MATHEMATICAL THINKING
PROBLEM-SOLVING AND PROOFS

SECOND EDITION (2000)

SOLUTION MANUAL
SUMMER 2005 VERSION

©John P. D'Angelo and Douglas B. West

MATHEMATICS DEPARTMENT

UNIVERSITY OF ILLINOIS

This is the 2005 version of the Instructor's Solution Manual for *Mathematical Thinking*, by John P. D'Angelo and Douglas B. West. Solutions are now included for all problems through the end of Chapter 15. Solutions are also present for more than 80% of the problems in Chapters 16 and 17.

The positions of solutions that have not yet been written into the files (in Chapter 16 and beyond) are occupied by the statements of the corresponding problems. These problems retain the (−), (!), (+) indicators. Also (•) has been added to introduce the statements of problems without other indicators. Thus every problem whose solution is not written out here should be marked by one of the indicators, for ease of identification.

We hope that the solutions contained herein will be useful to instructors. The level of detail in solutions varies. Instructors should feel free to write solutions with more or less detail according to the needs of the class. Due to time limitations, the solutions have not been proofread or edited as carefully as the text. Please send corrections to *west@math.uiuc.edu*.

Updated versions of the solution manual may be available in the future with the remaining solutions and perhaps with additional features. Inquiries regarding particular missing solutions may be sent to *west@math.uiuc.edu*. Meanwhile, we apologize for any inconvenience caused by the absence of some solutions.

John P. D'Angelo
Douglas B. West

INTRODUCTION

A course taught from *Mathematical Thinking: Problem-Solving and Proofs* should emphasize the pedagogical value and excitement of learning mathematics by solving problems. Also important is the clear presentation of solutions, which involves careful writing.

Even if the students are merely set the task of solving the motivating problems listed in the preface, much mathematics will be discovered. For the convenience of the instructor, we indicate here where the preface problems are treated in the text. Many of them are solved. Others appear in exercises; for these we provide further hints or sketches of solutions here.

The 950 or so exercises in the text provide substantial additional practice. The text provides hints to many of these, but not solutions. In the remainder of this manual we provide solutions. The exercises for which we have not yet had time to write the solutions appear as in the text; solutions will be inserted in later editions of this manual.

Some students may need further explicit discussions of the structure of proofs. Such discussion appear in many texts, such as

Eisenberg, *The Mathematical Method: A Transition to Advanced Mathematics*;
Fletcher/Patty, *Foundations of Higher Mathematics*;
Galovich, *Introduction to Mathematical Structures*;
Galovich, *Doing Mathematics: An Introduction to Proofs and Problem Solving*;
Solow, *How to Read and Do Proofs*.

For whatever assistance it may provide, we also offer a syllabus given to instructors at the University of Illinois in our course called "Fundamental Mathematics". This is based on the experience of the second author when teaching the course. Most students in the course have finished a (non-rigorous) calculus sequence.

Math 347, Fundamental Mathematics - Syllabus

Math 347 introduces students to mathematics, improving their ability to absorb and communicate it via practice in problem-solving and writing proofs. Topics that prepare for upperclass courses include logical reasoning, induction, equivalence relations, elementary counting, and limits.

When taught from *Mathematical Thinking: Problem-Solving and Proofs* by D'Angelo and West (second edition), the course samples discrete and continous topics from Parts III-IV to accompany the basic material in Parts I-II. Chapters 1-4 and 13 are the most crucial but don't fill the course. One can also present Chapters 5-10 and 14-15. In these, the central material appears early, and some later material (including all optional sections) can be skipped. It is not necessary to present all the examples.

Part II is more fundamental than Chapters 9-10; many instructors skip these to provide more time in other chapters. Skip Chapters 11-12 (and 16-18) to sharpen the focus of the course. A worthy goal is to end with the Intermediate Value Theorem and its applications in Chapter 15.

The course demands effort from the students; they must read the text and work many exercises. Encourage them to read the sections on *Approaches to Problems*. Unlike most of their earlier math courses, this course requires independent thinking and understanding; students cannot expect all questions to be instances of procedures listed in class.

Part I	Elementary Concepts	**12**	Part III	Discrete Mathematics	**5**
§1	Numbers, Sets, and Functions	2.5	§9	Probability	3
§2	Language and Proofs	2.5	§10	Two Principles of Counting	2
§3	Induction	4	§11	Graph Theory (skip)	0
§4	Bijections and Cardinality	3	§12	Recurrence Relations (skip)	0
Part II	Properties of Numbers	**10**	Part IV	Continuous Mathematics	**10**
§5	Combinatorial Reasoning	3	§13	The Real Numbers	3
§6	Divisibility	2	§14	Sequences and Series	4
§7	Modular Arithmetic	3	§15	Continuous Functions	3
§8	The Rational Numbers	2	*	Leeway and Exams	**6**
			*	Total	**43**

Notes: §1: allude to but don't present "The Real Number System"; some other elementary definitions can also be left as background reading. §2: treat lightly in class, emphasizing understanding rather than formality for quantifiers and conditionals - practice with logical statements comes throughout the course. §3: 3.26 and 3.27 are not both necessary; 3.27 can be done simply for powers of 2. §4: skip Schroeder-Bernstein.

§5: 5.30-31 optional. §6: Dart Board Problem very appealing but optional; skip the section on polynomials. §7: Newspaper Problem optional; skip "Congruence and Groups". §8: "Pythagorean triples" is appealing but optional; omit "Further Properties".

§9: the ideas of conditional probability and expectation are the most important if the chapter is covered; "Multinomial Coefficients" optional. §10: choose a few applications as time permits.

§13: cover completely. §14: the proofs of convergence tests apply Cauchy sequences but can be treated lightly; Exercise 14.58 is a valuable addition. §15: stress that the results on sequences imply the results on continuity; don't mention uniform continuity. §16: if reached in honors sections, treat lightly; state definitions, assume basic properties, perhaps prove chain rule and Rolle/MVT, skip Newton's method and convexity, aim to convey the idea of a continuous nowhere-differentiable function.

0. PREFACE PROBLEMS

0.1. Solved in Application 1.14 on page 8.

0.2. Solved in Solution 3.34 on page 66.

0.3. Solved in Solution 3.22 on page 58 for squares and in Solution 12.27 on pages 242–243 for triangles.

0.4. Solved in Solution 3.26 on pages 60–61.

0.5. Solved in Solution 4.8 on pages 79–80.

0.6. This is discussed on pages 109–110; see also Exercise 5.33 on page 120.

0.7. Solution 9.10 on page 109 presents the technique, with further analysis on page 110. Exercise 5.33 considers the next instances.

The result of Solution 9.10 yields another method when given a specific k. The answer is a polynomial of degree $k+1$ in the variable n. Using the method of undetermined coefficients, the easily-found values at $n = 0, 1, \ldots, k$ create a system of linear equations for the coefficients.

0.8. Solved in Solution 3.27 on pages 61–62.

0.9. Solved in Solution 5.24 on page 108.

0.10. Solved on pages 127–128.

0.11. Exercise 6.34 requests the standard proof of the first part. Exercises 6.41 and 7.22 request other proofs. Here we give the standard proof due to Euler. Suppose that there are finitely many primes. The integer that is one larger than the product of all these primes has no divisor in this set (since dividing it by each prime leaves a remainder of one). Hence there is another prime.

The second half is Exercise 6.35. For an integer $n > 1$, the $N - 1$ consecutive numbers $n! + 2, n! + 3, \ldots, n! + n$ are not prime, as they have the factors $2, 3, \ldots, n$, respectively.

0.12. Solved in two ways, in Solution 6.20 and Solution 6.22 on pages 130 and 131.

0.13. Solved on pages 146-147; see also Exercise 7.37 for extensions of this problem. Comment: This problem is valuable because the notion of equivalence relation used is so natural; two pairs are equivalent if they represent the same amount of money.

0.14. Exercise 7.16 on page 152. Here we sketch the solution. We associate with Friday the number 0, with Saturday the number 1, and so on up

through Thursday the number 6. If January 13 gets the number x, then February 13 gets the number $x + 3 \pmod{}7$ since January has 31 days. If we are not in a leap year, then the 13th day of the successive months from March through December get the numbers that are congruent modulo 7 to $x+3, x+6, x+1, x+4, x+6, x+2, x+5, x, x+3, x+5$. The congruence class shifts by k when leaving a month with $28 + k$ days. We obtain all congruence classes modulo 7. One of these is congruent to $0 \pmod{}7)$, which gives Friday the thirteenth.

The proof for leap years is the same, except the numbers from March 13 onward are increased by one. Again all 7 classes arise.

One can avoid separating into two cases by considering only the months after February, since again all congruences classes occur among these.

0.15. Exercise 7.4 on page 151 requests a stronger statement that immediately implies this.

0.16. This is Exercise 7.23 on page 153. Suppose that the decimal expansion of the number N is $a_1 a_2 \cdots a_k a_k \cdots a_2 a_1$. Since 10 is congruent to $(-1) \pmod 1)1$, 10^j is congruent to $(-1)\ j$. Thus N is congruent to $(-a_1 + a_1) + (a_2 - a_1) + \cdots \pm (a_k - a_{k-1})$, which is $0 \pmod 1)1$.

In base q we have q congruent to $-1 \pmod q) + 1$. Thus palindromatic numbers with an even number of digits in their base q expansions are divisible by $q + 1$. Comment: It is useful to ask students to explain the role of the hypothesis about an even number of digits.

0.17. For the first part we follow the method of Example 6.19 on page 128. First, z must be divisible by 21; otherwise there is no solution. Given such a z, let $z = 21w$. The reduced equation is $2x + 3y = w$. Since $x = 2, y = -1$ gives a solution to $2x + 3y = 1$, we see that $x = 2w, y = -w$ gives a particular solution to the given equation. All others are found to be $x = 2w - 3k, y = -w + 2k$ for some integer k. See Exercises 6.45–6.48 for more examples.

The second part is solved in Theorem 8.22 on pages 163-164.

0.18. This is Exercise 6.50 on page 137. If p is prime, then the values of k for which there exist $m, n \in \mathbb{N}$ such that $k/p = 1/m + 1/n$ are $2p, p, 2, 1$, and all divisors of $p+1$. Rewriting the equation as $kmn = p(m+n)$, we conclude that p must divide at least one of k, m, n. If $p|k$, then k/p is an integer, which is at most 2 since m, n are integers; the two possibilities are achieved by $(m, n) = (1, 1)$ and $(m, n) = (2, 2)$. If $p|m$ and $p|n$, then $k = 1/a + 1/b$ for some $a, b \in \mathbb{N}$, which implies $k \le 2$. We can achieve $k = 2$ by $(m, n) = (p, p)$ and $k = 1$ by $(m, n) = (2p, 2p)$. In the remaining cases, we may assume that p divides m but not n or k.

Suppose $m = ap$. From $kapn = p(ap + n)$, we obtain $(ka - 1)n = ap$. Hence $p|(ka - 1)$; let $ka - 1 = bp$. Now $nbp = ap$, which implies $nb = a$. The original equation becomes $knbpn = p(nbp + n)$, which reduces to $knb = bp + 1$. Since these are multiples of b, we conclude $b = 1$. Now $kn = p + 1$, which implies $k|(p + 1)$. Furthermore, setting $(m, n) = (p\frac{p+1}{k}, \frac{p+1}{k})$ makes any such k achievable.

0.19. Assuming that we define "B has more elements than A" to mean that there is no surjective function from A to B, then the answer to the first part is no, and the answer to the second is yes. For the first part, see Exercise 8.17; its solution parallels the proof of Theorem 4.44 on page 89. The second part is answered in the text in Theorem 13.27 on page 266.

0.20. See Example 9.20 for related discussion. To achieve the numbers here, player A hits one for three during the day and one for four at night; player B hits twenty-seven for ninety during the day and two for ten at night.

0.21. Solved in Application 9.31 on page 181.

0.22. Solved in Solution 9.10 on pages 172-173.

0.23. Solved in greater generality in Example 10.6 on page 191.

0.24. This is Exercise 10.23. Every two-colored square grid of order at least five has a rectangle with corner of the same color, but the grid of order four has no such rectangle, so the answer is five. Consider a subgrid of order five. Each column has a majority in some color. By the pigeonhole principle, some color is the majority in at least three of the columns. Suppose this is black. Each pair of these three columns is black in at least one common row. If they are all black in one common row, then they have at least 6 blacks in the other four rows; another row receiving two blacks completes the rectangle.

If they are not black in a common row, then the three pairs of columns have common blacks in three distinct rows. There remain at least three blacks among these columns in the remaining two rows. Thus one of these rows has at least two blacks, and this completes a black-cornered rectangle with one of the three rows described initially.

$$\begin{array}{cccc}
\bullet & \circ & \circ & \bullet \\
\circ & \circ & \bullet & \bullet \\
\circ & \bullet & \bullet & \circ \\
\bullet & \bullet & \circ & \circ
\end{array}$$

0.25. Solution 10.13 on page 196 provides the method.

0.26. Solved in Solution 10.14 on pages 196–197.

0.27. See Exercise 11.4. The minimum number of trails that together traverse each edge of a connected non-Eulerian graph exactly once is half the number of vertices with odd degree. Since each vertex of odd degree must be an endpoint of some trail, this many trails is necessary. By adding edges joining pairs of odd-degree vertices, we obtain an Eulerian supergraph (all degrees even). Deleting the added edges from an Eulerian circuit of this graph yields the desired trails.

0.28. Solved in Theorem 11.48 with discussion on pages 217-219.

0.29. Solved in Solution 11.69 on page 227. Also see Example 12.13 on page 251.

0.30. Solved in Solution 11.68 on page 226.

0.31. Solved in Solutions 12.20 and 12.23 on page 239–241.

0.32. See Exercises 14.23 and 14.24. The crucial idea in analyzing the recurrence $x_{n+1} = A{x_n}^2 + Bx_n + C$ is to change variables to reduce to a simpler situation.

0.33. See the discussion on page 264 and Exercise 13.35.

0.34. Solved in Solution 14.26 on page 281.

0.35. See the statement of problem 17.3 on page 337 and its solution in Solution 17.32 on page 307.

0.36. This is Exercise 16.64. The answer depends on p. When p is close to one, the singles hitters generate more runs, while when p is close to zero, the home run hitters generate more runs. The precise solution involves computing for each team the expected number of runs scored before there are three outs; one way to do this is to sum infinite series explicitly.

0.37. See the discussion on page 338 and material on convergence in general.

0.38. This is Problem 15.2, solved in Solution 15.23 on page 301.