Chapter 1

Systems of Linear Equations

1.1 Practice Problems

1. (a) $-2x_1 + 8x_2 = 5 \Rightarrow x_2 = \frac{5}{8} + \frac{1}{4}x_1$. Substitute into the second equation to obtain $3x_1 - 12\left(\frac{5}{8} + \frac{1}{4}x_1\right) = 4 \Rightarrow -\frac{15}{2} \neq 4$. Thus no solution exists.

   (b) $x_1 - 2x_2 = 3 \Rightarrow x_2 = \frac{1}{2}x_1 - \frac{3}{2}$. Substitute into the second equation to obtain $-3x_1 + 6\left(\frac{1}{2}x_1 - \frac{3}{2}\right) = -9 \Rightarrow -9 = -9$, which is true for all x_1. Therefore we may set x_1 as a free variable, $x_1 = s_1$ and then $x_2 = \frac{1}{2}s_1 - \frac{3}{2}$.

2. (a) The fourth equation, $0 = -2$, does not hold true, so no solutions exist.

   (b) x_2 and x_4 are free variables, so let $x_2 = s_1$ and $x_4 = s_2$. From the third equation, $x_5 = 4$. Substitute into the second equation to obtain

$$\begin{aligned} x_3 - 2s_2 + 4 &= 2 \\ x_3 &= 2s_2 - 2. \end{aligned}$$

   Now substitute into the first equation to obtain

$$\begin{aligned} x_1 - s_1 - 2(2s_2 - 2) + s_2 - 2(4) &= 1 \\ x_1 &= 5 + s_1 + 3s_2. \end{aligned}$$

3. (a) False, by Property (c) of triangular systems.

   (b) True. It will have 5 pivot variables, so it must have 3 free variables.

   (c) False. For example

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \\ x - y &= 1 \end{aligned}$$

   has exactly one solution.

   (d) False. In the system

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \end{aligned}$$

   there are no free variables or free parameters.

4. (a) There are 4 leading variables.

   (b) There are 5 free variables.

(c) There are 5 free parameters.

(d) There are infinitely many solutions.

5. Let x be the number of floor seats, and y the number of balcony seats. We have $x + y = 280$, because the theater capacity is 280. And we have $22x + 14y = 5320$, because the sales total \$5320. From the first equation, $y = 280 - x$. Substitute into the second equation to obtain

$$\begin{aligned} 22x + 14(280 - x) &= 5320 \\ 8x + 3920 &= 5320 \\ 8x &= 1400 \\ x &= 175 \end{aligned}$$

So there are 175 floor sears, and $y = 280 - 175 = 105$ balcony seats.

6. Let x be the number of nickels, y the number of dimes, and z the number of quarters. Because the quarters are worth \$2.75, we have $25z = 275$, so $z = 11$. The dimes and quarters are worth \$3.65, so we have

$$\begin{aligned} 10y + 25z &= 365 \\ 10y + 275 &= 365 \\ 10y &= 90 \\ y &= 9 \end{aligned}$$

There are 31 coins, so $x + y + z = 31$, and because $z = 11$ and $y = 9$, we have $x = 11$.

1.1 Lines and Linear Equations

1. $2(1) - 5(-2) = 12 \neq 9$, so $(1, -2)$ does not lie on the line $2x_1 - 5x_2 = 9$.
$2(-3) - 5(-3) = 9$, so $(-3, -3)$ lies on the line $2x_1 - 5x_2 = 9$.
$2(-2) - 5(-3) = 11 \neq 9$, so $(-2, -3)$ does not lie on the line $2x_1 - 5x_2 = 9$.

2. $(1) - 3(-2) + 4(0) = 7$, so $(1, -2, 0)$ lies on the plane $x_1 - 3x_2 + 4x_3 = 7$.
$(4) - 3(2) + 4(1) = 2 \neq 7$, so $(4, 2, 1)$ does not lie on the plane $x_1 - 3x_2 + 4x_3 = 7$.
$(2) - 3(-5) + 4(1) = 21 \neq 7$, so $(2, -5, 1)$ does not lie on the plane $x_1 - 3x_2 + 4x_3 = 7$.

3. $3(-1) + (2) = -1$ and $(-5)(-1) + 2(2) = 9 \neq 20$, so $(-1, 2)$ does not lie on both lines $3x_1 + x_2 = -1$ and $-5x_1 + 2x_2 = 20$.
$3(-2) + (5) = -1$ and $(-5)(-2) + 2(5) = 20 = 20$, so $(-2, 5)$ lies on both lines $3x_1 + x_2 = -1$ and $-5x_1 + 2x_2 = 20$.
$3(1) + (-5) = -2 \neq -1$ and $(-5)(1) + 2(-5) = -15 \neq 20$, so $(1, -5)$ does not lie on both lines $3x_1 + x_2 = -1$ and $-5x_1 + 2x_2 = 20$.

4. $2(3) - 5(1) = 1$ and $-4(3) + 10(1) = -2$, so $(3, 1)$ lies on both lines $2x_1 - 5x_2 = 1$ and $-4x_1 + 10x_2 = -2$.
$2(2) - 5(-4) = 24 \neq 1$ and $-4(2) + 10(-4) = -48 \neq -2$, so $(2, -4)$ does not lie on both lines $2x_1 - 5x_2 = 1$ and $-4x_1 + 10x_2 = -2$.
$2(-4) - 5(5) = -33 \neq 1$ and $-4(-4) + 10(5) = 66 \neq -2$, so $(-4, 5)$ does not lie on both lines $2x_1 - 5x_2 = 1$ and $-4x_1 + 10x_2 = -2$.

5. $-2(1) + 9(2) - (3) = 13 \neq -10$, so $(1, 2, 3)$ does not satisfy the first equation of the linear system.
$-2(1) + 9(-1) - (1) = -12 \neq -10$, so $(1, -1, 1)$ does not satisfy the first equation of the linear system.
$(-1) - 5(-2) + 2(-6) = -3 \neq 4$ so $(-1, -2, -6)$ does not satisfy the second equation of linear system.

6. $3(1) - (-2) + 2(-1) = 3 \neq 1$, so $(1, -2, -1, 3)$ does not satisfy the first equation of the linear system.
$3(-1) - (0) + 2(2) = 1$ and $2(-1) + 3(0) - (1) = -3$ so $(-1, 0, 2, 1)$ satisfies the linear system.
$3(-2) - (-1) + 2(4) = 3 \neq 1$, so $(-2, -1, 4, -3)$ does not satisfy the first equation of the linear system.

7. (a) Not a solution, since $-2(-3 + s_1 + s_2) + 3(s_1) + 2(s_2) = s_1 + 6 \neq 6$ for every s_1.

(b) A solution, since $-2(-3+3s_1+s_2)+3(2s_1)+2(s_2)=6$.

(c) A solution, since $-2(3s_1+s_2)+3(2s_1+2)+2(s_2)=6$.

(d) A solution, since $-2(s_1)+3(s_2)+2(3-3s_2/2+s_1)=6$.

8. (a) Not a solution, since $3(5-2s_1)+8(7+3s_1)-14(s_1)=4s_1+71\neq 6$ for every s_1.

(b) A solution, since $3(-5-5s_1)+8(s_1)-14(-(3+s_1)/2)=6$ and $(-5-5s_1)+3(s_1)-4(-(3+s_1)/2)=1$.

(c) A solution, since $3(10+10s_1)+8(-3-2s_1)-14(s_1)=6$ and $(10+10s_1)+3(-3-2s_1)-4(s_1)=1$.

(d) Not a solution, since $3((6-4s_1)/3)+8(s_1)-14(-(5-s_1)/4)=\frac{1}{2}s_1+\frac{47}{2}\neq 6$ for every s_1.

9. $3x_1+5x_2=4 \Rightarrow x_2=\frac{4}{5}-\frac{3}{5}x_1$. Substitute into the second equation to obtain $2x_1-7\left(\frac{4}{5}-\frac{3}{5}x_1\right)=13 \Rightarrow x_1=3$. Thus $x_2=(4-3(3))/5=-1$.

10. $-3x_1+2x_2=1 \Rightarrow x_2=\frac{3}{2}x_1+\frac{1}{2}$. Substitute into the second equation to obtain $5x_1+\left(\frac{3}{2}x_1+\frac{1}{2}\right)=-4 \Rightarrow x_1=-\frac{9}{13}$. Thus $x_2=\frac{3}{2}\left(-\frac{9}{13}\right)+\frac{1}{2}=-\frac{7}{13}$.

11. $-10x_1+4x_2=2 \Rightarrow x_2=\frac{5}{2}x_1+\frac{1}{2}$.Substitute into the second equation to obtain $15x_1-6\left(\frac{5}{2}x_1+\frac{1}{2}\right)=-3 \Rightarrow -3=-3$,which is true for all x_1. Hence we may set x_1 as a free variable, $x_1=s_1$ and then $x_2=\frac{5}{2}s_1+\frac{1}{2}$.

12. $-3x_1+4x_2=0 \Rightarrow x_2=\frac{3}{4}x_1$. Substitute into the second equation to obtain $9x_1-12\left(\frac{3}{4}x_1\right)=0\neq -2$. Thus no solution exists.

13. $7x_1-3x_2=-1 \Rightarrow x_2=\frac{7}{3}x_1+\frac{1}{3}$. Substitute into the second equation to obtain $-5x_1+8\left(\frac{7}{3}x_1+\frac{1}{3}\right)=0 \Rightarrow x_1=-\frac{8}{41}$. Thus $x_2=\frac{7}{3}\left(-\frac{8}{41}\right)+\frac{1}{3}=-\frac{5}{41}$.

14. $6x_1-3x_2=5 \Rightarrow x_2=2x_1-\frac{5}{3}$. Substitute into the second equation to obtain $-8x_1+4\left(2x_1-\frac{5}{3}\right)=-\frac{20}{3}\neq 1$. Thus no solution exists.

15. Echelon form. Leading variables: x_1 and x_2. No free variables.

16. Not in echelon form since x_1 is a leading variable in both equations.

17. Echelon form. Leading variables: x_1 and x_3. Free variable: x_2.

18. Not in echelon form, since the leading variable x_3 in equation 2 lies to the right of the leading variable x_2 in equation 3.

19. Not in echelon form since x_2 is a leading variable in both equations 2 and 3.

20. Echelon form. Leading variables: x_1, x_2, x_3, and x_4. No free variables.

21. Echelon form. Leading variables: x_1 and x_3. Free variables: x_2 and x_4.

22. Echelon form. Leading variables: x_1, x_3 and x_4. Free variables: x_2, x_5 and x_6.

23. Equation 2 $\Rightarrow x_2=5$. Substitute into equation 1, $-5x_1-3(5)=4 \Rightarrow x_1=-\frac{19}{5}$.

24. Equation 3 $\Rightarrow x_3=-3$. Substitute into equation 2, $-x_2+4(-3)=1 \Rightarrow x_2=-13$. Substitute into equation 1, $x_1+4(-13)-7(-3)=-3 \Rightarrow x_1=28$.

25. x_2 is a free variable, so let $x_2=s_1$. Substitute, $-3x_1+4s_1=2 \Rightarrow x_1=\frac{4}{3}s_1-\frac{2}{3}$.

26. x_2 is a free variable, so let $x_2=s_1$. Equation 2 $\Rightarrow x_3=2$. Substitute into equation 1, $3x_1-2(s_1)+2=4 \Rightarrow x_1=\frac{2}{3}s_1+\frac{2}{3}$.

27. x_3 is a free variable, so let $x_3=s_1$. Equation 3 $\Rightarrow x_4=5$. Substitute into equation 2, $-2x_2+s_1-5=-1 \Rightarrow x_2=\frac{1}{2}s_1-2$. Substitute into equation 1, $x_1+5\left(\frac{1}{2}s_1-2\right)-2s_1=0 \Rightarrow x_1=10-\frac{1}{2}s_1$.

28. x_2 and x_3 are free variables, so let $x_2 = s_1$ and $x_3 = s_2$. Substitute, $2x_1 - s_1 + 6s_2 = -3 \Rightarrow x_1 = \frac{1}{2}s_1 - 3s_2 - \frac{3}{2}$.

29. x_2 and x_4 are free variables, so let $x_2 = s_1$ and $x_4 = s_2$. Equation 2 $\Rightarrow -3x_3 + s_2 = -4 \Rightarrow x_3 = \frac{1}{3}s_2 + \frac{4}{3}$. Substitute into equation 1, $-2x_1 + s_1 + 2\left(\frac{1}{3}s_2 + \frac{4}{3}\right) = 1 \Rightarrow x_1 = \frac{1}{2}s_1 + \frac{1}{3}s_2 + \frac{5}{6}$.

30. x_2, x_5 and x_6 are free variables, so let $x_2 = s_1$, $x_5 = s_2$, and $x_6 = s_3$. Equation 3 $\Rightarrow 2x_4 + 5s_2 = 1 \Rightarrow x_4 = \frac{1}{2} - \frac{5}{2}s_2$. Substitute into equation 2, $-5x_3 - \left(\frac{1}{2} - \frac{5}{2}s_2\right) + 6s_2 + 3s_3 = 0 \Rightarrow x_3 = \frac{17}{10}s_2 + \frac{3}{5}s_3 - \frac{1}{10}$. Substitute into equation 1, $-7x_1 + 3s_1 + 8\left(\frac{1}{2} - \frac{5}{2}s_2\right) - 2s_2 + 13s_3 = -6 \Rightarrow x_1 = \frac{3}{7}s_1 - \frac{22}{7}s_2 + \frac{13}{7}s_3 + \frac{10}{7}$.

31. (a) Interchange equations 1 and 2, to obtain:

$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ -5x_2 &= 4 \end{aligned}$$

Equation 2 $\Rightarrow x_2 = -4/5$, and substituting into equation 1, $3x_1 + 2(-4/5) = 1 \Rightarrow x_1 = \frac{13}{15}$.

(b) Interchange equations 1 and 3 to obtain:

$$\begin{aligned} 3x_1 + 2x_2 + 7x_3 &= 0 \\ -x_2 - 4x_3 &= 13 \\ -3x_3 &= -3 \end{aligned}$$

Equation 3 $\Rightarrow x_3 = 1$. Substitute into equation 2, $-x_2 - 4(1) = 13 \Rightarrow x_2 = -17$. Substitute into equation 1, $3x_1 + 2(-17) + 7(1) = 0 \Rightarrow x_1 = 9$.

32. (a) Interchange equations 1 and 2 to obtain:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_4 &= -1 \\ 2x_2 + x_3 - 5x_4 &= 0 \end{aligned}$$

x_3 and x_4 are free variables, so let $x_3 = s_1$ and $x_4 = s_2$. Substitute into equation 2, $2x_2 + s_1 - 5s_2 = 0 \Rightarrow x_2 = \frac{5}{2}s_2 - \frac{1}{2}s_1$. Substitute into equation 1, $x_1 + 3\left(\frac{5}{2}s_2 - \frac{1}{2}s_1\right) - 2s_1 + 2s_2 = -1 \Rightarrow x_1 = \frac{7}{2}s_1 - \frac{19}{2}s_2 - 1$.

(b) Interchange equations 1 and 2, and also equations 3 and 4, to obtain:

$$\begin{aligned} x_1 - 5x_2 - 6x_3 + 3x_4 &= 3 \\ x_2 - 4x_3 + 3x_4 &= 2 \\ 5x_3 - 4x_4 &= 10 \\ -3x_4 &= 15 \end{aligned}$$

Equation 3 $\Rightarrow x_4 = -5$. Substitute into equation 2, $5x_3 - 4(-5) = 10 \Rightarrow x_3 = -2$. Substitute into equation 2, $x_2 - 4(-2) + 3(-5) = 2 \Rightarrow x_2 = 9$. Substitute into equation 1, $x_1 - 5(9) - 6(-2) + 3(-5) = 3 \Rightarrow x_1 = 51$.

33. x_3 is a free variable, so let $x_3 = s_1$. Equation 3 $\Rightarrow x_4 = 0$. Substitute into equation 2, $x_2 + 2s_1 - 2(0) = 2 \Rightarrow x_2 = 2 - 2s_1$. Substitute into equation 1, $x_1 + 2(2 - 2s_1) - s_1 + 0 = 1 \Rightarrow x_1 = 5s_1 - 3$.

34. Because the third equation, $0 = 1$, is not satisfied, there are no solutions.

35. x_3 is a free variable, so let $x_3 = s_1$. Substitute into equation 2, $x_2 + s_1 = 1 \Rightarrow x_2 = 1 - s_1$. Substitute into equation 1, $x_1 + (1 - s_1) - s_1 = 4 \Rightarrow x_1 = 2s_1 + 3$.

36. Because the third equation, $0 = -5$, is not satisfied, there are no solutions.

37. (a) From the first equation, $6x_1 - 5x_2 = 4$, we obtain $x_1 = \frac{5}{6}x_2 + \frac{2}{3}$. Substitute into equation 2, $9\left(\frac{5}{6}x_2 + \frac{2}{3}\right) + kx_2 = 1 \Rightarrow \left(\frac{15}{2} + k\right)x_2 = -5$. Hence the system is consistent provided $\frac{15}{2} + k \neq 0$, which means $k \neq -\frac{15}{2}$.

(b) From the second equation, $-9x_1 + 12x_2 = -1$, we obtain $x_1 = \frac{4}{3}x_2 + \frac{1}{9}$. Substitute into equation 1, $6\left(\frac{4}{3}x_2 + \frac{1}{9}\right) - 8x_2 = k \Rightarrow \frac{2}{3} = k$. The system is consistent provided $k = \frac{2}{3}$.

38. (a) Subtract the equations to obtain $(2-h)x_1 = -1-k$. If $h \neq 2$, the system will be consistent. If $h = 2$, then $0 = -1-k$, and the system has no solutions if $-1-k \neq 0$, *i.e.* $k \neq -1$. Hence, the system has no solutions if and only if $h = 2$ and $k \neq -1$.
Alternatively, there will be no solution if and only if the two lines are parallel and distinct. Thus we conclude that $h = 2$ and $k \neq -1$.

(b) If $h = 2$ and $k = 5$, then we have both $2x_1 + 5x_2 = -1$ and $2x_1 + 5x_2 = 3$. Because $-1 \neq 3$, there are no solutions.

39. There are 9 variables, as every variable is either a leading variable or free variable.

40. There are 3 free variables. Since there are 5 equations and the system is in echelon form, there are 5 leading variables. The number of free variables plus the number of leading variables must equal the total number of variables, 8.

41. There are 7 leading variables, since the number of leading variables of a system in echelon form is equal to the number of equations.

42. There are 5 equations. Since there are 4 free variables, there must be 5 leading variables, as there are 9 variables altogether. Since the number of leading variables of a system in echelon form is equal to the number of equations, we must have 5 equations.

43. For example,

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

44. For example,

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \\ x_1 & - & x_2 & & & = & 0 \end{array}$$

45. For example,

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & = & 0 \\ x_1 & + & x_2 & - & x_3 & = & 0 \\ & & & & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$$

46. For example

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & + & x_2 & & & & & = & 1 \\ & & & & x_3 & + & x_4 & = & 1 \end{array}$$

47. On Monday, I bought 3 apples and 4 oranges and spent \$0.55. On Tuesday I bought 6 oranges and spent \$0.60. How much does each apple and orange cost?
Solution: let x_1 be the price of an apple, and x_2 the price of an orange, then we have the following system in echelon form:

$$\begin{aligned} 3x_1 + 4x_2 &= 0.55 \\ 6x_2 &= 0.60 \end{aligned}$$

From equation 2, $x_2 = 0.10$. Substitute into equation 1, $3x_1 + 4(0.10) = 0.55, \Rightarrow x_1 = 0.05$. Hence apples cost 5 cents each and oranges cost 10 cents each.

48. The simplest such example,

$$\begin{array}{rcrcl} x_1 & & & = & -1 \\ & & x_2 & = & 3 \end{array}$$

49. For example,

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & -3 \\ 3x_1 & & & - & x_3 & = & 4 \end{array}$$

50. For example,

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 0 \\ 2x_1 & - & 4x_2 & = & 0 \end{array}$$

51. (a) False. Example:

$$\begin{array}{rcrcr} x_1 & & & = & 0 \\ & & x_2 & = & 0 \\ x_1 & + & x_2 & = & 0 \end{array}$$

(b) False. Example:

$$\begin{array}{rcrcrcrcrcr} x_1 & & & & & & & & & = & 0 \\ & & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 1 \end{array}$$

52. (a) False. Consider the equation $x_1 + x_2 = 1$. One can set $x_1 = s_1$ and then $x_2 = 1 - s_1$. Or one can set $x_2 = s_1$ and then $x_1 = 1 - s_1$.

(b) True. The last equation will uniquely determine the last variable. Substitution into the next to last equation will determine the next to last variable uniquely. And one can continue to determine uniquely all variables.

53. (a) True. The leading variable moves one column to the right each time you descend one row.

(b) False. Example:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

54. (a) False. Each equation in an echelon system has a unique leading variable, so back substitution is always possible. Hence a solution always exists.

(b) False. Each equation in an echelon system must have a unique leading variable, so the number of equations cannot exceed the number of variables.

55. (a) True. The last equation would be $c_n x_n = b_n$, so $x_n = \frac{b_n}{c_n}$ is rational. And then using back substitution, each proceeding variable would be rational, as it is determined from a sum of rational numbers, divided by an integer. In this manner, we see that each variable, x_i in the solution is a rational number.

(b) True. Example:

$$x_1 + x_2 = 0$$

56. (a) True, because the free parameter can be assigned any real value, so there will be correspondingly infinitely many solutions.

(b) False. Suppose one of the equations is $ax + by = c$, with either $a \neq 0$ or $b \neq 0$. Because $(1, 2)$ is a solution, $a+2b = c$. Also, $(4, 8)$ is a solution so $4a+8b = c$. We obtain $a+2b = 4a+8b = 4\,(a + 2b)$, so $a + 2b = 0$. Thus $a = -2b$ and $c = 0$, and the equation becomes $-2bx + by = 0$. Because $(-1, 5)$ is also a solution, $-2b\,(-1) + b\,(5) = 0$. This implies $b = 0$, so also $a = 0$. But this contradicts either $a \neq 0$ or $b \neq 0$, and we conclude that these three points cannot all be solutions.

57. Let x be the number of adults who attended, and y the number of children who attended. Since the total number of people who attended is 385, we have $x + y = 385$. The total revenue from the sale of tickets will be the revenue due to the adult tickets purchased and the children's tickets purchased. We obtain a second equation, $11x + 8y = 3974$. Solving the first equation for x, we have $x = 385 - y$. Substitute into the second equation, we have $11(385 - y) + 8y = 3974 \Rightarrow 4235 - 3y = 3974, \Rightarrow y = 87$. Solving now for x, we determine $x = 385 - 87 = 298$. So 298 adults and 87 children attended.

58. Let x be the number of coach tickets sold, and y the number of business class tickets sold. Because 150 people were sold, we have $x + y = 150$. Because the total revenue was $\$24,960$, we have $160x + 220y = 24,960$. We solve the system

$$\begin{aligned} x + y &= 150 \\ 160x + 220y &= 2,4960 \end{aligned}$$

and obtain $x = 134$ and $y = 16$. So 134 coach tickets were sold, and 16 business class tickets sold.

59. Using $f(0) = 5$, we have $5 = a_1e^{2(0)} + a_2e^{-3(0)} = a_1 + a_2$. Using $f'(0) = -1$,we have $-1 = 2a_1e^{2(0)} - 3a_2e^{-3(0)} = 2a_1 - 3a_2$. Solving the first equation for a_1, we have $a_1 = 5 - a_2$. Substitute into the second equation, $-1 = 2(5 - a_2) - 3a_2 \Rightarrow a_2 = \frac{11}{5}$. Therefore $a_1 = 5 - \frac{11}{5} = \frac{14}{5}$.

60. From $f(0) = 3$ we obtain $3 = a_1 + a_2$. And from $f'(0) = -1$ we obtain $-1 = -5a_1 + 2a_2$. Solve the first equation for a_1 to get $a_1 = 3 - a_2$. Substitute into the second equation, and get $-1 = -5(3 - a_2) + 2a_2 = 7a_2 - 15$. Thus $7a_2 = 14$, so $a_2 = 2$. Thus $a_1 = 3 - 2 = 1$.

61. The total amount of glycol needed is now $0.29(300) = 87.0$ liters. Thus the system of equations becomes

$$\begin{aligned} x + y &= 300 \\ 0.18x + 0.50y &= 87 \end{aligned}$$

Solving the first equation for x, we obtain $x = 300 - y$. Substitute into the second equation to get $0.18(300 - y) + 0.50y = 87, \Rightarrow y = 103.125$ liters. Hence $x = 300 - 103.125 = 196.875$ liters.

62. The total amount of glycol needed is now $0.46(300) = 138.0$ liters. Thus the system of equations becomes

$$\begin{aligned} x + y &= 300 \\ 0.18x + 0.50y &= 138.0 \end{aligned}$$

Solving the first equation for x, we obtain $x = 300 - y$. Substitute into the second equation to get $0.18(300 - y) + 0.50y = 138.0, \Rightarrow y = 262.5$ liters. Hence $x = 300 - 262.5 = 37.5$ liters.

63. Let x be the amount invested in the safe bond, and y the amount invested in the risky bond. Then $x + y = 100000$. The annual return on her investment is $1.03x + 1.09y$. We desire to have this be a 7% annual return, so $1.03x + 1.09y = 1.07(100000) = 107000$. From our first equation, $x = 100000 - y$. Substitute into the second equation, $1.03(100000 - y) + 1.09y = 107000, \Rightarrow y = 66667$. Thus, $x = 100000 - 66667 = 33333$.

64. Let x be the amount invested in the safe bond, and y the amount invested in the risky bond. Then $x + y = 200,000$. The annual return on the investment is $1.04x + 1.11y$. We desire to have this be a 8% annual return, so $1.04x + 1.11y = 1.08(200,000) = 216,000$. From our first equation, $x = 200,000 - y$. Substitute into the second equation, $1.04(200,000 - y) + 1.11y = 216,000, \Rightarrow y = 114,285.71$. Thus, $x = 200,000 - 114,285.71 = 85,714.29$.

65. Let x be the amount of hot water, and y the amount of cold water to be mixed. Then $x + y = 60$, since the 60-gallon bathtub is to be filled. The proportion of the water that is hot is $x/60$, and the proportion of water that is cold is $y/60$, and the final temperature of the water is determined by these proportions and the temperatures of the hot and cold water. Hence, $100 = (x/60)(125) + (y/60)(60)$. Solving the first equation, we have $x = 60 - y$. Substitute into the second equation, $100 = ((60 - y)/60)(125) + (y/60)(60), \Rightarrow y = \frac{300}{13} = 23.077$ gallons. Thus $x = 60 - \frac{300}{13} = \frac{480}{13} = 36.923$ gallons.

66. Let x be the amount of hot water, and y the amount of cold water to be mixed. Then $x+y=50$, because the 50-gallon bathtub is to be filled. The proportion of the water that is hot is $x/50$, and the proportion of water that is cold is $y/50$, and the final temperature of the water is determined by these proportions and the temperatures of the hot and cold water. Hence, $105=(x/50)(115)+(y/50)(70)$. Solving the first equation, we have $x=50-y$. Substitute into the second equation, $105=((50-y)/50)(115)+(y/50)(70)$, $\Rightarrow$ $y=\frac{100}{9}=11.11$ gallons. Thus $x=50-\frac{100}{9}=\frac{350}{9}=\frac{350}{9}=38.89$ gallons.

67. Using the freezing point of water, we have $0=a(32)+b$. From the boiling point of water, $100=a(212)+b$. From the first equation, $b=-32a$. Substitute into the second equation, $100=a(212)+(-32a)=180a \Rightarrow a=\frac{5}{9}$. Hence $b=-32\left(\frac{5}{9}\right)=-\frac{160}{9}$.

68. Let V be the value of the machine (in thousands of dollars), and t be the time in years since purchased. We are assuming that the relationship is linear, so $V=at+b$ for some values a and b . We are given that $V=800$ when $t=2$, so our first equation is $2a+b=800$. Using $V=440$ when $t=5$ we have a second equation $5a+b=440$. Subtracting equations, we get $-3a=360$, and so $a=-120$. Thus $b=800-2(-120)=1040$. So a formula for the value of the machine is $V=-120t+1040$.

69. After experimenting a bit, we get that 4 nickels and 8 quarters just about cover the long side. The short side is covered by either 9 nickels and 1 quarter, or 1 nickel and 8 quarters. Let n be the diameter of a nickel, and q the diameter of a quarter. The first equation becomes $4n+8q=11$, so $n=\frac{11}{4}-2q$. With the choice of 9 nickels and 1 quarter, the second equation is $9n+q=8.5$. Substituting for n, $9\left(\frac{11}{4}-2q\right)+q=8.5$, and hence $q=0.95588$ in. And thus $n=\frac{11}{4}-2\,(0.95588)=0.83824$ in.
Using instead 1 nickel and 8 quarters for the second equation, we have $n+8q=8.5$. Substituting, $\left(\frac{11}{4}-2q\right)+8q=8.5$, and we get $q=0.95833$ in. Thus $n=\frac{11}{4}-2\,(0.95833)=0.83334$ in. The published values from the United States Mint are $q=0.955$ in and $n=0.835$ in.

70. From Example 5, $A(t)=2a$, and since the acceleration is given as -9.8 m/s^2, we have $-9.8=2a \Rightarrow a=-4.9$. The velocity is given by $V(t)=2at+b$, and using that velocity is -34.4 m/s when $t=3$, we have $-34.4=2(-4.9)(3)+b$, $\Rightarrow$ $b=-5.0$. The height is $H(t)=at^2+bt+c$, and we know that the height is 25.9 meters when $t=3$. Thus, $25.9=(-4.9)(3^2)+(-5.0)(3)+c$, $\Rightarrow$ $c=85.0$ .Thus $H(t)=-4.9t^2-5.0t+85.0$. (*Note that we didn't use the initial velocity. Our result agrees with* $V(0)=-5.0$. *We could have used the initial velocity instead to determine* $b=-5.0$, *and then check that* $V(3)=-34.4$.)

71. $x_1=12$, $x_2=5$.

72. $x_1=\frac{10}{41}$, $x_2=\frac{6}{41}$. (*Solving numerically,* $x_1=0.24390244$, $x_2=0.14634146$.)

73. $x_1=0.625+41.25s_1$, $x_2=-1.75+2.25s_1$, $x_3=s_1$

74. $x_1=7s_1$, $x_2=-\frac{1}{3}$, $x_3=s_1+\frac{22}{21}$

75. $x_1=-8.417-4.125s_1$, $x_2=4.333-2s_1$, $x_3=1.5+1.75s_1$, $x_4=s_1$

76. $x_1=-\frac{139}{6}s_1$, $x_2=14s_1+\frac{668}{139}$, $x_3=-\frac{1}{2}s_1+\frac{552}{139}$, $x_4=s_1+\frac{425}{139}$

77. $x_1=\frac{2727}{88}+\frac{117}{8}s$; $x_2=\frac{173}{88}-\frac{17}{8}s$; $x_3=\frac{181}{22}-\frac{1}{2}s$; $x_4=s$; $x_5=\frac{2}{11}$

78. $x_1=-\frac{2}{3}s_1-\frac{119}{108}$; $x_2=-\frac{143}{36}$; $x_3=s_1$; $x_4=\frac{29}{9}$; $x_5=\frac{4}{3}$; $x_6=-\frac{5}{3}$

1.2 Practice Problems

1. (a)
$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 2 & -3 & 5 & 5 \\ -1 & 3 & 2 & -5 \end{bmatrix} \underset{\sim}{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ -1 & 3 & 2 & -5 \end{bmatrix}$$
$$\underset{\sim}{R_1+R_3\to R_3} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & -3 \end{bmatrix}$$
$$\underset{\sim}{-R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 3 & -1 & 1 & 2 & 1 \\ 2 & 6 & -1 & 5 & 3 & 6 \\ -1 & -3 & 3 & 6 & -4 & 4 \\ 1 & 3 & -2 & -1 & 4 & 1 \end{bmatrix} \underset{\sim}{\substack{-2R_1+R_2\to R_2 \\ R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}} \begin{bmatrix} 1 & 3 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 & 4 \\ 0 & 0 & 2 & 7 & -2 & 5 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{bmatrix}$$
$$\underset{\sim}{\substack{-2R_2+R_3\to R_3 \\ R_2+R_4\to R_4}} \begin{bmatrix} 1 & 3 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$
$$\underset{\sim}{-R_3+R_4\to R_4} \begin{bmatrix} 1 & 3 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

2. (a)
$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ -1 & 4 & -2 \end{bmatrix} \underset{\sim}{\substack{-2R_1+R_2\to R_2 \\ R_1+R_3\to R_3}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$
$$\underset{\sim}{\tfrac{1}{3}R_3\to R_3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\underset{\sim}{\substack{R_3+R_2\to R_2 \\ -3R_3+R_1\to R_1}} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\underset{\sim}{2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ -1 & -4 & -1 & 0 \\ 2 & 8 & 6 & 4 \end{bmatrix} \underset{\sim}{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}} \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. (a) $\begin{bmatrix} -1 & 2 & -3 & -1 \\ -1 & 3 & -1 & -3 \\ 2 & -2 & 10 & -2 \end{bmatrix} \underset{2R_1+R_3\to R_3}{\overset{-R_1+R_2\to R_2}{\sim}} \begin{bmatrix} -1 & 2 & -3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 2 & 4 & -4 \end{bmatrix}$

$\overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 2 & -3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Free variable, $x_3 = s$. Row 2 $\Rightarrow$ $x_2 + 2s = -2$ $\Rightarrow x_2 = -2 - 2s$. Row 1 $\Rightarrow$ $-x_1 + 2(-2-2s) - 3s = -1$ $\Rightarrow$ $x_1 = -3 - 7s$.

(b) $\begin{bmatrix} 1 & -1 & -2 & 1 & -2 & 0 \\ 1 & -1 & -1 & -1 & -1 & 0 \\ -1 & 1 & 1 & -3 & 6 & 0 \end{bmatrix} \underset{R_1+R_3\to R_3}{\overset{-R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -1 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 4 & 0 \end{bmatrix}$

$\overset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 5 & 0 \end{bmatrix}$

Free variables, $x_5 = s_1$ and $x_2 = s_2$. Row 3 $\Rightarrow$ $-4x_4 + 5s_1 = 0$ $\Rightarrow x_4 = \frac{5}{4}s_1$. Row 2 $\Rightarrow$ $x_3 - 2\left(\frac{5}{4}s_1\right) + s_1 = 0$ $\Rightarrow$ $x_3 = \frac{3}{2}s_1$. Row 1 $\Rightarrow$ $x_1 - (s_2) - 2\left(\frac{3}{2}s_1\right) + \left(\frac{5}{4}s_1\right) - 2(s_1) = 0$ $\Rightarrow$ $x_1 = \frac{15}{4}s_1 + s_2$.

4. (a) $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 1 \end{bmatrix} \overset{-R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & -3 \end{bmatrix}$

$\overset{2R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & -1 & 1 & -3 \end{bmatrix}$

$\overset{-R_2\to R_2}{\sim} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$

Free variable $x_3 = s$. Row 2 $\Rightarrow$ $x_2 - (s) = 3$ $\Rightarrow$ $x_2 = 3 + s$. Row 1 $\Rightarrow$ $x_1 + 3(s) = -2$ $\Rightarrow$ $x_1 = -2 - 3s$.

(b) $\begin{bmatrix} 1 & -1 & 3 & -1 \\ 2 & -1 & 4 & -1 \\ -1 & 3 & -6 & 4 \end{bmatrix} \underset{R_1+R_3\to R_3}{\overset{-2R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -3 & 3 \end{bmatrix}$

$\overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\underset{-3R_3+R_1\to R_1}{\overset{2R_3+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\overset{R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

From Row 1, $x_1 = -1$, from Row 2, $x_2 = 3$, and from Row 3, $x_3 = 1$.

5. (a) False. Every matrix can be transformed to reduced row echelon form.
   (b) True. Suppose the matrix is $m \times n$, with $m > n$. No column can have more than one pivot if the matrix is in echelon form, so the number of rows with a pivot is, at most, n. Since there are m rows, and $m > n$, the matrix must have a zero row.
   (c) True. The reverse operation of $R_i \leftrightarrow R_j$ is $R_j \leftrightarrow R_i$. The reverse operation of $cR_i \to R_i$ is $\frac{1}{c}R_i \to R_i$. And the reverse operation of $cR_i + R_j \to R_j$ is $-cR_i + R_j \to R_j$.
   (d) True. If there exists a solution, then there will be infinitely many solutions because any free variable can take infinitely many values in $\mathbb{R}$.

6. (a) 4
   (b) 3

(c) 6

(d) 4

1.2 Linear Systems and Matrices

1. $$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & - & x_3 & = & 2 \\ -x_1 & & & + & 5x_3 & = & 7 \end{array}$$

2. $$\begin{array}{rcrcr} -2x_1 & + & x_2 & = & 0 \\ 13x_1 & - & 3x_2 & = & 6 \\ -11x_1 & + & 7x_2 & = & -5 \end{array}$$

3. $$\begin{array}{rcrcrcrcr} & & 12x_2 & - & 3x_3 & - & 9x_4 & = & 17 \\ -12x_1 & + & 5x_2 & - & 3x_3 & + & 11x_4 & = & 0 \\ 6x_1 & + & 8x_2 & + & 2x_3 & + & 10x_4 & = & -8 \\ 17x_1 & & & & & + & 13x_4 & = & -1 \end{array}$$

4. $$\begin{array}{rcr} -x_1 & = & 2 \\ 5x_1 & = & -7 \\ 3x_1 & = & 0 \end{array}$$

5. Echelon form.

6. Reduced row echelon form.

7. Not echelon form.

8. Echelon form.

9. Echelon form.

10. Reduced row echelon form.

11. $-2R_1 \to R_1$

12. $3R_2 + R_1 \to R_1$

13. $-2R_2 + R_3 \to R_3$

14. $R_1 \leftrightarrow R_3$

15. $R_1 \leftrightarrow R_2$, $\begin{bmatrix} -1 & 4 & \mathbf{3} \\ 3 & 7 & -2 \\ 5 & 0 & -3 \end{bmatrix}$

16. $2R_1 + R_2$, $\begin{bmatrix} -2 & -2 & 1 & 6 \\ 0 & \mathbf{-5} & 2 & 7 \end{bmatrix}$

17. $2R_1 \to R_1$, $\begin{bmatrix} \mathbf{0} & 6 & -2 & 4 \\ -1 & -9 & 4 & 1 \\ 5 & 0 & 7 & 2 \end{bmatrix}$

18. $-2R_1 + R_3 \Rightarrow R_3$, $\begin{bmatrix} 1 & 7 & 2 & 0 \\ 0 & 4 & -8 & -3 \\ 1 & -14 & \mathbf{-4} & 1 \end{bmatrix}$.

19. $\begin{bmatrix} 2 & 1 & 1 \\ -4 & -1 & 3 \end{bmatrix} \overset{2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

    Row 2 $\Rightarrow$ $x_2 = 5$. Row 1 $\Rightarrow$ $2x_1 + (5) = 1$ $\Rightarrow$ $x_1 = -2$.

20. $\begin{bmatrix} 3 & -7 & 0 \\ 1 & 4 & 0 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 0 \\ 3 & -7 & 0 \end{bmatrix}$

$\underset{\sim}{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -19 & 0 \end{bmatrix}$

Row 2 $\Rightarrow$ $x_2 = 0$. Row 1 $\Rightarrow$ $x_1 + 4(0) = 0 \Rightarrow x_1 = 0$.

21. $\begin{bmatrix} -2 & 5 & -10 & 4 \\ 1 & -2 & 3 & -1 \\ 7 & -17 & 34 & -16 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 3 & -1 \\ -2 & 5 & -10 & 4 \\ 7 & -17 & 34 & -16 \end{bmatrix}$

$\underset{\sim}{\substack{2R_1+R_2 \to R_2 \\ -7R_1+R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & -3 & 13 & -9 \end{bmatrix}$

$\underset{\sim}{3R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

Row 3 $\Rightarrow$ $x_3 = -3$. Row 2 $\Rightarrow$ $x_2 - 4(-3) = 2 \Rightarrow x_2 = -10$. Row 1 $\Rightarrow$ $x_1 - 2(-10) + 3(-3) = -1 \Rightarrow x_1 = -12$.

22. $\begin{bmatrix} 2 & 8 & -4 & -10 \\ -1 & -3 & 5 & 4 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -3 & 5 & 4 \\ 2 & 8 & -4 & -10 \end{bmatrix}$

$\underset{\sim}{2R_1+R_2 \to R_2} \begin{bmatrix} -1 & -3 & 5 & 4 \\ 0 & 2 & 2 & -2 \end{bmatrix}$

Free variable, $x_3 = s_1$. Row 2 $\Rightarrow$ $2x_2 + 2s_1 = -2 \Rightarrow x_2 = -s_1 - 1$. Row 1 $\Rightarrow$ $-x_1 - 3(-s_1 - 1) + 5(s_1) = 4 \Rightarrow x_1 = 8s_1 - 1$.

23. $\begin{bmatrix} 2 & 2 & -1 & 8 \\ -1 & -1 & 0 & -3 \\ 3 & 3 & 1 & 7 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -1 & 0 & -3 \\ 2 & 2 & -1 & 8 \\ 3 & 3 & 1 & 7 \end{bmatrix}$

$\underset{\sim}{\substack{2R_1+R_2 \to R_2 \\ 3R_1+R_3 \Rightarrow R_3}} \begin{bmatrix} -1 & -1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$\underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} -1 & -1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Free variable, $x_2 = s_1$. Row 2 $\Rightarrow$ $x_3 = -2$. Row 1 $\Rightarrow$ $-x_1 - (s_1) = -3 \Rightarrow x_1 = 3 - s_1$.

24. $\begin{bmatrix} -5 & 9 & 13 \\ 3 & -5 & -9 \\ 1 & -2 & -2 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & -2 \\ 3 & -5 & -9 \\ -5 & 9 & 13 \end{bmatrix}$

$\underset{\sim}{\substack{-3R_1+R_2 \to R_2 \\ 5R_1+R_3 \Rightarrow R_3}} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix}$

$\underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

Row 2 $\Rightarrow$ $x_2 = -3$. Row 1 $\Rightarrow$ $x_1 - 2(-3) = -2 \Rightarrow x_1 = -8$.

25. $$\begin{bmatrix} 2 & 6 & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 2 & 6 & -9 & -4 & 0 \end{bmatrix}$$
$$\overset{\substack{3R_1+R_2\to R_2 \\ -2R_1+R_3\Rightarrow R_3}}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -2 & -5 & -6 & 0 \end{bmatrix}$$
$$\overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

Free variable, $x_4 = s_1$. Row 3 $\Rightarrow$ $x_3 - 2s_1 = 0 \Rightarrow x_3 = 2s_1$. Row 2 $\Rightarrow$ $x_2 + 3(2s_1) + 2s_1 = 0 \Rightarrow x_2 = -8s_1$. Row 1 $\Rightarrow$ $x_1 + 4(-8s_1) - 2(2s_1) + s_1 = 0 \Rightarrow x_1 = 35s_1$.

26. $$\begin{bmatrix} 1 & -1 & -3 & -1 & -1 \\ -2 & 2 & 6 & 2 & -1 \\ -3 & -3 & 10 & 0 & 5 \end{bmatrix} \overset{\substack{2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -1 & -3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & -6 & 1 & -3 & 2 \end{bmatrix}$$
$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & -1 & -3 & -1 & -1 \\ 0 & -6 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

The third row corresponds to the equation $0 = -3$, hence the system is inconsistent.

27. $$\begin{bmatrix} -2 & -5 & 0 \\ 1 & 3 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 3 & 1 \\ -2 & -5 & 0 \end{bmatrix}$$
$$\overset{2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$\overset{-3R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

Thus $x_1 = -5$ and $x_2 = 2$.

28. $$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 6 \\ -1 & 1 & 5 \end{bmatrix} \overset{\substack{-3R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$$
$$\overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\overset{-R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -2$, and $x_2 = 3$.

29. $$\begin{bmatrix} 2 & 1 & 0 & 2 \\ -1 & -1 & -1 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix}$$
$$\overset{2R_1+R_2\to R_2}{\sim} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 0 & -1 & -2 & 4 \end{bmatrix}$$
$$\overset{-R_2+R_1\to R_1}{\sim} \begin{bmatrix} -1 & 0 & 1 & -3 \\ 0 & -1 & -2 & 4 \end{bmatrix}$$
$$\overset{\substack{-R_1\to R_1 \\ -R_2\to R_2}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Free variable, $x_3 = s_1$. Row 1 $\Rightarrow$ $x_1 = 3 + s_1$. Row 2 $\Rightarrow$ $x_2 = -4 - 2s_1$.

30. $$\begin{bmatrix} -4 & 2 & -2 & 10 \\ 1 & 0 & 1 & -3 \\ 3 & -1 & 1 & -8 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & -3 \\ -4 & 2 & -2 & 10 \\ 3 & -1 & 1 & -8 \end{bmatrix}$$

$$\underset{\sim}{\substack{4R_1+R_2\to R_2 \\ -3R_1+R_3\Rightarrow R_3}} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$$\underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & -1 & -2 & 1 \\ 0 & 2 & 2 & -2 \end{bmatrix}$$

$$\underset{\sim}{2R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{\sim}{\substack{-R_3+R_2\to R_2 \\ (1/2)R_3+R_1\Rightarrow R_1}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{\sim}{\substack{-R_2\to R_2 \\ -(1/2)R_3\to R_3}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $x_1 = -3$, $x_2 = -1$, and $x_3 = 0$.

31. $$\begin{bmatrix} -3 & 2 & -1 & 6 & -7 \\ 7 & -3 & 2 & -11 & 14 \\ 1 & 0 & 0 & -1 & 1 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 7 & -3 & 2 & -11 & 14 \\ -3 & 2 & -1 & 6 & -7 \end{bmatrix}$$

$$\underset{\sim}{\substack{-7R_1+R_2\to R_2 \\ 3R_1+R_3\Rightarrow R_3}} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & -3 & 2 & -4 & 7 \\ 0 & 2 & -1 & 3 & -4 \end{bmatrix}$$

$$\underset{\sim}{\substack{2R_2\to R_2 \\ 3R_3\to R_3}} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & -6 & 4 & -8 & 14 \\ 0 & 6 & -3 & 9 & -12 \end{bmatrix}$$

$$\underset{\sim}{R_1+R_3\to R_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & -6 & 4 & -8 & 14 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\underset{\sim}{-4R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & -6 & 0 & -12 & 6 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\underset{\sim}{-(1/6)R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Free variable, $x_4 = s_1$. Thus, $x_1 = 1 + s_1$, $x_2 = -1 - 2s_1$, and $x_3 = 2 - s_1$.

32. $$\begin{bmatrix} 1 & 1 & 1 & -2 & 4 & -5 \\ -1 & 0 & -3 & 4 & -5 & 5 \\ 2 & 4 & -2 & 1 & 5 & -9 \end{bmatrix} \underset{\sim}{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}} \begin{bmatrix} 1 & 1 & 1 & -2 & 4 & -5 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & -4 & 5 & -3 & 1 \end{bmatrix}$$

$$\underset{\sim}{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & -2 & 4 & -5 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{\substack{-2R_3+R_2\to R_2 \\ 2R_3+R_1\to R_1}} \begin{bmatrix} 1 & 1 & 1 & 0 & 2 & -3 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{-R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & -1 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

Free variables, $x_3 = s_1$ and $x_5 = s_2$. Thus, $x_1 = -1 - 3s_1 - s_2$, $x_2 = -2 + 2s_1 - s_2$, and $x_4 = 1 + s_2$.

33. (a) $(1/5)R_1 \to R_1$
    (b) $(-1/2)R_3 \to R_3$

34. (a) $R_1 \leftrightarrow R_3$
    (b) $R_1 \leftrightarrow R_4$

35. (a) $5R_2 + R_6 \to R_6$
    (b) $3R_1 + R_3 \to R_3$

36. (a) $-4R_5 + R_1 \to R_1$
    (b) $4R_4 + R_2 \to R_2$

37. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

38. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$

41. $\begin{array}{ccccccc} x_1 & & & & & = & 0 \\ & & x_2 & & & = & 0 \\ & & & x_3 & + \; x_4 & = & 0 \end{array}$

42. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

43. (a) True, by definition of equivalent matrices.
    (b) True. For example, $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \overset{-(1/2)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}$.
    And $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \overset{-2R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

44. (a) False, by Theorem 1.6.
    (b) False, it could be inconsistent, and therefore have no solutions, as with the system

$$\begin{array}{ccccccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & = & 0 \\ x_1 & + & x_2 & + & x_3 & & & & & & & & & = & 1 \\ & & & & & & x_4 & + & x_5 & & & & & = & 1 \\ & & & & & & & & & & x_6 & + & x_7 & = & 1 \end{array}$$

45. (a) False. For example, all seven equations in the system could be $x_1 + x_2 + x_3 + x_4 = 0$, which is consistent, making the system consistent.

(b) False, a system can have free variables but also be inconsistent.

46. (a) True. If it is consistent, there will be at least one free variable, and hence infinitely many solutions.
(b) False. For example, the system

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 0 \\ x_1 & + & x_2 & = & 1 \end{array}$$

has no solutions. And the system

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ 2x_1 & + & 2x_2 & = & 2 \end{array}$$

has infinitely many solutions.

47. (a) $-R_2 \to R_2$, then $R_3 + R_2 \to R_2$
(b) $3R_1 \to R_1$, then $R_4 + R_1 \to R_1$

48. (a) $2R_4 \to R_4$, then $3R_2 + R_4 \to R_4$
(b) $-R_5 \to R_5$, then $R_4 + R_5 \to R_5$

49. (a) $-4R_6 \to R_6$, then $R_3 + R_6 \to R_6$
(b) Not a combination of elementary row operations. For example, the system

$$\begin{array}{rcr} 5x_1 & = & 5 \\ x_1 & = & 2 \\ -2x_1 & = & -2 \end{array}$$

has no solution, but if one applies the row operation

$$\begin{bmatrix} 5 & 5 \\ 1 & 2 \\ -2 & -2 \end{bmatrix} \underset{\sim}{\scriptstyle 2R_1+5R_3\to R_2} \begin{bmatrix} 5 & 5 \\ 0 & 0 \\ -2 & -2 \end{bmatrix}$$

then the system has a solution, $x_1 = 1$, which is a contradiction.

50. (a) Not a combination of elementary row operations. For example, the system

$$\begin{array}{rcr} x_1 & = & 1 \\ -5x_1 & = & -5 \\ x_1 & = & 2 \end{array}$$

has no solution, but if one applies the row operation

$$\begin{bmatrix} 1 & 1 \\ -5 & -5 \\ 1 & 2 \end{bmatrix} \underset{\sim}{\scriptstyle 5R_1+R_2\to R_3} \begin{bmatrix} 1 & 1 \\ -5 & -5 \\ 0 & 0 \end{bmatrix}$$

then the system has a solution, $x_1 = 1$, which is a contradiction.
(b) $-R_5 \to R_5$, then $2R_4 + R_5 \to R_5$

51. Exactly one solution. The last row produces a unique value for the last variable, and then back substitution produces a unique value for each preceding variable.

52. No solutions. The assumption implies that the last row consists of all zeros followed by a non-zero value. This corresponds to an inconsistent system.

53. Either the system has free variables or not. If there are no free variables and the system is consistent, then every variable is a leading variable, and there will be exactly one solution. If there exists a free variable, then there will be infinitely many solutions. Thus, if there are two distinct solutions then it follows that there must be infinitely many solutions.

54. Suppose there are no zero rows. The every row has a distinct leading term. Since there are more rows than columns, we have more leading terms than columns, which is a contradiction. Hence there must be a least one zero row.

55. Every homogeneous system is consistent. There will be free variables, as the number of leading variables is no greater than the number of equations, and there are more variables than equations. Since there are free variables, there must be infinitely many solutions.

56. (a) Clearly the set of solutions is not changed by simply writing the equations in a different order.

(b) Suppose that $(s_1, \ldots, s_k)$ is a solution to the linear equation

$$a_{j1}x_1 + \cdots + a_{jk}x_k = b_j$$

Then $a_{j1}s_1 + \cdots + a_{jk}s_k = b_j$, and if $c \neq 0$, then we also have $ca_{j1}s_1 + \cdots + ca_{jk}s_k = cb_j$, so that $(s_1, \ldots, s_k)$ satisfies

$$ca_{j1}x_1 + \cdots + ca_{jk}x_k = cb_j$$

Similarly, if $(t_1, \ldots, t_k)$ is a solution to $ca_{j1}x_1+\cdots+ca_{jk}x_k = cb_j$, then $ca_{j1}t_1+\cdots+ca_{jk}t_k = cb_j$. Dividing on both sides by c, we have $a_{j1}t_1 + \cdots + a_{jk}t_k = b_j$. Therefore it follows that the set of solutions is not changed by multiplying an equation by a nonzero constant.

(c) Suppose that $(s_1, \ldots, s_k)$ is a solution to the linear equations

$$\begin{aligned} a_{j1}x_1 + \cdots + a_{jk}x_k &= b_j \\ a_{i1}x_1 + \cdots + a_{ik}x_k &= b_i \end{aligned}$$

Then for $c \neq 0$, we have

$$c\big(a_{j1}s_1 + \cdots + a_{jk}s_k\big) + \big(a_{i1}s_1 + \cdots + a_{ik}s_k\big) = cb_j + b_i$$

so $(s_1, \ldots, s_k)$ is a solution to the new system obtained from adding c times equation j to equation i.

Now suppose that $(t_1, \ldots, t_k)$ is a solution to the system that results from adding c times equation j to equation i, so that

$$\begin{aligned} c\big(a_{j1}t_1 + \cdots + a_{jk}t_k\big) + \big(a_{i1}t_1 + \cdots + a_{ik}t_k\big) &= cb_j + b_i \\ a_{j1}t_1 + \cdots + a_{jk}t_k &= b_j \end{aligned}$$

Multiplying the second equation by $-c$ and adding it to the first yields

$$a_{i1}t_1 + \cdots + a_{ik}t_k = b_i$$

so that $(t_1, \ldots, t_k)$ is also a solution to the original system. Hence the two systems have the same set of solutions, so this equation operation does not change the solution set.

57. Apply $f(1) = 4$ to obtain $a(1)^2 + b(1) + c = 4 \Rightarrow a + b + c = 4$. From $f(2) = 7$, we have $a(2)^2+b(2)+c = 7 \Rightarrow 4a+2b+c = 7$.And $f(3) = 14 \Rightarrow a(3)^2+b(3)+c = 14 \Rightarrow 9a+3b+c = 14$.Write these equations as an augmented matrix and solve.

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \\ 9 & 3 & 1 & 14 \end{bmatrix} \underset{\sim}{\substack{-4R_1+R_2\to R_2 \\ -9R_1+R_3\to R_3}} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -9 \\ 0 & -6 & -8 & -22 \end{bmatrix}$$

$$\underset{\sim}{-3R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -9 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Row 3 $\Rightarrow$ $c = 5$. Row 2 $\Rightarrow$ $-2b - 3(5) = -9$ $\Rightarrow$ $b = -3$. Row 1 $\Rightarrow$ $a + (-3) + (5) = 4$ $\Rightarrow$ $a = 2$. Thus $f(x) = 2x^2 - 3x + 5$.

58. Evaluate $f(1) = 8$, $f(2) = 3$, $f(3) = 9$, $f(5) = 1$, and $f(7) = 7$ to produce the equations

$$\begin{array}{rcrcrcrcrcl} a & + & b & + & c & + & d & + & e & = & 8 \\ 16a & + & 8b & + & 4c & + & 2d & + & e & = & 3 \\ 81a & + & 27b & + & 9c & + & 3d & + & e & = & 9 \\ 625a & + & 125b & + & 25c & + & 5d & + & e & = & 1 \\ 2401a & + & 343b & + & 49c & + & 7d & + & e & = & 7 \end{array}$$

and solve using the corresponding augmented matrix using a computer algebra system. We obtain $a = \frac{43}{80}$, $b = -\frac{1949}{240}$, $c = \frac{3263}{80}$, $d = -\frac{18\,859}{240}$, and $e = \frac{427}{8}$.
Thus $f(x) = \frac{43}{80}x^4 - \frac{1949}{240}x^3 + \frac{3263}{80}x^2 - \frac{18\,859}{240}x + \frac{427}{8}$.

59. From a plot, the points do not appear linear, so we use a quadratic to model the data. Let $E(x) = ax^2 + bx + c$. Then $E(20) = 288$, $E(40) = 364$, and $E(60) = 360$. We obtain the three equations

$$\begin{array}{rcrcrcl} 400a & + & 20b & + & c & = & 288 \\ 1600a & + & 40b & + & c & = & 364 \\ 3600a & + & 60b & + & c & = & 360 \end{array}$$

and solve using the corresponding augmented matrix using a computer algebra system. We obtain $a = -\frac{1}{10}$, $b = \frac{49}{5}$, and $c = 132$. Thus $E(x) = -\frac{1}{10}x^2 + \frac{49}{5}x + 132$.

60. From a plot, the points do not appear linear, so we use a quadratic to model the data. Let $E(x) = ax^2 + bx + c$. Then $E(40) = 814$, $E(80) = 1218$, and $E(110) = 1311$. We obtain the three equations

$$\begin{array}{rcrcrcr} 1600a & + & 40b & + & c & = & 814 \\ 6400a & + & 80b & + & c & = & 1218 \\ 12100a & + & 110b & + & c & = & 1311 \end{array}$$

and solve using the corresponding augmented matrix using a computer algebra system. We obtain $a = -\frac{1}{10}$, $b = \frac{221}{10}$, and $c = 90$. Thus $E(x) = -\frac{1}{10}x^2 + \frac{221}{10}x + 90$.

61. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -\frac{157}{181} \\ 0 & 1 & 0 & \frac{20}{181} \\ 0 & 0 & 1 & -\frac{58}{181} \end{bmatrix}$. Hence $x_1 = -\frac{157}{181}$, $x_2 = \frac{20}{181}$, $x_3 = -\frac{58}{181}$. (*Or, as a decimal, we obtain* $\begin{bmatrix} 1 & 0 & 0 & -0.8674 \\ 0 & 1 & 0 & 0.1105 \\ 0 & 0 & 1 & -0.3204 \end{bmatrix}$, *so* $x_1 = -0.8674$, $x_2 = 0.1105$, *and* $x_3 = -0.3204$.)

62. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Hence $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

63. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{7}{9} \\ 0 & 1 & 0 & 1 & -\frac{23}{9} \\ 0 & 0 & 1 & -1 & -\frac{22}{27} \end{bmatrix}$. We have a free variable, $x_4 = s_1$. Thus $x_1 = \frac{7}{9} - s_1$, $x_2 = -\frac{23}{9} - s_1$, and $x_3 = -\frac{22}{27} + s_1$. (*Or, as a decimal, we obtain* $\begin{bmatrix} 1 & 0 & 0 & 1 & 0.77778 \\ 0 & 1 & 0 & 1 & -2.5556 \\ 0 & 0 & 1 & -1 & -0.81481 \end{bmatrix}$, *so* $x_1 = 0.77778 - s_1$, $x_2 = -2.5556 - s_1$, *and* $x_3 = -0.81\,81 + s_1$.)

64. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -\frac{71}{127} & \frac{116}{127} \\ 0 & 1 & 0 & -\frac{131}{254} & \frac{141}{127} \\ 0 & 0 & 1 & -\frac{663}{254} & \frac{606}{127} \end{bmatrix}$. We have a free variable, $x_4 = s_1$. Thus $x_1 = \frac{116}{127} + \frac{71}{127}s_1$, $x_2 = \frac{141}{127} + \frac{131}{254}s_1$, and $x_3 = \frac{606}{127} + \frac{663}{254}s_1$. (*Or, as a decimal, we obtain* $\begin{bmatrix} 1 & 0 & 0 & -0.5591 & 0.9134 \\ 0 & 1 & 0 & -0.5157 & 1.1102 \\ 0 & 0 & 1 & -2.6102 & 4.7717 \end{bmatrix}$, *so* $x_1 = 0.9134 + 0.5591s_1$, $x_2 = 1.1102 + 0.5157s_1$, *and* $x_3 = 4.7717 + 2.6102s_1$.)

65. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Since the last row corresponds to $0 = 1$, the linear system is inconsistent, and there are no solutions.

66. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Since the last row corresponds to $0 = 1$, the linear system is inconsistent, and there are no solutions.

67. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{46}{579} & 0 \\ 0 & 1 & 0 & 0 & -\frac{745}{579} & 0 \\ 0 & 0 & 1 & 0 & \frac{2264}{579} & 0 \\ 0 & 0 & 0 & 1 & \frac{655}{386} & 0 \end{bmatrix}$. We have a free variable, $x_5 = s_1$.
Thus $x_1 = \frac{46}{579}s_1$, $x_2 = -\frac{745}{579}s_1$, $x_3 = -\frac{2264}{579}s_1$, and $x_4 = -\frac{655}{386}s_1$.
(*Or, as a decimal, we obtain* $\begin{bmatrix} 1 & 0 & 0 & 0 & -0.07947 & 0 \\ 0 & 1 & 0 & 0 & -1.2867 & 0 \\ 0 & 0 & 1 & 0 & 3.9102 & 0 \\ 0 & 0 & 0 & 1 & 1.6969 & 0 \end{bmatrix}$, *so* $x_1 = 0.07947s_1$, $x_2 = 1.2867s_1$, $x_3 = -3.9102s_1$ *and* $x_4 = -1.6969s_1$.)

68. Using a computer algebra system, the row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{314}{71} & \frac{1167}{142} & -\frac{349}{142} \\ 0 & 1 & 0 & 0 & \frac{841}{71} & -\frac{1522}{71} & \frac{846}{71} \\ 0 & 0 & 1 & 0 & \frac{828}{71} & -\frac{1578}{71} & \frac{826}{71} \\ 0 & 0 & 0 & 1 & -\frac{431}{71} & \frac{1579}{142} & -\frac{851}{142} \end{bmatrix}$.
We have two free variables, $x_5 = s_1$ and $x_6 = s_2$.
Thus $x_1 = -\frac{349}{142} + \frac{314}{71}s_1 - \frac{1167}{142}s_2$, $x_2 = \frac{846}{71} - \frac{841}{71}s_1 + \frac{1522}{71}s_2$, $x_3 = \frac{826}{71} - \frac{828}{71}s_1 + \frac{1578}{71}s_2$, and $x_4 = -\frac{851}{142} + \frac{431}{71}s_1 - \frac{1579}{142}s_2$.
(*Or, as a decimal, we obtain* $\begin{bmatrix} 1 & 0 & 0 & 0 & -4.4225 & 8.2183 & -2.4577 \\ 0 & 1 & 0 & 0 & 11.845 & -21.437 & 11.915 \\ 0 & 0 & 1 & 0 & 11.662 & -22.225 & 11.634 \\ 0 & 0 & 0 & 1 & -6.0704 & 11.120 & -5.9930 \end{bmatrix}$, *so* $x_1 = -2.4577 + 4.4225s_1 - 8.2183s_2$, $x_2 = 11.915 - 11.845s_1 + 21.437s_2$, $x_3 = 11.634 - 11.662s_1 + 22.225s_2$, and $x_4 = -5.9930 + 6.0704s_1 - 11.120s_2$.)

1.3 Practice Problems

1. At equilibrium, we have

$$\begin{aligned} x_3 &= \frac{x_1 + x_2 + 40}{3} \\ x_2 &= \frac{x_1 + x_3 + 70}{3} \\ x_1 &= \frac{x_2 + x_3 + 30}{3} \end{aligned}$$

   Rearranging, we have

$$\begin{aligned} x_1 + x_2 - 3x_3 &= -40 \\ x_1 - 3x_2 + x_3 &= -70 \\ -3x_1 + x_2 + x_3 &= -30 \end{aligned}$$

   Row-reduce the augmented matrix, and obtain

$$\begin{bmatrix} 1 & 1 & -3 & -40 \\ 1 & -3 & 1 & -70 \\ -3 & 1 & 1 & -30 \end{bmatrix} \underset{\sim}{\overset{-R_1+R_2 \to R_2}{3R_1+R_3 \to R_3}} \begin{bmatrix} 1 & 1 & -3 & -40 \\ 0 & -4 & 4 & -30 \\ 0 & 4 & -8 & -150 \end{bmatrix}$$

$$\underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -3 & -40 \\ 0 & -4 & 4 & -30 \\ 0 & 0 & -4 & -180 \end{bmatrix}$$

   So Row 3 $\Rightarrow$ $x_3 = \frac{-180}{-4} = 45$. Row 2 $\Rightarrow$ $-4x_2 + 4(45) = -30 \;\Rightarrow\; x_2 = \frac{105}{2}$. Row 1 $\Rightarrow$ $x_1 + \left(\frac{105}{2}\right) - 3(45) = -40 \;\Rightarrow\; x_3 = \frac{85}{2}$.

2. As in Example 3, we determine output to satisfy consumer and between-industry demand, and obtain the equations

$$\begin{aligned} a &= 50 + 0.2b \\ b &= 80 + 0.35a \end{aligned}$$

   We may substitute the second equation into the first to obtain

$$\begin{aligned} a &= 50 + 0.2(80 + 0.35a) \\ &= 0.07a + 66 \\ 0.93a &= 66 \\ a &\approx 71.0 \end{aligned}$$

   Then substitute into equation 2 to obtain $b \approx 80 + 0.35(71.0) = 104.9$.

3. We consider $x_1\mathrm{CO_2} + x_2\mathrm{H_2O} \longrightarrow x_3\mathrm{C_6H_{12}O_6} + x_4\mathrm{O_2}$, which implies

$$\begin{array}{ccccccccc} x_1 & & & - & 6x_3 & & & = & 0 \\ 2x_1 & + & x_2 & - & 6x_3 & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 12x_3 & & & = & 0 \end{array}$$

   Row-reduce the augmented matrix

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{bmatrix} \underset{\sim}{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & -24 & 4 & 0 \end{bmatrix}$$

We set $x_4 = s_1$ as a free variable. From row 3, $-24x_3 + 4s_1 = 0 \;\Rightarrow\; x_3 = \frac{1}{6}s_1$. From row 2, $x_2 + 6\left(\frac{1}{6}s_1\right) - 2(s_1) = 0 \;\Rightarrow\; x_2 = s$. From row 1, $x_1 - 6\left(\frac{1}{6}s_1\right) = 0 \;\Rightarrow\; x_1 = s_1$. We set $s_1 = 6$ to obtain $x_1 = 6$, $x_2 = 6$, $x_3 = 1$, and the balanced equation

$$6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$$

4. Assuming $p = ad^b$, so that $\ln(p) = \ln(a) + b\ln(d)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for Earth and Neptune

$$\begin{aligned} a_1 + b\ln(149.6) &= \ln(365.2) \\ a_1 + b\ln(4495.1) &= \ln(59800) \end{aligned}$$

The solution to this system is $a_1 = -1.6029$ and $b = 1.4983$. Thus, $a = e^{a_1} = e^{-1.6029} = 0.2013$. Therefore, $p = (0.2013)\,d^{1.4983}$.

5. Multiply both sides of the equation by $(2x+1)(x-1)$ to obtain $(x+5) = A(x-1) + B(2x+1) = (A+2B)x + (-A+B)$. Equate coefficients of x and the constant terms to obtain

$$\begin{aligned} A + 2B &= 1 \\ -A + B &= 5 \end{aligned}$$

The solution to this system is $A = -3$ and $B = 2$.

6. (a) False. See Example 1.
   (b) True. Any positive integer multiple of a solution will also balance the equation.
   (c) False. For example, no parabola or the form $y = ax^2 + bx + c$ will pass through the three points $(0,0)$, $(0,1)$, $(0,2)$.
   (d) False. For example, $f(x) = 5e^x$ and $f(x) = 5e^{-2x}$ are both of the form $f(x) = ae^x + be^{-2x}$ and $f(0) = 5$.

1.3 Applications of Linear Systems

1. The number of cars entering and leaving each intersection must be the same, resulting in the three equations

$$\begin{array}{lrcl} \text{A:} & x_2 &=& x_3 + 20 \\ \text{B:} & x_3 + 35 + 50 &=& x_1 + 10 \\ \text{C:} & x_1 + 40 &=& x_2 + 45 + 50 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcr} & & x_2 & - & x_3 & = & 20 \\ -x_1 & & & + & x_3 & = & -75 \\ x_1 & - & x_2 & & & = & 55 \end{array}$$

The solution of this system, with $x_3 = s_1$ is $x_1 = 75 + s_1$ and $x_2 = 20 + s_1$. Restricting each $x_i \geq 0$ implies that $s_1 \geq 0$. Therefore the minimum volume of traffic from C to A is $x_2 = 20 + 0 = 20$ vehicles.

2. We obtain the following system of equations

$$\begin{array}{lrcl} \text{A:} & x_2 + 70 &=& x_1 + 40 + 85 \\ \text{B:} & x_3 + 40 + 20 + 25 &=& x_2 + 30 \\ \text{C:} & x_1 + 100 &=& x_4 + 70 \\ \text{D:} & x_4 + 30 &=& x_3 + 60 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcrcr} -x_1 & + & x_2 & & & & & = & 55 \\ & - & x_2 & + & x_3 & & & = & -55 \\ x_1 & & & & & - & x_4 & = & -30 \\ & & & - & x_3 & + & x_4 & = & 30 \end{array}$$

Using a computer algebra system, with free variable $x_4 = s_1$, we obtain $x_1 = -30 + s_1$, $x_2 = 25 + s_1$ and $x_3 = -30 + s_1$. The minimum traffic from C to D is determined by the restrictions $x_i \geq 0$, which implies $s_1 \geq 30$. Therefore the minimum volume of traffic from C to D is $x_4 = s_1 = 30$ vehicles.

3. We obtain the following system of equations

$$\begin{array}{lrcl} \text{A:} & x_4 + 30 + 40 & = & x_1 + 50 \\ \text{B:} & x_1 + x_3 + 25 & = & x_2 + 40 + 55 \\ \text{C:} & x_2 + 50 & = & x_4 + 25 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcrcr} -x_1 & & & & & + & x_4 & = & -20 \\ x_1 & - & x_2 & + & x_3 & & & = & 70 \\ & & x_2 & & & - & x_4 & = & -25 \end{array}$$

Using a computer algebra system, with free variable $x_4 = s_1$, we obtain $x_1 = 20 + 4s_1$, $x_2 = -25 + s_1$ and $x_3 = 25$. The minimum traffic from C to A is determined by the restrictions $x_i \geq 0$, which implies $s_1 \geq 25$. Therefore the minimum volume of traffic from C to A is $x_4 = s_1 = 25$ vehicles.

4. We obtain the following system of equations

$$\begin{array}{lrcl} \text{A:} & x_2 + 50 & = & x_1 + 20 + 40 \\ \text{B:} & x_3 + x_4 + 20 & = & x_2 + 45 \\ \text{C:} & 45 + 60 & = & x_4 + x_5 + 35 \\ \text{D:} & x_1 + 60 & = & 80 \\ \text{E:} & 80 & = & x_3 + 70 \\ \text{F:} & x_5 + 70 & = & x_6 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcrcrcrcr} -x_1 & + & x_2 & & & & & & & & & = & 10 \\ & - & x_2 & + & x_3 & + & x_4 & & & & & = & 25 \\ & & & & & - & x_4 & - & x_5 & & & = & -70 \\ x_1 & & & & & & & & & & & = & 20 \\ & & & - & x_3 & & & & & & & = & -10 \\ & & & & & & & & x_5 & + & x_6 & = & -70 \end{array}$$

Using a computer algebra system, there exists a unique solution, $x_1 = 20$, $x_2 = 30$, $x_3 = 10$, $x_4 = 45$, $x_5 = 25$, and $x_6 = -95$.

5. We obtain the system of equations

$$\begin{array}{rcl} x_1 & = & \dfrac{x_2 + 80}{2} \\ x_2 & = & \dfrac{x_1 + 30 + 40}{3} \end{array}$$

which reduces to

$$\begin{array}{rcl} 2x_1 - x_2 & = & 80 \\ -x_1 + 3x_2 & = & 70 \end{array}$$

We solve this system, and obtain $x_1 = 62$, and $x_2 = 44$.

6. We obtain the system of equations

$$\begin{array}{rcl} x_1 & = & \dfrac{x_2 + 60 + 90}{3} \\ x_2 & = & \dfrac{x_1 + 20 + 40}{3} \end{array}$$

which reduces to

$$\begin{aligned} 3x_1 - x_2 &= 150 \\ -x_1 + 3x_2 &= 60 \end{aligned}$$

We solve this system, and obtain $x_1 = \frac{255}{4}$, and $x_2 = \frac{165}{4}$.

7. We obtain the system of equations

$$\begin{aligned} x_1 &= \frac{x_2 + x_3 + 50}{3} \\ x_2 &= \frac{x_1 + x_3 + 90}{3} \\ x_3 &= \frac{x_1 + x_2 + 30}{3} \end{aligned}$$

which reduces to

$$\begin{aligned} 3x_1 - x_2 - x_3 &= 50 \\ -x_1 + 3x_2 - x_3 &= 90 \\ -x_1 - x_2 + 3x_3 &= 30 \end{aligned}$$

We solve this system, and obtain $x_1 = 55$, $x_2 = 65$, and $x_3 = 50$.

8. We obtain the system of equations

$$\begin{aligned} x_1 &= \frac{x_2 + x_4 + 30}{3} \\ x_2 &= \frac{x_1 + x_3 + 0}{3} \\ x_3 &= \frac{x_2 + x_4 + 90}{3} \\ x_4 &= \frac{x_1 + x_3 + 20}{3} \end{aligned}$$

which reduces to

$$\begin{aligned} 3x_1 - x_2 - x_4 &= 30 \\ -x_1 + 3x_2 - x_3 &= 0 \\ -x_2 + 3x_3 - x_4 &= 90 \\ -x_1 - x_3 + 3x_4 &= 20 \end{aligned}$$

We solve this system, and obtain $x_1 = 30$, $x_2 = \frac{80}{3}$, $x_3 = 50$ and $x_4 = \frac{100}{3}$.

9. Let a and b denote the total output from each of A and B, respectively. We obtain

$$\begin{aligned} a &= 60 + 0.30b \\ b &= 40 + 0.20a \end{aligned}$$

We solve this system, and obtain $a = 72.34$ and $b = 61.70$.

10. Let a and b denote the total output from each of A and B, respectively. We obtain

$$\begin{aligned} a &= 80 + 0.15b \\ b &= 50 + 0.25a \end{aligned}$$

We solve this system, and obtain $a = 96.10$ and $b = 64.42$.

11. Let a, b, and c denote the total output from each of A, B, and C, respectively. We obtain

$$\begin{array}{rcl} a & = & 30+0.15b+0.20c \\ b & = & 50+0.10a+0.10c \\ c & = & 60+0.15a+0.20b \end{array}$$

We solve this system, and obtain $a = 55.77$, and $b = 63.69$, and $c = 81.10$.

12. Let a, b, and c denote the total output from each of A, B, and C, respectively. We obtain

$$\begin{array}{rcl} a & = & 40+0.25b+0.10c \\ b & = & 30+0.20a+0.15c \\ c & = & 70+0.10a+0.10b \end{array}$$

We solve this system, and obtain $a = 61.82$, and $b = 54.61$, and $c = 81.64$.

13. We consider $x_1\mathrm{H}_2 + x_2\mathrm{O}_2 \longrightarrow x_3\mathrm{H}_2\mathrm{O}$, which implies

$$\begin{array}{rcrcrcl} 2x_1 & & & - & 2x_3 & = & 0 \\ & & 2x_2 & - & x_3 & = & 0 \end{array}$$

From the augmented matrix $\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$ we set $x_3 = s_1$ as a free variable, and thus $x_2 = \frac{1}{2}s_1$ and $x_1 = s_1$. We set $s_1 = 2$ to obtain $x_1 = 2$, $x_2 = 1$, $x_3 = 2$, and the balanced equation

$$2\mathrm{H}_2+\mathrm{O}_2 \longrightarrow 2\mathrm{H}_2\mathrm{O}$$

14. We consider $x_1\mathrm{H}_2 + x_2\mathrm{N}_2 \longrightarrow x_3\mathrm{NH}_3$, which implies

$$\begin{array}{rcrcrcl} 2x_1 & & & - & 3x_3 & = & 0 \\ & & 2x_2 & - & x_3 & = & 0 \end{array}$$

From the augmented matrix $\begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$ we set $x_3 = s_1$ as a free variable, and thus $x_2 = \frac{1}{2}s_1$ and $x_1 = \frac{3}{2}s_1$. We set $s_1 = 2$ to obtain $x_1 = 3$, $x_2 = 1$, $x_3 = 2$, and the balanced equation

$$3\mathrm{H}_2 + \mathrm{N}_2 \longrightarrow 2\mathrm{NH}_3$$

15. We consider $x_1\mathrm{Fe}+x_2\mathrm{O}_2 \longrightarrow x_3\mathrm{Fe}_2\mathrm{O}_3$, which implies

$$\begin{array}{rcrcrcl} x_1 & & & - & 2x_3 & = & 0 \\ & & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

From the augmented matrix $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix}$ we set $x_3 = s_1$ as a free variable, and thus $x_2 = \frac{3}{2}s_1$ and $x_1 = 2s_1$. We set $s_1 = 2$ to obtain $x_1 = 4$, $x_2 = 3$, $x_3 = 2$, and the balanced equation

$$4\mathrm{Fe} + 3\mathrm{O}_2 \longrightarrow 2\mathrm{Fe}_2\mathrm{O}_3$$

16. We consider $x_1\mathrm{Na}+x_2\mathrm{H}_2\mathrm{O} \longrightarrow x_3\mathrm{NaOH}+x_4\mathrm{H}_2$, which implies

$$\begin{array}{rcrcrcrcl} x_1 & & & - & x_3 & & & = & 0 \\ & & 2x_2 & - & x_3 & - & 2x_4 & = & 0 \\ & & x_2 & - & x_3 & & & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$$

$$\overset{-2R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

We set $x_4 = s_1$ as a free variable, and thus $x_3 = 2s_1$, $x_2 = x_3 = 2s_1$, and $x_1 = x_3 = 2s_1$. We set $s_1 = 1$ to obtain $x_1 = 2$, $x_2 = 2$, $x_3 = 2$, $x_4 = 1$, and the balanced equation

$$2\mathrm{Na} + 2\mathrm{H_2O} \longrightarrow 2\mathrm{NaOH} + \mathrm{H_2}$$

17. We consider $x_1\mathrm{C_3H_8} + x_2\mathrm{O_2} \longrightarrow x_3\mathrm{CO_2} + x_4\mathrm{H_2O}$, which implies

$$\begin{array}{rcrcrcrcl} 3x_1 & & & - & x_3 & & & = & 0 \\ 8x_1 & & & & & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \overset{(-8/3)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \end{bmatrix}$$

We set $x_4 = s_1$ as a free variable. From row 3, $\frac{8}{3}x_3 - 2s_1 = 0 \Rightarrow x_3 = \frac{3}{4}s_1$. From row 2, $2x_2 - 2x_3 - x_4 = 0 \Rightarrow 2x_2 - 2\left(\frac{3}{4}s_1\right) - s_1 = 0 \Rightarrow x_2 = \frac{5}{4}s_1$ From row 1, $3x_1 - x_3 = 0 \Rightarrow 3x_1 - \left(\frac{3}{4}s_1\right) = 0 \Rightarrow x_1 = \frac{1}{4}s_1$. We set $s_1 = 4$ to obtain $x_1 = 1$, $x_2 = 5$, $x_3 = 3$, $x_4 = 4$, and the balanced equation

$$\mathrm{C_3H_8} + 5\mathrm{O_2} \longrightarrow 3\mathrm{CO_2} + 4\mathrm{H_2O}$$

18. We consider $x_1\mathrm{C_2H_2} + x_2\mathrm{O_2} \longrightarrow x_3\mathrm{CO_2} + x_4\mathrm{H_2O}$, which implies

$$\begin{array}{rcrcrcrcl} 2x_1 & & & - & x_3 & & & = & 0 \\ 2x_1 & & & & & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \overset{-R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

We set $x_4 = s_1$ as a free variable. From row 3, $x_3 - 2s_1 = 0 \Rightarrow x_3 = 2s_1$. From row 2, $2x_2 - 2x_3 - x_4 = 0 \Rightarrow 2x_2 - 2(2s_1) - s_1 = 0 \Rightarrow x_2 = \frac{5}{2}s_1$. From row 1, $2x_1 - x_3 = 0 \Rightarrow 2x_1 - (2s_1) = 0 \Rightarrow x_1 = s_1$. We set $s_1 = 2$ to obtain $x_1 = 2$, $x_2 = 5$, $x_3 = 4$, $x_4 = 2$, and the balanced equation

$$2\mathrm{C_2H_2} + 5\mathrm{O_2} \longrightarrow 4\mathrm{CO_2} + 2\mathrm{H_2O}$$

19. We consider $x_1\mathrm{KO_2} + x_2\mathrm{CO_2} \longrightarrow x_3\mathrm{K_2CO_3} + x_4\mathrm{O_2}$, which implies

$$\begin{array}{rcrcrcrcl} x_1 & & & - & 2x_3 & & & = & 0 \\ 2x_1 & + & 2x_2 & - & 3x_3 & - & 2x_4 & = & 0 \\ & & x_2 & - & x_3 & & & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 2 & 2 & -3 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \underset{\sim}{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{(-1/2)R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 1 & -2 & 0 \\ 0 & 0 & -\frac{3}{2} & 1 & 0 \end{bmatrix}$$

We set $x_4 = s_1$ as a free variable. From row 3, $-\frac{3}{2}x_3 + s_1 = 0 \;\Rightarrow\; x_3 = \frac{2}{3}s_1$. From row 2, $2x_2 + x_3 - 2x_4 = 0 \;\Rightarrow\; 2x_2 + \left(\frac{2}{3}s_1\right) - 2s_1 = 0 \;\Rightarrow\; x_2 = \frac{2}{3}s_1$. From row 1, $x_1 - 2x_3 = 0 \;\Rightarrow\; x_1 - 2\left(\frac{2}{3}s_1\right) = 0 \;\Rightarrow\; x_1 = \frac{4}{3}s_1$. We set $s_1 = 3$ to obtain $x_1 = 4$, $x_2 = 2$, $x_3 = 2$, $x_4 = 3$, and the balanced equation

$$4\mathrm{KO_2} + 2\mathrm{CO_2} \longrightarrow 2\mathrm{K_2CO_3} + 3\mathrm{O_2}$$

20. We consider $x_1\mathrm{MnO_2} + x_2\mathrm{HCl} \longrightarrow x_3\mathrm{MnCl_2} + x_4\mathrm{H_2O} + x_5\mathrm{Cl_2}$, which implies

$$\begin{array}{rcrcrcrcrcl} x_1 & & & - & x_3 & & & & & = & 0 \\ 2x_1 & & & & & - & x_4 & & & = & 0 \\ & & x_2 & & & - & 2x_4 & & & = & 0 \\ & & x_2 & - & 2x_3 & & & - & 2x_5 & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \end{bmatrix} \underset{\sim}{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{\sim}{R_2\leftrightarrow R_4} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{-R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{-R_3+R_4\to R_4} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

We set $x_5 = s_1$ as a free variable. From row 4, $x_4 - 2s_1 = 0 \;\Rightarrow\; x_4 = 2s_1$. From row 3, $2x_3 - 2x_4 + 2s_1 = 0 \;\Rightarrow\; 2x_3 - 2\left(2s_1\right) + 2s_1 = 0 \;\Rightarrow\; x_3 = s_1$. From row 2, $x_2 - 2x_3 - 2s_1 = 0 \;\Rightarrow\; x_2 - 2\left(s_1\right) - 2s_1 = 0 \;\Rightarrow\; x_2 = 4s_1$. From row 2, $x_1 - x_3 = 0 \;\Rightarrow\; x_1 - \left(s_1\right) = 0 \;\Rightarrow\; x_1 = s_1$.We set $s_1 = 1$ to obtain $x_1 = 1$, $x_2 = 4$, $x_3 = 1$, $x_4 = 2$, $x_5 = 1$, and the balanced equation

$$\mathrm{MnO_2} + 4\mathrm{HCl} \longrightarrow \mathrm{MnCl_2} + 2\mathrm{H_2O} + \mathrm{Cl_2}$$

21. Assuming $p = ad^b$, so that $\ln(p) = \ln(a) + b\ln(d)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for Earth and Mars

$$a_1 + b\ln(149.6) = \ln(365.2)$$
$$a_1 + b\ln(227.9) = \ln(687)$$

The solution to this system is $a_1 = -1.6171$ and $b = 1.5011$. Thus $a = e^{a_1} = e^{-1.6171} = 0.19847$. Hence $p = (0.19847)\, d^{1.5011}$.

22. Assuming $p = ad^b$, so that $\ln(p) = \ln(a) + b\ln(d)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for Mercury and Uranus

$$a_1 + b\ln(57.9) = \ln(88)$$
$$a_1 + b\ln(2872.5) = \ln(30589)$$

The solution to this system is $a_1 = -1.60526$ and $b = 1.49865$. Thus $a = e^{-1.60526} = 0.20083$. Hence $p = (0.20083)\, d^{1.49865}$.

23. Assuming $p = ad^b$, so that $\ln(p) = \ln(a) + b\ln(d)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for Venus and Neptune

$$a_1 + b\ln(108.2) = \ln(224.7)$$
$$a_1 + b\ln(4495.1) = \ln(59800)$$

The solution to this system is $a_1 = -1.6035$ and $b = 1.49835$. Thus $a = e^{-1.6035} = 0.20120$. Hence $p = (0.20120)\, d^{1.49835}$.

24. Assuming $p = ad^b$, so that $\ln(p) = \ln(a) + b\ln(d)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for Jupiter and Saturn

$$a_1 + b\ln(778.6) = \ln(4331)$$
$$a_1 + b\ln(1433.5) = \ln(10747)$$

The solution to this system is $a_1 = -1.5392$ and $b = 1.48896$. Thus $a = e^{-1.5392} = 0.21455$. Hence $p = (0.21455)\, d^{1.48896}$.

25. Assuming $d = as^k$, so that $\ln(d) = \ln(a) + k\ln(s)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for $s = 10$ and $s = 20$

$$a_1 + k\ln(10) = \ln(4.5)$$
$$a_1 + k\ln(20) = \ln(18)$$

The solution to this system is $a_1 = -3.1010$ and $k = 2$. Thus $a = e^{-3.1010} = 0.04500$. Hence $d = (0.04500)\, s^2$. The predicted distance for each speed is as follows:

Speed (MPH)	10	20	30	40
Distance (Feet)	4.50	18.0	40.5	72.0

26. Assuming $d = as^k$, so that $\ln(d) = \ln(a) + k\ln(s)$, and letting $a_1 = \ln(a)$, we obtain the following equations using the data for $s = 10$ and $s = 20$

$$a_1 + k\ln(10) = \ln(20)$$
$$a_1 + k\ln(20) = \ln(80)$$

The solution to this system is $a_1 = -1.6094$ and $k = 2$. Thus $a = e^{-1.6094} = 0.20000$. Hence $d = (0.20000)\, s^2$. The predicted distance for each speed is as follows:

Speed (MPH)	10	20	30	40
Distance (Feet)	20.0	80.0	180.0	320.0

27. Multiply both sides of the equation by $x(x+1)$ to obtain $1 = A(x+1)+Bx = (A+B)x+A$. Equate coefficients of x to obtain the equation $0 = A+B$. Equate constant terms to obtain $A = 1$. Substitute into $0 = A+B$ to obtain $0 = 1+B \Rightarrow B = -1$.

28. Multiply both sides of the equation by $(x-1)(x+1)$ to obtain $(3x-1) = A(x+1)+B(x-1) = (A+B)x+(A-B)$. Equate coefficients of x and the constant terms to obtain

$$\begin{aligned} A + B &= 3 \\ A - B &= -1 \end{aligned}$$

The solution to this system is $A = 1$ and $B = 2$.

29. Multiply both sides of the equation by $x^2(x-1)$ to obtain $1 = A(x)(x-1)+B(x-1)+C(x^2) = (A+C)x^2+(-A+B)x-B$. Equate coefficients of x^2, x, and the constant terms to obtain

$$\begin{aligned} A + C &= 0 \\ -A + B &= 0 \\ -B &= 1 \end{aligned}$$

The solution to this system is $A = -1$, $B = -1$, and $C = 1$.

30. Multiply both sides of the equation by $x\left(x^2+1\right)$ to obtain $1 = A(x^2+1)+(Bx+C)x = (A+B)x^2+Cx+A$. Equate coefficients of x^2, x, and the constant terms to obtain

$$\begin{aligned} A + B &= 0 \\ C &= 0 \\ A &= 1 \end{aligned}$$

The solution to this system is $A = 1$, $B = -1$, and $C = 0$.

31. Let the line be given by the equation $y = ax+b$. Then using the point $(1,3)$ we have $3 = a(1)+b$; and the point $(-2,6)$ produces the equation $6 = a(-2)+b$. We solve the system

$$\begin{aligned} a + b &= 3 \\ -2a + b &= 6 \end{aligned}$$

and obtain $a = -1$ and $b = 4$. Thus $y = -x+4$. The point where this line crosses the x-axis is determined by setting $y = 0$ and then solving for x. Hence $0 = -x+4 \Rightarrow x = 4$.

32. Let the line be given by the equation $y = ax+b$. Then using the point $(5,-1)$ we have $-1 = a(5)+b$; and the point $(-8,3)$ produces the equation $3 = a(-8)+b$. We solve the system

$$\begin{aligned} 5a + b &= -1 \\ -8a + b &= 3 \end{aligned}$$

and obtain $a = -\frac{4}{13}$ and $b = \frac{7}{13}$. Thus $y = -\frac{4}{13}x+\frac{7}{13}$. The point where this line crosses the y-axis is determined by setting $x = 0$ and then solving for y. Hence $y = \frac{7}{13}$.

33. Let the plane be given by the equation $z = ax+by+c$. Using the points $(2,-1,-2)$, $(1,3,12)$, and $(4,2,3)$, we obtain the system

$$\begin{aligned} 2a - b + c &= -2 \\ a + 3b + c &= 12 \\ 4a + 2b + c &= 3 \end{aligned}$$

The solution to this system is $a = -2$, $b = 3$, and $c = 5$. Thus $z = -2x+3y+5$. The point where this plane crosses the z-axis is determined by setting $x = 0$ and $y = 0$, and then solving for z. Hence $z = -2(0)+3(0)+5 \Rightarrow z = 5$.

34. Let the plane be given by the equation $z = ax + by + c$. Using the points $(2, 2, -1)$, $(-1, -1, 0)$, and $(2, 1, 1)$, we obtain the system

$$\begin{aligned} 2a + 2b + c &= -1 \\ -a - b + c &= 0 \\ 2a + b + c &= 1 \end{aligned}$$

The solution to this system is $a = \frac{5}{3}$, $b = -2$, and $c = -\frac{1}{3}$. Thus $z = \frac{5}{3}x - 2y - \frac{1}{3}$. The point where this plane crosses the z-axis is determined by setting $x = 0$ and $y = 0$, and then solving for z. Hence $z = \frac{5}{3}(0) - 2(0) - \frac{1}{3} \Rightarrow z = -\frac{1}{3}$.

35. Substituting the points $(-1, -2)$, $(1, 4)$, and $(2, 4)$ into the equation $y = ax^2 + bx + c$ we obtain the equations

$$\begin{aligned} a - b + c &= -2 \\ a + b + c &= 4 \\ 4a + 2b + c &= 4 \end{aligned}$$

The solution to this system is $a = -1, b = 3$, $c = 2$. The equation of the parabola passing through all three points is $y = -x^2 + 3x + 2$.

36. Using the values $f(0) = -3$, $f(1) = 2$, $f(3) = 5$, and $f(4) = 0$ in the function $f(x) = ax^3 + bx^2 + cx + d$ we obtain the equations

$$\begin{aligned} d &= -3 \\ a + b + c + d &= 2 \\ 27a + 9b + 3c + d &= 5 \\ 64a + 16b + 4c + d &= 0 \end{aligned}$$

Using a computer algebra system we obtain $a = -\frac{1}{4}$, $b = -\frac{1}{6}$, $c = \frac{65}{12}$, and $d = -3$. Thus $f(x) = -\frac{1}{4}x^3 - \frac{1}{6}x^2 + \frac{65}{12}x - 3$.

37. Using the values $g(-2) = -17$, $g(-1) = 6$, $g(0) = 5$, $g(1) = 4$, and $g(2) = 3$ in the function $g(x) = ax^4 + bx^3 + cx^2 + dx + e$ we obtain the equations

$$\begin{aligned} 16a - 8b + 4c - 2d + e &= -17 \\ a - b + c - d + e &= 6 \\ e &= 5 \\ a + b + c + d + e &= 4 \\ 16a + 8b + 4c + 2d + e &= 3 \end{aligned}$$

Using a computer algebra system we obtain $a = -1$, $b = 2$, $c = 1$, $d = -3$, and $e = 5$. Thus $g(x) = -x^4 + 2x^3 + x^2 - 3x + 5$.

38. Using $f(0) = 2$ in the function $f(x) = ae^x + be^{2x} + ce^{-3x}$ we obtain $2 = a + b + c$. Using $f'(0) = 1$ in the derivative $f'(x) = ae^x + 2be^{2x} - 3ce^{-3x}$ we obtain $1 = a + 2b - 3c$. And using $f''(0) = 19$ in the second derivative $f''(x) = ae^x + 4be^{2x} + 9ce^{-3x}$ we obtain $19 = a + 4b + 9c$. Using a computer algebra system we solve the system of equations

$$\begin{aligned} a + b + c &= 2 \\ a + 2b - 3c &= 1 \\ a + 4b + 9c &= 19 \end{aligned}$$

to get $a = -2$, $b = 3$, and $c = 1$. Thus $f(x) = -2e^x + 3e^{2x} + e^{-3x}$.

39. Using $f(0)=-1$ in the function $f(x)=ae^{-2x}+be^{x}+cxe^{x}$ we obtain $-1=a+b$. Using $f'(0)=-2$ in the derivative $f'(x)=\frac{d}{dx}\left(ae^{-2x}+be^{x}+cxe^{x}\right)=-2ae^{-2x}+(b+c)e^{x}+cxe^{x}$ we obtain $-2=-2a+b+c$. And using $f''(0)=3$ in the second derivative $f''(x)=\frac{d}{dx}\left(-2ae^{-2x}+(b+c)e^{x}+cxe^{x}\right)=4ae^{-2x}+(b+2c)\,e^{x}+cxe^{x}$, we obtain $3=4a+b+2c$. Using a computer algebra system we solve the system of equations

$$\begin{aligned} a+b &= -1 \\ -2a+b+c &= -2 \\ 4a+b+2c &= 3 \end{aligned}$$

to get $a=\frac{2}{3}$, $b=-\frac{5}{3}$, and $c=1$. Thus $f(x)=\frac{2}{3}e^{-2x}-\frac{5}{3}e^{x}+xe^{x}$.

40. With these new LAI values, we obtain the three equations using the top three schools

$$\begin{aligned} 1482x_1+2699x_2+100x_3 &= 0.9655 \\ 1481x_1+2776x_2+89x_3 &= 0.9652 \\ 1408x_1+2616x_2+94x_3 &= 0.9237 \end{aligned}$$

Using a computer algebra system we solve this system and obtain $x_1=0.0003230$, $x_2=0.0001433$, and $x_3=0.0010009$ .Our LAI formula is now

$$\text{LAI}=0.0003230\,(\text{USA})+0.0001433\,(\text{Harris})+0.0010009\,(\text{Computer})$$

Testing this formula for all schools, we obtain the predicted values,

Team	LAI
Oklahoma	0.9655
Florida	0.9652
Texas	0.9237
Alabama	0.8538
Southern Cal	0.8436
Penn State	0.7646
Utah	0.7560
Texas Tech	0.7522

which agrees with the LAI values given.

1.4 Practice Problems

1. (a) Using Gaussian elimination with 3 significant digits of accuracy:

$$\begin{bmatrix} 1 & 562 & 52 \\ 49 & -78 & -11 \end{bmatrix} \underset{\sim}{-49R_1+R_2\to R_2} \begin{bmatrix} 1 & 562 & 52 \\ 0 & -27\,600 & -2560 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2=\frac{-2560}{-27\,600}=9.28\times10^{-2}$. Row 1 $\Rightarrow$ $x_1+562\left(9.28\times10^{-2}\right)=52$ $\Rightarrow$ $x_1=-0.154$.
Using partial pivoting:

$$\begin{bmatrix} 1 & 562 & 52 \\ 49 & -78 & -11 \end{bmatrix} \underset{\sim}{R_1\leftrightarrow R2} \begin{bmatrix} 49 & -78 & -11 \\ 1 & 562 & 52 \end{bmatrix}$$
$$\underset{\sim}{(-1/49)R_1+R_2\to R_2} \begin{bmatrix} 49 & -78 & -11 \\ 0 & 564 & 52.2 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2=\frac{52.2}{564}=9.26\times10^{-2}$. Row 1 $\Rightarrow$ $49x_1-78\left(9.26\times10^{-2}\right)=-11$ $\Rightarrow$ $x_1=-7.71\times10^{-2}$.

(b) Using Gaussian elimination with 3 significant digits of accuracy:

$$\begin{bmatrix} 2 & -8 & 598 & 15 \\ -3 & 7 & 913 & 5 \\ 67 & -39 & 84 & 11 \end{bmatrix} \underset{\sim}{\substack{(3/2)R_1+R_2\to R_2 \\ (-67/2)R_1+R_3\to R_3}}$$

$$\begin{bmatrix} 2 & -8 & 598 & 15 \\ 0 & -5.0 & 1810.0 & 27.5 \\ 0 & 229.0 & -19900.0 & -492.0 \end{bmatrix} \underset{\sim}{(229.0/5)R_2+R_3\to R_3}$$

$$\begin{bmatrix} 2 & -8 & 598 & 15 \\ 0 & -5.0 & 1810.0 & 27.5 \\ 0 & 0.00 & 63000.0 & 768.0 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{768.0}{63000.0} = 1.22\times 10^{-2}$. Row 2 $\Rightarrow$ $-5.0\,x_2+(1810.0)\left(1.22\times 10^{-2}\right) = 27.5 \Rightarrow x_2 = -1.08$. Row 1 $\Rightarrow$ $2x_1 - 8(-1.08) + 598\left(1.22\times 10^{-2}\right) = 15 \Rightarrow x_1 = -0.468$.
Using partial pivoting.

$$\begin{bmatrix} 2 & -8 & 598 & 15 \\ -3 & 7 & 913 & 5 \\ 67 & -39 & 84 & 11 \end{bmatrix} \underset{\sim}{R_1\leftrightarrow R_3} \begin{bmatrix} 67 & -39 & 84 & 11 \\ -3 & 7 & 913 & 5 \\ 2 & -8 & 598 & 15 \end{bmatrix}$$

$$\underset{\sim}{\substack{(3/67)R_1+R_2\to R_2 \\ (-2/67)R_1+R_3\to R_3}} \begin{bmatrix} 67 & -39 & 84 & 11 \\ 0 & 5.25 & 917.0 & 5.49 \\ 0 & -6.84 & 595.0 & 14.7 \end{bmatrix}$$

$$\underset{\sim}{R_2\leftrightarrow R_3} \begin{bmatrix} 67 & -39 & 84 & 11 \\ 0 & -6.84 & 595.0 & 14.7 \\ 0 & 5.25 & 917.0 & 5.49 \end{bmatrix}$$

$$\underset{\sim}{(5.25/6.84)R_2+R_3\to R_3} \begin{bmatrix} 67 & -39 & 84 & 11 \\ 0 & -6.84 & 595.0 & 14.7 \\ 0 & 0.00 & 1370.0 & 16.8 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{16.8}{1370.0} = 1.23\times 10^{-2}$. Row 2 $\Rightarrow$ $-6.84x_2 + (595.0)\left(1.23\times 10^{-2}\right) = 14.7 \Rightarrow x_2 = -1.08$. Row 1 $\Rightarrow$ $67x_1 - 39(-1.08) + 84\left(1.23\times 10^{-2}\right) = 11 \Rightarrow x_1 = -0.480$.

2. (a)

n	x_1	x_2
0	0	0
1	−2.25	0.385
2	−2.15	0.731
3	−2.07	0.716

Exact solution: $x_1 = -2.07$, $x_2 = 0.704$.

(b)

n	x_1	x_2	x_3
0	0	0	0
1	0.913	−3	1.38
2	0.935	−2.49	2.73
3	0.655	−1.89	2.54

Exact solution: $x_1 = 0.689$, $x_2 = -2.05$, $x_3 = 2.32$.

3. (a) Gauss–Seidel iteration of given linear system:

n	x_1	x_2
0	0	0
1	−2.25	0.731
2	−2.07	0.703
3	−2.07	0.704

Exact solution: $x_1 = -2.07$, $x_2 = 0.704$.

(b) Gauss–Seidel iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	0.913	−3.10	2.77
2	0.702	−1.85	2.24
3	0.684	−2.07	2.32

Exact solution: $x_1 = 0.689$, $x_2 = -2.05$, $x_3 = 2.32$.

1.4 Numerical Solutions

1. $$\begin{bmatrix} -2 & 3 & 4 \\ 5 & -2 & 1 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 3 & 4 \end{bmatrix} \overset{(2/5)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{11}{5} & \frac{22}{5} \end{bmatrix}$$

Row 2 $\Rightarrow$ $\frac{11}{5}x_2 = \frac{22}{5}$ $\Rightarrow$ $x_2 = 2$. Row 1 $\Rightarrow$ $5x_1 - 2(2) = 1$ $\Rightarrow$ $x_1 = 1$.

2. $$\begin{bmatrix} 1 & -2 & -1 \\ -3 & 7 & 5 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} -3 & 7 & 5 \\ 1 & -2 & -1 \end{bmatrix} \overset{(1/3)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} -3 & 7 & 5 \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Row 2 $\Rightarrow$ $\frac{1}{3}x_2 = \frac{2}{3}$ $\Rightarrow$ $x_2 = 2$. Row 1 $\Rightarrow$ $-3x_1 + 7(2) = 5$ $\Rightarrow$ $x_1 = 3$.

3. $$\begin{bmatrix} 1 & 1 & -2 & -3 \\ 3 & -2 & 2 & 9 \\ 6 & -7 & -1 & 4 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 6 & -7 & -1 & 4 \\ 3 & -2 & 2 & 9 \\ 1 & 1 & -2 & -3 \end{bmatrix}$$
$$\overset{\substack{(-1/2)R_1+R_2 \to R_2 \\ (-1/6)R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 6 & -7 & -1 & 4 \\ 0 & \frac{3}{2} & \frac{5}{2} & 7 \\ 0 & \frac{13}{6} & -\frac{11}{6} & -\frac{11}{3} \end{bmatrix}$$
$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 6 & -7 & -1 & 4 \\ 0 & \frac{13}{6} & -\frac{11}{6} & -\frac{11}{3} \\ 0 & \frac{3}{2} & \frac{5}{2} & 7 \end{bmatrix}$$
$$\overset{(-9/13)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 6 & -7 & -1 & 4 \\ 0 & \frac{13}{6} & -\frac{11}{6} & -\frac{11}{3} \\ 0 & 0 & \frac{49}{13} & \frac{124}{13} \end{bmatrix}$$

Row 3 $\Rightarrow$ $\frac{49}{13}x_3 = \frac{124}{13}$ $\Rightarrow$ $x_3 = \frac{124}{49}$. Row 2 $\Rightarrow$ $\frac{13}{6}x_2 - \frac{11}{6}\left(\frac{124}{49}\right) = -\frac{11}{3}$ $\Rightarrow$ $x_2 = \frac{22}{49}$. Row 1 $\Rightarrow$ $6x_1 - 7\left(\frac{22}{49}\right) - \left(\frac{124}{49}\right) = 4$ $\Rightarrow$ $x_1 = \frac{79}{49}$.

4. $$\begin{bmatrix} 1 & -3 & 2 & 4 \\ -2 & 7 & -2 & -7 \\ 4 & -13 & 7 & 12 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 4 & -13 & 7 & 12 \\ -2 & 7 & -2 & -7 \\ 1 & -3 & 2 & 4 \end{bmatrix}$$
$$\overset{\substack{(1/2)R_1+R_2 \to R_2 \\ (-1/4)R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 4 & -13 & 7 & 12 \\ 0 & \frac{1}{2} & \frac{3}{2} & -1 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 \end{bmatrix}$$
$$\overset{(-1/2)R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 4 & -13 & 7 & 12 \\ 0 & \frac{1}{2} & \frac{3}{2} & -1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Row 3 $\Rightarrow$ $-\frac{1}{2}x_3 = \frac{3}{2}$ $\Rightarrow$ $x_3 = -3$. Row 2 $\Rightarrow$ $\frac{1}{2}x_2 + \frac{3}{2}(-3) = -1$ $\Rightarrow$ $x_2 = 7$. Row 1 $\Rightarrow$ $4x_1 - 13(7) + 7(-3) = 12$ $\Rightarrow$ $x_1 = 31$.

5. Using Gaussian elimination with 3 significant digits of accuracy:

$$\begin{bmatrix} 2 & 975 & 41 \\ 53 & -82 & -13 \end{bmatrix} \underset{\sim}{\overset{(-53/2)R_1+R_2\to R_2}{}} \begin{bmatrix} 2 & 975 & 41 \\ 0 & -2.59\times10^4 & -1.10\times10^3 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{-1.10\times10^3}{-2.59\times10^4} = 4.25\times10^{-2}$. Row 1 $\Rightarrow$ $2x_1 + 975\left(4.25\times10^{-2}\right) = 41$ $\Rightarrow$ $x_1 = -0.219$.
Using partial pivoting:

$$\begin{bmatrix} 2 & 975 & 41 \\ 53 & -82 & -13 \end{bmatrix} \underset{\sim}{\overset{R_1\leftrightarrow R2}{}} \begin{bmatrix} 53 & -82 & -13 \\ 2 & 975 & 41 \end{bmatrix}$$
$$\underset{\sim}{\overset{(-2/53)R_1+R_2\to R_2}{}} \begin{bmatrix} 53 & -82 & -13 \\ 0 & 9.78\times10^2 & 4.15\times10^1 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{4.15\times10^1}{9.78\times10^2} = 4.24\times10^{-2}$. Row 1 $\Rightarrow$ $53x_1 - 82\left(4.24\times10^{-2}\right) = -13$ $\Rightarrow$ $x_1 = -0.180$.

6. Using Gaussian elimination with 3 significant digits of accuracy.

$$\begin{bmatrix} 3 & -813 & 32 \\ 71 & -93 & -5 \end{bmatrix} \underset{\sim}{\overset{(-71/3)R_1+R_2\to R_2}{}} \begin{bmatrix} 3 & -813 & 32 \\ 0 & 1.91\times10^4 & -7.62\times10^2 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{-7.62\times10^2}{1.91\times10^4} = -3.99\times10^{-2}$. Row 1 $\Rightarrow$ $3x_1 - 813\left(-3.99\times10^{-2}\right) = 32$ $\Rightarrow$ $x_1 = -0.146$.
Using partial pivoting:

$$\begin{bmatrix} 3 & -813 & 32 \\ 71 & -93 & -5 \end{bmatrix} \underset{\sim}{\overset{R_1\leftrightarrow R2}{}} \begin{bmatrix} 71 & -93 & -5 \\ 3 & -813 & 32 \end{bmatrix}$$
$$\underset{\sim}{\overset{(-3/71)R_1+R_2\to R_2}{}} \begin{bmatrix} 71 & -93 & -5 \\ 0 & -8.09\times10^2 & 3.22\times10^1 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{3.22\times10^1}{-8.09\times10^2} = -3.98\times10^{-2}$. Row 1 $\Rightarrow$ $71x_1 - 93\left(-3.98\times10^{-2}\right) = -5$ $\Rightarrow$ $x_1 = -0.123$,

7. Using Gaussian elimination with 3 significant digits of accuracy:

$$\begin{bmatrix} 3 & -7 & 639 & 12 \\ -2 & 5 & 803 & 7 \\ 56 & -41 & 79 & 10 \end{bmatrix} \underset{\sim}{\overset{\substack{(2/3)R_1+R_2\to R_2 \\ (-56/3)R_1+R_3\to R_3}}{}}$$
$$\begin{bmatrix} 3 & -7 & 639 & 12 \\ 0 & 0.333 & 1.23\times10^3 & 1.5\times10^1 \\ 0 & 8.97\times10^1 & -1.18\times10^4 & -2.14\times10^2 \end{bmatrix} \underset{\sim}{\overset{\left(-8.97\times10^1/0.333\right)R_2+R_3\to R_3}{}}$$
$$\begin{bmatrix} 3 & -7 & 639 & 12 \\ 0 & 0.333 & 1.23\times10^3 & 1.5\times10^1 \\ 0 & 0 & -3.43\times10^5 & -4.25\times10^3 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{-4.25\times10^3}{-3.43\times10^5} = 1.24\times10^{-2}$. Row 2 $\Rightarrow$ $0.333\,x_2 + 1.23\times10^3\left(1.24\times10^{-2}\right) = 1.5\times10$ $\Rightarrow$ $x_2 = -0.757$. Row 1 $\Rightarrow$ $3x_1 - 7(-0.757) + 639\left(1.24\times10^{-2}\right) = 12$ $\Rightarrow$ $x_1 = -0.407$.

Using partial pivoting.

$$\begin{bmatrix} 3 & -7 & 639 & 12 \\ -2 & 5 & 803 & 7 \\ 56 & -41 & 79 & 10 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 56 & -41 & 79 & 10 \\ -2 & 5 & 803 & 7 \\ 3 & -7 & 639 & 12 \end{bmatrix}$$

$$\underset{\sim}{\substack{(1/28)R_1+R_2 \to R_2 \\ (-3/56)R_1+R_3 \to R_3}} \begin{bmatrix} 56 & -41 & 79 & 10 \\ 0 & 3.54 & 8.06 \times 10^2 & 7.36 \\ 0 & -4.8 & 6.35 \times 10^2 & 1.15 \times 10^1 \end{bmatrix}$$

$$\underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} 56 & -41 & 79 & 10 \\ 0 & -4.8 & 6.35 \times 10^2 & 1.15 \times 10^1 \\ 0 & 3.54 & 8.06 \times 10^2 & 7.36 \end{bmatrix}$$

$$\underset{\sim}{(3.54/4.8)R_2+R_3 \to R_3} \begin{bmatrix} 56 & -41 & 79 & 10 \\ 0 & -4.8 & 6.35 \times 10^2 & 1.15 \times 10^1 \\ 0 & 0 & 1.27 \times 10^3 & 1.58 \times 10^1 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{1.58 \times 10^1}{1.27 \times 10^3} = 1.24 \times 10^{-2}$. Row 2 $\Rightarrow$ $-4.8x_2 + 6.35 \times 10^2 \left(1.24 \times 10^{-2}\right) = 1.15 \times 10^1$ $\Rightarrow$ $x_2 = -0.755$. Row 1 $\Rightarrow$ $56x_1 - 41\,(-0.755) + 79\left(1.24 \times 10^{-2}\right) = 10$ $\Rightarrow$ $x_1 = -0.392$

8. Using Gaussian elimination with 3 significant digits of accuracy.

$$\begin{bmatrix} 2 & -5 & 802 & -1 \\ -1 & 3 & -789 & -8 \\ 40 & 34 & 51 & 19 \end{bmatrix} \underset{\sim}{\substack{(1/2)R_1+R_2 \to R_2 \\ (-20)R_1+R_3 \to R_3}} \begin{bmatrix} 2 & -5 & 802 & -1 \\ 0 & 0.5 & -3.88 \times 10^2 & -8.5 \\ 0 & 1.34 \times 10^2 & -1.60 \times 10^4 & 3.9 \times 10^1 \end{bmatrix}$$

$$\underset{\sim}{\frac{-1.34 \times 10^2}{0.5}R_2+R_3 \to R_3} \begin{bmatrix} 2 & -5 & 802 & -1 \\ 0 & 0.5 & -3.88 \times 10^2 & -8.5 \\ 0 & 0 & 8.80 \times 10^4 & 2.32 \times 10^3 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{2.32 \times 10^3}{8.80 \times 10^4} = 2.64 \times 10^{-2}$. Row 2 $\Rightarrow$ $0.5x_2 - 3.88 \times 10^2 \left(2.64 \times 10^{-2}\right) = -8.5$ $\Rightarrow$ $x_2 = 3.49$. Row 1 $\Rightarrow$ $2x_1 - 5\,(3.49) + 802\left(2.64 \times 10^{-2}\right) = -1$ $\Rightarrow$ $x_1 = -1.86$.
Using partial pivoting.

$$\begin{bmatrix} 2 & -5 & 802 & -1 \\ -1 & 3 & -789 & -8 \\ 40 & 34 & 51 & 19 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 40 & 34 & 51 & 19 \\ -1 & 3 & -789 & -8 \\ 2 & -5 & 802 & -1 \end{bmatrix}$$

$$\underset{\sim}{\substack{(1/40)R_1+R_2 \to R_2 \\ (-2/40)R_1+R_3 \to R_3}} \begin{bmatrix} 40 & 34 & 51 & 19 \\ 0 & 3.85 & -7.88 \times 10^2 & -7.53 \\ 0 & -6.7 & 7.99 \times 10^2 & -1.95 \end{bmatrix}$$

$$\underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} 40 & 34 & 51 & 19 \\ 0 & -6.7 & 7.99 \times 10^2 & -1.95 \\ 0 & 3.85 & -7.88 \times 10^2 & -7.53 \end{bmatrix}$$

$$\underset{\sim}{(3.85/6.7)R_2+R_3 \to R_3} \begin{bmatrix} 40 & 34 & 51 & 19 \\ 0 & -6.7 & 7.99 \times 10^2 & -1.95 \\ 0 & 0 & -3.29 \times 10^2 & -8.65 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{-8.65}{-3.29 \times 10^2} = 2.63 \times 10^{-2}$. Row 2 $\Rightarrow$ $-6.7x_2 + 7.99 \times 10^2 \left(2.63 \times 10^{-2}\right) = -1.95$ $\Rightarrow$ $x_2 = 3.43$. Row 1 $\Rightarrow$ $40x_1 + 34\,(3.43) + 51\left(2.63 \times 10^{-2}\right) = 19$ $\Rightarrow$ $x_1 = -2.47$.

9.

n	x_1	x_2
0	0	0
1	-1.2	0.2
2	-1.12	0.56
3	-0.976	0.536

Exact solution: $x_1 = -1$, $x_2 = 0.5$.

10.

n	x_1	x_2
0	0	0
1	-2	2.2
2	-0.9	0.6
3	-1.7	1.48

Exact solution: $x_1 = -1.5$, $x_2 = 1.0$.

11.

n	x_1	x_2	x_3
0	0	0	0
1	-1.3	2.3	2.6
2	-2.295	3.34	1.42
3	-2.156	3.185	0.805

Exact solution: $x_1 = -2$, $x_2 = 3$, $x_3 = 1$.

12.

n	x_1	x_2	x_3
0	0	0	0
1	-2.5	1.6	1.6
2	-1.7	1.42	3.06
3	-0.97	1.872	2.792

Exact solution: $x_1 = -1$, $x_2 = 2$, $x_3 = 3$.

13.

n	x_1	x_2
0	0	0
1	-1.2	0.56
2	-0.976	0.4928
3	-1.0029	0.5009

Exact solution: $x_1 = -1$, $x_2 = 0.5$.

14.

n	x_1	x_2
0	0	0
1	-2	0.6
2	-1.7	0.84
3	-1.58	0.936

Exact solution: $x_1 = -1.5$, $x_2 = 1.0$.

15.

n	x_1	x_2	x_3
0	0	0	0
1	-1.3	2.56	1.316
2	-2.013	3.0974	0.9584
3	-2.0042	2.9884	1.0038

Exact solution: $x_1 = -2$, $x_2 = 3$, $x_3 = 1$.

16.

n	x_1	x_2	x_3
0	0	0	0
1	-2.5	1.1	2.76
2	-1.12	1.928	2.98
3	-1.0096	1.9942	2.9985

Exact solution: $x_1 = -1$, $x_2 = 2$, $x_3 = 3$.

17. Not diagonally dominant. Not possible to reorder to obtain diagonal dominance.

18. Diagonally dominant, since $|4| > |2| + |-1|$, $|7| > |-2| + |2|$, and $|-5| > |1| + |3|$.

19. Not diagonally dominant. Not possible to reorder to obtain diagonal dominance, since none of the coefficients in equation three has absolute value greater than the sum of the absolute values of the other coefficients.

20. Not diagonally dominant. Interchange rows to obtain diagonal dominance.

$$\begin{aligned} 5x_1 - x_2 &= -4 \\ -2x_1 + 6x_2 &= 12 \end{aligned}$$

Then $|5| > |-1|$ and $|6| > |-2|$.

21. Jacobi iteration of given linear system:

n	x_1	x_2
0	0	0
1	-1	-1
2	-3	-3
3	-7	-7
4	-15	-15

Diagonally dominant system: $\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 1 \\ x_1 & - & 2x_2 & = & -1 \end{array}$

Jacobi iteration of diagonally dominant system:

n	x_1	x_2
0	0	0
1	0.5	0.5
2	0.75	0.75
3	0.875	0.875
4	0.9375	0.9375

22. Jacobi iteration of given linear system:

n	x_1	x_2
0	0	0
1	-2	-2
2	-8	-8
3	-26	-26
4	-80	-80

Diagonally dominant system: $\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 2 \\ x_1 & - & 3x_2 & = & -2 \end{array}$

Jacobi iteration of diagonally dominant system:

n	x_1	x_2
0	0	0
1	0.6667	0.6667
2	0.8889	0.8889
3	0.9630	0.9630
4	0.9877	0.9877

23. Jacobi iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	-1	8	-0.3333
2	16.67	12.33	27
3	-111.3	-21.33	29.67
4	-192	624	2.778

Diagonally dominant system: $\begin{array}{rcrcrcr} 5x_1 & + & x_2 & - & 2x_3 & = & 8 \\ 2x_1 & - & 10x_2 & + & 3x_3 & = & -1 \\ x_1 & - & 2x_2 & + & 5x_3 & = & -1 \end{array}$

Jacobi iteration of diagonally dominant system:

n	x_1	x_2	x_3
0	0	0	0
1	1.6	0.1	-0.2
2	1.5	0.36	-0.48
3	1.336	0.256	-0.356
4	1.406	0.2604	-0.3648

24. Jacobi iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	-1.5	-7	3
2	27.5	-8.5	-17.25
3	-70.75	58.25	-36.25
4	-299.3	-255.5	213.1

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Diagonally dominant system: $\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 7 \\ -x_1 & + & 6x_2 & - & 2x_3 & = & -6 \\ 2x_1 & + & 4x_2 & - & 10x_3 & = & -3 \end{array}$

Jacobi iteration of diagonally dominant system:

n	x_1	x_2	x_3
0	0	0	0
1	2.333	-1	0.3
2	1.9	-0.5111	0.3667
3	2.041	-0.5611	0.4756
4	1.988	-0.5014	0.4837

25. Gauss-Seidel iteration of given linear system:

n	x_1	x_2
0	0	0
1	-1	-3
2	-7	-15
3	-31	-63
4	-127	-255

.

Diagonally dominant system: $\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 1 \\ x_1 & - & 2x_2 & = & -1 \end{array}$

Gauss-Seidel iteration of diagonally dominant system:

n	x_1	x_2
0	0	0
1	0.5	0.75
2	0.875	0.9375
3	0.9688	0.9844
4	0.9922	0.9961

26. Gauss-Seidel iteration of given linear system:

n	x_1	x_2
0	0	0
1	-2	-8
2	-26	-80
3	-242	-728
4	-2186	-6560

.

Diagonally dominant system: $\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 2 \\ x_1 & - & 3x_2 & = & -2 \end{array}$

Gauss-Seidel iteration of diagonally dominant system:

n	x_1	x_2
0	0	0
1	0.6667	0.8889
2	0.9630	0.9877
3	0.9959	0.9986
4	0.9995	0.9999

27. Gauss-Seidel iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	-1	13	43.67
2	-193.3	1062	3669
3	-1.622×10^4	8.844×10^4	3.056×10^5
4	-1.351×10^6	7.367×10^6	2.546×10^7

.

Diagonally dominant system: $\begin{array}{rcrcrcr} 5x_1 & + & x_2 & - & 2x_3 & = & 8 \\ 2x_1 & - & 10x_2 & + & 3x_3 & = & -1 \\ x_1 & - & 2x_2 & + & 5x_3 & = & -1 \end{array}$

Gauss-Seidel iteration of diagonally dominant system:

n	x_1	x_2	x_3
0	0	0	0
1	1.6	0.42	-0.352
2	1.375	0.2694	-0.3673
3	1.399	0.2697	-0.3720
4	1.397	0.2679	-0.3723

28. Jacobi iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	-1.5	-11.5	-30.75
2	-132.3	-434.5	-1.234×10^3
3	-5.304×10^3	-1.715×10^4	-4.881×10^3
4	-2.097×10^5	-6.780×10^5	-1.929×10^6

.

Diagonally dominant system:
$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 7 \\ -x_1 & + & 6x_2 & - & 2x_3 & = & -6 \\ 2x_1 & + & 4x_2 & - & 10x_3 & = & -3 \end{array}$$

Jacobi iteration of diagonally dominant system:

n	x_1	x_2	x_3
0	0	0	0
1	2.333	-0.6111	0.5222
2	1.955	-0.5000	0.4911
3	2.003	-0.5025	0.4996
4	1.999	-0.5003	0.4998

29. Let $x_i(n)$ be the value of the n^{th} iteration of x_i. Then we have $x_1(n+1) = b_1 - a_1 x_2(n)$. Applying this with $n = 0$ and $n = 1$, we obtain the 2 equations

$$\begin{aligned} 1 &= b_1 - a_1(0) \\ 5 &= b_1 - a_1(-2) \end{aligned}$$

Solve this system for the quantities $b_1$and a_1 to obtain $b_1 = 1$ and $a_1 = 2$. Thus $x_1(n+1) = 1 - 2x_2(n)$, and hence $x_1(3) = 1 - 2(2) = -3$. Similarly we have $x_2(n+1) = b_2 - a_2 x_1(n)$, and so with $n = 0$ and $n = 1$, we obtain the 2 equations

$$\begin{aligned} -2 &= b_2 - a_2\,(0) \\ 2 &= b_2 - a_2\,(1) \end{aligned}$$

Solve this system for the quantities $b_2$and a_2 to obtain $b_2 = -2$ and $a_2 = -4$. Thus $x_2(n+1) = -2 + 4x_1(n)$, and hence $x_2(3) = -2 + 4(5) = 18$.

30. Let $x_i(n)$ be the value of the n^{th} iteration of x_i. Then we have $x_1(n+1) = b_1 - a_{11}x_2(n) - a_{12}x_3(n)$. Applying this with $n = 0$, 1, and 2 to obtain the 3 equations

$$\begin{aligned} -2 &= b_1 - a_{11}(0) - a_{12}(0) \\ -4 &= b_1 - a_{11}(-1) - a_{12}(1) \\ -11 &= b_1 - a_{11}(-4) - a_{12}(5) \end{aligned}$$

Solve this system for the quantities b_1, a_{11},and a_{12} to obtain $b_1 = -2$, $a_{11} = -1$ and $a_{12} = 1$. Thus $x_1(n+1) = -2 + x_2(n) - x_3(n)$, and hence $x_1(4) = -2 + (-4) - (5) = -11$.
Similarly we have $x_2(n+1) = b_2 - a_{21}x_1(n) - a_{22}x_3(n)$, and so with $n = 0$, 1, and 2 we obtain the 3 equations

$$\begin{aligned} -1 &= b_2 - a_{21}(0) - a_{22}(0) \\ -4 &= b_2 - a_{21}(-2) - a_{22}(1) \\ -4 &= b_2 - a_{21}(-4) - a_{22}(5) \end{aligned}$$

Solve this system for the quantities b_2, a_{21},and a_{22} to obtain $b_2 = -1$, $a_{21} = -2$ and $a_{22} = -1$. Thus $x_2(n+1) = -1 + 2x_1(n) + x_3(n)$, , and hence $x_2(4) = -1 + 2(-11) + (5) = -18$.
Finally we have $x_3(n+1) = b_3 - a_{31}x_1(n) - a_{32}x_2(n)$, and so with $n = 1$, 2, and 3 we obtain the 3 equations

$$\begin{aligned} 1 &= b_3 - a_{31}(0) - a_{32}(0) \\ 5 &= b_3 - a_{31}(-2) - a_{32}(-1) \\ 5 &= b_3 - a_{31}(-4) - a_{32}(-4) \end{aligned}$$

Solve this system for the quantities b_3, a_{31},and a_{32} to obtain $b_3 = 1$, $a_{31} = 3$ and $a_{32} = -2$. Thus $x_3(n+1) = 1 - 3x_1(n) + 2x_3(n)$, and hence $x_3(4) = 1 - 3(-11) + 2(-4) = 26$.

31. Let $x_i(n)$ be the value of the n^{th} iteration of x_i. Then we have $x_1(n+1) = b_1 - a_1 x_2(n)$. Applying this with $n = 0$ and $n = 1$, we obtain the 2 equations

$$\begin{aligned} 3 &= b_1 - a_1(0) \\ -5 &= b_1 - a_1(4) \end{aligned}$$

Solve this system for the quantities $b_1$and a_1 to obtain $b_1 = 3$ and $a_1 = 2$. Thus $x_1(n+1) = 3-2x_2(n)$, and hence $x_1(3) = 3 - 2(-12) = 27$. With Gauss-Seidel iteration we have $x_2(n) = b_2 - a_2 x_1(n)$, and so using $n = 1$ and $n = 2$, we obtain the 2 equations

$$\begin{aligned} 4 &= b_2 - a_2\,(3) \\ -12 &= b_2 - a_2\,(-5) \end{aligned}$$

Solve this system for the quantities $b_2$and a_2 to obtain $b_2 = -2$ and $a_2 = -2$. Thus $x_2(n) = -2+2x_1(n)$, and hence $x_2(3) = -2 + 2(27) = 52$.

32. Let $x_i(n)$ be the value of the n^{th} iteration of x_i. Then we have $x_1(n+1) = b_1 - a_{11}x_2(n) - a_{12}x_3(n)$. Applying this with $n = 0$, 1, and 2 to obtain the 3 equations

$$\begin{aligned} 3 &= b_1 - a_{11}(0) - a_{12}(0) \\ 7 &= b_1 - a_{11}(4) - a_{12}(12) \\ -25 &= b_1 - a_{11}(-24) - a_{12}(-76) \end{aligned}$$

Solve this system for the quantities b_1, a_{11},and a_{12} to obtain $b_1 = 3$, $a_{11} = 2$ and $a_{12} = -1$. Thus $x_1(n+1) = 3 - 2x_2(n) + x_3(n)$, and hence $x_1(4) = 3 - 2(176) + (556) = 207$.
With Gauss-Seidel iteration we have $x_2(n+1) = b_2 - a_{21}x_1(n+1) - a_{22}x_3(n)$, and so with $n = 0$, 1, and 2 we obtain the 3 equations

$$\begin{aligned} 4 &= b_2 - a_{21}(3) - a_{22}(0) \\ -24 &= b_2 - a_{21}(7) - a_{22}(12) \\ 176 &= b_2 - a_{21}(-25) - a_{22}(-76) \end{aligned}$$

Solve this system for the quantities b_2, a_{21},and a_{22} to obtain $b_2 = -2$, $a_{21} = -2$ and $a_{22} = 3$. Thus $x_2(n+1) = -2 + 2x_1(n+1) - 3x_3(n)$, , and hence $x_2(4) = -2 + 2(207) - 3(556) = -1256$.
Finally with Gauss-Seidel iteration we have $x_3(n) = b_3 - a_{31}x_1(n) - a_{32}x_2(n)$, and so with $n = 1$, 2, and 3 we obtain the 3 equations

$$\begin{aligned} 12 &= b_3 - a_{31}(4) - a_{32}(3) \\ -76 &= b_3 - a_{31}(-24) - a_{32}(7) \\ 556 &= b_3 - a_{31}(176) - a_{32}(-25) \end{aligned}$$

Solve this system for the quantities b_3, a_{31},and a_{32} to obtain $b_3 = 3$, $a_{31} = -3$ and $a_{32} = 1$. Thus $x_3(n) = 3 + 3x_1(n) - x_3(n)$, and hence $x_3(4) = 3 + 3(207) - (-1256) = 1880$.

Chapter 1 Supplementary Exercises

1. $2x_1 - 5x_2 = 1 \Rightarrow x_2 = \frac{2}{5}x_1 - \frac{1}{5}$. Substitute into the second equation to obtain $-6x_1 + 7\left(\frac{2}{5}x_1 - \frac{1}{5}\right) = 3 \Rightarrow x_1 = -\frac{11}{8}$. Thus, $x_2 = \frac{2}{5}\left(-\frac{11}{8}\right) - \frac{1}{5} = -\frac{3}{4}$.

2. $3x_1 - x_2 = 2 \Rightarrow x_2 = 3x_1 - 2$. Substitute into the second equation to obtain $-5x_1 + 2\,(3x_1 - 2) = -1 \Rightarrow x_1 = 3$. Thus, $x_2 = 3\,(3) - 2 = 7$.

3. $x_1 - 4x_2 = 1 \Rightarrow x_2 = \frac{1}{4}x_1 - \frac{1}{4}$. Substitute into the second equation to obtain $-2x_1 + 8\left(\frac{1}{4}x_1 - \frac{1}{4}\right) = -2 \Rightarrow 0 = 0$. Thus, $x_1 = s$, and $x_2 = \frac{1}{4}s - \frac{1}{4}$.

4. $4x_1 - 2x_2 = 6 \Rightarrow x_2 = 2x_1 - 3$. Substitute into the second equation to obtain $6x_1 - 3(2x_1 - 3) = 9 \Rightarrow 0 = 0$. Thus, $x_1 = s$, and $x_2 = 2s - 3$.

5. $x_1 - 3x_2 = 5 \Rightarrow x_2 = \frac{1}{3}x_1 - \frac{5}{3}$. Substitute into the second equation to obtain $3x_1 - 9\left(\frac{1}{3}x_1 - \frac{5}{3}\right) = 7 \Rightarrow 15 = 7$. Thus, the system has no solutions.

6. $-6x_1 + 2x_2 = 3 \Rightarrow x_2 = 3x_1 + \frac{1}{2}$. Substitute into the second equation to obtain $15x_1 - 5\left(3x_1 + \frac{1}{2}\right) = 4 \Rightarrow 0 = \frac{13}{2}$. Thus, the system has no solutions.

7. Equation 3 $\Rightarrow x_3 = 3$. Substitute into equation 2, $-x_2 + 3(3) = -2 \Rightarrow x_2 = 11$. Substitute into equation 1, $x_1 + 2(11) - 4(3) = 0 \Rightarrow x_1 = -10$.

8. x_3 is a free variable, so let $x_3 = s$. Substitute into equation 2, $2x_2 - 6s = 4 \Rightarrow x_2 = 2 + 3s$. Substitute into equation 1, $x_1 - 4(s) = 3 \Rightarrow x_1 = 3 + 4s$.

9. x_2 and x_3 are free variables, so let $x_2 = s_1$ and $x_3 = s_2$. Substitute to obtain $-x_1 - 5s_1 + s_2 = -2 \Rightarrow x_1 = -5s_1 + s_2 + 2$.

10. x_2, x_3, and x_4 are free variables, so let $x_2 = s_1$, $x_3 = s_2$, and $x_4 = s_3$. Substitute to obtain $x_1 + 2s_1 + 4s_2 - s_3 = -2 \Rightarrow x_1 = -2s_1 - 4s_2 + s_3 - 2$.

11. x_3 is a free variable, so let $x_3 = s$. Equation 3 $\Rightarrow x_4 = -5$. Substitute into equation 2, $-x_2 + 4(-5) = 0 \Rightarrow x_2 = 20$. Substitute into equation 1, $-x_1 - 2(20) + 7(s) - 3(-5) = 7 \Rightarrow x_1 = 7s - 32$.

12. x_2 and x_4 are free variables, so let $x_2 = s_1$ and $x_3 = s_4$. Substitute to obtain $-x_1 - 5s_1 + s_2 = -2 \Rightarrow x_1 = -5s_1 - s_2 + 2$.

13. x_3 is a free variable, so let $x_3 = s$. Equation 4 $\Rightarrow x_5 = 4$. Substitute into equation 3, $-2x_4 - (4) = 0 \Rightarrow x_4 = -2$. Substitute into equation 2, $x_2 - 4(s) + 2(4) = -3 \Rightarrow x_2 = 4s - 11$. Substitute into equation 1, $x_1 + (4s - 11) + 3(s) - (-2) + (4) = 7 \Rightarrow x_1 = -7s + 12$.

14. x_3, x_4, and x_5 are free variables, so let $x_3 = s_1$, $x_4 = s_2$, and $x_5 = s_3$. Substitute into equation 2, $x_2 - (s_1) + (s_2) = 2 \Rightarrow x_2 = s_1 - s_2 + 2$. Substitute into equation 1, $2x_1 + 4(s_1) + 3(s_3) = -1 \Rightarrow x_1 = -2s_1 - \frac{3}{2}s_3 - \frac{1}{2}$.

15. $$\begin{array}{rcrcrcr} 2x_1 & - & 4x_2 & + & 3x_3 & = & 1 \\ -3x_1 & + & 5x_2 & + & 11x_3 & = & 0 \end{array}$$

16. $$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & 7 \\ & & 3x_2 & & & - & 2x_4 & = & 6 \end{array}$$

17. $$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & 5x_3 & = & 1 \\ 7x_1 & - & 2x_2 & & & = & 1 \\ 3x_1 & + & x_2 & + & 2x_3 & = & -4 \end{array}$$

18. $$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & = & 11 \\ 2x_1 & & & - & 5x_3 & = & 0 \\ & & 4x_2 & + & 4x_3 & = & -2 \\ 3x_1 & + & 2x_2 & + & 2x_3 & = & 1 \end{array}$$

19. $$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -6 & 5 & 5 \\ -1 & 6 & -7 & 3 \end{bmatrix} \underset{R_1+R_3 \to R_3}{\overset{-2R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & -2 & 3 & -1 \\ 0 & 4 & -6 & 6 \end{bmatrix}$$
$$\overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

20. $$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 2 & -5 & 6 & -1 \\ -1 & -8 & 6 & 6 \end{bmatrix} \underset{\sim}{\substack{2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}} \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & -9 & 8 & 5 \end{bmatrix}$$
$$\underset{\sim}{-3R_2+R_3\to R_3} \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

21. $$\begin{bmatrix} -3 & 2 & -2 & 0 & 2 \\ -9 & 6 & -3 & 2 & 5 \\ 6 & -4 & 10 & 6 & -7 \end{bmatrix} \underset{\sim}{\substack{-3R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}} \begin{bmatrix} -3 & 2 & -2 & 0 & 2 \\ 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 6 & 6 & -3 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_3\to R_3} \begin{bmatrix} -3 & 2 & -2 & 0 & 2 \\ 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix}$$

22. $$\begin{bmatrix} 2 & -8 & 4 & 2 & 1 \\ 1 & -3 & 0 & 2 & 2 \\ -1 & 2 & 2 & -4 & 5 \\ -3 & 11 & -4 & 2 & 2 \end{bmatrix} \underset{\sim}{\substack{-\frac{1}{2}R_1+R_2\to R_2 \\ \frac{1}{2}R_1+R_3\to R_3 \\ \frac{3}{2}R_1+R_4\to R_4}} \begin{bmatrix} 2 & -8 & 4 & 2 & 1 \\ 0 & 1 & -2 & 1 & \frac{3}{2} \\ 0 & -2 & 4 & -3 & \frac{11}{2} \\ 0 & -1 & 2 & 5 & \frac{7}{2} \end{bmatrix}$$
$$\underset{\sim}{\substack{2R_2+R_3\to R_3 \\ R_2+R_4\to R_4}} \begin{bmatrix} 2 & -8 & 4 & 2 & 1 \\ 0 & 1 & -2 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & -1 & \frac{17}{2} \\ 0 & 0 & 0 & 6 & 5 \end{bmatrix}$$
$$\underset{\sim}{6R_3+R_4\to R_4} \begin{bmatrix} 2 & -8 & 4 & 2 & 1 \\ 0 & 1 & -2 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & -1 & \frac{17}{2} \\ 0 & 0 & 0 & 0 & 56 \end{bmatrix}$$

23. $$\begin{bmatrix} 1 & -3 & 4 \\ -2 & 6 & -7 \\ 2 & -6 & 6 \end{bmatrix} \underset{\sim}{\substack{2R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$
$$\underset{\sim}{2R_2+R_3\to R_2} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\underset{\sim}{-4R_2+R_1\to R_1} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

24. $$\begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -3 & 4 & 3 \\ -3 & -4 & 5 & 5 \end{bmatrix} \underset{\sim}{\substack{2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}} \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_3\to R_2} \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\underset{\sim}{-2R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

25. $\begin{bmatrix} 1 & 2 & -3 & 2 & 12 \\ 1 & 3 & -5 & 1 & 12 \\ 2 & 2 & -2 & 4 & 18 \end{bmatrix} \underset{\substack{-R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 2 & -3 & 2 & 12 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & -2 & 4 & 0 & -6 \end{bmatrix}$

$\underset{2R_2+R_3\to R_2}{\sim} \begin{bmatrix} 1 & 2 & -3 & 2 & 12 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & -2 & -6 \end{bmatrix}$

$\underset{-\frac{1}{2}R_3\to R_3}{\sim} \begin{bmatrix} 1 & 2 & -3 & 2 & 12 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

$\underset{\substack{R_3+R_2\to R_2 \\ -2R_3+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & 2 & -3 & 0 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

$\underset{-2R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

26. $\begin{bmatrix} 1 & -2 & 7 & 1 & 3 \\ 3 & -5 & 19 & 1 & 2 \\ -2 & 6 & -18 & -5 & -17 \\ 1 & 0 & 3 & -3 & -10 \end{bmatrix} \underset{\substack{-3R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -2 & 7 & 1 & 3 \\ 0 & 1 & -2 & -2 & -7 \\ 0 & 2 & -4 & -3 & -11 \\ 0 & 2 & -4 & -4 & -13 \end{bmatrix}$

$\underset{\substack{-2R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -2 & 7 & 1 & 3 \\ 0 & 1 & -2 & -2 & -7 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\underset{\substack{-3R_4+R_3\to R_3 \\ 7R_4+R_2\to R_2 \\ -3R_4+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & -2 & 7 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\underset{\substack{2R_3+R_2\to R_2 \\ -R_3+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & -2 & 7 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\underset{2R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \underset{\frac{1}{2}R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & \frac{5}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Free variable, $x_3 = s$. Row 2 $\Rightarrow$ $\frac{5}{2}x_2 - \frac{1}{2}s = \frac{1}{2}$ $\Rightarrow$ $x_2 = \frac{1}{5}s + \frac{1}{5}$. Row 1 $\Rightarrow$ $2x_1 - \left(\frac{1}{5}s + \frac{1}{5}\right) + (s) = -1$ $\Rightarrow$ $x_1 = -\frac{2}{5}s - \frac{2}{5}$.

28. $\begin{bmatrix} 1 & -3 & 4 & 1 \\ -2 & 5 & -7 & 1 \\ 1 & -5 & 8 & 5 \end{bmatrix} \underset{\substack{2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & -2 & 4 & 4 \end{bmatrix}$

$\underset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

Row 3 $\Rightarrow$ $x_3 = -1$. Row 2 $\Rightarrow$ $-x_2 + (-1) = 3$ $\Rightarrow$ $x_2 = -4$. Row 1 $\Rightarrow$ $x_1 - 3(-4) + 4(-1) = 1$ $\Rightarrow$ $x_1 = -7$.

29. $\begin{bmatrix} 1 & -3 & 1 & 2 & 2 \\ -1 & 4 & -4 & -1 & -4 \\ 2 & -3 & -7 & 8 & -5 \end{bmatrix} \overset{\substack{R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & -3 & 1 & 2 & 2 \\ 0 & 1 & -3 & 1 & -2 \\ 0 & 3 & -9 & 4 & -9 \end{bmatrix}$

$\overset{-3R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & -3 & 1 & 2 & 2 \\ 0 & 1 & -3 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$

Free variable, $x_3 = s$. Row 3 $\Rightarrow$ $x_4 = -3$. Row 2 $\Rightarrow$ $x_2 - 3(s) + (-3) = -2 \Rightarrow x_2 = 3s + 1$. Row 1 $\Rightarrow$ $x_1 - 3(3s+1) + (s) + 2(-3) = 2 \Rightarrow x_1 = 8s + 11$.

30. $\begin{bmatrix} 2 & 4 & 9 & -5 & 2 & -5 \\ 1 & 2 & 4 & -1 & 2 & -1 \\ -3 & -6 & -14 & 9 & -3 & 14 \end{bmatrix} \overset{\substack{-\frac{1}{2}R_1+R_2 \to R_2 \\ \frac{3}{2}R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 2 & 4 & 9 & -5 & 2 & -5 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & \frac{13}{2} \end{bmatrix}$

$\overset{-R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 2 & 4 & 9 & -5 & 2 & -5 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & -1 & 5 \end{bmatrix}$

Free variables, $x_2 = s_1$ and $x_4 = s_2$. Row 3 $\Rightarrow$ $x_5 = -5$. Row 2 $\Rightarrow$ $-\frac{1}{2}x_3 + \frac{3}{2}(s_2) + 1(-5) = \frac{3}{2} \Rightarrow$ $x_3 = 3s_2 - 13$. Row 1 $\Rightarrow$ $2x_1 + 4(s_1) + 9(3s_2 - 13) - 5(s_2) + 2(-5) = -5 \Rightarrow x_1 = -2s_1 - 11s_2 + 61$.

31. We obtain the system of equations

$$\begin{aligned} x_1 &= \frac{x_2 + 80 + 120}{3} \\ x_2 &= \frac{x_1 + 60 + 30}{3} \end{aligned}$$

which reduces to

$$\begin{aligned} 3x_1 - x_2 &= 200 \\ -x_1 + 3x_2 &= 90 \end{aligned}$$

We solve this system, and obtain $x_1 = \frac{345}{4}$, and $x_2 = \frac{235}{4}$.

32. We obtain the system of equations

$$\begin{aligned} x_1 &= \frac{x_2 + x_3 + 80}{3} \\ x_2 &= \frac{x_1 + x_3 + 130}{3} \\ x_3 &= \frac{x_1 + x_2 + 50}{3} \end{aligned}$$

which reduces to

$$\begin{aligned} 3x_1 - x_2 - x_3 &= 80 \\ -x_1 + 3x_2 - x_3 &= 130 \\ -x_1 - x_2 + 3x_3 &= 50 \end{aligned}$$

We solve this system, and obtain $x_1 = 85$, $x_2 = \frac{195}{2}$, and $x_3 = \frac{155}{2}$.

33. Let a and b denote the total output from each of A and B, respectively. We obtain

$$\begin{aligned} a &= 50 + 0.50b \\ b &= 20 + 0.30a \end{aligned}$$

We solve this system, and obtain $a = 70.59$ and $b = 41.18$.

34. Let a, b, and c denote the total output from each of A, B, and C, respectively. We obtain

$$\begin{aligned} a &= 40 + 0.25b + 0.10c \\ b &= 70 + 0.20a + 0.30c \\ c &= 90 + 0.25a + 0.30b \end{aligned}$$

We solve this system, and obtain $a = 88.55$, and $b = 133.35$, and $c = 152.14$.

35. We consider $x_1 C_6H_{12}O_6 \longrightarrow x_2 C_2H_5OH + x_3 CO_2$, which implies

$$\begin{array}{rcrcrcl} 6x_1 & - & 2x_2 & - & x_3 & = & 0 \\ 12x_1 & - & 6x_2 & & & = & 0 \\ 6x_1 & - & x_2 & - & 2x_3 & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix} \underset{\sim}{\substack{-2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}} \begin{bmatrix} 6 & -2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\underset{\sim}{\tfrac{1}{2}R_2+R_3\to R_3} \begin{bmatrix} 6 & -2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We set $x_3 = s$ as a free variable. From row 2, $-2x_2 + 2s = 0 \Rightarrow x_2 = s$. From row 1, $6x_1 - 2(s) - (s) = 0 \Rightarrow x_1 = \frac{1}{2}s$. We set $s = 2$ to obtain $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, and the balanced equation

$$C_6H_{12}O_6 \longrightarrow 2C_2H_5OH + 2CO_2$$

36. We consider $x_1 HC_2H_3O_2 + x_2 NaHCO_3 \longrightarrow x_3 NaC_2H_3O_2 + x_4 H_2O + x_5 CO_2$, which implies

$$\begin{array}{rcrcrcrcrcl} 4x_1 & + & x_2 & - & 3x_3 & - & 2x_4 & & & = & 0 \\ 2x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ 2x_1 & + & 3x_2 & - & 2x_3 & - & x_4 & - & 2x_5 & = & 0 \\ & & x_2 & - & x_3 & & & & & = & 0 \end{array}$$

Row-reduce the augmented matrix

$$\begin{bmatrix} 4 & 1 & -3 & -2 & 0 & 0 \\ 2 & 1 & -2 & 0 & -1 & 0 \\ 2 & 3 & -2 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\substack{-\frac{1}{2}R_1+R_2\to R_2 \\ -\frac{1}{2}R_1+R_3\to R_3}} \begin{bmatrix} 4 & 1 & -3 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & -1 & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{\substack{-5R_2+R_3\to R_3 \\ -2R_2+R_4\to R_4}} \begin{bmatrix} 4 & 1 & -3 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & -1 & 0 \\ 0 & 0 & 2 & -5 & 3 & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 \end{bmatrix}$$

We set $x_5 = s$ as a free variable. From row 4, $-2x_4 + 2s = 0 \Rightarrow x_4 = s$. From row 3, $2x_3 - 5(s) + 3(s) = 0 \Rightarrow x_3 = s$. From row 2, $\frac{1}{2}x_2 - \frac{1}{2}(s) + (s) - (s) = 0 \Rightarrow x_2 = s$. From row 1, $4x_1 + (s) - 3(s) - 2(s) = 0 \Rightarrow x_1 = s$.We set $s = 1$ to obtain $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 1$, and the balanced equation

$$HC_2H_3O_2 + NaHCO_3 \longrightarrow NaC_2H_3O_2 + H_2O + CO_2$$

37. Using Gaussian elimination with 3 significant digits of accuracy.

$$\begin{bmatrix} 3 & 819 & 37 \\ 48 & -91 & -12 \end{bmatrix} \underset{\sim}{-16R_1+R_2\to R_2} \begin{bmatrix} 3 & 819 & 37 \\ 0 & -13{,}200 & -604 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{-604}{-13{,}200} = 4.58 \times 10^{-2}$. Row 1 $\Rightarrow$ $3x_1 + 819\left(4.58 \times 10^{-2}\right) = 37 \;\Rightarrow\; x_1 = -0.170$.
Using partial pivoting:

$$\begin{bmatrix} 3 & 819 & 37 \\ 48 & -91 & -12 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 48 & -91 & -12 \\ 3 & 819 & 37 \end{bmatrix}$$
$$\underset{\sim}{(-1/16)R_1 + R_2 \to R_2} \begin{bmatrix} 48 & -91 & -12 \\ 0 & 825 & 37.8 \end{bmatrix}$$

Row 2 $\Rightarrow$ $x_2 = \frac{37.8}{825} = 4.58 \times 10^{-2}$. Row 1 $\Rightarrow$ $48x_1 - 91\left(4.58 \times 10^{-2}\right) = -12 \;\Rightarrow\; x_1 = -0.163$

38. Using Gaussian elimination with 3 significant digits of accuracy:

$$\begin{bmatrix} 1 & -6 & 745 & 17 \\ -3 & 4 & 902 & 8 \\ 49 & -39 & 81 & 10 \end{bmatrix} \underset{\sim}{\begin{matrix} 3R_1 + R_2 \to R_2 \\ -49R_1 + R_3 \to R_3 \end{matrix}}$$
$$\begin{bmatrix} 1 & -6 & 745 & 17 \\ 0 & -14 & 3140 & 59 \\ 0 & 255 & -36,400 & -823 \end{bmatrix} \underset{\sim}{(255/14)R_2 + R_3 \to R_3}$$
$$\begin{bmatrix} 1 & -6 & 745 & 17 \\ 0 & -14 & 3140 & 59 \\ 0 & 0 & 20,800 & 252 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{252}{20{,}800} = 1.21 \times 10^{-2}$. Row 2 $\Rightarrow$ $-14\,x_2 + 3140\left(1.21 \times 10^{-2}\right) = 59 \;\Rightarrow\; x_2 = -1.50$.
Row 1 $\Rightarrow$ $x_1 - 6\,(-1.50) + 745\left(1.21 \times 10^{-2}\right) = 17 \;\Rightarrow\; x_1 = -1.01$.
Using partial pivoting.

$$\begin{bmatrix} 1 & -6 & 745 & 17 \\ -3 & 4 & 902 & 8 \\ 49 & -39 & 81 & 10 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_3} \begin{bmatrix} 49 & -39 & 81 & 10 \\ -3 & 4 & 902 & 8 \\ 1 & -6 & 745 & 17 \end{bmatrix}$$
$$\underset{\sim}{\begin{matrix} (3/49)R_1 + R_2 \to R_2 \\ (-1/49)R_1 + R_3 \to R_3 \end{matrix}} \begin{bmatrix} 49 & -39 & 81 & 10 \\ 0 & 1.61 & 907 & 8.61 \\ 0 & -5.20 & 743 & 16.8 \end{bmatrix}$$
$$\underset{\sim}{R_2 \leftrightarrow R_3} \begin{bmatrix} 49 & -39 & 81 & 10 \\ 0 & -5.20 & 743 & 16.8 \\ 0 & 1.61 & 907 & 8.61 \end{bmatrix}$$
$$\underset{\sim}{(1.61/5.20)R_2 + R_3 \to R_3} \begin{bmatrix} 49 & -39 & 81 & 10 \\ 0 & -5.20 & 743 & 16.8 \\ 0 & 0 & 1140 & 13.8 \end{bmatrix}$$

Row 3 $\Rightarrow$ $x_3 = \frac{13.8}{1140} = 1.21 \times 10^{-2}$. Row 2 $\Rightarrow$ $-5.20x_2 + 743\left(1.21 \times 10^{-2}\right) = 16.8 \;\Rightarrow\; x_2 = -1.50$. Row 1 $\Rightarrow$ $49x_1 - 39\,(-1.50) + 81\left(1.21 \times 10^{-2}\right) = 10 \;\Rightarrow\; x_1 = -1.01$

39.

n	x_1	x_2
0	0	0
1	-0.8	0.6
2	-0.62	0.92
3	-0.524	0.848

Exact solution: $x_1 = -31/56 \approx -0.554$, $x_2 = 23/28 \approx 0.821$.

40.

n	x_1	x_2	x_3
0	0	0	0
1	2.90	-1.05	2.20
2	2.45	-1.85	3.57
3	2.20	-1.55	3.54

Exact solution: $x_1 = 343/157 \approx 2.18$, $x_2 = -232/157 \approx -1.48$, $x_3 = 529/157 \approx 3.37$.

41. Gauss–Seidel iteration of given linear system:

n	x_1	x_2
0	0	0
1	-0.8	0.92
2	-0.524	0.810
3	-0.557	0.823

Exact solution: $x_1 = -31/56 \approx -0.554$, $x_2 = 23/28 \approx 0.821$.

42. Gauss–Seidel iteration of given linear system:

n	x_1	x_2	x_3
0	0	0	0
1	2.9	-2.07	3.77
2	2.18	-1.44	3.36
3	2.18	-1.48	3.37

Exact solution: $x_1 = 343/157 \approx 2.18$, $x_2 = -232/157 \approx -1.48$, $x_3 = 529/157 \approx 3.37$.