0999501, 0.10005, and 99.9999.

4.8.27 A preliminary sketch of the two curves seems to indicate that they intersect once near $x = 2$.

Let $f(x) = \sin x - x/2$. Then $f'(x_n) = \cos x_n - 1/2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{\sin x_n - x_n/2}{\cos x_n - 1/2}.$$

If we use an initial estimate of $x_0 = 2$, we obtain $x_1 = 1.901$, $x_2 = 1.89551$, $x_3 = 1.89549$ and $x_4 = 1.89549$, so the point of intersection appears to be at approximately $x = 1.89549$.

4.8.28 A preliminary sketch of the two curves seems to indicate that they intersect once near $x = 2$.

Let $f(x) = e^x - x^3$. Then $f'(x_n) = e^{x_n} - 3x_n^2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{e^{x_n} - x_n^3}{e^{x_n} - 3x_n^2}.$$

If we use an initial estimate of $x_0 = 2$, we obtain $x_1 = 1.8675$, $x_2 = 1.85725$, $x_3 = 1.85718$, so the point of intersection appears to be at approximately $x = 1.857$.

4.8.29 A preliminary sketch of the two curves seems to indicate that they intersect three times, once between -2.5 and -2, once between 0 and $1/2$, and once between 1.5 and 2.

Let $f(x) = 4 - x^2 - 1/x$. Then $f'(x_n) = -2x_n + (1/x_n^2)$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{4 - x_n^2 - (1/x_n)}{-2x_n + (1/x_n^2)}.$$

If we use an initial estimate of $x_0 = -2.25$, we obtain $x_1 = -2.11843$, $x_2 = -2.11491$, $x_3 = -2.11491$, so there appears to be a point of intersection near $x = -2.115$.

If we use an initial estimate of $x_0 = .25$, we obtain $x_1 = .254032$, $x_2 = .254102$, so there appears to be a point of intersection near $x = .254$.

If we use an initial estimate of $x_0 = 1.75$, we obtain $x_1 = 1.86535$, $x_2 = 1.86081$, so there appears to be another point of intersection near $x = 1.86$.

4.8.30 A preliminary sketch of the two curves seems to indicate that they intersect once, near $x = 1.5$.

Let $f(x) = x^3 - (x^2 + 1)$. Then $f'(x_n) = 3x_n^2 - 2x_n$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}.$$

If we use an initial estimate of $x_0 = 1.5$, we obtain $x_1 = 1.46667$, $x_2 = 1.46557$, $x_3 = 1.46557$, so there appears to be a point of intersection near $x = 1.46557$.

4.8.31 A preliminary sketch of the two curves seems to indicate that they intersect twice, once just to the right of 0, and once between 2 and 2.5.

Let $f(x) = 4\sqrt{x} - (x^2 + 1)$. Then $f'(x_n) = 2/\sqrt{x_n} - 2x_n$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{4\sqrt{x_n} - (x_n^2 + 1)}{2/\sqrt{x_n} - 2x_n}.$$

If we use an initial estimate of $x_0 = .1$, we obtain $x_1 = .0583788$, $x_2 = .0629053$, $x_3 = .0629971$, so there appears to be a point of intersection near $x = .06299$.

If we use an initial estimate of $x_0 = 2.25$, we obtain $x_1 = 2.23026$, $x_2 = 2.23012$, $x_3 = 2.23012$, so there appears to be a point of intersection near $x = 2.23012$.

4.8.32 A preliminary sketch of the two curves seems to indicate that they intersect twice, once just to the right of 0, and once between 1 and 1.5

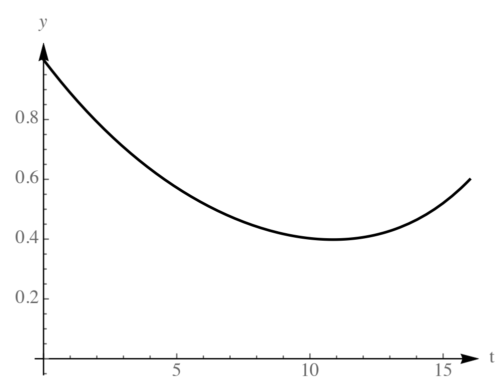
Let $f(x) = \ln x - (x^3 - 2)$. Then $f'(x_n) = 1/x_n - 3x_n^2$. The Newton's method formula becomes

$$x_{n+1} = x_n - \frac{\ln x_n - (x_n^3 - 2)}{1/x_n - 3x_n^2}.$$

If we use an initial estimate of $x_0 = .1$, we obtain $x_1 = .13045$, $x_2 = .13557$, $x_3 = .135674$, so there appears to be a point of intersection near $x = .13567$.

If we use an initial estimate of $x_0 = 1.4$, we obtain $x_1 = 1.32111$, $x_2 = 1.31501$, $x_3 = 1.31498$, so there appears to be a point of intersection near $x = 1.31498$.

4.8.33 The tumor decreases in size, then it starts growing again. Because the initial size of the tumor is 1, we need to find when $V(t) = \frac{1}{2}$. Using Newton's Method on the function $g(t) = V(t) - 1/2$, with an initial estimate of $t_0 = 6$, we find $t_1 = 6.40065$, $t_2 = 6.41774$, $t_3 = 6.41777$, $t_4 = 6.41777$. So the tumor reaches half its original size after about 6.4 days.



4.8.34 The derivative of V is $V'(t) = -0.120384e^{-0.1216t} + 0.00239e^{0.239t}$. Applying Newton's Method to this function to find where it is zero with an initial estimate of 8 yields

n	t_n
0	8
1	11.1212
2	10.8729
3	10.8691
4	10.8691

So the second treatment should be given after about 10.9 days.

4.8.35

a. We need $A = 1,000,000$, so $\dfrac{10,000((1+r)^{30}-1)}{r} = 1,000,000$. Multiplying both sides by r yields $10,000((1+r)^{30}-1) = 1,000,000r$, or

$$1,000,000r - 10,000(1+r)^{30} + 10,000 = 0.$$

b. Using Newton's Method with an initial estimate of 0.05 gives

n	r_n
0	0.05
1	0.121455
2	0.0969022
3	0.0811171
4	0.0743737
5	0.073191
6	0.0731527

The interest rate required is about 7.3%.

4.8.36

a. Let $f(r) = 2500\left(1+\dfrac{r}{12}\right)^{60} - 3200$. Applying Newton's method with an initial estimate of $r_0 = 0.05$ yields

n	r_n
0	0.05
1	0.04947
2	0.04947

So the interest rate should be about 4.95%.

b. We need $2500\left(1+\dfrac{r}{12}\right)^{60} = 3200$, so $\left(1+\dfrac{r}{12}\right)^{60} = \dfrac{3200}{2500}$, so $1+\dfrac{r}{12} = \left(\dfrac{32}{25}\right)^{1/60}$, so $r = 12\left(\dfrac{32}{25}\right)^{1/60} - 12 \approx 0.04947$.

4.8.37

a. We are seeking the time t when first $y(t) = 2.5e^{-t}\cos 2t$ is zero. This occurs first for $t = \pi/4$.

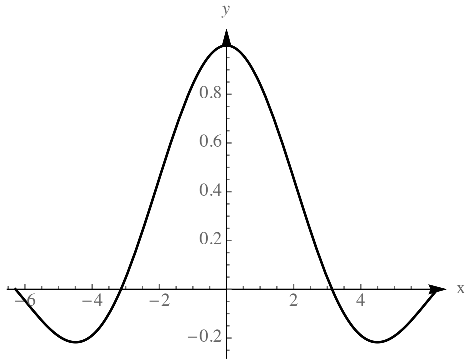
b. We are seeking the minimum value for y. We have $y'(t) = -2.5e^{-t}\cos 2t + 2.5e^{-t}(-2\sin 2t) = -2.5e^{-t}(\cos 2t + 2\sin 2t)$. This is zero when $\cos 2t = -2\sin 2t$, or $\tan 2t = \frac{-1}{2}$. Let $f(t) = \tan 2t + \frac{1}{2}$. If we apply Newton's method to $f(t)$ with a starting point of $t_0 = 1$, we obtain a root of 1.33897 after five iterations. An application of the First Derivative Test shows that there is a local minimum for y at this number. The displacement at this time is -0.586107. This local minimum is in fact an absolute minimum.

c. The second time that $y(t) = 2.5e^{-t}\cos 2t$ is zero is when $2t = \frac{3\pi}{2}$, or $t = 3\pi/4$.

d. Following our work in part b, we look for a root of $f(t) = \tan 2t + \frac{1}{2}$ that is bigger than 1.33897. From the graph, we are looking near $t = 3$. Applying Newton's method to $f(t)$ with an initial value of $x_0 = 3$ gives a root 2.90977 after three iterations. Applying the First Derivative Test, we see that there is a local maximum of 0.12181 at $x = 2.90977$.

4.8.38

a.



b. $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$. This is zero when $x\cos x - \sin x = 0$, or $x - \tan x = 0$. Let $g(x) = x - \tan x$. If we apply Newton's method to g with a starting value of $x_0 = 4.5$, we obtain a root at 4.49341 after 2 iterations. The First Derivative Test confirms that there is a local minimum of about -0.217234 at $x = 4.49341$. Applying Newton's method to g with a starting value of 7.8 yields the root 7.72525 after 5 iterations. The First Derivative Test confirms that there is a local maximum of about .128375 at $x = 7.2525$.

4.8.39

Let $p(x) = x^4 - 7$. Because $p'(x_n) = 4x_n^3$, we have

$$x_{n+1} = x_n - \frac{x_n^4 - 7}{4x_n^3} = \frac{3}{4}x_n + \frac{7}{4}\cdot\frac{1}{x_n^3}.$$

We use an initial guess of $x_0 = 1.7$.

n	x_n
0	1.7
1	1.6312
2	1.6266
3	1.62658
4	1.62658

It appears that $\sqrt[4]{7} \approx 1.62658$.

4.8.40

Let $p(x) = x^3 - 2$. Because $p'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3}x_n + \frac{2}{3}\cdot\frac{1}{x_n^2}.$$

We use an initial guess of $x_0 = 1.2$.

n	x_n
0	1.2
1	1.26296
2	1.25993
3	1.25992
4	1.25992

It appears that $\sqrt[3]{2} \approx 1.25992$.

4.8.41

Let $p(x) = x^3 + 9$. Because $p'(x_n) = 3x_n^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 + 9}{3x_n^2} = \frac{2}{3}x_n - \frac{3}{x_n^2}.$$

We use an initial guess of $x_0 = -2$.

n	x_n
0	−2.0
1	−2.08333
2	−2.08008
3	−2.08008

It appears that $\sqrt[3]{-9} \approx -2.08008$.

4.8.42 Let $p(x) = x^5 + 67$. Because $p'(x_n) = 5x_n^4$, we have

$$x_{n+1} = x_n - \frac{x_n^5 + 67}{5x_n^4} = \frac{4}{5}x_n - \frac{67}{5} \cdot \frac{1}{x_n^4}.$$

We use an initial guess of $x_0 = -2$.

n	x_n
0	-2.0
1	-2.4375
2	-2.3296
3	-2.31864
4	-2.31854
5	-2.31854

It appears that $\sqrt[5]{-67} \approx -2.31854$.

4.8.43

$$f'(x) = \frac{-x\sin x - \cos x}{x^2},$$

which is zero when $x\sin x + \cos x = 0$. Note that $f'(1) < 0$ and $f'(\pi) > 0$, so there must be a local minimum on the interval $(1, \pi)$. Let $g(x) = x\sin x + \cos x$. Then $g'(x_n) = \sin x_n + x_n \cos x_n - \sin x_n = x_n \cos x_n$, and the Newton's method formula becomes

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}.$$

If we use an initial estimate of $x_0 = 2.5$, we obtain $x_1 = 2.84702$, $x_2 = 2.79918$, $x_3 = 2.79839$, $x_4 = 2.79839$, so the smallest local minimum of f on $(0, \infty)$ occurs at approximately 2.79839.

4.8.44 $f'(x) = 12x^3 + 24x^2 + 24x + 48 = 12(x^3 + 2x^2 + 2x + 4)$. We are seeking values of x so that $x^3 + 2x^2 + 2x + 4 = 0$. Using Newton's method with an initial estimate of -1.5, we obtain $x_1 = -2.27273$, $x_2 = -2.04023$, $x_3 = -2.00104$, and $x_4 \approx -2$. We realize that $x = -2$ is a root of $f'(x)$. Using long division (by $x + 2$), we see that $f'(x) = 12(x + 2)(x^2 + 2)$, so $f'(x)$ has only the one root of -2.

Note that $f'(-3) < 0$ and $f'(0) > 0$, so there is a local (in fact, absolute) minimum at $x = -2$.

4.8.45 $f'(x) = 9x^4 - 30x^3 + 7x^2 + 60x$ and $f''(x) = 36x^3 - 90x^2 + 14x + 60 = 2(18x^3 - 45x^2 + 7x + 30)$. We are seeking roots of $f''(x)$. If we apply Newton's method to f'' we obtain the recursion

$$x_{n+1} = x_n - \frac{36x_n^3 - 90x_n^2 + 14x_n + 60}{108x_n^2 - 180x_n + 14}.$$

Starting with an initial estimate of $x_0 = 1$, we obtain $x_1 = 1.34483$, $x_2 = 1.45527$, $x_3 = 1.49284$, $x_4 = 1.49974$, and $x_5 \approx 1.5$. We check directly that 1.5 is a root of f'', so $2x - 3$ is a factor of f'', and using long division, we see that $f''(x) = 2(2x - 3)(9x^2 - 9x - 10) = 2(2x - 3)(3x + 2)(3x - 5)$. So the potential inflection points of f are located at $x = -2/3$, $x = 3/2$, and $x = 5/3$. A check of the sign of f'' on the various intervals confirms that these are all the locations of inflection points.

4.8.46 The domain of f is $(-\infty, 0) \cup (0, \infty)$.

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}.$$

This is 0 for $x = 1$. Note that $f'(1/2) < 0$ and $f'(2) > 0$, so there is a local minimum at $x = 1$. The point $(1, e)$ is the only extreme point. However, the questions calls for a solution via Newton's Method, so we attempt to solve the equation $\frac{xe^x - e^x}{x^2} = 0$ for x via Newton's Method. The recursion is given by

$$x_{n+1} = x_n - \frac{(x_n e^{x_n} - e^{x_n})/x_n^2}{e^{x_n}\left(x_n^2 - 2x_n + 2\right)/x_n^3}$$

. Starting with an initial estimate of $x_0 = 0.5$, we obtain $x_1 = 0.7$, $x_2 = 0.89266$, $x_3 = 0.98739$, $x_4 = 0.999839$, and $x_5 \approx 1$.

4.8.47

a. True.

b. False. The quadratic formula gives exact values.

c. False. It sometime fails depending on factors such as the shape of the curve and the closeness of the initial estimate.

4.8.48 A preliminary sketch suggests there are 3 intersection points, near $x = -0.5$, $x = 0.5$, and $x = 3$. Using Newton's Method on the function $g(x) = f(x) - x = x^3 - 3x^2 + x + 1 - x = x^3 - 3x^2 + 1$ gives.

n	x_n
0	-0.5
1	-0.53333
2	-0.53209
3	-0.53209

n	x_n
0	0.5
1	0.66666
2	0.65277
3	0.65270
4	0.65270

n	x_n
0	3.0
1	2.88888
2	2.87945
3	2.87939
4	2.87939

So the fixed points are approximately -0.53209, 0.65270, and 2.87939.

4.8.49 A preliminary sketch suggests there is 1 intersection point, near $x = 0.75$. Using Newton's Method on the function $g(x) = f(x) - x = \cos x - x$ gives.

n	x_n
0	0.75
1	0.73911
2	0.73908
3	0.73908

The fixed point is approximately 0.73908.

4.8.50 Let $g(x) = f(x) - x = \tan(x/2) - x$. Fixed points of f are roots of g. The Newton's method recursion for g is given by

$$x_{n+1} = x_n - \frac{\tan(x_n/2) - x_n}{(1/2)\sec^2(x_n/2) - 1}.$$

A preliminary sketch of g indicates that there are three roots, near $x = 0$ and $x = \pm 2.3$

n	x_n
0	-1.2
1	0.739493
2	-0.0887636
3	0.000116976
4	≈ 0

n	x_n
0	-2.3
1	-2.33281
2	-2.33113
3	-2.33112
4	-2.33112
5	-2.33112

n	x_n
0	2.3
1	2.33281
2	2.33113
3	2.33112
4	2.33112
5	2.33112

The fixed points are 0 and approximately ± 2.33112.

4.8.51 Let $g(x) = 2x\cos x - x$. Fixed points of f are roots of g. Clearly $x = 0$ is a root of g. The Newton's method recursion for g is given by

$$x_{n+1} = x_n - \frac{2x_n \cos x_n - x_n}{2\cos x_n - 2x\sin x_n - 1}.$$

A preliminary sketch of g indicates that there is only one nonzero root on $[0, 2]$, near $x = 1$. We have:

n	x_n
0	1
1	1.0503
2	1.04721
3	1.0472
4	1.0472

The fixed points are 0 and approximately 1.0472.

4.8.52

a. $f'(x) = \dfrac{1+x^2-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. Then the expression $x - \dfrac{f(x)}{f'(x)}$ can be written as

$$x - \frac{x}{1+x^2}\cdot\frac{(1+x^2)^2}{1-x^2} = x - \frac{x+x^3}{1-x^2} = \frac{2x^3}{x^2-1}.$$

Then note that starting with $x_0 = \dfrac{1}{\sqrt{3}}$, we have

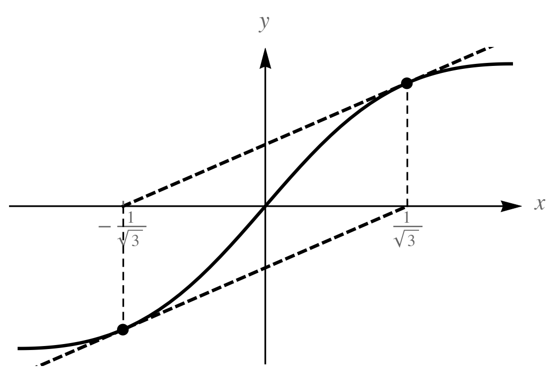
$$x_1 = \frac{\frac{2}{3\sqrt{3}}}{\frac{-2}{3}} = -\frac{1}{\sqrt{3}},$$

while

$$x_2 = \frac{\frac{-2}{3\sqrt{3}}}{\frac{-2}{3}} = \frac{1}{\sqrt{3}}.$$

b. By repetition, $x_3 = -\dfrac{1}{\sqrt{3}}$ and $x_4 = \dfrac{1}{\sqrt{3}}$, $x_5 = \dfrac{1}{\sqrt{3}}$, etc.

c.



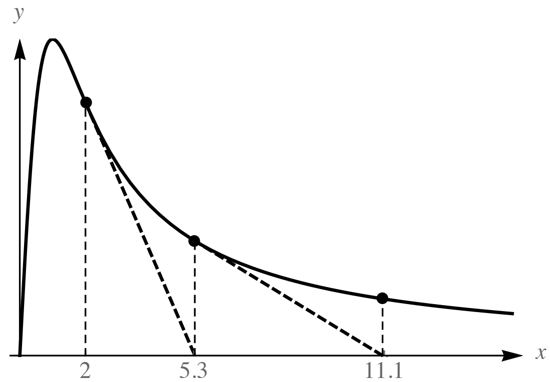
d. The sequence $x_0, x_1, x_2, \ldots$ doesn't converge at all because it just cycles between $\dfrac{1}{\sqrt{3}}$ and $-\dfrac{1}{\sqrt{3}}$.

4.8.53

a.

n	x_n
0	2
1	5.33333
2	11.0553
3	22.2931

b.



c. The tangent lines intersect the x-axis farther and farther away from the root r.

4.8.54

a. If r is a root of $x^2 - a$, then $r^2 - a = 0$, so $r^2 = a$, and $|r| = \sqrt{a}$, so either $r = \sqrt{a}$ or $r = -\sqrt{a}$. If we also insist that $r > 0$, then $r = \sqrt{a}$.

b. The Newton's method recursion is

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

c. Because $3^2 = 9 < 13$ and $4^2 = 16 > 13$, a good starting value for $\sqrt{13}$ would be a number between 3 and 4 (but closer to 4), like 3.6.

Because $8^2 = 64 < 73$ and $9^2 = 81 > 73$, a good starting value for $\sqrt{73}$ would be a number between 8 and 9, like 8.5.

d. The first chart is for $\sqrt{13}$ and the second is for $\sqrt{73}$.

n	x_n
0	3.6
1	3.60555555556
2	3.60555127547
3	3.6055127546
4	3.6055127546
5	3.6055127546
6	3.6055127546
7	3.6055127546
8	3.6055127546
9	3.6055127546
10	3.6055127546

n	x_n
0	8.5
1	8.54411764706
2	8.54400374608
3	8.54400374532
4	8.54400374532
5	8.54400374532
6	8.54400374532
7	8.54400374532
8	8.54400374532
9	8.54400374532
10	8.54400374532

4.8.55

a. The Newton's method formula would be:

$$x_{n+1} = x_n - \frac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2 = (2 - ax_n)x_n.$$

b. The approximation to $1/7$ is 0.14285714.

n	x_n
0	0.1
1	0.13
2	0.1417
3	0.14284777
4	0.14285714
5	0.14285714

4.8.56

a. Note that $f(0) = 0$, and $f'(x) = 2e^{2\sin x}\cos x - 2$, so $f'(0) = 2e^0 - 2 = 0$. Also, $f''(x) = 4e^{2\sin(x)}\cos^2(x) - 2e^{2\sin(x)}\sin(x)$, so $f''(0) = 4 \neq 0$. Thus, 0 is a root of multiplicity 2 for f.

b. The first chart is for the traditional Newton's method, and the second is for the modified version. Clearly, x_3 for the modified method is much closer to the actual root.

n	x_n
0	0.1
1	0.0511487
2	0.0258883
3	0.0130263

n	x_n
0	0.1
1	0.00229741
2	0.00000131725
3	-3.07429×10^{-11}

c. In Example 4, the value of x_3 was 0.0171665. For the modified version, we obtain:

n	x_n
0	0.15
1	-0.010125
2	0.00000311391
3	-9.05817×10^{-17}

Clearly the modified version converges much more quickly to the root of multiplicity two at 0.

4.8.57 Let $f(\lambda) = \tan(\pi\lambda) - \lambda$. We are looking for the first three positive roots of f. A preliminary sketch indicates that they are located near 1.4, 2.4, and 3.4. The Newton's method recursion is given by

$$x_{n+1} = x_n - \frac{\tan(\pi x_n) - x_n}{\pi\sec^2(\pi x_n) - 1}.$$

We obtain the following results:

n	x_n
0	1.4
1	1.34741
2	1.30555
3	1.29121
4	1.29012
5	1.29011
6	1.29011

n	x_n
0	2.4
1	2.37876
2	2.37331
3	2.37305
4	2.37305

n	x_n
0	3.4
1	3.4101
2	3.40919
3	3.40918
4	3.40918

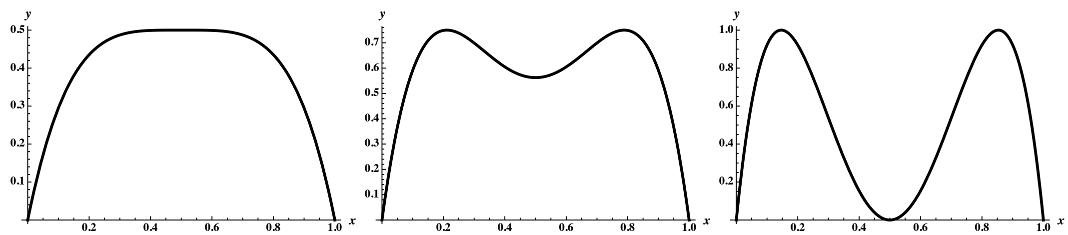
The first three positive eigenvalues are approximately 1.29011, 2.37305, and 3.40918.

4.8.58

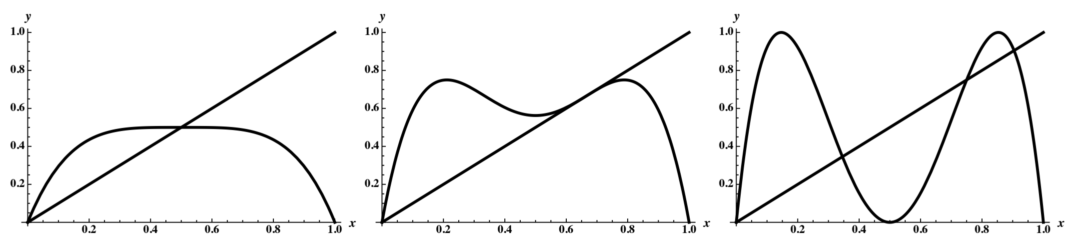
a. We are seeking solutions of $f(x) = ax(1-x) = x$. This can be written as $ax^2 + x(1-a) = 0$, or $x(ax + (1-a)) = 0$. The solutions of this equation are $x = 0$ and $x = \frac{a-1}{a}$. If $0 < a < 1$, this does not give a value of x in the range $(0, 1)$. If $1 \le a \le 4$, we do get a fixed point $x = \frac{a-1}{a}$.

b. $g(x) = f(f(x)) = f(ax(1-x)) = a(ax(1-x))(1 - ax(1-x)) = (a^2x - a^2x^2)(1 - ax + ax^2) = a^2x - a^2x^2 - a^3x^2 + a^3x^3 + a^3x^3 - a^3x^4 = a^2x - a^2x^2 - a^3x^2 + 2a^3x^3 - a^3x^4$. This is a fourth degree polynomial.

c. From left to right, with $a = 2$, then $a = 3$, then $a = 4$:



d. The graphs of $y = g(x)$ together with $y = x$ for $a = 2$, then $a = 3$, then $a = 4$.



When $a = 2$, we have $g(x) = -8x^4 + 16x^3 - 12x^2 + 4x$, so we are looking for a root of $g(x) - x = -8x^4 + 16x^3 - 12x^2 + 4x - x = -8x^4 + 16x^3 - 12x^2 + 3x = x(-8x^3 + 16x^2 - 12x + 3)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.5$. A quick check shows that $x = 0.5$ is a root of $g(x) - x$, so 0.5 is a fixed point of g.

When $a = 3$, we have $g(x) = -27x^4 + 54x^3 - 36x^2 + 9x$, so we are looking for a root of $g(x) - x = -27x^4 + 54x^3 - 36x^2 + 9x - x = -27x^4 + 54x^3 - 36x^2 + 8x = x(-27x^3 + 54x^2 - 36x + 8)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.6$. Applying Newton's method to $g(x) - x$ with an initial estimate of 0.6 yields a root of approximately $\overline{0.6} = 2/3$. A quick check shows that $2/3$ is a fixed point of g.

When $a = 4$, we have $g(x) = -64x^4 + 128x^3 - 80x^2 + 16x$, so we are looking for a root of $g(x) - x = -64x^4 + 128x^3 - 80x^2 + 16x - x = -64x^4 + 128x^3 - 80x^2 + 15x = x(-64x^3 + 128x^2 - 80x + 15)$. Clearly $x = 0$ is one root, and the diagram indicates that $g(x) = x$ near $x = 0.3$ and $x = 0.75$ and $x = 0.9$. Checking the value of $0.75 = 3/4$, we confirm that $g(3/4) = 3/4$. Applying Newton's method to $g(x) - x$ with an initial estimate of 0.3 yields a root of approximately 0.345492, and applying it with an initial estimate of 0.9 yields a root of approximately 0.904508.

Thus the fixed points of g with $a = 4$ are 0, 0.345492, 0.75, and 0.904508.

4.8.59 This problem can be solved (approximately) by setting up a computer or calculator program to run Newton's method, and then experimenting with different starting values. If this is done, it can be seen that any initial estimate between -4 and the local maximum at approximately -1.53 converges to the root at -2. Initial values between approximately -1.52 and -1.486 converge to the root at 3, while starting values between -1.485 and -1.475 converge to the root at -2. From -1.474 to approximately 0.841, starting values lead to convergence to the root at -1, while from 0.842 to 0.846 they lead to convergence to the root at -2. From about 0.847 to 0.862 they lead to convergence to the root at 3, while from 0.863 to the local minimum at 1.528 they lead to convergence to the root at -2. From about 1.528 to 4, the convergence is to the root at 3. Thus the approximate basis of convergence for -2 is $[-4,-1.53]\cup[-1.485,-1.475]\cup[0.842,0.846]$. For -1 the approximate basis of convergence is $[-1.474, 0.841]$, and for 3, it is $[-1.52,-1.486]\cup[0.847,0.862]\cup[1.53,4]$.

4.8.60 The recursion for $f(x)$ is

$$x_{n+1} = x_n - \frac{(x_n-1)^2}{2x_n-2} = \frac{2x_n^2-2x_n-(x_n^2-2x_n+1)}{2x_n-2} = \frac{x_n^2-1}{2(x_n-1)} = \frac{x_n+1}{2}.$$

The recursion for $g(x)$ is

$$y_{n+1} = y_n - \frac{y_n^2-1}{2y_n} = \frac{2y_n^2-(y_n^2-1)}{2y_n} = \frac{y_n^2+1}{2y_n}.$$

The comparison below shows that Newton's method converges much faster for $g(x) = x^2-1$. This is because it is steeper near the root $x = 1$ – the value of $g'(1) = 2$, while $f'(1) = 0$. The flatness of f near 1 causes slow convergence.

n	x_n
0	2
1	1.5
2	1.25
3	1.125
4	1.0625
5	1.03125
6	1.01563
7	1.00781
8	1.00391
9	1.00195
10	1.00098

n	y_n
0	2
1	1.25
2	1.025
3	1.0003
4	≈ 1

4.9 Antiderivatives

4.9.1 Derivative, antiderivative.

4.9.2 C, where C is any constant.

4.9.3 $x + C$, where C is any constant.

4.9.4 By Theorem 4.11, if two functions have the same derivative then they differ by a constant.

4.9.5 $\dfrac{x^{p+1}}{p+1} + C$, where C is any real number and $p \neq -1$.

4.9.6 $a\sin^{-1} x + C$, where C is any constant.

4.9.7 $\ln|x| + C$, where C is any constant.

4.9.8 $\int a\cos x\,dx = a\sin x + C$ and $\int a\sin x\,dx = -a\cos x + C$

4.9.9 Observe that $F(-1) = 4 + C = 4$, so $C = 0$.

4.9.10 First, find the general solution $F(t)$, which is the family of all antiderivatives of $f(t)$. Then use the initial condition to find the specific value of the constant in the formula for $F(t)$.

4.9.11 The antiderivatives of $5x^4$ are $x^5 + C$. Check: $\frac{d}{dx}(x^5 + C) = 5x^4$.

4.9.12 The antiderivatives of $11x^{10}$ are $x^{11} + C$. Check: $\frac{d}{dx}(x^{11} + C) = 11x^{10}$.

4.9.13 The antiderivatives of $2\sin x + 1$ are $-2\cos x + x + C$. Check:

$$\frac{d}{dx}(-2\cos x + x + C) = -2(-\sin x) + 1 = 2\sin x + 1.$$

4.9.14 The antiderivatives of $-4\cos x - x$ are $-4\sin x - \frac{x^2}{2} + C$. Check:

$$\frac{d}{dx}\left(-4\sin x - \frac{x^2}{2} + C\right) = -4\cos x - x.$$

4.9.15 The antiderivatives of $3\sec^2 x$ are $3\tan x + C$. Check: $\frac{d}{dx}(3\tan x + C) = 3\sec^2 x$.

4.9.16 The antiderivatives of $\csc^2 s$ are $-\cot s + C$. Check: $\frac{d}{ds}(-\cot s + C) = \csc^2 s$.

4.9.17 The antiderivatives of $-2/y^3 = -2y^{-3}$ are $y^{-2} + C$. Check: $\frac{d}{dy}(y^{-2} + C) = -2y^{-3}$.

4.9.18 The antiderivatives of $-6z^{-7}$ are $z^{-6} + C$. Check: $\frac{d}{dz}(z^{-6} + C) = -6z^{-7}$.

4.9.19 The antiderivatives of e^x are $e^x + C$. Check: $\frac{d}{dx}(e^x + C) = e^x$.

4.9.20 The antiderivatives of y^{-1} are $\ln|y| + C$. Check: $\frac{d}{dy}(\ln|y| + C) = \frac{1}{y}$.

4.9.21 The antiderivatives of $\frac{1}{s^2+1}$ are $\tan^{-1} s + C$. Check: $\frac{d}{ds}(\tan^{-1}(s) + C) = \frac{1}{s^2+1}$.

4.9.22 The antiderivatives of π are $\pi t + C$. Check: $\frac{d}{dt}(\pi t + C) = \pi$.

4.9.23 $\int (3x^5 - 5x^9)\,dx = 3\cdot\frac{x^6}{6} - 5\cdot\frac{x^{10}}{10} + C = \frac{1}{2}x^6 - \frac{1}{2}x^{10} + C$. Check: $\frac{d}{dx}(\frac{1}{2}x^6 - \frac{1}{2}x^{10} + C) = 3x^5 - 5x^9$.

4.9.24 $\int (3u^{-2} - 4u^2 + 1)\,du = 3\cdot\frac{u^{-1}}{-1} - 4\cdot\frac{u^3}{3} + u + C = -\frac{4}{3}u^3 + u - \frac{3}{u} + C$. Check: $\frac{d}{du}(-\frac{4}{3}u^3 + u - \frac{3}{u} + C) = -4u^2 + 1 + 3u^{-2}$.

4.9.25 $\int \left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right)dx = \int (4x^{1/2} - 4x^{-1/2})\,dx = 4\cdot\frac{x^{3/2}}{3/2} - 4\cdot\frac{x^{1/2}}{1/2} + C = \frac{8}{3}x^{3/2} - 8x^{1/2} + C$. Check: $\frac{d}{dx}(\frac{8}{3}x^{3/2} - 8x^{1/2} + C) = 4\sqrt{x} - \frac{4}{\sqrt{x}}$.

4.9.26 $\int\left(\frac{5}{t^2}+4t^2\right)dt = 5\cdot\frac{t^{-1}}{-1}+4\cdot\frac{t^3}{3}+C = \frac{4}{3}t^3-\frac{5}{t}+C$. Check: $\frac{d}{dt}(\frac{4}{3}t^3-\frac{5}{t}+C) = 4t^2+\frac{5}{t^2}$.

4.9.27 $\int(5s+3)^2\,ds = \int(25s^2+30s+9)\,ds = 25\frac{s^3}{3}+30\frac{s^2}{2}+9s+C = \frac{25s^3}{3}+15s^2+9s+C$. Check: $\frac{d}{ds}\left(\frac{25s^3}{3}+15s^2+9s+C\right) = 25s^2+30s+9 = (5s+3)^2$.

4.9.28 $\int 5m(12m^3-10m)\,dm = \int(60m^4-50m^2)\,dm = 60\cdot\frac{m^5}{5}-50\cdot\frac{m^3}{3}+C = 12m^5-\frac{50}{3}m^3+C$. Check: $\frac{d}{dm}(12m^5-\frac{50}{3}m^3+C) = 5m(12m^3-10m)$.

4.9.29 $\int(3x^{1/3}+4x^{-1/3}+6)\,dx = 3\cdot\frac{3}{4}x^{4/3}+4\cdot\frac{3}{2}x^{2/3}+6x+C = \frac{9}{4}x^{4/3}+6x^{2/3}+6x+C$. Check: $\frac{d}{dx}\left(\frac{9}{4}x^{4/3}+6x^{2/3}+6x+C\right) = 3x^{1/3}+4x^{-1/3}+6$.

4.9.30 $\int 6\sqrt[3]{x}\,dx = \int 6x^{1/3}\,dx = 6\cdot\frac{3}{4}x^{4/3}+C = \frac{9}{2}x^{4/3}+C$. Check $\frac{d}{dx}\left(\frac{9}{2}x^{4/3}+C\right) = 6\sqrt[3]{x}$.

4.9.31 $\int(3x+1)(4-x)\,dx = \int(12x-3x^2+4-x)\,dx = \int(-3x^2+11x+4)\,dx = -x^3+\frac{11}{2}x^2+4x+C$. Check: $\frac{d}{dx}\left(\frac{9}{2}x^{4/3}+C\right) = 6\sqrt[3]{x}$.

4.9.32 $\int(4z^{1/3}-z^{-1/3})\,dz = 3z^{4/3}-\frac{3}{2}z^{2/3}+C$. Check: $\frac{d}{dz}\left(3z^{4/3}-\frac{3}{2}z^{2/3}+C\right) = 4z^{1/3}-z^{-1/3}$.

4.9.33 $\int(3x^{-4}+2-3x^{-2})\,dx = -x^{-3}+2x+3x^{-1}+C$. Check: $\frac{d}{dx}\left(-x^{-3}+2x+3x^{-1}+C\right) = 3x^{-4}+2-3x^{-2}$.

4.9.34 $\int r^{2/5}\,dr = \frac{5}{7}r^{7/5}+C$. Check: $\frac{d}{dr}\left(\frac{5}{7}r^{7/5}+C\right) = r^{2/5}$.

4.9.35 $\int\frac{4x^4-6x^2}{x}\,dx = \int\left(\frac{4x^4}{x}-\frac{6x^2}{x}\right)dx = \int(4x^3-6x)\,dx = x^4-3x^2+C$.
Check: $\frac{d}{dx}\left(x^4-3x^2+C\right) = 4x^3-6x$.

4.9.36 $\int\frac{12t^8-t}{t^{3/2}}\,dt = \int\left(\frac{12t^8}{t^{3/2}}-\frac{t}{t^{3/2}}\right)dt = \int\left(12t^{13/2}-t^{-1/2}\right)dt = \frac{24}{15}t^{15/2}-2t^{1/2}+C = \frac{8}{5}t^{15/2}-2\sqrt{t}+C$. Check: $\frac{d}{dt}\left(\frac{8}{5}t^{15/2}-2\sqrt{t}+C\right) = 12t^{13/2}-t^{-1/2} = \frac{12t^8-t}{t^{3/2}}$.

4.9.37 $\int\frac{x^2-36}{x-6}\,dx = \int\frac{(x-6)(x+6)}{x-6}\,dx = \int(x+6)\,dx = \frac{x^2}{2}+6x+C$. Check: $\frac{d}{dx}\left(\frac{x^2}{2}+6x+C\right) = x+6 = \frac{x^2-36}{x-6}$.

4.9.38 $\int\frac{y^3-9y^2+20y}{y-4}\,dy = \int\frac{y(y-4)(y-5)}{y-4}\,dy = \int y(y-5)\,dy = \int(y^2-5y)\,dy = \frac{y^3}{3}-\frac{5y^2}{2}+C$.
Check: $\frac{d}{dy}\left(\frac{y^3}{3}-\frac{5y^2}{2}+C\right) = y^2-5y = \frac{(y^2-5y)(y-4)}{y-4} = \frac{y^3-9y^2+20y}{y-4}$.

4.9.39 $\int (\csc^2\theta + 2\theta^2 - 3\theta)\,d\theta = -\cot\theta + \frac{2}{3}\theta^3 - \frac{3}{2}\theta^2 + C$. Check: $\frac{d}{d\theta}\left(-\cot\theta + \frac{2}{3}\theta^3 - \frac{3}{2}\theta^2 + C\right) = \csc^2\theta + 2\theta^2 - 3\theta$.

4.9.40 $\int (\csc^2\theta + 1)\,d\theta = -\cot\theta + \theta + C$. Check: $\frac{d}{d\theta}(-\cot\theta + \theta + C) = \csc^2\theta + 1$.

4.9.41 $\int \frac{2 + 3\cos y}{\sin^2 y}\,dy = \int \left(\frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y}\right) dy = \int \left(2\csc^2 y + 3\cot y \csc y\right) dy = -2\cot y - 3\csc y + C$. Check: $\frac{d}{dy}(-2\cot y - 3\csc y + C) = -2(-\csc^2 y) - 3(-\csc y \cot y) = \frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y} = \frac{2 + 3\cos y}{\sin^2 y}$.

4.9.42 $\int \sin t(4\csc t - \cot t)\,dt = \int (4 - \cos t)\,dt = 4t - \sin t + C$. Check: $\frac{d}{dt}(4t - \sin t + C) = 4 - \cos t = \sin t\left(\frac{4}{\sin t} - \frac{\cos t}{\sin t}\right) = \sin t(4\csc t - \cot t)$.

4.9.43 $\int (\sec^2 x - 1)\,dx = \tan x - x + C$. Check: $\frac{d}{dx}(\tan x - x + C) = \sec^2 x - 1$.

4.9.44 $\int \frac{\sec^3 v - \sec^2 v}{\sec v - 1}\,dv = \int \frac{\sec^2 v(\sec v - 1)}{\sec v - 1}\,dv = \int \sec^2 v\,dv = \tan v + C$. Check: $\frac{d}{dv}(\tan v + C) = \sec^2 v = \frac{\sec^2 v(\sec v - 1)}{\sec v - 1} = \frac{\sec^3 v - \sec^2 v}{\sec v - 1}$.

4.9.45 $\int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$. Check: $\frac{d}{d\theta}(\tan\theta + \sec\theta + C) = \sec^2\theta + \sec\theta\tan\theta$.

4.9.46 $\int \frac{\sin\theta - 1}{\cos^2\theta}\,d\theta = \int (\sec\theta\tan\theta - \sec^2\theta)\,d\theta = \sec\theta - \tan\theta + C$. Check: $\frac{d}{d\theta}(\sec\theta - \tan\theta + C) = \sec\theta\tan\theta - \sec^2\theta = \frac{\sin\theta - 1}{\cos^2\theta}$.

4.9.47 $\int (3t^2 + 2\csc^2 t)\,dt = t^3 - 2\cot t + C$. Check: $\frac{d}{dt}(t^3 - 2\cot t + C) = 3t^2 + 2\csc^2 t$.

4.9.48 $\int \csc x(\cot x - \csc x)\,dx = \int (\csc x\cot x - \csc^2 x)\,dx = -\csc x + \cot x + C$. Check: $\frac{d}{dx}(-\csc x + \cot x + C) = \csc x\cot x - \csc^2 x = \csc x(\cot x - \csc x)$.

4.9.49 $\int \sec\theta(\tan\theta + \sec\theta + \cos\theta)\,d\theta = \int (\sec\theta\tan\theta + \sec^2\theta + 1)\,d\theta = \sec\theta + \tan\theta + \theta + C$. Check: $\frac{d}{d\theta}(\sec\theta + \tan\theta + \theta + C) = \sec\theta\tan\theta + \sec^2\theta + 1 = \sec\theta(\tan\theta + \sec\theta + \cos\theta)$.

4.9.50 $\int \frac{\csc^3 x + 1}{\csc x}\,dx = \int \frac{(\csc^3 x + 1)}{\csc x}\cdot\frac{\sin x}{\sin x}\,dx = \int (\csc^2 x + \sin x)\,dx = -\cot x - \cos x + C$. Check: $\frac{d}{dx}(-\cot x - \cos x + C) = \csc^2 x + \sin x = (\csc^2 x + \sin x)\cdot\frac{\csc x}{\csc x} = \frac{\csc^3 x + 1}{\csc x}$.

4.9.51 $\int \frac{1}{2y}\,dy = \frac{1}{2}\int y^{-1}\,dy = \frac{1}{2}\ln|y| + C$. Check: $\frac{d}{dy}\left(\frac{1}{2}\ln|y| + C\right) = \frac{1}{2y}$.

4.9.52 $\int \frac{e^{2t} - 1}{e^t - 1}\,dt = \int \frac{(e^t - 1)(e^t + 1)}{e^t - 1}\,dt = \int (e^t + 1)\,dt = e^t + t + C$. Check: $\frac{d}{dt}(e^t + t + C) = e^t + 1 = \frac{(e^t - 1)(e^t + 1)}{e^t - 1} = \frac{e^{2t} - 1}{e^t - 1}$.

4.9.53 $\int \frac{6}{\sqrt{4-4x^2}}\,dx = \frac{6}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx = 3\sin^{-1}x + C$. Check: $\frac{d}{dx}(3\sin^{-1}x + C) = \frac{3}{\sqrt{1-x^2}} = \frac{6}{2\sqrt{1-x^2}} = \frac{6}{\sqrt{4-4x^2}}$.

4.9.54 $\int \frac{v^3+v+1}{1+v^2}\,dv = \int \left(\frac{v^3+v}{1+v^2} + \frac{1}{1+v^2}\right)dv = \int \left(\frac{v(v^2+1)}{1+v^2} + \frac{1}{1+v^2}\right)dv = \int \left(v + \frac{1}{1+v^2}\right)dv = \frac{v^2}{2} + \tan^{-1}v + C$. Check: $\frac{d}{dv}\left(\frac{v^2}{2} + \tan^{-1}v + C\right) = v + \frac{1}{1+v^2} = \left(\frac{v(v^2+1)}{1+v^2} + \frac{1}{1+v^2}\right) = \frac{v^3+v+1}{1+v^2}$.

4.9.55 $\int \frac{4}{x\sqrt{x^2-1}}\,dx = 4\sec^{-1}|x| + C$. Check: $\frac{d}{dx}(4\sec^{-1}|x| + C) = \frac{4}{x\sqrt{x^2-1}}$.

4.9.56 $\int \frac{2}{25z^2+25}\,dz = \frac{2}{25}\int \frac{1}{z^2+1}\,dz = \frac{2}{25}\tan^{-1}z + C$. Check: $\frac{d}{dz}\left(\frac{2}{25}\tan^{-1}z + C\right) = \frac{2}{25}\cdot\frac{1}{z^2+1} = \frac{2}{25z^2+25}$.

4.9.57 $\int \frac{1}{x\sqrt{36x^2-36}}\,dx = \frac{1}{6}\int \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{1}{6}\sec^{-1}|x| + C$. Check: $\frac{d}{dx}\left(\frac{1}{6}\sec^{-1}|x| + C\right) = \frac{1}{6}\cdot\frac{1}{x\sqrt{x^2-1}} = \frac{1}{x\sqrt{36x^2-36}}$.

4.9.58 $\int (49-49x^2)^{-1/2}\,dx = \frac{1}{7}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{7}\sin^{-1}x + C$. Check: $\frac{d}{dx}\left(\frac{1}{7}\sin^{-1}x + C\right) = \frac{1}{7\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{49-49x^2}} = (49-49x^2)^{-1/2}$.

4.9.59 $\int \frac{t+1}{t}\,dt = \int \left(\frac{t}{t} + \frac{1}{t}\right)dt = \int \left(1 + \frac{1}{t}\right)dt = t + \ln|t| + C$. Check: $\frac{d}{dt}(t + \ln|t| + C) = 1 + \frac{1}{t} = \frac{t+1}{t}$.

4.9.60 $\int \frac{t^2-e^2t}{t+e^t}\,dt = \int \frac{(t-e^t)(t+e^t)}{t+e^t}\,dt = \int (t-e^t)\,dt = \frac{t^2}{2} - e^t + C$. Check: $\frac{d}{dt}\left(\frac{t^2}{2} - e^t + C\right) = t - e^t = \frac{(t-e^t)(t+e^t)}{t+e^t} = \frac{t^2-e^{2t}}{t+e^t}$.

4.9.61 $\int e^{x+2}\,dx = \int e^2e^x\,dx = e^2\int e^x\,dx = e^2e^x + C = e^{x+2} + C$. Check $\frac{d}{dx}(e^{x+2} + C) = e^{x+2}$.

4.9.62 $\int \frac{10t^5-3}{t}\,dt = \int \left(\frac{10t^5}{t} - \frac{3}{t}\right)dt = \int \left(10t^4 - \frac{3}{t}\right)dt = 2t^5 - 3\ln|t| + C$. Check: $\frac{d}{dt}(2t^5 - 3\ln|t| + C) = 10t^4 - \frac{3}{t} = \frac{10t^5-3}{t}$.

4.9.63 $\int \frac{e^{2w}-5e^w+4}{e^w-1}\,dw = \int \frac{(e^w-1)(e^w-4)}{e^w-1}\,dw = \int (e^w-4)\,dw = e^w - 4w + C$. Check: $\frac{d}{dw}(e^w - 4w + C) = e^w - 4 = \frac{(e^w-4)(e^w-1)}{e^w-1} = \frac{e^{2w}-5e^w+4}{e^w-1}$.

4.9.64 $\int \left(\sqrt[3]{x^2} + \sqrt{x^3}\right)dx = \int (x^{2/3} + x^{3/2})\,dx = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$. Check: $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C\right) = x^{2/3} + x^{3/2} = \sqrt[3]{x^2} + \sqrt{x^3}$.

4.9.65 $\int \frac{1+\sqrt{x}}{x}\,dx = \int (x^{-1}+x^{-1/2})\,dx = \ln|x| + 2x^{1/2} + C = \ln|x| + 2\sqrt{x} + C$. Check: $\frac{d}{dx}(\ln|x| + 2\sqrt{x} + C) = \frac{1}{x} + \frac{1}{\sqrt{x}} = \frac{1+\sqrt{x}}{x}$.

4.9.66 $\int \frac{16\cos^2 w - 81\sin^2 w}{4\cos w - 9\sin w}\,dw = \int \frac{(4\cos w - 9\sin w)(4\cos w + 9\sin w)}{4\cos w - 9\sin w}\,dw = \int (4\cos w + 9\sin w)\,dw = 4\sin w - 9\cos w + C$. Check: $\frac{d}{dw}(4\sin w - 9\cos w + C) = 4\cos w + 9\sin w = \frac{(4\cos w + 9\sin w)(4\cos w - 9\sin w)}{4\cos w - 9\sin w} = \frac{16\cos^2 w - 81\sin^2 w}{4\cos w - 9\sin w}$.

4.9.67 $\int \sqrt{x}(2x^6 - 4\sqrt[3]{x})\,dx = \int (2x^{13/2} - 4x^{5/6})\,dx = 2\cdot\frac{2}{15}x^{15/2} - 4\cdot\frac{6}{11}x^{11/6} + C = \frac{4}{15}x^{15/2} - \frac{24}{11}x^{11/6} + C$. Check: $\frac{d}{dx}\left(\frac{4}{15}x^{15/2} - \frac{24}{11}x^{11/6} + C\right) = 2x^{13/2} - 4x^{5/6} = \sqrt{x}(2x^6 - 4\sqrt[3]{x})$.

4.9.68 $\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{(1+x^2)+1}{1+x^2}\,dx = \int \left(1 + \frac{1}{1+x^2}\right)\,dx = x + \tan^{-1}x + C$. Check: $\frac{d}{dx}(x + \tan^{-1}x + C) = 1 + \frac{1}{x^2+1} = \frac{2+x^2}{1+x^2}$.

4.9.69 We have $F(x) = \int (x^5 - 2x^2 + 1)\,dx = \frac{x^6}{6} - \frac{2x^3}{3} + x + C$; substituting $F(0) = 1$ gives $C = 1$, and thus $F(x) = \frac{x^6}{6} - \frac{2x^3}{3} + x + 1$.

4.9.70 $F(x) = \int (4\sqrt{x} + 6)\,dx = \frac{8}{3}x^{3/2} + 6x + C$; substituting $F(1) = 8$ gives $\frac{8}{3} + 6 + C = 8$, so $C = -\frac{2}{3}$, and thus $F(x) = \frac{8}{2}x^{3/2} + 6x - \frac{2}{3}$.

4.9.71 $F(x) = \int (8x^3 + \sin x)\,dx = 2x^4 - \cos x + C$; substituting $F(0) = 2$ gives $-1 + C = 2$, so $C = 3$, and thus $F(x) = 2x^4 - \cos x + 3$.

4.9.72 We have $F(t) = \int \sec^2 t\,dt = \tan t + C$; substituting $F(\pi/4) = 1$ gives $\tan\frac{\pi}{4} + C = 1 + C = 1$, so $C = 0$, and thus $F(t) = \tan t$, $-\pi/2 < t < \pi/2$.

4.9.73 We have $F(v) = \int \sec v \tan v\,dv = \sec v + C$; substituting $F(0) = 2$ gives $\sec 0 + C = 1 + C = 2$, so $C = 1$, and thus $F(v) = \sec v + 1$, $-\pi/2 < t < \pi/2$.

4.9.74 We have $F(u) = \int (2e^u + 3)\,du = 2e^u + 3u + C$; substituting $F(0) = 8$ gives $2 + 0 + C = 8$, so $C = 6$, and thus $F(u) = 2e^u + 3u + 6$.

4.9.75 We have $F(y) = \int \frac{3y^3+5}{y}\,dy = \int \left(\frac{3y^3}{y} + \frac{5}{y}\right)\,dy = \int \left(3y^2 + \frac{5}{y}\right)\,dy = y^3 + 5\ln|y| + C$; substituting $F(1) = 3$ gives $1 + 0 + C = 3$, so $C = 2$, and thus $F(y) = y^3 + 5\ln y + 2$, $y > 0$.

4.9.76 $F(\theta) = \int (2\sin\theta - 4\cos\theta)\,d\theta = -2\cos\theta - 4\sin\theta + C$; substituting $F(\pi/4) = 2$ gives $-\sqrt{2} - 2\sqrt{2} + C = 2$, so $C = 2 + 3\sqrt{2}$, and thus $F(\theta) = -2\cos\theta - 4\sin\theta + 2 + 3\sqrt{2}$.

4.9.77 We have $f(x) = \int (2x-3)\,dx = x^2 - 3x + C$; substituting $f(0) = 4$ gives $C = 4$, so $f(x) = x^2 - 3x + 4$.

4.9.78 We have $g(x) = \int (7x^6 - 4x^3 + 12)\,dx = x^7 - x^4 + 12x + C$; substituting $g(1) = 24$ gives $1 - 1 + 12 + C = 24$, so $C = 12$, and thus $g(x) = x^7 - x^4 + 12x + 12$.

4.9.79 We have $g(x) = \int 7x\left(x^6 - \frac{1}{7}\right)\,dx = \int (7x^7 - x)\,dx = \frac{7}{8}x^8 - \frac{x^2}{2} + C$; substituting $g(1) = 2$ gives $\frac{7}{8} - \frac{1}{2} + C = 2$, so $C = \frac{13}{8}$, and thus $g(x) = \frac{7}{8}x^8 - \frac{x^2}{2} + \frac{13}{8}$.

4.9.80 $h(t) = \int (1 + 6\sin t)\,dt = t - 6\cos t + C$; substituting $h(\pi/3) = -3$ gives $\frac{\pi}{3} - 3 + C = -3$, so $C = -\frac{\pi}{3}$, and thus $h(t) = t - 6\cos t - \frac{\pi}{3}$.

4.9.81 $f(u) = \int 4(\cos u - \sin u)\,du = 4\sin u + 4\cos u + C$; substituting $f(\pi/2) = 0$ gives $4 + 0 + C = 0$, so $C = -4$, and thus $h(u) = 4\sin u + 4\cos u - 4$.

4.9.82 $p(t) = \int (10e^t + 70)\,dt = 10e^t + 70t + C$; substituting $p(0) = 100$ gives $10 + C = 100$, so $C = 90$, and thus $p(t) = 10e^t + 70t + 90$.

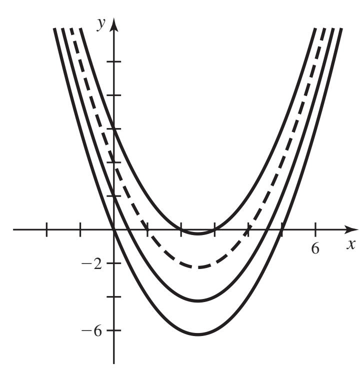
4.9.83 $y(t) = \int \left(\frac{3}{t} + 6\right)\,dt = 3\ln|t| + 6t + C$; substituting $y(1) = 8$ gives $0 + 6 + C = 8$, so $C = 2$, and thus $y(t) = 3\ln t + 6t + 2$, $t > 0$.

4.9.84 $u(x) = \int \frac{xe^{2x} + 4e^x}{xe^x}\,dx = \int \left(e^x + \frac{4}{x}\right)\,dx = e^x + 4\ln|x| + C$; substituting $u(1) = 0$ gives $e + C = 0$, so $C = -e$, and thus $u(x) = e^x + 4\ln x - e$, $x > 0$.

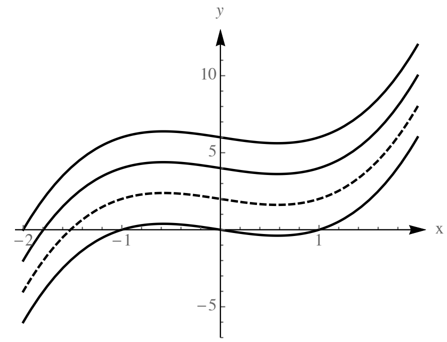
4.9.85 $y(\theta) = \int \frac{\sqrt{2}\cos^3\theta + 1}{\cos^2\theta}\,d\theta = \int \left(\frac{\sqrt{2}\cos^3\theta}{\cos^2\theta} + \frac{1}{\cos^2\theta}\right)\,d\theta = \int \left(\sqrt{2}\cos\theta + \sec^2\theta\right)\,d\theta = \sqrt{2}\sin\theta + \tan\theta + C$; substituting $y(\pi/4) = 3$ gives $1 + 1 + C = 3$, so $C = 1$, and thus $y(\theta) = \sqrt{2}\sin\theta + \tan\theta + 1$, $-\pi/2 < \theta < \pi/2$.

4.9.86 $v(x) = \int (4x^{1/3} + 2x^{-1/3})\,dx = 3x^{4/3} + 3x^{2/3} + C$, substituting $v(8) = 40$ gives $48 + 12 + C = 40$, so $C = -20$ and thus $v(x) = 3x^{4/3} + 3x^{2/3} - 20$, $x > 0$.

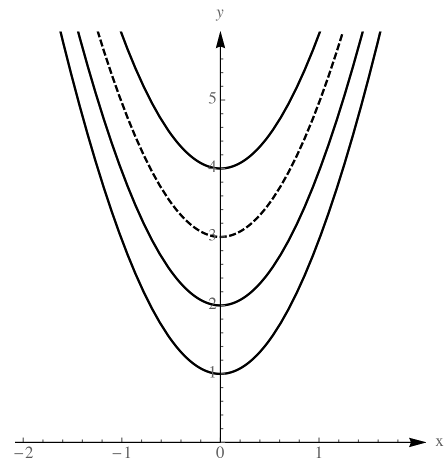
4.9.87 We have $f(x) = \int (2x - 5)\,dx = x^2 - 5x + C$; substituting $f(0) = 4$ gives $C = 4$, so $f(x) = x^2 - 5x + 4$.



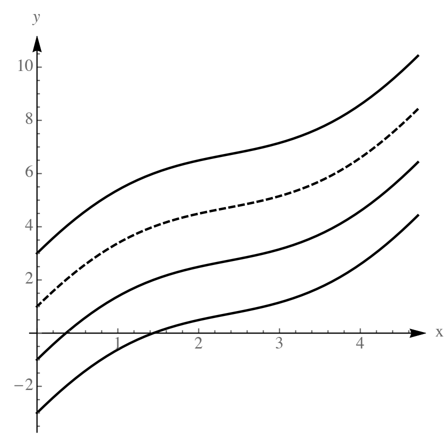
4.9.88 We have $f(x) = \int (3x^2 - 1)\,dx = x^3 - x + C$; substituting $f(1) = 2$ gives $C = 2$, so $f(x) = x^3 - x + 2$.



4.9.89 We have $f(x) = \int (3x + \sin x)\,dx = \frac{3}{2}x^2 - \cos x + C$; substituting $f(0) = 3$ gives $C = 4$, so $f(x) = \frac{3}{2}x^2 - \cos x + 4$.



4.9.90 We have $f(x) = \int (\cos x - \sin x + 2)\,dx = \sin x + \cos x + 2x + C$; substituting $f(0) = 1$ gives $C = 0$, so $f(x) = \sin x + \cos x + 2x$.



4.9.91 We have $s(t) = \int (2t + 4)\,dt = t^2 + 4t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = t^2 + 4t$.

4.9.92 We have $s(t) = \int (e^t + 4)\,dt = e^t + 4t + C$; substituting $s(0) = 2$ gives $1 + C = 2$, so $C = 1$, and thus $s(t) = e^t + 4t + 1$

4.9.93 We have $s(t) = \int 2\sqrt{t}\,dt = 2 \cdot \frac{2}{3}t^{3/2} + C = \frac{4}{3}t^{3/2} + C$; substituting $s(0) = 1$ gives $C = 1$, so $s(t) = \frac{4}{3}t^{3/2} + 1$.

4.9.94 We have $s(t) = \int 2\cos t\,dt = 2\sin t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = 2\sin t$.

4.9.95 We have $s(t) = \int (6t^2 + 4t - 10)\,dt = 2t^3 + 2t^2 - 10t + C$; substituting $s(0) = 0$ gives $C = 0$, so $s(t) = 2t^3 + 2t^2 - 10t$.

4.9.96 We have $s(t) = \int (4t + \sin t)\,dt = 2t^2 - \cos t + C$, substituting $s(0) = 0$ gives $-1 + C = 0$, so $C = 1$ and $s(t) = 2t^2 - \cos t + 1$.

4.9.97 $v(t) = \int a(t)\,dt = \int -32\,dt = -32t + C_1$. Because $v(0) = 20$, we have $0 + C_1 = 20$, so $v(t) = -32t + 20$.

$s(t) = \int v(t)\,dt = \int (-32t + 20)\,dt = -16t^2 + 20t + C_2$. Because $s(0) = 0$, we have $C_2 = 0$, and thus $s(t) = -16t^2 + 20t$.

4.9.98 $v(t) = \int a(t)\,dt = \int 4\,dt = 4t + C_1$. Because $v(0) = -3$, we have $C_1 = -3$.

$s(t) = \int v(t)\,dt = \int (4t - 3)\,dt = 2t^2 - 3t + C_2$. Because $s(0) = 2$, we have $C_2 = 2$, and thus $s(t) = 2t^2 - 3t + 2$.

4.9.99 $v(t) = \int a(t)\,dt = \int 0.2t\,dt = 0.1t^2 + C_1$. Because $v(0) = 0$, we have $C_1 = 0$.

$s(t) = \int v(t)\,dt = \int 0.1t^2\,dt = \frac{1}{30}t^3 + C_2$. Because $s(0) = 1$, we have $C_2 = 1$, and thus $s(t) = \frac{1}{30}t^3 + 1$.

4.9.100 $v(t) = \int a(t)\,dt = \int 2\cos t\,dt = 2\sin t + C_1$. Because $v(0) = 1$, we have $C_1 = 1$.

$s(t) = \int v(t)\,dt = \int (2\sin t + 1)\,dt = -2\cos t + t + C_2$. Because $s(0) = 0$, we have $-2 + 0 + C_2 = 0$, so $C_2 = 2$. Thus, $s(t) = -2\cos t + t + 2$.

4.9.101 $v(t) = \int a(t)\,dt = \int (2 + 3\sin t)\,dt = 2t - 3\cos t + C_1$. Because $v(0) = 1$, we have $0 - 3 + C_1 = 1$, so $C_1 = 4$.

$s(t) = \int v(t)\,dt = \int (2t - 3\cos t + 4)\,dt = t^2 - 3\sin t + 4t + C_2$. Because $s(0) = 10$, we have $C_2 = 10$, so $s(t) = t^2 + 4t - 3\sin t + 10$.

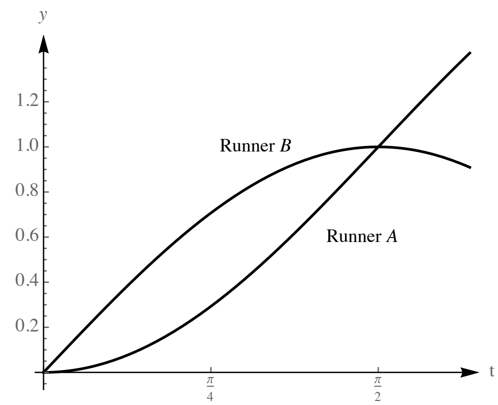
4.9.102 $v(t) = \int a(t)\,dt = \int (2e^t - 12)\,dt = 2e^t - 12t + C_1$. Because $v(0) = 1$, we have $2 + C_1 = 1$, so $C_1 = -1$.

$s(t) = \int v(t)\,dt = \int (2e^t - 12t - 1)\,dt = 2e^t - 6t^2 - t + C_2$. Because $s(0) = 0$, we have $C_2 = -2$, so $s(t) = 2e^t - 6t^2 - t - 2$.

4.9.103 $a(t) = 16$, so $v(t) = \int 16\,dt = 16t + C_1$. Because $v(0) = 0$, we have $C_1 = 0$, so $v(t) = 16t$. $s(t) = \int v(t)\,dt = \int 16t\,dt = 8t^2 + C_2$, where we can assume $s(0) = 0$ so $C_2 = 0$. Then the distance traveled in the first 5 seconds is $s(5) = 200$ feet.

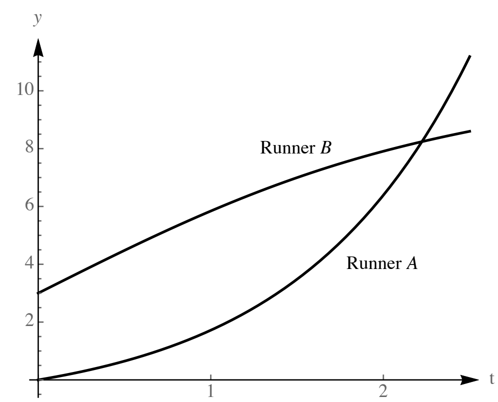
4.9.104 $a(t) = -10$ so $v(t) = \int -10\,dt = -10t + C_1$. Because $v(0) = 88$, we have $C_1 = 88$, so $v(t) = -10t + 88$. Then $s(t) = \int (-10t + 88)\,dt = -5t^2 + 88t + C_2$, where we can take $C_2 = 0$. The car travels $s(3) = -45 + 264 = 219$ ft in the 3 seconds.

4.9.105 Runner A has position function $s(y) = \int \sin t\,dt = -\cos t + C$; the initial condition $s(0) = 0$ gives $C = 1$, so $s(t) = 1 - \cos t$. Runner B has position function $S(t) = \int \cos t\,dt = \sin t + C$; the initial condition $S(0) = 0$ gives $C = 0$, so $S(t) = \sin t$. The smallest $t > 0$ where $s(t) = S(t)$ is $t = \pi/2$.



4.9.106 Runner A has position function $s(t) = \int e^t\,dt = e^t + C_1$, and because $s(0) = 0$ we have $s(t) = e^t - 1$.

Runner B has position function $S(t) = \int (2 + \cos t)\,dt = 2t + \sin t + C_2$, and because $S(0) = 3$ we have $C_2 = 3$, so $S(t) = 2t + \sin t + 3$. The smallest $t > 0$ where $s(t) = S(t)$ is $t \approx 2.22$.



4.9.107

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 30$, so $v(t) = -9.8t + 30$.

b. The height of the softball above ground is given by $s(t) = \int (-9.8t + 30)\,dt = -4.9t^2 + 30t + s_0 = -4.9t^2 + 30t$.

c. The ball reaches its maximum height when $v(t) = -9.8t + 30 = 0$, which gives $t = 30/9.8 \approx 3.06$ s; the maximum height is $s(30/9.8) \approx 45.92$ m.

d. The ball strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $t(30 - 4.9t) = 0$, so $t = 30/4.9 \approx 6.12$ s.

4.9.108

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 30$, so $v(t) = -9.8t + 30$.

b. The height of the stone above ground is given by $s(t) = \displaystyle\int (-9.8t + 30)\, dt = -4.9t^2 + 30t + s_0 = -4.9t^2 + 30t + 200$.

c. The stone reaches its maximum height when $v(t) = -9.8t + 30 = 0$, which gives $t = 30/9.8 \approx 3.06$ s; the maximum height is $s(30/9.8) \approx 245.92$ m.

d. The stone strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 + 30t + 200 = 0$, so $t \approx 10.15$ s.

4.9.109

a. We have $v(t) = -9.8t + v_0$ and $v_0 = 10$, so $v(t) = -9.8t + 10$.

b. The height of the payload above ground is given by $s(t) = \displaystyle\int (-9.8t + 10)\, dt = -4.9t^2 + 10t + s_0 = -4.9t^2 + 10t + 400$.

c. The payload reaches its maximum height when $v(t) = -9.8t + 10 = 0$, which gives $t = 10/9.8 \approx 1.02$ s; the maximum height is $s(10/9.8) \approx 405.10$ m.

d. The payload strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 + 10t + 400 = 0$, so $t \approx 10.11$ s.

4.9.110

a. We have $v(t) = -9.8t + v_0$ and $v_0 = -10$, so $v(t) = -9.8t - 10$.

b. The height of the payload above ground is given by $s(t) = \displaystyle\int (-9.8t - 10)\, dt = -4.9t^2 - 10t + s_0 = -4.9t^2 - 10t + 400$.

c. Because $v(t) < 0$ for $t > 0$, the maximum height occurs at $t = 0$ and is the initial height 400 m.

d. The payload strikes the ground when $s(t) = 0$ (and $t > 0$), which gives $-4.9t^2 - 10t + 400 = 0$, so $t \approx 8.07$ s.

4.9.111

a. True, because $F'(x) = G'(x)$.

b. False; f is the derivative of F.

c. True; $\int f(x)\, dx$ is the most general antiderivative of $f(x)$, which is $F(x) + C$.

d. False; a function cannot have more than one derivative.

e. False; one can only conclude that $F(x)$ and $G(x)$ differ by a constant.

4.9.112 We have $F'(x) = \displaystyle\int 1\, dx = x + C$; $F'(0) = 3$ so $C = 3$. Then $F(x) = \displaystyle\int (x + 3)\, dx = \frac{x^2}{2} + 3x + C$; $F(0) = 4$ so $C = 4$ and $F(x) = (1/2)x^2 + 3x + 4$.

4.9.113 We have $F'(x) = \displaystyle\int \cos x\, dx = \sin x + C$; $F'(0) = 3$ so $C = 3$. Then $F(x) = \displaystyle\int (\sin x + 3)\, dx = -\cos x + 3x + C$; $F(\pi) = 4$ gives $1 + 3\pi + C = 4$ so $C = 3 - 3\pi$ and $F(x) = -\cos x + 3x + 3 - 3\pi$.

4.9.114 We have $F''(x) = \int 4x\,dx = 2x^2 + C$; $F''(0) = 0$ so $C = 0$. Next $F'(x) = \int 2x^2\,dx = 2 \cdot \frac{x^3}{3} + C$; $F'(0) = 1$ so $C = 1$. Finally $F(x) = \int \left(\frac{2}{3}x^3 + 1\right) dx = \frac{2}{3} \cdot \frac{x^4}{4} + x + C$; $F(0) = 3$ so $C = 3$ and $F(x) = \frac{x^4}{6} + x + 3$.

4.9.115 We have $F''(x) = \int (672x^5 + 24x)\,dx = 672 \cdot \frac{x^6}{6} + 24 \cdot \frac{x^2}{2} + C$; $F''(0) = 0$ so $C = 0$. Next $F'(x) = \int (112x^6 + 12x^2)\,dx = 112 \cdot \frac{x^7}{7} + 12 \cdot \frac{x^3}{3} + C$; $F'(0) = 2$ so $C = 2$. Finally $F(x) = \int (16x^7 + 4x^3 + 2)\,dx = 16 \cdot \frac{x^8}{8} + 4 \cdot \frac{x^4}{4} + 2x + C$; $F(0) = 1$ so $C = 1$ and $F(x) = 2x^8 + x^4 + 2x + 1$.

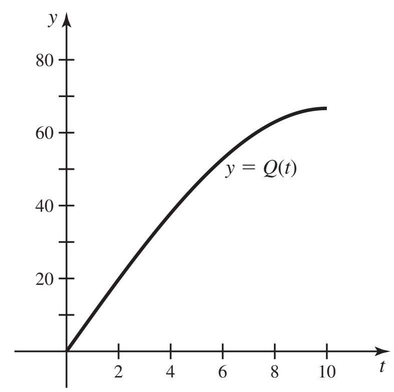
4.9.116 The velocity is given by $v(t) = \int 2\sin t\,dt = -2\cos t + C$; $v(0) = 3$ implies $-2 + C = 3$, so $C = 5$. The position s is given by $s(t) = \int (-2\cos t + 5)\;dt = -2\sin t + 5t + C_1$; $s(0) = 0$ implies that $C_1 = 0$, so $s(t) = -2\sin t + 5t$.

4.9.117

a. We have

$$Q(t) = \int 0.1(100 - t^2)\,dt = 0.1\left(100t - \frac{t^3}{3}\right) + C;$$

$Q(0) = 0$, so $C = 0$ and $Q(t) = 10t - t^3/30$ gal.

b. 

c. $Q(10) = 200/3 \approx 67$ gal.

4.9.118 Object A has position function $s(t)$ given by $s(t) = \int 2at\,dt = at^2 + s_0$, and we are given $s_0 = 0$, so $s(t) = at^2$. Object B has position function $S(t)$ given by $S(t) = \int b\,dt = bt + S_0$, and we are given $S_0 = c > 0$, so $S(t) = bt + c$. Therefore A will overtake B when $s(t) = S(t)$, which gives the quadratic equation $at^2 - bt - c = 0$; this equation has a unique positive root given by $t = \frac{b + \sqrt{b^2 + 4ac}}{2a}$.

4.9.119 Check that $\frac{d}{dx}(2\sin\sqrt{x}) = 2\cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{\sqrt{x}}$.

4.9.120 Check that $\frac{d}{dx}(\sqrt{x^2 + 1}) = \frac{2x}{2\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}$.

4.9.121 Check that $\frac{d}{dx}\left(\frac{1}{3}\sin x^3\right) = \frac{1}{3}\cos x^3(3x^2) = x^2\cos x^3$.

4.9.122 Check that $\frac{d}{dx}\left(-\frac{1}{2(x^2 - 1)}\right) = -\frac{1}{2}\frac{d}{dx}(x^2 - 1)^{-1} = -\frac{1}{2}(-1)(x^2 - 1)^{-2}(2x) = \frac{x}{(x^2 - 1)^2}$.

Chapter Four Review

1

a. False. The point $(c, f(c))$ is a critical point for f, but is not necessarily a local maximum or minimum. Example: $f(x) = x^3$ at $c = 0$.

b. False. The fact that $f''(c) = 0$ does not necessarily imply that f changes concavity at c. Example: $f(x) = x^4$ at $c = 0$.

c. True. Both are antiderivatives of $2x$.

d. True. The function has a maximum on the closed interval determined by the two local minima, and the only way the maximum can occur at the endpoints is if the function is constant, in which case every point is a local max and min.

e. True. The slope of the linearization is given by $f'(0) = \cos(0) = 1$, and the line has y-intercept $(0, f(0)) = (0, 0)$.

f. False. For example, $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} x = \infty$, but $\lim_{x\to\infty}(x^2 - x) = \infty$.

2

a. There is a local minimum at $(2, -3)$ and a local maximum at $(-1, 3)$.

b. The absolute minimum and maximum on $[-3, 3]$ occur at $(-3, -5)$ and $(-1, 3)$ respectively.

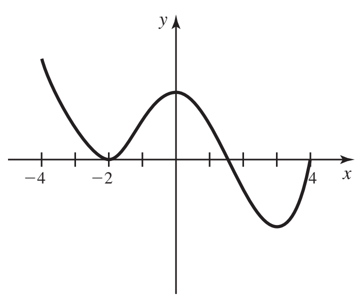
c. The inflection point has coordinates $(1/2, 0)$.

d. The function has zeros at $x \approx -2.2, 2.8$.

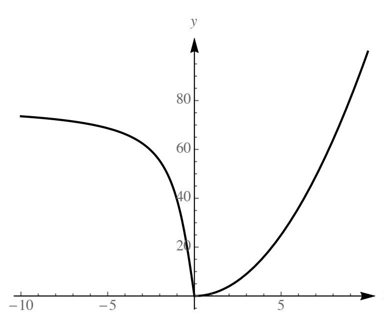
e. The function is concave up on the interval $(1/2, 3)$.

f. The function is concave down on the interval $(-3, 1/2)$.

3



4



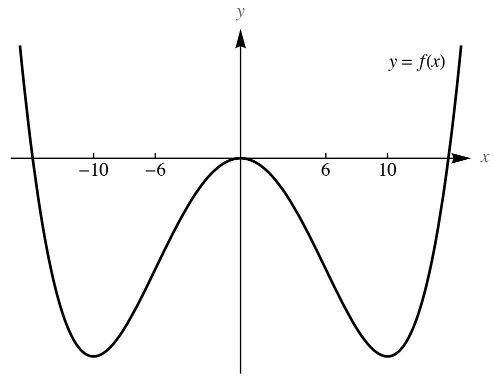
5

a. f' is zero for $x = 0$ and $x = \pm 10$, so those are the critical points. $f' > 0$ on $(-10, 0)$ and on $(10, \infty)$, so f is increasing on those intervals, while $f' < 0$ on $(-\infty, -10)$ and on $(0, 10)$, so f is decreasing on those intervals.

b. f'' is zero for $x = \pm 6$. $f'' > 0$ on $(-\infty, -6)$ and on $(6, \infty)$, so f is concave up on those intervals, while $f' < 0$ on $(-6, 6)$, so f is concave down on that interval.

c. There is a local minimum at $x = -10$ and $x = 10$, and a local maximum at $x = 0$.

d.



6 $f'(x) = 3x^2 - 12x = 3x(x-4)$, so f' is zero for $x = 0$ and $x = 4$. The critical points are $x = 0$ and $x = 4$. $f(0) = 0$ and $f(4) = -32$. Checking the endpoints, we have $f(-1) = -7$ and $f(5) = -25$. So the absolute maximum is 0 at $x = 0$ and the absolute minimum is -32 at $x = 4$.

7 $f'(x) = 12x^3 - 12x = 12x(x^2-1) = 12x(x-1)(x+1)$, which is zero for $x = 0, -1, 1$, so those are the critical points. $f(0) = 9$, $f(-1) = f(1) = 6$, and checking endpoints, we have $f(\pm 2) = 33$. So the absolute maximum is 33 at the endpoints and the absolute minimum is 6 at $x = \pm 1$.

8 $g'(x) = 4x^3 - 100x = 4x(x^2-25) = 4x(x-5)(x+5)$, which is zero for $x = 0, \pm 5$. However, -5 is not in the given domain, while 5 is an endpoint, so only $x = 0$ is a critical point. $g(0) = 0$, $g(5) = -625$, and $g(-1) = -49$. So the absolute maximum is 0 at $x = 0$ and the absolute minimum is -625 at $x = 5$.

9 $f'(x) = 6x^2 - 6x - 36 = 6(x-3)(x+2)$, which is zero for $x =$ and $x = -2$, so the critical points are $x = 3, -2$. Because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, this function has no absolute max or min on $(-\infty, \infty)$.

10 $f'(x) = 3x^2 \ln x + x^3 \cdot \dfrac{1}{x} = x^2(3\ln x + 1)$. This is zero on the given domain when $3\ln x = -1$, or $x = e^{-1/3}$. Note that $f' < 0$ on $(0, e^{-1/3})$ and $f' > 0$ on $(e^{-1/3}, \infty)$ and $\lim_{x\to\infty} f(x) = \infty$. So there is no absolute maximum, but there is an absolute minimum of $f(e^{-1/3}) = -\dfrac{1}{3e}$ at $x = e^{-1/3}$.

11 $f'(x) = \dfrac{2x-2}{x^2-2x+2}$, which is zero for $x = 1$. The only critical point is $x = 1$. $f(0) = f(2) = \ln 2$, while $f(1) = 0$. The absolute maximum is $\ln 2$ at the endpoints, while the absolute minimum is 0 at $x = 1$.

12 $f'(x) = 2\cos 2x$, which is zero on the given interval for $x = \pm\pi/4$ and $x = \pm 3\pi/4$. We have $f(-\pi) = 3$, $f(-3\pi/4) = 4$, $f(-\pi/4) = 2$, $f(\pi/4) = 4$, $f(3\pi/4) = 2$ and $f(\pi) = 3$; therefore the absolute minimum and maximum values of f on $[-\pi, \pi]$ are 2 and 4 respectively.

13

$$\begin{aligned} g'(x) &= -\frac{1}{2}\cos x + \frac{1}{2}\cos x \cos x - \frac{1}{2}\sin x \sin x = \frac{1}{2}\left(-\cos x + \cos^2 x - (1-\cos^2 x)\right) \\ &= \frac{1}{2}\left(2\cos^2 x - \cos x - 1\right) = \frac{1}{2}\left(2\cos x + 1\right)\left(\cos x - 1\right). \end{aligned}$$

This is zero on the interior of the given interval where $\cos x = -\dfrac{1}{2}$, which is $x = \dfrac{2\pi}{3}$ and $x = \dfrac{4\pi}{3}$, so those are the critical points. We have $g(2\pi/3) = -\dfrac{3\sqrt{3}}{8}$ and $g(4\pi/3) = \dfrac{3\sqrt{3}}{8}$, while $g(0) = g(2\pi) = 0$. So the absolute minimum is $-\dfrac{3\sqrt{3}}{8}$ and the absolute maximum is $\dfrac{3\sqrt{3}}{8}$.

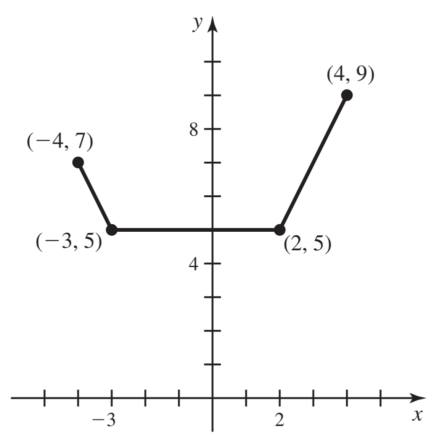
14 Observe that $f'(x) = 2x^{-1/2} - \frac{5}{2}x^{3/2}$, so the critical points satisfy $x^2 = 4/5$; hence $x = 2/\sqrt{5} \approx 0.894$ is the only critical point in the interval $(0, 4)$. We have $f(0) = 0$, $f(2/\sqrt{5}) \approx 3.026$, $f(4) = -24$; therefore the absolute minimum and maximum values are -24 and ≈ 3.026 respectively.

15 The critical points satisfy $f'(x) = 2\ln x + 2x \cdot \frac{1}{x} = 2\ln x + 2 = 0$, which has solution $x = 1/e$. The Second Derivative Test shows that this critical point is a local minimum, so by Theorem 4.5 the absolute minimum value on the interval $(0, \infty)$ is $f\left(\frac{1}{e}\right) = -\frac{2}{e} + 10$. Because $\lim_{x\to\infty} x \ln x = \infty$, this function does not have an absolute maximum on $(0, \infty)$.

16 The critical points satisfy $g'(x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x = 0$. Note that $g'(0) = 0$, so $x = 0$ is a critical point, and this turns out to be the only critical point. This can be seen by noting that for $x < 0$ both $\frac{x}{\sqrt{1-x^2}}$ and $\sin^{-1} x$ are negative, so their sum is negative, and for $x > 0$ both $\frac{x}{\sqrt{1-x^2}}$ and $\sin^{-1} x$ are positive, so their sum is positive. Note that $g(-1) = \frac{\pi}{2} = g(1)$, and $g(0) = 0$, so the absolute maximum of g is $\frac{\pi}{2}$ and the absolute minimum is 0.

17

All points in the interval $[-3, 2]$ are critical points. The absolute max occurs at $(4, 9)$; there are no local maxima. All points $(x, 5)$ for x in the interval $[-3, 2]$ are absolute and local minima.



18

a. $f'(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$. This is never zero, and is always positive on $(-\infty, \infty)$, so f is increasing on $(-\infty, \infty)$.

b. $f''(x) = \frac{(1+e^x)^2 e^x - e^x(2)(1+e^x)e^x}{(1+e^x)^4} = \frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^3} = \frac{e^x(1-e^x)}{(1+e^x)^3}$. This is zero when $e^x = 1$, or $x = 0$. $f'' > 0$ on $(-\infty, 0)$, so f is concave up there, while $f'' < 0$ on $(0, \infty)$, so f is concave down there.

19

a. $f'(x) = x^8 + 15x^4 - 16 = (x^4+16)(x^4-1) = (x^4+16)(x^2+1)(x^2-1) = (x^4+16)(x^2+1)(x+1)(x-1)$, which is zero for $x = \pm 1$. $f' > 0$ on $(-\infty, -1)$ and on $(1, \infty)$ so f is increasing there, while $f' < 0$ on $(-1, 1)$ so f is decreasing there.

b. $f''(x) = 8x^7 + 60x^3 = 4x^3(x^4 + 15)$, which is zero only for $x = 0$. $f'' < 0$ on $(-\infty, 0)$ so f is concave down there, while $f'' > 0$ on $(0, \infty)$ so f is concave up there.

20

a. Observe that the domain of f is $[-9, \infty)$.
$f'(x) = \sqrt{x+9} + \frac{x}{2\sqrt{x+9}} = \frac{2(x+9)}{2\sqrt{x+9}} + \frac{x}{2\sqrt{x+9}} = \frac{3(x+6)}{2\sqrt{x+9}}$. This is zero for $x = -6$. $f' > 0$ on $(-6, \infty)$, so f is increasing there, while $f' < 0$ on $(-9, -6)$, so f is decreasing there.

b. $f''(x) = \frac{3}{2}\left(\frac{\sqrt{x+9} - \frac{x+6}{2\sqrt{x+9}}}{x+9}\right) = \frac{3}{2}\left(\frac{\sqrt{x+9} - \frac{x+6}{2\sqrt{x+9}}}{x+9}\right) \cdot \frac{2\sqrt{x+9}}{2\sqrt{x+9}} = \frac{3}{4}\left(\frac{2x+18-x-6}{(x+9)^{3/2}}\right) = \frac{3(x+12)}{4(x+9)^{3/2}}$. This is always positive on the domain, so f is concave up on $(-9, \infty)$.

21 $f'(x) = 10x^4 - 40x^3 + 60x^2 + 1$, so $f''(x) = 40x^3 - 120x^2 + 120x = 40x(x^2 - 3x + 3)$. The quadratic $x^2 - 3x + 3$ has no real roots, so $x = 0$ is the only possible inflection point. The sign of $f''(x)$ changes at $x = 0$ so an inflection point occurs at $(0, 1)$.

22 $f'(x) = 3x^5 + 5x^3 - 30x$, so $f''(x) = 15x^4 + 15x^2 - 30 = 15(x^4 + x^2 - 2) = 15(x^2 + 2)(x^2 - 1) = 15(x^2 + 2)(x - 1)(x + 1)$. So the only potential location for inflection points are $x = \pm 1$. An analysis of the sign of f'' shows that it is positive on $(-\infty, -1)$, negative on $(-1, 1)$, and positive on $(1, \infty)$, so there are inflection points at both $x = 1$ and $x = -1$.

23 $f'(x) = (x+a)^3 + (x-a)(3)(x+a)^2 = (x+a)^2(x+a+3x-3a) = (x+a)^2(4x-2a) = 2(x+a)^2(2x-a)$. This is zero for $x = \frac{a}{2}$ and $x = -a$, so those are the critical points. $f''(x) = 4(x+a)(2x-a) + 2(x+a)^2(2) = 4(x+a)(2x-a+x+a) = 4(x+a)(3x) = 12x(x+a)$. This is zero for $x = 0$ and $x = -a$, and an analysis of the sign of f'' shows that these are the locations of inflection points.

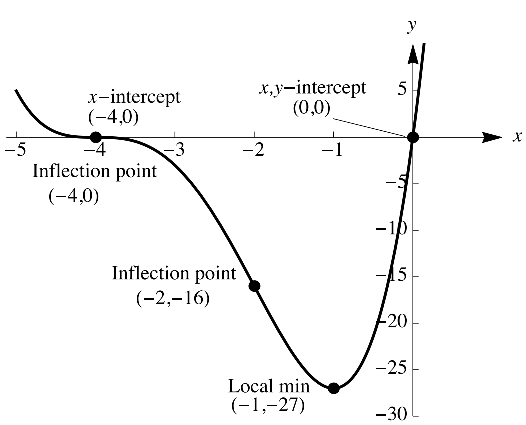
24 $f(x) = x(x+4)^3$ is a polynomial, so its domain is $(-\infty, \infty)$ and it has no asymptotes. $f(-x) = -x(4-x)^3 = x(x-4)^3$ which is neither $f(x)$ nor $-f(x)$, so there is neither even nor odd symmetry.

$$f'(x) = (x+4)^3 + x(3(x+4)^2) = (x+4)^2(x+4+3x) = 4(x+4)^2(x+1).$$

This is zero for $x = -1$ and $x = -4$. $f' < 0$ on $(-\infty, -4)$ and on $(-4, -1)$, so f is decreasing there, while $f'(x) > 0$ on $(-1, \infty)$, so f is increasing there. There is a local minimum of $f(-1) = -27$ at $x = -1$.

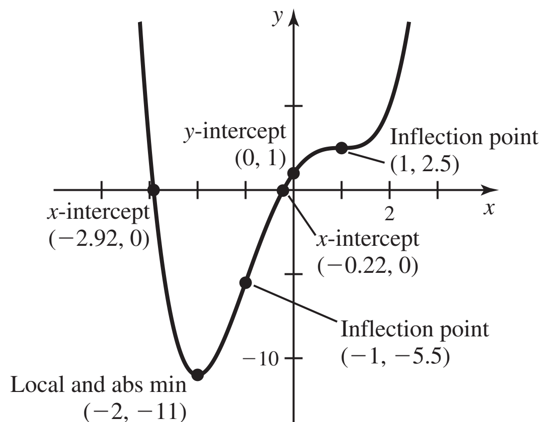
$$f''(x) = 8(x+4)(x+1) + 4(x+4)^2 = 4(x+4)(2x+2+x+4) = 4(x+4)(3x+6) = 12(x+4)(x+2).$$

This is zero for $x = -4$ and $x = -2$, so these are potential locations for inflection points. $f''(x) > 0$ on $(-\infty, -4)$ and on $(-2, \infty)$, so f is concave up there, while $f'' < 0$ on $(-4, -2)$, so f is concave down there. There are inflection points at $(-4, 0)$ and $(-2, -16)$. The x-intercepts are $(0, 0)$ and $(-4, 0)$. The y-intercept is $(0, 0)$.



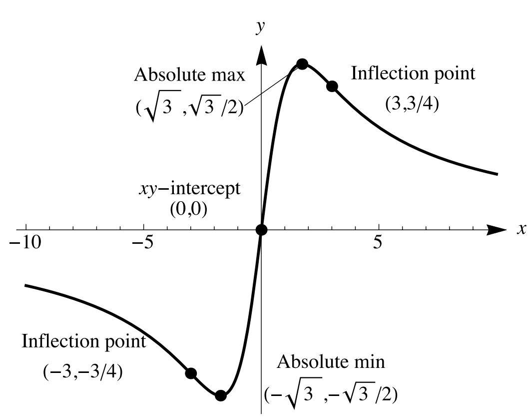
25 The derivatives of f are $f'(x) = 2x^3 - 6x + 4$ and $f''(x) = 6x^2 - 6$. Observe that $f'(x) = 2(x-1)^2(x+2)$, so we have critical points $x = 1, -2$. Solving $f''(x) = 6(x^2 - 1) = 0$ gives possible inflection points at $x = \pm 1$. Testing the sign of $f'(x)$ shows that f is decreasing on the interval $(-\infty, -2)$ and is increasing on $(-2, \infty)$. The First Derivative Test shows that a local minimum occurs at $x = -2$, and that the critical point $x = 1$ is neither a local max or min.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(-1, 1)$ and is concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$. Therefore inflection points occur at $x = \pm 1$. Using a numerical solver, we see that the graph has x-intercepts at $x \approx -2.917$, -0.215. We also observe that $\lim_{x \to \pm\infty} f(x) = \infty$, so f has no absolute maximum and an absolute minimum at $x = -2$.



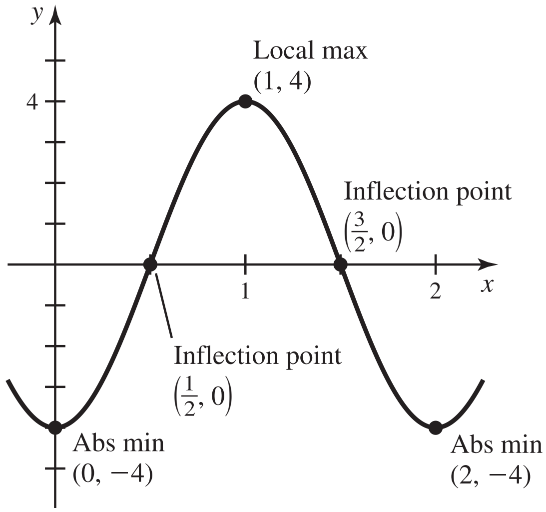
26 The derivatives of f are $f'(x) = 3 \cdot \dfrac{3 - x^2}{(x^2 + 3)^2}$ and $f''(x) = 6 \cdot \dfrac{x(x^2 - 9)}{(x^2 + 3)^3}$. Solving $f'(x) = 0$ gives critical points $x = \pm\sqrt{3}$, and solving $f''(x) = 0$ gives possible inflection points at $x = 0, \pm 3$. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$ and increasing on $(-\sqrt{3}, \sqrt{3})$. The First Derivative Test shows that a local minimum occurs at $x = -\sqrt{3}$ and a local maximum occurs at $x = \sqrt{3}$.

Testing the sign of $f''(x)$ shows that f is concave down on the intervals $(-\infty, -3)$ and $(0, 3)$ and concave up on the intervals $(-3, 0)$ and $(3, \infty)$. Therefore inflection points occur at $x = 0, \pm 3$. The graph has x-intercept at $x = 0$. We also observe that $\lim_{x \to \pm\infty} f(x) = 0$, so f has its absolute maximum and minimum at $x = \sqrt{3}, -\sqrt{3}$ respectively.



27 The derivatives of f are $f'(x) = -4\pi \sin[\pi(x - 1)]$ and $f''(x) = -4\pi^2 \cos[\pi(x - 1)]$. Solving $f'(x) = 0$ gives critical point $x = 1$, and solving $f''(x) = 0$ gives possible inflection points at $x = 1/2, 3/2$. Testing the sign of $f'(x)$ shows that f is decreasing on the interval $(1, 2)$ and increasing on $(0, 1)$. The First Derivative Test shows that a local maximum occurs at $x = 1$. By Theorem 4.5, this solitary local maximum must be the absolute maximum for f on the interval $[0, 2]$.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(1/2, 3/2)$ and concave up on the intervals $(0, 1/2)$ and $(3/2, 2)$. Therefore inflection points occur at $x = 1/2, 3/2$. These points are also the x-intercepts of the graph. We also observe that $f(0) = f(2) = -4$, so f takes its absolute minimum at these points.



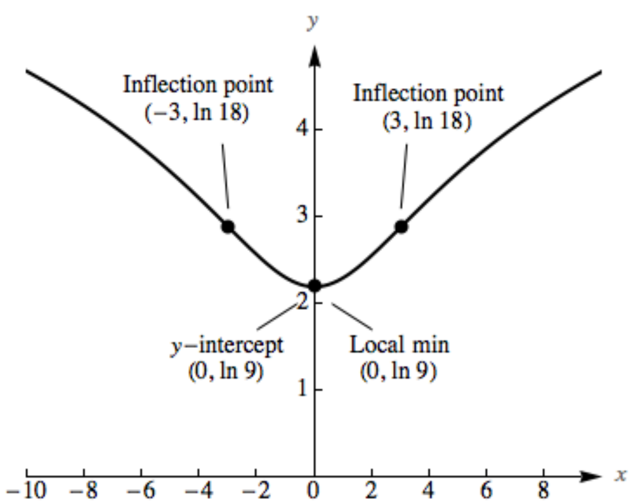
28 The domain of f is $(-\infty, \infty)$, and because $f(-x) = \ln((-x)^2 + 9) = \ln(x^2 + 9) = f(x)$, f has even symmetry.

$$f'(x) = \frac{1}{x^2 + 9} \cdot 2x = \frac{2x}{x^2 + 9},$$

which is zero for $x = 0$. $f' < 0$ on $(-\infty, 0)$ and $f' > 0$ on $(0, \infty)$ so f is decreasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. There is a local (and absolute) minimum of $\ln 9$ at $x = 0$.

$$f''(x) = \frac{(x^2 + 9)(2) - 2x(2x)}{(x^2 + 9)^2} = \frac{2(9 - x^2)}{(x^2 + 9)^2},$$

which is zero for $x = \pm 3$. $f'' > 0$ on $(-3, 3)$, so f is concave up there, while $f'' < 0$ on $(-\infty, -3)$ and on $(3, \infty)$, so f is concave down there. There are inflection points at $(\pm 3, \ln 18)$. The y-intercept is $(0, \ln 9)$.



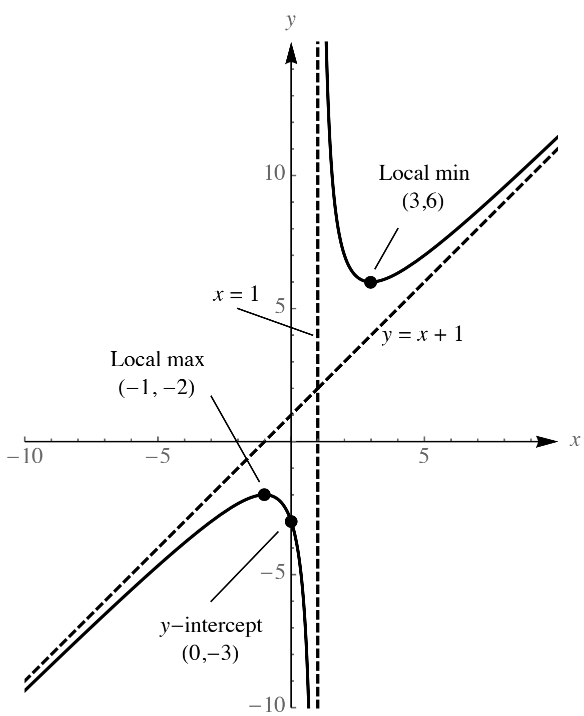
29 The domain of f is $(-\infty, 1) \cup (1, \infty)$. There is a vertical asymptote at $x = 1$. By long division, we can write $f(x) = x + 1 + \dfrac{4}{x - 1}$, so $y = x + 1$ is a slant asymptote. $f(-x) = \dfrac{x^2 + 3}{-x - 1} = -\dfrac{x^2 + 3}{x + 1}$ which is not the same as either $f(x)$ or $-f(x)$, so f has neither even nor odd symmetry.

$$f'(x) = \frac{(x-1)(2x) - (x^2+3)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2} = \frac{(x+1)(x-3)}{(x-1)^2},$$

which is zero for $x = -1$ and $x = 3$, so these are the critical points. $f' > 0$ on $(-\infty, -1)$ and $(3, \infty)$, so f is increasing on those intervals, while $f' < 0$ on $(-1, 1)$ and on $(1, 3)$ so f is decreasing on those intervals. There is a local minimum of 6 at $x = 3$ and a local maximum of -2 at $x = -1$.

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x-3)(2(x-1))}{(x-1)^4} = \frac{(2)(x-1)(x^2-2x+1-(x^2-2x-3))}{(x-1)^4} = \frac{8}{(x-1)^3}.$$

$f'' > 0$ on $(1, \infty)$ so f is concave up there, and $f'' < 0$ so f is concave down there. There are no inflection points as the only concavity change occurs at a vertical asymptote. The only intercept is the y-intercept at $f(0) = -3$.



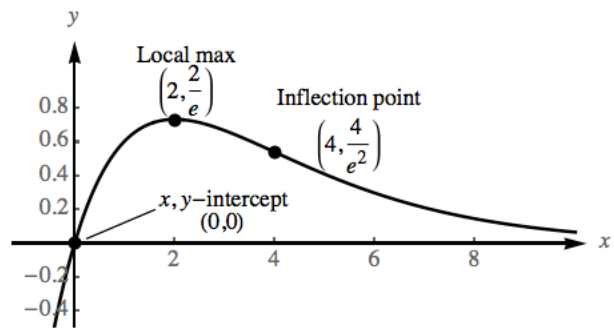
30 The domain of f is $(-\infty, \infty)$. $f(-x) = -xe^{x/2}$ which is neither $f(x)$ nor $-f(x)$, so this function has neither even nor odd symmetry. $\lim\limits_{x\to\infty} \dfrac{x}{e^{x/2}} = \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{2}e^{x/2}} = \lim\limits_{x\to\infty} \dfrac{2}{e^{x/2}} = 0$, so $y = 0$ is a horizontal asymptote as $x \to \infty$.

$$f'(x) = e^{-x/2} + x\left(\frac{1}{2}\right)(-e^{-x/2}) = e^{-x/2}\left(\frac{2-x}{2}\right),$$

which is zero for $x = 2$. $f' > 0$ on $(-\infty, 2)$ so f is increasing there, while $f' < 0$ on $(2, \infty)$, so f is decreasing there. There is a local (and absolute) maximum of $2/e$ at $x = 2$.

$$f''(x) = -\frac{1}{2}e^{-x/2}\left(\frac{2-x}{2}\right) + e^{-x/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-x/2}\left(1 - \frac{x}{2} + 1\right) = e^{-x/2}\left(\frac{x-4}{4}\right),$$

which is zero for $x = 4$. $f'' > 0$ on $(4, \infty)$, so f is concave up there, while $f'' < 0$ on $(-\infty, 4)$ so f is concave down there. There is an inflection point at $(4, 4/e^2)$. The only intercept is $(0, 0)$ which is both an x- and a y-intercept.



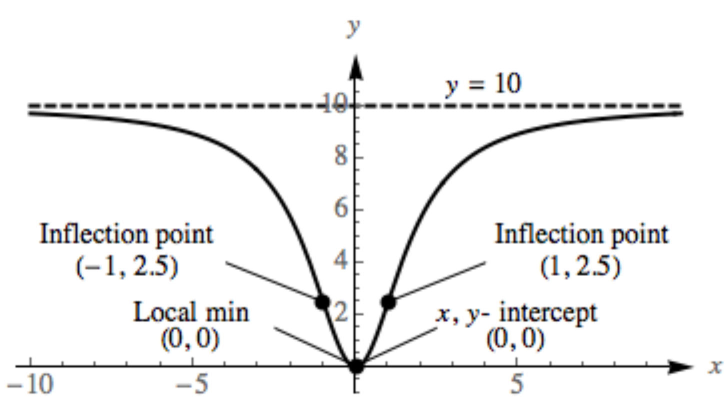
31 The domain of f is $(-\infty, \infty)$ and because $f(-x) = \dfrac{10(-x)^2}{(-x)^2 + 3} = f(x)$, f is symmetric with respect to the y-axis. Because $\displaystyle\lim_{x\to\pm\infty} \frac{10x^2}{x^3+3} = \lim_{x\to\pm\infty} \frac{20x}{2x} = 10$, the line $y = 10$ is a horizontal asymptote.

$$f'(x) = \frac{(x^2+3)(20x) - 10x^2(2x)}{(x^2+3)^2} = \frac{60x}{(x^2+3)^2},$$

which is zero for $x = 0$. $f' > 0$ on $(0, \infty)$, so f is increasing there, while $f' < 0$ on $(-\infty, 0)$, so f is decreasing there. There is a local (and absolute) minimum of 0 at $x = 0$.

$$f''(x) = \frac{(x^2+3)^2(60) - 60x(2)(x^2+3)(2x)}{(x^2+3)^4} = \frac{60(x^2+3)(x^2+3-4x^2)}{(x^2+3)^4} = \frac{180(1-x)(1+x)}{(x^2+3)^3},$$

which is zero for $x = -1$ and $x = 1$. $f'' < 0$ on $(-\infty, -1)$ and on $(1, \infty)$, so f is concave down there, while $f'' > 0$ on $(-1, 1)$, so f is concave up there. There are inflection points at $(\pm 1, 5/2)$. The only intercept is $(0, 0)$, which is both an x- and a y-intercept.



32 The domain of f is $(-\infty, \infty)$, and f has neither of our two types of symmetry.

$$f'(x) = \sqrt[3]{x+4} + \frac{x}{3\sqrt[3]{(x+4)^2}} = \frac{3(x+4)+x}{3\sqrt[3]{(x+4)^2}} = \frac{4(x+3)}{3\sqrt[3]{(x+4)^2}} = \frac{4(x+3)}{3(x+4)^{2/3}},$$

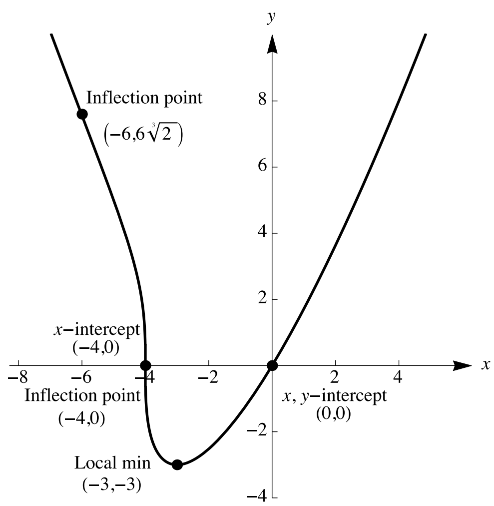
which is zero for $x = -3$ and is undefined for $x = -4$. Both $x = -3$ and $x = -4$ are critical points. $f'(x) < 0$ on $(-\infty, -4)$ and on $(-4, -3)$, so (using the fact that f is continuous at $x = -4$), f is decreasing on $(-\infty, -3)$, while $f' > 0$ on $(3, \infty)$, so f is increasing there. There is a local (and absolute) minimum of -3 at $x = -3$.

$$f''(x) = \frac{3(x+4)^{2/3}(4) - 4(x+3)(2)(x+4)^{-1/3}}{9\sqrt[3]{(x+4)^4}}.$$

This is perhaps best simplified by multiplying by $\frac{(x+4)^{1/3}}{(x+4)^{1/3}}$. Then we have

$$f''(x) = \frac{3(x+4)(4) - 8(x+3)}{9(x+4)^{5/3}} = \frac{4(x+6)}{9(x+4)^{5/3}},$$

which is zero for $x = -6$ and undefined for $x = -4$. $f'' > 0$ on $(-\infty, -6)$ and on $(-4, \infty)$, so f is concave up there, while $f'' < 0$ on $(-6, -4)$, so f is concave down there. There are inflection points at $(-4, 0)$ and at $(-6, 6\sqrt[3]{2})$. In addition to the intercept at $(-4, 0)$, there is the x- and y-intercept at $(0, 0)$.



33 Note that the domain of f is the interval $(0, \infty)$. The derivatives of f are

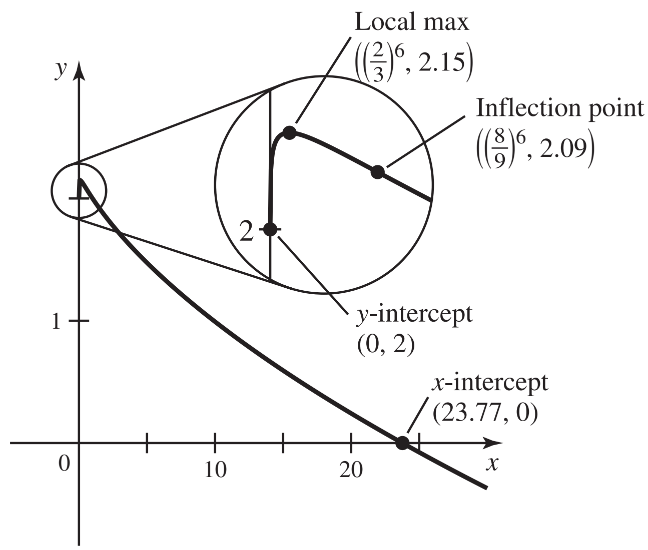
$$f'(x) = \frac{1}{3}x^{-2/3} - \frac{1}{2}x^{-1/2},$$

and

$$f''(x) = -\frac{2}{9}x^{-5/3} + \frac{1}{4}x^{-3/2}.$$

Solving $f'(x) = 0$ gives critical point $x = (2/3)^6$, and solving $f''(x) = 0$ gives a possible inflection point at $x = (8/9)^6$. Testing the sign of $f'(x)$ shows that f is increasing on the interval $(0, (2/3)^6)$ and decreasing on the interval $((2/3)^6, \infty)$. The First Derivative Test shows that a local maximum occurs at $x = (2/3)^6$ and By Theorem 4.5 this solitary local maximum must be the absolute maximum of f on the interval $(0, \infty)$.

Testing the sign of $f''(x)$ shows that f is concave down on the interval $(0, (8/9)^6)$ and concave up on the interval $((8/9)^6, \infty)$. Therefore an inflection point occurs at $x = (8/9)^6$. Using a numerical solver, we find that the graph has x-intercept at $x \approx 23.767$. Because $\lim_{x\to\infty} f(x) = -\infty$, f has no absolute minimum on the interval $(0, \infty)$.



34 The derivatives of f are

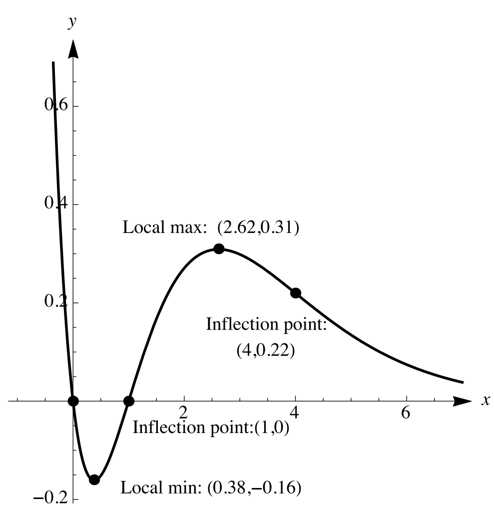
$$f'(x) = -(x^2 - 3x + 1)e^{-x},$$

and

$$f''(x) = (x^2 - 5x + 4)e^{-x}.$$

Solving $f'(x) = 0$ gives critical points $x \approx 0.382, 2.618$, and solving $f''(x) = 0$ gives possible inflection points at $x = 1, 4$. Testing the sign of $f'(x)$ shows that f is decreasing on the intervals $(-\infty, 0.382)$ and $(2.618, \infty)$ and increasing on $(0.382, 2.618)$. The First Derivative Test shows that a local minimum occurs at $x \approx 0.382$ and a local maximum occurs at $x \approx 2.618$.

Testing the sign of $f''(x)$ shows that f is concave up on the intervals $(-\infty, 1)$ and $(4, \infty)$ and concave down on the interval $(1, 4)$. Therefore inflection points occur at $x = 1, 4$. The graph has x-intercepts at $x = 0, 1$. Observe that $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to\infty} f(x) = 0$; therefore f has no absolute max, and the absolute min occurs at $x \approx 0.382$.



35 The volume of the box (where x is measured in inches) is given by $V(x) = x(6-x)(12-x)$, for $0 \le x \le 6$. Using the product rule for three functions, we have

$$V'(x) = (6-x)(12-x)+x(-1)(12-x)+x(6-x)(-1) = 72-6x-12x+x^2-12x+x^2-6x+x^2 = 3(x^2-12x+24).$$

By the quadratic formula, this is zero on the domain for $x = 6 - 2\sqrt{3} \approx 2.5$. Using the Second Derivative Test, we have $V''(a) = 3(2a - 12)$ which is negative for $a \approx 2.5$, so the solitary critical point on the domain gives a maximum. Therefore, the largest box is approximately $2.5 \times 3.5 \times 9.5$.

36 Let x be the distance from the fixed point P on the shore to the point where Hannah comes ashore, and let $T(x)$ be the total time for Hannah's trip to the beach house. Because time $= \dfrac{\text{distance}}{\text{rate}}$, we have

$$T(x) = \frac{\sqrt{x^2+1}}{2} + \frac{2-x}{6} \text{ for } 0 \le x \le 2.$$

Then $T'(x) = \dfrac{x}{2\sqrt{x^2+1}} - \dfrac{1}{6} = \dfrac{3x - \sqrt{x^2+1}}{6\sqrt{x^2+1}}$. This is zero when $3x = \sqrt{x^2+1}$, or $9x^2 = x^2+1$, so $x^2 = \dfrac{1}{8}$, so $x = \dfrac{1}{2\sqrt{2}}$. A quick check of the sign of T' on either side of this critical point shows that the critical point gives a minimum. So Hannah should come ashore at $\dfrac{1}{2\sqrt{2}} \approx 0.35$ miles down the beach and should make it home in approximately $T\left(\dfrac{1}{2\sqrt{2}}\right) \approx 0.805$ hours, which is about 48 minutes. So if she starts swimming at noon she should easily make it home by 1:00 pm (assuming she does her calculus correctly before she disembarks.)

37 Let x equal your distance from the louder speaker; then your distance from the other speaker is $100 - x$. So we want to minimize the function $I(x) = \dfrac{3}{x^2} + \dfrac{1}{(100-x)^2}$, for $0 < x < 100$. Then

$$\begin{aligned} I'(x) &= -\frac{6}{x^3} + \frac{2}{(100-x)^3} = \frac{-6(100-x)^3 + 2x^3}{x^3(100-x)^3} \\ &= \left(\frac{-6000000 + 180000x - 1800x^2 + 6x^3 + 2x^3}{x^3(100-x)^3}\right) \\ &= \frac{8(x^3 - 225x^2 + 22,500x - 750,000)}{(x^3)(100-x)^3}. \end{aligned}$$

Using a computer algebra system or a root finder on a calculator, we find that $I' = 0$ for $x \approx 59$. Because $I'(1) < 0$ and $I'(99) > 0$, we see that the critical number yields a minimum for $0 < x < 100$. So you should stand 59 m away from the louder speaker.

38 Let $\theta = \alpha - \beta$ as pictured. $\tan\alpha = \dfrac{13}{x}$, and $\tan\beta = \dfrac{7}{x}$, so $\theta = \tan^{-1}\left(\dfrac{13}{x}\right) - \tan^{-1}\left(\dfrac{7}{x}\right)$. Then

$$\begin{aligned} \theta' &= \frac{-13}{x^2(1+\frac{13^2}{x^2})} - \frac{-7}{x^2(1+\frac{7^2}{x^2})} \\ &= \frac{-13}{x^2+169} + \frac{7}{x^2+49} \\ &= \frac{7(x^2+169) - 13(x^2+49)}{(x^2+169)(x^2+49)} \\ &= \frac{546 - 6x^2}{(x^2+169)(x^2+49)} = \frac{-6(x^2-91)}{(x^2+169)(x^2+49)}. \end{aligned}$$

This is zero for $x = \sqrt{91} \approx 9.5$. It can be verified with either the First Derivative Test or Second Derivative Test that this is a maximum. So he should shoot the puck at the goal when he is about 9.5 feet from the goal line.

39 The objective function to be maximized is the volume of the cone, given by $V = \pi r^2 h/3$. By the Pythagorean Theorem, r and h satisfy the constraint $h^2 + r^2 = 16$, which gives $r^2 = 16 - h^2$. Therefore

$$V(h) = \frac{\pi}{3}h(16 - h^2) = \frac{\pi}{3}(16h - h^3).$$

We must maximize this function for $0 \le h \le 4$. The critical points of $V(h)$ satisfy $V'(h) = \dfrac{\pi}{3}(16 - 3h^2) = 0$, which has unique solution $h = 4/\sqrt{3} = 4\sqrt{3}/3$ in $(0, 4)$. Because $V(0) = V(4) = 0$, $h = 4\sqrt{3}/3$ gives the maximum value of $V(h)$ on $[0,4]$. The corresponding value of r satisfies $r^2 = 16 - \frac{16}{3} = \frac{32}{3}$, so $r = \frac{4\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}$.

40 The rectangle has dimensions x and $\cos x$, so the objective function to be maximized is $A(x) = x\cos x$, where $0 \leq x \leq \pi/2$. The critical points of this function satisfy

$$A'(x) = \cos x - x\sin x = 0,$$

which can be solved numerically to obtain $x \approx 0.860$. Note that $A(0) = A(\pi/2) = 0$, so the maximum area occurs at $x \approx 0.860$; the dimensions of the largest rectangle are 0.860 and $\cos 0.860 \approx 0.652$, which gives maximum area ≈ 0.561.

41 We have that $xy = 98$, and we want to maximize $p = (y-2)(x-1) = (y-2)(98/y-1) = 98-y-196/y+2$. Note that $p'(y) = -1 + 196/y^2$, which is zero when $y^2 = 196$, so $y = \sqrt{196} = 14$. Also note that $p'(13) > 0$ and $p'(15) < 0$, so there is a local (in fact, absolute) maximum at $y = 14$. The value of x when $y = 14$ is $x = 98/14 = 7$.

42 A point on the graph of $y = \frac{5}{2} - x^2$ has the form $(x, \frac{5}{2} - x^2)$; the square of its distance to the origin is given by

$$Q(x) = x^2 + \left(\frac{5}{2} - x^2\right)^2 = x^4 - 4x^2 + \frac{25}{4},$$

which we can take as our objective function to be minimized. The critical points of $Q(x)$ satisfy

$$Q'(x) = 4x^3 - 8x = 4x(x^2-2) = 0,$$

which has solutions $x = 0$ and $x = \pm\sqrt{2}$. The First Derivative Test shows that $x = 0$ is a local maximum and $x = \pm\sqrt{2}$ are local minima. Note that $Q(x)$ takes the same value at $x = \pm\sqrt{2}$, so the absolute minimum of $Q(x)$ occurs at $x = \pm\sqrt{2}$ and the points closest to the origin on the graph are $(\pm\sqrt{2}, \frac{1}{2})$.

43 The area of the triangle is $\frac{1}{2}pq$, and the constraint is $\sqrt{p^2+q^2} = 10$, or $p^2 + q^2 = 100$. So we can write the area of the triangle as $A(p) = (1/2)p\sqrt{100-p^2}$. We have

$$A'(p) = (1/2)\sqrt{100-p^2} + \frac{1}{2}p \cdot \frac{-p}{\sqrt{100-p^2}} = \frac{100-p^2-p^2}{2\sqrt{100-p^2}} = \frac{100-2p^2}{2\sqrt{100-p^2}}.$$

This is zero for $p = \sqrt{50} = 5\sqrt{2}$. An application of the First Derivative Test shows that there is a local (in fact, absolute) maximum at this value of p. The value of q for this value of p is $\sqrt{100-50} = 5\sqrt{2}$ as well. So the area of the triangle is maximized when $p = q = 5\sqrt{2}$.

44 The volume of the cistern is $\pi r^2 h$, so our constraint is $\pi r^2 h = 50$, so $h = \frac{50}{\pi r^2}$. The area of the painted surface is given by

$$A = 2\pi rh + \pi r^2 = 2\pi r \cdot \frac{50}{\pi r^2} + \pi r^2 = \frac{100}{r} + \pi r^2.$$

Thus we have

$$A'(r) = \frac{-100}{r^2} + 2\pi r,$$

which is zero when $2\pi r = \frac{100}{r^2}$, or $r^3 = \frac{50}{\pi}$, so $r = \sqrt[3]{50/\pi}$. This is a minimum because $A'(r) < 0$ for $r < \sqrt[3]{50\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{50/\pi}$. The dimension of the cistern are $r = \sqrt[3]{50/\pi}$ and $h = \frac{50}{\pi r^2} = \sqrt[3]{50/\pi}$.

45

a. $f'(x) = \frac{2}{3}x^{-1/3}$, so $f'(27) = \frac{2}{3\sqrt[3]{27}} = \frac{2}{9}$. Thus

$$L(x) - 9 = \frac{2}{9}(x - 27),$$

or $L(x) = 9 + \frac{2}{9}(x - 27) = \frac{2}{9}x + 3$.

b. $f(29) \approx L(29) = 9 + \frac{4}{9} \approx 9.44$. This is an overestimate, because $f''(27) < 0$. Note that the calculator value of $f(29)$ is about 9.43913.

46

a. $f'(x) = \frac{1}{\sqrt{1-x^2}}$, so $f'(1/2) = \frac{2}{\sqrt{3}}$. Thus

$$L(x) - \pi/6 = \frac{2}{\sqrt{3}}(x - 1/2),$$

or $L(x) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - 1/2)$.

b. $f(0.48) \approx L(0.48) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(-.02) \approx 0.5005$. This is an underestimate because $f''(1/2) > 0$. The calculator value of $\sin^{-1}(0.48) \approx 0.500655$.

47 Let $f(x) = 1/x^2$ and let $a = 4$. Then $f'(x) = \frac{-2}{x^3}$ so $f'(4) = \frac{-2}{64} = \frac{-1}{32}$. The linearization is

$$L(x) = \frac{1}{16} + (-1/32)(x - 4).$$

Then $f(4.2) = 1/(4.2)^2 \approx L(4.2) = \frac{1}{16} - \frac{1}{32} \cdot \frac{2}{10} = 0.05625$.

48 Let $f(x) = \tan^{-1}(x)$ and $a = 1$. Then $f'(x) = \frac{1}{x^2+1}$ so $f'(1) = \frac{1}{2}$. The linearization is

$$L(x) = \frac{\pi}{4} + \frac{1}{2}(x - 1).$$

Then $f(1.05) = \tan^{-1}(1.05) \approx L(1.05) = \frac{\pi}{4} + \frac{1}{2} \cdot \frac{1}{20} \approx 0.8104$.

49 $\Delta h \approx h'(a)\Delta t = -32 \cdot 5(0.7) = -112$ feet.

50 $\Delta E \approx E'(a)\Delta M = 25000 \cdot 10^{(1.5)(7)} \cdot 1.5(\ln 10) \cdot 0.5 \approx 1.365 \times 10^{15}$ J.

51 The value of c appears to be between 2 and 3. We are seeking c so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{10 - 0}{10 - 0} = 1,$$

so we want $\frac{\sqrt{10}}{2\sqrt{c}} = 1$. Thus $10 = 4c$ and $c = 2.5$, as projected.

52 The absolute value function is not differentiable at $x = 0$, so it is not differentiable on $(-1, 2)$, as required.

53

a. The average rate of change of $P(t)$ on the interval $[0, 8]$ is

$$\frac{P(8) - P(0)}{8 - 0} = \frac{800/9 - 0}{8} = \frac{100}{9} \text{cells/week.}$$

b. We solve

$$P'(t) = \frac{100}{(t+1)^2} = \frac{100}{9}$$

which gives $(t+1)^2 = 9$, so $t = 2$ weeks.

54

a. The average rate of change is $\dfrac{\text{change in growth}}{\text{elapsed time}} = \dfrac{15}{5} = 3$ cm per hour.

b. 3 cm per hour is equivalent to $30/3600 = 1/120$ mm per second. The Mean Value Theorem tells us that sometime between 10:00 a.m. and 3:00 p.m, there will be a time when the instantaneous growth rate is exactly $1/120$ mm per second.

55 It is possible to note that 1 is a root by inspection. Then by long division by $x - 1$, we have $f(x) = (x-1)(3x^2 - x - 1)$. We apply Newton's method to the function $g(x) = 3x^2 - x - 1$.

The Newton's method recursion is given by $x_{n+1} = x_n - \dfrac{3x_n^2 - x_n - 1}{6x_n - 1}$. Applying this recursion to the initial estimates of -0.5 and 0.8 yields:

n	x_n
0	-0.5
1	-0.4375
2	-0.434267
3	-0.434259
4	-0.434259

n	x_n
0	0.8
1	0.768421
2	0.767592
3	0.767592
4	0.767592

The roots are thus 1, and approximately -0.434259 and 0.767592.

56 The Newton's method recursion is given by $x_{n+1} = x_n - \dfrac{e^{-2x_n} + 2e^{x_n} - 6}{-2e^{-2x_n} + 2e^{x_n}}$. Applying this recursion to the initial estimates of -1 and 1 yields:

n	x_n
0	-1
1	-0.848685
2	-0.817331
3	-0.816164
4	-0.816162
5	-0.816162

n	x_n
0	1
1	1.08287
2	1.07918
3	1.07917
4	1.07917
5	1.07917

The roots are thus 1, and approximately -0.816162 and 1.07917.

57 First note that $f'(x) = 10x^4 - 18x^2 - 4$ and $f''(x) = 40x^3 - 36x = 4x(10x^2 - 9)$. This is clearly 0 when $x = 0$, and when $10x^2 - 9 = 0$. Applying Newton's method to the function $g(x) = 10x^2 - 9$ with initial estimates of -1 and 1 yields:

n	x_n
0	-1
1	-0.95
2	-0.948684
3	-0.948683
4	-0.948683
5	-0.948683

n	x_n
0	1
1	0.95
2	0.948684
3	0.948683
4	0.948683
5	0.948683

Checking the signs of $f''(x)$ on the appropriate intervals leads to the conclusion that these are all the locations of inflection points of f. So the inflection points of f are located at 0 and approximately ± 0.948683.

58 Observe that $\lim_{x\to\infty} \frac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}} = \lim_{x\to\infty} \frac{4 - x^{-7/2}}{2 + x^{-5}} = \frac{4}{2} = 2$. We can also apply L'Hôpital's rule four times: $\lim_{x\to\infty} \frac{4x^4 - x^{1/2}}{2x^4 + x^{-1}} = \lim_{x\to\infty} \frac{96 + (15/16)x^{-7/2}}{48 + 24x^{-5}} = 2$.

59 By L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{2x^5 - x + 1}{5x^6 + x} = \lim_{x\to\infty} \frac{10x^4 - 1}{30x^5 + 1} = \lim_{x\to\infty} \frac{40x^3}{150x^4} = \lim_{x\to\infty} \frac{4}{15x} = 0.$$

60 L'Hôpital's rule gives $\lim_{t\to 2} \frac{t^3 - t^2 - 2t}{t^2 - 4} = \lim_{t\to 2} \frac{3t^2 - 2t - 2}{2t} = \frac{3}{2}$.

61 L'Hôpital's rule gives $\lim_{t\to 0} \frac{1 - \cos 6t}{2t} = \lim_{t\to 0} \frac{6 \sin 6t}{2} = 0$.

62 If we factor out x^2 from the numerator and denominator, we see that

$$\frac{5x^2 + 2x - 5}{\sqrt{x^4 - 1}} = \frac{5 + 2/x - 5/x^2}{\sqrt{1 - 1/x^4}} \to 5$$

as $x \to \infty$.

63 Observe that $\lim_{\theta\to 0} \frac{3\sin^2 2\theta}{\theta^2} = 3\left(\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta}\right)^2$; L'Hôpital's rule gives $\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0} \frac{2\cos 2\theta}{1} = 2$, so $\lim_{\theta\to 0} \frac{3\sin^2 2\theta}{\theta^2} = 3 \cdot 2^2 = 12$.

64 First observe that

$$(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x})\frac{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})} = \frac{2x + 1}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})}.$$

Next, factor out x from both numerator and denominator to obtain

$$\frac{2x + 1}{(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x})} = \frac{2 + 1/x}{(\sqrt{1 + 1/x + 1/x^2} + \sqrt{1 - 1/x})}.$$

This expression converges to $2/2 = 1$ as $x \to \infty$, so $\lim_{x\to\infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}) = 1$.

65 Observe that $2\theta \cot 3\theta = 2\cos 3\theta \cdot \frac{\theta}{\sin 3\theta}$; $\lim_{\theta\to 0} \frac{\theta}{\sin 3\theta} = \lim_{\theta\to 0} \frac{1}{3\cos 3\theta} = \frac{1}{3}$ by L'Hôpital's rule, so

$$\lim_{\theta\to 0} 2\theta \cot 3\theta = 2 \cdot 1 \cdot \frac{1}{3} = \frac{2}{3}.$$

66 Apply L'Hôpital's rule twice: $\lim_{x\to 0}\dfrac{e^{-2x}-1+2x}{x^2}=\lim_{x\to 0}\dfrac{-2e^{-2x}+2}{2x}=\lim_{x\to 0}\dfrac{4e^{-2x}}{2}=2.$

67 Make the change of variables $x=1/y$; then $\ln^{10}y=(\ln y)^{10}=(-\ln x)^{10}=\ln^{10}x$, and so $\lim_{y\to 0^+}\dfrac{\ln^{10}y}{\sqrt{y}}=\lim_{x\to\infty}\sqrt{x}\ln^{10}x=\infty.$

68 Apply L'Hôpital's rule: $\lim_{\theta\to 0}\dfrac{3\sin 8\theta}{8\sin 3\theta}=\lim_{\theta\to 0}\dfrac{24\cos 8\theta}{24\cos 3\theta}=1.$

69 Apply L'Hôpital's rule twice:

$$\lim_{x\to 1}\frac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}=\lim_{x\to 1}\frac{4x^3-3x^2-6x+5}{3x^2+2x-5}=\lim_{x\to 1}\frac{12x^2-6x-6}{6x+2}=0.$$

70 The function $\ln x^{100}=100\ln x$ grows more slowly than $\sqrt{x}$ as $x\to\infty$, so $\lim_{x\to\infty}\dfrac{\ln x^{100}}{\sqrt{x}}=0.$

71 $\lim_{x\to 0}\csc x\sin^{-1}x=\lim_{x\to 0}\dfrac{\sin^{-1}x}{\sin x}=\lim_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{\cos x}=1$, by L'Hôpital's rule.

72 The function $\ln^3 x$ grows more slowly than $\sqrt{x}$ as $x\to\infty$, so $\lim_{x\to\infty}\dfrac{\ln^3 x}{\sqrt{x}}=0.$

73 Observe that $\lim_{x\to\infty}\dfrac{x+1}{x-1}=1$, by L'Hôpital's rule. Hence $\lim_{x\to\infty}\ln\left(\dfrac{x+1}{x-1}\right)=\ln 1=0.$

74 Let $z=(1+x)^{\ln x}$. Then $\ln z=(\ln x)\ln(x+1)$. Then

$$\begin{aligned}\lim_{x\to 0^+}\ln z&=\lim_{x\to 0^+}(\ln x)\ln(x+1)=\lim_{x\to 0^+}\frac{\ln(1+x)}{\ln^{-1}x}\\&=\lim_{x\to 0^+}\frac{1/(1+x)}{-(\ln x)^{-2}(1/x)}=\lim_{x\to 0^+}\frac{1}{1+x}\cdot\lim_{x\to 0^+}(x\ln x)^2=1\cdot\left(\lim_{x\to 0^+}x\ln x\right)^2.\end{aligned}$$

Now $\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\dfrac{\ln x}{1/x}=\lim_{x\to 0^+}\dfrac{1/x}{-1/x^2}=\lim_{x\to 0^+}-x=0$. Thus,

$$\lim_{x\to 0^+}\ln z=0,$$

so

$$\lim_{x\to 0^+}(1+x)^{\ln x}=e^0=1.$$

75 Note that $\ln(\sin x)^{\tan x}=\tan x\ln\sin x$, so we evaluate $L=\lim_{x\to\pi/2^-}\tan x\ln\sin x=\lim_{x\to\pi/2^-}\dfrac{\ln\sin x}{\cot x}=\lim_{x\to\pi/2^-}\dfrac{\cot x}{-\csc^2 x}=\lim_{x\to\pi/2^-}(-\cos x\sin x)=0$ by L'Hôpital's rule. Therefore $\lim_{x\to\pi/2^-}(\sin x)^{\tan x}=e^L=1.$

76 Note that $\ln(\sqrt{x}+1)^{1/x}=\dfrac{\ln(\sqrt{x}+1)}{x}$, so we evaluate $L=\lim_{x\to\infty}\dfrac{\ln(\sqrt{x}+1)}{x}=\lim_{x\to\infty}\dfrac{\frac{1}{(\sqrt{x}+1)}\cdot\frac{1}{2\sqrt{x}}}{1}=0$ by L'Hôpital's rule. Therefore, $\lim_{x\to\infty}(\sqrt{x}+1)^{1/x}=e^L=e^0=1.$

77 Note that for $0<x<1$, we have $\ln x<0$, so $|\ln x|=-\ln x$. Then $|\ln x|^x=(-\ln x)^x$. Consider the natural logarithm of this quantity, $x\ln(-\ln x)=\dfrac{\ln(-\ln x)}{\frac{1}{x}}$. Using L'Hôpital's rule we have

$$\lim_{x\to 0^+}\frac{\ln(-\ln x)}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x\ln x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}\frac{-x}{\ln x}=0.$$

Thus $\lim_{x\to 0^+}|\ln x|^x=e^0=1.$

78 Note that $\ln x^{1/x} = \ln x/x$, so we evaluate $L = \lim_{x\to\infty} \frac{\ln x}{x} = 0$ because $\ln x$ grows more slowly than x as $x \to \infty$. Therefore $\lim_{x\to\infty} x^{1/x} = e^L = 1$.

79 Note that $\ln\left(1-\frac{3}{x}\right)^x = x\ln\left(1-\frac{3}{x}\right) = \frac{\ln(1-3/x)}{1/x}$, so we evaluate

$$L = \lim_{x\to\infty} \frac{\ln(1-3/x)}{1/x} = \lim_{x\to\infty} \frac{(1-3/x)^{-1}(3/x^2)}{-1/x^2} = -3$$

by L'Hôpital's rule. Therefore $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x = e^L = e^{-3}$.

80 Note that $\ln\left(\left(\frac{2}{\pi}\tan^{-1}x\right)^x\right) = x\ln\left(\frac{2}{\pi}\tan^{-1}x\right) = \frac{\ln\left(\frac{2}{\pi}\tan^{-1}x\right)}{1/x}$. We evaluate
$L = \lim_{x\to\infty} \frac{\ln\left(\frac{2}{\pi}\tan^{-1}x\right)}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{(2/\pi)\tan^{-1}(x)}\cdot\frac{2}{\pi(x^2+1)}}{-1/x^2} = \lim_{x\to\infty} \frac{-x^2}{x^2+1}\cdot\frac{1}{\tan^{-1}(x)} = -\frac{2}{\pi}$.
Thus, $\lim_{x\to\infty}\left(\frac{2}{\pi}\tan^{-1}x\right)^x = e^L = e^{-2/\pi}$.

81 Let $y = (x-1)^{\sin\pi x}$. Then $\ln y = \sin\pi x\ln(x-1) = \frac{\ln(x-1)}{\csc\pi x}$. Then

$$\begin{aligned}\lim_{x\to1}\ln y &= \lim_{x\to1}\frac{\ln(x-1)}{\csc\pi x} = \lim_{x\to1}\frac{\frac{1}{x-1}}{-\pi\csc\pi x\cot\pi x} = \lim_{x\to1}\frac{\sin^2\pi x}{-\pi(x-1)\cos\pi x}\\ &= \lim_{x\to1}\frac{1}{-\pi\cos\pi x}\cdot\lim_{x\to1}\frac{\sin^2\pi x}{x-1} = \frac{1}{\pi}\lim_{x\to1}\frac{2\pi\sin\pi x\cos\pi x}{1} = \frac{1}{\pi}\cdot 0 = 0.\end{aligned}$$

Then $\lim_{x\to1} y = \lim_{x\to1} e^{\ln y} = e^0 = 1$.

82 By Theorem 4.14, 1.1^x grows faster than x^{100} as $x\to\infty$.

83 Observe that $\lim_{x\to\infty}\frac{x^{1/2}}{x^{1/3}} = \lim_{x\to\infty} x^{1/6} = \infty$, so $x^{1/2}$ grows faster than $x^{1/3}$ as $x\to\infty$.

84 Because $\log_{10}x = \ln x/\ln 10$, $\ln x$ and $\log_{10}x$ have comparable growth rates as $x\to\infty$.

85 By Theorem 4.14, $\sqrt{x}$ grows faster than $\ln^{10}x$ as $x\to\infty$.

86 Because $\ln x^2 = 2\ln x$, Theorem 4.14 shows that $10x$ grows faster than $\ln x^2$ as $x\to\infty$.

87 Observe that $\lim_{x\to\infty}\frac{e^x}{3^x} = \lim_{x\to\infty}\left(\frac{e}{3}\right)^x = 0$ because $e/3 < 1$. Therefore 3^x grows faster than e^x as $x\to\infty$.

88 Observe that $\lim_{x\to\infty}\frac{\sqrt{x^6+10}}{x^3} = \lim_{x\to\infty}\sqrt{1+\frac{10}{x^6}} = 1$, so $\sqrt{x^6+10}$ and x^3 have comparable growth rates as $x\to\infty$.

89 Observe that $4^{x/2} = (4^{1/2})^x = 2^x$, so these functions are identical and hence have comparable growth rates as $x\to\infty$.

90 $\int(x^8-3x^3+1)\,dx = \frac{x^9}{9} - 3\cdot\frac{x^4}{4} + x + C = \frac{x^9}{9} - \frac{3}{4}x^4 + x + C$.

91 $\int(2x+1)^2\,dx = \int(4x^2+4x+1)\,dx = \frac{4}{3}x^3 + 2x^2 + x + C$.

92 $\displaystyle\int \frac{x^5-3}{x}\,dx = \int\left(\frac{x^5}{x}-\frac{3}{x}\right)dx = \int\left(x^4-\frac{3}{x}\right)dx = \frac{x^5}{5} - 3\ln|x| + C.$

93 $\displaystyle\int\left(\frac{1}{x^2}-\frac{2}{x^{5/2}}\right)dx = \int(x^{-2}-2x^{-5/2})\,dx = -x^{-1} - 2\cdot\left(-\frac{2}{3}\right)x^{-3/2} = -\frac{1}{x}+\frac{4}{3}x^{-3/2}+C.$

94 $\displaystyle\int \frac{x^4-2\sqrt{x}+2}{x^2}\,dx = \int(x^2-2x^{-3/2}+2x^{-2})\,dx = \frac{x^3}{3} - 2\cdot(-2x^{-1/2}) - 2x^{-1} + C = \frac{x^3}{3}+\frac{4}{\sqrt{x}}-\frac{2}{x}+C.$

95 $\displaystyle\int(1+3\cos\theta)\,d\theta = \theta + 3\sin\theta + C.$

96 $\displaystyle\int 2\sec^2\theta\,d\theta = 2\tan\theta + C.$

97 $\displaystyle\int\frac{dx}{1-\sin^2 x} = \int\frac{dx}{\cos^2 x} = \int\sec^2 x\,dx = \tan x + C.$

98 $\displaystyle\int\frac{2e^{2x}+e^x}{e^x}\,dx = \int(2e^x+1)\,dx = 2e^x + x + C.$

99 $\displaystyle\int\frac{12}{x}\,dx = 12\ln|x| + C.$

100 $\displaystyle\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C.$

101 $\displaystyle\int\frac{x^2}{x^4+x^2}\,dx = \int\frac{1}{x^2+1}\,dx = \tan^{-1}x + C.$

102 Note that $\dfrac{1+\tan\theta}{\sec\theta} = \cos\theta+\sin\theta$, so $\displaystyle\int\frac{1+\tan\theta}{\sec\theta}\,d\theta = \sin\theta - \cos\theta + C.$

103 $\displaystyle\int\left(\sqrt[4]{x^3}+\sqrt{x^5}\right)dx = \int(x^{3/4}+x^{5/2})\,dx = \frac{4}{7}x^{7/4}+\frac{2}{7}x^{7/2}+C.$

104 We have $f(x) = \displaystyle\int(3x^2-1)\,dx = x^3 - x + C$; $f(0) = C = 10$, so $f(x) = x^3 - x + 10$.

105 We have $f(t) = \displaystyle\int(\sin t + 2t)\,dt = -\cos t + t^2 + C$; $f(0) = -1 + C = 5$, so $C = 6$ and $f(t) = -\cos t + t^2 + 6$.

106 We have $f(t) = \displaystyle\int(t^2+t^{-2})\,dt = \frac{t^3}{3} - \frac{1}{t} + C$; $f(1) = 1/3 - 1 + C = C - 2/3 = 1$, so $C = 5/3$ and $f(t) = t^3/3 - 1/t + 5/3$.

107 Note that by long division, we can write $\dfrac{x^4-2}{1+x^2} = x^2 - 1 - \dfrac{1}{1+x^2}$. Therefore $f(x) = \displaystyle\int\left(x^2-1-\frac{1}{1+x^2}\right)dx = \frac{x^3}{3} - x - \tan^{-1}x + C$. Because $f(1) = \dfrac{2}{3}$, we have $\dfrac{1}{3}-\dfrac{\pi}{4}+C = -\dfrac{2}{3}$, so $C = \dfrac{\pi}{4}$. Thus $f(x) = \dfrac{x^3}{3} - x - \tan^{-1}x + \dfrac{\pi}{4}$.

108 The difference in the positions of the objects is given by the function

$$f(t) = 2\sin t - \sin(t-\pi/2) = 2\sin t + \cos t.$$

The objects meet when $f(t) = 0$, which occurs at $t = \pi - \tan^{-1}(1/2) \approx 2.673$ and $t = 2\pi - \tan^{-1}(1/2) \approx 5.820$. The objects are furthest apart when f attains its absolute max or min; the critical points of f satisfy

$$f'(t) = 2\cos t - \sin t = 0,$$

or $\tan t = 2$; this gives $t = \tan^{-1}(2) \approx 1.11$ and $t = \pi + \tan^{-1}2 \approx 4.25$. The value of f is $\approx \pm 2.24$ at these points, so the objects are furthest apart at these two times.

109 The velocity of the rocket is given by

$$v(t) = -9.8t + v_0 = -9.8t + 120,$$

and the position function of the rocket is

$$s(t) = \int (-9.8t + 120)\, dt = -4.9t^2 + 120t + s_0 = -4.9t^2 + 120t + 125.$$

The rocket reaches its maximum height when $v(t) = 0$, which occurs at $t = 120/9.8 \approx 12.24$ s; the maximum height is $s(120/9.8) \approx 859.69$ m. The rocket hits the ground when $s(t) = 0$; solving this quadratic equation gives $t \approx 25.49$ s.

110 Because $a(t) = 20$, we have $v(t) = \displaystyle\int a(t)\, dt = \int 20\ dt = 20t + C$. Because the car is starting from rest, $v(0) = 0$, and so $C = 0$. So $v(t) = 20t$. Then

$$s(t) = \int v(t)\, dt = \int 20t\, dt = 10t^2 + C_1,$$

but we can assume the car's initial position is 0, so $C_1 = 0$ and $s(t) = 10t^2$. Then $s(5) = 250$ ft.

111

a. We have $v(t) = -32t + v_0$ and $v_0 = 64$, so $v(t) = -32t + 64$.

b. The height of the ball above the river is given by $s(t) = \displaystyle\int (-32t + 64)\, dt = -16t^2 + 64t + s_0$., and because $s(0) = 128$, we have $s_0 = 128$. So $s(t) = -16t^2 + 64t + 128$.

c. The ball reaches its maximum height when $v(t) = -32t + 64 = 0$, which occurs for $t = 2$. The height at this time is $s(2) = 192$.

d. The ball strikes the river when $s(t) = 0$ (and $t > 0$), which gives $-16t^2 + 64t + 128 = -16(t^2 - 4t - 8) = 0$. Using the quadratic formula, we see that this occurs for $t = 2(1 + \sqrt{3})$. The velocity with which the ball strikes the river is therefore $v((2(1 + \sqrt{3})) = -64\sqrt{3} \approx -110.9$ ft per s.

112 We have $\ln(\ln(\ln x)) << \ln(\ln x) << \ln x$ as $x \to \infty$. For the first relation, make the change of variables $y = \ln(\ln(x))$: then

$$\lim_{x\to\infty} \frac{\ln(\ln(\ln x))}{\ln(\ln x)} = \lim_{y\to\infty} \frac{\ln y}{y} = 0;$$

and for the second, let $y = \ln x$:

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{y\to\infty} \frac{\ln y}{y} = 0.$$

113 For the second limit, let $y = x^2$:

$$\lim_{x\to 0} \frac{x^2}{1 - e^{-x^2}} = \lim_{y\to 0} \frac{y}{1 - e^{-y}} = \lim_{y\to 0} \frac{1}{e^{-y}} = 1$$

by L'Hôpital's rule. For the first, observe that for $x > 0$

$$\frac{x}{\sqrt{1 - e^{-x^2}}} = \left(\frac{x^2}{1 - e^{-x^2}}\right)^{1/2},$$

so $\displaystyle\lim_{x\to 0^+} \frac{x}{\sqrt{1 - e^{-x^2}}} = 1$ as well.

114 Observe that

$$\ln\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r} = \frac{\ln(a^r+b^r+c^r)-\ln 3}{r}.$$

By L'Hôpital's rule, we have

$$L = \lim_{r\to 0}\frac{\ln(a^r+b^r+c^r)-\ln 3}{r} = \lim_{r\to 0}\frac{(a^r+b^r+c^r)^{-1}\left((\ln a)a^r+(\ln b)b^r+(\ln c)c^r\right)}{1}$$

$$= \frac{1}{3}(\ln a+\ln b+\ln c) = \ln(abc)^{1/3};$$

therefore $\lim_{r\to 0}\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r} = e^L = \sqrt[3]{abc}.$

115 Note that $\lim_{x\to 0+} x^x = 1$, as shown in Section 4.6. Therefore $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^x)^x = 1^0 = 1$, and $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^{(x^x)} = 0^1 = 0$.

116

a. Make the change of variables $y = x^n$ and apply L'Hôpital's rule twice:

$$\lim_{x\to 0}\frac{1-\cos x^n}{x^{2n}} = \lim_{y\to 0}\frac{1-\cos y}{y^2} = \lim_{y\to 0}\frac{\cos y}{2} = \frac{1}{2}.$$

b. Apply L'Hôpital's rule:

$$\lim_{x\to 0}\frac{1-\cos^n x}{x^2} = \lim_{x\to 0}\frac{n\cos^{n-1}x\sin x}{2x} = \frac{n}{2}\left(\lim_{x\to 0}\cos^{n-1}x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = \frac{n}{2}\cdot 1\cdot 1 = \frac{n}{2}.$$

117

a. First, observe that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+a} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{a} = e\cdot 1 = e.$$

Therefore $\lim_{x\to\infty}\ln g(x) = 1$. It suffices to determine whether $\ln g(x)-1$ is positive or negative as $x\to\infty$. To do this, consider

$$\lim_{x\to\infty} x(\ln g(x)-1) = \lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^2},$$

where we make the change of variables $t = 1/x$. This limit can be evaluated by using L'Hôpital's rule twice:

$$\lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^2} = a-\frac{1}{2}.$$

Therefore when $a > 1/2$ we have $g(x) > e$ as $x\to\infty$,

b. Using the analysis above, for $0 < a < 1/2$, $g(x) < e$ as $x\to\infty$. In the case $a = 1/2$ we consider the limit

$$\lim_{x\to\infty} x^2(\ln g(x)-1) = \lim_{t\to 0}\frac{(1+at)\ln(1+t)-t}{t^3},$$

which can be evaluated by using L'Hôpital's three times:

$$\lim_{t\to 0}\frac{(1+t/2)\ln(1+t)-t}{t^3} = \frac{1}{12}.$$

Therefore $g(x) > e$ as $x\to\infty$ in this case as well.

118

a. The domain is the interval $[-a, \infty)$.

b. Observe that $\lim_{x\to -a^+}(a+x)^x = \lim_{x\to -a^+}(a+x)^{a+x}(a+x)^{-a} = \lim_{y\to 0+} y^y \lim_{y\to 0+} y^{-a} = 1\cdot\infty = \infty$. We also have $\lim_{x\to\infty}(a+x)^x = \infty$ because $(a+x)^x > x^x$.

c. Using logarithmic differentiation, we find that $f'(x) = x(a+x)^{x-1} + \ln(a+x)(a+x)^x$.

d. Multiplying $f'(x)$ by $(a+x)^{1-x}$ shows that the critical point z for f satisfies the equation

$$z + (z+a)\ln(z+a) = 0,$$

which is equivalent to the equation

$$\ln(z+a) - \frac{a}{z+a} = -1.$$

The left side is an increasing function on $(-a, \infty)$ with range $(-\infty, \infty)$, so there exists a unique z satisfying this equation.

e. Graphical analysis shows that as $a \to \infty$, $z \to -\infty$ and $f(z) \to 0$.

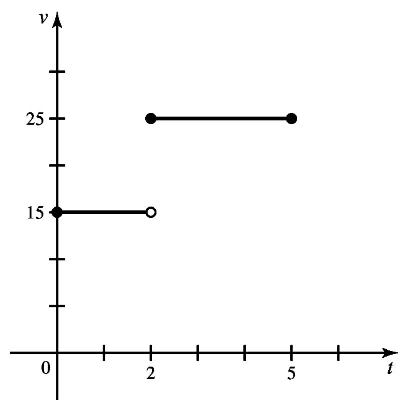
Chapter 5

Integration

5.1 Approximating Areas under Curves

5.1.1

In the first 2 seconds, the object moves $15 \cdot 2 = 30$ meters. In the next three seconds, the object moves $25 \cdot 3 = 75$ meters, so the total displacement is $75 + 30 = 105$ meters.



5.1.2 The area under the curve and above the t-axis, between $t = a$ and $t = b$ numerically represents the displacement.

5.1.3

a. The displacement is approximately $40(4) + 70(4) = 440$ ft.

b. The displacement is approximately $30(2) + 50(2) + 80(2) + 40(2) = 400$ ft.

5.1.4

a. The displacement is approximately $10(2) + 40(2) + 70(2) = 240$ ft.

b. The displacement is approximately $10(1) + 20(1) + 40(1) + 60(1) + 70(1) + 80(1) = 280$ ft.

5.1.5

a. The displacement is approximately $60(2) + 30(2) + 80(2) = 340$ ft.

b. The displacement is approximately $70(1) + 60(1) + 50(1) + 30(1) + 40(1) + 80(1) = 330$ ft.

5.1.6

a. The displacement is approximately $10(2) + 25(2) + 35(2) + 55(2) + 65(2) = 380$ ft.

b. The displacement is approximately $25(2) + 35(2) + 55(2) + 65(2) + 90(2) = 540$ ft.

c. The displacement is approximately $20(2) + 30(2) + 45(2) + 60(2) + 75(2) = 460$ ft.

5.1.7 Subdivide the interval from 0 to $\pi/2$ into subintervals. On each subinterval, pick a sample point (like the left endpoint, or the right endpoint, or the midpoint, for example) and call the first sample point x_1 and the second sample point x_2 and so on. For each sample point x_i, calculate the area of the rectangle which lies over the subinterval and has height $f(x_i) = \cos x_i$. Do this for each subinterval, and add the areas of the corresponding rectangles together. This will give an approximation to the area under the curve, with generally a better approximation occurring as n increases – where n is the number of subintervals used.

5.1.8 As the number of subintervals increases, the approximation to the area under the curve improves.

5.1.9 We have $\Delta x = \dfrac{7-1}{6} = 1$. The left Riemann sum is given by $1(10+9+7+5+2+1) = 34$ and the right Riemann sum is given by $1(9+7+5+2+1+0) = 24$.

5.1.10 Using 3 subintervals, we have $\Delta x = \dfrac{6}{3} = 2$. The left Riemann sum is $2(f(0)+f(2)+f(4)) = 2(1+6+9) = 32$. The right Riemann sum is given by $2(f(2)+f(4)+f(6)) = 2(6+9+11) = 52$. The midpoint Riemann sum is given by $2(f(1)+f(3)+f(5)) = 2(4+7+10) = 42$.

5.1.11 Because the interval $[1,3]$ has length 2, if we subdivide it into 4 subintervals, each will have length $\Delta x = \dfrac{2}{4} = \dfrac{1}{2}$. The grid points will be $x_0 = 1$, $x_1 = 1 + \Delta x = 1.5$, $x_2 = 1 + 2\Delta x = 2$, $x_3 = 1 + 3\Delta x = 2.5$, and $x_4 = 1 + 4\Delta x = 3$.

If we use the left-hand side of each subinterval, we will use 1, 1.5, 2, and 2.5.

If we use the right-hand side of each subinterval, we will use 1.5, 2, 2.5, and 3.

If we use the midpoint of each subinerval, we will use 1.25, 1.75, 2.25, and 2.75.

5.1.12

The left Riemann sum will be $\displaystyle\sum_{k=1}^{4} f(x_{k-1}) \cdot 1 = \sum_{k=1}^{4} (x_{k-1})^2$.

The right Riemann sum will be $\displaystyle\sum_{k=1}^{4} f(x_k) \cdot 1 = \sum_{k=1}^{4} (x_k)^2$.

The midpoint Riemann sum will be $\displaystyle\sum_{k=1}^{4} f\left(\frac{x_{k-1}+x_k}{2}\right) \cdot 1 = \sum_{k=1}^{4} \left(\frac{x_{k-1}+x_k}{2}\right)^2$.

5.1.13 It is an underestimate. If we use the right-hand side of each subinterval to determine the height of the rectangles, the height of each rectangle will be the minimum of f over the subinterval, so the sum of the areas of the rectangles will be less than the corresponding area under the curve.

5.1.14 It is an underestimate. If we use the left-hand side of each subinterval to determine the height of the rectangles, the height of each rectangle will be the minimum of f over the subinterval, so the sum of the areas of the rectangles will be less than the corresponding area under the curve.

5.1.15

a. On the first subinterval, the midpoint is 0.5, and $v(0.5) = 1.75$. On the 2nd subinterval, the midpoint is 1.5 and $v(1.5) = 7.75$. Continuing in this manner, we obtain the estimate to the displacement of

$$v(.5)\cdot 1 + v(1.5)\cdot 1 + v(2.5)\cdot 1 + v(3.5)\cdot 1 = 1.75 + 7.75 + 19.75 + 37.75 = 67\,\text{ft}.$$

b. This time the midpoints are at $0.25, 0.75, 1.25 \ldots$. Each subinterval has length $\frac{1}{2}$. Thus, the estimate is given by

$$\begin{aligned}
&v(0.25)\cdot 0.5 + v(0.75)\cdot 0.5 + v(1.25)\cdot 0.5 + v(1.75)\cdot 0.5 + v(2.25)\cdot 0.5 + v(2.75)\cdot 0.5 \\
&+ v(3.25)\cdot 0.5 + v(3.75)\cdot 0.5 \\
&= 0.5(1.1875 + 2.6875 + 5.6875 + 10.1875 + 16.1875 + 23.6875 + 32.6875 + 43.1875) \\
&= 0.5(135.5) = 67.75\,\text{ft..}
\end{aligned}$$

5.1.16

a. The midpoints are 2, 4, and 6. So the estimate is

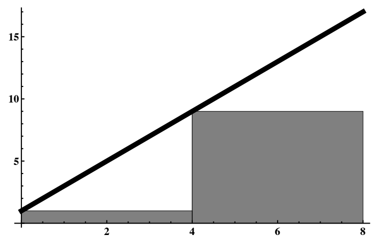
$$v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 \approx 37.085\,\text{ft}.$$

b. The midpoints are $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, $\frac{11}{2}$, and $\frac{13}{2}$. So the estimate is

$$v\left(\frac{3}{2}\right)+v\left(\frac{5}{2}\right)+v\left(\frac{7}{2}\right)+v\left(\frac{9}{2}\right)+v\left(\frac{11}{2}\right)+v\left(\frac{13}{2}\right) = \sqrt{15}+5+\sqrt{35}+3\sqrt{5}+\sqrt{55}+\sqrt{65} \approx 36.976\,\text{ft}.$$

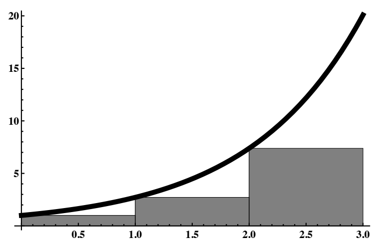
5.1.17

The left-hand grid points are 0 and 4. The length of each subinterval is $8/2 = 4$. So the left Riemann sum is given by $v(0)\cdot 4 + v(4)\cdot 4 = 4\cdot(1+9) = 40$ m.



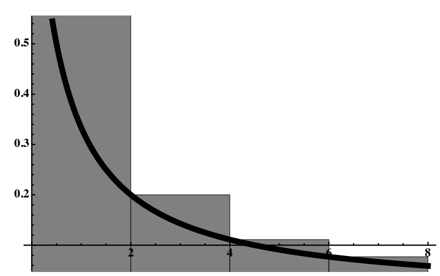
5.1.18

The left-hand grid points are 0, 1 and 2. The length of each subinterval is $3/3 = 1$. So the left Riemann sum is given by $v(0)\cdot 1 + v(1)\cdot 1 + v(2)\cdot 1 = 1 + e + e^2 \approx 11.1$ m.



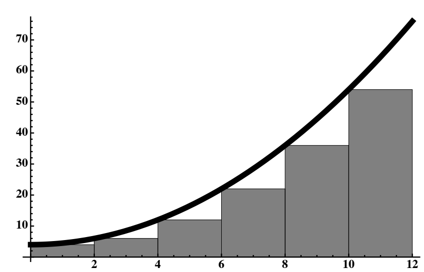
5.1.19

The left-hand grid points are 0, 2, 4, and 6. The length of each subinterval is 2. So the left Riemann sum is given by $v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 = 2\cdot\left(\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}\right) \approx 2.776$ m.



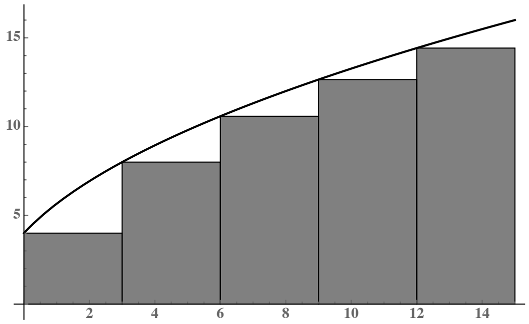
5.1.20

The left-hand grid points are 0, 2, 4, and 6, 8, and 10. The length of each subinterval is 2. So the left Riemann sum is given by $\sum_{k=1}^{6} v(2(k-1))\cdot 2 = 8 + 12 + 24 + 44 + 72 + 108 = 268$ ft.



5.1.21

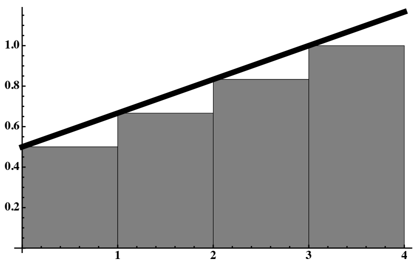
The left-hand grid points are 0, 3, 6, 9, and 12. The length of each subinterval is 3. So the left Riemann sum is given by $\sum_{k=1}^{5} v(3(k-1)) \cdot 3 = 12 + 24 + 12\sqrt{7} + 12\sqrt{10} + 12\sqrt{13} \approx 148.963$ mi.



5.1.22

The left-hand grid points are 0, 1, 2, and 3. The length of each subinterval is 1. So the left Riemann sum is given by

$$v(0)+v(1)+v(2)+v(3) = \frac{1}{2}+\frac{2}{3}+\frac{5}{6}+1 = 3\,\text{m}.$$

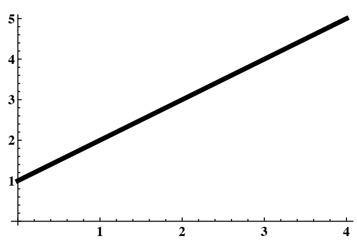


5.1.23 The left Riemann sum is given by $f(1)+f(2)+f(3)+f(4)+f(5) = 2+3+4+5+6 = 20$. The right Riemann sum is given by $f(2)+f(3)+f(4)+f(5)+f(6) = 3+4+5+6+7 = 25$.

5.1.24 The left Riemann sum is given by $f(1)+f(2)+f(3)+f(4) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{25}{12}$.
The right Riemann sum is given by $f(2)+f(3)+f(4)+f(5) = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{77}{60}$.

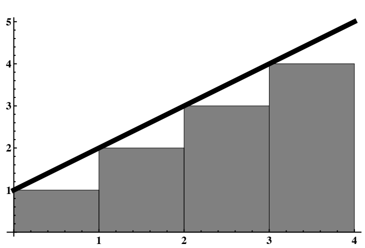
5.1.25

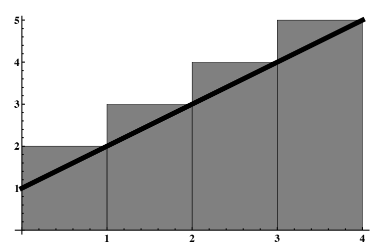
a.



b. We have $\Delta x = \frac{4-0}{4} = 1$. The grid points are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$.

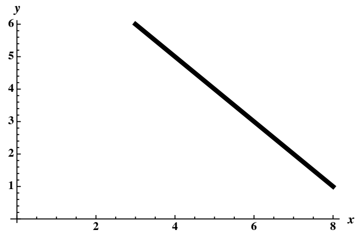
c.



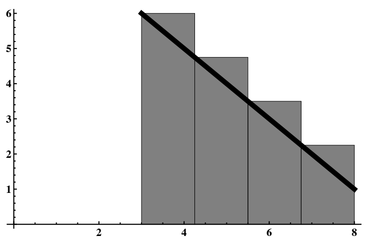


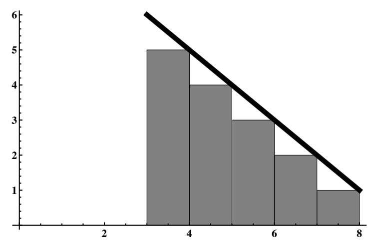
d. The left Riemann sum is $1\cdot 1+2\cdot 1+3\cdot 1+4\cdot 1 = 10$, which is an underestimate of the area under the curve. The right Riemann sum is $2\cdot 1+3\cdot 1+4\cdot 1+5\cdot 1 = 14$ which is an overestimate of the area under the curve.

5.1.26

a. 

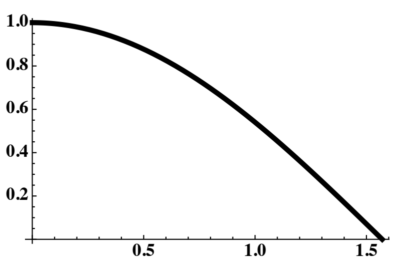
b. We have $\Delta x = \dfrac{8-3}{5} = 1$. The grid points are $x_0 = 3$, $x_1 = 4$, $x_2 = 5$, $x_3 = 6$, $x_4 = 7$, and $x_5 = 8$.

c. 

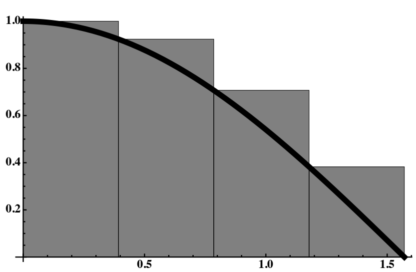


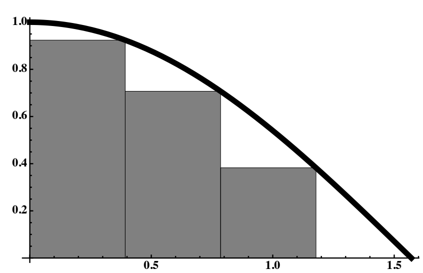
d. The left Riemann sum is $6 \cdot 1 + 5 \cdot 1 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 = 20$, which is an overestimate of the area under the curve. The right Riemann sum is $5 \cdot 1 + 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 15$ which is an underestimate of the area under the curve.

5.1.27

a. 

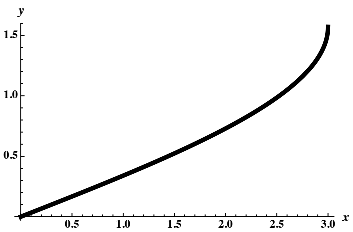
b. We have $\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}$. The grid points are $x_0 = 0$, $x_1 = \dfrac{\pi}{8}$, $x_2 = \dfrac{\pi}{4}$, $x_3 = \dfrac{3\pi}{8}$, and $x_4 = \dfrac{\pi}{2}$.

c. 

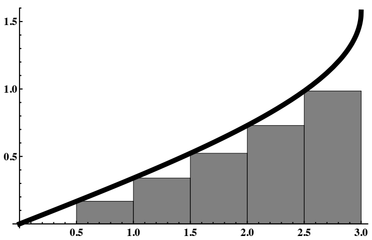


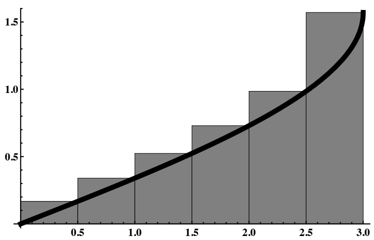
d. The left Riemann sum is $1 \cdot \dfrac{\pi}{8} + \cos(\pi/8) \cdot \dfrac{\pi}{8} + \cos(\pi/4) \cdot \dfrac{\pi}{8} + \cos(3\pi/8) \cdot \dfrac{\pi}{8} \approx 1.185$, which is an overestimate of the area under the curve. The right Riemann sum is $\cos(\pi/8) \cdot \dfrac{\pi}{8} + \cos(\pi/4) \cdot \dfrac{\pi}{8} + \cos(3\pi/8) \cdot \dfrac{\pi}{8} + 0 \cdot \dfrac{\pi}{8} \approx 0.791$ which is an underestimate of the area under the curve.

5.1.28

a. 

b. We have $\Delta x = \dfrac{3-0}{6} = \dfrac{1}{2}$. The grid points are $x_0 = 0$, $x_1 = \dfrac{1}{2}$, $x_2 = 1$, $x_3 = \dfrac{3}{2}$, $x_4 = 2$, $x_5 = \dfrac{5}{2}$, and $x_6 = 3$.

c. 



d. The left Riemann sum is

$$\frac{1}{2}\left(0 + \sin^{-1}(1/6) + \sin^{-1}(1/3) + \sin^{-1}(1/2) + \sin^{-1}(2/3) + \sin^{-1}(5/6)\right) \approx 1.373,$$

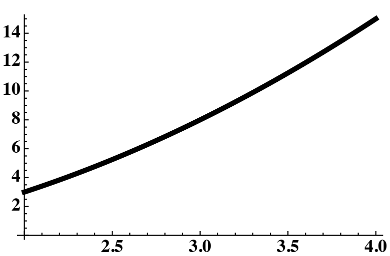
which is an underestimate of the area under the curve.

The right Riemann sum is

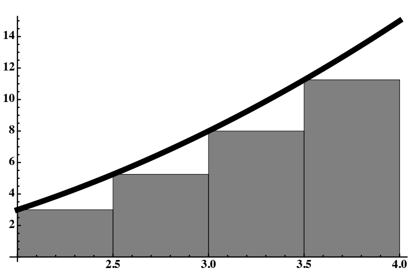
$$\frac{1}{2}\left(\sin^{-1}(1/6) + \sin^{-1}(1/3) + \sin^{-1}(1/2) + \sin^{-1}(2/3) + \sin^{-1}(5/6) + \sin^{-1}1\right) \approx 2.158,$$

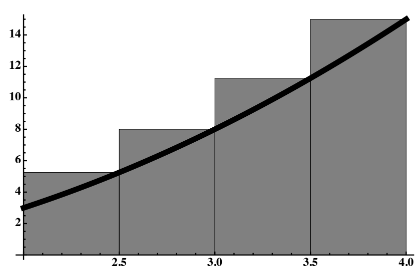
which is an overestimate of the area under the curve.

5.1.29

a. 

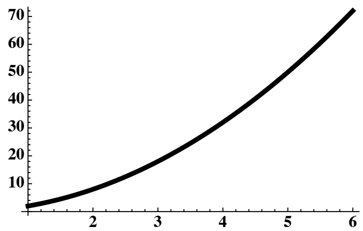
b. We have $\Delta x = \dfrac{4-2}{4} = \dfrac{1}{2}$. The grid points are $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, and $x_4 = 4$.

c. 

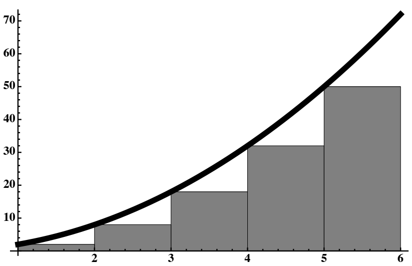


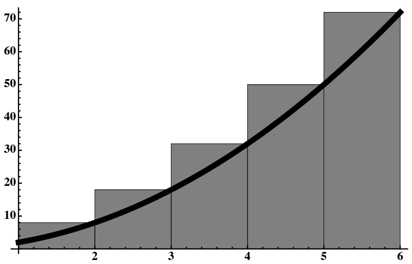
d. The left Riemann sum is $(3+5.25+8+11.25)\cdot 0.5 = 13.75$, which is an underestimate of the area under the curve. The right Riemann sum is $(5.25+8+11.25+15)\cdot 0.5 = 19.75$ which is an overestimate of the area under the curve.

5.1.30

a. 

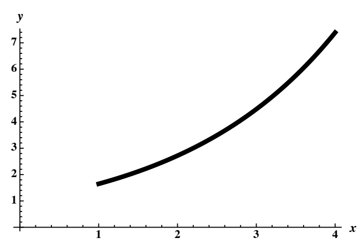
b. We have $\Delta x = \dfrac{6-1}{5} = 1$. The grid points are $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$, and $x_5 = 6$.

c. 

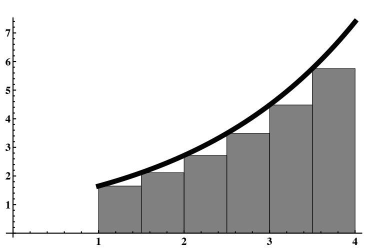


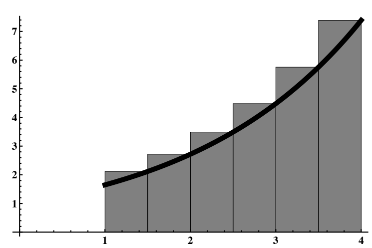
d. The left Riemann sum is $2 + 8 + 18 + 32 + 50 = 110$, which is an underestimate of the area under the curve. The right Riemann sum is $8 + 18 + 32 + 50 + 72 = 180$ which is an overestimate of the area under the curve.

5.1.31

a. 

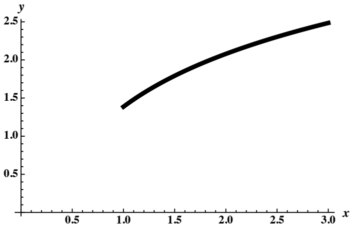
b. We have $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$. The grid points are $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$, $x_5 = 3.5$, and $x_6 = 4$.

c. 

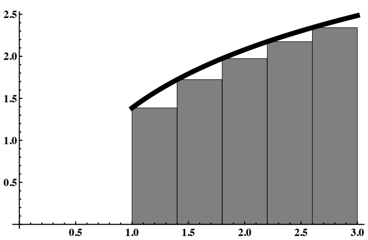


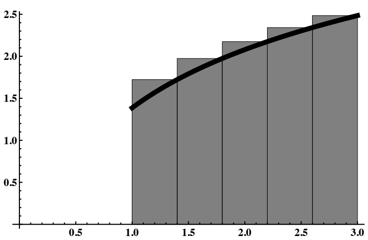
d. The left Riemann sum is $\frac{1}{2}\left(e^{1/2} + e^{3/4} + e + e^{5/4} + e^{3/2} + e^{7/4}\right) \approx 10.105$, which is an underestimate of the area under the curve. The right Riemann sum is $\dfrac{1}{2}\left(e^{3/4} + e + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right) \approx 12.975$ which is an overestimate of the area under the curve.

5.1.32

a. 

b. We have $\Delta x = \dfrac{3-1}{5} = \dfrac{2}{5} = 0.4$. The grid points are $x_0 = 1$, $x_1 = 1.4$, $x_2 = 1.8$, $x_3 = 2.2$, $x_4 = 2.6$, and $x_5 = 3$.

c. 



d. The left Riemann sum is

$$.4\,(\ln 4 + \ln 5.6 + \ln 7.2 + \ln 8.8 + \ln 10.4) \approx 3.840,$$

which is an underestimate of the area under the curve. The right Riemann sum is

$$0.4\,(\ln 5.6 + \ln 7.2 + \ln 8.8 + \ln 10.4 + \ln 12) \approx 4.279,$$

which is an overestimate of the area under the curve.

5.1.33 We have $\Delta x = 2$, so the midpoints are 1, 3, 5, 7, and 9. So the midpoint Riemann sum is $2(f(1) + f(3) + f(5) + f(7) + f(9)) = 670$.

5.1.34 We have $\Delta x = \pi/4$, so the midpoints are $\pi/8$, $3\pi/8$, $5\pi/8$, and $7\pi/8$. So the midpoint Riemann sum is $\frac{\pi}{4} \cdot (f(\pi/8) + f(3\pi/8) + f(5\pi/8) + f(7\pi/8)) \approx 2.013$.

5.1.35

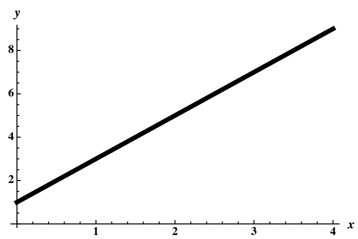
a. The left Riemann sum is $315(6) + 350(6) + 365(6) + 370(6) + 350(6) = 10,500$ m. The right Riemann sum is $350(6) + 365(6) + 370(6) + 350(6) + 290(6) = 10,350$ m. Answers will vary.

b. The left Riemann sum is more accurate.

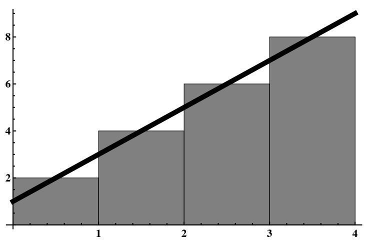
c. More accurate approximations are obtained by increasing the number of subintervals.

5.1.36 Felix fell approximately $\dfrac{1}{2}(20)(200) = 2000$ m.

5.1.37

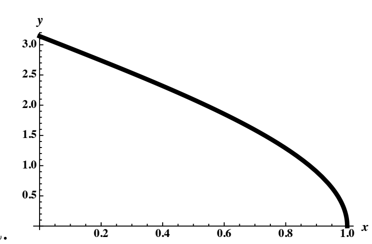
a. 

b. We have $\Delta x = \dfrac{4-0}{4} = 1$. The gridpoints are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$, so the midpoints are .5, 1.5, 2.5, and 3.5.

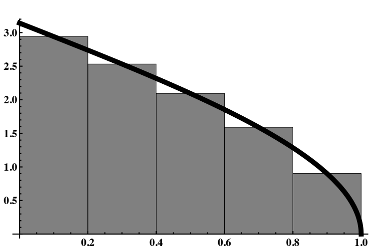
c. 

d. The midpoint Riemann sum is $1(2+4+6+8) = 20$.

5.1.38

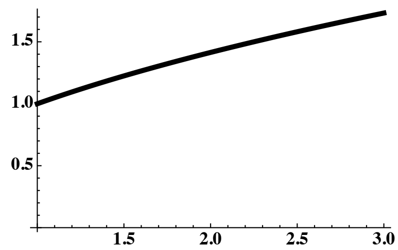
a. 

b. We have $\Delta x = \dfrac{1-0}{5} = 0.2$. The gridpoints are $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, and $x_5 = 1$, so the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9.

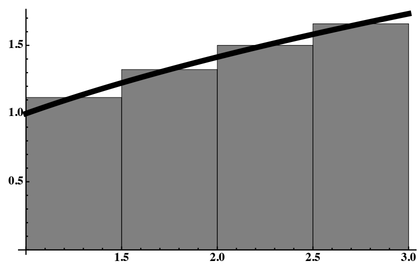
c. 

d. The midpoint Riemann sum is $.4(\cos^{-1} 0.1 + \cos^{-1} 0.3 + \cos^{-1} 0.5 + \cos^{-1} 0.7 + \cos^{-1} 0.9) \approx 2.012$.

5.1.39

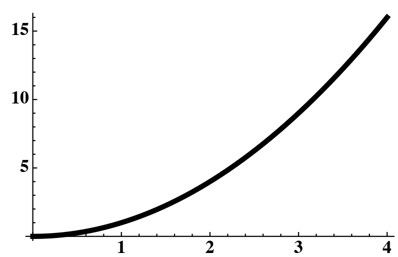
a. 

b. We have $\Delta x = \dfrac{3-1}{4} = \dfrac{1}{2}$. So the midpoints are 1.25, 1.75, 2.25, and 2.75.

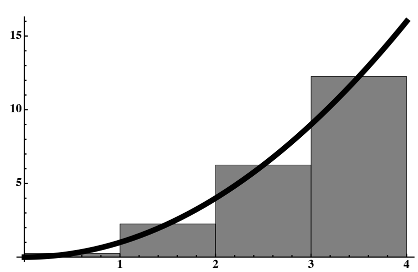
c. 

d. The midpoint Riemann sum is $.5(\sqrt{1.25}+\sqrt{1.75}+\sqrt{2.25}+\sqrt{2.75}) \approx 2.800$.

5.1.40

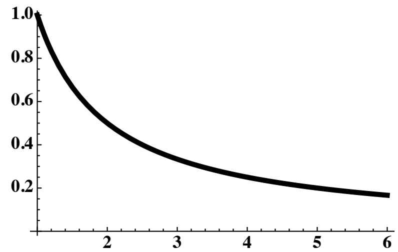
a. 

b. We have $\Delta x = \dfrac{4-0}{4} = 1$. So the midpoints are 0.5, 1.5, 2.5, and 3.5.

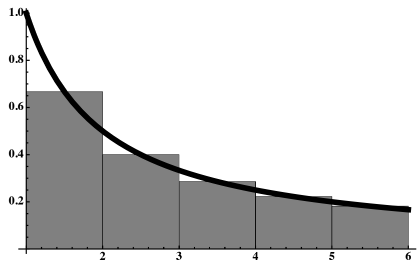
c. 

d. The midpoint Riemann sum is $0.5^2+1.5^2+2.5^2+3.5^2 = 21$.

5.1.41

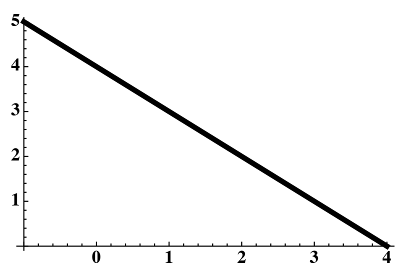
a. 

b. We have $\Delta x = \dfrac{6-1}{5} = 1$. So the midpoints are 1.5, 2.5, 3.5, 4.5, and 5.5.

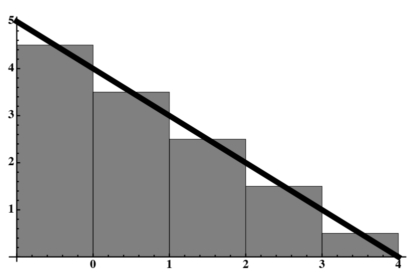
c. 

d. The midpoint Riemann sum is $2/3 + 2/5 + 2/7 + 2/9 + 2/11 \approx 1.756$.

5.1.42

a. 

b. We have $\Delta x = \dfrac{4-(-1)}{5} = 1$. So the midpoints are -0.5, 0.5, 1.5, 2.5, and 3.5.

c. 

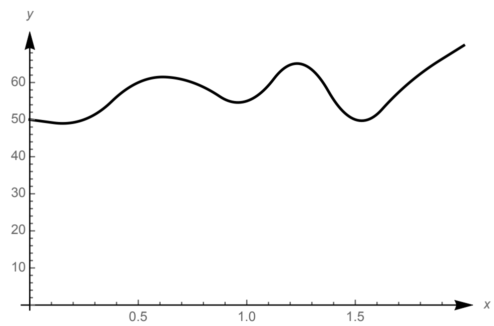
d. The midpoint Riemann sum is $(4-(-0.5)) + (4-0.5) + (4-1.5) + (4-2.5) + (4-3.5) = 20 - 7.5 = 12.5$.

5.1.43 Note that $\Delta x = \dfrac{2-0}{4} = .5$. So the left Riemann sum is given by $0.5(5+3+2+1) = 5.5$ and the right Riemann sum is given by $0.5(3+2+1+1) = 3.5$.

5.1.44 Note that $\Delta x = \dfrac{5-1}{8} = 0.5$. So the left Riemann sum is given by $0.5(0+2+3+2+2+1+0+2) = 6$ and the right Riemann sum is given by $0.5(2+3+2+2+1+0+2+3) = 7.5$.

5.1.45

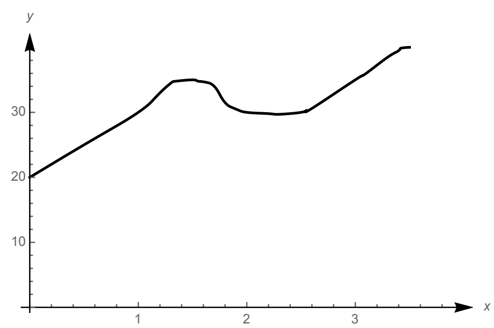
a.



b. With $n = 2$, we have $\Delta x = \frac{2-0}{2} = 1$, so the midpoints are .5 and 1.5. So the midpoint Riemann sum is $60 + 50 = 110$. For $n = 4$, we have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, so the midpoints are 0.25, 0.75, 1.25, and 1.75. The midpoint Riemann sum in this case is $0.5(50 + 60 + 65 + 60) = \frac{235}{2} = 117.5$.

5.1.46

a.



b. With $n = 2$, we have $\Delta x = \frac{4-0}{2} = 2$, so the midpoints are 1 and 3. So the midpoint Riemann sum is $2(30 + 35) = 130$. For $n = 4$, we have $\Delta x = \frac{4-0}{4} = 1$, so the midpoints are 0.5, 1.5, 2.5, and 3.5. The midpoint Riemann sum in this case is $25 + 35 + 30 + 40 = 130$.

5.1.47

a. $\displaystyle\sum_{k=1}^{5} k.$ b. $\displaystyle\sum_{k=1}^{6}(k+3).$ c. $\displaystyle\sum_{k=1}^{4} k^2.$ d. $\displaystyle\sum_{k=1}^{4}\frac{1}{k}.$

5.1.48

a. $\displaystyle\sum_{k=1}^{50}(2k-1).$ b. $\displaystyle\sum_{k=1}^{9}(5k-1).$ c. $\displaystyle\sum_{k=1}^{13}(5k-2).$ d. $\displaystyle\sum_{k=1}^{49}\frac{1}{k(k+1)}.$

5.1.49

a. $\displaystyle\sum_{k=1}^{10} k = 1 + 2 + 3 + \ldots + 10 = 55.$

b. $\displaystyle\sum_{k=1}^{6}(2k+1) = 3 + 5 + 7 + 9 + 11 + 13 = 48.$

c. $\displaystyle\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30.$

d. $\displaystyle\sum_{n=1}^{5}(1+n^2) = 2 + 5 + 10 + 17 + 26 = 60.$

e. $\displaystyle\sum_{m=1}^{3}\frac{2m+2}{3} = \frac{4}{3} + \frac{6}{3} + \frac{8}{3} = 6.$

f. $\displaystyle\sum_{j=1}^{3}(3j-4) = -1 + 2 + 5 = 6.$

g. $\displaystyle\sum_{p=1}^{5}(2p+p^2) = 3 + 8 + 15 + 24 + 35 = 85.$

h. $\displaystyle\sum_{n=0}^{4}\sin\frac{n\pi}{2} = 0 + 1 + 0 + (-1) + 0 = 0.$

5.1.50

a. $\displaystyle\sum_{k=1}^{45} k = \frac{45 \cdot 46}{2} = 45 \cdot 23 = 1035.$

b. $\displaystyle\sum_{k=1}^{45}(5k-1) = 5\sum_{k=1}^{45} k - \sum_{k=1}^{45} 1 = 5 \cdot 1035 - 45 = 5130.$

c. $\displaystyle\sum_{k=1}^{75} 2k^2 = 2\sum_{k=1}^{75} k^2 = 2 \cdot \frac{75 \cdot 76 \cdot 151}{6} = 286,900.$

d. $\displaystyle\sum_{n=1}^{50}(1+n^2) = \sum_{n=1}^{50} 1 + \sum_{n=1}^{50} n^2 = 50 + \frac{50 \cdot 51 \cdot 101}{6} = 50 + 42925 = 42975.$

e. $\displaystyle\sum_{m=1}^{75} \frac{2m+2}{3} = \frac{2}{3}\sum_{m=1}^{75} m + \frac{2}{3}\sum_{m=1}^{75} 1 = \frac{2}{3} \cdot \frac{75 \cdot 76}{2} + \frac{2}{3} \cdot 75 = 1900 + 50 = 1950.$

f. $\displaystyle\sum_{j=1}^{20}(3j-4) = 3\sum_{j=1}^{20} j - \sum_{j=1}^{20} 4 = 3 \cdot \frac{20 \cdot 21}{2} - 20 \cdot 4 = 550.$

g. $\displaystyle\sum_{p=1}^{35}(2p+p^2) = 2\sum_{p=1}^{35} p + \sum_{p=1}^{35} p^2 = 2 \cdot \frac{35 \cdot 36}{2} + \frac{35 \cdot 36 \cdot 71}{6} = 16170.$

h. $\displaystyle\sum_{n=0}^{40}(n^2+3n-1) = \sum_{n=0}^{40} n^2 + \sum_{n=0}^{40} 3n - \sum_{n=0}^{40} 1 = \frac{40 \cdot 41 \cdot 81}{6} + 3 \cdot \frac{40 \cdot 41}{2} - 41 = 24559.$

5.1.51 Note that $\Delta x = \dfrac{1}{10}$, and $x_k = a + k\Delta x = \dfrac{k}{10}$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{39} 3\sqrt{k/10} \cdot (1/10) \approx 15.6809.$

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{40} 3\sqrt{k/10} \cdot (1/10) \approx 16.2809.$

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{39} 3\sqrt{(1/20)+(k/10)} \cdot (1/10) \approx 16.0055.$

b. It appears that the actual area is about 16.

5.1.52 Note that $\Delta x = \dfrac{1}{25}$, and $x_k = -1 + k\Delta x = -1 + (k/25)$. So $f(x_k) = (-1+(k/25))^2 + 1$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{49}[(-1+(k/25))^2+1] \cdot (1/25) \approx 2.667.$

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{50}[(-1+(k/25))^2+1] \cdot (1/25) \approx 2.667.$

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{49}[(-(49/50)+(k/25))^2+1] \cdot (1/25) \approx 2.666.$

b. It appears that the actual area is about $2 + 2/3$.

5.1.53 Note that $\Delta x = \dfrac{1}{25}$, and $x_k = 2 + k\Delta x = 2 + (k/25)$. So $f(x_k) = (2 + (k/25))^2 - 1$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{74}[(2 + (k/25))^2 - 1] \cdot (1/25) \approx 35.5808$.

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{75}[(2 + (k/25))^2 - 1] \cdot (1/25) \approx 36.2408$.

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{74}[((61/30) + (k/25))^2 - 1] \cdot (1/25) \approx 35.9996$.

b. It appears that the actual area is about 36.

5.1.54 Note that $\Delta x = \dfrac{\pi}{240}$, and $x_k = k\Delta x = (k\pi/240)$. So $f(x_k) = \cos(k\pi/120)$.

a. The left Riemann sum is given by $\displaystyle\sum_{k=0}^{59}\cos(k\pi/120) \cdot (\pi/240) \approx 0.507$.

The right Riemann sum is given by $\displaystyle\sum_{k=1}^{60}\cos(k\pi/120) \cdot (\pi/240) \approx 0.493$.

The midpoint Riemann sum is given by $\displaystyle\sum_{k=0}^{59}\cos((\pi/240) + k\pi/120) \cdot (\pi/240) \approx 0.500$.

b. It appears that the actual area is about 0.5.

5.1.55 $\Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}$. $x_k = -1 + k\Delta x = -1 + \dfrac{2k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2}{n}\sum_{k=1}^{n}\left(12 - 3\left(-1 + \frac{2k}{n}\right)^2\right).$$

n	Right Riemann Sum
10	21.96
30	21.9956
60	21.9989
80	21.9994

The sum approaches 22.

5.1.56 $\Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}$. $x_k = -1 + k\Delta x = -1 + \dfrac{2k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(-1 + \frac{2k}{n}\right)^2 + 1\right).$$

n	Right Riemann Sum
10	4.04
30	4.00444
60	4.00111
80	4.00063

The sum approaches 4.

5.1.57 $\Delta x = \dfrac{\pi - (-\pi)}{n} = \dfrac{2\pi}{n}$. $x_k = -\pi + k\Delta x = -\pi + \dfrac{2\pi k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{2\pi}{n}\sum_{k=1}^{n}\left(\frac{1-\cos\left(-\pi+\frac{2\pi k}{n}\right)}{2}\right).$$

n	Right Riemann Sum
10	3.14159
30	3.14159
60	3.14159
80	3.14159

The sum approaches π.

5.1.58 $\Delta x = \dfrac{2-(-2)}{n} = \dfrac{4}{n}$. $x_k = -2 + k\Delta x = -2 + \dfrac{4k}{n}$. The right Riemann sum is given by

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{4}{n}\ln 2\sum_{k=1}^{n}\left(2^{-2+\frac{4k}{n}} + 2^{-(-2+\frac{4k}{n})}\right).$$

n	Right Riemann Sum
10	7.54798
30	7.50534
60	7.50133
80	7.50075

The sum approaches 7.5.

5.1.59

a. True. Because the curve is a straight line, the region under the curve and over each subinterval is a trapezoid. The formula for the area of such a trapezoid over $[x_i, x_{i+1}]$ is $\dfrac{f(x_i)+f(x_{i+1})}{2}\cdot\Delta x = \dfrac{2x_i+5+2x_{i+1}+5}{2}\cdot\Delta x = (x_i+x_{i+1}+5)\Delta x$ and the area given by using the midpoint formula is $f\left(\dfrac{x_i+x_{i+1}}{2}\right)\Delta x = (x_i+x_{i+1}+5)\Delta x$. So the area under the curve is exactly given by the midpoint Riemann sum. Note that this holds for any straight line.

b. False. The left Riemann sum will underestimate the area under an increasing function.

c. True. The value of f at the midpoint will always be between the value of f at the endpoints, if f is monotonic increasing or monotonic decreasing.

5.1.60

a. Note that if $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$, which represents a circle of radius one. Note that for the original function $y > 0$ for all x, so this represents the top semicircle.

b. We have $\displaystyle\sum_{k=1}^{25} f(x_k^*)\frac{2}{25}$ where x_k^* represents the midpoint of the kth subinterval. This sum is

$$\sum_{k=0}^{24}\sqrt{1-\left(-.96+\frac{2k}{25}\right)^2}\cdot\frac{2}{25} \approx 1.575.$$

c. We have $\displaystyle\sum_{k=1}^{75} f(x_k^*)\frac{2}{75}$ where x_k^* represents the midpoint of the kth subinterval. This sum is

$$\sum_{k=0}^{74} \sqrt{1 - \left((-98\overline{6}) + \frac{2k}{75}\right)^2} \cdot \frac{2}{75} \approx 1.572.$$

d. The sums approach $\pi/2$ as $n \to \infty$. Note that this confirms what we already know about the area of half a circle of radius one.

5.1.61 $\displaystyle\sum_{k=1}^{50} \left(\frac{2k}{25} + 1\right) \cdot \frac{2}{25} = 12.16.$

5.1.62 $\displaystyle\sum_{k=0}^{39} e^{k(\ln 2)/40} \cdot \frac{\ln 2}{40} \approx 0.991.$

5.1.63 $\displaystyle\sum_{k=0}^{31} \left(3 + \frac{1}{8} + \frac{k}{4}\right)^3 \cdot \frac{1}{4} \approx 3639.125.$

5.1.64 $\displaystyle\sum_{k=0}^{49} \left[1 + \cos\left(\pi\left(\frac{1}{50} + \frac{k}{25}\right)\right)\right] \cdot \frac{1}{25} = 2.$

5.1.65 This is the right Riemann sum for f on the interval $[1, 5]$ for $n = 4$.

5.1.66 This is the right Riemann sum for f on the interval $[2, 6]$ for $n = 4$.

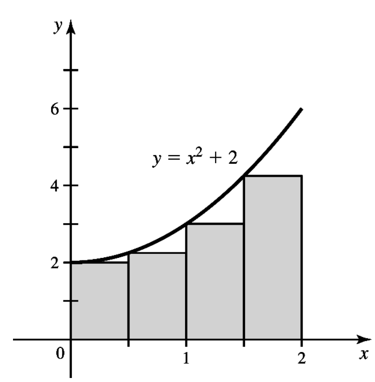
5.1.67 This is the midpoint Riemann sum for f on the interval $[2, 6]$ for $n = 4$.

5.1.68 This is the right Riemann sum for f on the interval $[1.5, 5.5]$ for $n = 8$.

5.1.69 For all of the calculations below, we have $\Delta x = \dfrac{1}{2}$, and grid points $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 2.5$, and $x_4 = 2$.

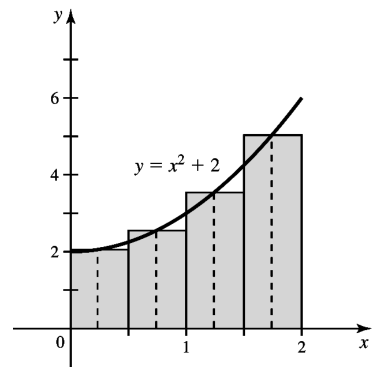
a.

The left Riemann sum is given by $\frac{1}{2}\left(f(0) + f(0.5) + f(1) + f(1.5)\right)$ which is equal to $\frac{1}{2}(2 + 2.25 + 3 + 4.25) = 5.75$.



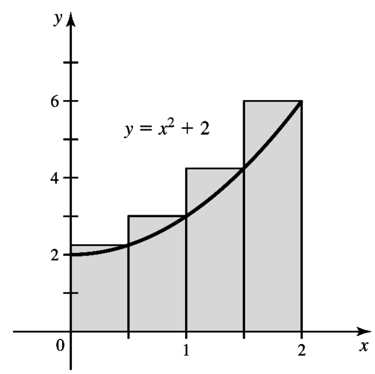
b.

The midpoint Riemann sum is given by $\frac{1}{2}(f(.25)+f(.75)+f(1.25)+f(1.75))$ which is equal to $\frac{1}{2}(2.0625+2.5625+3.5625+5.0625) = 6.625$.



c.

The right Riemann sum is given by $\frac{1}{2}(f(0.5)+f(1)+f(1.5)+f(2))$ which is equal to $\frac{1}{2}(2.25+3+4.25+6) = 7.75$.



5.1.70

a. The object's velocity decreases during the first second, then remains constant between time $t = 1$ and $t = 3$, and then steadily increases until $t = 4$, and then stays constant after that.

b. The displacement is given by the area under the curve, which between $t = 0$ and $t = 3$ is 35, so the displacement is 35 meters.

c. Between $t = 3$ and $t = 5$ the area under the curve is 50, so the displacement is 50 meters.

d. Between $t = 0$ and $t = 4$ the displacement is 55, and between 4 and t for $t > 4$, the displacement is $30(t-4)$. So the displacement between 0 and t for $t > 4$ is $55 + 30(t-4)$.

5.1.71

a. During the first second, the velocity steadily increases from 0 to 20, then it remains constant until $t = 3$. From $t = 3$ until $t = 5$ it steadily decreases, and then remains constant until $t = 6$.

b. Between $t = 0$ and $t = 2$ the area under the curve is $\frac{1}{2} \cdot 1 \cdot 20 + 1 \cdot 20 = 30$.

c. Between $t = 2$ and $t = 5$ the displacement is the sum of the area of a rectangle with area 20 and a trapezoid with area 30, so the displacement is 50 meters.

d. Between $t = 0$ and $t = 5$ the displacement is 80. Between $t = 5$ and any time $t \geq 5$ the displacement is $10(t-5)$ so the displacement between $t = 0$ and $t \geq 5$ is $80 + 10(t-5)$.

5.1.72

a. Between 0 and 4, the area under the curve is given by $\frac{1}{2} \cdot 4000 \cdot 4 = 8000$ cubic feet.

b. Between 8 and 10, the area under the curve is given by $2 \cdot 5000 = 10,000$ cubic feet.

c. Between 4 and 6 the amount is 9500 cubic feet, which is more than between 0 and 4.

d. When we multiply $\text{ft}^3/\text{hr} \cdot \text{hr}$ the result is ft^3.

5.1.73

a. Between 0 and 5, the area under the curve is given by the area of a square of area 4 and the area of a trapezoid of area 10.5, so the total area is 14.5.

b. Between 5 and 10, the area under the curve is given by the area of a trapezoid of area 5.5 and the area of a rectangle of area $4 \cdot 6 = 24$, so the total area is 29.5.

c. The mass of the entire rod would be the total area under the curve from 0 to 10, which would be $14.5 + 29.5 = 44$ grams.

d. At $x = \dfrac{19}{3}$ there is a mass of 22 on each side. Note that from 0 to 6 the mass is 20 grams, so the center of mass is a little greater than 6.

5.1.74 If $0 \le t \le 1.5$, the displacement is $40t$. If $1.5 \le t \le 3$, the displacement is $60 + 50(t - 1.5)$.

Thus, $d(t) = \begin{cases} 40t & \text{if } 0 \le t \le 1.5, \\ 50t - 15 & \text{if } 1.5 \le t \le 3. \end{cases}$

5.1.75 If $0 \le t \le 2$, the displacement is $30t$. If $2 \le t \le 2.5$, the displacement is $60 + 50(t - 2)$. If $2.5 \le t \le 3$, the displacement is $85 + 44(t - 2.5)$.

Thus, $d(t) = \begin{cases} 30t & \text{if } 0 \le t \le 2, \\ 50t - 40 & \text{if } 2 \le t \le 2.5 \\ 44t - 25 & \text{if } 2.5 \le t \le 3. \end{cases}$

5.1.76 Using the left Riemann sum

$$\sum_{k=0}^{n-1} \left| 25 - \left(\frac{10k}{n}\right)^2 \right| \cdot \frac{10}{n},$$

we have

n	16	32	64
A_n	234.375	242.188	246.094

It appears that the areas are approaching 250.

5.1.77 Using the left Riemann sum

$$\sum_{k=0}^{n-1} \left| 1 - \left(-1 + \frac{3k}{n}\right)^3 \right| \cdot \frac{3}{n},$$

we have

n	16	32	64
A_n	4.33054	4.52814	4.63592

It appears that the areas are approaching 4.75.

5.1.78 Because the function f is constant, its value is c at each grid point. Thus the left Riemann sum is

$$\sum_{k=0}^{n-1} f(x_k)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=0}^{n-1} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

For the right Riemann sum we have

$$\sum_{k=1}^{n} f(x_k)\cdot\frac{b-a}{n} = \sum_{k=1}^{n} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=1}^{n} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

For the midpoint Riemann sum we have

$$\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{2n}+\frac{(b-a)k}{n}\right)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1} c\cdot\frac{b-a}{n} = \frac{c(b-a)}{n}\cdot\sum_{k=0}^{n-1} 1 = \frac{c(b-a)}{n}\cdot n = c(b-a).$$

So all three rules give the exact area of $(b-a)\cdot c$.

5.1.79 The midpoint Riemann sum gives

$$\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{2n}+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n} = \sum_{k=0}^{n-1}\left(m\left(a+\frac{b-a}{2n}+\frac{k(b-a)}{n}\right)+c\right)\cdot\frac{b-a}{n} =$$

$$m\cdot a\cdot n\cdot\frac{b-a}{n}+\frac{m(b-a)^2 n}{2n^2}+\frac{(n-1)n}{2}\cdot\frac{m(b-a)^2}{n^2}+\frac{cn(b-a)}{n} =$$

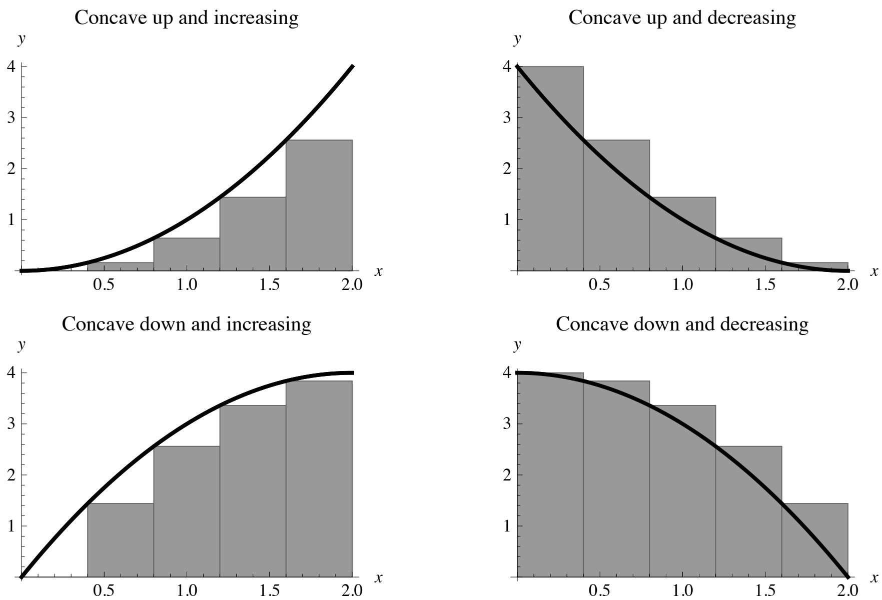
$$m\cdot a\cdot(b-a)+\frac{m(b-a)^2}{2n}+\frac{m(b-a)^2}{2}-\frac{m(b-a)^2}{2n}+c(b-a) =$$

$$m\cdot a\cdot(b-a)+\frac{m(b-a)^2}{2}+c(b-a) = (b-a)\cdot\left(\frac{m(a+b)}{2}+c\right).$$

This proves that the midpoint Riemann sum is independent of n. Because the region in question is a trapezoid, we know that the exact area is given by the width of the subinterval times the average value at the endpoints, which is

$$(b-a)\left(\frac{f(a)+f(b)}{2}\right) = (b-a)\left(\frac{ma+c+mb+c}{2}\right) = (b-a)\left(\frac{m(a+b)}{2}+c\right).$$

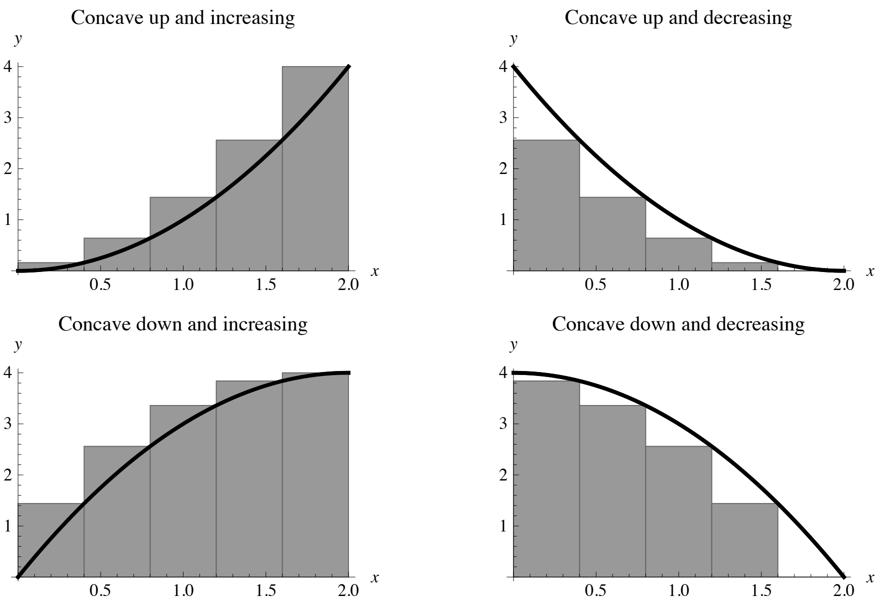
5.1.80 For a function that is concave up and increasing, each rectangle of the left Riemann sum will lie wholly below the curve, since the value of the function at the left edge of the rectangle will be smaller than at any other point in the rectangle. Thus this will be an underestimate. For a function that is concave up and decreasing, however, each rectangle will have its top edge above the curve, since the value of the function at the left edge will be larger than at any other point in the rectangle. Thus this will be an overestimate. For a function that is concave down and increasing, each rectangle of the left Riemann sum will lie wholly below the curve, since the value of the function at the left edge of the rectangle will be smaller than at any other point in the rectangle. Thus this will be an underestimate. Finally, for a function that is concave down and decreasing, each rectangle will have its top edge above the curve, since the value of the function at the left edge will be larger than at any other point in the rectangle. Thus this will be an overestimate. Graphs of each of the four situations are below:



So the answer is

	Increasing on $[a, b]$	Decreasing on $[a, b]$
Concave up on $[a, b]$	Underestimate	Overestimate
Concave down on $[a, b]$	Underestimate	Overestimate

5.1.81 For a function that is concave up and increasing, each rectangle of the right Riemann sum will have its top edge above the curve, since the value of the function at the right edge of the rectangle will be larger than at any other point in the rectangle. Thus this will be an overestimate. For a function that is concave up and decreasing, however, each rectangle will lie wholly below the curve, since the value of the function at the right edge will be smaller than at any other point in the rectangle. Thus this will be an underestimate. For a function that is concave down and increasing, each rectangle of the right Riemann sum will have its top edge above the curve, since the value of the function at the right edge of the rectangle will be larger than at any other point in the rectangle. Thus this will be an overestimate. Finally, for a function that is concave down and decreasing, however, each rectangle will lie wholly below the curve, since the value of the function at the right edge will be smaller than at any other point in the rectangle. Thus this will be an underestimate. Graphs of each of the four situations are below:



So the answer is

	Increasing on $[a, b]$	Decreasing on $[a, b]$
Concave up on $[a, b]$	Overestimate	Underestimate
Concave down on $[a, b]$	Overestimate	Underestimate

5.2 Definite Integrals

5.2.1 The net area is the difference between the area above the x-axis and below the curve, and below the x-axis and above the curve.

5.2.2 When the function is strictly above the x-axis, the net area is equal to the area. The net area differs from the area when the function dips below the x-axis so that the area below the x-axis and above the curve is nonzero.

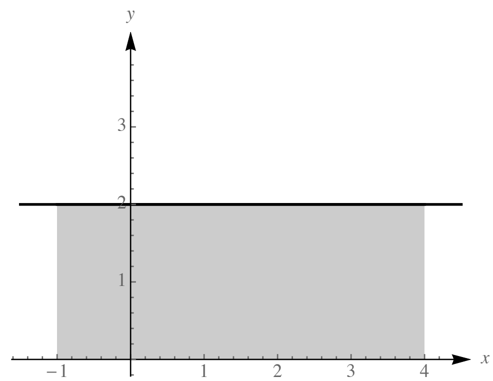
5.2.3 The left Riemann sum is: $40(1) + 30(1) + 20(1) + 0(1) - 10(1) - 20(1) = 60$. The right Riemann sum is $30(1) + 20(1) + 0(1) - 10(1) - 20(1) - 20(1) = 0$.

5.2.4 The left Riemann sum is: $-40(2)+0(2)+20(2)+50(2) = 60$. The right Riemann sum is $0(2)+20(2)+50(2) + 20(2) = 180$. The midpoint Riemann sum is $(-25)(2) + 10(2) + 30(2) + 45(2) = 120$.

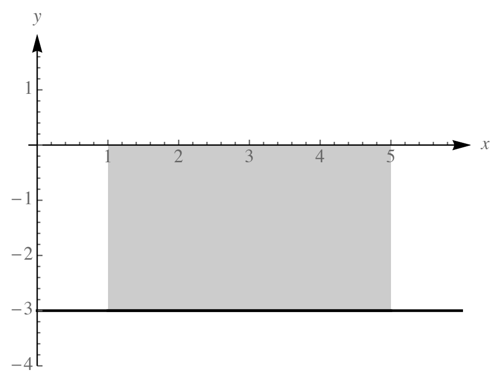
5.2.5 The left Riemann sum is: $2(2)-3(2)-5(2) = -12$. The right Riemann sum is $(-3)(2)-5(2)-1(2) = -18$. The midpoint Riemann sum is $-2(2) - 4(2) - 2(2) = -16$.

5.2.6 The left Riemann sum is: $(-5)(2) - 3(2) + 2(2) + 5(2) = -2$. The right Riemann sum is $(-3)2 + 2(2) + 5(2) - 1(2) = 6$. The midpoint Riemann sum is $-4(2) + 0(2) + 4(2) + 0(2) = 0$.

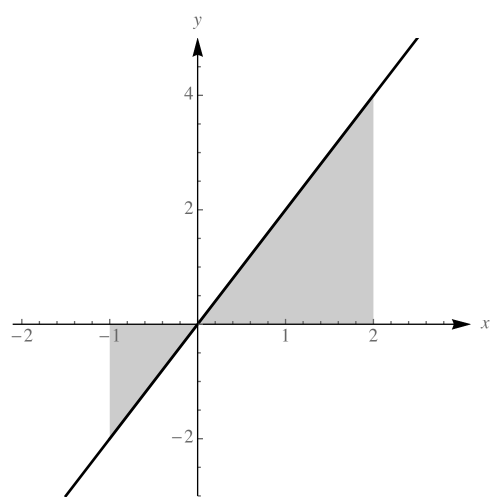
5.2.7 The integral represents the area of a 5 by 2 rectangle, so the value is 10.



5.2.8 The integral represents the negative of the area of a 4 by 3 rectangle, so the value is -12.



5.2.9 The integral represents the difference in areas of a triangle with base 2 and height 4 (so its area is 4) and one with base 1 and height 2 (so its area is 1), so the value of the integral is $4 - 1 = 3$.



5.2.10 $\displaystyle\int_1^3 (2f(x) - 4g(x))\,dx = 2\int_1^3 f(x)\,dx - 4\int_1^3 g(x)\,dx = 2(10) - 4(-20) = 100.$

$\displaystyle\int_3^1 (2f(x) - 4g(x))\,dx = -\int_1^3 (2f(x) - 4g(x))\,dx = -100.$

5.2.11 Because each of the functions $\sin x$ and $\cos x$ have the same amount of area above the x-axis as below between 0 and 2π, these both have value 0.

5.2.12 The greek letter $\sum$ and the integral sign $\displaystyle\int$ both remind us of the letter S, which stands for sum. The differential dx is analogous to Δx, helping us think of a small width. In both cases, the product of some form of $f(x)$ with either dx or Δx should make us think of an area – a height times a width. So both symbols are evocative of a sum of areas of rectangles, or a limit of such things.

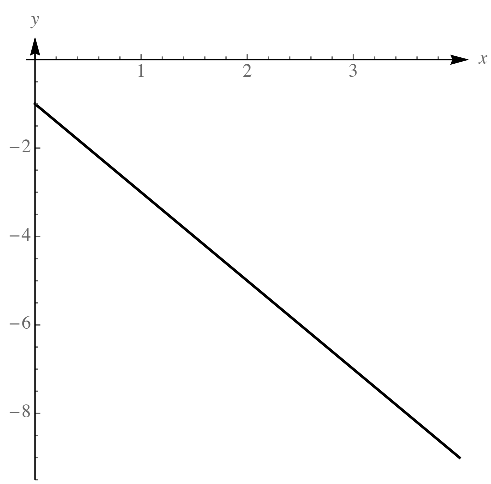
5.2.13 Because a region "from $x = a$ to $x = a$" has no width, its area is zero. This is akin to asking for the area of a one-dimensional object.

5.2.14 $\displaystyle\int_1^6 (2x^3 - 4x)\,dx = \int_1^6 2x^3\,dx - 4\int_1^6 x\,dx.$

5.2.15 This integral represents the area under $y = x$ between $x = 0$ and $x = a$, which is a right triangle. The length of the base of the triangle is a and the height is a, so the area is $\frac{1}{2} \cdot a^2$, so $\displaystyle\int_0^a x\,dx = \frac{a^2}{2}$.

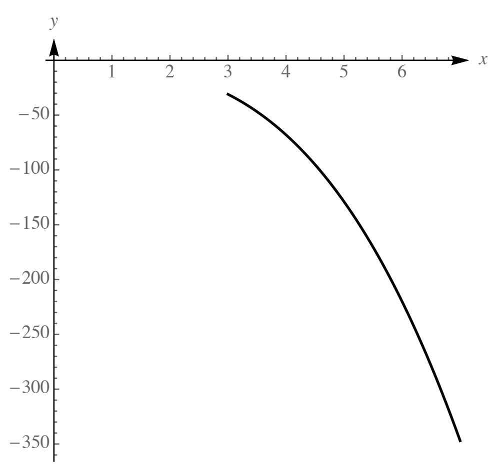
5.2.16 Because the function $|f|$ never goes below the x axis, the definite integral of $|f|$ does represent the area between $|f|$ and the x-axis. If this area is zero, then f must strictly lie on the x axis, so f must be the constant function with value 0.

5.2.17



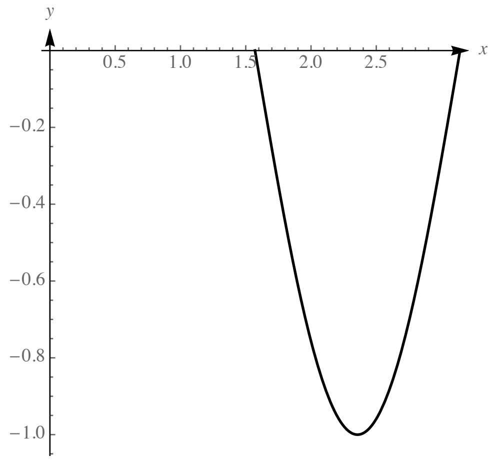
The left Riemann sum is $f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = -1 - 3 - 5 - 7 = -16$.
The right Riemann sum is $f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = -3 - 5 - 7 - 9 = -24$.
The midpoint Riemann sum is $f(.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = -2 - 4 - 6 - 8 = -20$.

5.2.18



We have $\Delta x = \dfrac{7-3}{4} = 1$. The left Riemann sum is $f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1 + f(6) \cdot 1 = -31 - 68 - 129 - 220 = -448$.
The right Riemann sum is $f(4) \cdot 1 + f(5) \cdot 1 + f(6) \cdot 1 + f(7) \cdot 1 = -68 - 129 - 220 - 347 = -764$.
The midpoint Riemann sum is $f(3.5) \cdot 1 + f(4.5) \cdot 1 + f(5.5) \cdot 1 + f(6.5) \cdot 1 = -46.875 - 95.125 - 170.375 - 278.625 = -591$.

5.2.19

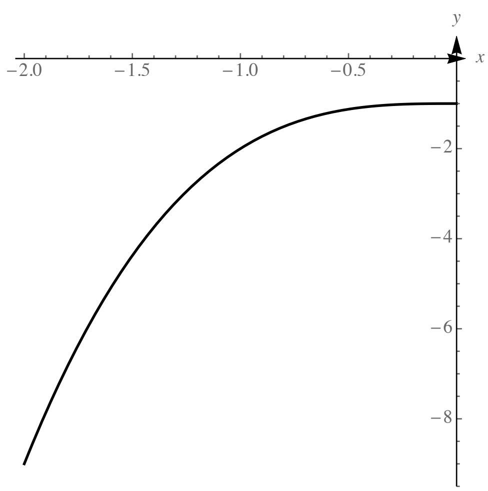


We have $\Delta x = \dfrac{\pi/2}{4} = \dfrac{\pi}{8}$. The left Riemann sum is $f(\pi/2)\cdot\dfrac{\pi}{8}+f(5\pi/8)\cdot\dfrac{\pi}{8}+f(6\pi/8)\cdot\dfrac{\pi}{8}+f(7\pi/8)\cdot\dfrac{\pi}{8} = \left(0-\sqrt{2}/2-1-\sqrt{2}/2\right)\cdot\dfrac{\pi}{8} = \dfrac{\pi}{8}\cdot(-1-\sqrt{2}) \approx -0.948$.

The right Riemann sum is $f(5\pi/8)\cdot\dfrac{\pi}{8} + f(6\pi/8)\cdot\dfrac{\pi}{8} + f(7\pi/8)\cdot\dfrac{\pi}{8} + f(\pi)\cdot\dfrac{\pi}{8} = \left(-\sqrt{2}/2-1-\sqrt{2}/2-0\right)\cdot\dfrac{\pi}{8} = \dfrac{\pi}{8}\cdot(-1-\sqrt{2}) \approx -0.948$.

The midpoint Riemann sum is $f(9\pi/16)\cdot\dfrac{\pi}{8} + f(11\pi/16)\cdot\dfrac{\pi}{8} + f(13\pi/16)\cdot\dfrac{\pi}{8} + f(15\pi/16)\cdot\dfrac{\pi}{8} \approx \dfrac{\pi}{8}\cdot(-0.382683-0.92388-0.92388-0.382683) \approx -1.026$.

5.2.20

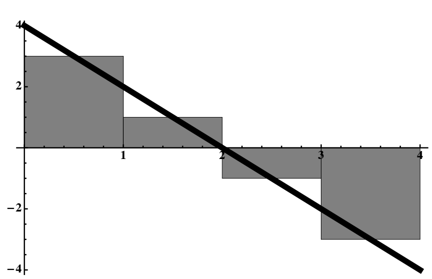


We have $\Delta x = \dfrac{2}{4} = \dfrac{1}{2}$. The left Riemann sum is $f(-2)\cdot\dfrac{1}{2}+f(-1.5)\cdot\dfrac{1}{2}+f(-1)\cdot\dfrac{1}{2}+f(-.5)\cdot\dfrac{1}{2} = .5(-9-4.375-2-1.125) = -8.25$.

The right Riemann sum is $f(-1.5)\cdot\dfrac{1}{2}+f(-1)\cdot\dfrac{1}{2}+f(-.5)\cdot\dfrac{1}{2}+f(0)\cdot\dfrac{1}{2} = .5(-4.375-2-1.125-1) = -4.25$.

The midpoint Riemann sum is $f(-1.75)\cdot\dfrac{1}{2} + f(-1.25)\cdot\dfrac{1}{2} + f(-.75)\cdot\dfrac{1}{2} + f(-.25)\cdot\dfrac{1}{2} = .5(-6.35938-2.95313-1.42188-1.10563) \approx -5.875$.

5.2.21

a. 

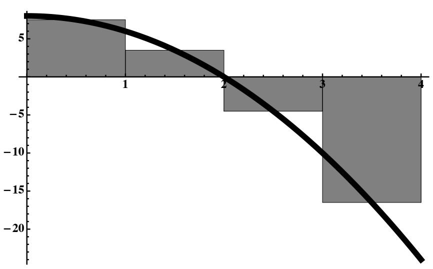
b. The left Riemann sum $\displaystyle\sum_{k=0}^{3} f(x_k)\cdot 1 = 4$.

The right Riemann sum $\displaystyle\sum_{k=1}^{4} f(x_k)\cdot 1 = -4$.

The midpoint Riemann sum $\displaystyle\sum_{k=1}^{4} f(x_k^*)\cdot 1 = 0$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < 2$ and negatively when $x_k > 2$.

5.2.22

a. 

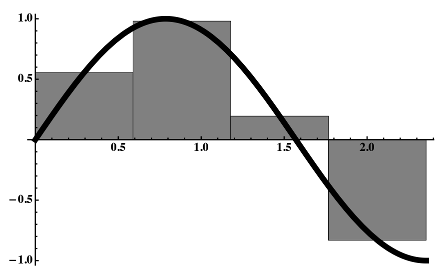
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k)\cdot 1 = 4$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k)\cdot 1 = -28$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right)\cdot 1 = -10$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < 2$ and negatively when $x_k > 2$.

5.2.23

a. 

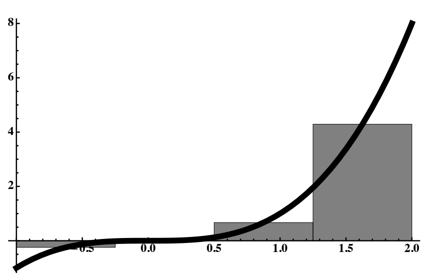
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k)\cdot \frac{3\pi}{16} \approx 0.7353$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k)\cdot \frac{3\pi}{16} \approx 0.146$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right)\cdot \frac{3\pi}{16} \approx 0.530$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k < \frac{\pi}{2}$ and negatively when $x_k > \frac{\pi}{2}$.

5.2.24

a. 

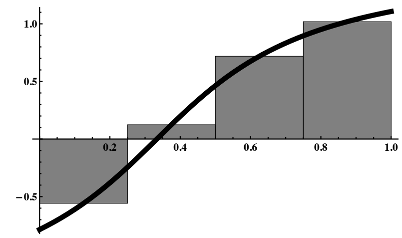
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k)\cdot \frac{3}{4} \approx 0.797$.

The right Riemann sum $\sum_{k=1}^{4} f(x_k)\cdot \frac{3}{4} \approx 7.547$.

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right)\cdot \frac{3}{4} \approx 3.539$.

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 0$ and negatively when $x_k < 0$.

5.2.25

a. 

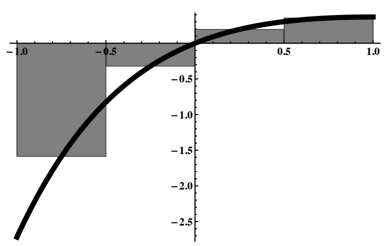
b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{1}{4} \approx 0.082.$

The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{1}{4} \approx 0.555.$

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{1}{4} \approx 0.326.$

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 1/3$ and negatively when $x_k < 1/3$.

5.2.26

a. 

b. The left Riemann sum $\sum_{k=0}^{3} f(x_k) \cdot \frac{1}{2} \approx -1.620.$

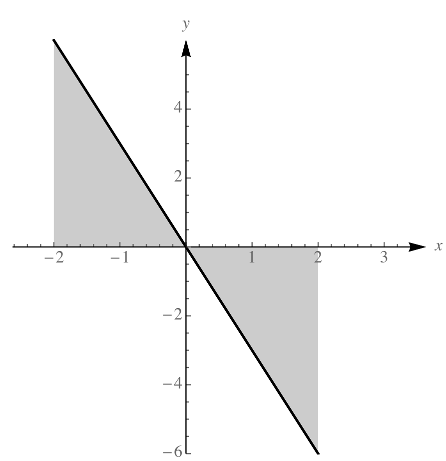
The right Riemann sum $\sum_{k=1}^{4} f(x_k) \cdot \frac{1}{2} \approx -0.077.$

The midpoint Riemann sum $\sum_{k=1}^{4} f\left(x_k^*\right) \cdot \frac{1}{2} \approx -0.6780.$

c. The rectangles whose height is $f(x_k)$ contribute positively when $x_k > 0$ and negatively when $x_k < 0$.

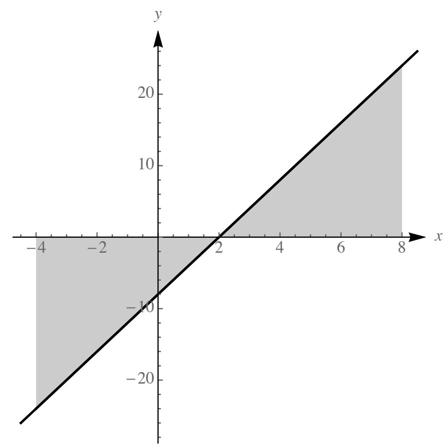
5.2.27

The region above the axis is a triangle with base 2 and height $f(-2) = 6$, and the region below the axis is a triangle with base 2 and height $-f(2) = 6$, so the net area is 0, and the area is 12.



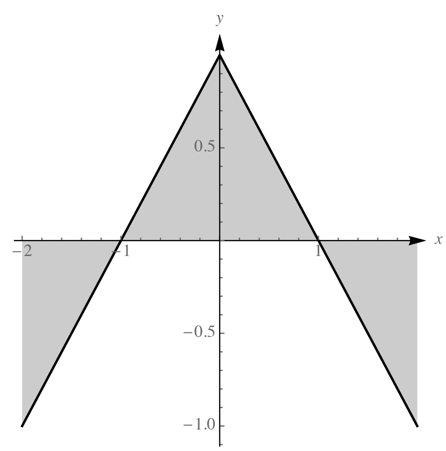
5.2.28

The region above the axis is a triangle with base $8-2=6$ and height $f(8)=24$, while the region below the axis is a triangle with base $2-(-4)=6$ and height $-f(-4)=24$, so the net area is 0, and the area is 144.



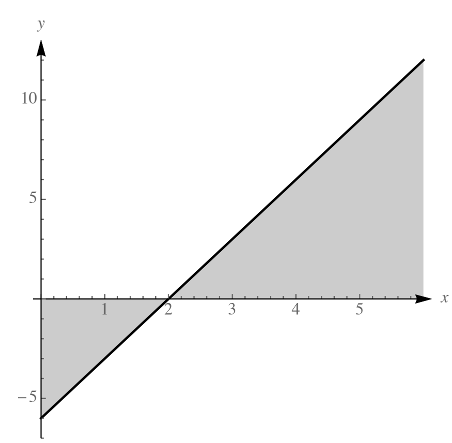
5.2.29

The region above the axis is a triangle with base 2 and height $f(0)=1$, while the region below the axis consists of two triangles each with base 1 and height 1, so the net area is 0, and the area is 2.



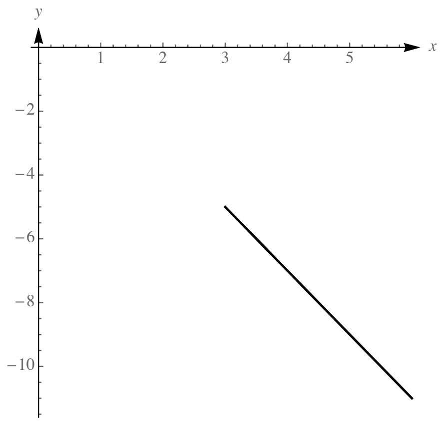
5.2.30

The region above the axis is a triangle with base $6-2=4$ and height $f(6)=12$, while the region below the axis is a triangle with base 2 and height $-f(-0)=6$, so the net area is $\frac{1}{2}\cdot 4\cdot 12-\frac{1}{2}\cdot 2\cdot 6 = 24-6=18$, while the area is 30.



5.2.31

a.



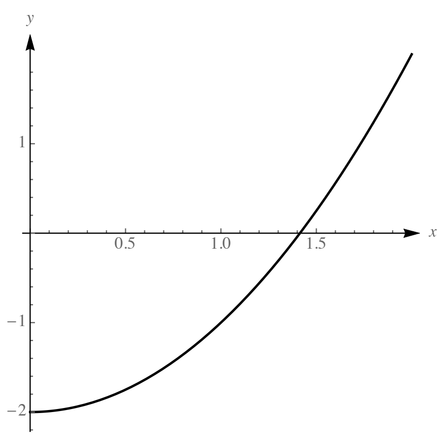
b. $\Delta x = \frac{1}{2}$, so the grid points are at 3, 3.5, 4, 4.5, 5, 5.5, and 6.

c. The left Riemann sum is $0.5(-5-6-7-8-9-10) = -22.5$. The right Riemann sum is $-.5(-6-7-8-9-10-11) = -25.5$.

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.32

a.



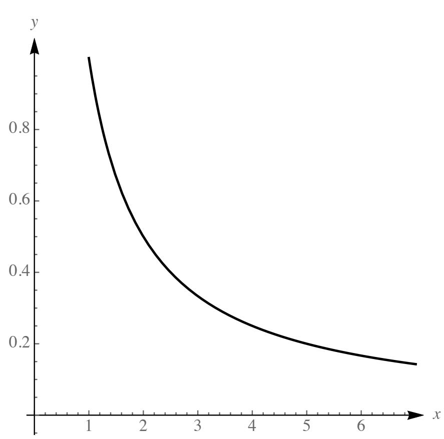
b. $\Delta x = \frac{1}{2}$, so the grid points are at 0, 0.5, 1, 1.5, and 2.

c. The left Riemann sum is $0.5(-2-1.75-1+0.25) = -2.25$. The right Riemann sum is $0.5(-1.75-1+.25+2) = -0.25$.

d. The left Riemann sum underestimates the true value, while the right Riemann sum overestimates it.

5.2.33

a.

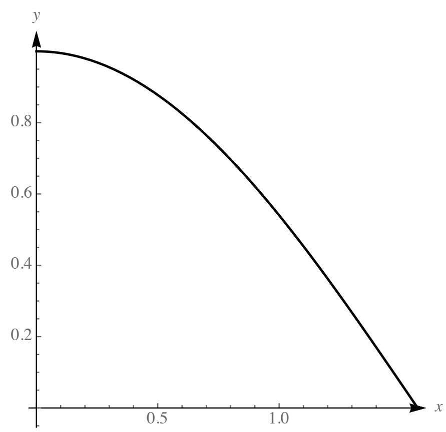


b. $\Delta x = 1$, so the grid points are at 1, 2, 3, 4, 5, 6, and 7.

c. The left Riemann sum is approximately $1 + 0.5 + 0.333333 + 0.25 + -.2 + 0.166666 = 2.45$. The right Riemann sum is approximately $.5 + 0.333333 + 0.25 + 0.2 + 0.166666 + 0.142857 \approx 1.593$.

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.34

a. 

b. $\Delta x = \frac{\pi}{8}$, so the grid points are at 0, $\frac{\pi}{8}$, $\frac{\pi}{4}$, $\frac{3\pi}{8}$, and $\frac{\pi}{2}$.

c. The left Riemann sum is approximately

$$\frac{\pi}{8}(1 + 0.92388 + 0.707107 + 0.382683) \approx 1.183.$$

The right Riemann sum is about

$$\frac{\pi}{8}(0.92388 + 0.707107 + 0.382683 + 0) \approx 0.791.$$

d. The left Riemann sum overestimates the true value, while the right Riemann sum underestimates it.

5.2.35 This is $\int_0^2 (x^2 + 1)\, dx$.

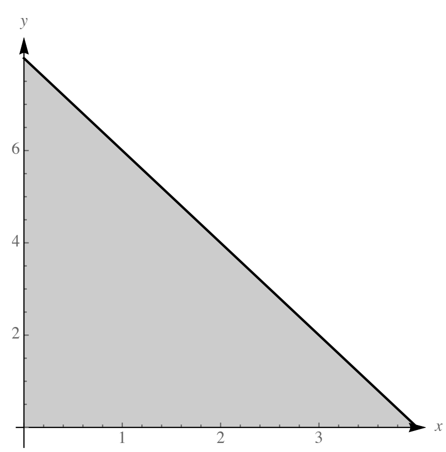
5.2.36 This is $\int_{-2}^2 (4 - x^2)\, dx$.

5.2.37 This is $\int_1^2 x \ln(x)\, dx$.

5.2.38 This is $\int_{-2}^2 |x^2 - 1|\, dx$.

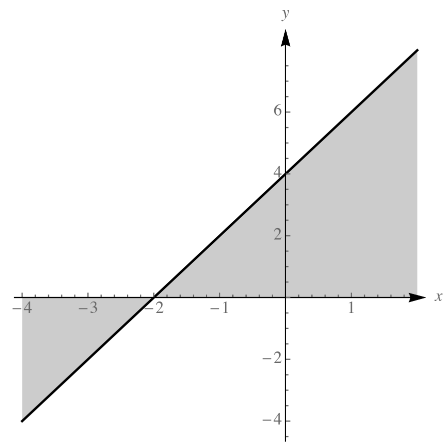
5.2.39

The region in question is a triangle with base 4 and height 8, so the area is $\frac{1}{2} \cdot 8 \cdot 4 = 16$, and this is therefore the value of the definite integral as well.



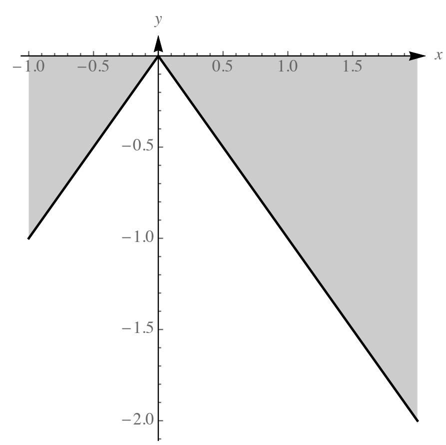
5.2.40

The region in question is a triangle with base 4 and height 8, above the axis, and a triangle with base 2 and height 4 below the axis, so the net area is $\frac{1}{2} \cdot 4 \cdot 8 - \frac{1}{2} \cdot 2 \cdot 4 = 16 - 4 = 12$.



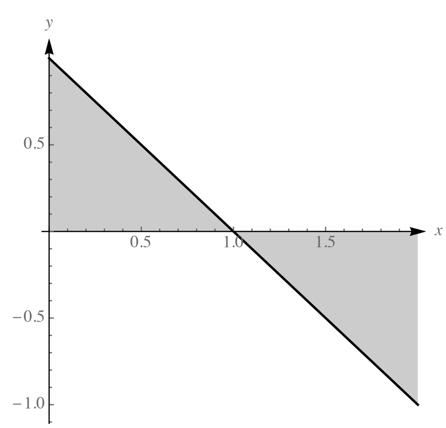
5.2.41

The region consists of two triangles, both below the axis. One has base 1 and height 1, the other has base 2 and height 2, so the net area is $-\dfrac{1}{2} \cdot 1 \cdot 1 - \dfrac{1}{2} \cdot 2 \cdot 2 = -2.5$.



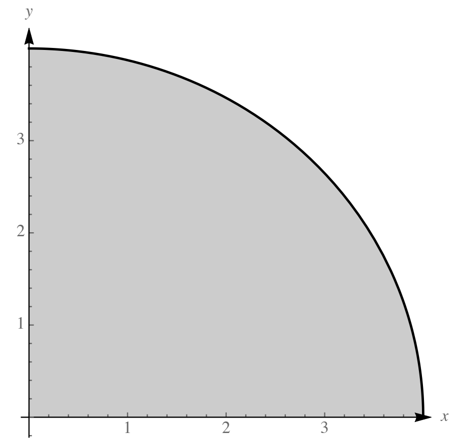
5.2.42

The region consists of two triangles of equal area, one of which is above the axis and one below, so the net area is 0.



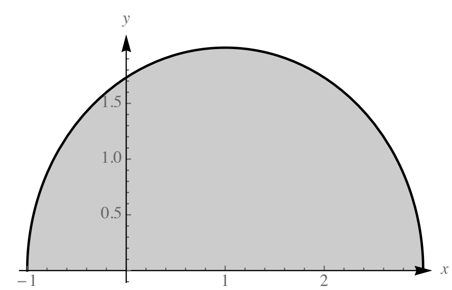
5.2.43

The region consists of a quarter circle of radius 4, situated above the axis. So the net area is $\frac{\pi \cdot 4^2}{4} = 4\pi$.



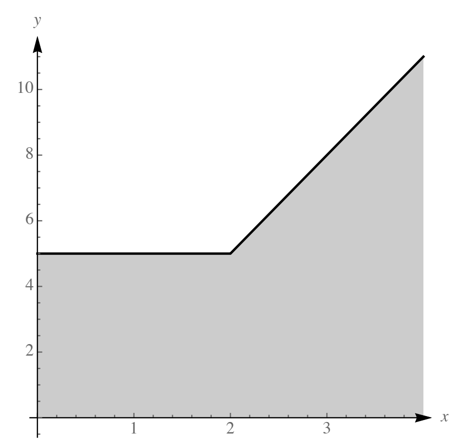
5.2.44

The region consists of a semicircle situated above the axis, of radius 2. The area is thus $\frac{4\pi}{2} = 2\pi$.



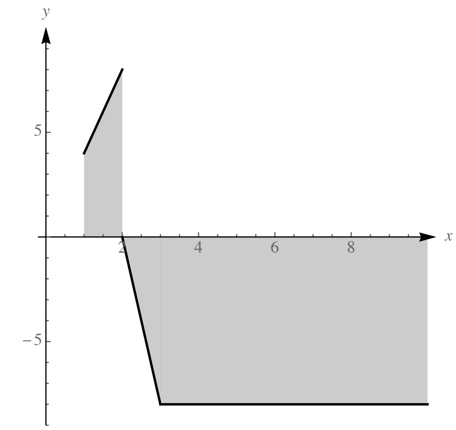
5.2.45

The region consists of a rectangle of area 10 above the axis, and a trapezoid of area 16 above the axis, so the net area is $10 + 16 = 26$.



5.2.46

The region consists of a trapezoid of area 6 above the axis, a triangle of area 4 below the axis, and a rectangle of area 56 below the axis. So the net area is $6 - 4 - 56 = -54$.



5.2.47 $\displaystyle\int_0^{\pi} x\sin x\,dx = A(R_1) + A(R_2) = 1 + \pi - 1 = \pi.$

5.2.48 $\displaystyle\int_0^{3\pi/2} x\sin x\,dx = A(R_1) + A(R_2) - A(R_3) = 1 + \pi - 1 - \pi - 1 = -1.$

5.2.49 $\displaystyle\int_0^{2\pi} x\sin x\,dx = A(R_1) + A(R_2) - A(R_3) - A(R_4) = 1 + \pi - 1 - \pi - 1 - 2\pi + 1 = -2\pi.$

5.2.50 $\displaystyle\int_{\pi/2}^{2\pi} x\sin x\,dx = A(R_2) - A(R_3) - A(R_4) = \pi - 1 - \pi - 1 - 2\pi + 1 = -2\pi - 1.$

5.2.51

a. $\displaystyle\int_4^0 3x(4-x)\,dx = -\int_0^4 3x(4-x)\,dx = -32.$

b. $\displaystyle\int_0^4 x(x-4)\,dx = -\frac{1}{3}\int_0^4 3x(4-x)\,dx = -\frac{1}{3}\cdot 32 = -\frac{32}{3}.$

c. $\displaystyle\int_4^0 6x(4-x)\,dx = -2\cdot\int_0^4 3x(4-x)\,dx = -2\cdot 32 = -64.$

d. $\displaystyle\int_0^8 3x(4-x)\,dx = \int_0^4 3x(4-x)\,dx + \int_4^8 3x(4-x)\,dx = 32 + \int_4^8 3x(4-x)\,dx$. It is not possible to evaluate the given integral from the information given.

5.2.52

a. $\displaystyle\int_1^4 (-3f(x))\,dx = -3\int_1^4 f(x)\,dx = -3\cdot 8 = -24.$

b. $\displaystyle\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot 8 = 24.$

c. $\displaystyle\int_6^4 12f(x)\,dx = -12\int_4^6 f(x)\,dx = -12\left(\int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx\right) = -12(5-8) = 36.$

d. $\displaystyle\int_4^6 3f(x)\,dx = 3\left(\int_1^6 f(x)\,dx - \int_1^4 f(x)\,dx\right) = 3(5-8) = -9.$

5.2.53

a. $\displaystyle\int_0^3 5f(x)\,dx = 5\int_0^3 f(x)\,dx = 5\cdot 2 = 10.$

b. $\displaystyle\int_3^6 (-3g(x))\,dx = -3\int_3^6 g(x)\,dx = -3\cdot 1 = -3.$

c. $\displaystyle\int_3^6 (3f(x)-g(x))\,dx = 3\int_3^6 f(x)\,dx - \int_3^6 g(x)\,dx = 3(-5)-1 = -16.$

d. $\displaystyle\int_6^3 [f(x)+2g(x)]\,dx = -\left[\int_3^6 f(x)\,dx + 2\int_3^6 g(x)\,dx\right] = -[-5+2\cdot 1] = 3.$

5.2.54

a. $\displaystyle\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = 6+(-8) = -2.$

b. $\displaystyle\int_0^5 |f(x)|\,dx = \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx = 6+8 = 14.$

c. $\displaystyle\int_2^5 4|f(x)|\,dx = 4\int_2^5 (-f(x))\,dx = 4\cdot 8 = 32.$

d. $\displaystyle\int_0^5 (f(x)+|f(x)|)\,dx = \int_0^2 (f(x)+f(x))\,dx + \int_2^5 f(x)-f(x)\,dx = 2\int_0^2 f(x)\,dx = 2\cdot 6 = 12.$

5.2.55

a. $\displaystyle\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot\left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) = 3\cdot(10-5) = 15.$

b. $\displaystyle\int_1^6 (f(x)-g(x))\,dx = \int_1^6 f(x)\,dx - \int_1^6 g(x)\,dx = 10-5 = 5.$

c. $\displaystyle\int_1^4 (f(x)-g(x))\,dx = \int_1^4 f(x)\,dx - \int_1^4 g(x)\,dx = \left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) - 2 = (10-5)-2 = 3.$

d. $\displaystyle\int_4^6 (g(x)-f(x)\,dx = \int_4^6 g(x)\,dx - \int_4^6 f(x)\,dx = \left(\int_1^6 g(x)\,dx - \int_1^4 g(x)\,dx\right) - 5 = (5-2)-5 = -2.$

e. $\displaystyle\int_4^6 8g(x)\,dx = 8\left(\int_1^6 g(x)\,dx - \int_1^4 g(x)\,dx\right) = 8(5-2) = 24.$

f. $\displaystyle\int_4^1 2f(x)\,dx = -2\int_1^4 f(x)\,dx = -2\cdot\left(\int_1^6 f(x)\,dx - \int_4^6 f(x)\,dx\right) = -2(10-5) = -10.$

5.2.56 Because f is bounded above by the function $g(x)=3$ on the interval, we must have

$$\int_1^5 f(x)\,dx \le \int_1^5 3\,dx = 12,$$

so our integral is bounded above by 12. On the the other hand, because $2 \le f(x)$ on the interval, we must have

$$\int_1^5 f(x)\,dx \ge \int_1^5 2\,dx = 8,$$

so our integral is bounded below by 8. Therefore

$$8 \le \int_1^5 f(x)\,dx \le 12.$$

5.2.57

a. $\int_0^1 (4x - 2x^3)\,dx = -2\int_0^1 (x^3 - 2x)\,dx = -2 \cdot \left(-\frac{3}{4}\right) = \frac{3}{2}$.

b. $\int_1^0 (2x - x^3)\,dx = \int_0^1 (x^3 - 2x)\,dx = -\frac{3}{4}$.

5.2.58

a. $\int_0^{\pi/2} (2\sin\theta - \cos\theta)\,d\theta = -\int_0^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -(-1) = 1$.

b. $\int_{\pi/2}^0 (4\cos\theta - 8\sin\theta)\,d\theta = -4\int_0^{\pi/2} (\cos\theta - 2\sin\theta)\,d\theta = -4(-1) = 4$.

5.2.59 $\int_0^a f(x)\,dx = 16$.

5.2.60 $\int_0^b f(x)\,dx = 16 - 5 = 11$.

5.2.61 $\int_a^c f(x)\,dx = 11 - 5 = 6$.

5.2.62 $\int_0^c f(x)\,dx = 16 - 5 + 11 = 22$.

5.2.63 $\int_0^c |f(x)|\,dx = 16 + 5 + 11 = 32$.

5.2.64 $\int_0^c (2|f(x)| + 3f(x))\,dx = 2(32) + 3(22) = 130$.

5.2.65 $\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx = -16$.

5.2.66 $\int_c^0 |f(x)|\,dx = -\int_0^c |f(x)|\,dx = -32$.

5.2.67 $\int_0^1 (2x + \sqrt{1-x^2} + 1)\,dx = \int_0^1 2x\,dx + \int_0^1 \sqrt{1-x^2}\,dx + \int_0^1 1\,dx$. The first integral in this sum represents the area of a triangle with base 1 and height 2 (which has value 1), the second represents the area of a quarter of a circle of radius 1 (which has value $\frac{\pi}{4}$, and the third represents the area of a 1×1 square (which has value 1). So the integral's value is $1 + \frac{\pi}{4} + 1 = 2 + \frac{\pi}{4}$.

5.2.68 $\int_1^5 \left(|x-2| + \sqrt{-x^2+6x-5}\right)\,dx = \int_1^5 |x-2|\,dx + \int_1^5 \sqrt{-x^2+6x+5}\,dx$. The first integral in this last sum represents the areas of two triangles, one with base 1 and height 1 (over the interval $[1, 2]$) and one with base 3 and height 3 (over the interval $[2, 5]$). So the value of the first integral is $\frac{1}{2} + \frac{9}{2} = 5$. For the second integral, note that under the radical we have $-x^2 + 6x - 5 = -(x^2 - 6x + 5) = -(x^2 - 6x + 9 - 4) = -((x-3)^2 - 2^2) = 2^2 - (x-3)^2$. So the function $y = \sqrt{2^2 - (x-3)^2}$ represents a circle of radius 2 centered at $(3, 0)$, so the value of the second integral is $\frac{1}{2}(4\pi) = 2\pi$. So the sum of the two integrals is $5 + 2\pi$.

5.2.69

a. True. See problem 78 in the previous section for a proof.

b. True. See problem 79 in the previous section for a proof.

c. True. Because both of those function are periodic with period $\frac{2\pi}{a}$, and both have the same amount of area above the axis as below for one period, the net area of each between 0 and $\frac{2\pi}{a}$ is zero.

d. False. For example $\int_0^{2\pi} \sin x\,dx = 0 = \int_{2\pi}^0 \sin x\,dx$, but $\sin x$ is not a constant function.

e. False. Because x is not a constant, it can not be factored outside of the integral. For example $\int_0^1 x \cdot 1\,dx \neq x\int_0^1 1\,dx$.

5.2.70

a. $\displaystyle\sum_{k=1}^{20} 3\sqrt{4+\frac{5(k-1)}{20}}\cdot\frac{5}{20} \approx 37.624.$ $\displaystyle\sum_{k=1}^{20} 3\sqrt{4+\frac{5k}{20}}\cdot\frac{5}{20} \approx 38.374.$

$\displaystyle\sum_{k=1}^{50} 3\sqrt{4+\frac{5(k-1)}{50}}\cdot\frac{5}{50} \approx 37.850.$ $\displaystyle\sum_{k=1}^{50} 3\sqrt{4+\frac{5k}{50}}\cdot\frac{5}{50} \approx 38.150.$

$\displaystyle\sum_{k=1}^{100} 3\sqrt{4+\frac{5(k-1)}{100}}\cdot\frac{5}{100} \approx 37.925.$ $\displaystyle\sum_{k=1}^{100} 3\sqrt{4+\frac{5k}{100}}\cdot\frac{5}{100} \approx 38.075.$

b. It appears that the integral's value is 38.

5.2.71

a. $\displaystyle\sum_{k=1}^{20}\left(\left(\frac{k-1}{20}\right)^2+1\right)\cdot\frac{1}{20} \approx 1.309.$ $\displaystyle\sum_{k=1}^{20}\left(\left(\frac{k}{20}\right)^2+1\right)\cdot\frac{1}{20} \approx 1.359.$

$\displaystyle\sum_{k=1}^{50}\left(\left(\frac{k-1}{50}\right)^2+1\right)\cdot\frac{1}{50} \approx 1.323.$ $\displaystyle\sum_{k=1}^{50}\left(\left(\frac{k}{50}\right)^2+1\right)\cdot\frac{1}{50} \approx 1.343.$

$\displaystyle\sum_{k=1}^{100}\left(\left(\frac{k-1}{100}\right)^2+1\right)\cdot\frac{1}{100} \approx 1.328.$ $\displaystyle\sum_{k=1}^{100}\left(\left(\frac{k}{100}\right)^2+1\right)\cdot\frac{1}{100} \approx 1.338.$

b. It appears that the integral's value is about $\frac{4}{3}$.

5.2.72

a. $\displaystyle\sum_{k=1}^{20}\ln\left(1+(k-1)\frac{(e-1)}{20}\right)\cdot\frac{e-1}{20} \approx 0.957.$ $\displaystyle\sum_{k=1}^{20}\ln\left(1+k\cdot\frac{(e-1)}{20}\right)\cdot\frac{e-1}{20} \approx 1.043.$

$\displaystyle\sum_{k=1}^{50}\ln\left(1+(k-1)\frac{(e-1)}{50}\right)\cdot\frac{e-1}{50} \approx 0.983.$ $\displaystyle\sum_{k=1}^{50}\ln\left(1+k\cdot\frac{(e-1)}{50}\right)\cdot\frac{e-1}{50} \approx 1.017.$

$\displaystyle\sum_{k=1}^{100}\ln\left(1+(k-1)\frac{(e-1)}{100}\right)\cdot\frac{e-1}{100} \approx 0.991.$ $\displaystyle\sum_{k=1}^{100}\ln\left(1+k\cdot\frac{(e-1)}{100}\right)\cdot\frac{e-1}{100} \approx 1.009.$

b. It appears that the integral's value is 1.

5.2.73

a. $\displaystyle\sum_{k=1}^{20}\cos^{-1}\left((k-1)/20\right)\cdot\frac{1}{20} \approx 1.036.$ $\displaystyle\sum_{k=1}^{20}\cos^{-1}\left(k/20\right)\cdot\frac{1}{20} \approx 0.958.$

$\displaystyle\sum_{k=1}^{50}\cos^{-1}\left((k-1)/50\right)\cdot\frac{1}{50} \approx 1.015.$ $\displaystyle\sum_{k=1}^{50}\cos^{-1}\left(k/50\right)\cdot\frac{1}{50} \approx 0.983.$

$\displaystyle\sum_{k=1}^{100}\cos^{-1}\left((k-1)/100\right)\cdot\frac{1}{100} \approx 1.008.$ $\displaystyle\sum_{k=1}^{100}\cos^{-1}\left(k/100\right)\cdot\frac{1}{100} \approx 0.992.$

b. It appears that the integral's value is 1.

5.2.74

a. $\displaystyle\sum_{k=1}^{20} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{20}\right)\right)\cdot\frac{2}{20} \approx 3.992.$ $\displaystyle\sum_{k=1}^{20} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{20}\right)\right)\cdot\frac{2}{20} \approx 3.992.$

$\displaystyle\sum_{k=1}^{50} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{50}\right)\right)\cdot\frac{2}{50} \approx 3.999.$ $\displaystyle\sum_{k=1}^{50} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{50}\right)\right)\cdot\frac{2}{50} \approx 3.999.$

$\displaystyle\sum_{k=1}^{100} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2(k-1)}{100}\right)\right)\cdot\frac{2}{100} \approx 4.000.$ $\displaystyle\sum_{k=1}^{100} \pi \cos\left(\frac{\pi}{2}\left(-1+\frac{2k}{100}\right)\right)\cdot\frac{2}{100} \approx 4.000.$

b. It appears that the integral's value is 4.

5.2.75

a. $\displaystyle\sum_{k=1}^{n} 2\sqrt{1+\frac{3}{2n}+\frac{3(k-1)}{n}}\cdot\frac{3}{n}.$

b.

n	20	50	100
Midpoint Sum	9.3338	9.33341	9.33335

It appears that the integral's value is about $\dfrac{28}{3}$.

5.2.76

a. $\displaystyle\sum_{k=1}^{n} \sin\left(\frac{\pi}{4}\left(-1+\frac{3}{2n}+\frac{3(k-1)}{n}\right)\right)\cdot\frac{3}{n}.$

b.

n	20	50	100
Midpoint Sum	0.900837	0.9004	0.900337

It appears that the integral's value is about 0.9.

5.2.77

a. $\displaystyle\sum_{k=1}^{n} \left(4\left(\frac{2}{n}+\frac{4(k-1)}{n}\right)-\left(\frac{2}{n}+\frac{4(k-1)}{n}\right)^2\right)\cdot\frac{4}{n}.$

b.

n	20	50	100
Midpoint Sum	10.68	10.6688	10.6672

It appears that the integral's value is about $\dfrac{32}{3}$.

5.2.78

a. $\displaystyle\sum_{k=1}^{n} \left(\sin^{-1}\left(\frac{k-0.5}{n}\right)+1\right)\cdot\frac{1}{\pi n}.$

b.

n	20	50	100
Midpoint Sum	0.49973	0.49993	0.49997

It appears that the integral's value is about $\dfrac{1}{2}$.

5.2.79

$$\begin{aligned}\int_0^2 (2x+1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[2\left(\frac{2k}{n}\right)+1\right]\frac{2}{n}\\ &= \lim_{n\to\infty}\left[\frac{8}{n^2}\sum_{k=1}^n k + \frac{2}{n}\sum_{k=1}^n 1\right]\\ &= \lim_{n\to\infty}\left[\frac{8}{n^2}\cdot\frac{n(n+1)}{2}+\frac{2}{n}\cdot n\right]\\ &= \lim_{n\to\infty}\left[\frac{4(n+1)}{n}+2\right] = 4+2=6.\end{aligned}$$

5.2.80

$$\begin{aligned}\int_1^5 (1-x)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[1-\left(1+\frac{4k}{n}\right)\right]\frac{4}{n}\\ &= \lim_{n\to\infty}\left[\frac{-16}{n^2}\sum_{k=1}^n k\right]\\ &= \lim_{n\to\infty}\left[\frac{-16}{n^2}\cdot\frac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty}\left[\frac{-8(n+1)}{n}\right] = -8.\end{aligned}$$

5.2.81

$$\begin{aligned}\int_3^7 (4x+6)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[4\left(3+\frac{4k}{n}\right)+6\right]\frac{4}{n}\\ &= \lim_{n\to\infty}\left[\frac{64}{n^2}\sum_{k=1}^n k + \frac{72}{n}\sum_{k=1}^n 1\right]\\ &= \lim_{n\to\infty}\left[\frac{64}{n^2}\cdot\frac{n(n+1)}{2}+\frac{72}{n}\cdot n\right]\\ &= \lim_{n\to\infty}\left[\frac{32(n+1)}{n}+72\right] = 104.\end{aligned}$$

5.2.82

$$\begin{aligned}\int_0^2 (x^2-1)\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[\left(\frac{2k}{n}\right)^2-1\right]\frac{2}{n}\\ &= \lim_{n\to\infty}\left[\frac{2}{n}\left(\sum_{k=1}^n -1\right) + \frac{8}{n^3}\sum_{k=1}^n k^2\right]\\ &= \lim_{n\to\infty}\left[\frac{2}{n}\cdot(-n)+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]\\ &= \lim_{n\to\infty}\left[-2+\frac{8n^2+12n+4}{3n^2}\right] = -2+\frac{8}{3}=\frac{2}{3}.\end{aligned}$$

5.2.83

$$\int_1^4 (x^2-1)\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \left[\left(1+\frac{3k}{n}\right)^2 - 1\right]\frac{3}{n}$$
$$= \lim_{n\to\infty}\left[\frac{18}{n^2}\sum_{k=1}^n k + \frac{27}{n^3}\sum_{k=1}^n k^2\right]$$
$$= \lim_{n\to\infty}\left[\frac{18}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \lim_{n\to\infty}\left[\frac{9(n+1)}{n} + \frac{18n^2+27n+9}{2n^2}\right] = 9+9 = 18.$$

5.2.84

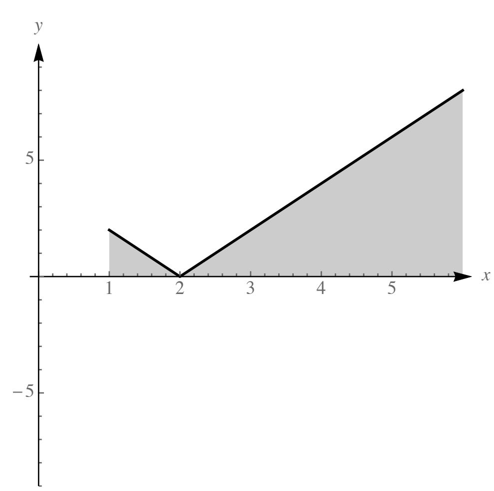
$$\int_0^2 (x^3+x+1)\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^n \left(\left(\frac{2k}{n}\right)^3 + \left(\frac{2k}{n}\right)+1\right)$$
$$= \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{k=1}^n k^3 + \frac{4}{n^2}\sum_{k=1}^n k + 2\right)$$
$$= \lim_{n\to\infty}\left(\frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) + \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right)+2\right)$$
$$= \lim_{n\to\infty}\left(4\left(\frac{n^2(n+1)^2}{n^4}\right) + 2\left(\frac{n(n+1)}{n^2}\right)+2\right)$$
$$= 4+2+2 = 8.$$

5.2.85

$$\int_0^1 (4x^3+3x^2)\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \left(4\left(\frac{k}{n}\right)^3 + 3\left(\frac{k}{n}\right)^2\right)$$
$$= \lim_{n\to\infty}\left(\frac{4}{n^4}\sum_{k=1}^n k^3 + \frac{3}{n^3}\sum_{k=1}^n k^2\right)$$
$$= \lim_{n\to\infty}\left(\frac{4}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) + \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$$
$$= 1+1 = 2.$$

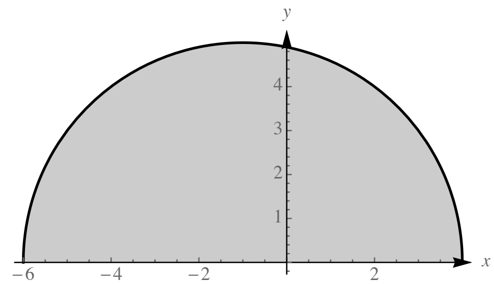
5.2.86

The region in question consists of two triangles above the axis, one with base 1 and height 2, and one with base 4 and height 8, so the net area is $\frac{1}{2}\cdot 1\cdot 2 + \frac{1}{2}\cdot 4\cdot 8 = 17.$



5.2.87

The region in question is a semicircle above the axis with radius 5, so the area is $\frac{1}{2}\pi \cdot 5^2 = \frac{25\pi}{2}$.



5.2.88 Let the grid points for the interval $[a, p]$ be $x_i = a + i \cdot \frac{p-a}{n}$, where $1 \le i \le n$. Let the grid points for the interval $[p, b]$ be $x_j^* = p + j \cdot \frac{b-p}{m}$ where $1 \le j \le m$. Note that if we take the union of both of these sets of grid points, we get a set of grid points for $[a, b]$.

One Riemann sum for f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k) \cdot \frac{p-a}{n} + \sum_{j=1}^{m} f(x_j^*) \cdot \frac{b-p}{m}$, which naturally splits into a right Riemann sum for f on $[a, p]$ plus a right Riemann sum for f on $[p, b]$.

By the definition of definite integral, taking limits as m, $n \to \infty$ shows that $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$.

5.2.89 $\int_0^{10} f(x)\,dx = \int_0^5 2\,dx + \int_5^{10} 3\,dx = 10 + 15 = 25$.

5.2.90 $\int_1^6 f(x)\,dx = \int_1^4 2x\,dx + \int_4^6 (10-2x)\,dx = 15 + 0 = 15$.

5.2.91 $\int_1^5 x\lfloor x \rfloor\,dx = \int_1^2 x\lfloor x \rfloor\,dx + \int_2^3 x\lfloor x \rfloor\,dx + \int_3^4 x\lfloor x \rfloor\,dx + \int_4^5 x\lfloor x \rfloor\,dx = \int_1^2 x\,dx + \int_2^3 2x\,dx + \int_3^4 3x\,dx + \int_4^5 4x\,dx$. Each of these integrals represents the area of a trapezoid with base 1. The value of the integral is $\frac{1+2}{2} + \frac{4+6}{2} + \frac{9+12}{2} + \frac{16+20}{2} = 35$.

5.2.92 $\int_0^4 \frac{x}{\lceil x \rceil}\,dx = \int_0^1 \frac{x}{\lceil x \rceil}\,dx + \int_1^2 \frac{x}{\lceil x \rceil}\,dx + \int_2^3 \frac{x}{\lceil x \rceil}\,dx + \int_3^4 \frac{x}{\lceil x \rceil}\,dx = \int_0^1 x\,dx + \int_1^2 \frac{x}{2}\,dx + \int_2^3 \frac{x}{3}\,dx + \int_3^4 \frac{x}{4}\,dx$. The first of these represents the area of a triangle with base one and height 1, while the others represent the area of trapezoids with base 1. The value of the integral is $\frac{1}{2} + \frac{\frac{1}{2}+1}{2} + \frac{\frac{2}{3}+1}{2} + \frac{\frac{3}{4}+1}{2} = \frac{71}{24}$.

5.2.93

$$
\begin{aligned}
\int_a^b cf(x)\,dx &= \lim_{\Delta \to 0} \sum_{k=1}^{n} cf(x_k^*)\Delta x_k \\
&= \lim_{\Delta \to 0} c \sum_{k=1}^{n} f(x_k^*)\Delta x_k \\
&= c \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = c \int_a^b f(x)\,dx.
\end{aligned}
$$

5.2.94 Note that $\displaystyle\int_c^d x\,dx = \frac{d^2-c^2}{2}$, because it represents the area of a trapezoid with base of length $d-c$ and heights c and d. Also note that $\displaystyle\int_c^d b\,dx = b(d-c)$ because it represents the area of a rectangle with base $d-c$ and height b.

Therefore, $\displaystyle\int_c^d (x+b)\,dx = \int_c^d x\,dx + \int_c^d b\,dx = \frac{d^2-c^2}{2} + b(d-c) = (d-c)\cdot\left(\frac{d+c}{2}+b\right)$. Because $c \neq d$, this is zero exactly when $b = -\frac{c+d}{2}$.

5.2.95 Let n be a positive integer. Let $\Delta x = \dfrac{1}{n}$. Note that each grid point $\dfrac{k}{n}$ for $0 \le k \le n$ where i is an integer is a rational number. So $f(x_k) = 1$ for each grid point. So the right Riemann sum is $\displaystyle\sum_{k=1}^n f(x_k)\frac{1}{n} = \frac{1}{n}\sum_{k=1}^n 1 = \frac{1}{n}\cdot n = 1$. The left Riemann sum calculation is similar, as is the midpoint Riemann sum calculation (because the grid midpoints are also rational numbers – they are the average of two rational numbers and hence are rational as well).

5.2.96

a. The left Riemann sum for I(p) is $\displaystyle\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p\cdot\frac{1}{n}$.

b. We have $\displaystyle I(p) = \int_0^1 x^p\,dx = \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^p\cdot\frac{1}{n} = \frac{1}{p+1}$.

5.2.97

a. Note that for all values of $k = 1, 2, \ldots, n$, we have $x_{k-1}x_{k-1} \le x_{k-1}x_k$, so $\sqrt{x_{k-1}x_{k-1}} \le \sqrt{x_{k-1}x_k}$, and thus $x_{k-1} \le \sqrt{x_{k-1}x_k}$. Similarly, $x_{k-1}x_k \le x_kx_k$, so $\sqrt{x_{k-1}x_k} \le \sqrt{x_kx_k} = x_k$, so $\sqrt{x_{k-1}x_k} \le x_k$. Thus $x_{k-1} \le \sqrt{x_{k-1}x_k} \le x_k$ for all $k = 1, 2, \ldots, n$.

b. $\displaystyle\frac{1}{x_{k-1}} - \frac{1}{x_k} = \frac{x_k}{x_{k-1}x_k} - \frac{x_{k-1}}{x_{k-1}x_k} = \frac{x_k - x_{k-1}}{x_{k-1}x_k} = \frac{\Delta x_k}{x_{k-1}x_k}$, for all $k = 1, 2, \ldots, n$.

c. The Riemann sum is $\displaystyle\sum_{k=1}^n \frac{\Delta x_k}{\overline{x_k}^2}$. Using $\overline{x_k} = \sqrt{x_{k-1}x_k}$, we have

$$\sum_{k=1}^n \frac{\Delta x_k}{x_{k-1}x_k} = \sum_{k=1}^n\left(\frac{1}{x_{k-1}} - \frac{1}{x_k}\right),$$

where the last equality follows from part (b). Now note that the sum telescopes (that is, has many canceling terms).

$$\begin{aligned}
\sum_{k=1}^n\left(\frac{1}{x_{k-1}} - \frac{1}{x_k}\right) &= \left(\frac{1}{x_0} - \frac{1}{x_1}\right) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \cdots + \left(\frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}\right) + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right)\\
&= \frac{1}{x_0} + \left(-\frac{1}{x_1} + \frac{1}{x_1}\right) + \left(-\frac{1}{x_2} + \frac{1}{x_2}\right) + \cdots + \left(-\frac{1}{x_{n-1}} + \frac{1}{x_{n-1}}\right) - \frac{1}{x_n}\\
&= \frac{1}{x_0} - \frac{1}{x_n}\\
&= \frac{1}{a} - \frac{1}{b}.
\end{aligned}$$

d. The integral is the limit of the Riemann sum as $n \to \infty$. Thus we have

$$\int_a^b \frac{dx}{x^2} = \lim_{n\to\infty}\sum_{k=1}^n \frac{\Delta x_k}{\overline{x_k}^2} = \lim_{n\to\infty}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{a} - \frac{1}{b}.$$

5.2.98 Suppose $f(x) \geq g(x)$ on $[a,b]$. Then $f(x)-g(x) \geq 0$ on $[a,b]$. Then by Property 7, $\int_a^b (f(x) - g(x))\,dx \geq 0$. But by Property 3, $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$. Therefore $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$, so $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

5.3 Fundamental Theorem of Calculus

5.3.1 A is also an antiderivative of f.

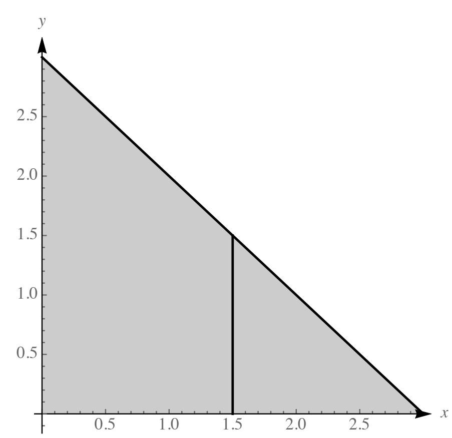
5.3.2 Because F and A are both antiderivatives of f, we have $A(x) = F(x) + C$, where C is a constant.

5.3.3 The fundamental theorem says that $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f. So to evaluate $\int_a^b f(x)\,dx$, one could find an antiderivative $F(x)$, and then evaluate this at a and b and then subtract, obtaining $F(b) - F(a)$.

5.3.4 An area function has the form $\int_a^x c\,dt$, and gives the area between a and x and under c, which is the area of a rectangle with base $x-a$ and height c. As x increases, the base $x-a$ increases while the height c remains constant, so the area increases.

5.3.5

$A(x) = \int_0^x (3-t)\,dt$ represents the area between 0 and x and below this curve. As x increases (but remains less than 3), the trapezoidal region's area increases, so the area function increases until x is 3.



5.3.6 $\int_0^2 3x^2 = x^3 \Big|_0^2 = 8 - 0 = 8.$

$\int_{-2}^2 3x^2 = x^3 \Big|_{-2}^2 = 8 - -8 = 16.$

5.3.7 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, and $\int f'(x)\,dx = f(x) + C$.

5.3.8 It can be omitted because it doesn't change the value of $F(b)-F(a)$. For example, suppose $F(x)$ is an antiderivative of f, and so is $G(x) = F(x) + C$. Then $G(b) - G(a) = F(b) + C - (F(a) + C) = F(b) - F(a)$.

5.3.9 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, and $\frac{d}{dx}\int_a^b f(t)\,dt = 0$. The latter is the derivative of a constant, the former follows from the Fundamental Theorem.

5.3.10 Because f is an antiderivative of f', the Fundamental Theorem assures us that $\int_a^b f'(x)\,dx = f(b) - f(a)$.

5.3.11 $\int_3^8 f'(t)\,dt = f(8) - f(3) = 20 - 4 = 16$.

5.3.12 $\int_2^7 3\,dx = 3x\Big|_2^7 = 21 - 6 = 15$. The integral represents the area of a 5×3 rectangle, which is 15.

5.3.13

a. $A(-2) = \int_{-2}^{-2} f(t)\,dt = 0$.

b. $F(8) = \int_4^8 f(t)\,dt = -9$.

c. $A(4) = \int_{-2}^{4} f(t)\,dt = 8 + 17 = 25$.

d. $F(4) = \int_4^4 f(t)\,dt = 0$.

e. $A(8) = \int_{-2}^{8} f(t)\,dt = 25 - 9 = 16$.

5.3.14

a. $A(2) = \int_0^2 f(t)\,dt = 8$.

b. $F(5) = \int_2^5 f(t)\,dt = -5$.

c. $A(0) = \int_0^0 f(t)\,dt = 0$.

d. $F(8) = \int_2^8 f(t)\,dt = -16$.

e. $A(8) = \int_0^8 f(t)\,dt = 8 - 16 = -8$.

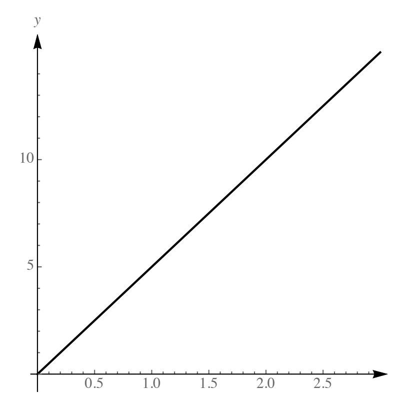
f. $A(5) = \int_0^5 f(t)\,dt = 8 - 5 = 3$.

g. $F(2) = \int_2^2 f(t)\,dt = 0$.

5.3.15

a. $A(x) = \int_0^x f(t)\,dt = \int_0^x 5\,dt = 5x$.

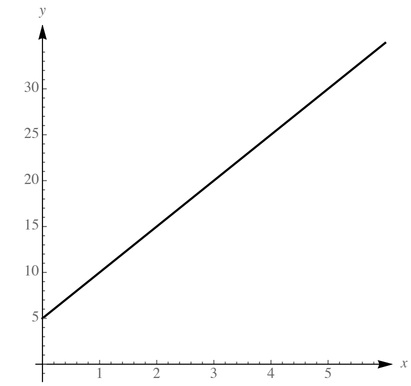
b. $A'(x) = 5 = f(x)$.



5.3.16

a. $A(x) = \int_{-5}^{x} f(t)\,dt = \int_{-5}^{x} 5\,dt = 5(x+5).$

b. $A'(x) = 5 = f(x)$.



5.3.17

a. $A(2) = \int_0^2 t\,dt = 2$. $A(4) = \int_0^4 t\,dt = 8$. Because the region whose area is $A(x) = \int_0^x t\,dt$ is a triangle with base x and height x, its value is $\frac{1}{2}x^2$.

b. $F(4) = \int_2^4 t\,dt = 6$. $F(6) = \int_2^6 t\,dt = 16$. Because the region whose area is $A(x) = \int_2^x t\,dt$ is a trapezoid with base $x-2$ and $h_1 = 2$ and $h_2 = x$, its value is $(x-2)\frac{2+x}{2} = \frac{x^2-4}{2} = \frac{x^2}{2} - 2$.

c. We have $A(x) - F(x) = \dfrac{x^2}{2} - (\dfrac{x^2}{2} - 2) = 2$, a constant.

5.3.18

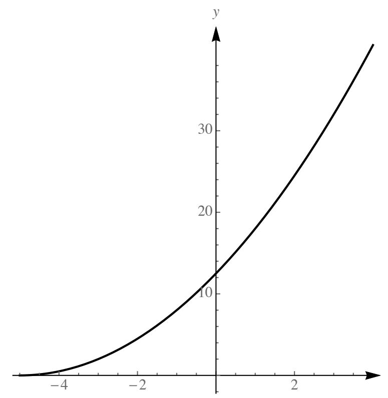
a. $A(2) = \int_1^2 (2t-2)\,dt = 1$. $A(3) = \int_1^3 (2t-2)\,dt = 4$. Because the region whose area is $A(x) = \int_1^x (2t-2)\,dt$ is a triangle with base $x-1$ and height $2x-2$, its value is $\dfrac{1}{2} \cdot (x-1)(2(x-1)) = (x-1)^2$.

b. $F(5) = \int_4^5 (2t-2)\,dt = 7$. $F(6) = \int_4^6 (2t-2)\,dt = 16$. Because the region whose area is $A(x) = \int_2^x t\,dt$ is a trapezoid with base $x-4$ and $h_1 = 6$ and $h_2 = 2x-2$, its value is $(x-4)\left(\dfrac{6+2x-2}{2}\right) = (x-4)(x+2) = x^2 - 2x - 8$.

c. We have $A(x) - F(x) = x^2 - 2x + 1 - (x^2 - 2x - 8) = 9$, a constant.

5.3.19

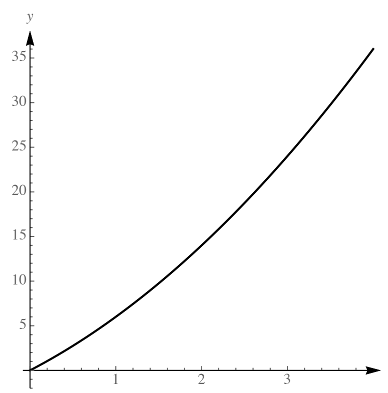
a. The region is a triangle with base $x+5$ and height $x+5$, so its area is $A(x) = \frac{1}{2}(x+5)^2$.



b. $A'(x) = x + 5 = f(x)$.

5.3.20

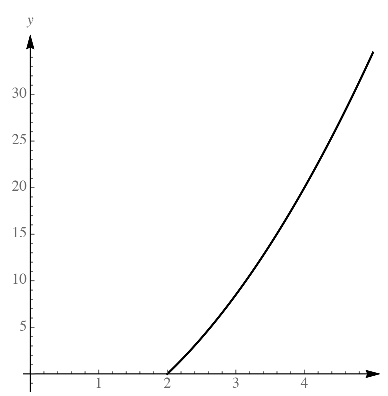
a. The region is a trapezoid with base x and heights $h_1 = f(0) = 5$ and $h_2 = f(x) = 2x+5$, so its area is $A(x) = x \cdot \dfrac{5+2x+5}{2} = x \cdot (x+5) = x^2 + 5x$.



b. $A'(x) = 2x + 5 = f(x)$.

5.3.21

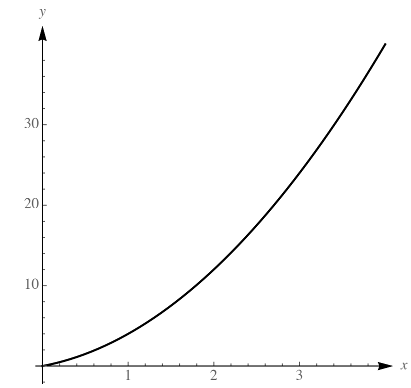
a. The region is a trapezoid with base $x-2$ and heights $h_1 = f(2) = 7$ and $h_2 = f(x) = 3x+1$, so its area is $A(x) = (x-2) \cdot \dfrac{7+3x+1}{2} = (x-2) \cdot (\frac{3}{2}x + 4) = \frac{3}{2}x^2 + x - 8$.



b. $A'(x) = 3x + 1 = f(x)$.

5.3.22

a. The region is a trapezoid with base x and heights $h_1 = f(0) = 2$ and $h_2 = f(x) = 4x + 2$, so its area is $A(x) = (x) \cdot \dfrac{2 + 4x + 2}{2} = (x) \cdot (2x + 2) = 2x^2 + 2x$.



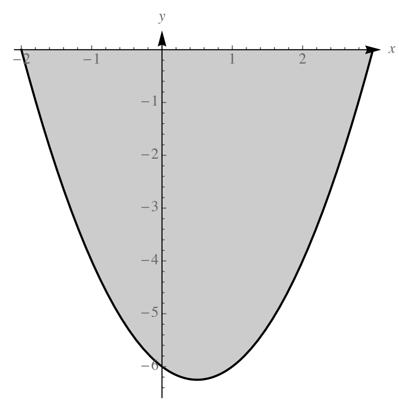
b. $A'(x) = 4x + 2 = f(x)$.

5.3.23 $\displaystyle\int_0^1 (x^2 - 2x + 3)\,dx = \left(\frac{x^3}{3} - x^2 + 3x\right)\Big|_0^1 = \frac{1}{3} - 1 + 3 - (0 - 0 + 0) = \frac{7}{3}$. It does appear that the area is between 2 and 3.

5.3.24 $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx = (-\cos x + \sin x)\Big|_{-\pi/4}^{7\pi/4} = -\sqrt{2}/2 + -\sqrt{2}/2 - (-\sqrt{2}/2 + -\sqrt{2}/2) = 0$. It does appear that the area above the axis is equal to the area below, so the net area is 0.

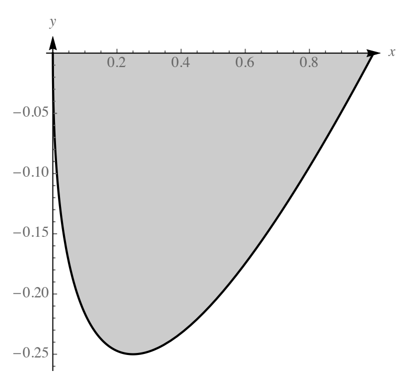
5.3.25

$$\int_{-2}^{3} (x^2 - x - 6)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2} - 6x\right)\Big|_{-2}^{3} = -\frac{125}{6}.$$



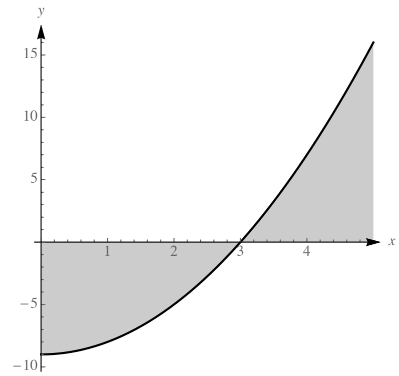
5.3.26

$$\int_0^1 (x - \sqrt{x})\,dx = \left(\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right)\Big|_0^1 = \frac{1}{2} - \frac{2}{3} - (0 - 0) = -\frac{1}{6}.$$



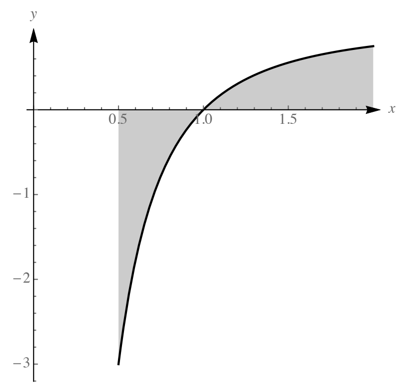
5.3.27

$$\int_0^5 (x^2-9)\,dx = \left(\frac{x^3}{3} - 9x\right)\Big|_0^5 = \frac{125}{3} - 45 - (0-0) = -\frac{10}{3}.$$



5.3.28

$$\int_{1/2}^2 \left(1 - \frac{1}{x^2}\right)\,dx = \left(x + \frac{1}{x}\right)\Big|_{1/2}^2 = 2 + \frac{1}{2} - \left(\frac{1}{2} + 2\right) = 0.$$



5.3.29 $\displaystyle\int_0^2 4x^3\,dx = x^4\Big|_0^2 = 16 - 0 = 16.$

5.3.30 $\displaystyle\int_0^2 (3x^2+2x)\,dx = \left(x^3+x^2\right)\Big|_0^2 = (8+4)-(0+0) = 12.$

5.3.31 $\displaystyle\int_1^8 8x^{1/3}\,dx = 6x^{4/3}\Big|_1^8 = 6(16-1) = 90.$

5.3.32 $\displaystyle\int_1^{16} x^{-5/4}\,dx = -4x^{-1/4}\Big|_1^{16} = -4\left(\frac{1}{2}-1\right) = 2.$

5.3.33 $\displaystyle\int_0^1 (x+\sqrt{x})\,dx = \left(\frac{x^2}{2} + \frac{2x^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{2} + \frac{2}{3} - (0+0) = \frac{7}{6}.$

5.3.34 $\displaystyle\int_0^{\pi/4} 2\cos x\,dx = 2\sin x\Big|_0^{\pi/4} = \frac{2\sqrt{2}}{2} - 0 = \sqrt{2}.$

5.3.35 $\displaystyle\int_1^9 \frac{2}{\sqrt{x}}\,dx = \int_1^9 2x^{-1/2}\,dx = 4x^{1/2}\Big|_1^9 = 12 - 4 = 8.$

5.3.36 $\displaystyle\int_4^9 \frac{2+\sqrt{t}}{\sqrt{t}}\,dt = \int_4^9 \left(2t^{-1/2}+1\right)\,dt = (4t^{1/2}+t)\Big|_4^9 = 12+9-(8+4) = 9.$

5.3.37 $\displaystyle\int_{-2}^{2}(x^2-4)\,dx = \left(\frac{x^3}{3}-4x\right)\Bigg|_{-2}^{2} = \frac{8}{3}-8-\left(-\frac{8}{3}+8\right) = \frac{16}{3}-16 = -\frac{32}{3}.$

5.3.38 $\displaystyle\int_{0}^{\ln 8} e^x\,dx = e^x\Bigg|_{0}^{\ln 8} = e^{\ln 8}-e^0 = 8-1 = 7.$

5.3.39 $\displaystyle\int_{1/2}^{1}(x^{-3}-8)\,dx = \left(\frac{x^{-2}}{-2}-8x\right)\Bigg|_{1/2}^{1} = -\frac{1}{2}-8-(-2-4) = -\frac{5}{2}.$

5.3.40 $\displaystyle\int_{0}^{4} x(x-2)(x-4)\,dx = \int_{0}^{4}(x^3-6x^2+8x)\,dx = \left(\frac{x^4}{4}-2x^3+4x^2\right)\Bigg|_{0}^{4} = 64-128+64-0 = 0.$

5.3.41 $\displaystyle\int_{1}^{4}(1-x)(x-4)\,dx = \int_{1}^{4}(-x^2+5x-4)\,dx = \left(-\frac{x^3}{3}+\frac{5x^2}{2}-4x\right)\Bigg|_{1}^{4} = \frac{9}{2}.$

5.3.42 $\displaystyle\int_{0}^{1/2}\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Bigg|_{0}^{1/2} = \sin^{-1}(1/2)-\sin^{-1}0 = \pi/6-0 = \pi/6.$

5.3.43 $\displaystyle\int_{-2}^{-1} x^{-3}\,dx = \frac{x^{-2}}{-2}\Bigg|_{-2}^{-1} = -\frac{1}{2x^2}\Bigg|_{-2}^{-1} = -\frac{1}{2}-\left(-\frac{1}{8}\right) = -\frac{3}{8}.$

5.3.44 $\displaystyle\int_{0}^{\pi}(1-\sin x)\,dx = (x+\cos x)\Bigg|_{0}^{\pi} = \pi-1-(0+1) = \pi-2.$

5.3.45 $\displaystyle\int_{0}^{\pi/4}\sec^2\theta\,d\theta = \tan\theta\Bigg|_{0}^{\pi/4} = 1-0 = 1.$

5.3.46 $\displaystyle\int_{-\pi/2}^{\pi/2}(\cos x-1)\,dx = (\sin x-x)\Bigg|_{-\pi/2}^{\pi/2} = 1-\frac{\pi}{2}-\left(-1-\left(-\frac{\pi}{2}\right)\right) = 2-\pi.$

5.3.47 $\displaystyle\int_{1}^{2}\frac{3}{t}\,dt = 3\ln|t|\Bigg|_{1}^{2} = 3\ln 2-3\ln 1 = \ln 8.$

5.3.48 $\displaystyle\int_{4}^{9}\frac{x-\sqrt{x}}{x^2}\,dx = \int_{4}^{9}(x^{-1}-x^{-3/2})\,dx = (\ln|x|+2x^{-1/2})\Bigg|_{4}^{9} = \ln 9+\frac{2}{3}-(\ln 4+1) = \ln\left(\frac{9}{4}\right)-\frac{1}{3}.$

5.3.49 $\displaystyle\int_{1}^{8}\sqrt[3]{y}\,dy = \frac{3}{4}y^{4/3}\Bigg|_{1}^{8} = 12-\frac{3}{4} = \frac{45}{4}.$

5.3.50 $\displaystyle\frac{1}{2}\int_{0}^{\ln 2} e^x\,dx = \frac{1}{2}\left(e^x\Bigg|_{0}^{\ln 2}\right) = \frac{1}{2}(2-1) = \frac{1}{2}.$

5.3.51

$$\begin{aligned}\int_{1}^{4}\frac{x-2}{\sqrt{x}}\,dx &= \int_{1}^{4}\left(\frac{x}{\sqrt{x}}-\frac{2}{\sqrt{x}}\right)dx = \int_{1}^{4}\left(x^{1/2}-2x^{-1/2}\right)dx\\ &= \left(\frac{2}{3}x^{3/2}-4x^{1/2}\right)\Bigg|_{1}^{4} = \frac{16}{3}-8-\left(\frac{2}{3}-4\right) = \frac{14}{3}-\frac{12}{3} = \frac{2}{3}.\end{aligned}$$

5.3.52 $\displaystyle\int_{1}^{2}\left(\frac{2}{s}-\frac{4}{s^3}\right)ds = \left(2\ln|s|+\frac{2}{s^2}\right)\Bigg|_{1}^{2} = 2\ln 2+\frac{1}{2}-(0+2) = \ln 4-\frac{3}{2}.$

5.3.53 $\displaystyle\int_0^{\pi/3} \sec x \tan x\, dx = \sec x \,\Big|_0^{\pi/3} = 2 - 1 = 1.$

5.3.54 $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta = -\cot\theta \,\Big|_{\pi/4}^{\pi/2} = 0 + 1 = 1.$

5.3.55 $\displaystyle\int_{\pi/4}^{3\pi/4} (\cot^2 x + 1)\, dx = \int_{\pi/4}^{3\pi/4} \csc^2 x\, dx = -\cot x\Big|_{\pi/4}^{3\pi/4} = -(-1-1) = 2.$

5.3.56 $\displaystyle\int_0^1 10e^{x+3}\, dx = 10e^{x+3}\Big|_0^1 = 10e^4 - 10e^3 = 10e^3(e-1).$

5.3.57 $\displaystyle\int_1^{\sqrt{3}} \frac{1}{1+x^2}\, dx = \tan^{-1}\Big|_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \pi/3 - \pi/4 = \pi/12.$

5.3.58 $\displaystyle\int_0^{\pi/4} \sec x(\sec x + \cos x)\, dx = \int_0^{\pi/4} (\sec^2 x + 1)\, dx = (\tan x + x)\Big|_0^{\pi/4} = 1 + \pi/4 - (0+0) = 1 + \pi/4.$

5.3.59 $\displaystyle\int_1^2 \frac{z^2+4}{z}\, dz = \int_1^2 \left(z + \frac{4}{z}\right)\, dz = \left(\frac{z^2}{2} + 4\ln z\right)\Big|_1^2 = 2 + 4\ln 2 - \left(\frac{1}{2} + 0\right) = \ln 16 + \frac{3}{2}.$

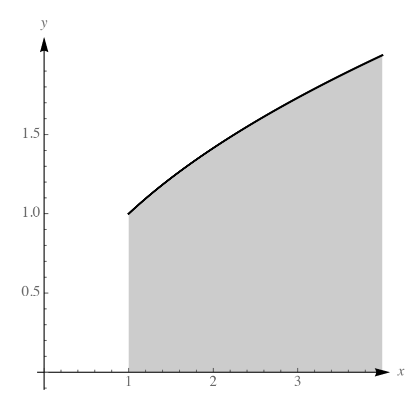
5.3.60 $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\,\Big|_{\sqrt{2}}^2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$

5.3.61 $\displaystyle\int_0^{\pi} f(x)\, dx = \int_0^{\pi/2} (\sin x + 1)\, dx + \int_{\pi/2}^{\pi} (2\cos x + 2)\, dx = (-\cos x + x)\Big|_0^{\pi/2} + (2\sin x + 2x)\Big|_{\pi/2}^{\pi} =$ $(0 + \pi/2) - (-1+0) + (0 + 2\pi) - (2+\pi) = 3\pi/2 - 1.$

5.3.62 $\displaystyle\int_1^3 g(x)\, dx = \int_1^2 (3x^2 + 4x + 1)\, dx + \int_2^3 (2x+5)\, dx = (x^3 + 2x^2 + x)\Big|_1^2 + (x^2 + 5x)\Big|_2^3 = (8+8+2) -$ $(1+2+1) + (9+15) - (4+10) = 24.$

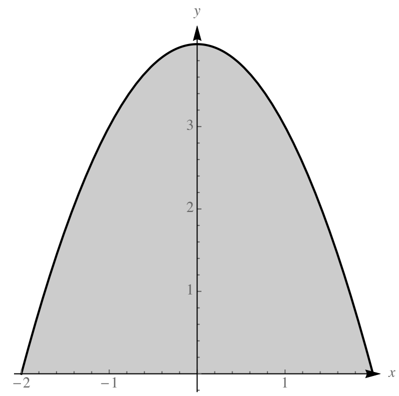
5.3.63

The area (and net area) of this region is given by $\displaystyle\int_1^4 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\,\Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$



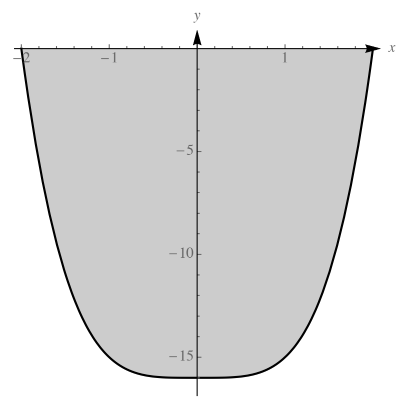
5.3.64

The area (and net area) of this region is given by $\int_{-2}^{2}(4-x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$.



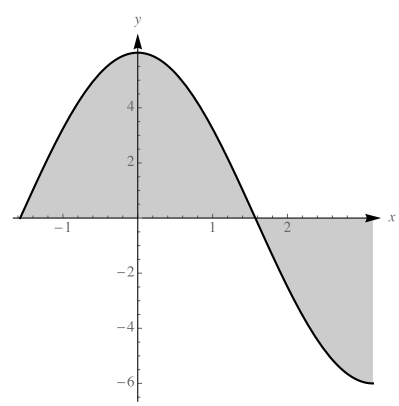
5.3.65

The net area of this region is given by $\int_{-2}^{2}(x^4 - 16)\,dx = \left(\frac{x^5}{5} - 16x\right)\Big|_{-2}^{2} = \frac{32}{5} - 32 - \left(-\frac{32}{5} + 32\right) = \frac{64}{5} - 64 = -\frac{256}{5}$. Thus the area is $\frac{256}{5}$.



5.3.66

The net area of this region is given by $\int_{-\pi/2}^{\pi} 6\cos x\,dx = 6\sin x\Big|_{-\pi/2}^{\pi} = 0 - -6 = 6$. The area is given by $\int_{-\pi/2}^{\pi/2} 6\cos x\,dx - \int_{\pi/2}^{\pi} 6\cos x\,dx = 6\sin x\Big|_{-\pi/2}^{\pi/2} - 6\sin x\Big|_{\pi/2}^{\pi} = 6 - -6 - (0-6) = 18$.



5.3.67 Because this region is below the axis, the area of it is given by $-\int_{2}^{4}(x^2-25)\,dx = -\left(\frac{x^3}{3} - 25x\right)\Big|_{2}^{4} = -\left(\frac{64}{3} - 100 - \left(\frac{8}{3} - 50\right)\right) = 50 - \frac{56}{3} = \frac{94}{3}$.

5.3.68 Because the function is below the axis between −1 and 1, and is above the axis between 1 and 2, the area of the bounded region is given by $-\int_{-1}^{1}(x^3-1)\,dx+\int_{1}^{2}(x^3-1)\,dx = -\left(\frac{x^4}{4}-x\right)\Big|_{-1}^{1}+\left(\frac{x^4}{4}-x\right)\Big|_{1}^{2} = -\left(\frac{1}{4}-1-\left(\frac{1}{4}+1\right)\right)+\left(4-2-\left(\frac{1}{4}-1\right)\right) = 2+2.75 = 4.75.$

5.3.69 Because this region is below the axis, the area of it is given by $-\int_{-2}^{-1}\frac{1}{x}\,dx = -\left(\ln|x|\,\Big|_{-2}^{-1}\right) = \ln 2-\ln 1 = \ln 2.$

5.3.70 Because the function is above the axis between -1 and 0 and is below the axis between 0 and 2, the area is given by $\int_{-1}^{0}(x^3-x^2-2x)\,dx-\int_{0}^{2}(x^3-x^2-2x)\,dx = \left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Big|_{-1}^{0}-\left(\frac{x^4}{4}-\frac{x^3}{3}-x^2\right)\Big|_{0}^{2} = \left(0-\left(\frac{1}{4}+\frac{1}{3}-1\right)\right)-\left(4-\frac{8}{3}-4-0\right) = \frac{5}{12}+\frac{8}{3} = \frac{37}{12}.$

5.3.71 The area is given by

$$-\int_{-\pi/4}^{0}\sin x\,dx+\int_{0}^{3\pi/4}\sin x\,dx = \left(\cos x\,\Big|_{-\pi/4}^{0}\right)+\left(-\cos x\,\Big|_{0}^{3\pi/4}\right) = \left(1-\frac{\sqrt{2}}{2}\right)+\left(1+\frac{\sqrt{2}}{2}\right) = 2.$$

5.3.72 Because this region is below the axis, the area is given by $-\int_{\pi/2}^{\pi}\cos x\,dx = -\left(\sin x\,\Big|_{\pi/2}^{\pi}\right) = \sin(\pi/2)-\sin(\pi) = 1.$

5.3.73 By a direct application of the Fundamental Theorem, this is x^2+x+1.

5.3.74 $\frac{d}{dx}\int_{x}^{1}e^{t^2}\,dt = -\frac{d}{dx}\int_{1}^{x}e^{t^2}\,dt = -e^{x^2}.$

5.3.75 This is $-\frac{d}{dx}\int_{1}^{x}\sqrt{t^4+1}\,dt = -\sqrt{x^4+1}.$

5.3.76 This is $-\frac{d}{dx}\int_{0}^{x}\frac{dp}{p^2+1} = \frac{-1}{x^2+1}.$

5.3.77 By the Fundamental Theorem and the chain rule, this is $\frac{1}{x^6}\cdot 3x^2 = \frac{3}{x^4}.$

5.3.78 $\frac{d}{dx}\int_{0}^{x^2}\frac{1}{t^2+4}\,dt = \frac{2x}{x^4+4}.$

5.3.79 $\frac{d}{dx}\int_{0}^{\cos x}(t^4+6)\,dt = -(\cos^4 x+6)\sin x.$

5.3.80 $\frac{d}{dw}\int_{0}^{\sqrt{w}}\ln(x^2+1)\,dx = \ln(w+1)\cdot\frac{1}{2\sqrt{w}} = \frac{\ln(w+1)}{2\sqrt{w}}.$

5.3.81 $\frac{d}{dz}\int_{\sin z}^{10}\frac{dt}{t^4+1} = -\frac{d}{dz}\int_{10}^{\sin z}\frac{dt}{t^4+1} = -\frac{1}{\sin^4+1}\cdot\cos z = -\frac{\cos z}{\sin^4 z+1}.$

5.3.82 $\frac{d}{dy}\int_{y^3}^{10}\sqrt{x^6+1}\,dx = -\frac{d}{dy}\int_{10}^{y^3}\sqrt{x^6+1}\,dx = -\sqrt{y^{18}+1}\cdot 3y^2 = -3y^2\sqrt{y^{18}+1}.$

5.3.83 $\frac{d}{dt}\left(\int_{1}^{t}\frac{3}{x}\,dx-\int_{t^2}^{1}\frac{3}{x}\,dx\right) = \frac{d}{dt}\int_{1}^{t}\frac{3}{x}\,dx+\frac{d}{dt}\int_{1}^{t^2}\frac{3}{x}\,dx = \frac{3}{t}+\frac{6t}{t^2} = \frac{9}{t}.$

5.3.84 $\frac{d}{dt}\left(\int_0^t \frac{dx}{1+x^2} + \int_0^{1/t} \frac{dx}{1+x^2}\right) = \frac{1}{1+t^2} + \frac{1}{1+(1/t)^2}\left(-\frac{1}{t^2}\right) = \frac{1}{1+t^2} - \frac{1}{1+t^2} = 0.$

5.3.85 This can be written as

$$\frac{d}{dx}\left(\int_{-x}^0 \sqrt{1+t^2}\,dt + \int_0^x \sqrt{1+t^2}\,dt\right) = \frac{d}{dx}\left(-\int_0^{-x} \sqrt{1+t^2}\,dt + \int_0^x \sqrt{1+t^2}\,dt\right)$$
$$= -\sqrt{1+(-x)^2}(-1) + \sqrt{1+x^2} = 2\sqrt{1+x^2}.$$

5.3.86 This can be written as

$$\frac{d}{dx}\left(\int_{e^x}^0 \ln(t^2)\,dt + \int_0^{e^{2x}} \ln(t^2)\,dt\right) = \frac{d}{dx}\left(-\int_0^{e^x} \ln(t^2)\,dt + \int_0^{e^{2x}} \ln(t^2)\,dt\right)$$
$$= -\ln((e^x)^2)\cdot e^x + \ln((e^{2x})^2)\cdot 2e^{2x} = -2xe^x + 8xe^{2x} = 2xe^x(4e^x - 1).$$

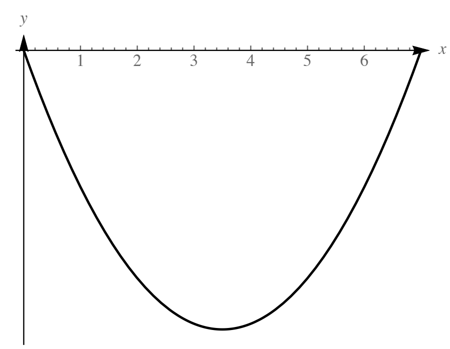
5.3.87

(a) matches with (C) – its area function is increasing linearly.
(b) matches with (B) – its area function increases then decreases.
(c) matches with (D) – its area function is always increasing on $[0, b]$, although not linearly.
(d) matches with (A) – its area function decreases at first and then eventually increases.

5.3.88

a. It appears that $A(x) = 0$ for $x = 0$ and $x = 10$.
b. A has a local minimum at $x = 5$ where the area function changes from decreasing to increasing.

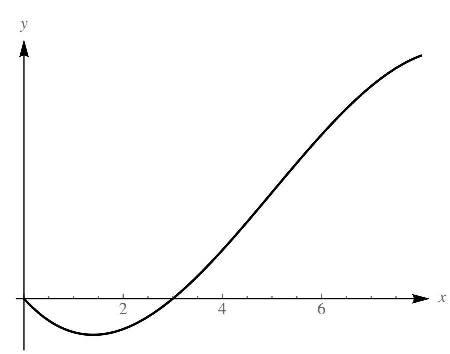
c.



5.3.89

a. It appears that $A(x) = 0$ for $x = 0$ and at about $x = 3$.
b. A has a local minimum at about $x = 1.5$ where the area function changes from decreasing to increasing, and a local max at around $x = 8.5$ where the area function changes from increasing to decreasing.

c.

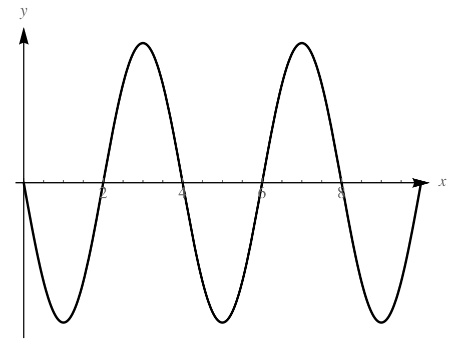


5.3.90

a. It appears that $A(x) = 0$ for $x = 0$, $x = 2$, $x = 4$, $x = 6$, $x = 8$, and $x = 10$.

b. A has a local minimum at $x = 1$, $x = 5$, and $x = 9$ where the area function changes from decreasing to increasing, and a local maximum at $x = 3$ and $x = 7$ where the area function changes from increasing to decreasing.

c.

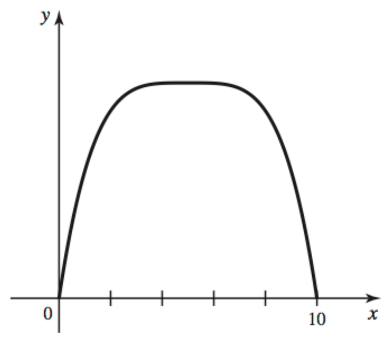


5.3.91

a. It appears that $A(x) = 0$ for $x = 0$ and $x = 10$.

b. A has a local maximum at $x = 5$ where the area function changes from increasing to decreasing.

c.



5.3.92 $A(1) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. $A(2) = \frac{1}{2} \cdot 2 \cdot 2 = 2$. $A(4) = 2 + 2^2 = 6$. $A(6) = 6 + \frac{1}{4}\pi \cdot 2^2 = 6 + \pi$.

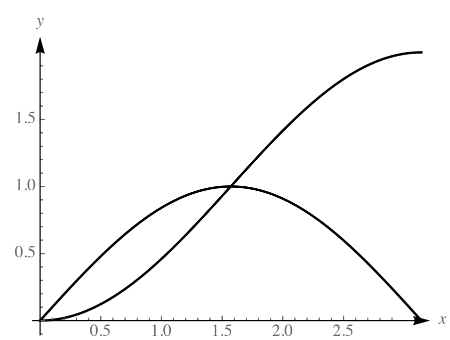
5.3.93 $A(2) = -\frac{1}{4}\pi \cdot 2^2 = -\pi$. $A(5) = -\pi + \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} - \pi$. $A(8) = \frac{9}{2} - \pi + \frac{1}{2} \cdot 3 \cdot 3 = 9 - \pi$. $A(12) = 9 - \pi - \frac{1}{2} \cdot 4 \cdot 2 = 5 - \pi$.

5.3.94

a. $A(x) = \int_0^x \sin t \, dt = -\cos t \Big|_0^x = -\cos x - (-1) = 1 - \cos x$.

c. $A(\pi/2) = 1 - \cos(\pi/2) = (1 - 0) = 1$ and $A(\pi) = 1 - \cos \pi = 1 - (-1) = 2$. The area under the curve between 0 and $\pi/2$ is the same as the area under the curve between $\pi/2$ and π.

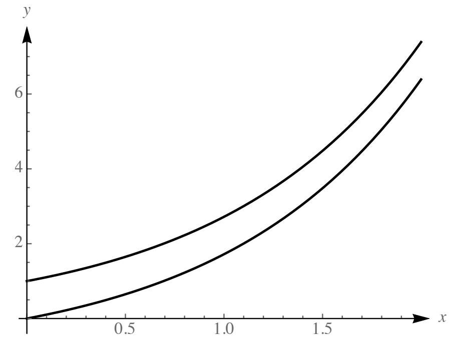
b.



5.3.95

a. $A(x) = \int_0^x e^t\,dt = e^t \Big|_0^x = e^x - (1)$.

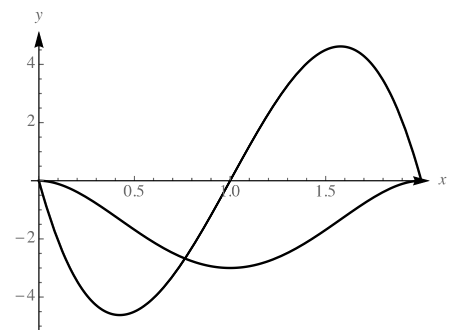
c. $A(\ln 2) = e^{\ln 2} - 1 = 2 - 1 = 1$. $A(\ln 4) = e^{\ln 4} - 1 = 4 - 1 = 3$. There is twice as much area under the curve between $\ln 2$ and $\ln 4$ as there is between 0 and $\ln 2$.

b.


5.3.96

a. $A(x) = \int_0^x -12t(t-1)(t-2)\,dt = \int_0^x -12t^3 + 36t^2 - 24t\,dt = \left(-3t^4 + 12t^3 - 12t^2\right) \Big|_0^x = -3x^4 + 12x^3 - 12x^2$.

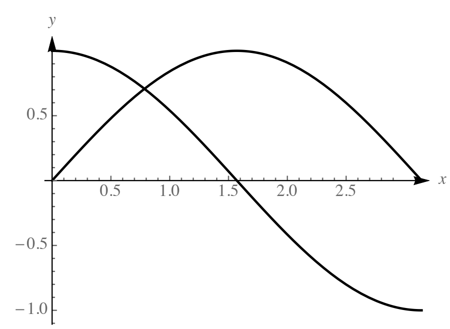
c. $A(1) = -3 + 12 - 12 = -3$. $A(2) = -48 + 96 - 48 = 0$. The area bounded between the x-axis and the curve on $[0, 1]$ is equal to the area bounded between the x-axis and the curve on $[1, 2]$.

b.


5.3.97

a. $A(x) = \int_0^x \cos t\,dt = \sin t \Big|_0^x = \sin x$.

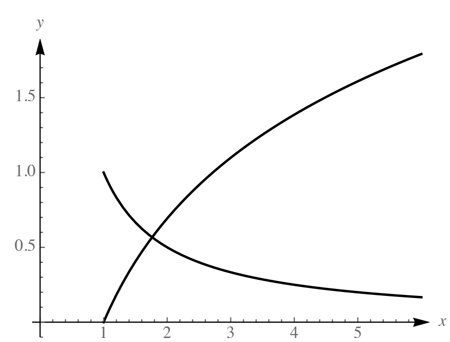
c. $A(\pi/2) = 1$. $A(\pi) = 0$.

b.


5.3.98

a. $A(x) = \int_1^x 1/t\,dt = \ln t \Big|_1^x = \ln x$.

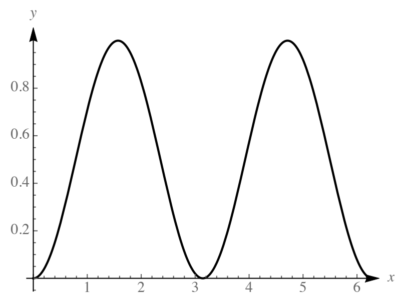
c. $A(4) = \ln 4$ and $A(6) = \ln 6$.

b.


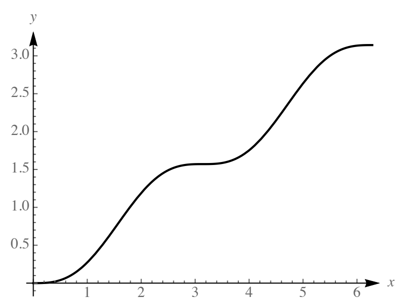
5.3.99 $f'(x) = x^2(x-3)(x-4)$, so the critical points of f are $x = 0$, $x = 3$, and $x = 4$. $f' > 0$ on $(-\infty, 0)$ and on $(0,3)$ and on $(4,\infty)$, so f is increasing on those intervals, while $f' < 0$ on $(3,4)$, so f is decreasing on that interval.

5.3.100 $g(x) = -\int_0^x \frac{t}{t^2+1}\,dt$, so $g'(x) = -\frac{x}{x^2+1}$ and $g''(x) = -\frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{x^2-1}{(x^2+1)^2}$. $g''(x) < 0$ on $(-1,1)$ (so g is concave down there), while $g''(x) > 0$ on $(-\infty,-1)$ and on $(1,\infty)$, so g is concave up on those intervals.

5.3.101

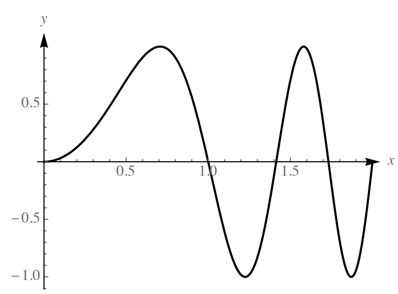
a. 

b. $g'(x) = \sin^2 x$.

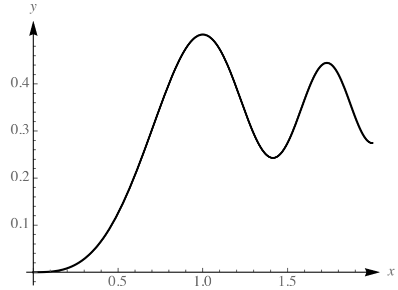
c. 

Note that g' is always positive, so g is always increasing. There are inflection points where g' changes from increasing to decreasing, and vice versa.

5.3.102

a. 

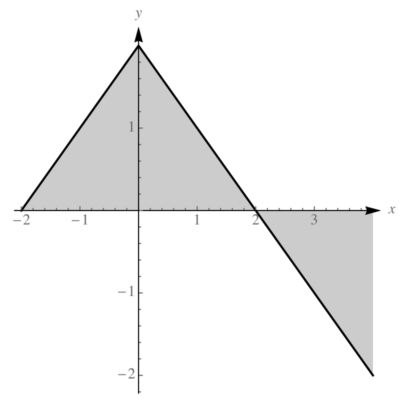
b. $g'(x) = \sin(\pi x^2)$.

c. 

Note that g is increasing where $g' > 0$ and g is decreasing when $g' < 0$. Also, where g' is increasing, g is concave up and where g' is decreasing, g is concave down.

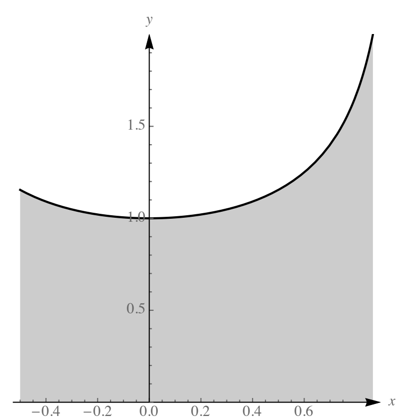
5.3.103

We can use geometry – there is a triangle with base 4 and height 2 and a triangle with base 2 and height 2, so the total area is $\frac{1}{2}\cdot 4\cdot 2+\frac{1}{2}\cdot 2\cdot 2 = 6$.



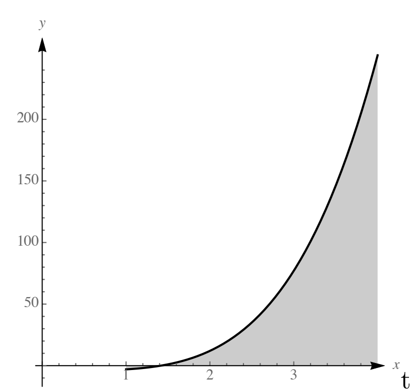
5.3.104

Because the region is above the axis, we can simply compute $\displaystyle\int_{-1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \,\Big|_{-1/2}^{\sqrt{3}/2} = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2}$.



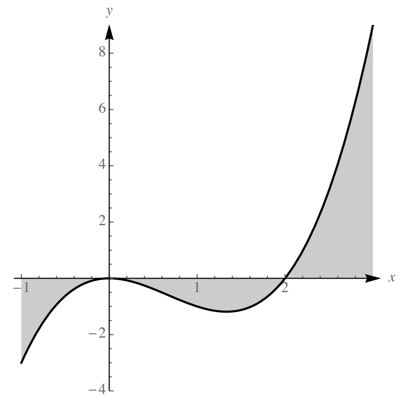
5.3.105

Because the region is below the axis on $[1, \sqrt{2}]$ and above on $[\sqrt{2}, 4]$ we need to compute $\displaystyle\int_{\sqrt{2}}^{4} (x^4 - 4)\,dx - \int_{1}^{\sqrt{2}} (x^4 - 4)\,dx = \left(\frac{x^5}{5} - 4x\right)\Big|_{\sqrt{2}}^{4} - \left(\frac{x^5}{5} - 4x\right)\Big|_{1}^{\sqrt{2}} = \frac{1024}{5} - 16 - (4\sqrt{2}/5 - 4\sqrt{2}) - (4\sqrt{2}/5 - 4\sqrt{2}) + \frac{1}{5} - 4 = 185 + \frac{32\sqrt{2}}{5}$



5.3.106

Because the function is below (or touching) the axis on $[-1,2]$ and above on $[2,3]$, the area is given by $\int_2^3 (x^3 - 2x^2)\,dx - \int_{-1}^2 (x^3 - 2x^2)\,dx = \left(\frac{x^4}{4} - \frac{2x^3}{3}\right)\Big|_2^3 - \left(\frac{x^4}{4} - \frac{2x^3}{3}\right)\Big|_{-1}^2 = \left(\frac{81}{4} - 18\right) - \left(4 - \frac{16}{3}\right) - \left(4 - \frac{16}{3}\right) + \left(\frac{1}{4} + \frac{2}{3}\right) = \frac{41}{2} - 26 + \frac{34}{3} = \frac{35}{6}$.



5.3.107

a. True. The net area under the curve increases as x increases, as long as f is above the axis.

b. True. The net area decreases as x increases, as long as f is below the axis.

c. False. These do not have the same derivative, so they are not antiderivatives of the same function.

d. True, because the two functions differ by a constant, and thus have the same derivative.

e. True, because the derivative of a constant is zero.

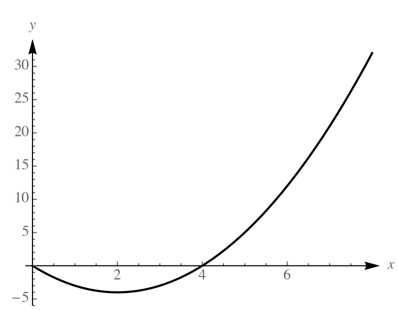
5.3.108

$$\lim_{x\to 2} \frac{\int_2^x \sqrt{t^2+t+3}\,dt}{x^2-4} = \lim_{x\to 2} \frac{\sqrt{x^2+x+3}}{2x} = \frac{3}{4}.$$

5.3.109 Because $\frac{d}{db}\int_{-1}^b x^2(3-x)\,dx = b^2(3-b)$ we see that this function of b has critical points at $b=0$ and $b=3$. Note also that the integrand is positive on $[0,3]$, but is negative on $[3,\infty)$. So the maximum for this area function occurs at $b=3$.

5.3.110 The function $f(x) = 8+2x-x^2 = (4-x)(2+x)$ is 0 for $x=4$ and $x=-2$, and is positive on $(-2,4)$ and negative on $(-\infty,-2)$ and on $(4,\infty)$. Thus, the largest possible value for the area $\int_a^b f(x)\,dx$ is when $a=-2$ and $b=4$.

5.3.111

a. 

b. We seek b so that $\int_0^b (x^2-4x) = 0$ for $b>0$. We have $\left(\frac{x^3}{3} - 2x^2\right)\Big|_0^b = \frac{b^3}{3} - 2b^2 = 0$, which occurs for $\frac{b}{3} = 2$, or $b=6$.

c. We seek b so that $\int_0^b (x^2-ax) = 0$ for $b>0$. We have $\left(\frac{x^3}{3} - \frac{ax^2}{2}\right)\Big|_0^b = \frac{b^3}{3} - \frac{ab^2}{2} = 0$, which occurs for $\frac{b}{3} = \frac{a}{2}$, or $b = \frac{3a}{2}$.

5.3.112 If $0 < x < a$, then $x > 0$, $x - a < 0$, and $x - b < 0$, so the product of these three quantities is positive. If $a < x < b$, then $x > 0$, $x - a > 0$, and $x - b < 0$, so the product of these three quantities is negative. The region between $x = 0$ and $x = a$, which is above the x-axis, has area

$$\int_0^a x(x-a)(x-b)\,dx = \int_0^a (x^3 - (a+b)x^2 + abx)\,dx = \left(\frac{x^4}{4} - \frac{a+b}{3}x^3 + \frac{ab}{2}x\right)\Bigg|_0^a = \frac{a^3(2b-a)}{12},$$

while the region between $x = a$ and $x = b$, which is below the x-axis, has area

$$-\int_a^b x(x-a)(x-b)\,dx = -\int_a^b (x^3 - (a+b)x^2 + abx)\,dx = -\left(\frac{x^4}{4} - \frac{a+b}{3}x^3 + \frac{ab}{2}x\right)\Bigg|_a^b = \frac{(b-a)^3(a+b)}{12}.$$

These are equal when $a^3(2b-a) = (b-a)^3(a+b)$. Divide through by a^4 to obtain

$$2 \cdot \frac{b}{a} - 1 = \left(\frac{b}{a} - 1\right)^3 \left(1 + \frac{b}{a}\right).$$

Let $c = \dfrac{b}{a}$; then $2c - 1 = (c-1)^3(c+1) = c^4 - 2c^3 + 2c - 1$, so that $c^4 - 2c^3 = 0$. Because $b > 0$ we must have $c > 0$, and the only nonzero root is $c = 2$. Thus $c = \dfrac{b}{a} = 2$, so $b = 2a$.

5.3.113 Differentiating both sides of the given equation yields $f(x) = -2\sin x + 3$.

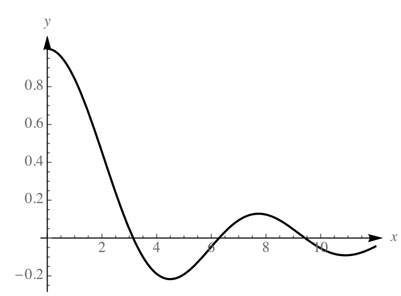
5.3.114 Suppose that a maximum of A occurs at $x = c$, and that A is not a constant function near c. Then A changes from increasing to decreasing at c. But because A is the net area from 0 to x, the only way for A to change from increasing to decreasing is for f to change from above the axis to below, so it must be the case that $f > 0$ to the left of c and $f < 0$ to the right of c, but because f is continuous, this implies that $f(c) = 0$. An analogous argument holds for the case when A has a minimum at c.

For $f(x) = x^2 - 10x$, note that $A(x) = \displaystyle\int_0^x (t^2 - 10t)\,dt = \frac{x^3}{3} - 5x^2$, and that this function has a minimum at $x = 10$, because $A'(x) = x^2 - 10x = f(x)$ changes from negative to positive at $x = 10$, so that A changes from decreasing to increasing there.

5.3.115 Using a computer or calculator, we obtain:

x	500	1000	1500	2000
$S(x)$	1.5726	1.57023	1.57087	1.57098

This appears to be approaching $\frac{\pi}{2}$.
Note that between 0 and π, the area is approximately half the area of a rectangle with height 1 and base π, and then from π on there is approximately as much area above the axis as below.



5.3.116 By the Fundamental Theorem, $S'(x) = \dfrac{\sin x}{x}$, so $S''(x) = \dfrac{x\cos x - \sin x}{x^2} = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$. Thus, $S'''(x) = \dfrac{-x\sin x - \cos x}{x^2} - \dfrac{x^2\cos x - 2x\sin x}{x^4} = \dfrac{-\sin x}{x} - \dfrac{2\cos x}{x^2} + \dfrac{2\sin x}{x^3}$.

$$xS'(x) + 2S''(x) + xS'''(x) = \sin x + \frac{2\cos x}{x} - \frac{2\sin x}{x^2} + -\sin x - \frac{2\cos x}{x} + \frac{2\sin x}{x^2} = 0.$$

5.3.117 By the Fundamental Theorem, $S'(x) = \sin x^2$, so $S''(x) = 2x\cos x^2$, so

$$S'(x)^2 + \left(\frac{S''(x)}{2x}\right)^2 = \sin^2 x^2 + \cos^2 x^2 = 1.$$

5.3.118

a. If m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[a,x]$ of width h_i, we have $m^* h_i \le f(x_i^*)h_i \le M^* h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[a,x]$ and taking the limit as $n \to \infty$ gives $m^*(x-a) \le \int_a^x f(t)\,dt \le M^*(x-a)$, so $m^*(x-a) \le A(x) \le M^*(x-a)$. Now consider $\lim_{x\to a^+} m^*(x-a) = \lim_{x\to a^+} M^*(x-a) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x\to a^+} A(x) = 0 = A(a)$, so A is continuous from the right at $x = a$.

b. First note that $A(b) - A(x) = \int_a^b f(t)\,dt - \int_a^x f(t)\,dt = \int_x^b f(t)\,dt$. Now if m^* is the minimum and M^* is the maximum of f on $[a,b]$, then for every possible subinterval $[x_i, x_{i+1}]$ of $[b,x]$ of width h_i, we have $m^* h_i \le f(x_i^*)h_i \le M^* h_i$ where x_i^* is any value on $[x_i, x_{i+1}]$. Adding these up over any partition of $[b,x]$ and taking the limit as $n \to \infty$ gives $m^*(b-x) \le \int_x^b f(t)\,dt \le M^*(b-x)$, so $m^*(b-x) \le A(b) - A(x) \le M^*(b-x)$. Now consider $\lim_{x\to b^-} m^*(b-x) = \lim_{x\to b^-} M^*(b-x) = 0$. Thus by the Squeeze Theorem we must have $\lim_{x\to b^-}(A(b) - A(x)) = 0$, so $\lim_{x\to b^-} A(x) = A(b)$, and A is continuous from the left at $x = b$.

5.3.119

a. By definition of Reimann sums, $\int_a^b f'(x)\,dx$ is approximated by $\sum_{k=1}^n f'(x_{k-1})\Delta x$. But $f'(x_{k-1}) = \lim_{h\to 0} \frac{f(x_{k-1}+h) - f(x_{k-1})}{h}$. If $h = \Delta x$, then we have

$$f'(x_{k-1}) \approx \frac{f(x_{k-1}+\Delta x) - f(x_{k-1})}{\Delta x} = \frac{f(x_k) - f(x_{k-1})}{\Delta x},$$

so that

$$\int_a^b f'(x)\,dx \approx \sum_{k=1}^n \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x.$$

b. Canceling the Δx factors we obtain

$$\begin{aligned}\int_a^b f'(x)\,dx &\approx \sum_{k=1}^n \frac{f(x_k) - f(x_{k-1})}{\Delta x} \cdot \Delta x \\ &= \sum_{k=1}^n (f(x_k) - f(x_{k-1})) \\ &= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \cdots + (f(x_{n-1}) - f(x_{n-2})) + (f(x_n) - f(x_{n-1})) \\ &= f(x_n) - f(x_0) = f(b) - f(a).\end{aligned}$$

c. The analogy between the two situations is that both (a) the sum of difference quotients and (b) integral of a derivative are equal to the difference in function values at the endpoints.

5.4 Working with Integrals

5.4.1 If f is odd, it is symmetric about the origin, which guarantees that between $-a$ and a, there is as much area above the axis and under f as there is below the axis and above f, so the net area must be 0.

5.4.2 If f is even, it is symmetric about the the $y-$axis, which guarantees that the region between $-a$ and 0 has the same net area as the region between 0 and a, so $\int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

5.4.3

a. Because $\int_{-8}^{8} f(x)\,dx = 18 = 2\int_{0}^{8} f(x)\,dx$, we have $\int_{0}^{8} f(x)\,dx = \frac{18}{2} = 9$.

b. Because $xf(x)$ is an odd function when $f(x)$ is even, we have $\int_{-8}^{8} xf(x)\,dx = 0$.

5.4.4

a. Because $f(x)$ is odd, $\int_{-4}^{0} f(x)\,dx = -\int_{0}^{4} f(x)\,dx$. We have

$$\int_{-4}^{8} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{8} f(x)\,dx = -\int_{0}^{4} f(x)\,dx + \int_{0}^{8} f(x)\,dx = -3 + 9 = 6.$$

b. $\int_{-8}^{4} f(x)\,dx = \int_{-8}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx = -9 + 3 = -6$.

5.4.5 The integrand can be written as $(5x^4 + 2x^2 + 1) + (3x^3 + x)$. The function $f(x) = 5x^4 + 2x^2 + 1$ is an even function and the function $g(x) = 3x^3 + x$ is an odd function. Thus,

$$\int_{-4}^{4} (f(x) + g(x))\,dx = \int_{-4}^{4} f(x)\,dx + \int_{-4}^{4} g(x)\,dx = 2\int_{0}^{4} f(x)\,dx + 0 = 2\int_{0}^{4} (5x^4 + 2x^2 + 1)\,dx.$$

5.4.6 The number 2 should go in the first blank and the function $\cos x$ in the second.

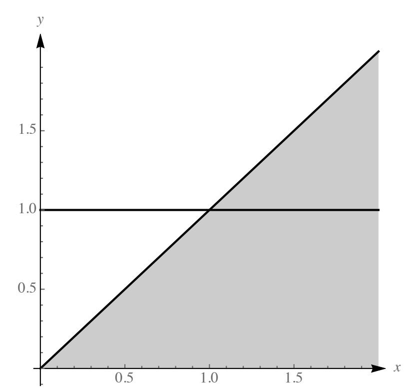
$$\int_{-\pi}^{\pi} (\sin x + \cos x)\,dx = \int_{-\pi}^{\pi} \sin x\,dx + \int_{-\pi}^{\pi} \cos x\,dx = 0 + 2\int_{0}^{\pi} \cos x\,dx = 2\int_{0}^{\pi} \cos x\,dx.$$

5.4.7 $f(x) = x^{12}$ is an even function, because $f(-x) = (-x)^{12} = x^{12} = f(x)$. $g(x) = \sin x^2$ is also even, because $g(-x) = \sin((-x)^2) = \sin x^2 = g(x)$.

5.4.8 The average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_{a}^{b} f(x)\,dx$. This is analogous to "adding up all the value of f and dividing by how many there are" – in the sense that computing the interval is like adding up all the values of the function, and dividing by $b - a$ is like dividing by how many x values there are.

5.4.9 The average value of a continuous function on a closed interval $[a, b]$ will always be between the maximum and the minimum value of f on that interval. Because the function is continuous, the Intermediate Value Theorem assures us that the function will take on each value between the maximum and the minimum somewhere on the interval.

5.4.10 Note that the area of the triangle is $\frac{1}{2} \cdot 2 \cdot 2 = 2$, so the rectangle needs to have a height of 1 and a base of 2 so that its area is 2.



5.4.11 Because x^9 is an odd function, $\int_{-2}^{2} x^9\,dx = 0$.

5.4.12 Because $2x^5$ is an odd function, $\int_{-200}^{200} 2x^5\,dx = 0$.

5.4.13 $\int_{-2}^{2}(3x^8-2)\,dx = 2\int_0^2(3x^8-2)\,dx = 2\left(\frac{x^9}{3}-2x\right)\Big|_0^2 = \left(\frac{1024}{3}\right)-8=\frac{1000}{3}$.

5.4.14 $\int_{-\pi/4}^{\pi/4}\cos x\,dx = 2\int_0^{\pi/4}\cos x\,dx = 2\,(\sin x)\Big|_0^{\pi/4} = 2\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}$.

5.4.15 $\int_{-2}^{2}(x^2+x^3)\,dx = 2\int_0^2 x^2\,dx = 2\left(\frac{x^3}{3}\right)\Big|_0^2 = 2\left(\frac{8}{3}-0\right)=\frac{16}{3}$.

5.4.16 Because $x^2\sin x$ is an odd function, $\int_{-\pi}^{\pi} x^2\sin x\,dx = 0$

5.4.17 Note that the first two terms of the integrand form an odd function, and the last two terms form an even function. $\int_{-2}^{2}(x^9-3x^5+2x^2-10)\,dx = 2\int_0^2(2x^2-10)\,dx = 2\left(\frac{2x^3}{3}-10x\right)\Big|_0^2 = \frac{32}{3}-40=-\frac{88}{3}$.

5.4.18 $\int_{-\pi/2}^{\pi/2} 5\sin x\,dx = 0$ because the integrand is an odd function.

5.4.19 Because the integrand is an odd function and the interval is symmetric about 0, this integral's value is 0.

5.4.20 $\int_{-1}^{1}(1-|x|)\,dx = 2\int_0^1(1-x)\,dx = 2\left(x-\frac{x^2}{2}\right)\Big|_0^1 = 2\left(1-\frac{1}{2}\right)=1$.

5.4.21 $\sec^2 x$ is even, so the value of this integral is $2\int_0^{\pi/4}\sec^2 x\,dx = 2\,(\tan x)\Big|_0^{\pi/4} = 2\cdot(1-0)=2$.

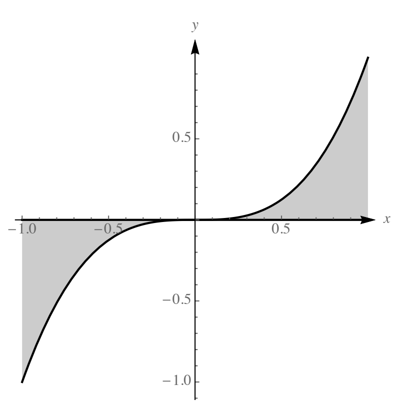
5.4.22 Recall that the tangent function is an odd function, so the value of this integral is 0.

5.4.23 The integrand is an odd function, so the value of this integral is zero.

5.4.24 The function $1-|x|^3$ is even, so the value of this integral is $2\int_0^2(1-x^3)\,dx = 2\left(x-\frac{x^4}{4}\right)\Big|_0^2 = 2(2-4)=-4$.

5.4.25

The average value is $\frac{1}{1-(-1)}\int_{-1}^{1}x^3\,dx = \frac{1}{2}\left(x^4/4\right)\Big|_{-1}^{1} = 0$.

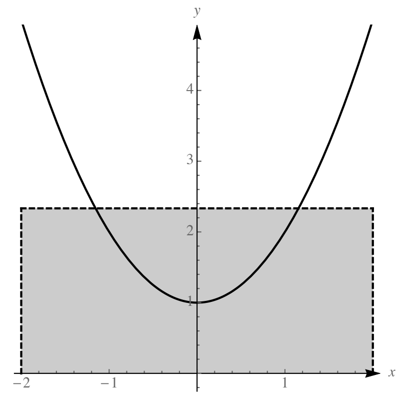


5.4.26

The average value is

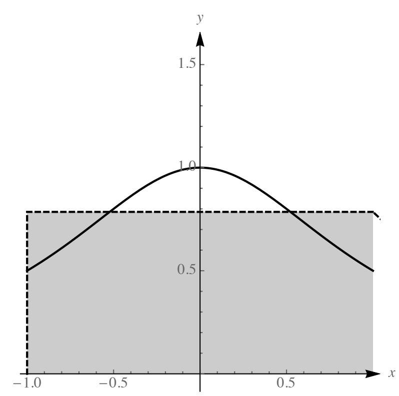
$$\frac{1}{2-(-2)}\int_{-2}^{2}(x^2+1)\,dx$$

$$= \frac{1}{4}\left(x^3/3+x\right)\Big|_{-2}^{2} = \frac{8/3+2-(-8/3-2)}{4} = \frac{7}{3}.$$



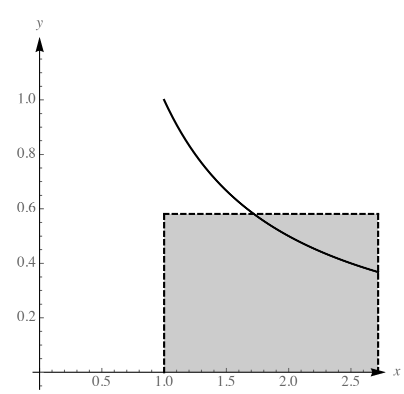
5.4.27

The average value is
$$\frac{1}{1-(-1)}\int_{-1}^{1}\frac{1}{x^2+1}\,dx \;=\; \frac{1}{2}\tan^{-1}x\,\Big|_{-1}^{1} \;=\; \frac{\pi/4-(-\pi/4)}{2} = \frac{\pi}{4}.$$



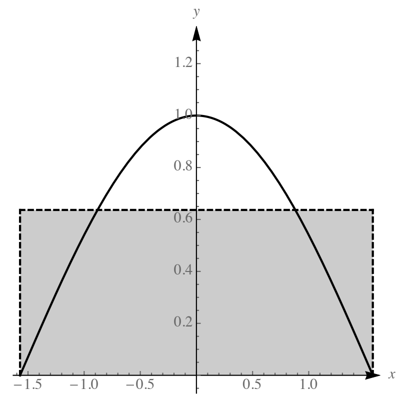
5.4.28

The average value is $\displaystyle\frac{1}{e-1}\int_{1}^{e}\frac{1}{x}\,dx = \frac{1}{e-1}\left(\ln|x|\right)\Big|_{1}^{e} = \frac{1}{e-1} \approx 0.582.$



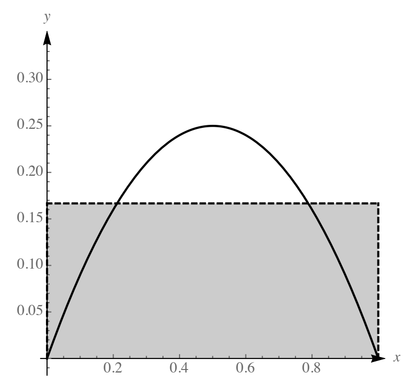
5.4.29

The average value is
$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{1}{\pi}\,(\sin x)\Big|_{-\pi/2}^{\pi/2}$
$= \frac{1}{\pi}\cdot(1 - -1) = \frac{2}{\pi} \approx 0.6366.$



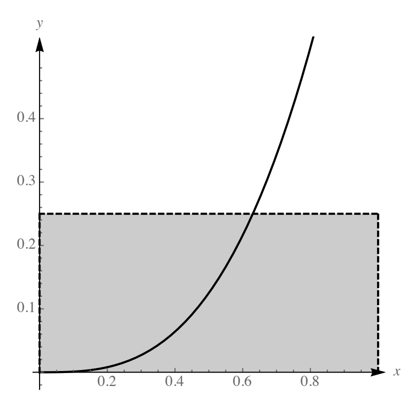
5.4.30

The average value is $\frac{1}{1}\int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \approx 0.1667.$



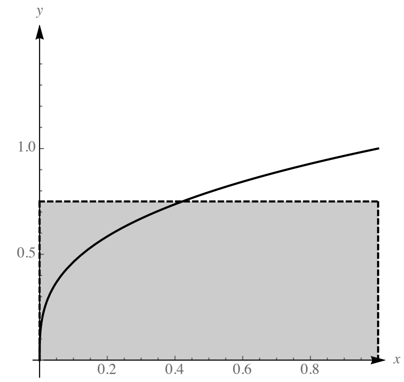
5.4.31

The average value is $\frac{1}{1}\int_0^1 x^n\,dx = \left(\frac{x^{n+1}}{n+1}\right)\Big|_0^1 = \frac{1}{n+1}$. The picture shown is for the case $n = 3$.



5.4.32

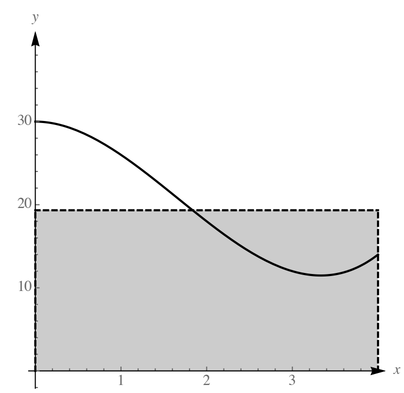
The average value is $\frac{1}{1}\int_0^1 x^{1/n}\,dx = \left(\frac{x^{(n+1)/n}}{(n+1)/n}\right)\Big|_0^1 = \frac{n}{n+1}$. The picture shown is for the case $n = 3$.



5.4.33 The average distance to the axis is given by $\frac{1}{20}\int_0^{20} 30x(20-x)\,dx$. This is equal to $\frac{1}{20}\int_0^{20}(600x - 30x^2)\,dx = \frac{1}{20}\left(300x^2 - 10x^3\right)\Big|_0^{20} = 2000$.

5.4.34

The average value is $\frac{1}{4-0}\int_0^4 (x^3 - 5x^2 + 30)\,dx = \frac{1}{4}\left(\frac{x^4}{4} - \frac{5x^3}{3} + 30x\right)\Big|_0^4 = \frac{1}{4}(64 - \frac{320}{3} + 120) - 0 = \frac{58}{3}$.



5.4.35 The average velocity is

$$\frac{1}{6-0}\int_0^6 (t^2+3t)\,dt = \frac{1}{6}\left(\frac{t^3}{3}+\frac{3t^2}{2}\right)\Big|_0^6 = \frac{1}{6}(72+54-0) = \frac{1}{6}(126) = 21\,\text{m/s}.$$

5.4.36 The average velocity is

$$\frac{1}{4-0}\int_0^4 (-32t+64)\,dt = \frac{1}{4}\left(-16t^2+64t\right)\Big|_0^4 = \frac{1}{4}(-256+256) = 0\,\text{ft/s}.$$

5.4.37 The average height is $\frac{1}{\pi}\int_0^{\pi} 10\sin x\,dx = \frac{1}{\pi}(-10\cos x)\Big|_0^{\pi} = \frac{1}{\pi}(10 - -10) = \frac{20}{\pi}$.

5.4.38 The average height is $\frac{1}{2\pi}\int_{-\pi}^{\pi}(5+5\cos x)\,dx = \frac{1}{2\pi}(5x+5\sin x)\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}(5\pi - -5\pi) = 5$.

5.4.39 The average value is $\frac{1}{4}\int_0^4 (8-2x)\,dx = \frac{1}{4}\left(8x - x^2\right)\Big|_0^4 = 4$. The function has a value of 4 when $8 - 2x = 4$, which occurs when $x = 2$.

5.4.40 The average value is $\frac{1}{2}\int_0^2 e^x\,dx = \frac{1}{2}\left(e^x\right)\Big|_0^2 = \frac{e^2-1}{2}$.The function attains this value when $\frac{e^2-1}{2} = e^x$, which is when $x = \ln\left(\frac{e^2-1}{2}\right) \approx 1.1614$.

5.4.41 The average value is $\frac{1}{a}\int_0^a \left(1-\frac{x^2}{a^2}\right)\,dx = \frac{1}{a}\left(x - \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{2}{3}$. The function attains this value when $\frac{2}{3} = 1 - \frac{x^2}{a^2}$, which is when $x^2 = \frac{a^2}{3}$, which on the given interval occurs for $x = \sqrt{3}a/3$.

5.4.42 The average value is $\frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x\,dx = \frac{1}{4}\left(-\cos x\right)\Big|_0^\pi = \frac{1}{4}(1--1) = \frac{1}{2}$. The function attains this value when $\frac{1}{2}\cdot\frac{4}{\pi} = \sin x$, which is when $x = \sin^{-1}\frac{2}{\pi} \approx 0.690107$ and for $x \approx 2.45149$.

5.4.43 The average value is $\frac{1}{2}\int_{-1}^1 (1-|x|)\,dx = \frac{1}{2}\int_{-1}^0 (1+x)\,dx + \frac{1}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}\left(x+\frac{x^2}{2}\right)\Big|_{-1}^0 + \frac{1}{2}\left(x - \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$. The function attains this value twice, once on $[-1,0]$ when $1+x = \frac{1}{2}$ which occurs when $x = -\frac{1}{2}$, and once on $[0,1]$ when $1-x = \frac{1}{2}$ which occurs when $x = \frac{1}{2}$.

5.4.44 The average value is given by $\frac{1}{3}\int_1^4 1/x\,dx = \frac{1}{3}\left(\ln x\right)\Big|_1^4 = \frac{1}{3}(\ln 4)$. The function attains this value when $x = \frac{3}{\ln 4} \approx 2.164$.

5.4.45

a. True. Because of the symmetry, the net area between 0 and 4 will be twice the net area between 0 and 2.

b. True. This follows because the symmetry implies that the net area from a to $a+2$ is the opposite of the net area from $a-2$ to a.

c. True. If $f(x) = cx+d$ on $[a,b]$ the value at the midpoint is $c\cdot\frac{a+b}{2}+d$, and the average value is $\frac{1}{b-a}\int_a^b (cx+d)\,dx = \frac{1}{b-a}\left(\frac{cx^2}{2}+dx\right)\Big|_a^b = \frac{1}{b-a}\left(\frac{cb^2}{2}+db - \left(\frac{ca^2}{2}+da\right)\right) = \frac{c}{2}\cdot(a+b)+d$.

d. False, for example, when $a = 1$, we have that the maximum value of $x-x^2$ on $[0,1]$ occurs at $\frac{1}{2}$ and is equal to $\frac{1}{4}$, but the average value is $\int_0^1 (x-x^2)\,dx = \left(\frac{x^2}{2}-\frac{x^2}{3}\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$.

5.4.46

a. $d^2 = x^2+y^2 = x^2+b^2(1-(x^2/a^2))$. The average value of d^2 is $\frac{1}{2a}\int_{-a}^a \left(b^2 + \left(1-\frac{b^2}{a^2}\right)x^2\right)\,dx = \frac{1}{2a}\left(b^2x + \frac{\left(1-(b^2/a^2)\right)x^3}{3}\right)\Big|_{-a}^a = \frac{1}{2a}\left(b^2a+\frac{a^3}{3}-\frac{b^2a}{3} - \left(-b^2a - \frac{a^3}{3}+\frac{b^2a}{3}\right)\right) = \frac{2b^2}{3}+\frac{a^2}{3}$.

b. If $a = b = R$, the above becomes $\frac{2R^2}{3}+\frac{R^2}{3} = R^2$.

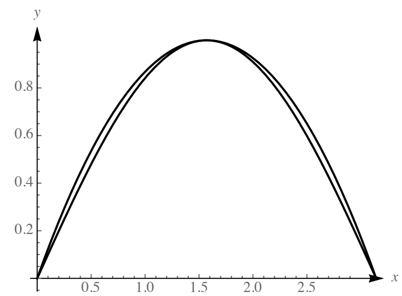
c. $D^2 = (x-\sqrt{a^2-b^2})^2 + y^2 = x^2 - 2x\sqrt{a^2-b^2}+y^2+a^2-b^2 = \left(1-\frac{b^2}{a^2}\right)x^2 - 2\sqrt{a^2-b^2}x + a^2$. So the average value of D^2 is $\frac{1}{2a}\int_{-a}^a D^2\,dx = \frac{1}{2a}\int_{-a}^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]\,dx - \frac{1}{a}\int_{-a}^a x\sqrt{a^2-b^2}\,dx = \frac{1}{a}\int_0^a \left[\left(1-\frac{b^2}{a^2}\right)x^2+a^2\right]\,dx + 0 = \frac{1}{3}(a^2-b^2)+a^2 = \frac{4a^2-b^2}{3}$.

5.4.47 The average height of the arch is given by

$$\frac{1}{630}\int_{-315}^{315}\left(630-\frac{630}{315^2}x^2\right)\,dx=\frac{630}{630}\left(x-\frac{x^3}{3\cdot 315^2}\right)\Bigg|_{-315}^{315}=(315-105-(-315+105))=420 \text{ ft.}$$

5.4.48

a. Note that $\dfrac{d}{dx}\sin x = \cos x$, which is zero when $x=\pi/2$, and because the derivative is positive on $(0,\pi/2)$ and negative on $(\pi/2,0)$, there is a maximum at $x=\pi/2$. Similarly, $\dfrac{d}{dx}\dfrac{4\pi x-4x^2}{\pi^2}=\dfrac{4\pi-8x}{\pi^2}$, which is zero when $x=\pi/2$, and this function is increasing on $(0,\pi/2)$ and decreasing on $(\pi/2,\pi)$, so it also has a maximum at $\pi/2$. Also, both functions have the value 1 at $x=\pi/2$.



b. On $(0,\pi)$, the sine function is always less than or equal to the other function.

c. The average values are

$$\frac{1}{\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}\left(-\cos x\right)\Big|_0^{\pi}=\frac{2}{\pi}$$

and

$$\frac{1}{\pi}\cdot\frac{4}{\pi^2}\int_0^{\pi}\pi x-x^2\,dx=\frac{4}{\pi^3}\left(\frac{\pi x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^{\pi}=\frac{4}{\pi^3}\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)=\frac{2}{3}.$$

5.4.49 $f(g(-x))=f(g(x))$, so $f(g(x))$ is an even function, and $\displaystyle\int_{-a}^{a}f(g(x))\,dx=2\int_0^a f(g(x))\,dx$.

5.4.50 $f(p(-x))=f(-p(x))=f(p(x))$, and thus $f(p(x))$ is an even function. Therefore, $\displaystyle\int_{-a}^{a}f(p(x))\,dx=2\int_0^a f(p(x))\,dx$.

5.4.51 $p(g(-x))=p(g(x))$, so $p(g(x))$ is an even function, and $\displaystyle\int_{-a}^{a}p(g(x))\,dx=2\int_0^a p(g(x))\,dx$.

5.4.52 $p(q(-x))=p(-q(x))=-p(q(x))$, so $p(q(x))$ is an odd function, and $\displaystyle\int_{-a}^{a}p(q(x))\,dx=0$.

5.4.53

a. The average value is $\displaystyle\int_0^1(ax-ax^2)\,dx=\left(\frac{ax^2}{2}-\frac{ax^3}{3}\right)\Bigg|_0^1=\frac{a}{2}-\frac{a}{3}=\frac{a}{6}$.

b. The function is equal to its average value when $\frac{a}{6}=ax-ax^2$ which occurs when $6x-6x^2=1$, so when $6x^2-6x+1=0$. On the given interval, this occurs for $x=\dfrac{6\pm\sqrt{12}}{12}=\dfrac{3\pm\sqrt{3}}{6}$.

5.4.54

a. $f(0)=\dfrac{\int_a^b x\,dx}{\int_a^b 1\,dx}=\dfrac{\frac{b^2-a^2}{2}}{b-a}=\dfrac{a+b}{2}$.

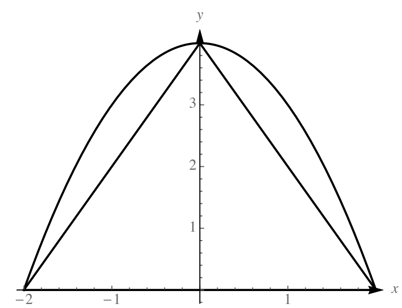
b. $f\left(-\dfrac{3}{2}\right)=\dfrac{\int_a^b x^{-1/2}\,dx}{\int_a^b x^{-3/2}\,dx}=\dfrac{\frac{2}{1}(b^{1/2}-a^{1/2})}{-\frac{2}{1}\cdot(b^{-1/2}-a^{-1/2})}\cdot\dfrac{\sqrt{ab}}{\sqrt{ab}}=\dfrac{b^{1/2}-a^{1/2}}{b^{1/2}-a^{1/2}}\cdot\sqrt{ab}=\sqrt{ab}$.

c. $f(-3) = \dfrac{\int_a^b x^{-2}\,dx}{\int_a^b x^{-3}\,dx} = \dfrac{-(b^{-1} - a^{-1})}{\frac{-1}{2} \cdot (b^{-2} - a^{-2})} \cdot \dfrac{a^2b^2}{a^2b^2} = \dfrac{2(a^2b - b^2a)}{a^2 - b^2} = \dfrac{2ab(a-b)}{(a-b)(a+b)} = \dfrac{2ab}{a+b}.$

d. $f(-1) = \dfrac{\int_a^b 1\,dx}{\int_a^b x^{-1}\,dx} = \dfrac{b-a}{\ln b - \ln a}.$

5.4.55

a. The area of the triangle is $\frac{1}{2} \cdot 2a \cdot a^2 = a^3$. The area under the parabola is $\int_{-a}^{a} (a^2 - x^2)\,dx = \left(a^2x - \frac{x^3}{3}\right)\Big|_{-a}^{a} = a^3 - \frac{a^3}{3} - \left(-a^3 + \frac{a^3}{3}\right) = 2a^3 - \frac{2a^3}{3} = \frac{4a^3}{3}$, as desired. The diagram shown is for $a = 2$.



b. The area of the rectangle described is $2a \cdot a^2 = 2a^3$, and $\frac{2}{3}$ of this is $\frac{4a^3}{3}$, which is the area under the parabola derived above.

5.4.56 The average value of f' is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f'(x)\,dx = \dfrac{f(b)-f(a)}{b-a}$. This result tells us that for a function with a continuous derivative, the average slope of the tangent line over an interval is the slope of the secant line through the endpoints of the interval.

5.4.57 Suppose f is even, so that $f(-x) = f(x)$. Then $f^n(x) = f^n(-x)$, so that f^n is an even function, no matter what the parity of n is.

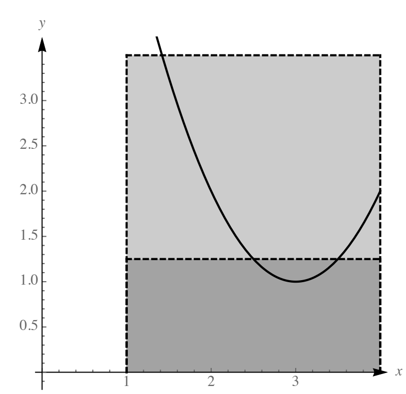
Suppose g is an odd function, so that $g(-x) = -g(x)$. Then $g^n(-x) = (-1)^n g^n(x)$, so g^n is even when n is even, and is odd when n is odd.

Summarizing, we have:

	f is even	f is odd
n is even	f^n is even	f^n is even
n is odd	f^n is even	f^n is odd

5.4.58

a. The smallest expression is the area of a rectangle on the x-axis over $[a, b]$ and height given by the value of f at the midpoint of the interval. The biggest expression is the area of a rectangle with that same base, but height equal to the average of the values of the function at the endpoints. The middle quantity represents the area under the curve.



b. After dividing, we have that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

This says that the average value of f over the interval is greater than or equal to the value of f at the average of the endpoints, and is less than or equal to the average of the values of f at the endpoints.

5.4.59

a. Note that h is continuous and differentiable on $[a, b]$, and that $h(a) = (a-b)\int_a^a f(t)\,dt + (a-a)\int_a^b g(t)\,dt = 0+0 = 0$. Also, $h(b) = (b-b)\int_a^b f(t)\,dt + (b-a)\int_b^b g(t)\,dt = 0+0 = 0$. So by Rolle's theorem, there exists c between a and b so that $h'(c) = 0$. By the Product and Sum Rules, and the Fundamental Theorem of Calculus, we have $h'(x) = \int_a^x f(t)\,dt + (x-b)f(x) + \int_x^b g(t)\,dt - (x-a)g(x)$. So at the promised number c we have $h'(c) = \int_a^c f(t)\,dt + (c-b)f(c) + \int_c^b g(t)\,dt - (c-a)g(c) = 0$, so

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = (b-c)f(c) + (c-a)g(c).$$

b. Given f continuous on $[a, b]$, let g be the constant zero function, that is, $g(x) = 0$ for all x. Applying the result of part (a), we have that there exists a c between a and b with

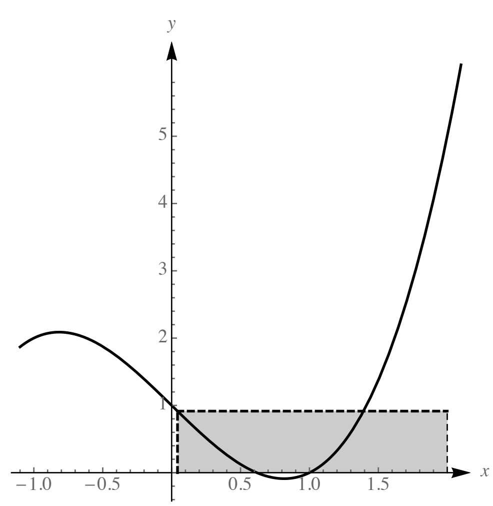
$$\int_a^c f(t)\,dt + 0 = f(c)(b-c) + 0,$$

so

$$\int_a^c f(t)\,dt = f(c)(b-c),$$

as desired.

c. There exists a rectangle with base from c to b and height $f(c)$ so that the area of the rectangle is equal to the value of the integral of f from a to c.



d. Given a function f continuous on $[a, b]$, let $g = f$. Then by part (a) there exists c between a and b so that

$$\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = f(c)(b-c) + f(c)(c-a),$$

so

$$\int_a^b f(t)\,dt = f(c)(b-c+c-a) = f(c)(b-a),$$

so

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c).$$

5.4.60

a. The left Riemann sum is given by $\frac{\pi}{2n}\sum_{k=0}^{n-1}\sin((k\pi)/(2n))$.

b.

$$\begin{aligned}\lim_{\theta\to 0}\theta\left(\frac{\cos\theta+\sin\theta-1}{2(1-\cos\theta)}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right) &= \lim_{\theta\to 0}\frac{\theta}{2}\left(\frac{(1+\cos\theta)(\cos\theta+\sin\theta-1)}{\sin^2\theta}\right)\\ &= \left(\frac{1}{2}\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1}\right)\left(\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}+\lim_{\theta\to 0}\frac{\sin\theta}{\sin\theta}\right)\\ &= \frac{1}{2}\cdot 1\cdot 2\left(\lim_{\theta\to 0}\frac{\frac{\cos -1}{\theta}}{\frac{\sin\theta}{\theta}}+1\right) = 1(0+1) = 1.\end{aligned}$$

c. Using the previous result, the left Riemann sum is given by $\frac{\pi}{2n}\left(\frac{\cos(\pi/(2n))+\sin(\pi/(2n))-1}{2(1-\cos(\pi/(2n)))}\right)$. Let $\theta = \frac{\pi}{2n}$. Then as $n\to\infty$, $\theta\to 0$, and the limit of the left Riemann sum as $n\to\infty$ is 1.

5.5 Substitution Rule

5.5.1 It is based on the Chain Rule for differentiation.

5.5.2 After making a substitution, one obtains an integral in terms of a different variable, so the variable has "changed."

5.5.3 Typically u is substituted for the inner function, so $u = g(x)$.

5.5.4 One can either let $u = \tan x$, which is a good choice because the derivative is then $\sec^2 x$ which is a factor of the integrand, or one can let $u = \sec x$, because then the derivative is $\tan x \sec x$ which is also a factor of the integrand.

5.5.5 The new integral is $\int_{g(a)}^{g(b)} f(u)\,du$.

5.5.6 The new limits of integration are $2^2-4=0$ and $4^2-4=12$.

5.5.7 Because $u = x^2+1$, $du = 2x\,dx$. Substituting yields $\int u^4\,du = \frac{u^5}{5}+C = \frac{(x^2+1)^5}{5}+C$.

5.5.8 Because $u = 4x^2+3$, $du = 8x\,dx$. Substituting yields $\int \cos u\,du = \sin u + C = \sin(4x^2+3)+C$.

5.5.9 Because $u = \sin x$, $du = \cos x\,dx$. Substituting yields $\int u^3\,du = \frac{u^4}{4}+C = \frac{\sin^4(x)}{4}+C$.

5.5.10 Because $u = 3x^2+x$, $du = 6x+1\,dx$. Substituting yields $\int \sqrt{u}\,du = \frac{2}{3}\cdot u^{3/2}+C = \frac{2}{3}\cdot\sqrt{(3x^2+x)^3}+C$.

5.5.11 Let $u = x + 1$. Then $du = dx$, and $\int (x+1)^{12}\,dx = \int u^{12}\,du = \frac{u^{13}}{13} + C = \frac{(x+1)^{13}}{13} + C$. Check: $\frac{d}{dx}\left(\frac{(x+1)^{13}}{13} + C\right) = (x+1)^{12}$.

5.5.12 Let $u = 3x + 1$. Then $du = 3dx$, and $\int e^{3x+1}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{e^{3x+1}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{e^{3x+1}}{3} + C\right) = e^{3x+1}$.

5.5.13 Let $u = 2x + 1$. Then $du = 2dx$ and $\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$. Check: $\frac{d}{dx}\left(\frac{(2x+1)^{3/2}}{3} + C\right) = \frac{3}{2}\cdot\frac{1}{3}\cdot(2x+1)^{1/2}\cdot 2 = \sqrt{2x+1}$.

5.5.14 Let $u = 2x+5$. Then $du = 2dx$, and $\int \cos(2x+5)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{\sin(2x+5)}{2} + C$. Check: $\frac{d}{dx}\left(\frac{\sin(2x+5)}{2} + C\right) = \cos(2x+5)$.

5.5.15

a. $\int e^{10x}\,dx = \frac{1}{10}e^{10x} + C$.

b. $\int \sec 5x \tan 5x\,dx = \frac{1}{5}\sec 5x + C$.

c. $\int \sin 7x\,dx = -\frac{1}{7}\cos 7x + C$.

d. $\int \cos\frac{x}{7}\,dx = 7\sin\frac{x}{7} + C$.

e. $\int \frac{dx}{81+9x^2} = \frac{1}{9}\int\frac{dx}{9+x^2} = \frac{1}{27}\tan^{-1}\frac{x}{3} + C$.

f. $\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\frac{x}{6} + C$.

5.5.16

a. $\int_0^1 10^x\,dx = \frac{1}{\ln 10}\cdot 10^x\Big|_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$.

b. $\int_0^{\pi/40} \cos 20x\,dx = \frac{1}{20}\sin 20x\Big|_0^{\pi/40} = \frac{1}{20}(1-0) = \frac{1}{20}$.

c. $\int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2-9}} = \frac{1}{3}\sec^{-1}\frac{x}{3}\Big|_{3\sqrt{2}}^6 = \frac{1}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{36}$.

d. $\int_0^{\pi/16} \sec^2 4x\,dx = \frac{1}{4}\tan 4x\Big|_0^{\pi/16} = \frac{1}{4}(1-0) = \frac{1}{4}$.

5.5.17 Let $u = x^2 - 1$. Then $du = 2x\,dx$. Substituting yields $\int u^{99}\,du = \frac{u^{100}}{100} + C = \frac{(x^2-1)^{100}}{100} + C$.

5.5.18 Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$. Substituting yields $\frac{1}{2}\int e^u\,du = \frac{1}{2}\cdot e^u + C = \frac{1}{2}\cdot e^{x^2} + C$.

5.5.19 Let $u = 1 - 4x^3$. Then $du = -12x^2\,dx$, so $-\frac{1}{6}du = 2x^2\,dx$. Substituting yields $-\frac{1}{6}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{3}\cdot\sqrt{u} + C = -\frac{1}{3}\cdot\sqrt{1-4x^3} + C$

5.5.20 Let $u = \sqrt{x} + 1$. Then $du = \frac{1}{2\sqrt{x}}\,dx$. Substituting yields $\int u^4\,du = \frac{u^5}{5} + C = \frac{(\sqrt{x}+1)^5}{5} + C$.

5.5.21 Let $u = x^2 + x$. Then $du = (2x+1)\,dx$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(x^2+x)^{11}}{11} + C$.

5.5.22 Let $u = 10x - 3$. Then $du = 10\,dx$, so $\frac{1}{10}du = dx$. Substituting yields $\frac{1}{10}\int \frac{1}{u}\,du = \frac{1}{10}\cdot \ln|u| + C = \frac{1}{10}\ln|10x-3| + C$.

5.5.23 Let $u = x^4 + 16$. Then $du = 4x^3\,dx$, so $\frac{1}{4}du = x^3\,dx$. Substituting yields $\frac{1}{4}\int u^6\,du = \frac{1}{4}\cdot\frac{u^7}{7} + C = \frac{(x^4+16)^7}{28} + C$.

5.5.24 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Substituting yields $\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(\sin\theta)^{11}}{11} + C$.

5.5.25 $\int \frac{dx}{\sqrt{36-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{9-x^2}} = \frac{1}{2}\sin^{-1}\frac{x}{3} + C$ by equation 10 in Table 5.6.

5.5.26 $\int \frac{dx}{\sqrt{1-(3x)^2}} = \frac{1}{3}\sin^{-1}3x + C$, by equation 10 in Table 5.6.

5.5.27 Let $u = x^3$. Then $du = 3x^2\,dx$. Then $\int 6x^2 4^{x^3}\,dx = 2\int 4^u\,du = 2\cdot\frac{4^u}{\ln 4} + C = 2\cdot\frac{4^{x^3}}{\ln 4} + C = \frac{4^{x^3}}{\ln 2} + C$.

5.5.28 Let $u = x^{10}$. Then $du = 10x^9\,dx$, so $\frac{1}{10}du = x^9 dx$. Substituting yields $\frac{1}{10}\int \sin u\,du = -\frac{1}{10}\cos u + C = -\frac{1}{10}\cos x^{10} + C$.

5.5.29 Let $u = x^6 - 3x^2$. Then $du = (6x^5 - 6x)\,dx$, so $\frac{1}{6}du = (x^5 - x)\,dx$. Substituting yields $\frac{1}{6}\int u^4\,du = \frac{1}{6}\cdot\frac{u^5}{5} + C = \frac{(x^6-3x^2)^5}{30} + C$.

5.5.30 Let $u = 2x$, so that $du = 2dx$. Substituting yields $\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}2x + C$.

5.5.31 Let $u = 5x$ so that $du = 5dx$. Substituting yields $\frac{3}{5}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{3}{5}\sin^{-1}u + C = \frac{3}{5}\sin^{-1}x + C$.

5.5.32 Let $u = 2x$, so that $du = 2dx$. Substituting yields $2\int \frac{1}{u\sqrt{u^2-1}}\,du = 2\sec^{-1}u + C = 2\sec^{-1}2x + C$.

5.5.33 The integral can be rewritten as $\int \frac{e^w}{36+(e^w)^2}\,dw$. Let $u = e^w$, so that $du = e^w\,dw$. Substituting yields $\int \frac{du}{36+u^2} = \frac{1}{6}\tan^{-1}\frac{u}{6} + C = \frac{1}{6}\tan^{-1}\frac{e^w}{6} + C$.

5.5.34 Let $u = 2x^2 + 3x$, so that $du = (4x+3)\,dx = \frac{1}{2}(8x+6)\,dx$. Substituting yields $2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln|2x^2+3x| + C$.

5.5.35 Let $u = x^2$ so that $du = 2x\,dx$. Substitution yields $\frac{1}{2}\int \csc u \cot u\,du = -\frac{1}{2}\csc u + C = -\frac{1}{2}\csc x^2 + C$.

5.5.36 Let $u = 4w$. Then $du = 4\,dw$. Substituting yields $\frac{1}{4}\int \sec u \tan u\,du = \frac{1}{4}\sec u + C = \frac{1}{4}\sec 4w + C$.

5.5.37 Let $u = 10x+7$ so that $du = 10\,dx$. Substituting yields $\frac{1}{10}\int \sec^2 u\,du = \frac{1}{10}\tan u + C = \frac{1}{10}\tan(10x+7) + C$.

5.5.38 Let $u = \tan^{-1} w$ so that $du = \frac{1}{1+w^2}\,dw$. Substituting yields $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\tan^{-1} w\right)^2 + C$.

5.5.39 Let $u = 4t+1$ so that $du = 4dt$. Substituting yields $\frac{1}{4}\int 10^u\,du = \frac{1}{4}\cdot\frac{10^u}{\ln 10} + C = \frac{10^u}{4\ln 10} + C$.

5.5.40 Let $u = \sin x$. Then $du = \cos x\,dx$. Substituting yields $\int u^5 + 3u^3 - u\,du = \frac{u^6}{6} + \frac{3u^4}{4} - \frac{u^2}{2} + C = \frac{\sin^6 x}{6} + \frac{3\sin^4 x}{4} - \frac{\sin^2 x}{2} + C$.

5.5.41 Let $u = \cot x$. Then $du = -\csc^2 x\,dx$. Substituting yields $-\int u^{-3}\,du = \frac{1}{2u^2} + C = \frac{1}{2\cot^2 x} + C$.

5.5.42 Let $u = x^{3/2} + 8$. Then $du = \frac{3}{2}\cdot\sqrt{x}\,dx$. Substituting gives $\frac{2}{3}\int u^5\,du = \frac{2}{3}\frac{u^6}{6} + C = \frac{(x^{3/2}+8)^6}{9} + C$.

5.5.43 Note that $\sin x \sec^8 x = \frac{\sin x}{\cos^8 x}$. Let $u = \cos x$, so that $du = -\sin x\,dx$. Substituting yields $-\int u^{-8}\,du = \frac{1}{7u^7} + C = \frac{1}{7\cos^7 x} + C = \frac{\sec^7 x}{7} + C$.

5.5.44 Let $u = e^{2x} + 1$. Then $du = 2e^{2x}\,dx$. Substituting yields $\frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln|u|}{2} + C = \frac{\ln(e^{2x}+1)}{2} + C$.

5.5.45 $\int_0^{\pi/8} \cos 2x\,dx = \left(\frac{\sin 2x}{2}\right)\Big|_0^{\pi/8} = \frac{\sqrt{2}/2 - 0}{2} = \frac{\sqrt{2}}{4}$.

5.5.46 $\int_0^1 10e^{2x}\,dx = \left(5e^{2x}\right)\Big|_0^1 = 5(e^2 - 1)$.

5.5.47 Let $u = 4 - x^2$. Then $du = -2x\,dx$. Also, when $x = 0$ we have $u = 4$ and when $x = 1$ we have $u = 3$. Substituting yields $-\int_4^3 u\,du = \int_3^4 u\,du = \left(\frac{u^2}{2}\right)\Big|_3^4 = 8 - 4.5 = 3.5$.

5.5.48 Let $u = x^2 + 1$. Then $du = 2x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = 2$ we have $u = 5$. Substituting yields $\int_1^5 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_1^5 = -\frac{1}{5} + 1 = \frac{4}{5}$.

5.5.49 Let $u = 2^x + 4$ so that $du = 2^x \ln 2$. Also, when $x = 1$, $u = 6$ and when $x = 3$, $u = 12$. Substituting yields

$$\frac{1}{\ln 2}\int_6^{12} \frac{du}{u} = \frac{1}{\ln 2}\ln u\Big|_6^{12} = \frac{1}{\ln 2}(\ln 12 - \ln 6) = \frac{1}{\ln 2}(\ln 2) = 1.$$

5.5.50 Let $u = \dfrac{\theta}{8}$ so that $du = \dfrac{1}{8}\,d\theta$. Also, when $\theta = -2\pi$, $u = -\dfrac{\pi}{4}$, and when $\theta = 2\pi$, $u = \dfrac{\pi}{4}$. Substituting yields

$$8\int_{-\pi/4}^{\pi/4} \cos u\,du = 8\sin u\Big|_{-\pi/4}^{\pi/4} = 8\left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = 8\sqrt{2}.$$

5.5.51 Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Also, when $\theta = 0$ we have $u = 0$ and when $\theta = \pi/2$ we have $u = 1$. Substituting yields $\displaystyle\int_0^1 u^2\,du = \left(\frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.52 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $\displaystyle -\int_1^{\sqrt{2}/2} \frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^1 u^{-2}\,du = \left(-\frac{1}{u}\right)\Big|_{\sqrt{2}/2}^1 = -1 + \frac{2}{\sqrt{2}} = \sqrt{2} - 1$.

5.5.53 Let $u = e^w$. Then $du = e^w\,dw$. Also, when $w = \ln\pi/4$, $u = \pi/4$, and when $w = \ln\pi/2$, $u = \pi/2$. Substituting yields

$$\int_{\pi/4}^{\pi/2} \cos u\,du = \sin u\Big|_{\pi/4}^{\pi/2} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}.$$

5.5.54 Let $u = 4x$ so that $du = 4\,dx$. Also, when $x = \pi/16$, $u = \pi/4$ and when $x = \pi/8$, $u = \pi/2$. Substituting yields

$$2\int_{\pi/4}^{\pi/2} \csc^2 u\,du = -2\cot u\Big|_{\pi/4}^{\pi/2} = -2(0-1) = 2.$$

5.5.55 Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Also, when $x = -1$ we have $u = 0$ and when $x = 2$ we have $u = 9$. Substituting yields $\displaystyle\frac{1}{3}\int_0^9 e^u\,du = \left(\frac{e^u}{3}\right)\Big|_0^9 = \frac{e^9 - 1}{3}$.

5.5.56 Let $u = 9 + p^2$. Then $du = 2p\,dp$. Also, when $p = 0$ we have $u = 9$ and when $p = 4$ we have $u = 25$. Substituting yields $\displaystyle\frac{1}{2}\int_9^{25} u^{-1/2}\,du = \sqrt{u}\,\Big|_9^{25} = 5 - 3 = 2$.

5.5.57 Let $u = \sin x$. Then $du = \cos x\,dx$. Also, when $x = \pi/4$ we have $u = \sqrt{2}/2$ and when $x = \pi/2$ we have $u = 1$. Substituting yields $\displaystyle\int_{\sqrt{2}/2}^1 \frac{1}{u^2}\,du = \left(\frac{-1}{u}\right)\Big|_{\sqrt{2}/2}^1 = \left(-1 - \left(-\frac{2}{\sqrt{2}}\right)\right) = \sqrt{2} - 1$.

5.5.58 Let $u = \cos x$. Then $du = -\sin x\,dx$. Also, when $x = 0$ we have $u = 1$ and when $x = \pi/4$ we have $u = \sqrt{2}/2$. Substituting yields $\displaystyle -\int_1^{\sqrt{2}/2} \frac{1}{u^3}\,du = \int_{\sqrt{2}/2}^1 u^{-3}\,du = \left(-\frac{1}{2u^2}\right)\Big|_{\sqrt{2}/2}^1 = -\frac{1}{2} + 1 = \frac{1}{2}$.

5.5.59 Let $u = 5x$, so that $du = 5\,dx$. Also, when $x = 2/(5\sqrt{3})$ we have $u = 2/\sqrt{3}$ and when $x = 2/5$ we have $u = 2$. Substituting yields $\displaystyle\int_{2/\sqrt{3}}^2 \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u\,\Big|_{2/\sqrt{3}}^2 = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

5.5.60 Let $u = v^4 + 4v + 4$, so that $du = (4v^3 + 4)\,dv$, so that $\dfrac{1}{4}\cdot du = (v^3+1)\,dv$. Also, when $v = 0$ we have $u = 4$ and when $v = 1$ we have $u = 9$. Substituting yields $\displaystyle\frac{1}{4}\int_4^9 u^{-1/2}\,du = \frac{1}{4}\left(2\sqrt{u}\right)\Big|_4^9 = \frac{1}{2}(3-2) = \frac{1}{2}$.

5.5.61 Let $u = x^2 + 1$, so that $du = 2x\,dx$. Substituting yields $\displaystyle\frac{1}{2}\int_1^{17} \frac{1}{u}\,du = \frac{1}{2}\,\ln|u|\,\Big|_1^{17} = \frac{\ln 17}{2}$.

5.5.62 Let $u = 1 - 16x^2$, so that $du = -32x\,dx$. Substituting yields $\displaystyle -\frac{1}{32}\int_1^0 \frac{1}{\sqrt{u}}\,du = \frac{1}{16}\,\sqrt{u}\,\Big|_0^1 = \frac{1}{16}$.

5.5.63 Let $u = 3x$, so that $du = 3\,dx$. Substituting yields $\frac{4}{3}\int_1^{3/\sqrt{3}} \frac{1}{u^2+1}\,du = \frac{4}{3}\tan^{-1} u\,\Big|_1^{3/\sqrt{3}} = \frac{4}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{4}{3}\cdot\frac{\pi}{12} = \frac{\pi}{9}$.

5.5.64 Let $u = 3 + 2e^x$, so that $du = 2e^x\,dx$. Substituting yields $\frac{1}{2}\int_5^{11} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\,\Big|_5^{11} = \frac{\ln(11/5)}{2}$.

5.5.65 Let $u = 1 - x^2$. Then $du = -2x\,dx$. Also note that when $x = 0$ we have $u = 1$, and when $x = 1$ we have $u = 0$. Substituting yields $-\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \left(\frac{u^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{3}$.

5.5.66 Let $u = \ln p$. Then $du = \frac{1}{p}\,dp$. Also note that when $p = 1$ we have $u = 0$, and when $p = e^2$ we have $u = 2$. Substituting yields $\int_0^2 u\,du = \left(\frac{u^2}{2}\right)\Big|_0^2 = 2$.

5.5.67 Let $u = x^2 - 1$, so that $du = 2x\,dx$. Also note that when $x = 2$ we have $u = 3$, and when $x = 3$ we have $u = 8$. Substituting yields $\frac{1}{2}\int_3^8 u^{-1/3}\,du = \frac{1}{2}\left(\frac{3u^{2/3}}{2}\right)\Big|_3^8 = \frac{3}{4}\left(4 - \sqrt[3]{9}\right)$.

5.5.68 Let $u = 5x/6$ so that $du = \frac{5}{6}\,dx$. Also note that when $x = 0$ we have $u = 0$ and when $x = 6/5$ we have $u = 1$. Substituting yields $\frac{6}{5\cdot 36}\int_0^1 \frac{1}{u^2+1}\,du = \frac{1}{30}\left(\tan^{-1} u\right)\Big|_0^1 = \frac{\pi}{120}$.

5.5.69 Let $u = 16 - x^4$. Then $du = -4x^3\,dx$. Also note that when $x = 0$ we have $u = 16$, and when $x = 2$ we have $u = 0$. Substituting yields $\frac{1}{4}\int_0^{16} \sqrt{u}\,du = \frac{1}{4}\left(\frac{2u^{3/2}}{3}\right)\Big|_0^{16} = \frac{32}{3}$.

5.5.70 Let $u = x^2 - 2x$. Then $du = 2(x-1)\,dx$. Also note that when $x = -1$ we have $u = 3$ and when $x = 1$ we have $u = -1$. Substituting yields $\frac{1}{2}\int_3^{-1} u^7\,du = \frac{1}{16}\left(u^8\right)\Big|_3^{-1} = \frac{1}{16}\left(1 - 3^8\right) = -\frac{6560}{16} = -410$.

5.5.71 Let $u = 2 + \cos x$ so that $du = -\sin x\,dx$. Note that when $x = -\pi$, $u = 1$ and when $x = 0$, $u = 3$. Substituting yields $\int_1^3 -\frac{1}{u}\,du = (-\ln|u|)\Big|_1^3 = -(\ln 3 - \ln 1) = -\ln 3$.

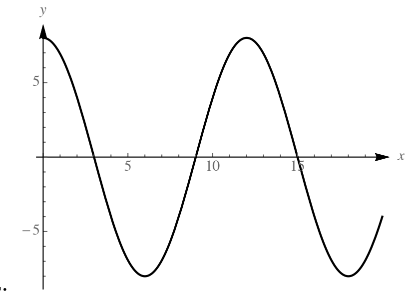
5.5.72 Let $u = 2v^3 + 9v^2 + 12v + 36$, so that $du = (6v^2 + 18v + 12)\,dv = 6(v+1)(v+2)\,dv$. Note that $u = 36$ when $v = 0$ and $u = 59$ when $v = 1$. Substituting yields $\frac{1}{6}\int_{36}^{59} \frac{1}{u}\,du = \frac{1}{6}\left(\ln|u|\right)\Big|_{36}^{59} = \frac{1}{6}(\ln 59 - \ln 36) = \frac{1}{6}\ln(59/36)$.

5.5.73 Let $u = 3x + 1$ so that $du = 3\,dx$. Note that $9x^2 + 6x + 1 = (3x+1)^2 = u^2$, and also that when $x = 1$, $u = 4$ and when $x = 2$, $u = 7$. Substituting yields $\frac{4}{3}\int_4^7 \frac{1}{u^2}\,du = \frac{4}{3}\left(-\frac{1}{u}\right)\Big|_4^7 = \frac{4}{3}\left(-\frac{1}{7} - \left(-\frac{1}{4}\right)\right) = \frac{4}{3}\left(\frac{3}{28}\right) = \frac{1}{7}$.

5.5.74 Let $u = \sin^2 x$, so that $du = 2\sin x\cos x\,dx = \sin 2x\,dx$. Note that when $x = 0$, $u = 0$, and when $x = \pi/4$, $u = 1/2$. Substituting yields $\int_0^{1/2} e^u\,du = e^u\,\Big|_0^{1/2} = \sqrt{e} - 1$.

5.5.75 The average velocity is given by $\frac{1}{10-0}\int_0^{10} (8\sin\pi t + 2t)\,dt = \frac{1}{10}\left(-\frac{8}{\pi}\cos\pi t + t^2\right)\Big|_0^{10} = \frac{1}{10}\left(-\frac{8}{\pi}\cos 10\pi + 100 + \frac{8}{\pi}\cos 0 - 0\right) = 10$.

5.5.76

a. 

b. $\int_0^t 8\cos(\pi y/6)\,dy = \left(\frac{48}{\pi}\sin(\pi y/6)\right)\Big|_0^t = \frac{48}{\pi}\sin(\pi t/6)$.

c. The period is $\frac{2\pi}{\pi/6} = 12$.

5.5.77

a. $\int_0^4 \frac{200}{(t+1)^2}\,dt = \left(\frac{-200}{t+1}\right)\Big|_0^4 = -40 + 200 = 160$.

b. $\int_0^6 \frac{200}{(t+1)^3}\,dt = \left(\frac{-200}{2(t+1)^2}\right)\Big|_0^6 = \frac{-100}{49} + 100 = \frac{4800}{49}$.

c. $\Delta P = \int_0^T \frac{200}{(t+1)^r}\,dt$. This decreases as r increases, because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$.

d. Suppose $\int_0^{10} \frac{200}{(t+1)^r}\,dt = 350$. Then $\left(\frac{200(t+1)^{-r+1}}{1-r}\right)\Big|_0^{10} = 350$, so $11^{1-r} - 1 = \frac{350(1-r)}{200}$, and thus $\frac{11}{11^r} = \frac{7-7r}{4} + \frac{4}{4} = \frac{11-7r}{4}$, and $11^r = \frac{44}{11-7r}$. Using trial and error to find r, we arrive at $r \approx 1.278$.

e. $\int_0^T \frac{200}{(t+1)^3}\,dt = \left(-\frac{200}{2(t+1)^2}\right)\Big|_0^T = -\frac{100}{(T+1)^2} + 100$. As $T \to \infty$, this expression $\to 100$, so in the long run, the bacteria approaches a finite limit.

5.5.78 Let $u = x-2$, so that $u+2 = x$. Then $du = dx$. Substituting yields $\int \frac{u+2}{u}\,du = \int \left(1 + \frac{2}{u}\right)du = u + 2\ln|u| + D = x - 2 + 2\ln|x-2| + D$. The constant $-2+D$ could be renamed as a different constant C, yielding $x + 2\ln|x-2| + C$.

5.5.79 Let $u = x-4$, so that $u+4 = x$. Then $du = dx$. Substituting yields $\int \frac{u+4}{\sqrt{u}}\,du = \int \left(\frac{u}{\sqrt{u}} + \frac{4}{\sqrt{u}}\right)du = \int u^{1/2} + 4u^{-1/2}\,du = \frac{2}{3}u^{3/2} + 8u^{1/2} + C = \frac{2}{3}\cdot(x-4)^{3/2} + 8\sqrt{x-4} + C$.

5.5.80 Let $u = y+1$, so that $u-1 = y$. Then $du = dy$. Substituting yields $\int \frac{(u-1)^2}{u^4}\,du = \int \frac{u^2-2u+1}{u^4}\,du = \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)du = -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} + C = -\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{1}{3(y+1)^3} + C$.

5.5.81 Let $u = x+4$, so that $u-4 = x$. Then $du = dx$. Substituting yields

$$\int \frac{u-4}{\sqrt[3]{u}}\,du = \int \left(u^{2/3} - 4u^{-1/3}\right)du = \frac{3}{5}u^{5/3} + -6u^{2/3} + C$$
$$= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C.$$

5.5.82 Let $u = e^x + e^{-x}$. Then $du = (e^x - e^{-x})\,dx$. Substituting yields $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C = \ln(e^x + e^{-x}) + C$.

5.5.83 Let $u = 2x+1$. Then $du = 2dx$ and $x = \frac{u-1}{2}$. Substituting yields $\displaystyle\frac{1}{2}\int \frac{u-1}{2}\cdot\sqrt[3]{u}\,du = \frac{1}{4}\int (u^{4/3} - u^{1/3})\,du = \frac{1}{4}\left(\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right) + C = \frac{3(2x+1)^{7/3}}{28} - \frac{3(2x+1)^{4/3}}{16} + C = (2x+1)^{4/3}\left(\frac{3(2x+1)}{28} - \frac{3}{16}\right) = \frac{3}{112}(2x+1)^{4/3}(8x+4-7) = \frac{3}{112}(2x+1)^{4/3}(8x-3)$.

5.5.84 Let $u = 3z+2$. Then $du = 3dz$ and $z = \frac{u-2}{3}$. Substituting yields $\displaystyle\frac{1}{3}\int \frac{u+1}{3}\cdot\sqrt{u}\,du = \frac{1}{9}\int (u^{3/2} + u^{1/2})\,du = \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{2(3z+2)^{5/2}}{45} + \frac{2(3z+2)^{3/2}}{27} + C. = \frac{2}{9}(3z+2)^{3/2}\left(\frac{3z+2}{5} + \frac{1}{3}\right) = \frac{2}{9}(3z+2)^{3/2}\left(\frac{9z+6+5}{15}\right) = \frac{2}{135}(3z+2)^{3/2}(9z+11)$.

5.5.85 Let $u = x+10$. Then $du = dx$ and $x = u - 10$. Substituting gives $\displaystyle\int (u-10)u^9\,du = \int (u^{10} - 10u^9)\,du = \frac{u^{11}}{11} - u^{10} + C = \frac{1}{11}(x+10)^{11} - (x+10)^{10} + C = (x+10)^{10}\left(\frac{x+10}{11} - 1\right) + C = \frac{(x+10)^{10}(x-1)}{11} + C$.

5.5.86 Using Table 5.6: $\displaystyle\int_0^{\sqrt{3}} \frac{3dx}{9+x^2} = \tan^{-1}(x/3)\Big|_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3}/3) = \pi/6$.

5.5.87 $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \left(x + \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \pi$.

5.5.88 $\displaystyle\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

5.5.89 $\displaystyle\int \sin^2\left(\theta + \frac{\pi}{6}\right)\,d\theta = \frac{1}{2}\int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right)\,d\theta = \frac{\theta}{2} - \frac{\sin\left(2\theta + \frac{\pi}{3}\right)}{4} + C$.

5.5.90 $\displaystyle\int_0^{\pi/4} \cos^2 8\theta\,d\theta = \int_0^{\pi/4} \frac{1+\cos 16\theta}{2}\,d\theta = \left(\frac{\theta}{2} + \frac{\sin 16\theta}{32}\right)\Big|_0^{\pi/4} = \frac{\pi}{8}$.

5.5.91 $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \sin^2 2\theta\,d\theta = 2\int_0^{\pi/4} \frac{1-\cos 4\theta}{2}\,d\theta = \left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4} = \frac{\pi}{4}$.

5.5.92 Let $u = x^2$, so that $du = 2x\,dx$. Substituting yields

$$\begin{aligned}\frac{1}{2}\int \cos^2 u\,du = \frac{1}{2}\int \frac{1+\cos 2u}{2}\,du &= \frac{1}{4}\left(u + \frac{\sin 2u}{2}\right) + C\\ &= \frac{x^2}{4} + \frac{\sin 2x^2}{8} + C.\end{aligned}$$

5.5.93 Let $u = \sin^2 y + 2$ so that $du = 2\sin y\cos y\,dy = \sin 2y\,dy$. Substituting yields $\displaystyle\int_2^{9/4} \frac{1}{u}\,du = (\ln|u|)\Big|_2^{9/4} = \ln(9/4) - \ln 2 = \ln(9/8)$.

5.5.94 Because $\sin^4\theta = (\sin^2\theta)^2 = \left(\frac{1-\cos 2\theta}{2}\right)^2 = \frac{1-2\cos 2\theta+\cos^2 2\theta}{4}$, we have

$$\int \sin^4\theta\,d\theta = \int \left(\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}\right)d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{4}\int \cos^2 2\theta\,d\theta.$$

Because $\frac{1}{4}\cos^2 2\theta = \frac{1+\cos 4\theta}{8}$, we have

$$\int \sin^4\theta\,d\theta = \frac{1}{4}\theta - \frac{\sin 2\theta}{4} + \frac{1}{8}\theta + \frac{\sin 4\theta}{32} = \frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}.$$

Thus, $\int_0^{\pi/2} \sin^4\theta\,d\theta = \left(\frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right)\Big|_0^{\pi/2} = \frac{3\pi}{16}$.

5.5.95

a. True. This follows by substituting $u = f(x)$ to obtain the integral $\int u\,du = \frac{u^2}{2} + C = \frac{f(x)^2}{2} + C$.

b. True. Again, this follows from substituting $u = f(x)$ to obtain the integral $\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$ where $n \neq -1$.

c. False. If this were true, then $\sin 2x$ and $2\sin x$ would have to differ by a constant, which they do not. In fact, $\sin 2x = 2\sin x\cos x$.

d. False. The derivative of the right hand side is $(x^2+1)^9 \cdot 2x$ which is not the integrand on the left hand side.

e. False. If we let $u = f'(x)$, then $du = f''(x)\,dx$. Substituting yields $\int_{f'(a)}^{f'(b)} u\,du = \left(\frac{u^2}{2}\right)\Big|_{f'(a)}^{f'(b)} = \frac{(f'(b))^2}{2} - \frac{(f'(a))^2}{2}$.

5.5.96 $A(x) = \int_4^5 \frac{x}{\sqrt{x^2-9}}\,dx$. Let $u = x^2 - 9$, so that $du = 2x\,dx$. Also, when $x = 4$ we have $u = 7$ and when $x = 5$ we have $u = 16$. Substituting yields $\frac{1}{2}\int_7^{16} u^{-1/2}\,du = \sqrt{u}\,\Big|_7^{16} = 4 - \sqrt{7}$.

5.5.97 $A(x) = \int_0^{\sqrt{\pi}} x\sin x^2\,dx$. Let $u = x^2$, so that $du = 2x\,dx$. Also, when $x = 0$ we have $u = 0$ and when $x = \sqrt{\pi}$ we have $u = \pi$. Substituting yields $\frac{1}{2}\int_0^{\pi} \sin u\,du = \frac{1}{2}(-\cos u)\Big|_0^{\pi} = 1$.

5.5.98 $A(x) = \int_2^6 (x-4)^4\,dx = \frac{(x-4)^5}{5}\Big|_2^6 = \frac{2^5}{5} - \left(-\frac{(2)^5}{5}\right) = \frac{64}{5}$.

5.5.99 $A(a) = \int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3}\right)dx = \left(\frac{x}{a} - \frac{x^3}{3a^3}\right)\Big|_0^a = 1 - \frac{1}{3} = \frac{2}{3}$. This is a constant function.

5.5.100

a. Let $u = x^2$, so that $du = 2x\,dx$. Note that when $x = 1$ or $x = -1$, we have $u = 1$. Substituting gives $\frac{1}{2}\int_1^1 f(u)\,du = 0$. Alternatively, we could note that when f is even, $xf(x^2)$ is odd, so $\int_{-1}^1 xf(x^2)\,dx = 0$.

b. Let $u = x^3$ so that $du = 3x^2\,dx$. Note that when $x = -2$, $u = -8$, and when $x = 2$, $u = 8$. Substituting yields $\frac{1}{3}\int_{-8}^{8} f(u)\,du = \frac{2}{3}\int_{0}^{8} f(u)\,du = \frac{2}{3}\cdot 9 = 6$.

5.5.101

a. Let $u = \sin px$, so that $du = p\cos px\,dx$. Note that when $x = 0$, $u = 0$, and when $x = \pi/2p$, $u = 1$. Substituting yields $\frac{1}{p}\int_0^1 f(u)\,du = \frac{\pi}{p}$.

b. Let $u = \sin x$ so that $du = \cos x\,dx$. Note that when $x = -\pi/2$, $u = -1$ and when $x = \pi/2$, $u = 1$. Substituting yields $\int_{-1}^{1} f(u)\,du = 0$, because f is an odd function. Alternatively, we could note that when f is odd, $\cos x \cdot f(\sin x)$ is also odd, because $\sin x$ is odd and $\cos x$ is even. Thus the given integral must be zero because it is the definite integral of an odd function over a symmetric interval about 0.

5.5.102 The average vertical distance is given by

$$\frac{1}{a}\int_0^a \left(b - \frac{b}{a}x\right)\,dx = \frac{1}{a}\left(bx - \frac{b}{2a}x^2\right)\Big|_0^a = \frac{1}{a}\left(ba - \frac{ba^2}{2a}\right) = b - \frac{b}{2} = \frac{b}{2}.$$

5.5.103 $\displaystyle \frac{1}{\pi/k - 0}\int_0^{\pi/k} \sin kx\,dx = \frac{k}{\pi}\cdot\left(\frac{-\cos kx}{k}\right)\Big|_0^{\pi/k} = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}.$

5.5.104 The area on the left is given by $\int_0^{\pi/2} 2\sin 2x\,dx$. If we let $u = 2x$ so that $du = 2\,dx$, we obtain the equivalent integral $\int_0^{\pi} \sin u\,du$ which represents the area on the right.

5.5.105 The area on the left is given by $\int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}\,dx$. If we let $u = \sqrt{x} - 1$ so that $du = \frac{1}{2\sqrt{x}}\,dx$, we obtain the equivalent integral $\int_1^2 u^2\,du$ which represents the area on the right.

5.5.106 Let $u = f(x)$, so that $du = f'(x)\,dx$. Substituting yields $\int (5u^3+7u^2+u)\,du = \frac{5u^4}{4} + \frac{7u^3}{3} + \frac{u^2}{2} + C = \frac{5f^4(x)}{4} + \frac{7f^3(x)}{3} + \frac{f^2(x)}{2} + C$.

5.5.107 Let $u = f(x)$, so that $du = f'(x)\,dx$. Substituting yields

$$\int_4^5 (5u^3 + 7u^2 + u)\,du = \left(\frac{5u^4}{4} + \frac{7u^3}{3} + \frac{u^2}{2}\right)\Big|_4^5 = \frac{7297}{12}.$$

5.5.108 Let $u = f^{(p)}(x)$ so that $du = f^{(p+1)}(x)\,dx$. Substituting yields

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{1}{n+1}\left(f^{(p)}(x)\right)^{n+1} + C.$$

5.5.109 If we let $u = \sqrt{x+a}$, then $u^2 = x + a$ and $2u\,du = dx$. Substituting yields $\int_{\sqrt{a}}^{\sqrt{1+a}} (u^2 - a)\cdot u \cdot 2u\,du = \int_{\sqrt{a}}^{\sqrt{1+a}} (2u^4 - 2au^2)\,du = \left(\frac{2u^5}{5} - \frac{2au^3}{3}\right)\Big|_{\sqrt{a}}^{\sqrt{1+a}} = \frac{2(\sqrt{1+a})^5}{5} - \frac{2a(\sqrt{1+a})^3}{3} - \frac{2a^{5/2}}{5} + \frac{2a^{5/2}}{3}$.

If we let $u = x + a$, then $u - a = x$ and $du = dx$. Substituting yields $\int_a^{a+1} (u-a)\sqrt{u}\,du = \left(\frac{2u^{5/2}}{5} - \frac{2au^{3/2}}{3}\right)\Big|_a^{a+1} = \frac{2(a+1)^{5/2}}{5} - \frac{2a(a+1)^{3/2}}{3} - \frac{2a^{5/2}}{5} + \frac{2a^{5/2}}{3}$.

Note that the two results are the same.

5.5.110 If we let $u = \sqrt[p]{x+a}$, then $u^p = x + a$ and $pu^{p-1}\,du = dx$. Substituting yields

$$p\int_{\sqrt[p]{a}}^{\sqrt[p]{1+a}} (u^{2p} - au^p)\,du = \left(\frac{pu^{2p+1}}{2p+1} - \frac{pau^{p+1}}{p+1}\right)\Bigg|_{\sqrt[p]{a}}^{\sqrt[p]{1+a}}$$
$$= \frac{p(1+a)^2\sqrt[p]{1+a}}{2p+1} - \frac{ap(1+a)\sqrt[p]{1+a}}{p+1} - \frac{pa^2\sqrt[p]{a}}{2p+1} + \frac{a^2p\sqrt[p]{a}}{p+1}.$$

If we let $u = x + a$, then $u - a = x$ and $du = dx$. Substituting yields

$$\int_a^{a+1} (u-a)\sqrt[p]{u}\,du = \int_a^{a+1}\left(u^{(p+1)/p} - au^{1/p}\right)du = \left(\frac{u^{(2p+1)/p}}{(2p+1)/p} - \frac{au^{(p+1)/p}}{(p+1)/p}\right)\Bigg|_a^{a+1}$$
$$= \left(\frac{pu^2u^{1/p}}{2p+1} - \frac{pau\cdot u^{1/p}}{p+1}\right)\Bigg|_a^{a+1}$$
$$= \frac{p(1+a)^2\sqrt[p]{1+a}}{2p+1} - \frac{ap(1+a)\sqrt[p]{1+a}}{p+1} - \frac{pa^2\sqrt[p]{a}}{2p+1} + \frac{a^2p\sqrt[p]{a}}{p+1}.$$

Note that the two results are the same.

5.5.111 If we let $u = \cos\theta$, then $du = -\sin\theta\,d\theta$. Substituting yields $\int -u^{-4}\,du = \frac{1}{3u^3} + C = \frac{1}{3\cos^3\theta} + C = \frac{\sec^3\theta}{3} + C$.

If we let $u = \sec\theta$, then $du = \sec\theta\tan\theta\,d\theta$. Substituting yields $\int u^2\,du = \frac{u^3}{3} + C = \frac{\sec^3\theta}{3} + C$. Note that the two results are the same.

5.5.112 Let $u = ax$, so that $\frac{1}{a}du = dx$. Substituting yields $\frac{1}{a}\int \sin^2 u\,du = \frac{1}{a}\int \frac{1-\cos 2u}{2}\,du = \frac{1}{2a}\int(1 - \cos 2u)\,du = \frac{1}{2a}\left(u - \frac{\sin 2u}{2}\right) + C = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$.

For the second integral, we use the same substitution to obtain $\frac{1}{a}\int \cos^2 u\,du = \frac{1}{a}\int \frac{1+\cos 2u}{2}\,du = \frac{1}{2a}\int(1+\cos 2u)\,du = \frac{1}{2a}\left(u + \frac{\sin 2u}{2}\right) + C = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$.

5.5.113

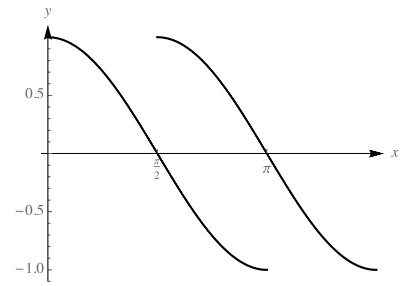
a. Because $\sin 2x = 2\sin x\cos x$, we can write $(\sin x\cos x)^2 = \left(\frac{\sin 2x}{2}\right)^2 = \frac{\sin^2 2x}{4}$. Then we have $I = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$. Note that we used the result of the previous problem during this derivation.

b. $I = \frac{1}{4}\int(1-\cos 2x)(1+\cos 2x)\,dx = \frac{1}{4}\int(1-\cos^2 2x)\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C$.

c. The results are consistent. The work involved is similar in each method.

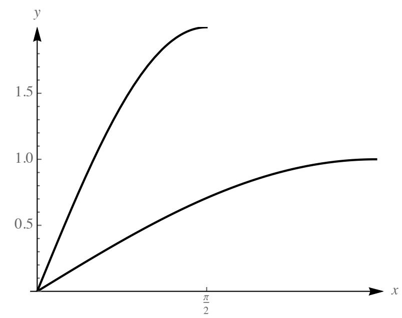
5.5.114

a. Let $u = x + c$. Note that $du = dx$. Substitution yields $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du$.

b. 

5.5.115

a. Let $u = cx$. Note that $du = c \cdot dx$. Substitution yields $\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(u)\,du$.

b. 

5.5.116 First let $u = x^2$, so that $du = 2x\,dx$. Substituting yields $\frac{1}{2}\int \sin^4 u \cos u\,du$. Now let $v = \sin u$, so that $dv = \cos u\,du$. This substitution yields $\frac{1}{2}\int v^4\,dv = \frac{1}{2}\cdot\frac{v^5}{5} + C = \frac{\sin^5 u}{10} + C = \frac{\sin^5 x^2}{10} + C$.

5.5.117 Let $u = \sqrt{x+1}$ so that $u^2 = x+1$. Then $2u\,du = dx$. Substituting yields $\int 2\cdot\frac{u\,du}{\sqrt{1+u}}$. Now let $v = \sqrt{1+u}$ so that $v^2 = 1+u$ and $2v\,dv = du$. Now a substitution yields $4\int \frac{(v^2-1)v}{v}\,dv = 4\int (v^2-1)\,dv =$
$\frac{4v^3}{3} - 4v + C = \frac{4}{3}(1+u)^{3/2} - 4\sqrt{1+u} + C = \frac{4}{3}\left(1+\sqrt{x+1}\right)^{3/2} - 4\sqrt{1+\sqrt{x+1}} + C =$
$\frac{4}{3}\sqrt{1+\sqrt{x+1}}\left(1+\sqrt{x+1}-3\right) = \frac{4}{3}\sqrt{1+\sqrt{x+1}}(\sqrt{x+1}-2) + C.$

5.5.118 Let $u = 4x$, so that $du = 4\,dx$. Substituting yields $\frac{1}{4}\int \tan^{10} u \sec^2 u\,du$. Now let $v = \tan u$, so that $dv = \sec^2 u\,du$. This leads to $\frac{1}{4}\int v^{10}\,dv = \frac{v^{11}}{44} + C = \frac{\tan^{11} 4x}{44} + C$.

5.5.119 Let $u = \cos\theta$, so that $du = -\sin\theta\,d\theta$. This substitution yields $\int_0^1 \frac{u}{\sqrt{u^2+16}}\,du$. Now let $v = u^2+16$, so that $dv = 2u\,du$. Now a substitution yields $\frac{1}{2}\int_{16}^{17} v^{-1/2}\,dv = \sqrt{v}\,\Big|_{16}^{17} = \sqrt{17} - 4$.

Chapter Five Review

1

a. True. The antiderivative of a linear function is a quadratic function.

b. False. $A'(x) = f(x)$, not $F(x)$.

c. True, Note that f is an antiderivative of f', so this follows from the Fundamental Theorem.

d. True. Because $|f(x)| \geq 0$ for all x, this integral must be positive, unless f is constantly 0.

e. False. For example, the average value of $\sin x$ on $[0, 2\pi]$ is zero.

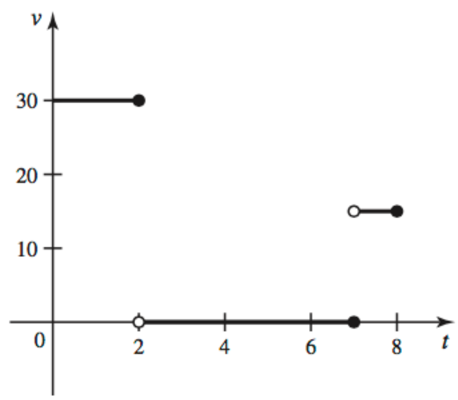
f. True. This is equal to $2\int_a^b f(x)\,dx - 3\int_a^b g(x)\,dx = 2\int_a^b f(x)\,dx + 3\int_b^a g(x)\,dx$.

g. True. The derivative of the right hand side is $f'(g(x))g'(x)$ by the Chain Rule.

2 The left Riemann sum gives $30(4)+25(4)+25(4) = (30+25+25)(4) = 80(4) = 320$ ft. The right Riemann sum gives $25(4) + 25(4) + 20(4) = (25 + 25 + 20)(4) = 70(4) = 280$ ft. The midpoint Riemann sum gives $25(4) + 20(4) + 30(4) = (25 + 20 + 30)(4) = 75(4) = 300$ ft.

3

a.



b. The area is $2 \cdot 30 + 0 + 15 \cdot 1 = 75$.

c. The area represents the distance that the diver ascends.

4 $\Delta x = \dfrac{6-0}{3} = 2$. The left Riemann sum is $f(0)(2) + f(2)(2) + f(4)(2) = 20(2) + 26(2) + 34(2) = (20 + 26 + 34)(2) = 80(2) = 160$. The right Riemann sum is $f(2)(2) + f(4)(2) + f(6)(2) = 26(2) + 34(2) + (40)(2) = (26 + 34 + 40)(2) = 100(2) = 200$. The midpoint Riemann sum is $f(1)(2) + f(3)(2) + f(5)(2) = 22(2) + 30(2) + 38(2) = (22 + 30 + 38)(2) = 90(2) = 180$.

5 $\Delta x = \dfrac{4-1}{6} = \dfrac{1}{2}$. The left Riemann sum is given by $(f(1)+f(1.5)+f(2)+f(2.5)+f(3)+f(3.5))\left(\dfrac{1}{2}\right) = (\sqrt{5} + \sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} + \sqrt{15})\left(\dfrac{1}{2}\right) \approx 9.34$. The right Riemann sum is given by $(f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4))\left(\dfrac{1}{2}\right) = (\sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} + \sqrt{15} + \sqrt{17})\left(\dfrac{1}{2}\right) \approx 10.28$. The midpoint Riemann sum is given by $(f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75))\left(\dfrac{1}{2}\right) = (\sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14} + \sqrt{16})\left(\dfrac{1}{2}\right) \approx 9.82$.

6 The midpoint Riemann sum is $(3\cdot 3.5+4)+(3\cdot 4.5+4)+(3\cdot 5.5+4)+(3\cdot 6.5+4) = 3\cdot 20+16 = 76$. The exact area of the region is given by $\int_3^7 (3x+4)\,dx = \left(\frac{3x^2}{2}+4x\right)\Big|_3^7 = \frac{147}{2}+28-\left(\frac{27}{2}+12\right) = 60+16 = 76$.

7

n	Midpoint Riemann sum
10	114.167
30	114.022
60	114.006

From the table, it appears that $\int_1^{25} \sqrt{2x-1}\,dx = 114$. This result can be verified using the Fundamental Theorem of Calculus and the substitution $u = 2x-1$.

8 Suppose $f(x) = x^3 + x$. Then the given expression represents $\int_3^8 f(x)\,dx$.

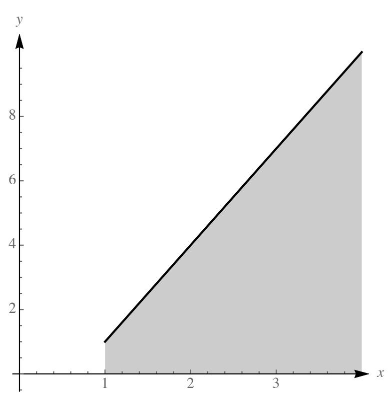
9

a. The right Riemann sum is $4\cdot 1 + 7\cdot 1 + 10\cdot 1 = 21$.

b. The right Riemann sum is $\sum_{k=1}^{n} \left(3\left(1+\frac{3k}{n}\right)-2\right)\cdot\frac{3}{n}$.

c. The sum evaluates as $\sum_{k=1}^{n}\frac{3}{n} + \sum_{k=1}^{n}\frac{27k}{n^2} = 3 + \frac{27}{n^2}\cdot\frac{n(n+1)}{2}$. As $n\to\infty$, the limit of this expression is $3+13.5 = 16.5$.

d.



The area consists of a trapezoid with base 3 and heights 1 and 10, so the value is $(3)\left(\frac{1+10}{2}\right) = \frac{33}{2} = 16.5$. The Fundamental Theorem assures us that $\int_1^4 (3x-2)\,dx = \left(\frac{3x^2}{2}-2x\right)\Big|_1^4 = (24-8)-(3/2-2) = 16+1/2 = 16.5$.

10 Let $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Let $x_k = 0 + k\Delta x = \frac{k}{n}$. Then $f(x_k) = \frac{4k}{n} - 2$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k}{n}-2\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\left(\frac{4}{n^2}\sum_{k=1}^{n}k - \frac{2}{n}\sum_{k=1}^{n}1\right) =$$

$$\lim_{n\to\infty}\left(\frac{4}{n^2}\left(\frac{n^2+n}{2}\right)-2\right) = 2-2 = 0.$$

11 Let $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Let $x_k = 0 + k\Delta x = \frac{2k}{n}$. Then $f(x_k) = \frac{4k^2}{n^2} - 4$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2} - 4\right)\cdot\frac{2}{n} = \lim_{n\to\infty}\left(\frac{8}{n^3}\sum_{k=1}^{n}k^2 - \frac{8}{n}\sum_{k=1}^{n}1\right) =$$

$$\lim_{n\to\infty}\left(\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - 8\right) = \frac{8}{3} - 8 = \frac{-16}{3}.$$

12 Let $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. Let $x_k = 1 + k\Delta x = 1 + \frac{k}{n} = \frac{n+k}{n}$. Then $f(x_k) = 3\frac{(n+k)^2}{n^2} + \frac{n+k}{n}$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(3\left(\frac{(n+k)^2}{n^2}\right) + \frac{n+k}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3}{n} + \frac{6k}{n^2} + \frac{3k^2}{n^3} + \frac{1}{n} + \frac{k}{n^2}\right) =$$

$$\lim_{n\to\infty}\left(\frac{4}{n}\sum_{k=1}^{n}1 + \frac{7}{n^2}\sum_{k=1}^{n}k + \frac{3}{n^3}\sum_{k=1}^{n}k^2\right) = \lim_{n\to\infty}\left(4 + \frac{7}{2}\cdot\frac{n^2+n}{n^2} + \frac{3}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) = 4 + \frac{7}{2} + 1 = 8.5$$

13 Let $\Delta x = \frac{4-0}{n} = \frac{4}{n}$. Let $x_k = 0 + k\Delta x = 0 + \frac{4k}{n}$. Then $f(x_k) = \frac{64k^3}{n^3} - \frac{4k}{n}$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{64k^3}{n^3} - \frac{4k}{n}\right)\cdot\frac{4}{n} = \lim_{n\to\infty}\left(\frac{256}{n^4}\sum_{k=1}^{n}k^3 - \frac{16}{n^2}\sum_{k=1}^{n}k\right) =$$

$$\lim_{n\to\infty}\left(\frac{256}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{16}{n^2}\cdot\frac{n(n+1)}{2}\right) = 64 - 8 = 56.$$

14 This sum is equal to $\int_0^4 (x^5+1)\,dx = \left(\frac{x^6}{6} + x\right)\Big|_0^4 = \frac{4^6}{6} + 4 = \frac{2060}{3}$.

15

a. $\int_{-4}^{4} f(x)\,dx = 2\int_0^4 f(x)\,dx = 2\cdot 10 = 20.$

b. $\int_{-4}^{4} 3g(x)\,dx = 3\cdot 0 = 0.$

c. $\int_{-4}^{4} 4f(x) - 3g(x)\,dx = 2\cdot 4\cdot\int_0^4 f(x)\,dx - 3\cdot 0 = 8\cdot 10 - 0 = 80.$

d. Let $u = 4x^2$, so that $du = 8x\,dx$. Substituting yields $\int_0^4 f(u)\,du = 10$.

e. Because f is an even function, $3xf(x)$ is an odd function. Thus, $\int_{-2}^{2} 3xf(x)\,dx = 0$.

16

a. $\int_a^c f(x)\,dx = 20 - 12 = 8.$

b. $\int_b^d f(x)\,dx = 15 - 12 = 3.$

c. $2\int_c^b f(x)\,dx = -2\int_b^c f(x)\,dx = -2(-12) = 24.$

d. $4\int_a^d f(x)\,dx = 80 - 48 + 60 = 92.$

e. $3\int_a^b f(x)\,dx = 3\cdot 20 = 60.$

f. $2\int_b^d f(x)\,dx = 2(15-12) = 6.$

g. $\int_a^c |f(x)|\,dx = 20 + 12 = 32.$

h. $\int_d^a |f(x)|\,dx = -(20 + 12 + 15) = -47.$

17 $\int_1^4 3f(x)\,dx = 3\int_1^4 f(x)\,dx = 3\cdot 6 = 18.$

18 $-\int_4^1 2f(x)\,dx = 2\int_1^4 f(x)\,dx = 2\cdot 6 = 12.$

19 $\int_1^4 (3f(x) - 2g(x))\,dx = 3\int_1^4 f(x)\,dx - 2\int_1^4 g(x)\,dx = 3\cdot 6 - 2\cdot 4 = 18 - 8 = 10.$

20 There is not enough information to compute this integral.

21 There is not enough information to compute this integral.

22 $\int_4^1 (f(x) - g(x))\,dx = \int_1^4 (g(x) - f(x))\,dx = \int_1^4 g(x)\,dx - \int_1^4 f(x)\,dx = 4 - 6 = -2.$

23

a. This region can be divided up into a 4×2 rectangle and a right triangle with base and height equal to 1. Thus, the integral is equal to $8 + \frac{1}{2} = 8.5$.

b. $\int_6^4 f(x)\,dx = -\int_4^6 f(x)\,dx$. The region whose area is $\int_4^6 f(x)\,dx$ consists of a 1×3 rectangle, together with a right triangle with base 1 and height 3, so $\int_4^6 f(x)\,dx = 3 + \frac{3}{2} = 4.5$, and $\int_6^4 f(x)\,dx = -4.5$.

c. $\int_5^7 f(x)\,dx = \int_5^6 f(x)\,dx + \int_6^7 f(x)\,dx$. The region lying over $[5,6]$ is a right triangle with height 3 and base 1, so its area is $\frac{3}{2}$. The region lying under $[6,7]$ has the same area, but is below the x-axis, so $\int_6^7 f(x)\,dx = -\frac{3}{2}$. So $\int_5^7 f(x)\,dx = \frac{3}{2} + -\frac{3}{2} = 0$.

d. Note that $\int_4^5 f(x)\,dx = 3$, because the area represented is that of a 1×3 rectangle. Now by the work above, $\int_0^7 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx + \int_5^7 f(x)\,dx = 8.5 + 3 + 0 = 11.5$.

24

i. We are seeking $\int_0^5 g(t)\,dt$. Because this represents the area of a region which can be divided into a 5×1 rectangle and a right triangle with base 2 and height 2, its value is $5 + 2 = 7$.

ii. We are seeking $\int_3^7 g(t)\,dt$. Because this represents the area of a region which can be divided into a 4×1 rectangle and a right triangle with base 4 and height 2, its value is $4 + 4 = 8$.

iii. We are seeking $\int_0^8 g(t)\,dt = \int_0^3 g(t)\,dt + \int_3^7 g(t)\,dt + \int_7^8 g(t)\,dt = 3 + 8 + 1 = 12$. The first term in this sum is represented by a 1×3 rectangle, the second term is from part ii), and the third is represented by a 1×1 square.

25 $\int_0^4 \sqrt{8x - x^2}\,dx = \int_0^4 \sqrt{16 - (x^2 - 8x + 16)}\,dx = \int_0^4 \sqrt{16 - (x-4)^2}\,dx$. This represents one quarter of the area inside the circle centered at $(4,0)$ with radius 4, so its value is $\frac{1}{4}\cdot 16\pi = 4\pi$.

26 $\displaystyle\int_4^0 (2x+\sqrt{16-x^2})\,dx = -2\int_0^4 x\,dx - \int_0^4 \sqrt{16-x^2}\,dx$. The first integral in this sum is the area of a triangle with base 4 and height 4, and the second is the area of a quarter of a circle of radius 4. Therefore the given integral is equal to

$$-2\left(\frac{1}{2}\cdot 4\cdot 4\right) - \frac{1}{4}\left(\pi\cdot 4^2\right) = -16-4\pi = -4(4+\pi).$$

27 It appears that B is the derivative of A, and C is the derivative of B. Thus we must have $A = \displaystyle\int_0^x f(t)\,dt$, $B=f(x)$, and $C=f'(x)$. Note that A is decreasing where B is negative and increasing where B is positive, and has a minimum where B is zero. Note also that B is increasing where C is positive, and is decreasing where C is negative, and has a maximum where C is zero.

28

a. $F(2) = \displaystyle\int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = -1+1 = 0.$
$F(-2) = \displaystyle\int_0^{-2} f(t)\,dt = -\int_0^{-2} f(t)\,dt = \frac{1}{4}\cdot\pi\cdot 4 = \pi.$
$F(4) = \displaystyle\int_0^4 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^4 f(t)\,dt = 0 + \frac{1}{4}\cdot\pi\cdot 4 = \pi.$

b. $G(-2) = \displaystyle\int_1^{-2} f(t)\,dt = -\int_{-2}^1 f(t)\,dt = -\left(\int_{-2}^0 f(t)\,dt + \int_0^1 f(t)\,dt\right) = \pi+1.$
$G(0) = \displaystyle\int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = 1.$
$G(4) = \displaystyle\int_1^4 f(t)\,dt = \int_1^2 f(t)\,dt + \int_2^4 f(t)\,dt = 1+\pi.$

c. One can either reason that the two functions differ by $\displaystyle\int_0^1 f(t)\,dt$ which is a constant, or using the Fundamental Theorem of Calculus, that the two functions have the same derivative (namely $f(x)$) and therefore differ by a constant. Note that $F(x) = G(x) - 1$ for $-2 \le x \le 4$.

29 $\displaystyle\frac{d}{dx}\int_7^x \sqrt{1+t^4+t^6}\,dt = \sqrt{1+x^4+x^6}.$

30 $\displaystyle\frac{d}{dx}\int_3^{e^x} \cos t^2\,dt = \cos(e^x)^2\cdot\frac{d}{dx}e^x = e^x\cos e^{2x}.$

31 $\displaystyle\frac{d}{dx}\int_x^5 \sin w^6\,dw = -\frac{d}{dx}\int_5^x \sin w^6\,dw = -\sin x^6.$

32 $\displaystyle\frac{d}{dx}\int_{x^2}^5 \sin w^6\,dw = -\frac{d}{dx}\int_5^{x^2} \sin w^6\,dw = -\sin(x^2)^6\cdot\frac{d}{dx}(x^2) = -\sin x^{12}\cdot 2x = -2x\sin x^{12}.$

33

$$\begin{aligned}\frac{d}{dx}\int_{-x}^x \frac{dt}{t^{10}+1} &= \frac{d}{dx}\left(\int_{-x}^0 \frac{dt}{t^{10}+1} + \int_0^x \frac{dt}{t^{10}+1}\right)\\ &= \frac{d}{dx}\left(-\int_0^{-x} \frac{dt}{t^{10}+1} + \int_0^x \frac{dt}{t^{10}+1}\right) = -\frac{1}{(-x)^{10}+1}\cdot(-1) + \frac{1}{x^{10}+1} = \frac{2}{x^{10}+1}.\end{aligned}$$

Alternatively, if we realize that the integrand is an even function, we could write

$$\frac{d}{dx}2\int_0^x \frac{dt}{t^{10}+1} = \frac{2}{x^{10}+1}.$$

34 $\frac{d}{dx}\int_{x^2}^{e^x}\sin^3 t\,dt = \frac{d}{dx}\left(-\int_0^{x^2}\sin^3 t\,dt+\int_0^{e^x}\sin^3 t\,dt\right) = -(\sin^3 x^2)2x+(\sin^3 e^x)\cdot e^x = e^x\sin^3 e^x - 2x\sin^3 x^2$.

35 $f(x) = -\int_1^x (t-3)(t-6)^{11}\,dt$, so $f'(x) = -(x-3)(x-6)^{11}$. $f' = 0$ for $x = 3$ and $x = 6$. We have $f' > 0$ on $(3,6)$, so f is increasing there, while $f' < 0$ on $(-\infty, 3)$ and on $(6,\infty)$, so f is decreasing on those intervals.

36 The area represented is a triangle with base $x-2$ and height $2x-4$, so its area is $\frac{(x-2)(2x-4)}{2} = x^2-4x+4$. If we call this quantity $A(x)$, then $A'(x) = 2x-4$, as desired.

37 Let $u = ax+b$. Then $du = a\,dx$, or $dx = \frac{du}{a}$. The substitution gives

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(u)\,du = \frac{1}{a}F(u)+C = \frac{1}{a}F(ax+b)+C.$$

38 $\int_1^5 dx = x\Big|_1^5 = 5-1 = 4$.

39 $\int_{-2}^2 (3x^4-2x+1)\,dx = \left(\frac{3x^5}{5}-x^2+x\right)\Big|_{-2}^2 = \frac{96}{5}-4+2-\left(-\frac{96}{5}-4-2\right) = \frac{192}{5}+4 = \frac{212}{5}$.

40 $\int_0^1 (4x^{21}-2x^{16}+1)\,dx = \left(\frac{4x^{22}}{22}-\frac{2x^{17}}{17}+x\right)\Big|_0^1 = \frac{2}{11}-\frac{2}{17}+1 = \frac{199}{187}$.

41 $\int (9x^8-7x^6)\,dx = x^9-x^7+C$.

42 $\int \frac{\sqrt{x}+1}{\sqrt{x}}\,dx = \int\left(1+\frac{1}{\sqrt{x}}\right)dx = \int(1+x^{-1/2})\,dx = x+2x^{1/2}+C = x+2\sqrt{x}+C$.

43 $\int_0^1 (x+\sqrt{x})\,dx = \left(\frac{x^2}{2}+\frac{2x^{3/2}}{3}\right)\Big|_0^1 = \frac{1}{2}+\frac{2}{3} = \frac{7}{6}$.

44 Let $u = 3x^2+2x+1$, so that $du = (6x+2)\,dx = 2(3x+1)\,dx$. Then $\int(3x+1)(3x^2+2x+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8}+C = \frac{(3x^2+2x+1)^4}{8}+C$.

45 $\int_{\pi/6}^{\pi/3}(\sec^2 t+\csc^2 t)\,dt = (\tan t-\cot t)\Big|_{\pi/6}^{\pi/3} = \left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)-\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right) = 2\sqrt{3}-\frac{2}{\sqrt{3}} = \frac{6-2}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$.

46 $\int_{\pi/12}^{\pi/9}(\csc 3x\cot 3x+\sec 3x\tan 3x)\,dx = \frac{1}{3}(-\csc 3x+\sec 3x)\Big|_{\pi/12}^{\pi/9} = \frac{1}{3}\left(\left(-\frac{2}{\sqrt{3}}+2\right)-\left(-\sqrt{2}+\sqrt{2}\right)\right) = \frac{2}{3}-\frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}-2}{3\sqrt{3}}$.

47 $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x\Big|_{\sqrt{2}}^2 = \sec^{-1}2-\sec^{-1}\sqrt{2} = \frac{\pi}{3}-\frac{\pi}{4} = \frac{\pi}{12}$.

48 $\int_1^4\left(\frac{\sqrt{v}+v}{v}\right)dv = \int_1^4(v^{-1/2}+1)\,dv = \left(2\sqrt{v}+v\right)\Big|_1^4 = 8-3 = 5$.

49 Let $u = \sin x$. Then $du = \cos x\, dx$. Then $\displaystyle\int \frac{\cos x}{\sin^{7/4} x}\, dx = \int u^{-7/4}\, du = -\frac{4}{3}u^{-3/4}\, du = -\frac{4}{3}(\sin x)^{-3/4} + C = -\frac{4}{3\sin^{3/4} x} + C$.

50 Let $u = 1 + \ln x$. Then $du = \dfrac{1}{x}\, dx$. Also, when $x = 1$, $u = 1$, and when $x = e$, $u = 2$. Substituting yields $\displaystyle\int_1^e \frac{dx}{x(1+\ln x)}\, dx = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2$.

51 Let $u = x^3$, so that $du = 3x^2\, dx$. Substituting gives $\displaystyle\int x^2 \cos x^3\, dx = \frac{1}{3}\int \cos u\, du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin x^3 + C$.

52 Let $u = 1 + \sin 3t$, so that $du = 3\cos 3t\, dt$. Substituting yields $\displaystyle\int \frac{\cos 3t}{1+\sin 3t}\, dt = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+\sin 3t| + C$. Note that because $-1 \le \sin 3t$, we have $0 \le 1 + \sin 3t$, so the absolute value sign isn't necessary, so we can write the answer as $\dfrac{1}{3}\ln(1+\sin 3t) + C$.

53 Let $u = \sin 7w$. Then $du = 7\cos 7w$. Substituting yields $\displaystyle\frac{1}{7}\int \frac{du}{16+u^2} = \frac{1}{7}\cdot\frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) = \frac{1}{28}\tan^{-1}\left(\frac{\sin 7w}{4}\right) + C$.

54 Let $u = 1 + \tan 2t$. Then $du = 2\sec^2 2t\, dt$. Substituting gives

$$\frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + C = \frac{1}{3}(1+\tan 2t)^{3/2} + C.$$

55 Let $u = x^2 + 1$. Then $du = 2x\, dx$ and when $x = 0$, $u = 1$ and when $x = 1$, $u = 2$. Substituting gives

$$\frac{1}{2}\int_1^2 2^u\, du = \frac{2^u}{2\ln 2}\Big|_1^2 = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2}.$$

56 $\displaystyle\int \cos 3x\, dx = \frac{\sin 3x}{3} + C$.

57 $\displaystyle\int_0^2 (2x+1)^3\, dx = \left(\frac{(2x+1)^4}{8}\right)\Big|_0^2 = \frac{625}{8} - \frac{1}{8} = 78$.

58 $\displaystyle\int_{-2}^2 e^{4x+8}\, dx = \left(\frac{1}{4}\cdot e^{4x+8}\right)\Big|_{-2}^2 = \frac{1}{4}\left(e^{16}-1\right)$.

59 Let $u = 3r^2 + 2$. Then $du = 6r\, dr$. When $r = 0$, $u = 2$ and when $r = 1$, $u = 5$. Substituting gives

$$\frac{5}{6}\int_2^5 e^u\, du = \frac{5}{6}e^u\Big|_2^5 = \frac{5}{6}(e^5 - e^2) = \frac{5}{6}e^2(e^3 - 1).$$

60 Let $u = \cos z$. Then $du = -\sin z\, dz$. Substituting gives $\displaystyle -\int \sin u\, du = \cos u + C = \cos(\cos z) + C$.

61 Let $u = e^x$, so that $du = e^x\, dx$. We can rewrite the given integral as $\displaystyle\int e^{e^x} e^x\, dx$, so that substituting gives $\displaystyle\int e^u\, du = e^u + C = e^{e^x} + C$.

62 Let $u = y^3+27$, and note that $du = 3y^2\,dy$. Substituting yields $\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u|+C = \frac{\ln|y^3+27|}{3}+C$.

63 Let $u = 2x$ so that $du = 2\,dx$. Substituting gives

$$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}2x + C.$$

64 Let $u = 3y^3+1$, and note that $du = 9y^2\,dy$. Substituting yields $\frac{1}{9}\int u^4\,du = \frac{u^5}{45}+C = \frac{(3y^3+1)^5}{45}+C$.

65 Let $u = \frac{x}{6}$, then $du = \frac{1}{6}\,dx$. When $x = 0$, $u = 0$ and when $x = 2\pi$, $u = \pi/3$. Substituting gives $6\int_0^{\pi/3}\cos^2 u\,du$. Using Example 6 of section 5.6, we have

$$6\int_0^{\pi/3}\cos^2 u\,du = 6\left(\frac{u}{2}+\frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/3} = \pi + \frac{3\sqrt{3}}{4}.$$

66 Let $u = \cos x^2$ and note that $du = -\sin x^2\cdot 2x\,dx$. Substituting yields $-\frac{1}{2}\int u^8\,du = -\frac{u^9}{18}+C = -\frac{\cos^9 x^2}{18}+C$.

67 $\int_0^{\pi}\sin^2 5\theta\,d\theta = \int_0^{\pi}\frac{1-\cos 10\theta}{2}\,d\theta = \left(\frac{\theta}{2}-\frac{\sin 10\theta}{20}\right)\Bigg|_0^{\pi} = \frac{\pi}{2}$.

68 $\int_0^{\pi}(1-\cos^2 3\theta)\,d\theta = \int_0^{\pi}\sin^2 3\theta\,d\theta = \int_0^{\pi}\frac{1-\cos 6\theta}{2}\,d\theta = \left(\frac{\theta}{2}-\frac{\sin 6\theta}{12}\right)\Bigg|_0^{\pi} = \frac{\pi}{2}$.

69 Let $u = x^3+3x^2-6x$, and note that $du = 3x^2+6x-6\,dx = 3(x^2+2x-2)\,dx$. Substituting yields $\frac{1}{3}\int_8^{36}\frac{1}{u}\,du = \left(\frac{1}{3}\ln u\right)\Bigg|_8^{36} = \frac{1}{3}(\ln 36 - \ln 8) = \frac{1}{3}\ln\left(\frac{9}{2}\right)$.

70 Let $u = e^x$ so that $du = e^x\,dx$. Substituting yields $\int_1^2 \frac{1}{1+u^2}\,du = \tan^{-1}u\,\Bigg|_1^2 = \tan^{-1}2 - \frac{\pi}{4}$.

71 Note that the integrand is an odd function (as it is an odd function divided by an even function). Therefore $\int_{-5}^{5}\frac{w^3}{\sqrt{w^{50}+w^{20}+1}} = 0$.

72 Because x^{17}, x^{13}, x^9, and x^3 are all odd functions, this integral reduces to $2\int_0^3 x^2\,dx = \frac{2x^3}{3}\Bigg|_0^3 = 18$.

73 Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$. Substituting yields $-\int \sin u\,du = \cos u + C = \cos\left(\frac{1}{x}\right)+C$.

74 Let $u = \tan^{-1}x$. Then $du = \frac{1}{1+x^2}\,dx$. Substituting yields $\int u^5\,du = \frac{u^6}{6}+C = \frac{(\tan^{-1}x)^6}{6}+C$.

75 Let $u = \tan^{-1}x$. Then $du = \frac{1}{1+x^2}\,dx$. Substituting yields $\int \frac{1}{u}\,du = \ln|u|+C = \ln|\tan^{-1}x|+C$.

76 Let $u = \sin^{-1}x$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$. Substituting yields $\int u\,du = \frac{u^2}{2}+C = \frac{(\sin^{-1}x)^2}{2}+C$.

77 Let $u = x + 3$. Then $du = dx$ and $u - 3 = x$. Substituting gives $\int (u-3)u^{10}\,du = \int (u^{11} - 3u^{10})\,du = \frac{u^{12}}{12} - \frac{3u^{11}}{11} + C = u^{11}\left(\frac{u}{12} - \frac{3}{11}\right) = (x+3)^{11}\left(\frac{x}{12} - \frac{1}{44}\right) = (x+3)^{11}\left(\frac{11x-3}{132}\right)$.

78 Let $u = x^4 + 1$. Then $du = 4x^3\,dx$ and $u - 1 = x^4$. Substituting gives $\int x^3 x^4 \sqrt{x^4+1}\,dx = \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{u^{3/2}}{4}\left(\frac{2}{5}u - \frac{2}{3}\right) + C = \frac{(x^4+1)^{3/2}}{4}\left(\frac{2}{5}(x^4+1) - \frac{2}{3}\right) + C = \frac{(x^4+1)^{3/2}}{4}\left(\frac{2}{5}x^4 - \frac{4}{15}\right) + C = \frac{1}{30}\left(x^4+1\right)^{3/2}(3x^4-2) + C$.

79 Let $u = 25 - x^2$. Then $du = -2x\,dx$. Substituting yields $-\frac{1}{2}\int_{25}^{16} u^{-1/2}\,du = -\sqrt{u}\,\Big|_{25}^{16} = 5 - 4 = 1$.

80 Using Table 5.6, we have

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2}\Big|_0^1 = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$$

81 Let $u = 5x$ so that $du = 5dx$. When $x = \sqrt{2}/5$, $u = \sqrt{2}$, and when $x = 2/5$, $u = 2$. Substituting then gives

$$\int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|\Big|_{\sqrt{2}}^{2} = \left(\sec^{-1} 2 - \sec^{-1}\sqrt{2}\right) = \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}.$$

82 Let $u = 1 + \cos^2 x$. Then $du = -2\sin x\cos x\,dx$. Substituting yields $-\int \frac{1}{u}\,du = -\ln|u| + C = -\ln(1+\cos^2 x) + C$.

83 $\int_{-10}^{10} \frac{x}{\sqrt{200-x^2}}\,dx = 0$ because the integrand is an odd function.

84 Note that the first term of the integrand is an even function, and the other two terms are odd functions. Thus, $\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x\sin x - 3\sin x^5)\,dx = 2\int_0^{\pi/2} \cos 2x\,dx = 2\left(\frac{\sin 2x}{2}\right)\Big|_0^{\pi/2} = 0$.

85 $\int_0^4 f(x)\,dx = \int_0^3 (2x+1)\,dx + \int_3^4 (3x^2+2x-8)\,dx = (x^2+x)\Big|_0^3 + (x^3+x^2-8x)\Big|_3^4 = 12 + (48-12) = 48$.

86 $\int_0^5 |2x-8|\,dx = \int_0^4 (8-2x)\,dx + \int_4^5 (2x-8)\,dx = (8x-x^2)\Big|_0^4 + (x^2-8x)\Big|_4^5 = 16 + (-15+16) = 17$.

87 The area is given by $\int_{-4}^{4} (16-x^2)\,dx = \left(16x - \frac{x^3}{3}\right)\Big|_{-4}^{4} = 64 - \frac{64}{3} - (-64 + \frac{64}{3}) = \frac{256}{3}$.

88 The area is given by $\int_{-1}^{0} (x^3 - x)\,dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$.

89 The area is given by $2\int_0^{2\pi} \sin(x/4)\,dx = 2\left(-4\cos(x/4)\right)\Big|_0^{2\pi} = -8(0-1) = 8$.

90 The area is given by $\int_{-1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \tan^{-1} x\,\Big|_{-1}^{\sqrt{3}} = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}$.

91

i. $\displaystyle\int_{-1}^{1}(x^4-x^2)\,dx = \left(\frac{x^5}{5}-\frac{x^3}{3}\right)\Big|_{-1}^{1} = \left(\frac{1}{5}-\frac{1}{3}\right)-\left(-\frac{1}{5}-\left(-\frac{1}{3}\right)\right) = -\frac{4}{15}$.

ii. Because the region lies completely below the x-axis, the area bounded by the curve and the x-axis is $\displaystyle -\int_{-1}^{1}(x^4-x^2)\,dx = \frac{4}{15}$.

92

i. $\displaystyle\int_0^3 (x^2-x)\,dx = \left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_0^3 = 9-\frac{9}{2}=\frac{9}{2}$.

ii. Because the region is below the x-axis between 0 and 1 and above between 1 and 3, the area bounded by the curve and the x-axis is $\displaystyle -\int_0^1 (x^2-x)\,dx + \int_1^3 (x^2-x)\,dx = -\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_1^3 = -\left(\frac{1}{3}-\frac{1}{2}\right)+\left(9-\frac{9}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right) = \frac{9}{2}+2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{9}{2}+\frac{1}{3} = \frac{29}{6}$.

93 The average height of the arch is given by

$$\begin{aligned}&\frac{1}{630}\int_{-315}^{315}\left(1260-315\left(e^{0.00418x}+e^{-.00418x}\right)\right)\,dx\\ &=\frac{1}{630}\left(1260x-\frac{315}{0.00418}(e^{0.00418x}-e^{-.00418x})\right)\Big|_{-315}^{315} \approx 431.5 \text{ ft.}\end{aligned}$$

94 Let $T=\dfrac{2\pi k}{\omega}$ where k is an integer. The RMS is given by $\displaystyle\sqrt{\frac{\omega}{2\pi k}\int_0^{2\pi k/\omega} A^2\sin^2(\omega t)\,dt} =$ $\displaystyle A\sqrt{\frac{\omega}{2\pi k}\int_0^{2\pi k/\omega}\frac{1-\cos(2\omega t)}{2}\,dt} = A\sqrt{\frac{\omega}{2\pi k}\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]_0^{2\pi k/\omega}} = \frac{A}{\sqrt{2}}$.

95 The displacement is $\displaystyle\int_0^2 5\sin\pi t\,dt = -\frac{5\cos\pi t}{\pi}\Big|_0^2 = -\frac{5}{\pi}(\cos 2\pi-\cos 0)=0$. The distance traveled is

$$\begin{aligned}\int_0^1 5\sin\pi t\,dt+\int_1^2 -5\sin\pi t\,dt &= -\frac{5\cos\pi t}{\pi}\Big|_0^1+\frac{5\cos\pi t}{\pi}\Big|_1^2\\ &=\frac{5}{\pi}+\frac{5}{\pi}+\frac{5}{\pi}+\frac{5}{\pi}=\frac{20}{\pi}.\end{aligned}$$

96

a. The distance traveled is given by $\displaystyle\int_0^4 (2t+5)\,dt = \left(t^2+5t\right)\Big|_0^4 = 36$.

b. The average value is $\displaystyle\frac{1}{4}\int_0^4 (2t+5)\,dt = \frac{1}{4}\cdot 36 = 9$.

c. True. If it traveled at a rate of 9 for a time of 4, it would have gone 36 units.

97 The average value is

$$\frac{1}{\ln 2}\int_0^{\ln 2} e^{2x}\,dx = \frac{1}{2\ln 2}e^{2x}\Big|_0^{\ln 2} = \frac{4-1}{2\ln 2} = \frac{3}{2\ln 2}.$$

98 The baseball is in the air for x in the interval $(0, 200)$. The average height is

$$\frac{1}{200}\int_0^{200}(2x - 0.01x^2)\,dx = \frac{1}{200}\left(x^2 - \frac{0.01x^3}{3}\right)\Big|_0^{200} = 200 - \frac{400}{3} = \frac{200}{3}.$$

99

a. The average value is 2.5. This is because for a straight line, the average value occurs at the midpoint of the interval, which is at the point $(3.5, 2.5)$, so $c = 3.5$.

b. The average value is 3 over the interval $[2, 4]$ and 3 over the interval $[4, 6]$, so is 3 over the interval $[2, 6]$. The function takes on this value at $c = 3$ and $c = 5$.

100 Differentiating both sides of the given equation gives $12x^3 = f(x)$. To check, we compute $\int_2^x 12t^3\,dt = 3t^4\Big|_2^x = 3x^4 - 48$, which gives the original equation.

101 Let $u = 2x$, so that $du = 2\,dx$. We have $\frac{1}{2}\int_2^4 f'(u)\,du = \frac{1}{2}\cdot(f(4) - f(2)) = \frac{f(4)}{2} - 2$. Because we are given that this quantity is 10, we have $f(4) = 24$.

102 Note that $H'(x) = \sqrt{4 - x^2}$ by the Fundamental Theorem.

a. $H(0) = \int_0^0 \sqrt{4 - t^2}\,dt = 0$.

b. $H'(1) = \sqrt{3}$.

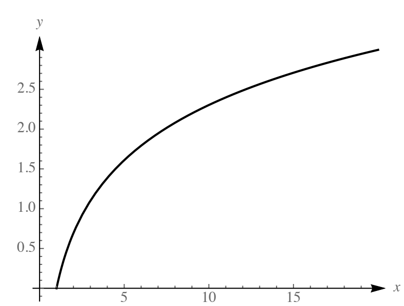
c. $H'(2) = \sqrt{4 - 4} = 0$.

d. $H(2) = \int_0^2 \sqrt{4 - t^2}\,dt = \frac{1}{4}\cdot\pi\cdot 2^2 = \pi$. This follows because the given area represents $\frac{1}{4}$ of the area inside a circle of radius 2.

e. $H(-x) = \int_0^{-x} \sqrt{4 - t^2}\,dt = -\int_{-x}^0 \sqrt{4 - t^2}\,dt = -H(x)$, because $\sqrt{4 - t^2}$ is an even function. So $s = -1$.

103

By the Fundamental Theorem, $f'(x) = \frac{1}{x}$, which is always positive for $x > 1$. Thus f is always increasing. Also, $f(1) = \int_1^1 \frac{1}{t}\,dt = 0$. Also, $f''(x) = -\frac{1}{x^2}$ which is always negative, so f is always concave down.



104 If we let $u^3 = x^2 - 1$, then $3u^2\,du = 2x\,dx$, so $x\,dx = \frac{3u^2}{2}\,du$. Also, when $x = 1$ we have $u = 0$ and when $x = 3$ we have $u = 2$. Substituting gives $\int_1^3 x\sqrt[3]{x^2 - 1}\,dx = \frac{3}{2}\int_0^2 u^2\cdot u\,du = \frac{3}{2}\left(\frac{u^4}{4}\right)\Big|_0^2 = 6$.

105

a. $F(-2) = \int_{-1}^{-2} f(t)\,dt = \int_{-1}^{-2} t\,dt = \left.\frac{t^2}{2}\right|_{-1}^{-2} = \frac{3}{2}$.
$F(2) = \int_{-1}^{2} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt = \int_{-1}^{0} t\,dt + \int_{0}^{2} \frac{t^2}{2}\,dt = \left.\frac{t^2}{2}\right|_{-1}^{0} + \left.\frac{t^3}{6}\right|_{0}^{2} = \left(0 - \frac{1}{2}\right) + \left(\frac{8}{6} - 0\right) = \frac{5}{6}$.

b. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, so for $-2 \le x < 0$ we have $F''(x) = x$.

c. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, so for $0 \le x < 2$ we have $F''(x) = \frac{x^2}{2}$.

d. $F'(-1) = -1$ and $F'(1) = \frac{1}{2}$. These represent the rate of change of F at the given points. Because the graph in the exercise is the derivative of F, this is just the value of f at the given points.

e. Note that $F''(x) = \begin{cases} 1 & \text{if } -2 \le x < 0, \\ x & \text{if } 0 \le x \le 2. \end{cases}$ Thus we have $F''(-1) = 1$ and $F''(1) = 1$.

f. The difference $F(x) - G(x) = \int_{-1}^{-2} f(t)\,dt = \frac{3}{2}$, as noted in part (a).

106

a. $G(-1) = \int_{-2}^{-1} f(t)\,dt = \int_{-2}^{-1} t\,dt = \left.\frac{t^2}{2}\right|_{-2}^{-1} = -\frac{3}{2}$.
$G(1) = \int_{-2}^{1} f(t)\,dt = \int_{-2}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt = \int_{-2}^{0} t\,dt + \int_{0}^{1} \frac{t^2}{2}\,dt = \left.\frac{t^2}{2}\right|_{-2}^{0} + \left.\frac{t^3}{6}\right|_{0}^{1} = -\frac{11}{6}$.

b. By the Fundamental Theorem, $G'(x) = f(x)$, so that for $-2 \le x < 0$, we have $G'(x) = f(x) = x$.

c. By the Fundamental Theorem, $G'(x) = f(x)$, so that for $0 \le x \le 2$, we have $G'(x) = f(x) = \frac{x^2}{2}$.

d. $G'(0) = f(0) = 0$ and $G'(1) = f(1) = \frac{1}{2}$. These represent the rate of change of G at the given points. Because the graph in the exercise is the derivative of G, this is just the value of f at the given points.

e. The difference $F(x) - G(x) = \int_{-1}^{-2} f(t)\,dt = \frac{3}{2}$, as noted in part (f) of the previous problem.

107 By L'hôpital's rule, we have

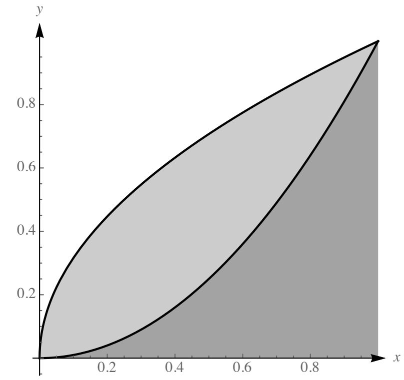
$$\lim_{x\to 2} \frac{\int_2^x e^{t^2}\,dt}{x-2} = \lim_{x\to 2} \frac{e^{x^2}}{1} = e^4.$$

108 By L'hôpital's rule, we have

$$\lim_{x\to 1} \frac{\int_1^{x^2} e^{t^3}\,dt}{x-1} = \lim_{x\to 1} \frac{2xe^{x^6}}{1} = 2e.$$

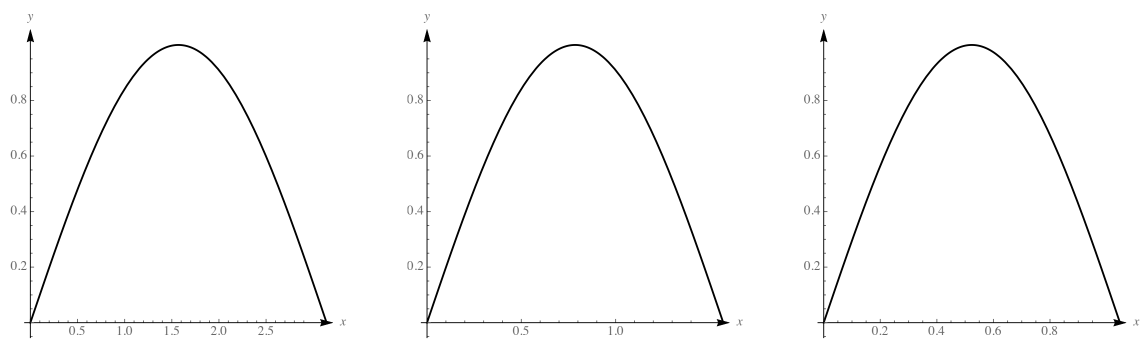
109

Because x^n and $\sqrt[n]{x}$ are inverse functions of each other, they are symmetric in the square $[0,1] \times [0,1]$ about the line $y = x$. Together, the two regions completely fill up the 1×1 square, so these two areas add to one.



110

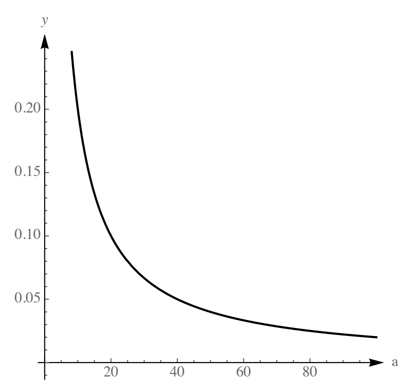
a.



b.

$$\int_0^{\pi/a} \sin(ax)\,dx = \left(\frac{-\cos ax}{a}\right)\Bigg|_0^{\pi/a} = \frac{1}{a} + \frac{1}{a} = \frac{2}{a}.$$

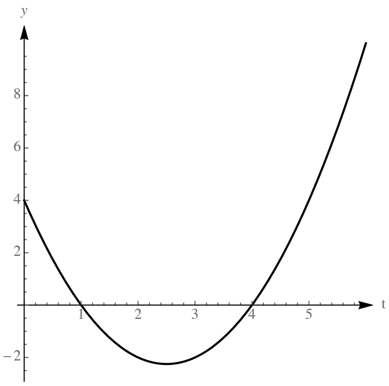
This is a decreasing function of a.



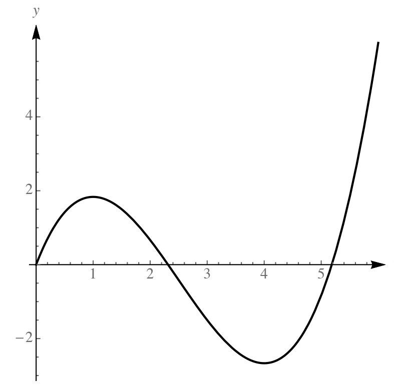
111 Factoring out $\frac{1}{b^2}$ gives $\frac{1}{b^2}\int \frac{dx}{(ax/b)^2+1}$. Now let $u = \frac{ax}{b}$, so that $du = \frac{a}{b}\,dx$.

Substituting yields $\frac{1}{b^2} \cdot \frac{b}{a} \int \frac{du}{u^2+1} = \frac{1}{ab}\left(\tan^{-1}(u)\right) + C = \frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right) + C.$

112



a.



b.

c. The zeros of f are at 1 and 4, and A has a local maximum at $x = 1$ and a local minimum at $x = 4$.

d. Geometric: Because f is above the axis from 0 to 1 and then crosses below the axis at 1, the net area from 0 to x will switch from increasing to decreasing as x moves from the left of 1 to the right of 1. A similar (but opposite) thing can be said near 4, because f switches from below the axis to above, the net area switches from decreasing to increasing at $x = 4$.

Analytic: By the Fundamental Theorem, $A'(x) = f(x)$, so the zeros of f are critical points of A, and in this example, lead to extrema.

e. Because $A(x) = \frac{x}{6} \cdot \left(2x^2 - 15x + 24\right)$, the non-zero zeros of A occur at $x = \frac{15 \pm \sqrt{225 - 4 \cdot 2 \cdot 24}}{4} = \frac{15 \pm \sqrt{33}}{4}$. So $x_1 = \frac{15 - \sqrt{33}}{4} \approx 2.31386$ and $x_2 = \frac{15+\sqrt{33}}{4} \approx 5.18614$.

f. Because $f(x) = A'(x)$, the area bounded by the graph of f and the x-axis on $[0, x_1]$ is

$$\int_0^1 f(x)\,dx - \int_1^{x_1} f(x)\,dx = A(1) - A(0) - (A(x_1) - A(1)) = 2A(1) = \frac{11}{3}.$$

Now, note that the area bounded by the graph of f on $[x_1, 4]$ is $-(A(4) - A(x_1)) = \frac{8}{3}$, so that $b > 4$. Then the area bounded by the graph of f and the x-axis on $[x_1, b]$ is

$$-\int_{x_1}^4 f(x)\,dx + \int_4^b f(x)\,dx = -(A(4) - A(x_1)) + A(b) - A(4) = A(b) - 2A(4) = A(b) + \frac{16}{3}.$$

Thus we want to solve $A(b) = -\frac{5}{3}$; using technology, we obtain $b \approx 4.756$.

g. No. For example, consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ -1 & \text{if } 1 \le x \le 2. \end{cases}$$

Then $A(x) = \int_0^x f(t)\,dt$ has a maximum at $x = 1$, even though f is never zero. An extreme point of A can occur at points of discontinuity of f.

113 Note that $f'(x) = (x-1)^{15}(x-2)^9$, and that the zeros of f' are at $x = 1$ and $x = 2$.

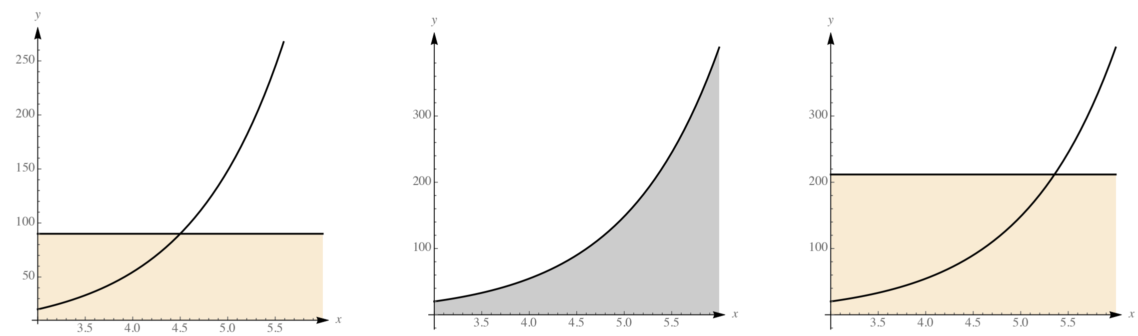
a. f' is positive and thus f is increasing on $(-\infty, 1)$ and on $(2, \infty)$, while f' is negative and f is decreasing on $(1, 2)$.

b. $f''(x) = 15(x-1)^{14}(x-2)^9 + (x-1)^{15} \cdot 9(x-2)^8 = 3(x-1)^{14}(x-2)^8(8x-13)$.
f is concave up on $\left(\dfrac{13}{8}, \infty\right)$ and concave down on $\left(-\infty, \dfrac{13}{8}\right)$.

c. f has a local maximum at $x = 1$ and a local minimum at $x = 2$.

d. f has an inflection point at $x = \dfrac{13}{8}$.

114



The first graph on the left above shows that the area of the rectangle with height e^m where m is the midpoint of $[a, b]$ is less than the area under the curve of e^x over $[a, b]$. The last graph on the right above shows that the area of the rectangle whose height is the average of e^a and e^b is greater than the area under the curve of e^x over that interval. Note that the area under the curve is $\int_a^b e^x\, dx = (e^x)\Big|_a^b = e^b - e^a$. Putting these ideas together, we have

$$e^{(a+b)/2}(b-a) < e^b - e^a < \left(\frac{e^a + e^b}{2}\right)(b-a),$$

and dividing through by $(b-a)$ yields

$$e^{(a+b)/2} < \frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}.$$

115 This follows by differentiating each side of the equation. $\dfrac{d}{dx}\left(u(x) + 2\int_0^x u(t)\, dt\right) = u'(x) + 2u(x)$, and $\dfrac{d}{dx}10 = 0$. The reverse is not true, because if $u(x) + 2\int_0^x u(t)\, dt = C$ for any constant C, then it would satisfy the second equation, even if $C \neq 10$.

116 The area bounded by $c \sin x$ over the stated interval is $\int_0^\pi c \sin x\, dx = (-c\cos x)\Big|_0^\pi = (c - -c) = 2c$. This is equal to 1 when $c = \dfrac{1}{2}$.

117 $\int_0^c x(x-c)^2\, dx = \int_0^c (x^3 - 2cx^2 + c^2x)\, dx = \left(\dfrac{x^4}{4} - 2c\dfrac{x^3}{3} + \dfrac{c^2x^2}{2}\right)\Big|_0^c = \dfrac{c^4}{4} - \dfrac{2c^4}{3} + \dfrac{c^4}{2} = \dfrac{c^4}{12}$. This is one when $c = \sqrt[4]{12}$.