Solutions Manual to
Optoelectronics and Photonics: Principles and Practices, Second Edition
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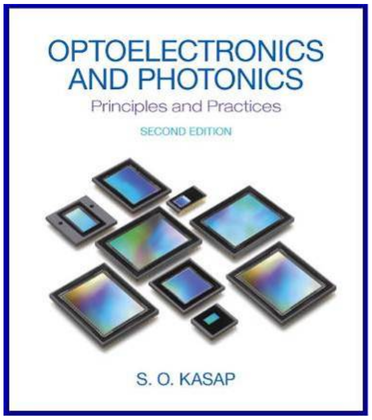

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Preliminary Solutions to Problems and Questions
Chapter 1

Note: Printing errors and corrections are indicated in dark red. See Question 1.47. These are correct in the e-version of the textbook

1.1 Maxwell's wave equation and plane waves

(a) Consider a traveling sinusoidal wave of the form $E_x = E_o \cos(\omega t - kz + \phi_o)$. The latter can also be written as $E_x = E_o \cos[k(vt - z) + \phi_o]$, where $v = \omega/k$ is the velocity. Show that this wave satisfies Maxwell's wave equation, and show that $v = (\mu_o\varepsilon_o\varepsilon_r)^{-1/2}$.

(b) Consider a traveling function of any shape, even a very short delta pulse, of the form $E_x = f[k(vt - z)]$, where f is any function, which can be written is $E_x = f(\phi)$, $\phi = k(vt - z)$. Show that this traveling function satisfies Maxwell's wave equation. What is its velocity? What determines the form of the function f?

Solution

(a)

$$E_x = E_o \cos(\omega t - kz + \phi_o)$$

∴ $$\frac{\partial^2 E_x}{\partial x^2} = 0$$

and $$\frac{\partial^2 E_x}{\partial y^2} = 0$$

and $$\frac{\partial^2 E_x}{\partial z^2} = -k^2 E_0 \cos(\omega t - kz + \phi_0)$$

∴ $$\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_0 \cos(\omega t - kz + \phi_0)$$

Substitute these into the wave equation $\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \varepsilon_o\varepsilon_r\mu_o \frac{\partial^2 E}{\partial t^2} = 0$ to find

$$-k^2 E_0 \cos(\omega t - kz + \phi_0) + \varepsilon_o\varepsilon_r\mu_o + \omega^2 E_0 \cos(\omega t - kz + \phi_0) = 0$$

∴ $$\frac{\omega^2}{k^2} = \frac{1}{\varepsilon_o\varepsilon_r\mu_o}$$

∴ $$\frac{\omega}{k} = (\varepsilon_o\varepsilon_r\mu_o)^{-1/2}$$

∴ $$v = (\varepsilon_o\varepsilon_r\mu_o)^{-1/2}$$

(b) Let

$$E_x = f[k(vt - z)] = f(\phi)$$

Take first and second derivatives with respect to x, y, z and t.

$$\frac{\partial^2 E_x}{\partial x^2} = 0$$

$$\frac{\partial^2 E_x}{\partial y^2} = 0$$

$$\frac{\partial E_x}{\partial z} = -k\frac{df}{d\phi}$$

$$\frac{\partial^2 E_x}{\partial z^2} = k^2\frac{d^2 f}{d\phi^2}$$

$$\frac{\partial E_x}{\partial t} = k\mathrm{v}\frac{df}{d\phi}$$

$$\frac{\partial^2 E_x}{\partial t^2} = k^2\mathrm{v}^2\frac{d^2 f}{d\phi^2}$$

Substitute these into the wave equation $\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \varepsilon_o\varepsilon_r\mu_o\frac{\partial^2 E}{\partial t^2} = 0$ to find

$$k^2\frac{d^2 f}{d\phi^2} - \varepsilon_o\varepsilon_r\mu_o k^2\mathrm{v}^2\frac{d^2 f}{d\phi^2} = 0$$

$\therefore$ $$\mathrm{v}^2 = \frac{1}{\varepsilon_o\varepsilon_r\mu_o}$$

$\therefore$ $$\mathrm{v} = (\varepsilon_o\varepsilon_r\mu_o)^{-1/2}$$

1.2 Propagation in a medium of finite small conductivity An electromagnetic wave in an isotropic medium with a dielectric constant ε_r and a finite conductivity σ and traveling along z obeys the following equation for the variation of the electric field E perpendicular to z,

$$\frac{d^2 E}{dz^2} - \varepsilon_o\varepsilon_r\mu_o\frac{\partial^2 E}{\partial t^2} = \mu_o\sigma\frac{\partial E}{\partial t} \qquad (1)$$

Show that one possible solution is a plane wave whose amplitude decays exponentially with propagation along z, that is $E = E_o\exp(-\alpha z)\exp[j(\omega t - kz)]$. Here $\exp(-\alpha z)$ causes the envelope of the amplitude to decay with z (attenuation) and $\exp[j(\omega t - kz)]$ is the traveling wave portion. Show that in a medium in which α is small, the wave velocity and the attenuation coefficient are given by

$$\mathrm{v} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_o\varepsilon_o\varepsilon_r}} \quad \text{and} \quad \alpha = \frac{\sigma}{2\varepsilon_o cn}$$

where n is the refractive index ($n = \varepsilon_r^{1/2}$). (Metals with high conductivities are excluded.)

Solution

We can write $E = E_o\exp(-\alpha z)\exp[j(\omega t - kz)]$ as $E = E_o\exp[j\omega t - j(k - j\alpha)z]$. Substitute this into the wave resonance condition

$$[-j(k-j\alpha)]^2E_o\exp[j\omega t - j(k-j\alpha)z] - (j\omega)^2\varepsilon_o\varepsilon_r\mu_oE_o\exp[j\omega t - j(k-j\alpha)z] = j\omega\mu_o\sigma E_o\exp[j\omega t - j(k-j\alpha)z]$$

$\therefore$ $$-(k-j\alpha)^2 + \omega^2\varepsilon_o\varepsilon_r\mu_o = j\omega\mu_o\sigma$$

$\therefore$ $$-k^2 + 2jk\alpha - \alpha^2 + \omega^2\varepsilon_o\varepsilon_r\mu_o = j\omega\mu_o\sigma$$

Rearrange into real and imaginary parts and then equating the real parts and imaginary parts

$\therefore$ $$-k^2 - \alpha^2 + \omega^2\varepsilon_o\varepsilon_r\mu_o + 2jk\alpha = j\omega\mu_o\sigma$$

Real parts

$$-k^2 - \alpha^2 + \omega^2\varepsilon_o\varepsilon_r\mu_o = 0$$

Imaginary parts

$$2k\alpha = \omega\mu_o\sigma$$

Thus, $$\alpha = \frac{\omega\mu_o\sigma}{2k} = \frac{\omega}{k}\cdot\frac{\mu_o\sigma}{2} = \frac{\mu_o c\sigma}{2n} = \frac{\sigma}{2\varepsilon_o n}$$

where we have assumed ω/k = velocity = c/n (see below).

From the imaginary part

$$k^2 = \omega^2\mu_o\varepsilon_o\varepsilon_r - \alpha^2$$

Consider the small α case (otherwise the wave is totally attenuated with very little propagation). Then

$$k^2 = \omega^2\mu_o\varepsilon_o\varepsilon_r$$

and the velocity is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_o\varepsilon_o\varepsilon_r}}$$

1.3 Point light source What is the irradiance measured at a distance of 1 m and 2 m from a 1 W light point source?

Solution

Then the irradiance I at a distance r from O is

$$I = \frac{P_o}{4\pi r^2} = \frac{1\text{W}}{4\pi(1\text{m})^2} = 8.0\ \mu\text{W cm}^{-2}$$

which drops by a factor of 4 at r = 2 m to become 2.0 μW cm^{-2}

1.4 Gaussian beam A particular HeNe laser beam at 633 nm has a spot size of 0.8 mm. Assuming a Gaussian beam, what is the divergence of the beam? What are its Rayleigh range and beam width at 10 m?

Solution

Using Eq. (1.1.7), we find,

$$2\theta = \frac{4\lambda}{\pi(2w_o)} = \frac{4(633\times10^{-9}\text{ m})}{\pi(0.8\times10^{-3}\text{ m})} = 1.01\times10^{-3}\text{ rad} = 0.058°$$

The Rayliegh range is

$$z_o = \frac{\pi w_o^2}{\lambda} = \frac{\pi[\frac{1}{2}(0.8\times10^{-3}\text{ m})]^2}{(633\times10^{-9}\text{ m})} = 0.79\text{ m}$$

The beam width at a distance of 10 m is

$$2w = 2w_o[1 + (z/z_o)^2]^{1/2} = (0.8\times10^{-3}\text{ m})\{1 + [(10\text{ m})/(0.79\text{ m})]^2\}^{1/2}$$
$$= 0.01016\text{ m or }10.16\text{ mm.}$$

1.5 Gaussian beam in a cavity with spherical mirrors Consider an optical cavity formed by two aligned spherical mirrors facing each other as shown in Figure 1.54. Such an optical cavity is called a *spherical mirror resonator*, and is most commonly used in gas lasers. Sometimes, one of the reflectors is a plane mirror. The two spherical mirrors and the space between them form an optical resonator because only certain light waves with certain frequencies can exist in this optical cavity. The radiation inside a spherical mirror cavity is a *Gaussian beam*. The actual or particular Gaussian beam that fits into the cavity is that beam whose wavefronts at the mirrors match the curvature of the mirrors. Consider the symmetric resonator shown in Figure 1.54 in which the mirrors have the same radius of curvature R. When a wave starts at A, its wavefront is the same as the curvature of A. In the middle of the cavity it has the minimum width and at B the wave again has the same curvature as B. Such a wave in the cavity can replicate itself (and hence exist in the cavity) as it travels between the mirrors provided that it has right beam characteristics, that is the right curvature at the mirrors. The radius of curvature R of a Gaussian beam wavefront at a distance z along its axis is given by

$$R(z) = z[1 + (z_o/z)^2]\ ;\ \ z_o = \pi w_o^2/\lambda$$

is the Rayleigh range

Consider a confocal symmetric optical cavity in which the mirrors are separated by $L = R$.

(a) Show that the cavity length L is $2z_o$, that is, it is the same as the Rayleigh range, which is the reason the latter is called the **confocal length**.

(b) Show that the waist of the beam $2w_o$ is fully determined only by the radius of curvature R of the mirrors, and given by

$$2w_o = (2\lambda R/\pi)^{1/2}$$

(c) If the cavity length $L = R = 50$ cm, and $\lambda = 633$ nm, what is the waist of the beam at the center and also at the mirrors?

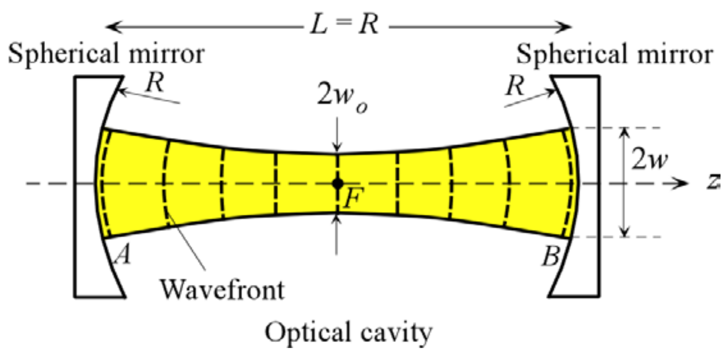


Figure 1.54 Two spherical mirrors reflect waves to and from each other. The optical cavity contains a Gaussian beam. This particular optical cavity is symmetric and confocal; the two focal points coincide at F.

Solution

(a) At $z = R/2$ we have $R(z) = R$. Substitute these into $R(z) = z[1 + (z_o/z)^2]$ to find

$$R = (R/2)[1 + (2z_o/R)^2]$$

∴ $$2 = 1 + \left(\frac{2z_o}{R}\right)^2$$

∴ $$\left(\frac{2z_o}{R}\right) = 1$$

∴ $$L = 2z_o$$

(b) $R = (R/2)[1 + (2z_o/R)^2]$

∴ $$2 = 1 + \left(\frac{2z_o}{R}\right)^2$$

∴ $$\left(\frac{2z_o}{R}\right) = 1$$

Now use $z_o = \pi w_o^2/\lambda$,

∴ $$\left(\frac{2\pi w_o^2}{R\lambda}\right) = 1$$

∴ $$2w_o = \sqrt{\frac{2R\lambda}{\pi}}$$

(c) Substitute λ = 633 nm, $L = R$ = 50 cm into the above equation to find $2w_o$ = 449 μm or **0.449 mm**. At the mirror, $z = R/2$, and also $z_o = R/2$ so that

$$2w = 2w_o\left[1 + \left(\frac{z}{z_o}\right)^2\right]^{1/2} = 2w_o\left[1 + \left(\frac{R/2}{R/2}\right)^2\right]^{1/2} = 2w_o(2^{1/2}) = \mathbf{0.635\ mm}$$

1.6 Cauchy dispersion equation Using the Cauchy coefficients and the general Cauchy equation, calculate refractive index of a silicon crystal at 200 μm and at 2 μm, over two orders of magnitude wavelength change. What is your conclusion?

Solution

At λ = 200μm, the photon energy is

$$h\upsilon = \frac{hc}{\lambda} = \frac{(6.62\times10^{-34}\ \mathrm{J\,s})(3\times10^{8}\ \mathrm{m\,s^{-1}})}{(200\times10^{-6}\ \mathrm{m})} \times \frac{1}{1.6.\times10^{-19}\ \mathrm{J\,eV^{-1}}} = 6.2062\times10^{-3}\,\mathrm{eV}$$

Using the Cauchy dispersion relation for silicon with coefficients from Table 9.2,

$$\begin{aligned} n &= n_{-2}(h\upsilon)^{-2} + n_0 + n_2(h\upsilon)^2 + n_4(h\upsilon)^4 \\ &= (-2.04\times10^{-8})(6.2062\times10^{-3})^{-2} + 3.4189 + (8.15\times10^{-2})(6.2062\times10^{-3})^2 \\ &\qquad + (1.25\times10^{-2})(6.2062\times10^{-3})^4 \\ &= 3.4184 \end{aligned}$$

At λ = 2μm, the photon energy is

$$h\upsilon = \frac{hc}{\lambda} = \frac{(6.62\times10^{-34}\ \mathrm{J\,s})(3\times10^{8}\ \mathrm{m\,s^{-1}})}{(2\times10^{-6}\ \mathrm{m})} \times \frac{1}{1.6\times10^{-19}\ \mathrm{J\,eV^{-1}}} = 0.6206\mathrm{eV}$$

Using the Cauchy dispersion relation for silicon with coefficients from Table 9.2,

$$n = n_{-2}(h\upsilon)^{-2} + n_0 + n_2(h\upsilon)^2 + n_4(h\upsilon)^4$$
$$= (-2.04\times10^{-8})(0.6206)^{-2} + 3.4189 + (8.15\times10^{-2})(0.6206)^2 + (1.25\times10^{-2})(0.6206)^4$$
$$= 3.4521$$

1.7 Sellmeier dispersion equation Using the Sellmeier equation and the coefficients, calculate the refractive index of fused silica (SiO_2) and germania GeO_2 at 1550 nm. Which is larger, and why?

Solution

The Sellmeier dispersion relation for fused silica is

$$n^2 = 1 + \frac{0.696749\lambda^2}{\lambda^2 - 0.0690660^2\,\mu\text{m}^2} + \frac{0.408218\lambda^2}{\lambda^2 - 0.115662^2\,\mu\text{m}^2} + \frac{0.890815\lambda^2}{\lambda^2 - 9.900559^2\,\mu\text{m}^2}$$

$$n^2 = 1 + \frac{0.696749(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (69.0660\,\text{nm})^2} + \frac{0.408218(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (115.662\,\text{nm})^2} + \frac{0.890815(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (9900.559\,\text{nm})^2}$$

so that

n = 1.4443

The Sellmeier dispersion relation for germania is

$$n^2 = 1 + \frac{0.8068664\lambda^2}{\lambda^2 - (0.0689726\,\mu\text{m})^2} + \frac{0.7181585\lambda^2}{\lambda^2 - (0.1539661\,\mu\text{m})^2} + \frac{0.8541683\lambda^2}{\lambda^2 - (11.841931\,\mu\text{m})^2}$$

$$n^2 = 1 + \frac{0.8068664(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (68.9726\,\text{nm})^2} + \frac{0.7181585(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (153.9661\text{nm})^2} + \frac{0.8541683(1550\,\text{nm})^2}{(1550\,\text{nm})^2 - (11841.931\text{nm})^2}$$

so that n = **1.5871**

1.8 Sellmeier dispersion equation The Sellmeier dispersion coefficient for pure silica (SiO_2) and 86.5%SiO_2-13.5 mol.% GeO_2 re given in Table 1.2 Write a program on your computer or calculator, or use a math software package or even a spread sheet program (*e.g.* Excel) to obtain the refractive index n as a function of λ from 0.5 μm to 1.8 μm for both pure silica and 86.5%SiO_2-13.5%GeO_2. Obtain the group index, N_g, vs. wavelength for both materials and plot it on the same graph. Find the wavelength at which the material dispersion becomes zero in each material.

TABLE 1.2 Sellmeier and Cauchy coefficients

Sellmeier	A_1	A_2	A_3	λ_1 (μm)	λ_2 (μm)	λ_3 (μm)
SiO_2 (fused silica)	0.696749	0.408218	0.890815	0.0690660	0.115662	9.900559
86.5%SiO_2-13.5%GeO_2	0.711040	0.451885	0.704048	0.0642700	0.129408	9.425478
GeO_2	0.80686642	0.71815848	0.85416831	0.068972606	0.15396605	11.841931
Sapphire	1.023798	1.058264	5.280792	0.0614482	0.110700	17.92656
Diamond	0.3306	4.3356	–	0.1750	0.1060	–

Solution

Excel program to plot *n* and differentiate and find N_g

$$n^2 = 1 + \frac{\boxed{A_1}\lambda^2}{\lambda^2 - \boxed{\lambda_1^2}} + \frac{A_2\lambda^2}{\lambda^2 - \lambda_2^2} + \frac{\boxed{A_3}\lambda^2}{\lambda^2 - \boxed{\lambda_3^2}}$$

Enter the Sellmeier coefficients for SiO_2 and 86.5%SiO_2-13.5%GeO

Worksheet 1: Data

	A	B	C	D	E	F	G
1		A1	A2	A3	λ1	λ2	λ3
2					μm	μm	μm
3	SiO2	0.696749	0.408218	0.890815	0.069066	0.115662	9.900559
4	86.5%SiO2-13.5%GeO2	0.71104	0.451885	0.704048	0.06427	0.129408	9.425478

Worksheet 2: SiO_2

Use Sellmeier equation and data from worksheet "Data"

$$n = \sqrt{1 + \frac{A_1\lambda^2}{\lambda^2 - \lambda_1^2} + \frac{A_2\lambda^2}{\lambda^2 - \lambda_2^2} + \frac{A_3\lambda^2}{\lambda^2 - \lambda_3^2}}$$

=SQRT(1+(Data!$B$3*SiO2!A2^2)/(SiO2!A2^2-Data!$E$3^2)+(Data!$C$3*SiO2!A2^2)/(SiO2!A2^2-Data!$F$3^2)+(Data!$D$3*SiO2!A2^2)/(SiO2!A2^2-Data!$G$3^2))!

Enter the first wavelength (0.50 μm) and then increment by 0.001

= A2+0.001

	A	B	C
1	λ(μm)	n	Ng
2	0.5	1.462642	1.49024
3	0.501	1.462587	1.49008
4	0.502	1.462532	1.48992
5	0.503	1.462477	1.489762
6	0.504	1.462423	1.489605
7	0.505	1.462369	1.489448
8	0.506	1.462315	1.489293
9	0.507	1.462262	1.489139
10	0.508	1.462209	1.488986
11	0.509	1.462156	1.488834
12	0.51	1.462104	1.488682

=B2-(A2*(B3-B2)/0.001)

$$N_g = n - \lambda_o \frac{dn}{d\lambda_o}$$

$$N_{g1} = n_1 - \lambda_1 \frac{n_2 - n_1}{\lambda_2 - \lambda_1} = n_1 - \lambda_1 \frac{n_2 - n_1}{\underbrace{\Delta\lambda}_{0.001}}$$

Differentiate by using finite difference

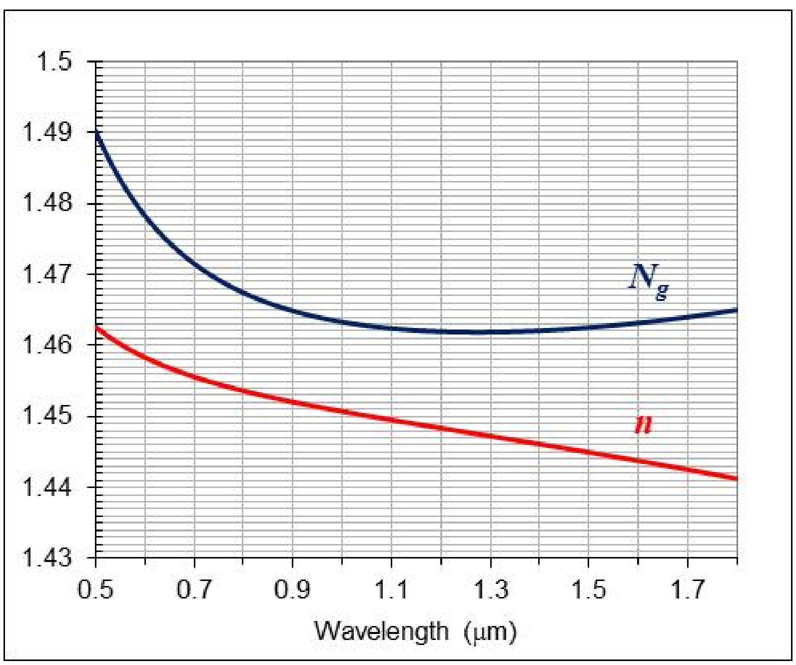


Figure 1Q8-1 Refractive index n and the group index N_g of pure SiO_2 (silica) glass as a function of wavelength (Excel). The minimum in N_g is around 1.3 μm. Note that the *smooth line* option used in Excel to pass a continuous smooth line through the data points. Data points are exactly on the line and are not shown for clarity.

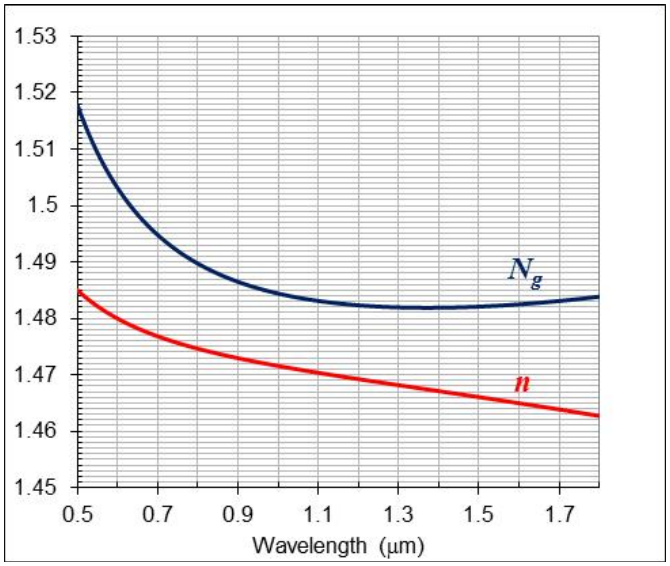


Figure 1Q8-2 Refractive index n and the group index N_g of 86.5%$SiO_2$13.5%GeO as a function of wavelength (Excel). The minimum in N_g is around 1.4 μm. Note that the *smooth line* option used in Excel to pass a continuous smooth line through the data points. Data points are exactly on the line and are not shown for clarity.

Material dispersion is proportional to derivative of group velocity over wavelength. The corresponding values are close to 1.3 and 1.4 μm.

1.9 The Cauchy dispersion relation for zinc selenide ZnSe is a II-VI semiconductor and a very useful optical material used in various applications such as optical windows (especially high power laser windows), lenses, prisms etc. It transmits over 0.50 to 19 μm. n in the 1 – 11 μm range described by a Cauchy expression of the form

$$n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2 \qquad \textit{ZnSe dispersion relation}$$

in which λ in μm. What are the n_{-2}, n_0, n_2 and n_4 coefficients? What is ZnSe's refractive index n and group index N_g at 5 μm?

Solution

$$h\upsilon = \frac{hc}{\lambda}$$

$$hc = (6.62\times10^{-34}\ \text{J s}\times\frac{1}{1.6\times10^{-19}\ \text{J eV}^{-1}})(3\times10^{8}\ \text{m s}^{-1}) = 1.24\times10^{-6}\,\text{eVm}$$

so that

$$n = 2.4365 + \frac{0.0485}{(hc)^2}(h\nu)^2 + \frac{0.0061}{(hc)^4}(h\nu)^4 - 0.0003(hc)^2(h\nu)^{-2}$$

Comparing with Cauchy dispersion equation in photon energy: $n = n_{-2}(h\upsilon)^{-2} + n_0 + n_2(h\upsilon)^2 + n_4(h\upsilon)^4$, we have

$$n_0 = 2.4365$$

$$n_2 = \frac{0.0485}{(hc)^2} = \frac{0.0485}{(1.24\times10^{-6})^2} = 3.15\times10^{10}\,\text{eV}^{-2}$$

$$n_{-2} = 0.0003(hc)^2 = 0.0003\times(1.24\times10^{-6})^2 = 4.62\times10^{-16}\,\text{eV}^2$$

and $$n_4 = \frac{0.0061}{(hc)^4} = \frac{0.0061}{(1.24\times10^{-6})^4} = 2.58\times10^{21}\,\text{eV}^{-4}$$

At λ = 5 μm

$$n = 2.4365 + \frac{0.0485}{(5\mu m)^2} + \frac{0.0061}{(5\mu m)^4} - 0.0003(5\mu m)^2$$

$$= 2.4365 + \frac{0.0485}{25} + \frac{0.0061}{625} - 0.0003(25) = 2.43$$

Group index $$N_g = n - \lambda\frac{dn}{d\lambda}$$

and $$n = 2.4365 + \frac{0.0485}{\lambda^2} + \frac{0.0061}{\lambda^4} - 0.0003\lambda^2$$

∴ $$\frac{dn}{d\lambda} = \frac{-2\lambda\times0.0485}{\lambda^4} + \frac{-4\lambda^3\times0.0061}{\lambda^8} - 2\times0.0003\lambda$$

∴ $$\frac{dn}{d\lambda} = \frac{-0.097}{\lambda^3} + \frac{-0.0244}{\lambda^5} - 0.0006\lambda$$

At λ = 5 μm

$$\frac{dn}{d\lambda} = \frac{-0.097}{(5\mu\text{m})^3} + \frac{-0.0244}{(5\mu\text{m})^5} - 0.0006\times(5\mu\text{m})$$

∴ $$\frac{dn}{d\lambda} = -0.003783\mu\text{m}^{-1}$$

∴ $$N_g = n - \lambda\frac{dn}{d\lambda} = 2.43 - 5\mu\text{m}\times(-0.003783\mu\text{m}^{-1}) = 2.45$$

1.10 Refractive index, reflection and the Brewster angle

(a) Consider light of free-space wavelength 1300 nm traveling in pure silica medium. Calculate the phase velocity and group velocity of light in this medium. Is the group velocity ever greater than the phase velocity?

(b) What is the Brewster angle (the polarization angle θ_p) and the critical angle (θ_c) for total internal reflection when the light wave traveling in this silica medium is incident on a silica/air interface. What happens at the polarization angle?

(c) What is the reflection coefficient and reflectance at normal incidence when the light beam traveling in the silica medium is incident on a silica/air interface?

(d) What is the reflection coefficient and reflectance at normal incidence when a light beam traveling in air is incident on an air/silica interface? How do these compare with part (c) and what is your conclusion?

Solution

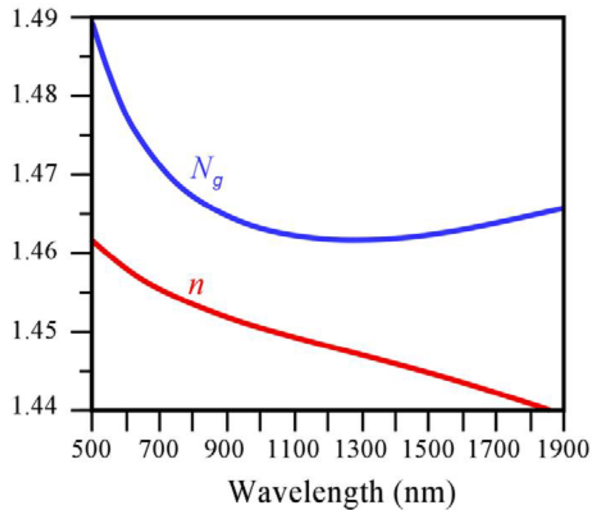


Figure 1.8 Refractive index n and the group index N_g of pure SiO_2 (silica) glass as a function of wavelength.

(a) From Figure 1.8, at λ = 1300 nm, n = 1.447, N_g = 1.462, so that
The phase velocity is given by

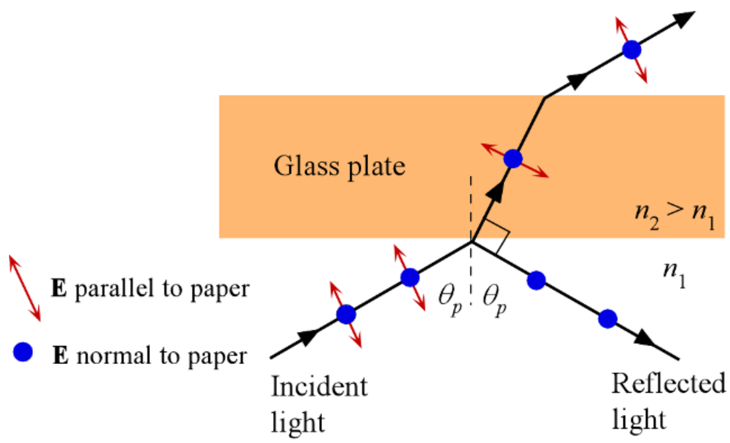
$$v = c/n = (3\times10^8 \text{ m s}^{-1})/(1.447) = 2.073\times10^8 \text{ m s}^{-1}.$$

The group velocity is given by

$$v_g = c/N_g = (3\times10^8 \text{ ms}^{-1})/(1.462) = 2.052\times10^8 \text{ m s}^{-1}.$$

The group velocity is about ~1% smaller than the phase velocity.

(b)

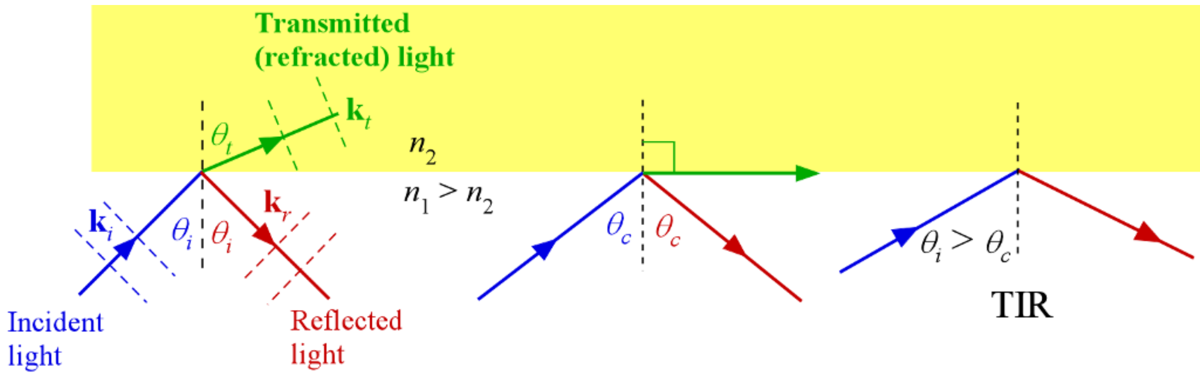


The Brewster angle θ_p is given by

$$\tan\theta_p = \frac{n_2}{n_1} = \frac{1}{1.447} = 0.691$$

∴ $$\theta_p = \tan^{-1} 0.691 = 34.64$$

At the Brewster angle of incidence $\theta_i = \theta_p$, the reflected light contains only field oscillations normal to the plane of incidence (paper).



The critical angle is

$$\sin\theta_c = \frac{n_2}{n_1} = \frac{1}{1.447} = 0.691$$

∴ $$\theta_c = \sin^{-1}(0.691) = 43.7^\circ$$

(c) Given

$$n_1 = 1.447$$

$$n_2 = 1$$

∴ $$r_{//} = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.447 - 1}{1.447 + 1} = 0.1827$$

and $$R_\perp = R_{//} = |r_\perp|^2 = |r_{//}|^2 = 0.0333$$

(d) Given

$$n_1 = 1$$

$$n_2 = 1.447$$

∴ $$r_{//} = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.447}{1.447 + 1} = -0.1827$$

and $$R_\perp = R_{//} = |r_\perp|^2 = |r_{//}|^2 = 0.0333$$

Reflection coefficients are negative, which means that in external reflection at normal incidence there is a phase shift of 180°.

1.11 Snell's law and lateral beam displacement What is the displacement of a laser beam passing through a glass plate of thickness 2 mm and refractive index 1.570 if the angle of incidence is 40°? (See Figure 1.14)

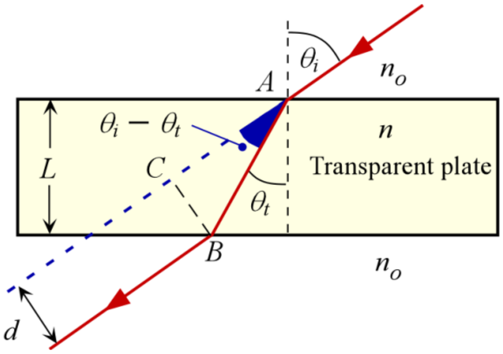


Figure 1.14 Lateral displacement of light passing obliquely through a transparent plate

Solution

The problem is sketched in Figure 1Q12-1

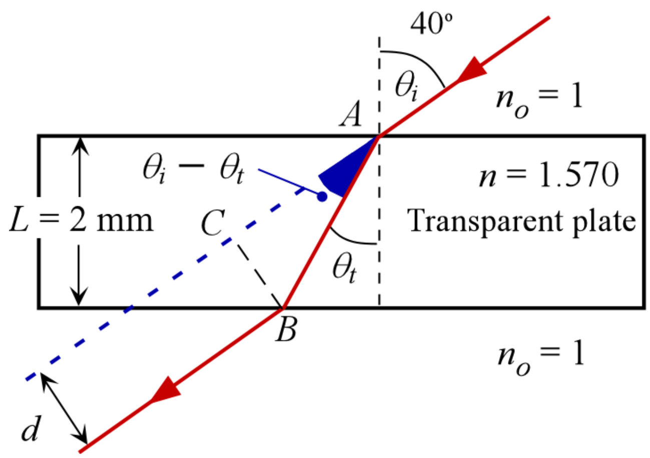


Figure 1Q12-1 Light beam deflection through a glass plate of thickness L = 2 mm. The angle of incidence is 40° and the glass has a refractive index of 1.570

$$\frac{d}{L} = \sin\theta_i\left[1 - \frac{\cos\theta_i}{\sqrt{(n/n_o)^2 - \sin^2\theta_i}}\right]$$

$\therefore$
$$\frac{d}{L} = \sin 40^\circ\left[1 - \frac{\cos 40^\circ}{\sqrt{(1.570/1)^2 - \sin^2 40^\circ}}\right]$$

$$= 0.6428\left[1 - \frac{0.7660}{\sqrt{2.46 - 0.4132}}\right] = 0.2986$$

$\therefore$
$$\frac{d}{2\,\text{mm}} = 0.2986$$

$\therefore$ $d = 0.60$ mm

This is a significant displacement that can be easily measured by using a photodiode array.

1.12 Snell's law and lateral beam displacement An engineer wants to design a refractometer (an instrument for measuring the refractive index) using the lateral displacement of light through a glass plate. His initial experiments involve using a plate of thickness L, and measuring the displacement of a laser beam when the angle of incidence θ_i is changed, for example, by rotating (tilting) the sample. For $\theta_i = 40°$, he measures a displacement of 0.60 mm, and when $\theta_i = 80°$ he measures 1.69 mm. Find the refractive index of the plate and its thickness. (Note: You need to solve a nonlinear equation for n numerically.)

Solution

Figure 1.14 shows the lateral beam deflection through a transparent plate.

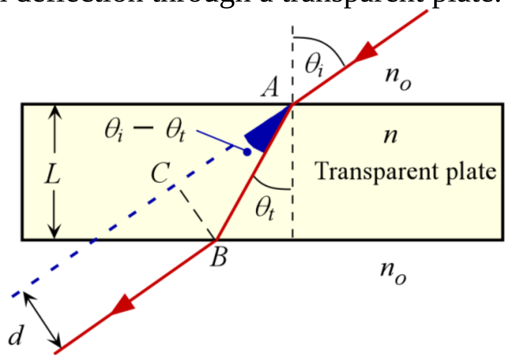


Figure 1.14 Lateral displacement of light passing obliquely through a transparent plate

Apply $d = L\sin\theta_i\left[1 - \dfrac{\cos\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}\right]$

$$0.60\,\text{mm} = L\sin 40°\left[1 - \frac{\cos 40°}{\sqrt{n^2 - \sin^2 40°}}\right] \quad \text{and} \quad 1.69\,\text{mm} = L\sin 80°\left[1 - \frac{\cos 80°}{\sqrt{n^2 - \sin^2 80°}}\right]$$

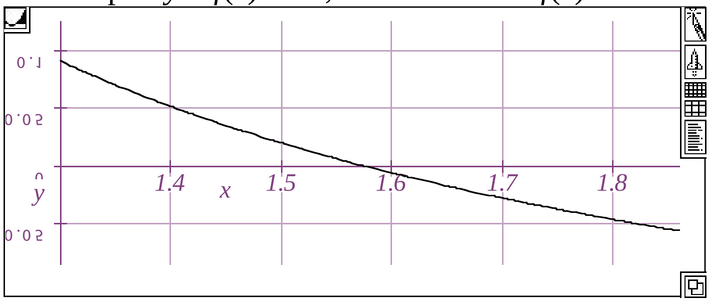
Divide one by the other

$$\frac{0.60}{1.69} = \left(\frac{\sin 40°}{\sin 80°}\right)\frac{\left[1 - \dfrac{\cos 40°}{\sqrt{n^2 - \sin^2 40°}}\right]}{\left[1 - \dfrac{\cos 80°}{\sqrt{n^2 - \sin^2 80°}}\right]} \quad \therefore \quad 0 = \frac{0.60}{1.69} - \left(\frac{\sin 40°}{\sin 80°}\right)\frac{\left[1 - \dfrac{\cos 40°}{\sqrt{n^2 - \sin^2 40°}}\right]}{\left[1 - \dfrac{\cos 80°}{\sqrt{n^2 - \sin^2 80°}}\right]}$$

Define $y = \dfrac{0.60}{1.69} - \left(\dfrac{\sin 40°}{\sin 80°}\right)\dfrac{\left[1 - \dfrac{\cos 40°}{\sqrt{x^2 - \sin^2 40°}}\right]}{\left[1 - \dfrac{\cos 80°}{\sqrt{x^2 - \sin^2 80°}}\right]}$

$$\therefore \qquad y = 0.35503 - 0.6527\frac{\left[1 - \dfrac{0.76606}{\sqrt{x^2 - 0.41318}}\right]}{\left[1 - \dfrac{0.17365}{\sqrt{x^2 - 0.96985}}\right]} = f(x)$$

We can plot $y = f(x)$ vs. x, and find where $f(x)$ cross the x-axis, which will give $x = n$



The above graph was generated in LiveMath (Theorist) (http://livemath.com)

Clearly, the x-axis is cut at $\boldsymbol{n \approx 1.575}$

Substitute n = 1.575 into one of the equations *i.e.*

$$0.60\,\text{mm} = L\sin 40°\left[1 - \frac{\cos 40°}{\sqrt{1.575^2 - \sin^2 40°}}\right]$$

Solving for L we find $\boldsymbol{L \approx 2.0}$ **mm.**

1.13 Snell's law and prisms Consider the quartz prism shown in Figure 1.55 that has an apex angle $\alpha = 60°$. The prism has a refractive index of n and it is in air.

(a) What are Snell's law at interfaces at A (incidence and transmittance angles of θ_i and θ_t) and B (incidence and transmittance angles of θ_i' and θ_t')?

(b) Total deflection $\delta = \delta_1 + \delta_2$ where $\delta_1 = \theta_i - \theta_t$ and $\delta_2 = \theta_t' - \theta_i'$. Now, $\beta + \theta_i' + \theta_t = 180°$ and $\alpha + \beta = 180°$. Find the deflection of the beam for an incidence angle of 45° for the following three colors at which n is known: Blue, n = 1.4634 at λ = 486.1 nm; yellow, n = 1.4587 at λ = 589.2 nm; red, n = 1.4567 at λ = 656.3 nm. What is the separation in distance between the rays if the rays are projected on a screen 1 m away.

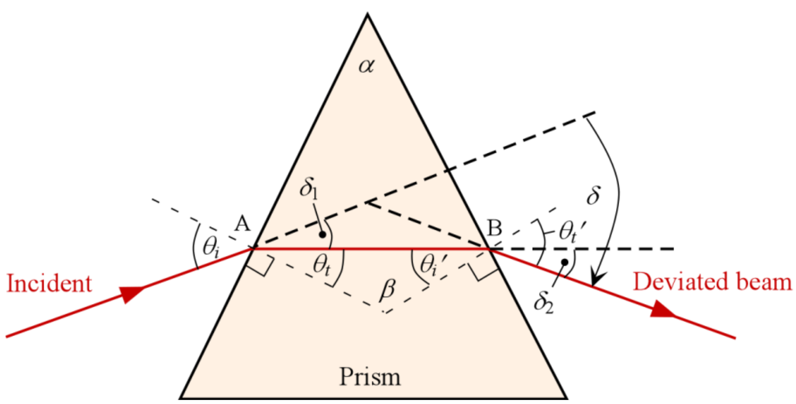


Figure 1.55 A light beam is deflected by a prism through an angle δ. The angle of incidence is υ_i. The apex angle of the prism is α.

Solution

(a) Snell's law at interfaces at A:

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n}{1}$$

Snell's law at interfaces at B:

$$\frac{\sin\theta_i'}{\sin\theta_t'} = \frac{1}{n}$$

(b) Consider the deflection angle δ,

$\delta = \delta_1 + \delta_2$ where $\delta_1 = \theta_i - \theta_t$ and $\delta_2 = \theta_t' - \theta_i'$, i.e. $\delta = \theta_i - \theta_t + \theta_t' - \theta_i'$. Now,

$$\theta_t = \arcsin\left[\frac{\sin\theta_i}{n}\right]$$

and from Figure 1.55

$$\theta_i' = 180^\circ - \beta - \theta_t = \alpha - \theta_t = \alpha - \arcsin\left[\frac{\sin\theta_i}{n}\right]$$

so that

$$\theta_t' = \arcsin[n\sin\theta_i'] = \arcsin\left[n\sin\left(\alpha - \arcsin\left[\frac{\sin\theta_i}{n}\right]\right)\right]$$

and the deflection is,

$$\delta = \theta_i - \arcsin\left[\frac{\sin\theta_i}{n}\right] + \arcsin\left[n\sin\left(\alpha - \arcsin\left[\frac{\sin\theta_i}{n}\right]\right)\right] - \left\{\alpha - \arcsin\left[\frac{\sin\theta_i}{n}\right]\right\}$$

so that finally,

$$\delta = \theta_i - \alpha + \arcsin\left[n\sin\left(\alpha - \arcsin\left[\frac{\sin\theta_i}{n}\right]\right)\right]$$

Substituting the values, and keeping n as a variable, the deflection $\delta(n)$ as a function of n is

$$\delta(n) = (45^\circ - 60^\circ) + \arcsin\left[n\sin\left(60^\circ - \arcsin\left[\frac{1}{\sqrt{2}n}\right]\right)\right]$$

where sin(45°) = 1/√2.

The separation δL for two wavelengths λ_1 and λ_2 corresponding to n_1 and n_2 at the screen at a distance L away is therefore

$\delta L = L[\delta(n_1) - \delta(n_2)]$

where the deflections must be in radians.

Consider the deflection of blue light

$$\delta_{\text{blue}} = (45^\circ - 60^\circ) + \arcsin\left[(1.4634)\sin\left(60^\circ - \arcsin\left[\frac{1}{\sqrt{2}(1.4634)}\right]\right)\right]$$

$\therefore \quad \delta_{\text{blue}} = 34.115^\circ$

Similarly, $\delta_{\text{yellow}} = 33.709^\circ$

The separation of blue and yellow beams at the screes is

$\delta L_{\text{blue-yellow}} = L(\delta_{\text{blue}} - \delta_{\text{blue}}) = (1\text{m})(\pi/180)(34.115^\circ - 33.709^\circ) = 7.08$ mm

Table 1Q13-1 summarizes the results of the calculations for blue, yellow and red light.

Table 1Q13-1 Deflection of blue, yellow and red light through a prism with apex angle 60°. The angle of incidence is 45°.

	Blue 486.1 nm		Yellow 589.2 nm		Red 656.3 nm

n	1.4634		1.4587		1.4567
δ (Deflection angle)	0.5954 rad 34.115°		0.5883 rad 33.709°		5853 rad 33.537°
δL between colors		7.08 mm		3.00 mm	
δL between blue and rede		10.1 mm			

1.14 Fermat's principle of least time Fermat's principle of least time in simple terms states that *when light travels from one point to another it takes a path that has the shortest time*. In going from a point A in some medium with a refractive index n_1 to a point B in a neighboring medium with refractive index n_2 as in Figure 1.56 the light path is AOB that involves refraction at O and satisfies Snell's law. The time it takes to travel from A to B is minimum only for the path AOB such that the incidence and refraction angles θ_i and θ_t satisfy Snell's law. Let's draw a straight line from A to B cutting the x-axes at O'. The line $AO'B$ will be our reference line and we will place the origin of x and y coordinates at O'. Without invoking Snell's law, we will vary point O along the x-axis (hence OO' is a variable labeled x), until the time it takes to travel AOB is minimum, and thereby derive Snell's law. The time t it takes for light to travel from A to B through O is

$$t = \frac{AO}{c/n_1} + \frac{OB}{c/n_2} = \frac{[(x_1 - x)^2 + y_1^2]^{1/2}}{c/n_1} + \frac{[(x_2 + x)^2 + y_2^2]^{1/2}}{c/n_2} \tag{1}$$

The incidence and transmittance angles are given by

$$\sin\theta_i = \frac{x_1 - x}{[(x_1 - x)^2 + y_1^2]^{1/2}} \quad \text{and} \quad \sin\theta_t = \frac{(x_2 + x)}{[(x_2 + x)^2 + y_2^2]^{1/2}} \tag{2}$$

Differentiate Eq. (1) with respect to x to find the condition for the "least time" and then use Eq. (2) in this condition to derive Snell's law.

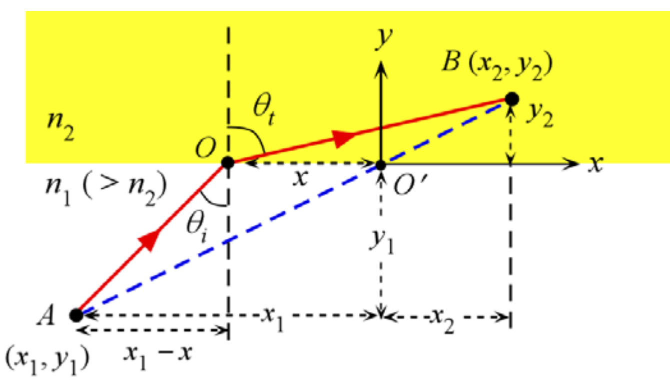


Figure 1.56 Consider a light wave traveling from point A (x_1, y_2) to B (x_1, y_2) through an arbitrary point O at a distance x from O'. The principle of least time from A to B requires that O is such that the incidence and refraction angles obey Snell's law.

Pierre de Fermat (1601–1665) was a French mathematician who made many significant contributions to modern calculus, number theory, analytical geometry, and probability. *(Courtesy of Mary Evans Picture Library/Alamy.)*

Solution

Differentiate t with respect to x

$$\frac{dt}{dx}=\frac{-1/2\times 2(x_1-x)[(x_1-x)^2+y_1^2]^{-1/2}}{c/n_1}+\frac{1/2\times 2(x_2+x)[(x_2+x)^2+y_2^2]^{-1/2}}{c/n_2}$$

The time should be minimum so

$$\frac{dt}{dx}=0 \qquad \text{condition for the "least time"}$$

$$\therefore \qquad \frac{-(x_1-x)[(x_1-x)^2+y_1^2]^{-1/2}}{c/n_1}+\frac{(x_2+x)[(x_2+x)^2+y_2^2]^{-1/2}}{c/n_2}=0$$

$$\therefore \qquad \frac{(x_1-x)}{c/n_1[(x_1-x)^2+y_1^2]^{1/2}}=\frac{(x_2+x)}{c/n_2[(x_1-x)^2+y_1^2]^{1/2}}$$

Use Eq. (2) in the above expression to find

$$n_1\sin\theta_i=n_2\sin\theta_t$$

$$\therefore \qquad \frac{\sin\theta_i}{\sin\theta_t}=\frac{n_2}{n_1} \qquad \text{Snell's law}$$

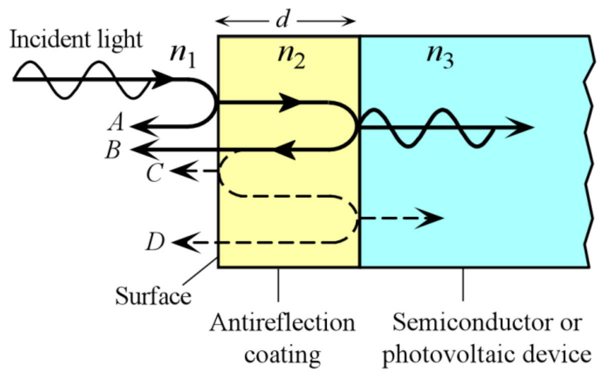
1.15 Antireflection (AR) coating

(a) Consider three dielectric media with flat and parallel boundaries with refractive indices n_1, n_2, and n_3. Show that for normal incidence the reflection coefficient between layers 1 and 2 is the same as that between layers 2 and 3 if $n_2 = \sqrt{[n_1 n_3]}$. What is the significance of this result?

(b) Consider a Si photodiode that is designed for operation at 900 nm. Given a choice of two possible antireflection coatings, SiO_2 with a refractive index of 1.5 and TiO_2 with a refractive index of 2.3, which would you use and what would be the thickness of the antireflection coating? The refractive index of Si is 3.5. Explain your decision.

(c) Consider a Ge photodiode that is designed for operation around 1200 nm. What are the best AR refractive index and coating thickness if the refractive index of Ge is about 4.0?

Solution



(a) Start with the reflection coefficient r_{12} between n_1 and n_2,

$$r_{12}=\frac{n_1-n_2}{n_1+n_2}=\frac{n_1-\sqrt{n_1n_3}}{n_1+\sqrt{n_1n_3}}=\frac{n_1-\sqrt{n_1n_3}}{n_1+\sqrt{n_1n_3}}\times\frac{\sqrt{n_1n_3}+n_3}{\sqrt{n_1n_3}+n_3}$$

$$=\frac{n_1\sqrt{n_1n_3}+n_1n_3-n_3\sqrt{n_1n_3}-n_1n_3}{n_1\sqrt{n_1n_3}+n_1n_3+n_1n_3+n_3\sqrt{n_1n_3}}=\frac{n_1\sqrt{n_1n_3}-n_3\sqrt{n_1n_3}}{n_1\sqrt{n_1n_3}+2n_1n_3+n_3\sqrt{n_1n_3}}$$

Now consider r_{23} between n_2 and n_3,

$$r_{23}=\frac{n_2-n_3}{n_2+n_3}=\frac{\sqrt{n_1n_3}-n_3}{\sqrt{n_1n_3}+n_3}=\frac{\sqrt{n_1n_3}-n_3}{\sqrt{n_1n_3}+n_3}\times\frac{n_1+\sqrt{n_1n_3}}{n_1+\sqrt{n_1n_3}}$$

$$=\frac{n_1\sqrt{n_1n_3}-n_1n_3+n_1n_3-n_3\sqrt{n_1n_3}}{n_1\sqrt{n_1n_3}+n_1n_3+n_1n_3+n_3\sqrt{n_1n_3}}=\frac{n_1\sqrt{n_1n_3}-n_3\sqrt{n_1n_3}}{n_1\sqrt{n_1n_3}+2n_1n_3+n_3\sqrt{n_1n_3}}$$

$\therefore$ $$r_{12}=r_{23}$$

To reduce the reflected light, waves *A* and *B* must interfere destructively. To obtain a good degree of destructive interference between waves *A* and *B*, the amplitudes of reflection coefficients must be comparable. When $n_2 = (n_1n_3)^{1/2}$, then the reflection coefficient between the air and coating is equal to that between the coating and the semiconductor. So the reflection is minimum.

(b) We use $n_1 = 1$ for air, n_2 for the antireflection coefficient and $n_3 =3.5$ for Si photodiode,

$$R_{\min}=\left(\frac{n_2^2-n_1n_3}{n_2^2+n_1n_3}\right)^2$$

For SiO_2 $n_2 = 1.5$

$$R_{\min}(\mathrm{SiO_2})=\left(\frac{1.5^2-1\times3.5}{1.5^2+1\times3.5}\right)^2=0.047$$

For TiO_2 $n_2 = 2.3$

$$R_{\min}(\mathrm{TiO_2})=\left(\frac{2.3^2-1\times3.5}{2.3^2+1\times3.5}\right)^2=0.041$$

$\therefore$ $$R_{\min}(\mathrm{TiO_2})<R_{\min}(\mathrm{SiO_2})$$

So, TiO_2 is a better choice

The thickness of the AR layer should be

$$d = \lambda/(4n_2) = (900 \text{ nm})/[4(2.3)] = \mathbf{97.8\ nm}$$

(c) Consider the Ge photodiode. Ge has $n = 4.0$. We use $n_1 = 1$ for air, n_2 for the antireflection coefficient and $n_3 = 4.0$ for Ge photodiode,
The ideal AR coating would have $n_2 = (n_1 n_3)^{1/2} = 2.0$
The thickness of the AR layer should be

$$d = \lambda/(4n_2) = (1200 \text{ mm})/[4(2)] = \mathbf{150\ nm}$$

1.16 Single and double layer antireflection V-coating For a single layer AR coating of index n_2 on a material with index n_3 ($> n_2 > n_1$), as shown in Figure 1.57(a), the minimum reflectance at normal incidence is given by

$$R_{\min} = \left[\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right]^2 \qquad \textit{Single layer AR coating}$$

when the reflections *A*, *B*, … all interfere as destructively as possible. $R_{\min} = 0$ when $n_2 = (n_1 n_3)^{1/2}$. The choice of materials may not always be the best for a single layer antireflection coating. *Double layer AR coatings*, as shown in Figure 1.57(b) can achieve lower and sharper reflectance at a specified wavelength as in Figure 1.57(c). To reduce the reflection of light at the n_1/n_4, interface, two layers n_2 and n_3, each quarter wavelength in the layer (λ/n_2 and λ/n_3) are interfaced between n_1 and n_4. The reflections *A*, *B* and *C* for normal incidence result in a minimum reflectance given by

$$R_{\min} = \left[\frac{n_3^2 n_1 - n_4 n_2^2}{n_3^2 n_1 + n_4 n_2^2}\right]^2 \qquad \textit{Double layer AR coating}$$

Double layer reflectance vs. wavelength behavior usually has V-shape, and they are called *V-coatings.*

(a) Show that double layer reflectance vanishes when

$$(n_2/n_3)^2 = n_1/n_4 \qquad \textit{Best double layer AR coating}$$

(b) Consider an InGaAs, a semiconductor crystal with an index 3.8, for use in a photodetector. What is the reflectance without any AR coating?

(c) What is the reflectance when InGaAs is coated with a thin AR layer of Si_3N_4? Which material in the table would be ideal as an AR coating?

(d) What two materials would you choose to obtain a V-coating? Note: The choice of an AR coating also depends on the technology involved in depositing the AR coating and its effects on the interface states between the AR layer and the semiconductor. $Si_{1-x}N_x$ is a common AR coating on devices inasmuch as it is a good passive dielectric layer, its deposition technology is well established and changing its composition (x) changes its index.

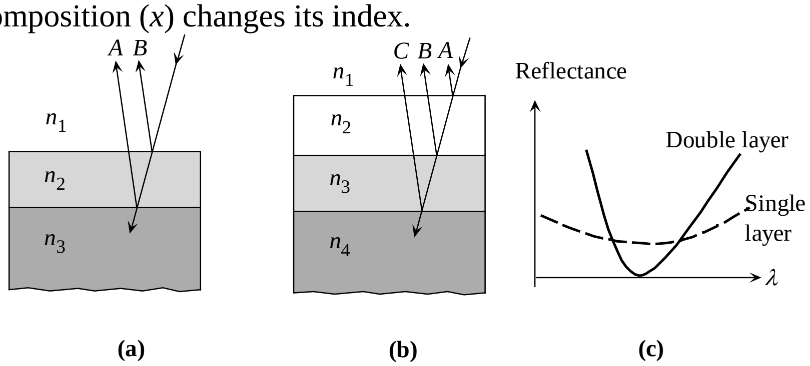


Figure 1.57(a) A single layer AR coating. (b) A double layer AR coating and its V-shaped reflectance spectrum over a wavelength range.

TABLE 1.3 Typical AR materials and their approximate refractive indices over the visible wavelengths

	MgF_2	SiO_2	Al_2O_3	CeF_3	Sb_2O_3	Si_3N_4	SiO	ZrO_2	ZnS	TiO_2	CdS
n	1.38	1.46	1.65	1.65	1.9–2.1	1.95	2.0	2.05	2.35	2.35	2.60

Solution

(a) The minimum reflectance,

$$R_{\min} = \left[\frac{n_3^2 n_1 - n_4 n_2^2}{n_3^2 n_1 + n_4 n_2^2}\right]^2 = 0$$

$\therefore$ $$n_3^2 n_1 - n_4 n_2^2 = 0$$

$\therefore$ $$n_3^2 n_1 = n_4 n_2^2$$

$\therefore$ $$\frac{n_1}{n_4} = \frac{n_2^2}{n_3^2}$$

$\therefore$ $$\frac{n_1}{n_4} = \left(\frac{n_2}{n_3}\right)^2$$ *Best double layer AR coating*

(b) Without an AR coating, the reflectance is

$$R = [(n_1 - n_3)/ (n_1 + n_3)]^2 = [(1 - 3.8)/ 1 + 3.8)]^2 = 0.34 \text{ or } \mathbf{34\%}$$

(c) Take $n_3 = 3.8$, $n_2 = 1.95$, $n_1 = 1$, and find the minimum reflectance from

$$R_{\min} = \left[\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right]^2$$

$\therefore$ $$R_{\min} = \left[\frac{1.95^2 - 3.8}{1.95^2 + 3.8}\right]^2 = 1.08\times10^{-7}$$

For ideal an AR coating:

$$n_2 = \sqrt{n_1 n_3}$$

$\therefore$ $$\sqrt{n_1 n_3} = \sqrt{1\times 3.8} = 1.9493$$

$\therefore$ $$n_2 = 1.9493$$

Looking at table, Si_3N_4 (n_2 = 1.95) would be ideal.

(d) Two find 2 materials for a V-coating, consider first,

$$(n_2 / n_3)^2 = n_1 / n_4$$ *Best double layer AR coating*

$$(n_2 / n_3) = \sqrt{n_1 / n_4}$$

$(n_2 / n_3) = \sqrt{1/3.8} = 0.51$

This is the ratio we need. From the table MgF_2 (n_2 = 1.38) and CdS (n_3 = 2.60) are the best two materials for V-coating: $(n_2 / n_3) = \dfrac{1.38}{2.60} = 0.53$.

The minimum reflectance would be

$$R_{\min} = \left[\frac{n_3^2 n_1 - n_4 n_2^2}{n_3^2 n_1 + n_4 n_2^2}\right]^2 = \left[\frac{2.60^2 - 3.8\times 1.38^2}{2.60^2 + 3.8\times 1.38^2}\right]^2 = 0.001$$

1.17 Single, double and triple layer antireflection coatings

Figure shows the reflectance of an uncoated glass, and glass that has a single (1), double (2) and triple (3) layer antireflection coatings? The coating details are in the figure caption. Each layer in single and double layer AR coatings has a thickness of λ/4, where *λ* is the wavelength in the layer. The triple layer AR layer has three coatings with thicknesses *λ*/4, *λ*/2 and *λ*/4. Can you qualitatively explain the results by using interference? What applications would need single, double and triple layer coatings?

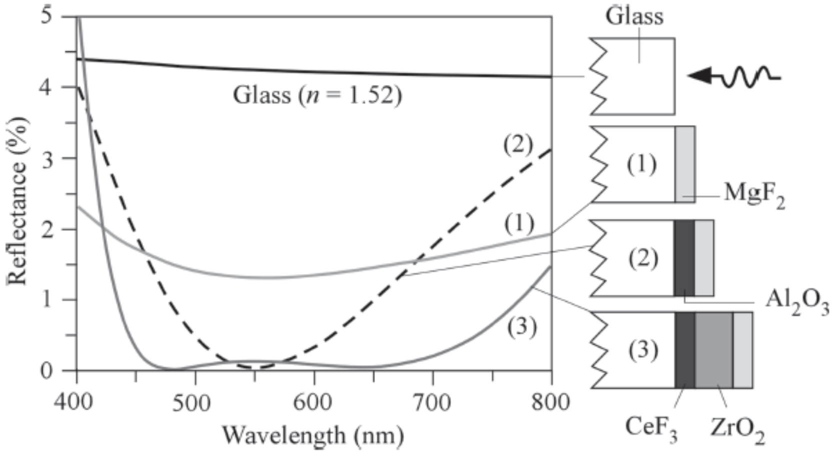


FIGURE 1.58 Reflectance vs. wavelength for a glass plate, $n = 1.52$, with and without AR coatings. (1) Single-layer AR coating is a quarter wavelength ($\lambda/4$) thick MgF_2, $n = 1.38$. (2) Double-layer coating is $\lambda/4$ thick MgF_2 and $\lambda/4$ thick Al_2O_3, $n = 1.69$. (3) Triple-layer coating is $\lambda/4$ thick MgF_2, $\lambda/2$ thick ZrO_2, $n = 2.05$, and a $\lambda/4$ thick CeF_3, $n = 1.64$. (*Source:* Plotted from data appearing in Figure 2.2 in S. Chattopadhyay *et al.*, *Mater. Sci. Engin. R*, *69*, 1, 2010.)

Solution

Instructor's choice of answers. Can be given out as a short project to students.

1.18 Reflection at glass-glass and air-glass interfaces

A ray of light that is traveling in a glass medium of refractive index n_1 = 1.460 becomes incident on a less dense glass medium of refractive index n_2 = 1.430. Suppose that the free space wavelength of the light ray is 850 nm.

(a) What should the minimum incidence angle for TIR be?

(b) What is the phase change in the reflected wave when the angle of incidence θ_i = 85° and when θ_i = 90°?

(c) What is the penetration depth of the evanescent wave into medium 2 when θ_i = 85° and when θ_i = 90°?

(d) What is the reflection coefficient and reflectance at normal incidence (θ_i = 0°) when the light beam traveling in the silica medium (n = 1.460) is incident on a silica/air interface?

(e) What is the reflection coefficient and reflectance at normal incidence when a light beam traveling in air is incident on an air/silica (n = 1.460) interface? How do these compare with part (d) and what is your conclusion?

Solution

(a) The critical angle θ_c for TIR is given by

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{1.430}{1.460} = 78.36^\circ$$

(b) Since the incidence angle $\theta_i > \theta_c$, there is a phase shift in the reflected wave. The phase change in $E_{r,\perp}$ is given by $\phi_\perp$. With n_1 = 1.460, n_2 = 1.430, and θ_i = 85°,

$$\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i} = \frac{\left[\sin^2(85^\circ) - \left(\dfrac{1.430}{1.460}\right)^2\right]^{1/2}}{\cos(85^\circ)}$$

$$= 2.08675$$

$$\tan\left(\tfrac{1}{2}\varphi_\perp\right) = 2.08675$$

$$\varphi_\perp = 2\tan^{-1}(2.08673) = 128.79^\circ$$

so that the phase change is 128.79°. For the $E_{r,//}$ component, the phase change is

$$\tan\left(\tfrac{1}{2}\varphi_{//} + \tfrac{1}{2}\pi\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{n^2\cos\theta_i} = \frac{1}{n^2}\tan\left(\tfrac{1}{2}\varphi_\perp\right)$$

so that $\tan(^1/_2\phi_{//} + \tfrac{1}{2}\pi) = (n_1/n_2)^2\tan(\phi_\perp/2) = (1.460/1.430)^2\tan(^1/2(128.79°)) = 2.17522$

which gives $\varphi_{//} = 2\tan^{-1}(2.17522) - \pi = -49.38^\circ$

We can repeat the calculation with θ_i = 90°.
The phase change in $E_{r,\perp}$ is given by $\phi_\perp$. With n_1 = 1.460, n_2 = 1.430, and θ_i = 90°,

$$\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i} = \frac{\left[\sin^2(90^\circ) - \left(\dfrac{1.430}{1.460}\right)^2\right]^{1/2}}{\cos(90^\circ)}$$

$\therefore$ $\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \infty$

$\therefore$ $\varphi_\perp = 2\tan^{-1}(\infty) = 180^\circ$

so that the phase change is 180°. For the $E_{r,//}$ component, the phase change is

$$\tan\left(\tfrac{1}{2}\varphi_{//} + \tfrac{1}{2}\pi\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{n^2\cos\theta_i} = \frac{1}{n^2}\tan\left(\tfrac{1}{2}\varphi_\perp\right)$$

so that $\tan(^1/_2\phi_{//} + \tfrac{1}{2}\pi) = \infty$

which gives $\phi_{//} = 2\tan^{-1}(\infty) - \pi = 0°$

(c) The amplitude of the evanescent wave as it penetrates into medium 2 is

$$E_{t,\perp}(y,t) = E_{to,\perp}\exp(-\alpha_2 y)$$

We ignore the z-dependence, $\exp j(\omega t - k_z z)$, as this represents propagation along z. The field strength drops to $\boldsymbol{e}^{-1}$ when $y = 1/\alpha_2 = \delta$, the *penetration depth*. The attenuation constant α_2 for $\theta_i = 85°$ is

$$\alpha_2 = \frac{2\pi n_2}{\lambda_o}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

$$\therefore \qquad \alpha_2 = \frac{2\pi(1.430)}{(850\times10^{-9})}\left[\left(\frac{1.460}{1.430}\right)^2 \sin^2(85^\circ) - 1\right]^{1/2} = 1.96\times10^6\,\text{m} .$$

so the penetration depth $\delta = 1/\alpha_2 = 1/(1.96\times10^6\,\text{m}) = 5.1\times10^{-7}$ m, or **0.51μm.**

For 90°, repeating the calculation above,

$$\alpha_2 = \frac{2\pi(1.430)}{(850\times10^{-9})}\left[\left(\frac{1.460}{1.430}\right)^2 \sin^2(90^\circ) - 1\right]^{1/2} = 2.175\times10^6\,\text{m}$$

so the penetration depth $\delta = 1/\alpha_2 = 1/(2.175\times10^6\ \text{m}) = 4.5\times10^{-7}$ m, or **0.45 μm.**

Conclusion: The penetration depth increases as the angle of incidence decreases

(d)

$$r_{//} = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.460 - 1}{1.460 + 1} = \mathbf{0.187}$$

and $R_{//} = R_\perp = r_{//}^2 = 0.035$ or **3.5%**

(e)

$$r_{//} = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.460}{1 + 1.460} = \mathbf{-0.187}$$

and $R_{//} = R_\perp = r_{//}^2 = 0.035$ or **3.5%**

When $r_\perp$ is a negative number, then there is a phase shift of 180° (or π) which is in agreement with part **(b)**.

1.19 Dielectric mirror Consider a dielectric mirror that is made up of quarter wave layers of GaAs with $n_H = 3.382$ and AlAs with $n_L = 2.912$, both around 1500 nm. The GaAs-AlAs dielectric mirror is inside a vertical cavity surface emitting laser diode operating at 1.5 μm. The substrate is GaAs with $n_3 = n_{\text{substrate}} = 3.382$. The light is incident on the mirror from another semiconductor that is GaAlAs with an index $n_0 = 3.40$. Calculate how many pairs of layers N would be needed to get a reflectance above 95%. What would be the bandwidth?

Solution

$$R_N = \left[\frac{n_1^{2N} - \left(n_0/n_3\right) n_2^{2N}}{n_1^{2N} + \left(n_0/n_3\right) n_2^{2N}}\right]^2 = \left[\frac{\left(n_1/n_2\right)^{2N} - \left(n_0/n_3\right)}{\left(n_1/n_2\right)^{2N} + \left(n_0/n_3\right)}\right]^2 \qquad \therefore \quad \left(n_1/n_2\right)^{2N} = \left(n_0/n_3\right)\frac{1+\sqrt{R_N}}{1-\sqrt{R_N}}$$

Insert R_N = 0.95 and solve,

$$N = \frac{1}{2}\frac{\ln\left(\dfrac{n_0}{n_3}\dfrac{1+\sqrt{R_N}}{1-\sqrt{R_N}}\right)}{\ln\left(n_1/n_2\right)} = 14.58$$ *i.e.* **15 pairs are needed**

$$\frac{\Delta\lambda}{\lambda_0} \approx \frac{4}{\pi}\arcsin\left(\frac{n_1 - n_2}{n_1 + n_2}\right) = 0.095167$$

But, λ_0 = 1500 nm, $\therefore$ $\Delta\lambda$ = **142 nm**

1.20 TIR and polarization at water-air interface

(a) Given that the refractive index of water is about 1.33, what is the polarization angle for light traveling in air and reflected from the surface of the water?

(b) Consider a diver in sea pointing a flashlight towards the surface of the water. What is the critical angle for the light beam to be reflected from the water surface?

Solution

(a) Apply $\tan\theta_p = \dfrac{n_2}{n_1}$

with $n_1 = 1$, $n_2 = 1.33$

$\therefore$ $\theta_p = \tan^{-1}(n_2 / n_1) = \tan^{-1}(1.33/1) = 53.06$

(b) Given

and $n_1 = 1.33$, $n_2 = 1$

The critical angle is

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{1}{1.33} = 48.75^\circ$$

1.21 Reflection and transmission at a semiconductor-semiconductor interface A light wave with a free space wavelength of 890 nm (free space wavelength) that is propagating in GaAs becomes incident on AlGaAs. The refractive index of GaAs is 3.60, that of AlGaAs is 3.30.

(a) Consider normal incidence. What are the reflection and transmission coefficients and the reflectance and transmittance? (From GaAs into AlGaAs)

(b) What is the Brewster angle (the polarization angle θ_p) and the critical angle (θ_c) for total internal reflection for the wave in **(a)**; the wave that is traveling in GaAs and incident on the GaAs/AlGaAs interface.

(c) What is the reflection coefficient and the phase change in the reflected wave when the angle of incidence θ_i = 79°?

(d) What is the penetration depth of the evanescent wave into medium 2 when θ_i = 79° and when θ_i = 89°? What is your conclusion?

Solution

(a) Given,

$$n_1 = 3.60$$

$$n_2 = 3.30$$

we have

$$r_{//} = r_{\perp} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{3.60 - 3.30}{3.60 + 3.30} = 0.043$$

∴

$$R_{\perp} = R_{//} = |r_{\perp}|^2 = |r_{//}|^2 = 0.0018$$

and

$$t_{//} = t_{\perp} = \frac{2n_1}{n_1 + n_2} = 2 \times \frac{3.60}{3.60 + 3.30} = 1.043$$

∴

$$T = T_{\perp} = T_{//} = \frac{4n_1 n_2}{(n_1 + n_2)^2} = \frac{4 \times 3.60 \times 3.30}{(3.60 + 3.30)^2} = 0.998$$

(b) Apply

$$\tan\theta_p = \frac{n_2}{n_1}$$

and

$$\sin\theta_c = \frac{n_2}{n_1}$$

Take

$$n_1 = 3.60$$

$$n_2 = 3.30$$

∴

$$\theta_p = \tan^{-1}(n_2 / n_1) = \tan^{-1}(3.30/3.60) = 42.51^{\circ}$$

∴

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{3.30}{3.60} = 66.44^{\circ}$$

(c) Take

$$n = \frac{n_2}{n_1} = \frac{3.30}{3.60} = 0.9166$$

$$r_{\perp} = \frac{\cos\theta_i - [n^2 - \sin^2\theta_i]^{1/2}}{\cos\theta_i + [n^2 - \sin^2\theta_i]^{1/2}}$$

then,

$$= \frac{\cos(79^{\circ}) - [(0.9166)^2 - \sin^2(79^{\circ})]^{1/2}}{\cos(79^{\circ}) + [(0.9166)^2 - \sin^2(79^{\circ})]^{1/2}}$$

$$= \frac{0.1908 - 0.3513j}{0.1908 + 0.3513j}$$

$$r_\perp = \frac{0.1908 - 0.3513j}{0.1908 + 0.3513j} \times \frac{0.1908 - 0.3513j}{0.1908 - 0.3513j}$$

and

$$= \frac{0.1908^2 - 0.3513^2 - 2j0.1908 \times 0.3513}{0.1598}$$

$$= \frac{-0.087 - 0.1340j}{0.1598}$$

$$= -0.5444 - 0.8385j$$

$\therefore$ $r_\perp = -0.5444 + j(-0.8385)$

$\therefore$ $r_\perp = 0.9997\exp(j237^\circ)$

We know that $r_\perp = |r_\perp| \exp(-j\phi_\perp)$, thus

$$0.9997\exp(j57^\circ) = 0.9997\exp(-j\phi_\perp)$$

$\therefore$ $\exp(-j\phi_\perp) = \exp(j237^\circ)$

$\therefore$ $\phi_\perp = -237^\circ$

$\phi_\perp = 123^\circ$

or $\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \dfrac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i} = \dfrac{\left[\sin^2(79°) - (0.9166)^2\right]^{1/2}}{\cos(79°)}$

$\therefore$ $\varphi_\perp = 122.98 = 123°$

Now the parallel component, $r_{//}$,

$$r_{//} = \frac{[n^2 - \sin^2\theta_i]^{1/2} - n^2\cos\theta_i}{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$$

$$= \frac{[(0.9166)^2 - \sin^2(79^\circ)]^{1/2} - (0.9166)^2\cos(79^\circ)}{[(0.9166)^2 - \sin^2(79^\circ)]^{1/2} + (0.9166)^2\cos(79^\circ)}$$

$$= \frac{0.3513j - 0.1603}{0.3513j + 0.1603}$$

$$r_{//} = \frac{0.3513j - 0.1603}{0.3513j + 0.1603} \times \frac{0.3513j - 0.1603}{0.3513j - 0.1603}$$

$\therefore$

$$= \frac{-0.0977 - j0.1126}{-0.1491}$$

$$= 0.6552 + j0.7552$$

$\therefore$ $r_{//} = 0.6552 + j(0.7552)$

$r_{//} = 1.088 \times \exp(j49.05^\circ)$

We know that $r_{//} = |r_{//}| \exp(-j\phi_{//})$, thus

$$\exp(-j\phi_{//}) = \exp(j49.05^\circ)$$

$\therefore$ $\phi_{//} = -49.05^\circ$

or,

$$\tan\left(\tfrac{1}{2}\varphi_{//} + \tfrac{1}{2}\pi\right) = \frac{1}{n^2}\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \frac{1}{0.9166^2}\tan\left(\tfrac{1}{2}122.98^\circ\right) = 2.1913$$

$\therefore$ $\tfrac{1}{2}(\varphi_{//} + \pi) = \tan^{-1}(2.1913) = 65.47$

$\therefore$ $(\varphi_{//} + \pi) = 130.94^\circ$

$\therefore$ $\varphi_{//} = 130.94 - 180 = -49.05^\circ$

(d)

The attenuation coefficient α_2 for θ_i =79° is

$$\alpha_2 = \frac{2\pi n_2}{\lambda_o}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

i.e. $$\alpha_2 = \frac{2\pi(3.30)}{(890\times10^{-9}\ \text{m})}\left[\left(\frac{3.60}{3.30}\right)^2 \sin^2(79°) - 1\right]^{1/2} = 8.92\times10^6\, m^{-1}.$$

so the penetration depth is $\delta = 1/\alpha_2 = 1/(8.92\times10^6\ \text{m}^{-1}) = 1.12\times10^{-7}$ m, or 0.112 μm

For 89°, repeating the calculation

$$\alpha_2 = \frac{2\pi(3.30)}{(890\times10^{-9}\ \text{m})}\left[\left(\frac{3.60}{3.30}\right)^2 \sin^2(89°) - 1\right]^{1/2} = 1.01\times10^7\, m^{-1}$$

So, the penetration depth is $\delta = 1/\alpha_2 = 1/(1.01\times10^7\, m^{-1}) = 9.9\times10^{-8}$ m, or 990nm.

The conclusion is that the penetration depth decreases as the incidence angle increases

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1.22 Phase changes on TIR Consider a light wave of wavelength 870 nm traveling in a semiconductor medium (GaAs) of refractive index 3.60. It is incident on a different semiconductor medium (AlGaAs) of refractive index 3.40, and the angle of incidence is 80°. Will this result in total internal reflection? Calculate the phase change in the parallel and perpendicular components of the reflected electric field.

Solution

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{3.40}{3.60} = 70.81^\circ$$

Clearly, $\theta_i = 80^\circ > \theta_c$

So, this results in total internal reflection.

$$n = \frac{n_2}{n_1} = \frac{3.40}{3.60} = 0.9444$$

$$\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \frac{\left[\sin^2\theta_i - n^2\right]^{1/2}}{\cos\theta_i} = \frac{\left[\sin^2(80°) - (0.9444)^2\right]^{1/2}}{\cos(80°)} = 1.6078$$

∴ $\varphi_\perp = 2\tan^{-1}(1.6078) = 116.24^\circ$

$$\tan\left(\tfrac{1}{2}\varphi_{//} + \tfrac{1}{2}\pi\right) = \frac{1}{n^2}\tan\left(\tfrac{1}{2}\varphi_\perp\right) = \frac{1}{0.9444^2}\tan\left(\tfrac{1}{2}116.24^\circ\right) = 1.8027$$

∴ $\tfrac{1}{2}(\varphi_{//} + \pi) = \tan^{-1}(1.8027) = 60.98$

∴ $(\varphi_{//} + \pi) = 121.96^\circ$

∴ $\varphi_{//} = 121.96 - 180 = -58.04^\circ$

1.23 Fresnel's equation Fresnel's equations are sometimes given as follows:

$$r_\perp = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$$

$$r_{//} = \frac{E_{r0,//}}{E_{i0,//}} = \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$$

$$t_\perp = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}$$

and
$$t_{//} = \frac{E_{t0,//}}{E_{i0,//}} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$$

Show that these reduce to Fresnel's equation given in Section 1.6.

Using Fresnel's equations, find the reflection and transmission coefficients for normal incidence and show that

$$r_\perp + 1 = t_\perp \qquad \text{and} \qquad r_{//} + nt_{//} = 1$$

where $n = n_2/n_1$.

Solution

$$n_2 / n_1 = n$$

From Snell's law

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} = n$$

∴ $$\sin^2\theta_t = \frac{\sin^2\theta_i}{n^2}$$

Perpendicular component

$$r_\perp = \frac{E_{r0,\perp}}{E_{i0,\perp}} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{\cos\theta_i - n_2/n_1\cos\theta_t}{\cos\theta_i + n_2/n_1\cos\theta_t}$$

∴ $$r_\perp = \frac{\cos\theta_i - n\cos\theta_t}{\cos\theta_i + n\cos\theta_t} = \frac{\cos\theta_i - n[1-\sin^2\theta_t]^{1/2}}{\cos\theta_i + n[1-\sin^2\theta_t]^{1/2}}$$

$$\therefore \qquad r_\perp = \frac{\cos\theta_i - n[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2}}{\cos\theta_i + n[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2}}$$

$$\therefore \qquad r_\perp = \frac{\cos\theta_i - [n^2-\sin^2\theta_i]^{1/2}}{\cos\theta_i + [n^2-\sin^2\theta_i]^{1/2}}$$

Parallel component

$$r_{//} = \frac{E_{r0,//}}{E_{i0,//}} = \frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i} = \frac{\cos\theta_t - n_2/n_1\cos\theta_i}{\cos\theta_t + n_2/n_1\cos\theta_i} = \frac{\cos\theta_t - n\cos\theta_i}{\cos\theta_t + n\cos\theta_i}$$

$$\therefore \qquad r_{//} = \frac{[1-\sin^2\theta_t]^{1/2} - n\cos\theta_i}{[1-\sin^2\theta_t]^{1/2} + n\cos\theta_i} = \frac{[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2} - n\cos\theta_i}{[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2} + n\cos\theta_i}$$

$$\therefore \qquad r_{//} = \frac{[n^2-\sin^2\theta_i]^{1/2} - n^2\cos\theta_i}{[n^2-\sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$$

$$t_\perp = \frac{E_{t0,\perp}}{E_{i0,\perp}} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + n_2/n_1\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + n\cos\theta_t}$$

$$\therefore \qquad t_\perp = \frac{2\cos\theta_i}{\cos\theta_i + n[1-\sin^2\theta_t]^{1/2}} = \frac{2\cos\theta_i}{\cos\theta_i + n[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2}}$$

$$\therefore \qquad t_\perp = \frac{2\cos\theta_i}{\cos\theta_i + [n^2-\sin^2\theta_i]^{1/2}}$$

$$t_{//} = \frac{E_{t0,//}}{E_{i0,//}} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_t + n_2/n_1\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_t + n\cos\theta_i}$$

$$\therefore \qquad t_{//} = \frac{2\cos\theta_i}{\cos\theta_t + n\cos\theta_i} = \frac{2\cos\theta_i}{[1-\dfrac{\sin^2\theta_i}{n^2}]^{1/2} + n\cos\theta_i}$$

$$\therefore \qquad t_{//} = \frac{2n\cos\theta_i}{[n^2-\sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$$

$$r_\perp + 1 = \frac{\cos\theta_i - [n^2-\sin^2\theta_i]^{1/2}}{\cos\theta_i + [n^2-\sin^2\theta_i]^{1/2}} + 1$$

$$= \frac{\cos\theta_i - [n^2-\sin^2\theta_i]^{1/2} + \cos\theta_i + [n^2-\sin^2\theta_i]^{1/2}}{\cos\theta_i + [n^2-\sin^2\theta_i]^{1/2}}$$

$\therefore \quad r_{\perp} + 1 = \dfrac{2\cos\theta_i}{\cos\theta_i + [n^2 - \sin^2\theta_i]^{1/2}}$

$\therefore \quad r_{\perp} + 1 = t_{\perp}$

$$r_{//} + nt_{//} = \frac{[n^2 - \sin^2\theta_i]^{1/2} - n^2\cos\theta_i}{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i} + n\frac{2n\cos\theta_i}{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$$

$\therefore \quad r_{//} + nt_{//} = \dfrac{[n^2 - \sin^2\theta_i]^{1/2} - n^2\cos\theta_i + 2n^2\cos\theta_i}{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$

$\therefore \quad r_{//} + nt_{//} = \dfrac{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i}{[n^2 - \sin^2\theta_i]^{1/2} + n^2\cos\theta_i}$

$\therefore \quad r_{//} + nt_{//} = 1$

1.24 Fresnel's equations Consider a light wave traveling in a glass medium with an index $n_1 = 1.440$ and it is incident on the glass-air interface. Using Fresnel equations only *i.e.* Eqs (6a) and 6(b) in §1.6, calculate the reflection coefficients $r_{\perp}$ and $r_{//}$ and hence reflectances $R_{\perp}$ and $R_{//}$ for (a) $\theta_i = 25°$ and (b) $\theta_i = 50°$. In the case of $\theta_i = 50°$, find the phase change $\phi_{\perp}$ and $\phi_{//}$ from the reflection coefficients by writing $r = |r|\exp(-j\phi)$. Compare $\phi_{\perp}$ and $\phi_{//}$ from $r_{\perp}$ and $r_{//}$ calculations with those calculated from Eqs (11) and (12).

Solution

The above problem is solved using LiveMath and reproduced below. It should be relatively straightforward to follow.

$n_a = n_1$; $n_b = n_2$; $r_\pi = r_{//}$; $r_\sigma = r_\perp$; subscript σ means $\perp$; π means //.

Declarations

Graphics Primitives

Constants

$\theta = 25\frac{\pi}{180}$

$\theta = 0.43633$ Calculate

$na = 1.44$

$nb = 1$

$n = \frac{nb}{na}$

$n = 0.69444$ Calculate

Critical angle is

$\sin(\theta c) = n$

$\theta c = \arcsin(n)$ Isolate

$\theta c = 0.76765$ Calculate

$\theta c\frac{180}{\pi}$

$\theta c\frac{180}{\pi} = 43.983$ Calculate

Parallel reflection coefficient is $r\pi$

Perpendicular reflection coefficient is $r\sigma$

$$r\pi = \frac{\sqrt{n^2-(\sin[\theta])^2} - n^2\cos(\theta)}{\sqrt{n^2-(\sin[\theta])^2} + n^2\cos(\theta)}$$

$r\pi = 0.11534$ Calculate

$R\pi = r\pi^2$

$R\pi = 0.013304$ Calculate

Rπ = 1.33%

$$r\sigma = \frac{\cos(\theta)-\sqrt{n^2-(\sin[\theta])^2}}{\sqrt{n^2-(\sin[\theta])^2} + \cos(\theta)}$$

$r\sigma = 0.24378$ Calculate

$R\sigma = r\sigma^2$

$R\sigma = 0.059427$ Calculate

Rσ = 5.9%

Now consider $\theta' = 50$

$\theta' = 50\frac{\pi}{180}$

$\theta' = 0.87266$ Calculate

We find the perpendicular reflection coefficient first, $r\sigma'$

$$r\sigma' = \frac{\cos(\theta')-\sqrt{n^2-(\sin[\theta'])^2}}{\sqrt{n^2-(\sin[\theta'])^2} + \cos(\theta')}$$

$r\sigma' = 0.59605-0.80294i$ Calculate

Magnitude is

$|r\sigma'|$

$|r\sigma'| = 1$ Calculate

Now, we have defined the phase change ϕ by $r = |r|\exp(-j\phi)$. Thus

$$\phi\sigma' = -\arctan\left(\frac{\mathrm{Im}[r\sigma']}{\mathrm{Re}[r\sigma']}\right)\frac{180}{\pi}$$

$\phi\sigma' = -(-53.412)$ Calculate

$\phi\sigma' = 53.412$ Calculate

We can also find it by using Equation (1.6.11)

$$\phi\sigma'' = \arctan\left(\frac{\sqrt{-n^2+[\sin\{\theta'\}]^2}}{\cos[\theta']}\right)2\frac{180}{\pi}$$

$\phi\sigma'' = 53.412$ Calculate

Now consider the parallel reflection coefficient

$$r\pi' = \frac{\sqrt{n^2-(\sin[\theta'])^2} - n^2\cos(\theta')}{\sqrt{n^2-(\sin[\theta'])^2} + n^2\cos(\theta')}$$

$r\pi' = 0.042257+0.99911i$ Calculate

This is complex. The magnitude (modulus) is 1

$|r\pi'|$

$|r\pi'| = 1$ Calculate

The argument as before is based on defining ϕ by $r = |r|\exp(-j\phi)$.

$$\phi\pi' = -\arctan\left(\frac{\mathrm{Im}[r\pi']}{\mathrm{Re}[r\pi']}\right)\frac{180}{\pi}$$

$\phi\pi' = -87.578$ Calculate

Alternatively

$$\phi\pi'' = \arctan\left(\frac{\sqrt{-n^2+[\sin\{\theta'\}]^2}}{\cos[\theta']n^2}\right)2\frac{180}{\pi} - 180$$

$\phi\pi'' = -87.578$ Calculate

These match Figure 1.16(b)

Reflectance is obviously 1

END OF CALCULATIONS

1.25 Goos-Haenchen phase shift A ray of light which is traveling in a glass medium (1) of refractive index $n_1 = 1.460$ becomes incident on a less dense glass medium (2) of refractive index $n_2 = 1.430$. Suppose that the free space wavelength of the light ray is 850 nm. The angle of incidence $\theta_i = 85°$. Estimate the lateral Goos-Haenchen shift in the reflected wave for the perpendicular field component. Recalculate the Goos-Haenchen shift if the second medium has $n_2 = 1$ (air). What is your conclusion? Assume that the virtual reflection occurs from a virtual plane in medium B at a distance d that is the same as the penetration depth. Note that d actually depends on the polarization, the direction of the field, but we will ignore this dependence.

Solution

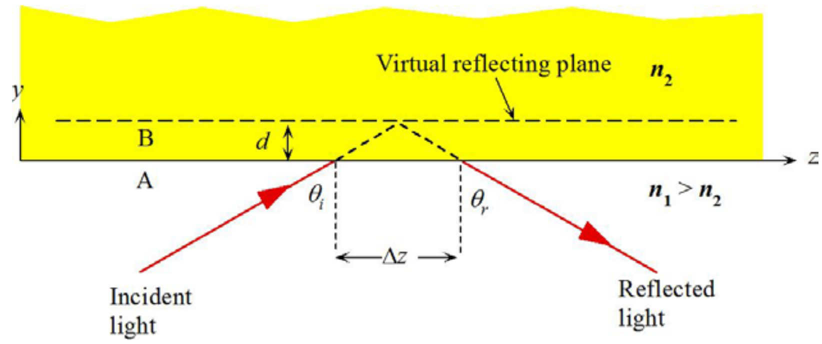


Figure 1.20 The reflected light beam in total internal reflection appears to have been laterally shifted by an amount Δz at the interface. It appears as though it is reflected from a virtual plane at a depth d in the second medium from the interface.

The problem is shown in Figure 1.20. When $\theta_i = 85°$,

$$\alpha_2 = \frac{2\pi n_2}{\lambda_o}\left[\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i - 1\right]^{1/2}$$

The penetration depth is $\delta = 1/\alpha_2 = 5.09\times10^{-7}$ m.

As an estimate, we can assume that $d \sim \delta$ so that the Goose-Haenchen shift is

$$\Delta z \approx 2d\tan\theta = 2(5.09\times10^{-7}\text{ m})(\tan 85°) = 11.6\times10^{-6}\text{ m} = \mathbf{11.6\ \mu m}$$

We can repeat the calculation using $n_2 = 1$ (air), then we find $\delta = 1/\alpha_2 = 1.28\times10^{-7}$ m, and $\Delta z \approx 2d\tan\theta = 2(1.28\times10^{-7}\text{ m})(\tan 85°) = 2.93\times10^{-6}\text{ m} = 2.93\ \mu\text{m}$. The shift is small when the refractive index difference is large. The wave penetrates more into the second medium when the refractive index difference is smaller. *Note:* The use of $d \approx \delta$ is a rough approximation to estimate Δz.

1.26 Evanescent wave Total internal reflection (TIR) of a plane wave from a boundary between a more dense medium (1) n_1 and a less dense medium (2) n_2 is accompanied by an evanescent wave propagating in medium 2 near the boundary. Find the functional form of this wave and discuss how its magnitude varies with the distance into medium 2.

Solution

The transmitted wave has the general form

$$E_{t,\perp} = t_\perp E_{io,\perp}\exp j(\omega t - \mathbf{k}_t\cdot\mathbf{r})$$

in which $t_\perp$ is the transmission coefficient. The dot product, examining

$$\mathbf{k}_t\cdot\mathbf{r} = yk_t\cos\theta_t + zk_t\sin\theta_t.$$

However, from Snell's law, when $\theta_i > \theta_c$, $\sin\theta_t = (n_1/n_2)\sin\theta_i > 1$ and $\cos\theta_t = \sqrt{[1 - \sin^2\theta_t]} = \pm jA_2$ is a purely imaginary number. Thus, taking $\cos\theta_t = -jA_2$

$$E_{t,\perp} = t_\perp E_{io,\perp}\exp j(\omega t - zk_t\sin\theta_t + jyk_tA_2) = t_\perp E_{io,\perp}\exp(-yk_tA_2)\exp j(\omega t - zk_t\sin\theta_t)$$

which has an amplitude that decays along y as $\exp(-\alpha_2 y)$ where $\alpha_2 = k_t A_2$. Note that $+jA_2$ is ignored because it implies a wave in medium 2 whose amplitude and hence intensity grows. Consider the traveling wave part $\exp j(\omega t - z k_t \sin\theta_t)$. Here, $k_t \sin\theta_t = k_i \sin\theta_i$ (by virtue of Snell's law). But $k_i \sin\theta_i = k_{iz}$, which is the wave vector along z, that is, along the boundary. Thus the evanescent wave propagates along z at the same speed as the incident and reflected waves along z. Furthermore, for TIR we need $\sin\theta_i > n_2/n_1$. This means that the transmission coefficient,

$$t_\perp = \frac{n_i \cos\theta_i}{\cos\theta_i + \left[\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_i\right]^{1/2}} = t_{\perp 0}\exp(j\psi_\perp)$$

must be a complex number as indicated by $t_{\perp 0}\exp(j\psi_\perp)$ in which $t_{\perp 0}$ is a real number and $\psi_\perp$ is a phase change. Note that $t_\perp$ does not, however, change the general behavior of propagation along z and the penetration along y.

1.27 TIR and FTIR

(a) By considering the electric field component in medium in Figure 1.22(b), explain how you can adjust the amount of transmitted light.

(b) What is the critical angle at the hypotenuse face of a beam splitter cube (Figure 1.22 (b)) made of glass with n_1 = 1.6 and having a thin film of liquid with n_2 =1.3. Can you use 45° prisms with normal incidence?

(c) Explain how a light beam can propagate along a layer of material between two different media as shown in Figure 1.59 (a). Explain what the requirements are for the indices n_1, n_2, n_3. Will there be any losses at the reflections?

(d) Consider the prism coupler arrangement in Figure 1.59(b). Explain how this arrangement works for coupling an external light beam from a laser into a thin layer on the surface of a glass substrate. Light is then propagated inside the thin layer along the surface of the substrate. What is the purpose of the adjustable coupling gap?

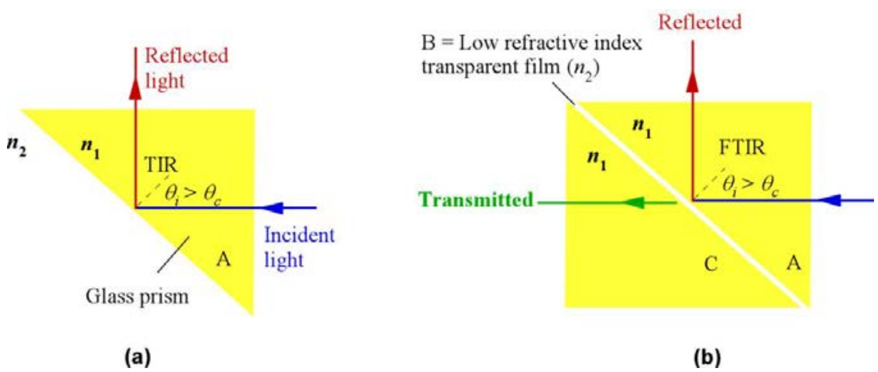


Figure 1.22 (a) A light incident at the long face of a glass prism suffers TIR; the prism deflects the light. (b) Two prisms separated by a thin low refractive index film forming a beam-splitter cube. The incident beam is split into two beams by FTIR.

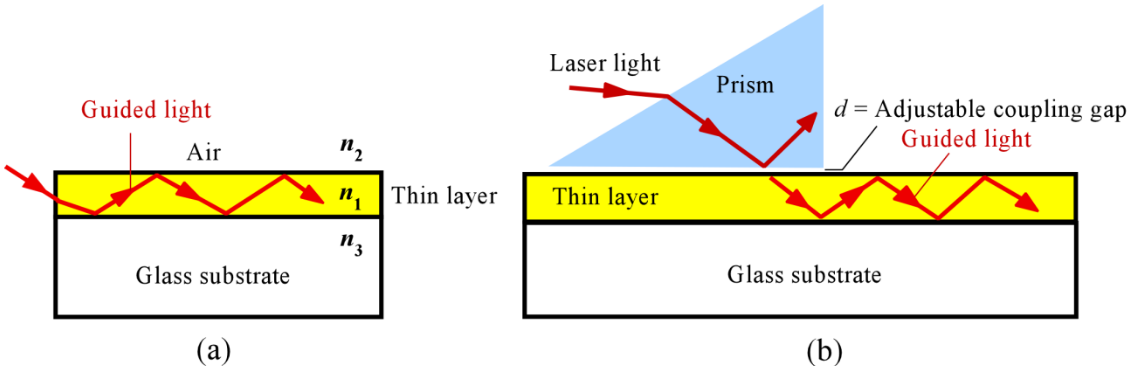


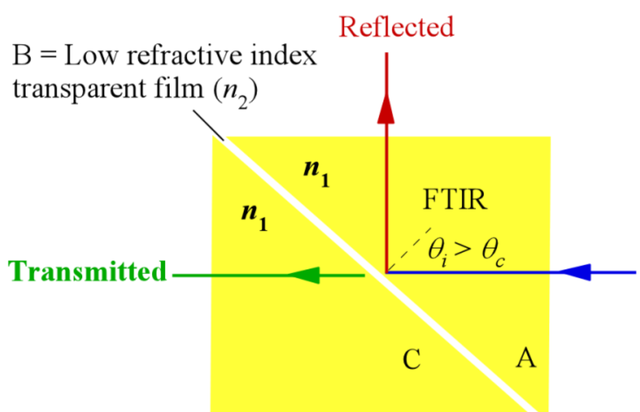
Figure 1.59 (a) Light propagation along an optical guide. (b) Coupling of laser light into a thin layer - optical guide - using a prism. The light propagates along the thin layer.

Solution

(a) Consider the prism A when the neighboring prism C in Figure 1.22 (b) in far away. When the light beam in prism A is incident on the A/B interface, hypotenuse face, it suffers TIR as $\theta_i > \theta_c$. There is however an evanescent wave whose field decays exponentially with distance in medium *B*. When we bring prism C close to A, the field in B will reach C and consequently penetrates C. (The tangential field must be continuous from B to C). One cannot just use the field expression for the evanescent wave because this was derived for a light beam incident at an interface between two media only; no third medium. The transmitted light intensity from A to C depends on the thickness of B.

(b) For the prism *A* in Figure 1.22 (b), $n_1 = 1.6$ and $n_2 = 1.3$ so that the critical angle for TIR at the hypotenuse face is

$$\theta_c = \arcsin(n_2/n_1) = \arcsin(1.3/1.6) = \mathbf{54.3°}$$



In this case, one *cannot* use a 45 ° prism.

(d) If the angle of incidence θ_i at the n_1/n_2 layer is more than the critical angle θ_{c12} and if angle of incidence θ_i at the n_1/n_3 layer is more than the critical angle θ_{13} then the light ray will travel by TIR, zigzagging between the boundaries as sketched in Figure 1.59(a). For example, suppose that $n_1 = 2$ (thin layer); $n_2 = 1$ (air) and $n_3 = 1.6$ (glass),

$$\theta_{c12} = \arcsin(n_2/n_1) = \arcsin(1/2) = 38.8°,$$

and θ_{c13} = arcsin(n_3/n_1) = arcsin(1.6/2) = 53.1°,

so that θ_i > 53.1° will satisfy TIR. There is no loss in TIR as the magnitude of the amplitude of the reflected way is the same as that of the incident wave.

Note: There is an additional requirement that the waves entering the thin film interfere constructively, otherwise the waves will interfere destructively to cancel each other. Thus there will be an additional requirement, called the *waveguide condition*, which is discussed in Chapter 2.

(e) The light ray entering the prism is deflected towards the base of the prism as shown in Figure 1.59 (b). There is a small gap between the prism and the thin layer. Although the light arriving at the prism base/gap interface is reflected, because of the close proximity of the thin layer, some light is coupled into the thin layer per discussion in Part (a) due to frustrated TIR. This arrangement is a much more efficient way to couple the light into the thin layer because the incident light is received by the large hypotenuse face compared with coupling the light directly into the thin layer.

1.28 Complex refractive index and dielectric constant The complex refractive index N = $n - jK$ can be defined in terms of the *complex relative permittivity* $\varepsilon_r = \varepsilon_{r1} - j\varepsilon_{r2}$ as

$$\mathrm{N} = n - jK = \varepsilon_r^{1/2} = (\varepsilon_{r1} - j\varepsilon_{r2})^{1/2}$$

where ε_{r1} and ε_{r2} are the real and imaginary parts of ε_r. Show that

$$n = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} + \varepsilon_{r1}}{2}\right]^{1/2} \quad \text{and} \quad K = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} - \varepsilon_{r1}}{2}\right]^{1/2}$$

Solution

Given $\mathrm{N} = n - jK = \sqrt{\varepsilon_r} = \sqrt{\varepsilon_{r1} - j\varepsilon_{r2}}$

we have $n^2 - 2jnK - K^2 = \varepsilon_{r1} - j\varepsilon_{r2}$

$\therefore$ $2nK = \varepsilon_{r2}$ (1)

and $n^2 - K^2 = \varepsilon_{r1}$ (2)

$\therefore$ $K = \varepsilon_{r2}/2n$

and substituting into the second equation above,

$$n^2 - \left(\frac{\varepsilon_{r2}}{2n}\right)^2 = \varepsilon_{r1}$$

$\therefore$ $n^4 - n^2\varepsilon_{r1} - \tfrac{1}{4}\varepsilon_{r2}^2 = 0$

$\therefore$ $n^2 = \dfrac{\varepsilon_{r1} \pm \sqrt{\varepsilon_{r1}^2 - 4(-\tfrac{1}{4}\varepsilon_{r2}^2)}}{2} = \dfrac{\varepsilon_{r1} \pm \sqrt{\varepsilon_{r1}^2 + \varepsilon_{r2}^2}}{2}$

It is apparent that n^2 has two solutions. The negative sign has to be excluded because this would make the numerator negative and lead to a complex number for n. By definition, n is a real number, and not imaginary. Thus,

$$n^2 = \frac{\varepsilon_{r1} + \sqrt{\varepsilon_{r1}^2 + \varepsilon_{r2}^2}}{2}$$

$$\therefore \qquad n = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} + \varepsilon_{r1}}{2}\right]^{1/2}$$

From Eq. (1), $n = \varepsilon_{r2}/2K$, Substitute this into Eq. (2),

$$\left(\frac{\varepsilon_{r2}}{2K}\right)^2 - K^2 = \varepsilon_{r1}$$

$$\therefore \qquad K^4 + K^2\varepsilon_{r1} - \tfrac{1}{4}\varepsilon_{r2}^2 = 0$$

$$\therefore \qquad K^2 = \frac{-\varepsilon_{r1} \pm \sqrt{\varepsilon_{r1}^2 - 4(-\tfrac{1}{4}\varepsilon_{r2}^2)}}{2} = \frac{-\varepsilon_{r1} + \sqrt{\varepsilon_{r1}^2 + \varepsilon_{r2}^2}}{2}$$

Where the negative sign is excluded as K cannot imaginary. Thus,

$$\therefore \qquad K = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} - \varepsilon_{r1}}{2}\right]^{1/2}$$

1.29 Complex refractive index Spectroscopic ellipsometry measurements on a germanium crystal at a photon energy of 1.5 eV show that the real and imaginary parts of the complex relative permittivity are 21.56 and 2.772 respectively. Find the complex refractive index. What is the reflectance and absorption coefficient at this wavelength? How do your calculations match with the experimental values of $n = 4.653$ and $K = 0.298$, $R = 0.419$ and $\alpha = 4.53 \times 10^6$ m^{-1}?

Solution

From problem 1.28 we have

$$n = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} + \varepsilon_{r1}}{2}\right]^{1/2} = \left[\frac{(21.56^2 + 2.772^2)^{1/2} + 21.56}{2}\right]^{1/2} = \mathbf{4.653}$$

Similarly

$$K = \left[\frac{(\varepsilon_{r1}^2 + \varepsilon_{r2}^2)^{1/2} - \varepsilon_{r1}}{2}\right]^{1/2} = \left[\frac{(21.56^2 + 2.772^2)^{1/2} - 21.56}{2}\right]^{1/2} = \mathbf{0.298}$$

Almost an exact agreement (not surprisingly).

The reflectance R is given by

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2} = \frac{(4.653-1)^2 + 0.298^2}{(4.653+1)^2 + 0.298^2} = \mathbf{0.42} \text{ or } \mathbf{42\%}$$

The absorption coefficient α is $2k''$ as in Eq. (1.8.67) so that

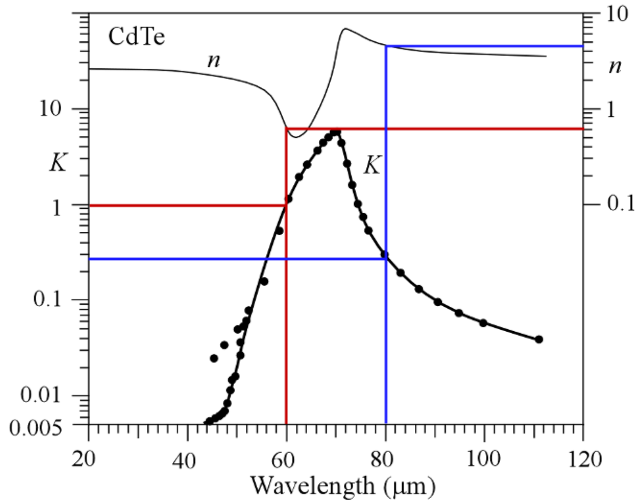
$$\alpha = 2k'' = 2k_oK = 2(2\pi/\lambda_o)K = 2(2\pi\upsilon/c)K$$

$$\therefore \qquad \alpha = \frac{2(2\pi)h\upsilon K}{hc} = \frac{2(2\pi)(1.5\,\text{eV})(0.298)}{(4.136\times10^{-15}\,\text{eV s})(3\times10^8\,\text{m s}^{-1})} = \mathbf{4.53\times10^{-6}\ m^{-1}}$$

which agrees with the measurements.

1.30 Complex refractive index Figure 1.26 shows the infrared extinction coefficient K of CdTe . Calculate the absorption coefficient α and the reflectance R of CdTe at 60 μm and 80 μm.

Solution



At 60 μm:

$K \approx 1$, and $n \approx 0.6$, so that the corresponding free-space wave vector is

$$k_o = 2\pi/\lambda = 2\pi/(60\times10^{-6}\text{ m}) = 1.05\times10^{5}\text{ m}^{-1}.$$

The absorption coefficient α is $2k''$ as in Eq. (1.8.6) so that

$$\alpha = 2k'' = 2k_oK = 2(1.05\times10^{5}\text{ m}^{-1})(1) = \mathbf{2.1\times10^{5}\ m^{-1}}$$

which corresponds to an *absorption depth* $1/\alpha$ of about 4.8 micron. The reflectance is

$$R = \frac{(n-1)^2+K^2}{(n+1)^2+K^2} = \frac{(0.6-1)^2+1^2}{(0.6+1)^2+1^2} = 0.32 \text{ or } \mathbf{32\%}$$

At 80 μm:
$K \approx 0.27$, and $n \approx 4.5$, so that the corresponding free-space wave vector is

$$k_o = 2\pi/\lambda = 2\pi/(80\times10^{-6}\text{ m}) = 7.85\times10^{4}\text{ m}^{-1}.$$

The absorption coefficient α is $2k''$ so that

$$\alpha = 2k'' = 2k_oK = 2(7.85\times10^{4}\text{ m}^{-1})(0.27) = \mathbf{4.2\times10^{4}\ m^{-1}}$$

which corresponds to an *absorption depth* $1/\alpha$ of about 24 micron. The reflectance is

$$R = \frac{(n-1)^2+K^2}{(n+1)^2+K^2} = \frac{(0.6-1)^2+1^2}{(0.6+1)^2+1^2} = 0.41 \text{ or } \mathbf{41\%}$$

1.31 Refractive index and attenuation in the infrared region - Reststrahlen absorption Figure 1.26 shows the refractive index n and the attenuation (absorption) coefficient K as a function of wavelength λ in the infrared for a CdTe crystal due to lattice absorption, called Reststrahlen absorption. It results from the ionic polarization of the crystal induced by the optical field in the light wave. The relative permittivity ε_r due to positive (Cd^{2+}) and negative (Te^{2-}) ions being made to oscillate by the optical field about their equilibrium positions is given in its simplest form by

$$\varepsilon_r = \varepsilon_r' - j\varepsilon_r'' = \varepsilon_{rH} + \frac{\varepsilon_{rH} - \varepsilon_{rL}}{\left(\frac{\omega}{\omega_T}\right)^2 - 1 + j\frac{\gamma}{\omega_T}\left(\frac{\omega}{\omega_T}\right)} \quad (1)$$

where ε_{rL} and ε_{rH} are the relative permittivity at low (L) and high (H) frequencies, well below and above the infrared peak, γ is a loss coefficient characterizing the rate of energy transfer from the EM wave to lattice vibrations (phonons), and ω_T is a transverse optical lattice vibration frequency that is related to the nature of bonding between the ions in the crystal. Table 1.3 provides some typical values for CdTe and GaAs. Eq. (1) can be used to obtain a reasonable approximation to the infrared refractive index n and absorption K due to Reststrahlen absorption.

(a) Consider CdTe, and plot n and K vs. λ from 40 μm to 90 μm and compare with the experimental results in Figure 1.26 in terms of the peak positions and the width of the absorption peak.

(b) Consider GaAs, and plot n and K vs. λ from 30 μm to 50 μm.

(c) Calculate n and K for GaAs at λ = 38.02 μm and compare with the experimental values n = 7.55 and K = 0.629.

TABLE 1.4 Ionic polarization resonance parameters for CdTe and GaAs

	ε_{rL}	ε_{rH}	ω_T (rad s^{-1})	γ (rad s^{-1})
CdTe	10.20	7.10	2.68×10^{13}	0.124×10^{13}
GaAs	13.0	11.0	5.07×10^{13}	0.045×10^{13}

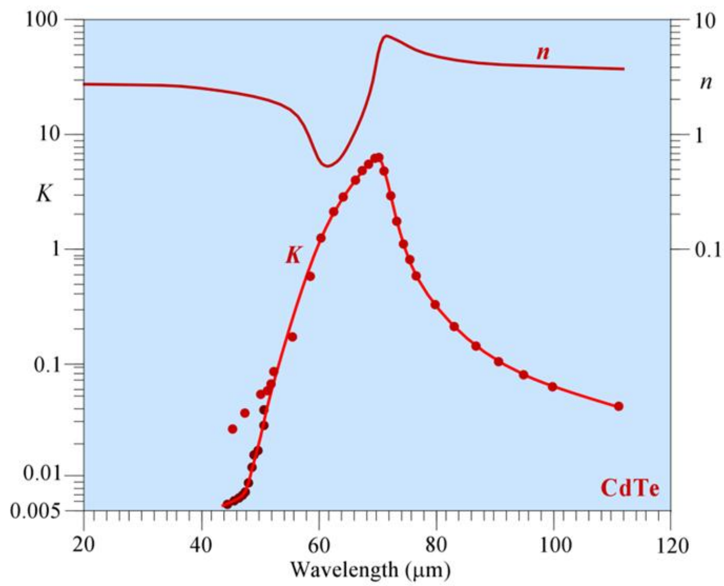


Figure 1.26 Optical properties of CdTe as a function of wavelength in the infrared region.

(c) Calculate n and K for GaAs at λ = 38.02 μm and compare with the experimental values n = 7.55 and K = 0.629.

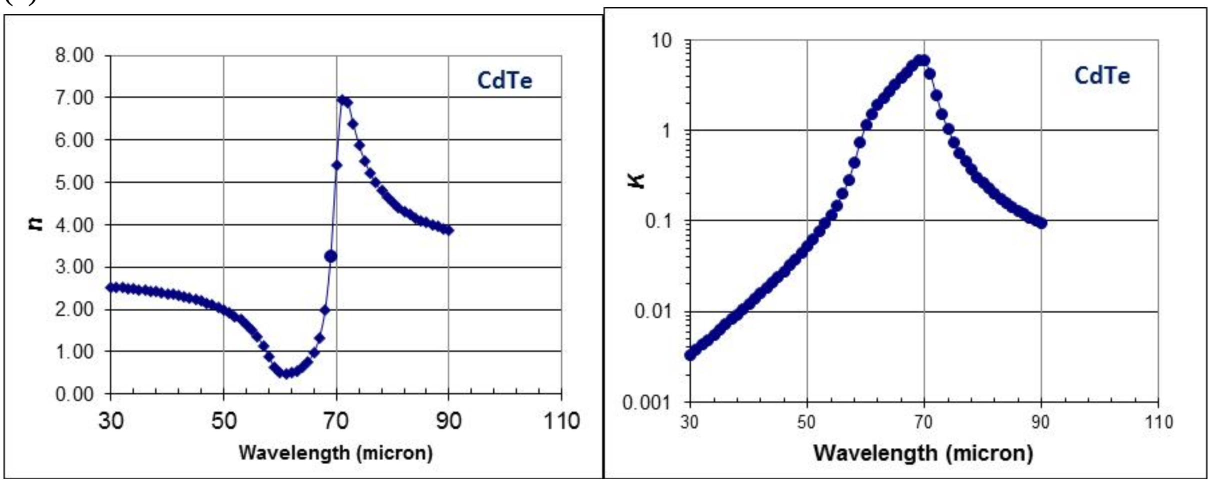
Solution

From Question 1.28

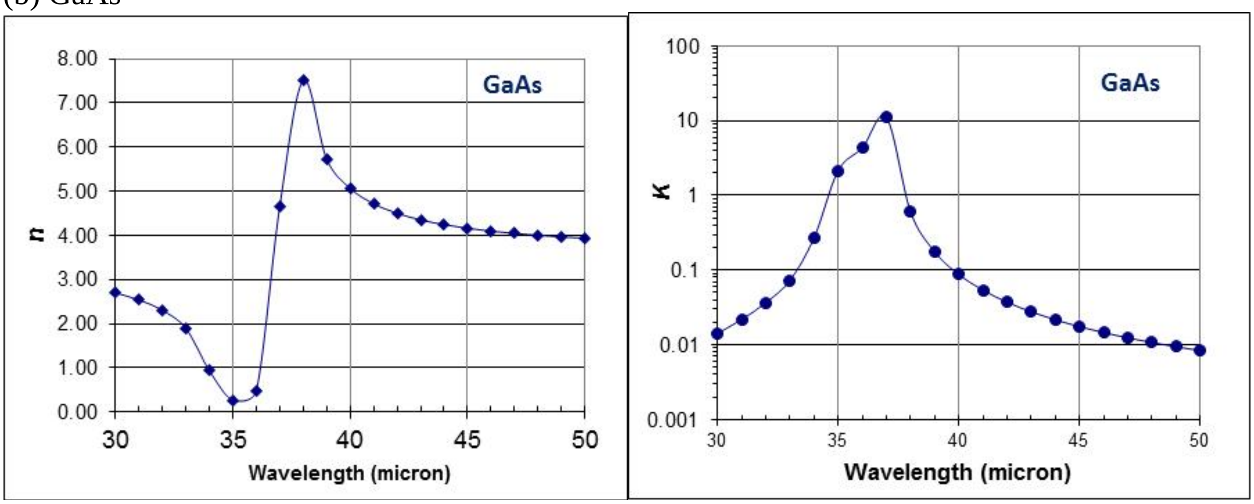
$$n = \sqrt{\frac{1}{2}\left(\varepsilon_r' + \sqrt{(\varepsilon_r')^2 + (\varepsilon_r'')^2}\right)} \qquad K = \sqrt{n^2 - \varepsilon_r'}$$

which means that we can substitute the values of ε_r' and ε_r'' from Eq. (1) into the above two equations and plot n and K as a function of *wavelength*.

(a) CdTe



(b) GaAs



More points can also be used

(c) For GaAs at 38.02 μm, the calculated values are n = 7.44 and K = 0.586, which compare reasonable well with experimental values of n = 7.55 and K = 0.629.

1.32 Coherence length A particular laser is operating in single mode and emitting a continuous wave lasing emission whose spectral width is 1 MHz. What is the coherence time and coherence length?

Solution

The spectral width in frequency $\Delta\upsilon$ and the coherence time Δt are related by

$$\Delta\upsilon \approx \frac{1}{\Delta t}$$

Thus, the coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(1\times10^{6}\text{ Hz}) = 10^{-6}\text{ s or }1\mu\text{s}.$$

The coherence length is

$$l_c = c\Delta t = (3\times10^{8}\text{ m s}^{-1})\times10^{-6}\text{ s} = 300\text{m}$$

1.33 Spectral widths and coherence

(a) Suppose that frequency spectrum of a radiation emitted from a source has a central frequency υ_o and a spectral width $\Delta\upsilon$. The spectrum of this radiation in terms of wavelength will have a central wavelength λ_o and a spectral width $\Delta\lambda$. Clearly, $\lambda_o = c/\upsilon_o$. Since $\Delta\lambda << \lambda_o$ and $\Delta\upsilon << \upsilon_o$, using $\lambda = c/\upsilon$, show that the line width $\Delta\lambda$ and hence the coherence length l_c are

$$\Delta\lambda = \Delta\upsilon\frac{\lambda_o}{\upsilon_o} = \Delta\upsilon\frac{\lambda_o^2}{c} \qquad \text{and} \qquad l_c = c\Delta t = \frac{\lambda_o^2}{\Delta\lambda}$$

(b) Calculate $\Delta\lambda$ for a lasing emission from a He-Ne laser that has λ_o = 632.8 nm and $\Delta\upsilon \approx$ 1.5 GHz. Find its coherence time and length.

Solution

(a) See Example 1.9.3

(b) Consider the width in wavelength,

$\Delta\lambda = \Delta\upsilon(\ \lambda^2/c) = (1.5\times10^{9}\text{ s}^{-1}\ (632/8\times10^{-9}\text{ m})^2/(3\times10^{8}\text{ m s}^{-1}) = \mathbf{3.16\times10^{-6}\ m}$.

The coherence time is

$\Delta t \approx 1/\ \Delta\upsilon = 1/(1.5\times10^{9}\text{ Hz}) = 0.666\times10^{-9}\text{ s}$

The coherence length is

$l_c = c\Delta t = (3\times10^{8}\text{ m s}^{-1})(0.666\times10^{-9}\text{ s}) = 0.20\text{ m} = 20\text{ cm}$

1.34 Coherence lengths Find the coherence length of the following light sources

(a) An LED emitting at 1550 nm with a spectral width 150 nm

(b) A semiconductor laser diode emitting at 1550 nm with a spectral width 3 nm

(c) A quantum well semiconductor laser diode emitting at 1550 nm with a spectral with of 0.1 nm

(d) A multimode HeNe laser with a spectral frequency with of 1.5 GHz

(e) A specially designed single mode and stabilized HeNe laser with a spectral width of 100 MHz

Solution

(a) $$\frac{\Delta\upsilon}{\Delta\lambda} \approx \left|\frac{d\upsilon}{d\lambda}\right| = \left|-\frac{c}{\lambda^2}\right|$$

so that $\Delta\upsilon = \Delta\lambda(c/\lambda^2) = (150\times10^{-9}\text{ m})(3\times10^{8}\text{ m s}^{-1})/(1550\times10^{-9}\text{ m})^2 = 1.873\times10^{13}\text{ Hz}$

Thus, the coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(1.873\times10^{13}\text{ Hz}) = 5.34\times10^{-14}\text{ s}$$

The coherence length is

$$l_c = c\Delta t = 1.6\times10^{-5}\text{ m or } \mathbf{16\ \mu m}$$

(b)

$\Delta\upsilon = \Delta\lambda(c/\lambda^2) = (3\times10^{-9}\text{ m})(3\times10^{8}\text{ m s}^{-1})/(1550\times10^{-9}\text{ m})^2 = 3.746\times10^{11}\text{ Hz}$

The coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(3.746\times10^{11}\text{Hz}) = 2.67\times10^{-12}\text{ s or } 2.67\text{ fs}$$

The coherence length is

$$l_c = c\Delta t = 8.01\times10^{-4}\text{ m or } \mathbf{0.8\ mm}$$

(c) Apply $\Delta\upsilon = \Delta\lambda(c/\lambda^2)$, that is,

$\Delta\upsilon = \Delta\lambda(c/\lambda^2) = (0.1\times10^{-9}\text{ m})(3\times10^{8}\text{ m s}^{-1})/(1550\times10^{-9}\text{ m})^2 = 1.248\times10^{10}$ Hz

The coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(1.248\times10^{10}\text{ Hz}) = 8.01\times10^{-11}\text{ s or } 80.1\text{ fs}$$

The coherence length is

$$l_c = c\Delta t = 2.4\times10^{-2}\text{ m or } \mathbf{24\ mm}$$

(d) Apply

$$\Delta\upsilon \approx \frac{1}{\Delta t}$$

Thus, the coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(1.5\times10^{9}\text{ Hz}) = 6.66\times10^{-10}\text{ s or } 666\text{ fs.}$$

The coherence length is

$$l_c = c\Delta t = (3\times10^{8}\text{ m s}^{-1})\times6.66\times10^{-10}\text{ s} = 0.2\text{ m} = \mathbf{20\ cm}$$

(e)

$$\Delta\upsilon \approx \frac{1}{\Delta t}$$

Thus, the coherence time is

$$\Delta t \approx 1/\ \Delta\upsilon = 1/(100\times10^{6}\text{ Hz}) = 10^{-8}\text{ s}$$

The coherence length is

$$l_c = c\Delta t = (3\times10^{8}\text{ m s}^{-1})\times10^{-8}\text{ s} = \mathbf{3\ m}$$

1.35 Fabry-Perot optical cavity Consider an optical cavity formed between two identical mirrors. The cavity length is 50 cm and the refractive index of the medium is 1. The mirror reflectances are 0.97 each. What is the nearest mode number that corresponds to a radiation of wavelength 632.8 nm? What is the actual wavelength of the mode closest to 632.8 nm? What is the mode separation in frequency and wavelength? What are the finesse F and Q-factors for the cavity?

Solution

For $\lambda = \lambda_o = 632.8$ nm, the corresponding mode number m_o is,

$$m_o = 2L\ /\ \lambda_o = (2\times0.5\text{ m})\ /\ (632.8\times10^{-9}\text{ m}) = 1580278.1$$

and actual m_o has to be the closest integer value to 1580278.1, that is **1580278**

The actual wavelength of the mode closest to 632.8 nm is $\lambda_o = 2L\ /\ m_o = (2\times0.5\text{ m})\ /\ (1580278) =$ 632.80005 nm

The frequency separation $\Delta\upsilon_m$ of two consecutive modes is

$$\Delta\upsilon_m = \upsilon_{m+1} - \upsilon_m = \frac{c}{\lambda_{m+1}} - \frac{c}{\lambda_m} = \frac{c}{\dfrac{2L}{(m+1)}} - \frac{c}{\dfrac{2L}{m}} = \frac{c}{2L}$$

or $$\Delta\upsilon_m = \frac{c}{2L} = \frac{3\times10^8}{2(0.5)} = \mathbf{3\times10^8\ Hz}.$$

The wavelength separation of two consecutive modes is

$$\Delta\lambda_m = \frac{\lambda_m^2}{2L} = \frac{(632.8\times10^{-9})^2}{2(0.5)} = 4.004\times10^{-13}\text{ m or } \mathbf{0.400\ pm}.$$

Finesse is

$$F = \frac{\pi R^{1/2}}{1-R} = \mathbf{103.1}$$

Quality factor is

$$Q = m_0F = 1580278\times103.14 = \mathbf{1.63\times10^8}$$

1.36 Fabry-Perot optical cavity from a ruby crystal Consider a ruby crystal of diameter 1 cm and length 10 cm. The refractive index is 1.78. The ends have been silvered and the reflectances are 0.99 and 0.95 each. What is the nearest mode number that corresponds to a radiation of wavelength 694.3 nm? What is the actual wavelength of the mode closest to 694.3 nm? What is the mode separation in frequency and wavelength? What are the finesse F and Q-factors for the cavity?

Solution

Number mode nearest to the emission wavelength is

$$m = \frac{2L}{\lambda/n} = (2\times10\text{ cm})\times(1.78)/(694.3\text{ nm}) = 512746.65 \text{ i.e. } i.e.\ m_0 = \mathbf{512746}.$$

The actual wavelength of the mode closest to 694.3 nm is

$$\lambda_0 = \frac{2Ln}{m_0} = (2\times10\text{ cm})\times(1.78)/(512746) = \mathbf{694.3008819}\text{ nm}$$

The frequency separation $\Delta\upsilon_m$ of two consecutive modes is

$$\Delta\upsilon_m = \upsilon_{m+1} - \upsilon_m = \frac{c}{\lambda_{m+1}} - \frac{c}{\lambda_m} = \frac{c}{2Ln/(m+1)} - \frac{c}{2Ln/m} = \frac{c}{2Ln} = \mathbf{8.43\times10^9\ Hz}.$$

The wavelength separation of two consecutive modes is

$$\Delta\lambda = \frac{\lambda_m^2}{2Ln} = \mathbf{0.00135408\ nm = 1.35\ pm}$$

Average geometric reflectance is $R = (R_1R_2)^{1/2}$=0.96979

Finesse, $F = \dfrac{\pi R^{1/2}}{1-R} = \mathbf{102.42}$

Quality factor, $Q = m_0F$ =1580278×103.14= $\mathbf{5.25\times10^7}$

1.37 Fabry-Perot optical cavity spectral width Consider an optical cavity of length 40 cm. Assume the refractive index is 1, and use Eq. (1.11.3) to plot the peak closest to 632.8 nm for 4 values of R = 0.99, 0.90, 0.75 and 0.6. For each case find the spectral width $\delta\lambda_m$,, the finesse F and Q. How accurate is Eq.(1.11.5) in predicting $\delta\lambda_m$? (You may want to use a graphing software for this problem.)

Solution

$$\upsilon_m = m\left(\frac{c}{2L}\right) = m\upsilon_f\,; \qquad \upsilon_f = c/(2L) \qquad \textit{Cavity resonant frequencies} \qquad (2)$$

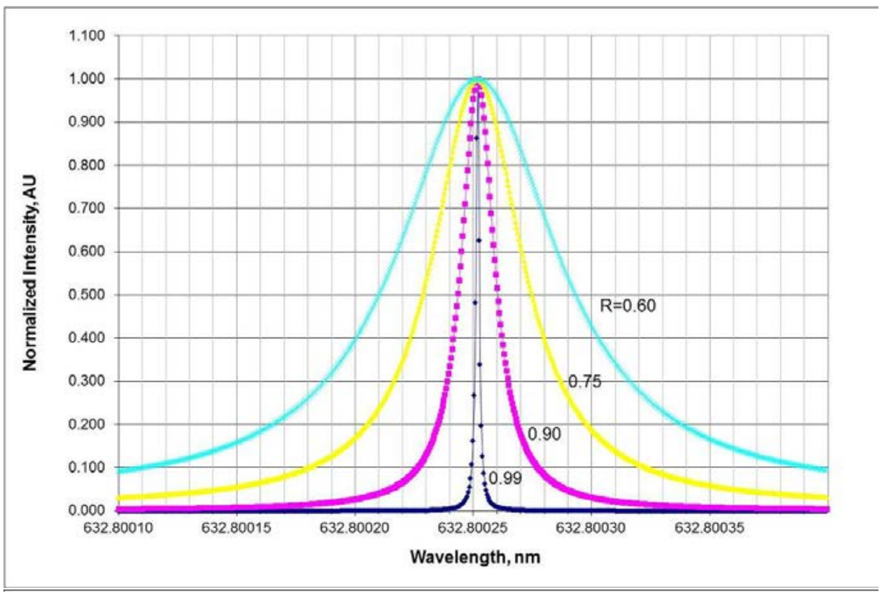
$$I_{\text{cavity}} = \frac{I_o}{(1-R)^2 + 4R\sin^2(kL)} \qquad \textit{Cavity intensity} \qquad (3)$$

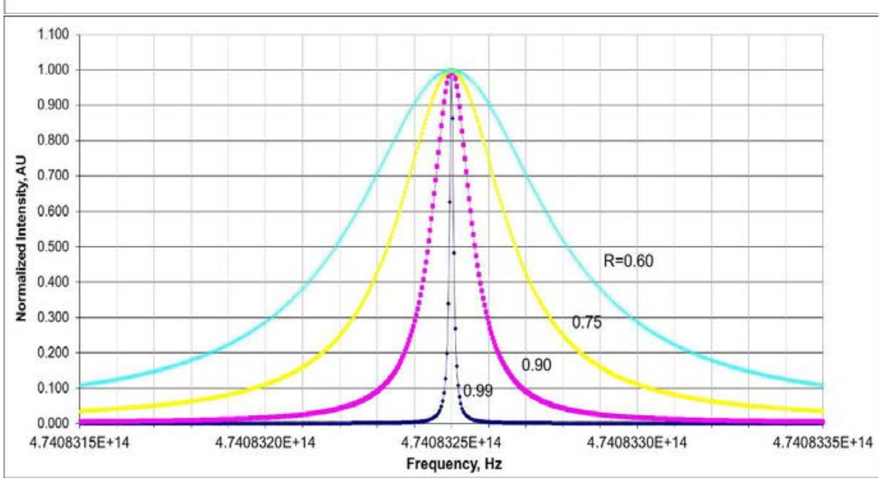
$$I_{\max} = \frac{I_o}{(1-R)^2}\,; \qquad k_m L = m\pi \qquad \textit{Maximum cavity intensity} \qquad (4)$$

$$\delta\upsilon_m = \frac{\upsilon_f}{F}\,; \qquad F = \frac{\pi R^{1/2}}{1-R} \qquad \textit{Spectral width} \qquad (5)$$

$$\text{Quality factor, } Q = \frac{\text{Resonant frequency}}{\text{Spectral width}} = \frac{\upsilon_m}{\delta\upsilon_m} = mF \qquad (6)$$

Cavity fundamental mode is $\upsilon_f = c/(2L) = 3.75\times10^8$ Hz. The Graph below shows that the peak closest to 632.8 nm is 632.80025 nm which corresponds to $\upsilon_m = 4.7408325\times10^{14}$ Hz.





AU: Arbitrary Units

R	0.99	0.9	0.75	0.6	Source
$\delta\lambda$, nm	1.5000E-06	1.6750E-05	4.6000E-05	8.3000E-05	From Graph
$\delta\nu_{exp}$, MHz	1.12	12.55	34.46	62.18	From Graph
$\delta\nu_{calc}$, MHz	1.20	12.58	34.46	61.64	$\delta\upsilon_m = \dfrac{\upsilon_f}{F}$
$F_{calculation}$	312.58	29.80	10.88	6.08	$F = \dfrac{\pi R^{1/2}}{1-R}$
$F_{experiment}$	333.70	29.88	10.88	6.03	$F = \dfrac{\upsilon_f}{\delta\upsilon_m}$
Q	4.219E+14	3.778E+13	1.376E+13	7.624E+12	$Q = \dfrac{\upsilon_m}{\delta\upsilon_m}$

1.38 Diffraction Suppose that a collimated beam of light of wavelength 600 nm is incident on a circular aperture of diameter of 200 μm. What is the divergence of the transmitted beam? What is the diameter at a distance 10 m? What would be the divergence if the aperture were a single slit of width 200 μm?

Solution

(a)

$$\sin\theta_o = 1.22\frac{\lambda}{D} = 1.22\frac{600\times10^{-9}}{200\times10^{-6}} = 3.66\times10^{-3}$$

$$\theta_o = 0.209^\circ$$

The divergence angle is

$$2\theta_o = 0.418^\circ$$

If *R* is the distance of the screen from the aperture, then the radius of the Airy disk, approximately *b*, can be calculated from $b/R = \tan\theta_o \approx \theta_o$

$$b = R\tan\theta_o = 10\times\tan(0.209^\circ) = 0.036m = 3.6cm$$

Thus, the diameter is

2*b* = 0.072 m or **7.2 cm**

(b)

Divergence from a single slit of width *a* is

$$\Delta\theta = 2\theta_o \approx \frac{2\lambda}{a} = \frac{2\times600\times10^{-9}}{200\times10^{-6}} = 6\times10^{-3}rad = 6\times10^{-3}\times\frac{360}{2\pi} = 0.34^\circ$$

1.39 Diffraction Consider diffraction from a uniformly illuminated circular aperture of diameter *D*. The far field diffraction pattern is given by a Bessel function of the first kind and first order, J_1, and the intensity at a point *P* on the angle θ with respect to the central axis through aperture is

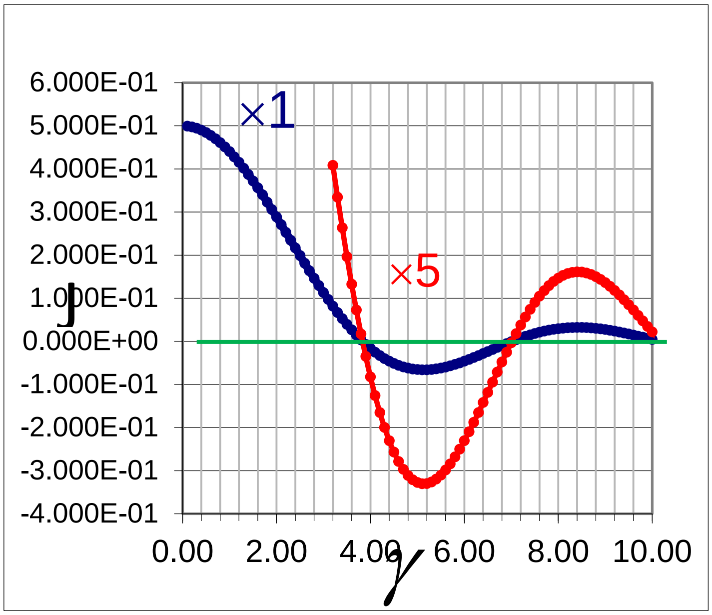
$$I(\gamma) = I_o\left(\frac{2J_1(\gamma)}{\gamma}\right)^2$$

where I_o is the maximum intensity, $\gamma = (1/2)kD\sin\theta$ is a variable quantity that represents the angular position θ on the screen as well as the wavelength ($k = 2\pi/\lambda$) and the aperture diameter D. $J_1(\gamma)$ can be calculated from

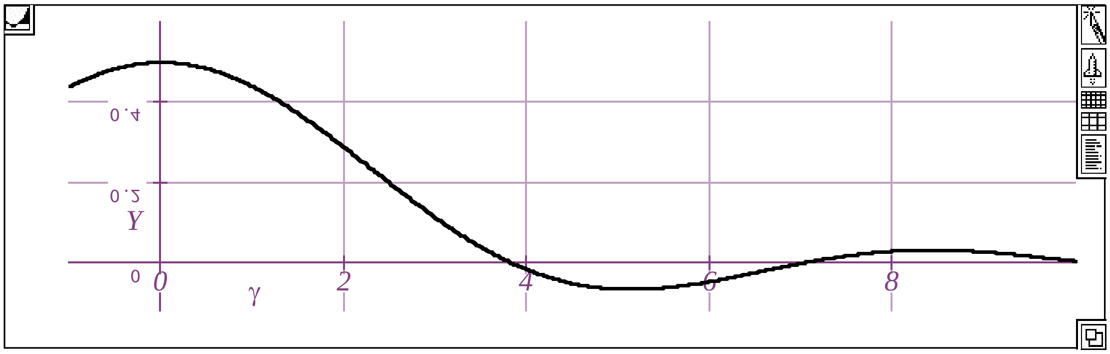
$$J_1(\gamma) = \frac{1}{\pi}\int_0^{\pi} \cos(\alpha - \gamma\sin\alpha)d\alpha$$

Using numerical integration (or a suitable math software package), plot $[J_1(\gamma)/\gamma]$ vs. γ for $\gamma = 0$ to 10 using suitable number of points, and then find the zeros. What are the first two γ that lead to dark rings?

Solution



There are two zeros at $\gamma = 3.80$ and 7.00



The above is from Livemath (Theorist)

The ratio of the intensity of first bright ring to the intensity at the center of the Airy disk

$$= \frac{\left(\left[\dfrac{J_1(\gamma)}{\gamma}\right]_{\gamma\to 0}\right)^2}{\left(\dfrac{J_1(5.14)}{5.14}\right)} = 0.017$$

Additional Problem

Consider an aperture that is 50 μm in diameter and illuminated by a 550 nm green laser light beam. If the screen is 2 m away, what are the radius of the first dark ring?The first dark ring occurs when $\gamma = 3.83$ so that $\gamma = (1/2)kD\sin\theta = (1/2)(2\pi/\lambda)D\sin\theta$

$$\gamma = 3.83 \ = (1/2)kD\sin\theta = (1/2)[2\pi/(0.550\ \mu\text{m})](50\ \mu\text{m})\sin\theta$$

gives $\theta = 1.16^{\text{o}}$.

Let R be the distance from the aperture to the screen. If the radius of the dark ring is r then $r/R = \tan\theta$. Thus substituting R = 2 m, and $\theta = 1.16^{\text{o}}$ we find r = 0.024 m and $2r$ = 0.048 m.

1.40 Bragg diffraction Suppose that parallel grooves are etched on the surface of a semiconductor to act as a reflection grating and that the periodicity (separation) of the grooves is 1 micron. If light of wavelength 1.3 μm is incident at an angle 89° to the normal, find the diffracted beams.

Solution

When the incident beam is not normal to the diffraction grating, then the diffraction angle θ_m for the m-th mode is given by,

$$d(\sin\theta_m - \sin\theta_i) = m\lambda \ ; m = 0, \pm1, \pm2, \ldots$$

so that for first order

$$(1\ \mu\text{m})(\sin\theta_m - \sin(89°) = (+1)(1.3\ \mu\text{m})$$

and

$$(1\ \mu\text{m})(\sin\theta_m - \sin(89°) = (-1)(1.3\ \mu\text{m})$$

Solving these two equations, we find θ_m = complex number for m = 1, and θ_m = 17.5° for m = –1. θ_m = 17.5° for m = –1, in fact, is the only solution.

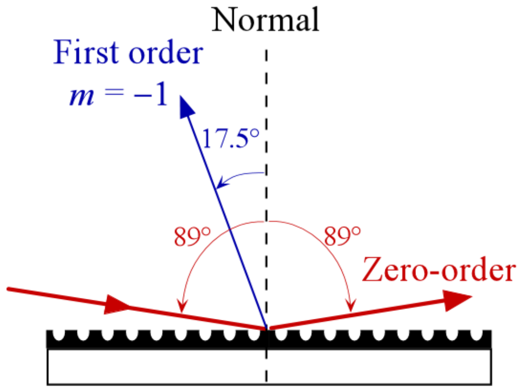


Figure 1Q40-1 There is only one diffracted beam, which corresponds to m = –1.

1.41 Diffraction grating for WDM Consider a transmission diffraction grating. Suppose that we wish to use this grating to separate out different wavelengths of information in a WDM signal at 1550 nm. (WDM stands of wavelength division multiplexing.) Suppose that the diffraction grating has a periodicity of 2 μm. The angle of incidence is 0° with respect to the normal to the diffraction grating.

What is the angular separation of the two wavelength component s at 1.550 μm and 1.540 μm? How would you increase this separation?

Solution

Consider the transmission grating shown in Figure 1Q41-1 with normal incidence, $\theta_i = 0$

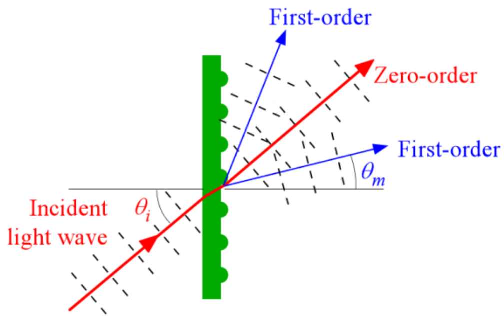


Figure 1Q41-1 Transmission gratings.

The grating equation for normal incidence with the grating in air is given by

$$d\sin\theta = m\lambda \quad ; m = 0, \pm1, \pm2, \ldots$$

in which we need to set

$$d = 2\,\mu\text{m}$$

For $\lambda = 1.550\,\mu\text{m}$

$$2\,\mu\text{m} \times \sin\theta = m \times 1.550\,\mu\text{m}$$

∴ $$\sin\theta = 0.775m$$

For $m = 1$

$$\theta = \sin^{-1}(0.775) = 50.08^\circ$$

For $\lambda = 1.540\,\mu\text{m}$

$$2\,\mu\text{m} \times \sin\theta = m \times 1.540\,\mu\text{m}$$

∴ $$\sin\theta = 0.770\,m$$

For $m = 1$

$$\theta = \sin^{-1}(0.770) = 50.35^\circ$$

∴ $$\Delta\theta = 50.35^\circ - 50.08^\circ = 0.28^\circ$$

1.42 A monochromator Consider an incident beam on a reflection diffraction grating as in Figure 1.60. Each incident wavelength will result in a diffracted wave with a different diffraction angle. We can place a small slit and allow only one diffracted wave λ_m to pass through to the photodetector. The diffracted beam would consist of wavelengths in the incident beam separated (or fanned) out after diffraction. Only one wavelength λ_m will be diffracted favourably to pass through the slit and reach the photodetector. Suppose that the slit width is s = 0.1 mm, and the slit is at a distance R = 5 cm from the grating. Suppose that the slit is placed so that it is at right angles to the incident beam: $\theta_i + \theta_m = \pi/2$. The grating has a corrugation periodicity of 1 μm.

(a) What is the range of wavelengths that can be captured by the photodetector when we rotate the grating from θ_i = 1° to 40°?

(b) Suppose that θ_i = 15°. What is the wavelength that will be detected? What is the resolution, that is, the range of wavelengths that will pass through the slit? How can you improve the resolution? What would be the advantage and disadvantage in decreasing the slit width s?

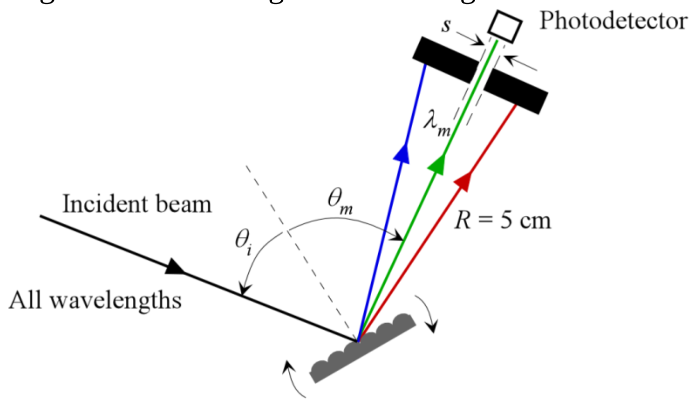


Figure 1.60 A monochromator based on using a diffraction grating

Solution

Grating equation is $d(\sin(\theta_m) - \sin(\theta_i)) = m\lambda$ where m = 0, ±1, ±2, … Moreover, in this particular case $\theta_m + \theta_i = 90°$. Therefore the grating equation may be transformed as

$$d(\sin(90° - \theta_i) - \sin(\theta_i)) = 2d\left[\sin\frac{1}{2}(90° - \theta_i - \theta_i)\cos\frac{1}{2}(90° - \theta_i + \theta_i)\right] = \sqrt{2}d\sin(45° - \theta_i) = m\lambda \text{ or}$$

$$\therefore \qquad \lambda = \frac{\sqrt{2}d}{m}\sin(45° - \theta_i)$$

(a) The equation shows that the largest λ may be achieved for m = 1 which is usually done in monochromator. Substituting 40° and 1° into above formula we get the values of **123.3 nm** and **982.4 nm** that can be captured by the photodetector when we rotate the grating from θ_i = 1° to 40°.

(b) At θ_i=15° and m = 1 the above formula gives λ = **707.1 nm**

Spectral resolution may be found by differentiating the above formula

$$|\delta\lambda| = \left|\sqrt{2}d\cos(45° - \theta_i)\right|\delta\theta_i \approx \left|\sqrt{2}d\cos(45° - \theta_i)\frac{s}{R}\right|$$

which gives $\delta\lambda$ = **2.45** nm

1.43 Thin film optics Consider light incident on a thin film on a substrate, and assume normal incidence for simplicity.

(a) Consider a thin soap film in air, $n_1 = n_2 = 1$, $n_2 = 1.40$. If the soap thickness $d = 1$ μm, plot the reflectance vs. wavelengt from 0.35 to 0.75 μm, which includes the visible range. What is your conclusion?

(b) MgF_2 thin films are used on glass plates for the reduction of glare. Given that $n_1 = 1$, $n_2 = 1.38$ and $n_3 = 1.70$. (n for glass depends on the type of glass but 1.6 is a reasonable value), plot the reflectance as a function of wavelength from 0.35 to 0.75 μm for a thin film of thickness 0.10 μm. What is your conclusion?

Solution

(a) Substitute $\phi = 2dn_2(2\pi/\lambda)$ in

$$r = \frac{r_1 + r_2 e^{-j\phi}}{1 + r_1 r_2 e^{-j\phi}} = \frac{r_1 + r_2 e^{-j(4\pi n_2 d/\lambda)}}{1 + r_1 r_2 e^{-j(4\pi n_2 d/\lambda)}}$$

and plot $R = |\, r\,|^2$ as a function of wavelength from 0.35 to 0.75 μm as in the figure. Clearly, certain wavelengths, in this case, violet, green, orange-red are reflected more than others (blue and yellow).

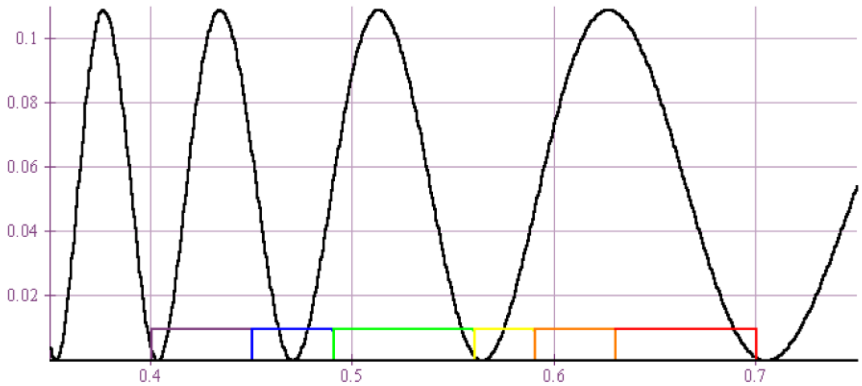


Figure: Reflectance vs wavelength in the visible range for a soap film

(b)

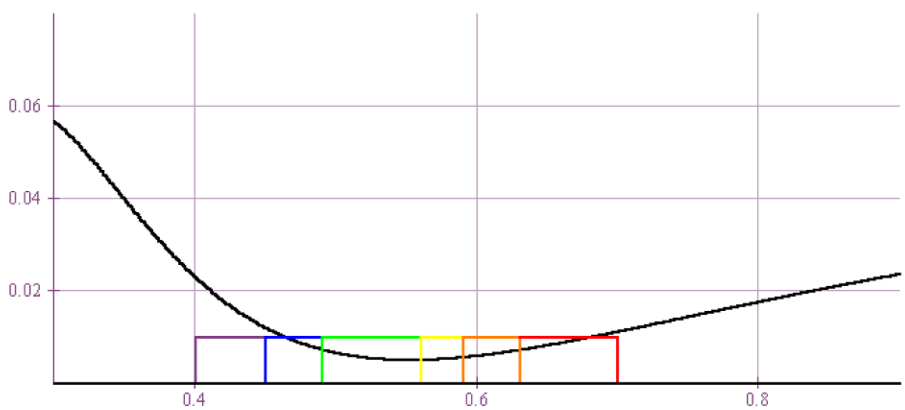


Figure: Reflectance vs wavelength in the visible range for a MgF_2 tin film coating on glass

The reflectance is lowered substantially by the thin film coating, and remains low over the visible spectrum. Without the coating, the reflectance is 6.0%. With the coating, it is below 1%
Authors comment: The above are for normal incidence. Obviously reflections at other angles will have R vs λ shifted in wavelength. This will affect the spectrum of the reflected light from the soap film but

the MgF_2 coating will still result in a relatively low reflectance over the visible because the minimum in the reflectance is very broad over the visible range.

1.44 Thin film optics Consider a glass substrate with n_3 = 165 that has been coated with a transparent optical film (a dielectric film) with n_2 = 2.50, n_1 = 1 (air). If the film thickness is 500 nm, find the minimum and maximum reflectances and transmittances and their corresponding wavelengths in the visible range for normal incidence. (Assume normal incidence.) Note that the thin n_2-film is not an AR coating, and for $n_1 < n_3 < n_2$'

$$R_{\max} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right)^2 \text{ and } R_{\min} = \left(\frac{n_3 - n_1}{n_3 + n_1}\right)^2$$

Solution

Minimum reflectance $R_{\min}$ occurs at $\phi = 2\pi$ or multiples of 2π, and maximum reflectance $R_{\max}$ occurs at $\phi = \pi$ or an odd integer multiple of 2π. The corresponding equations to Eq. (1.11.8) are

$$R_{\min} = \left(\frac{n_3 - n_1}{n_3 + n_1}\right)^2 = \left(\frac{1.65 - 1}{1.65 + 1}\right)^2 = 0.060 \text{or } 6.0\ \%$$

and
$$R_{\max} = \left(\frac{n_2^2 - n_1 n_3}{n_2^2 + n_1 n_3}\right)^2 = \left(\frac{2.5^2 - (1)(1.65)}{2.5^2 + (1)(1.65)}\right)^2 = 0.34 \text{ or } 34\%$$

Corresponding transmittances are,

$T_{\max} = 1 - R_{\min}$ = 0.94 or 94%

and $T_{\min} = 1 - R_{\max}$ = 0.66 or 66%.

Since n_2 is not an intermediate index between n_1 and n_3, the n_2-film does not reduce the reflection that would have occurred at the n_1-n_3 interface had there been no n_2-layer. Indeed R_{13} in the absence of n_2, is the same as $R_{\min}$ and the n_2-layer increases reflection

Since $\phi = 2dn_2(2\pi/\lambda)$, and d = 500 nm, and the wavelengths for maximum reflectance are given by the condition $\phi = (2m+1)\pi$, m = 0, 1,2… we can calculate the maximum reflectance wavelengths

$$\lambda_{\max} = 4dn_2/\phi = 4dn_2/[(2m+1)\pi]$$

ϕ/π	1	3	5	7	9	11	13
$\lambda_{\max}$ (nm)	5000	1,667	1000	**714**	**555**	**455**	385

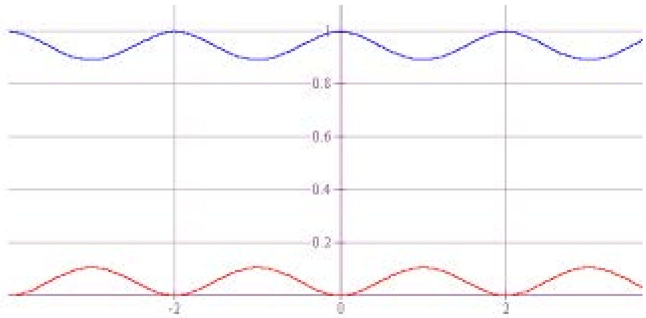
1.45 Thin film optics Consider light incident on a thin film on a substrate, and assume normal incidence for simplicity. Plot the reflectance R and transmittance T as a function of the phase change ϕ from $\phi = -4\pi$ to $+4\pi$ for the following cases

(a) Thin soap film in air, $n_1 = n_2 = 1$, n_2 = 1.40. If the soap thickness d = 1 μm, what are the maxima and minima in the reflectance in the visible range?

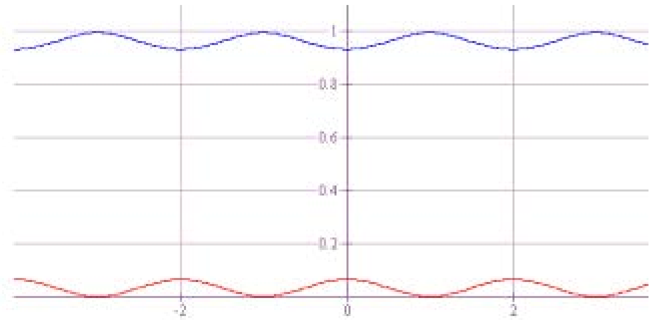
(b) A thin film of MgF_2 on a glass plate for the reduction of glare, where that n_1 = 1, n_2 = 1.38 and n_3 = 1.70. (n for glass depends on the type of glass but 1.7 is a reasonable value). What should be the thickness od MgF2 to for minimum reflection at 550 nm?

(c) A thin film of semiconductor on glass where n_1 = 1, n_2 = 3.5 and n_3 = 1.55.

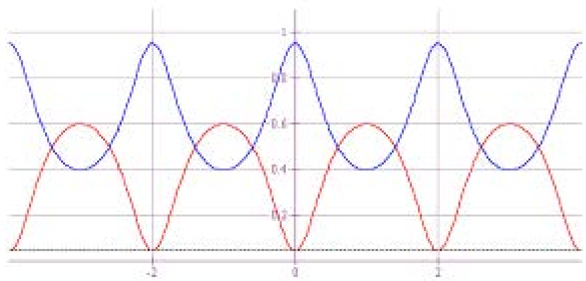
Solution



(a) R and T vs ϕ for a thin soap film in air, $n_1 = n_3 = 1$, $n_2 = 1.4$ (LiveMath)



(b) R and T vs ϕ for a thin film of MgF_2 on glass $n_1 = 1$, $n_2 = 1.38$, $n_3 = 1.70$ (LiveMath)



R and T vs ϕ for a thin film of semiconductor on glass $n_1 = 1$, $n_2 = 3.5$, $n_3 = 1.55$ (LiveMath)

1.46 Transmission through a plate Consider the transmittance of light through a partially transparent glass plate of index n_2 in which light experiences attenuation (either by absorption or scattering). Suppose that the plate is in a medium of index n_1,the reflectance at each n_1-n_2 interface is R and the attenuation coefficient is α.

(a) Show that

$$T_{\text{plate}} = \frac{(1-R)^2 e^{-\alpha d}}{(1-R^2)e^{-2\alpha d}}$$

(b) If T is transmittance of a glass plate of refractive index n in a medium of index n_o show that, in the absence of any absorption in the glass plate,

$$n/n_o = T^{-1} + (T^{-2} - 1)^{1/2}$$

if we neglect any losses in the glass plate.

(c) If the transmittance of a glass plate in air has been measured to be 89.96%. What is its refractive index? Do you think this is a good way to measure the refractive index?

Solution

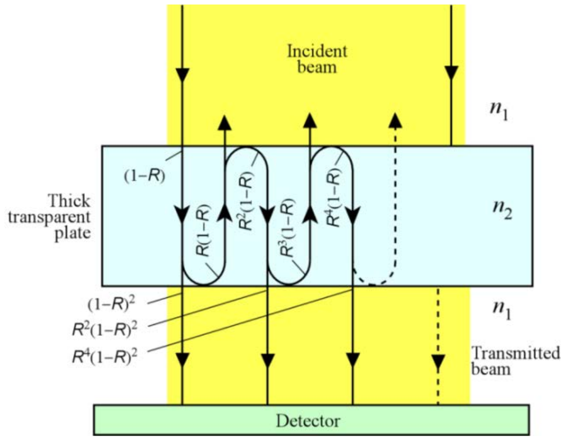


Figure 1.47 Transmitted and reflected light through a slab of material in which there is no interference.

(a) Consider a light beam of unit intensity that is passed through a thick plate of partially transparent material of index n_2 in a medium of index n_1 as in Figure 1.47. The first transmitted light intensity into the plate is $(1–R)$, and the first transmitted light out is $(1–R)\times(1–R)\ e^{-\alpha d} = (1–R)^2 e^{-\alpha d}$. However, there are internal reflections as shown, so that the second transmitted light is $(1–R)\times\ e^{-\alpha d}\ \times R\times e^{-\alpha d}\ \times R\times e^{-\alpha d}\ \times(1–R) = R^2(1–R)^2 e^{-3\alpha d}$ so that the transmitted intensity through the plate is

$$T_{\text{plate}} = (1–R)^2 e^{-\alpha d} + R^2(1–R)^2 e^{-3\alpha d} + R^4(1–R)^2 e^{-5\alpha d} + \ldots = (1–R)^2 e^{-\alpha d}\ [1 + R^2 e^{-2\alpha d} + R^4\ e^{-4\alpha d} + \ldots]$$

or
$$T_{\text{plate}} = \frac{(1-R)^2 e^{-\alpha d}}{1-R^2 e^{-2\alpha d}}$$

(b) For the transparent plate α=0 and the transmittance of plate becomes

$$T_{\text{plate}} = \frac{(1-R)^2 e^{-\alpha d}}{1-R^2 e^{-2\alpha d}} = \frac{(1-R)^2}{1-R^2} = \frac{1-R}{1+R}$$

Assuming that $R = \left(\dfrac{n-n_0}{n+n_0}\right)^2$ the equation above becomes

$$T_{plate} = \frac{1-\left(\dfrac{n-n_0}{n+n_0}\right)^2}{1+\left(\dfrac{n-n_0}{n+n_0}\right)^2} = \frac{(n+n_0)^2-(n-n_0)^2}{(n+n_0)^2+(n-n_0)^2} = \frac{(n/n_0+1)^2-(n/n_0-1)^2}{(n/n_0+1)^2+(n/n_0-1)^2} = \frac{2(n/n_0)}{(n/n_0)^2+1}$$

which leads to quadratic equation $(n/n_0)^2 - {}^2\!/_T(n/n_0)+1=0$ with the solution $n/n_o = T^{-1} + (T^{-2} – 1)^{1/2}$
.

(c) The transmittance of 89.96% leads to refractive index of 1.5967. In practice, this is not very good method because it does not give sufficient precision.

1.47 Scattering Consider Rayleigh scattering. If the incident light is unpolarized, the intensity I_s of the scattered light a point at a distance r at an angle θ to the original light beam is given by

$$I_s \propto \frac{1+\cos^2\theta}{r^2}$$

Plot a polar plot of the intensity I_s at at a fixed distance r from the scatter as we change the angle θ around the scattered. In a polar plot, the radial coordinate (OP in Figure 1.48 b)) is I_s. Construct a contour plot in the xy plane in which a contour represents a constant intensity. You need to solve vary r and θ or x and y such that I_s remains constant. Note $x = r\cos\theta$ and $y = r\sin\theta$; $\theta = \arctan(y/x)$, $r = (x^2 + y^2)^{1/2}$.

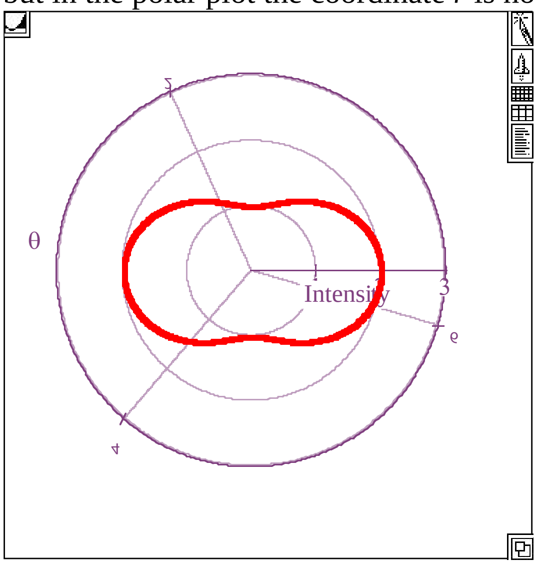
Author's Note: There is a printing error. The minus sign should have been plus as in the above expression. This should have been obvious from Figure 1.48(b). The error will be corrected in the next reprint. The e-version of the book is correct.

Solution

(a) Polar plot

Take $I_s = 1+\cos^2\theta$ as we are interested in the **angular dependence** only (set r = 1).

In the polar plot, the distance from the origin is the intensity. Do not confuse this with r. r is contant but in the polar plot the coordinate r is now the intensity.



Polar plot on LiveMath (Theorist)

(b) Contour plot

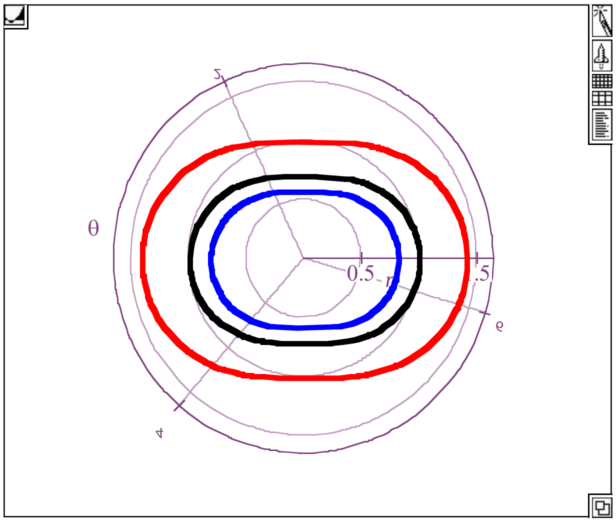
We can set the proportionality constant to 1, and write

$$I_s = \frac{1+\cos^2\theta}{r^2}$$

$$\therefore \qquad r = \frac{\sqrt{1+\cos^2\theta}}{\sqrt{I_s}}$$

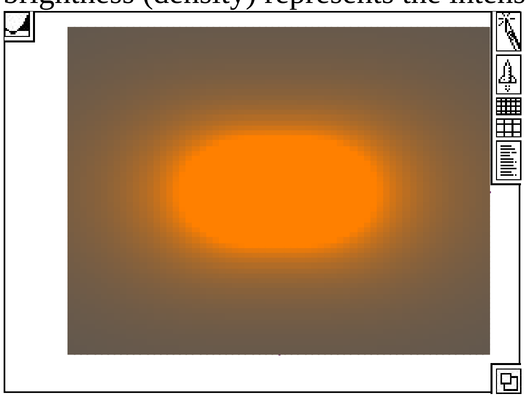
We can now plot the above on a polar plot in which the distance from the center is r, and r and θ pairs of coordinates are such that they always yield a constant I_s because we have set I_s = constant. We can arbitrarily set I = 1, 2 or 3 to get 3 contour lines.

Blue, I_s = 1, black, I_s = 2 and red, I_s = 3 in AU (arbitrary units)



Contour plots on LiveMath (Theorist)

Another interesting plot is the **density plot** in which the density represents the intensity I_s. The brightness (density) represents the intensity at a point r, θ.



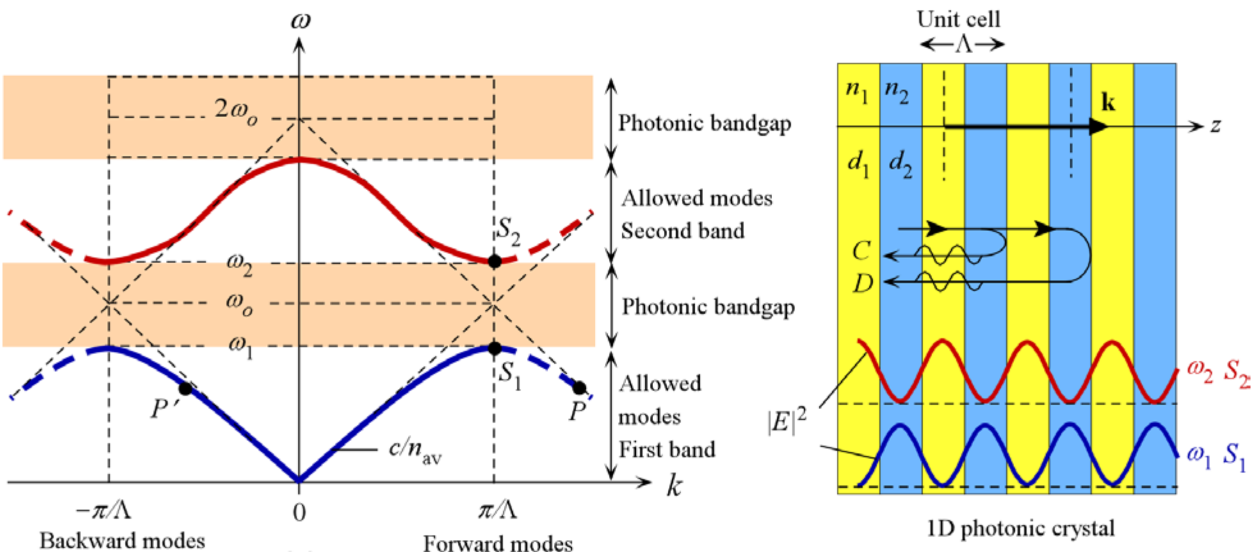
Density plot of Rayleigh scattering. Brightness represents more light intensity at the point r, θ

1.48 One dimensional photonic crystal (a Bragg mirror) The 1D photonic crystal in Figure 1.50(a), which is essentially a Bragg reflector, has the dispersion behavior shown in Figure 1.51. The stop-band $\Delta\omega$ for normal incidence and for all polarizations of light is given by (R.H. Lipson and C. Lu, Eur. J. Phys. **30**, S33, 2009)

$$\frac{\Delta\omega}{\omega_o} = 2(2/\pi)\arcsin\left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

where $\Delta\omega$ is the stop-band, ω_o is the center frequency defined in Figure 1.50(a) and n_2 and n_1 are the high and low refractive indices. Calculate the lowest stop band in terms of photon energy in eV, and wavelength (nm) for a structure in which $n_2 = 4$ and $n_1 = 1.5$, and $n_1 d_1 = n_2 d_2 = \lambda/4$ and $d_1 = 2$ μm.

Solution



$$\Delta\omega = \omega_o 2(2/\pi)\arcsin\left(\frac{n_2 - n_1}{n_2 + n_1}\right)$$

$n_2 = 4$ and $n_1 = 1.5$, $d_1 = 2$ μm

$$n_1 d_1 = n_2 d_2 = \lambda / 4$$

$$1.5 \times 2\mu m = 4 d_2 = \lambda / 4$$

$$d_2 = 0.75\mu m$$

$$\lambda_0 = 12\mu m$$

$$\omega_0 = 2\pi \frac{c}{\lambda_0} = 1.57 \times 10^{14}$$

$$\Delta\omega = 1.57 \times 10^{14} \times 2(2/\pi)\arcsin\left(\frac{4 - 1.5}{4 + 1.5}\right) = 5.41 \times 10^{15}$$

$$\Delta\lambda = 2\pi c / \Delta\omega = \frac{2\pi \times 3 \times 10^8}{5.41 \times 10^{15}} = 3.48 \times 10^{-7} m = 348 nm$$

$$\Delta E = h\Delta\omega / 2\pi = 6.62 \times 10^{-34}\ \mathrm{J\,s} \times \frac{1}{1.6 \times 10^{-19}\ \mathrm{J\,eV^{-1}}}) \frac{5.41 \times 10^{15}}{2\pi} = 3.56\mathrm{eV}$$

1.49 Photonic crystals Concepts have been borrowed from crystallography, such as a unit cell, to define a photonic crystal. What is the difference between a unit cell used in a photonic crystal and that used in a real crystal? What is the size limit on the unit cell of a photonic crystal? Is the refractive index a microscopic or a macroscopic concept? What is the assumption on the refractive index?

Solution

The size limit on the unit cell of a photonic crystal is that it must be longer than the wavelength scale. Refractive index is a macroscopic concept.

NOTES FROM THE AUTHOR

Some of the problems have been solved by using LiveMath (previously Mathview and Theorist).
http://livemath.com

LiveMath interpretation

Critical angle is

This is a comment, and is not used in calculations

□ $\sin(\theta c) = n$

A square represents a **mathematical statement**

□ $\sin(\theta c) = n$

△ $\theta c = \arcsin(n)$ *Isolate*

△ $\theta c = 0.76765$ *Calculate*

The first line with a square is a mathematical statement.

The second line with a triangle isolates θ_c. It is a **mathematical conclusion** from the first line. The dot at the center of the triangle represents a working conclusion, something that will be used elsewhere in calculations.

The third line with a triangle is a mathematical conclusion from the second line. It calculates θ_c

Italic text next to mathematical conclusions explains the mathematical operation *e.g. isolate*

LIVEMATH Former names: MATHVIEW THEORIST

This is a comment. It is not used in calculations,

□ $q = 1.6022\times10^{-19}$ □ $c = 299792458$ □ $\varepsilon o = 8.8542\times10^{-12}$

The above are working statements. Squares with a dot mean that these are used in calculations in this sheet (anywhere in the sheet)

The statement below is a mathematical working statement that assigns 4 to *a*

□ $a = 4$

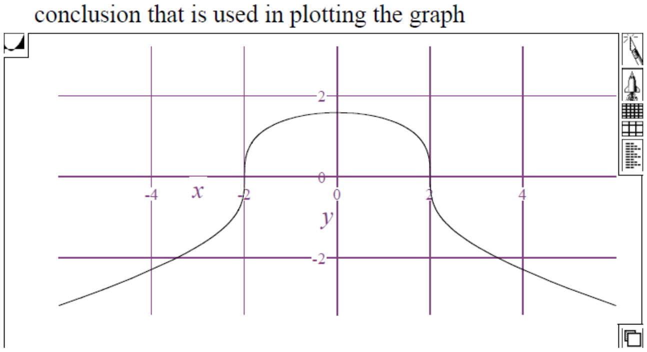
The following is a mathematical statement, an equation

□ $y^3 + x^2 = a$

The triangle below is a conclusion or a derivation from the mathematical statement above. It *isolates y*

△ $y = \left(-x^2 + a\right)^{\frac{1}{3}}$ *Isolate*

The graph below is plotted by using "Graph" from the menu. The above triangle has a dot because it is a working derivation/ conclusion that is used in plotting the graph



The following finds the wavelength λ of radiation emission (photon) when an electron transits down from one energy level to another, and changes its energy by $\Delta E = 2$ eV.

□ $\Delta E = 2q$

Relationship between ΔE and photon energy $h\upsilon$

□ $\Delta E = h\,\upsilon$

Substitute for υ from below

△ $\Delta E = \dfrac{c\,h}{\lambda}$ *Substitute*

Isolate for λ

△ $\lambda = \dfrac{c\,h}{\Delta E}$ *Isolate*

△ $\lambda = 6.1992\times10^{-7}$ *Calculate*

Relationship between λ and υ

□ $\lambda = \dfrac{c}{\upsilon}$

△ $\upsilon = \dfrac{c}{\lambda}$ *Isolate*

Drag this above and substitute for υ