Chapter 0

Preliminaries

0.1 Polynomials and Rational Fuctions

1. $3x + 2 < 8$

$3x < 8 - 2$

$3x < 6$

$x < 2$

2. $3 - 2x < 7$

$-2x < 4$

$x > -2$

3. $1 \le 2 - 3x < 6$

$1 - 2 \le -3x < 6 - 2$

$-1 \le -3x < 4$

$\frac{1}{3} \ge x > -\frac{4}{3}$

4. $-2 < 2x - 3 \le 5$

$1 < 2x \le 8$

$\frac{1}{2} < x \le 4$

5. $\frac{x+2}{x-4} \ge 0$

$x + 2 \ge 0, x - 4 > 0$ or $x + 2 \le 0, x - 4 < 0$

$x \ge -2, x > 4$ or $x \le -2, x < 4$

$x > 4$ or $x \le -2$

6. $\frac{2x+1}{x+2} < 0$

$2x + 1 < 0, x + 2 > 0$ or $2x + 1 > 0, x + 2 < 0$

$x < -\frac{1}{2}, x > -2$ or $x > -\frac{1}{2}, x < -2$

$-2 < x < -\frac{1}{2}$ (Since $x > -\frac{1}{2}, x < -2$ is not possible).

7. $x^2 + 2x - 3 \ge 0$

$(x + 3)(x - 1) \ge 0$

$x \ge 1$ or $x \le -3$

8. $x^2 - 5x - 6 < 0$

$(x - 6)(x + 1) < 0$

$-1 < x < 6$

9. $|x + 5| < 2 - 2 < x + 5 < 2$

$-2 - 5 < x < 2 - 5$

$-7 < x < -3$.

10. $|2x + 1| < 4$

$-4 < 2x + 1 < 4$

$-4 - 1 < 2x < 4 - 1$

$-5 < 2x < 3$

$-\frac{5}{2} < x < \frac{3}{2}$

11. Yes. The slope of the line joining the points (2, 1) and (0, 2) is $-\frac{1}{2}$, which is also the slope of the line joining the Points (0,2) and (4, 0).

12. No. The slope of the line joining the points (3, 1) and (4, 4) is 3, while the slope of the line joining the points (4, 4) and (5, 8) is 4.

13. No. The slope of the line joining the points (4, 1) and (3, 2) is −1, while the slope of the line joining the points (3, 2) and (1, 3) is $-\frac{1}{2}$.

14. No. The slope of the line joining the points (1, 2) and (2, 5) is 3, but the slope of line joining the points (2, 5) and (4, 8) is $\frac{3}{2}$.

15. (a) $d\{(1, 2), (3, 6)\}$

$$= \sqrt{(3-1)^2 + (6-2)^2}$$
$$= \sqrt{4+16} = \sqrt{20}$$

(b) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-2}{3-1} = 2$

(c) The equation of line is

$y = m(x - x_0) + y_0$

$y = 2(x - 1) + 2$

$y = 2x$

16. (a) $d\{(1, -2), (-1, -3)\}$

$$= \sqrt{(-1-1)^2 + (-3+2)^2}$$
$$= \sqrt{4+1} = \sqrt{5}$$

(b) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3+2}{-1-1} = \dfrac{1}{2}$

(c) The equation of line is

$y = m(x - x_0) + y_0$

$y = \dfrac{1}{2}(x-1) + (-1)$

$y = \dfrac{x-3}{2}$

17. (a) $d\{(0.3, -1.4), (-1.1, -0.4)\}$

$$= \sqrt{(-1.1-0.3)^2 + (-0.4+1.4)^2}$$
$$= \sqrt{(-1.4)^2 + 1} = \sqrt{2.96}$$

(b) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-0.4+1.4}{-1.1-0.3} = -\dfrac{1}{1.4}$

(c) The equation of line is

$y = m(x - x_0) + y_0$

$y = -\dfrac{1}{1.4}(x-0.3) - 1.4$

$1.4y = -x - 1.66$

$x + 1.4y = -1.66$

18. (a) $d\{(1.2, 2.1), (3.1, 2.4)\}$

$$= \sqrt{(3.1-1.2)^2 + (2.4-2.1)^2}$$
$$= \sqrt{(1.9)^2 + (0.3)^2}$$
$$= \sqrt{3.61+0.09} = \sqrt{3.7}$$

(b) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2.4-2.1}{3.1-1.2} = \dfrac{0.3}{1.9} \approx 0.16$

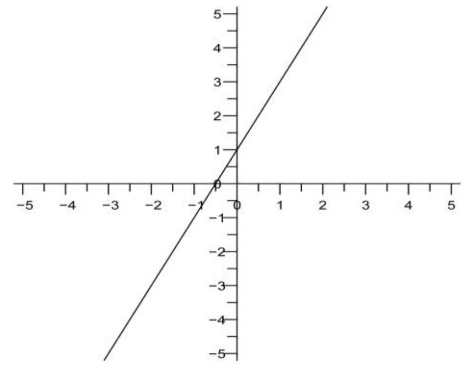
(c) The equation of line is

$y = m(x - x_0) + y_0$

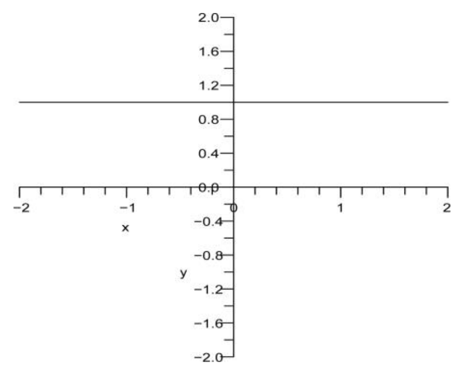
$y = (0.16)(x - 1.2) - 2.1$

$y = 0.16x - 2.292$

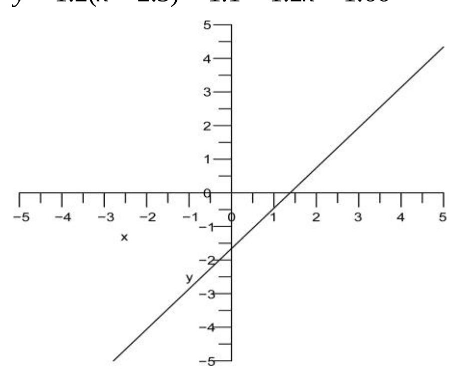
19. $y = 2(x - 1) + 3 = 2x + 1$



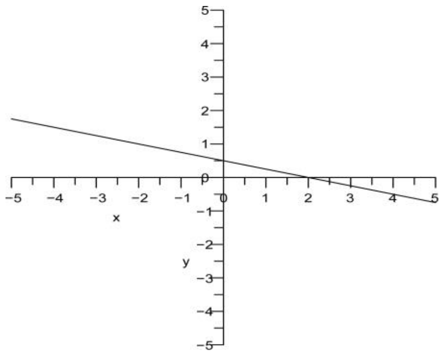
20. $y = 1$



21. $y = 1.2(x - 2.3) + 1.1 = 1.2x - 1.66$



22. $y = -\frac{1}{4}(x+2)+1 = -\frac{1}{4}x + \frac{1}{2}$



23. Parallel. Both have slope 3.

24. Neither. Slopes are 2 and 4.

25. Perpendicular. Slopes are −2 and $\frac{1}{2}$.

26. Neither. Slopes are 2 and −2.

27. Perpendicular. Slopes are 3 and $-\frac{1}{3}$.

28. Parallel. Both have slope $-\frac{1}{2}$.

29. (a) $y = 2(x - 2) + 1$

(b) $y = -\frac{1}{2}(x-2)+1$

30. (a) $y = 3x + 3$

(b) $y = -\frac{1}{3}x + 3$

31. (a) $y = 2(x - 3) + 1$

(b) $y = -\frac{1}{2}(x-3)+1$

32. (a) $y = -1$

(b) $x = 0$

33. Slope $m = \frac{3-1}{2-1} = \frac{2}{1} = 2$

Equation of line is $y = 2(x - 1) + 1 = 2x - 1$. When $x = 4$, $y = 7$.

34. Slope $m = \frac{1}{2}$

Equation of line is $y = \frac{1}{2}(x+2)+1$.

When $x = 4$, $y = 4$.

35. Yes, passes vertical line test.

36. Yes, passes vertical line test.

37. No. The vertical line $x = 0$ meets the curve twice; nearby vertical lines meet it three times.

38. No, does not pass vertical line test.

39. Both: This is clearly a cubic polynomial, and also a rational function because it can be written as

$$f(x) = \frac{x^3 - 4x + 1}{1}.$$

This shows that all polynomials are rational.

40. Rational.

41. Rational.

42. Neither: Contains square root.

43. We need the function under the square root to be non-negative. $x + 2 \geq 0$ when $x \geq -2$. The domain is $\{x \in \mathbb{R} \mid x \geq -2\} = [-2, \infty)$.

44. Negatives are permitted inside the cube root. There are no restrictions, so the domain is $(-\infty, \infty)$ or all real numbers.

45. The function is defined only if

$x^2 - x - 6 \geq 0$ and $x \neq 5$

$(x - 3)(x + 2) \geq 0$ and $x \neq 5$

$x \leq -2$ or $x \geq 3$ and $x \neq 5$

$(-\infty, -2] \cup [3, 5) \cup (5, \infty)$

46. We need the numerator function under square root be non-negative. $x^2 - 4 \geq 2$, when $|x| \geq 2$ Also the denominator cannot be zero. $9 - x^2 > 0$, when $|x| < 3$ The domain is $(-3, -2] \cup [2, 3)$.

47. The denominator cannot be zero. $x^2 - 1 = 0$ when $x = \pm 1$. The domain is

$\{x \in \mathbb{R} \mid x \neq \pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

48. The denominator cannot be zero.

$x^2 + 2x - 6 = 0$ when $x = -1 \pm \sqrt{7}$.

The domain is $\left\{x \in \mathbb{R} \mid x \neq -1 \pm \sqrt{7}\right\}$

$= \left(-\infty, -1-\sqrt{7}\right) \cup \left(-1-\sqrt{7}, -1+\sqrt{7}\right)$

$\cup \left(-1+\sqrt{7}, \infty\right)$

49. $f(0) = 0^2 - 0 - 1 = -1$

$f(2) = 2^2 - 2 - 1 = 1$

$f(-3) = (-3)^2 - (-3) - 1 = 11$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 1 = -\frac{5}{4}$$

50. $f(1) = \frac{3}{1} = 1$

$f(10) = \frac{3}{10} = 0.3$

$f(100) = \frac{3}{100} = 0.03$

$f\left(\frac{1}{3}\right) = \frac{3}{\frac{1}{3}} = 9$

51. Again, the only constraint we know for sure is that x should not be negative, i.e., a reasonable domain would be $\{x | x \geq 0\}$.

52. Width can be anywhere from 0 to 70 meters. A reasonable domain is $\{x | 0 \leq x \leq 70\}$.

53. Answers vary. There may well be a positive correlation (more study hours = better grade), but not necessarily a functional relation.

54. Answers vary. Evidence supports a relationship.

55. Answers vary. While not denying a negative correlation (more exercise = less weight), there are too many other factors (metabolic rate, diet) to be able to quantify a person's weight as a function just of the amount of exercise.

56. Answers vary. Objects of all weights fall at the same speed unless friction affects them differently.

57. A flat interval corresponds to an interval of constant speed; going up means that the speed is increasing while the graph going down means that the speed is decreasing. It is likely that the bicyclist is going uphill when the graph is going down and going downhill when the graph is going up.

58. Influxes of immigrants occur where graph rises. War and plague occur where graph falls.

59. The x–intercept occurs where $0 = x^2 - 2x - 8 = (x - 4)(x + 2)$, so $x = 4$ or

$x = -2$; y–intercept at $y = 0^2 - 2(0) - 8 = -8$

60. The x– intercept occurs where $0 = x^2 + 4x + 4 = (x + 2)^2$, so $x = -2$; y–intercept at $y = 0^2 + 4(0) + 4 = 4$.

61. The x–intercept occurs where $0 = x^3 - 8 = (x - 2)(x^2 + 2x + 4)$, so $x = 2$ (using the quadratic formula on the quadratic factor gives the solutions $x = -1 \pm \sqrt{-3}$, neither of which is real so neither contributes a solution); y–intercept at $y = 0^3 - 8 = -8$.

62. The x–intercept occurs where $0 = x^3 - 3x^2 + 3x - 1 = (x - 1)^3$, so $x = 1$ y–intercept at $y = (0)^3 - 3(0)^2 + 3(0) - 1 = -1$.

63. The x–intercept occurs where the numerator is zero, at $0 = x^4 - 4 = (x - 2)(x + 2)$, so $x = \pm 2$; y–intercept at $y = \frac{(0)^2 - 4}{0 + 1} = -4$.

64. The x-intercept occurs where the numerat or is zero, at $x = \frac{1}{2}$; y–intercept at $y = \frac{2(0) - 1}{(0)^2 - 4} = \frac{1}{4}$.

65. $x^2 - 4x + 3 = (x - 3)(x - 1)$, so the zeros are $x = 1$ and $x = 3$.

66. $x^2 + x - 12 = (x + 4)(x - 3)$, so the zeros are $x = -4$ and $x = 3$.

67. Quadratic formula gives

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$
$$= 2 \pm \sqrt{2}$$

68. Quadratic formula gives

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{-2 \pm \sqrt{6}}{2}$$

69. $x^3 - 3x^2 + 2x = x(x - 2)(x - 1)$.

So, the zeros are $x = 0$, 1 and 2.

70. $x^3 - 2x^2 - x + 2 = (x - 2)(x - 1)(x + 1)$. So, the zeros are $x = -1$, 1 and 2.

71. With $t = x^3$, $x^6 + x^3 - 2$ becomes $t^2 + t - 2$ and factors as $(t + 2)(t - 1)$. The expression is zero only if one of the factors is zero, i.e., if $t = 1$ or $t = -2$. With $x = t^{1/3}$, the first occurs only if $x = (1)^{1/3} = 1$. The latter occurs only if $x = (-2)^{1/3}$, about -1.2599.

72. $x^3 + x^2 - 4x - 4 = (x - 2)(x + 1)(x + 2)$.

So, the zeros are $x = -2, -1$ and 2.

73. Substitute $y = x^2 + 2x + 3$ into $y = x + 5$

$x^2 + 2x + 3 = x + 5$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

When $x = -2$, $y = 3$

When $x = 1$, $y = 6$

The points of intersection are $(-2, 3)$ and $(1, 6)$.

74. Substitute $y = x^2 + 4x - 2$ into $y = 2x^2 + x - 6$

$x^2 + 4x - 2 = 2x^2 + x - 6$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$x = 4$ or $x = -1$

When $x = 4$, $y = 30$

When $x = -1$, $y = -5$

The points of intersection are $(4, 30)$ and $(-1, -5)$.

Applications

1. If $B(h) = -0.0034h + 100$, then we can solve $B(h) = 73.6$ for h as follows:

$73.6 = -0.0034h + 100$

$0.0034h = 26.4$

$$h = \frac{26.4}{0.0034} \approx 7765\,m$$

2. Let x represent compression and $L(x)$ represent spin rate. Given the points (120, 9100) and (60, 10, 000), the linear function is $y = -15(x - 60) + 10{,}000$.

The spin rate of a 90-compression ball is 9550, and the spin rate of a 100-compression ball is 9400.

3. This is a two-point line-fitting problem. If a point is interpreted as (x, y) = (temperature, chirp rate), then the two given points are (20, 110) and (28, 166). The slope being $\frac{166-110}{28-20} = 7$, we could write $y - 110 = 7(x - 20)$ or $y = 7x - 30$.

4. From problem 3 we know the temperature is a function of chirping rate, where r is measured in chirps per minute. The number of chirps in 15 seconds will then be $\frac{1}{4}r$. In this case, the temperature may not be found conveniently.

5. Her winning percentage is calculated by the formula $P = \frac{100w}{t}$, where P is the winning percentage, w is the number of games won and t is the total number of games. Plugging in $w = 415$ and $t = 415 + 120 = 535$, we find her winning percentage is approximately $P \approx 77.57$, so we see that the percentage displayed is rounded up from the actual percentage. Let x be the number of games won in a row. If she doesn't lose any games, her new winning percentage will be given by the formula $P = \frac{100(415+x)}{535+x}$. In order to have her winning percentage displayed as 80%, she only needs a winning percentage of 79.5 or greater. Thus, we must solve the inequality

$$79.5 \le \frac{100(415+x)}{535+x}$$
$$79.5 \le \frac{100(415+x)}{535+x}$$

$79.5(535 + x) \le 41500 + 100x$

$42532.5 + 79.5x \le 41500 + 100x$

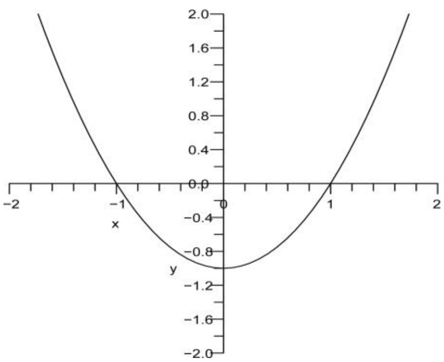
$1032.5 \le 20.5x$

$50.4 \le x$

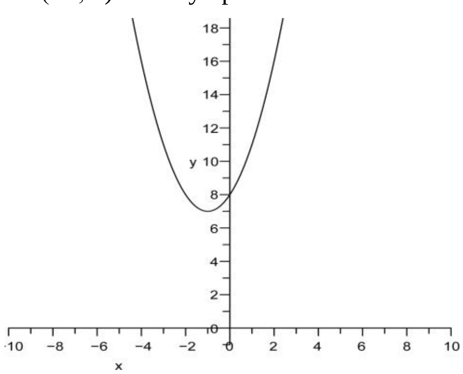
(In the above, we are allowed to multiply both sides of the inequality by 535 + x because we assume x (the number of wins in a row) is positive.) Thus she must win at least 50.4 times in a row to get her winning percentage to display as 80% Since she can't win a fraction of a game, she must win at least 51 games in a row.

0.2 Graphing Calculators and Computer Algebra Systems

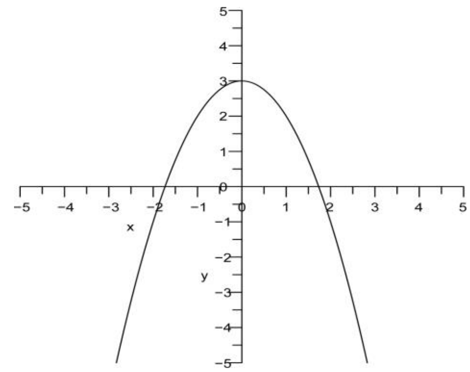
1. (a) Intercepts: $x = \pm 1$, $y = -1$. Minimum occur at $(0, -1)$. No asymptotes.



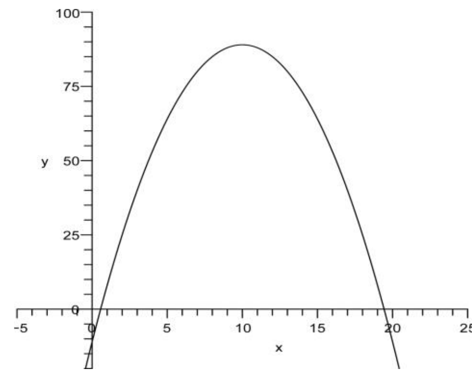
(b) Intercepts: $y = 8$ (No x-intercepts). Minimum at $(-1, 7)$. No asymptotes.



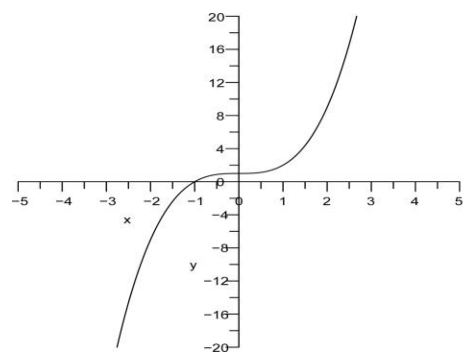
2. (a) Intercepts: $x = \sqrt{3} = \pm 1.73$, $y = 3$. Maximum at $(0, 3)$. No asymptotes.



(b) Intercepts: $x \approx 0.566$, 19.434, $y = -11$. Maximum at $(10, 89)$. No asymptotes.

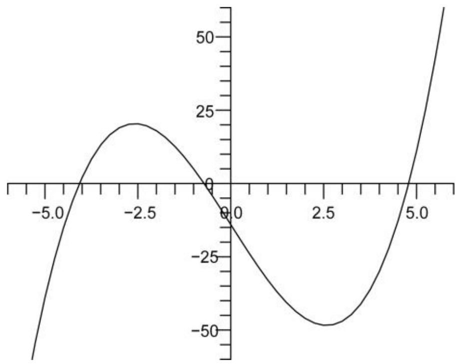


3. (a) Intercepts: $x = -1$, $y = 1$. No extrema or asymptotes.

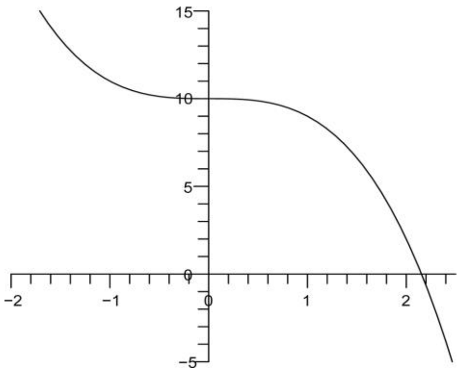


(b) Intercepts:

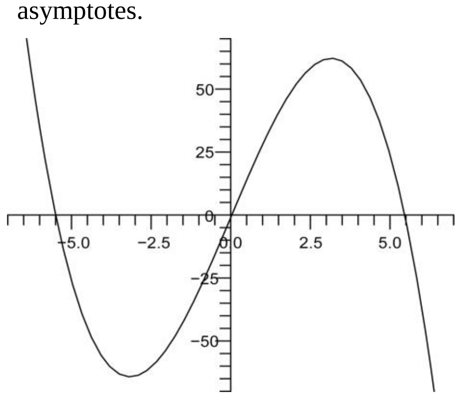
$x \approx -4.066$, -0.72 and 4.788, $y = -14$. Local minimum: Approximately at $(2.58, -48.427)$. Local maximum: Approximately at $(-2.58, 20.4225)$. No asymptotes.



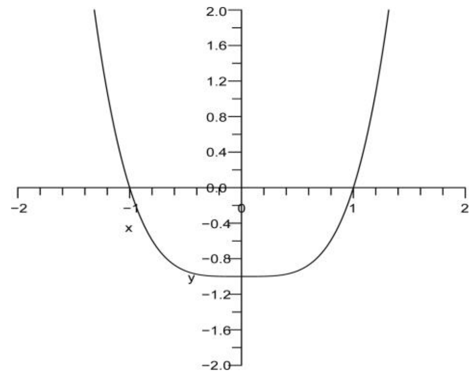
4. (a) Intercepts: $x = \sqrt[3]{10} \approx 2.1544$, $y = 10$. No extrema or asymptotes.



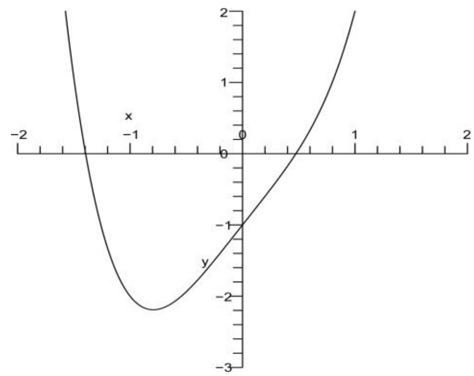
(b) Intercepts: $x \approx 0.0334$, -5.494 and 5.46, $y = -1$. Local minimum: Approximately at $(-3.16, -64.24)$. Local maximum: Approximately at $(3.16, 62.245)$. No asymptotes.



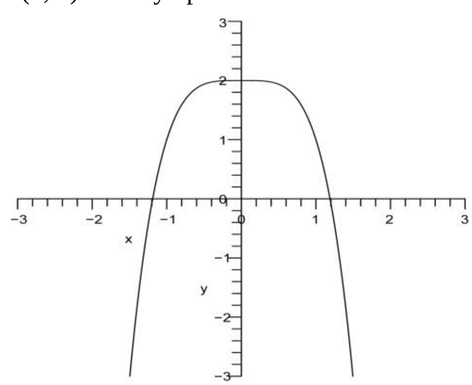
5. (a) Intercepts: $x = \pm 1$, $y = -1$. Minimum at $(0, -1)$. No asymptotes.



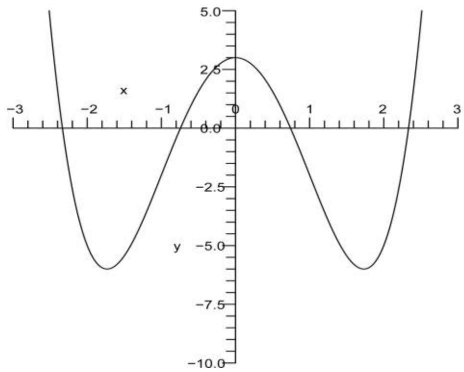
(b) Intercepts: $x \approx 0.475$, -1.395, $y = -1$. Minimum at (approximately) $(-1/\sqrt[3]{2}$, $-2.191)$. No asymptotes.



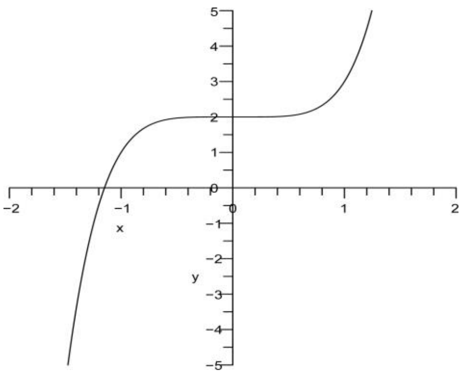
6. (a) Intercepts: $x = \pm\sqrt[4]{2}$, $y = 2$. Maximum at $(0, 2)$. No asymptotes.



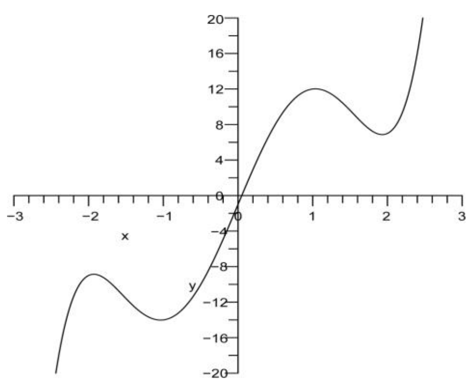
(b) Intercepts: $x \approx \pm 2.33$, and ± 0.74, $y = 3$. Local maximum at $(0, 3)$. Minima at $(\pm\sqrt{3}, -6)$. No asymptotes.



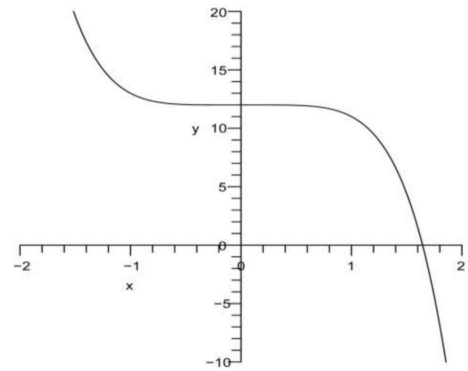
7. (a) Intercepts: $x \approx -1.149$, $y = 2$. No extrema or asymptotes



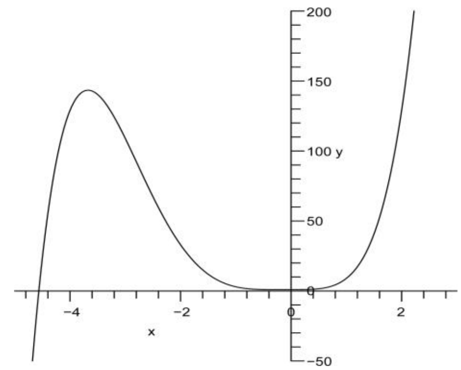
(b) Intercepts: $x \approx 0.050$, $y = -1$. The two local maxima occur at $x = \sqrt{\frac{24 - \sqrt{176}}{10}}$ and $x = -\sqrt{\frac{24 + \sqrt{176}}{10}}$, while the two local minima occur at $x = \sqrt{\frac{24 + \sqrt{176}}{10}}$ and $x = -\sqrt{\frac{24 - \sqrt{176}}{10}}$. No asymptotes.



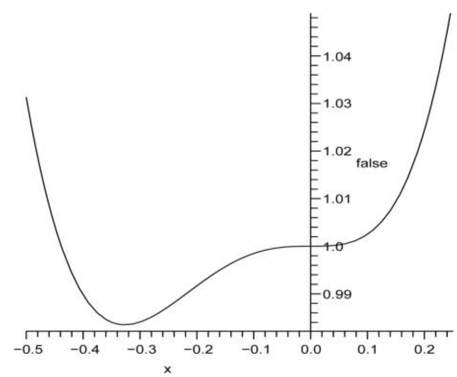
8. (a) Intercepts: $x = \sqrt[5]{12}$, $y = 12$. No extrema or asymptotes



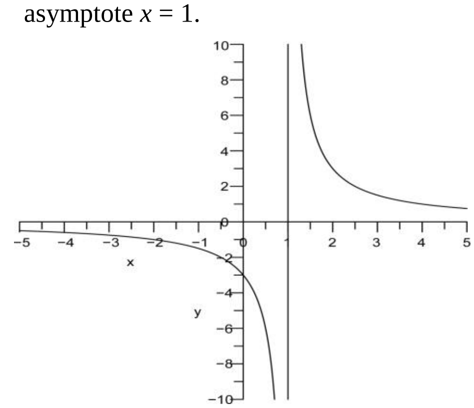
(b) Intercepts: $x \approx -4.56$, $y = 1$. Local maximum at approximately $(-3.67,\ 143.42)$. Local Minimum at approximately $(-0.33,\ 0.98)$, No asymptotes.



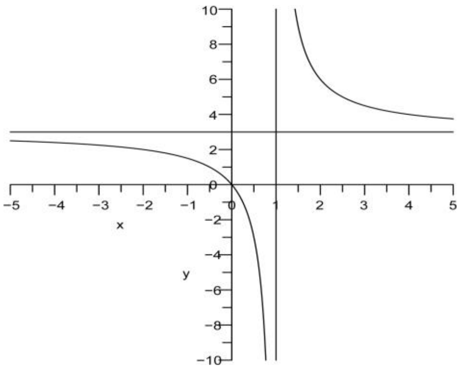
Close up of the behavior near the origin:



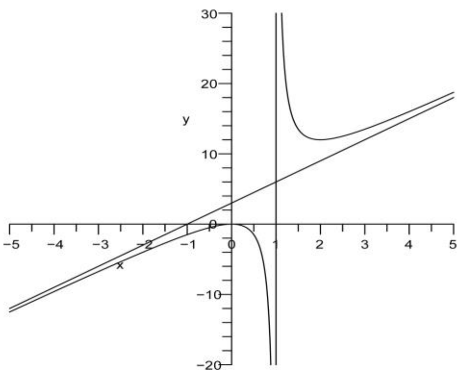
9. (a) Intercepts: $y = -3$ (no x-intercepts). No extrema. Horizontal asymptote $y = 0$. Vertical asymptote $x = 1$.



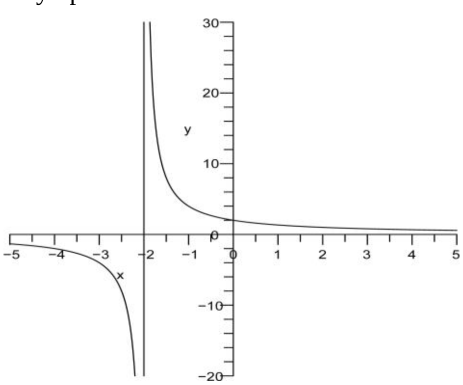
(b) Intercepts $y = 0$ (and $x = 0$). No extrema. Horizontal asymptote $y = 3$. Vertical asymptote $x = 1$.



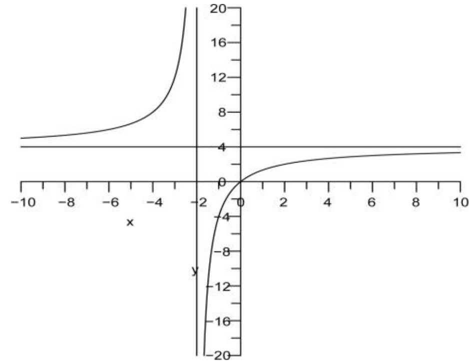
(c) Intercepts $y = 0$ (and $x = 0$). Local maximum at (0, 0). Local minimum at (2, 12).Vertical asymptote $x = 1$. Slant asymptote $y = 3x + 3$.



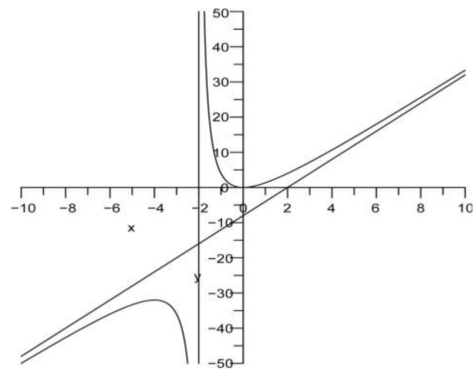
10. (a) No x-intercept. y-intercept at $y = 2$. No extrema. Horizontal asymptote $y = 0$. Vertical asymptote $x = -2$.



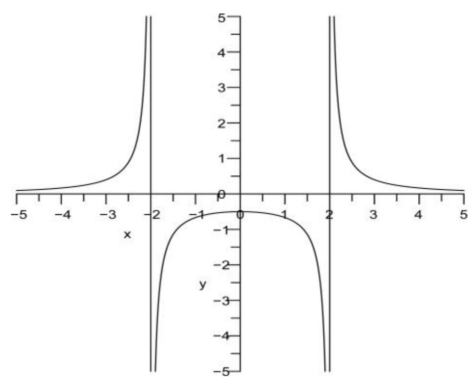
(b) Intercepts $x = 0$, $y = 0$. No extrema. Horizontal asymptote $y = 4$. Vertical asymptote $x = -2$.



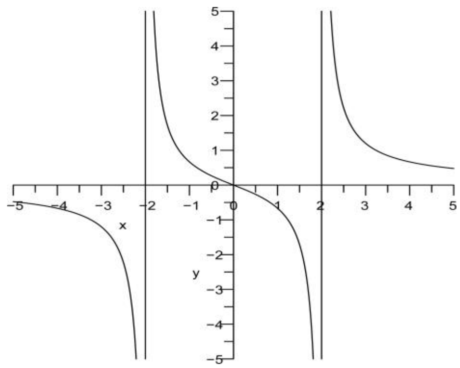
(c) Intercepts $x = 0$, $y = 0$. Local maximum at $(-4, -32)$. Local minimum at (0, 0). Vertical asymptote $x = -2$. Slant asymptote $y = 4x - 8$.



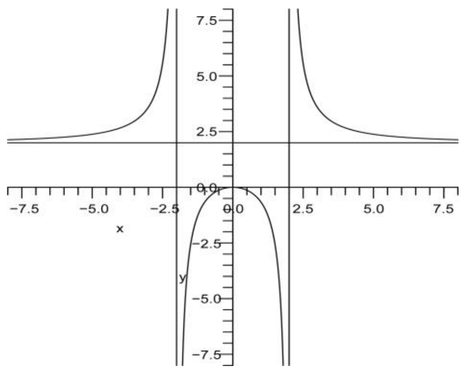
11. (a) Intercepts are $y = -\frac{1}{2}$ (No x-intercepts). Local maximum: At $(0, -\frac{1}{2})$. Horizontal asymptote: $y = 0$. Vertical asymptotes: $x = \pm 2$



(b) Intercepts: $x = 0$, $y = 0$. No extrema. Vertical asymptotes:$x = \pm 2$. No horizontal asymptotes.

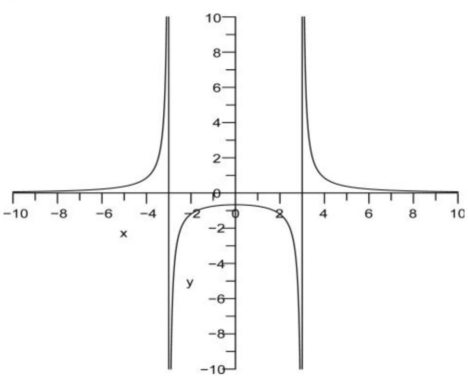


(c) Intercepts: $x = 0$, $y = 0$. Local maximum: At (0, 0). Vertical asymptotes: $x = \pm 2$. Horizontal asymptotes: $y = 2$

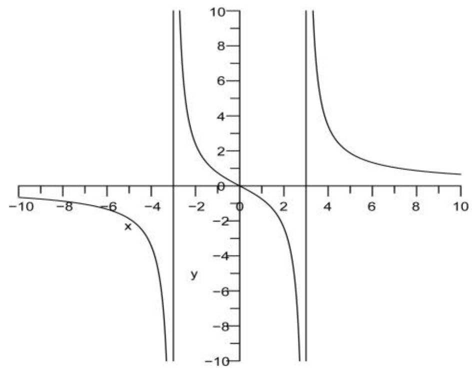


12. (a) No intercepts.

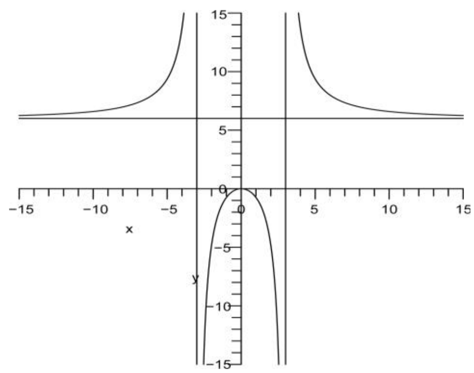
Local maximum: At $(0, -\frac{2}{3})$. Vertical asymptotes: At $x = \pm 3$. Horizontal asymptote: $y = 0$



(b) Intercepts: $x = 0$, $y = 0$. No extrema. Vertical asymptotes: $x = \pm 3$. No horizontal asymptotes.

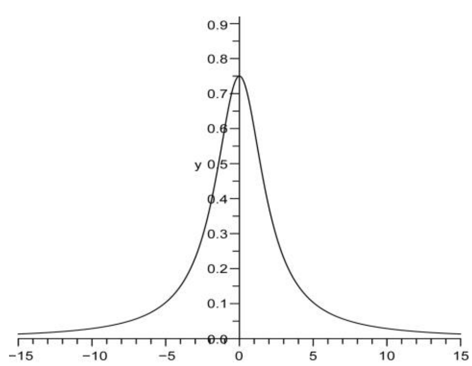


(c) Intercepts: $x = 0$, $y = 0$. Local maximum: At (0, 0). Vertical asymptotes: $x = \pm 3$. Horizontal asymptote: $y = 6$



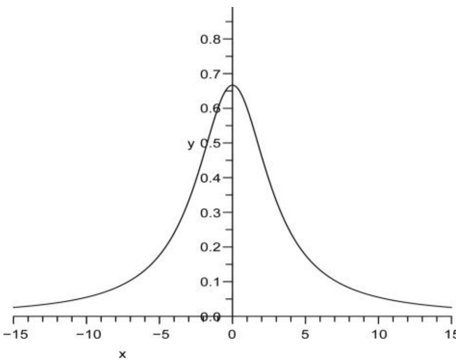
13. (a) Intercepts: $y = \frac{3}{4}$ (no x-intercepts).

Maximum at $(0, \frac{3}{4})$. Horizontal asymptote $y = 0$.

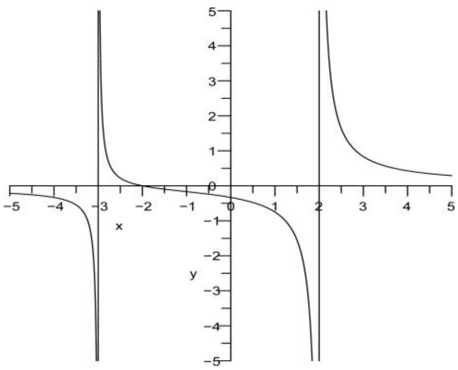


(b) No x-intercept. y-intercept at $y = \frac{2}{3}$.

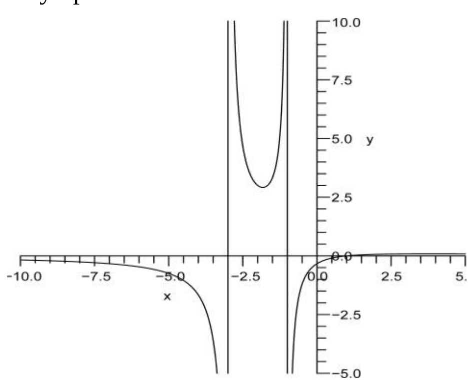
Maximum at $(0, \frac{2}{3})$. Horizontal asymptote $y = 0$.



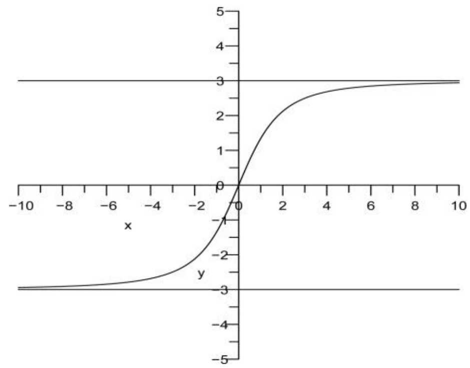
14. (a) Intercepts: $x = -2$, $y = -\frac{1}{3}$. No extrema. Horizontal asymptotes at $y = 0$. Vertical asymptotes at $x = -3$ and $x = 2$.



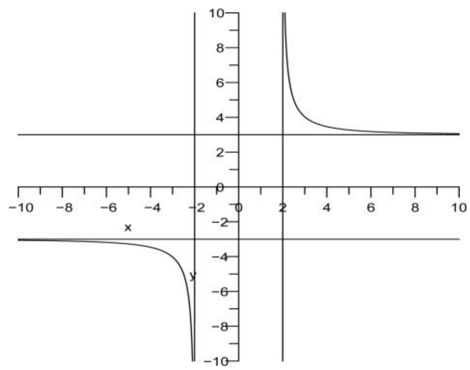
(b) Intercepts at $x = 1$, $y = -\frac{1}{3}$. Local maximum at approximately (3.83, 0.09). Local minimum at approximately (−1.83, 2.91). Horizontal asymptote $y = 0$. Vertical asymptotes $x = -3$ and $x = -1$.



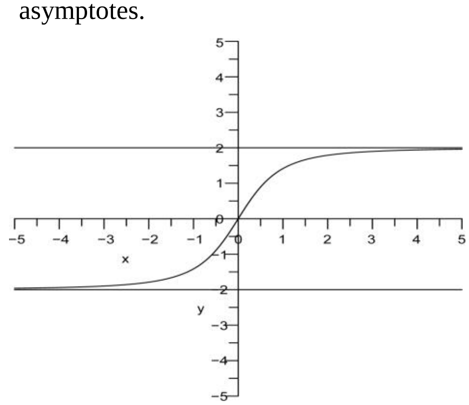
15. (a) Intercepts: $x = 0$, $y = 0$. No extrema. Horizontal asymptotes: $x = \pm 3$. No vertical asymptotes.



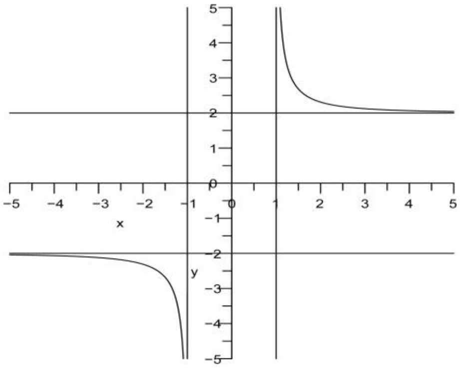
(b) No extrema or intercepts. Vertical asymptotes: $x = \pm 2$. Horizontal asymptotes: $y = \pm 3$



16. (a) Intercepts: $x = 0$, $y = 0$. No extrema. Horizontal asymptotes: $y = \pm 2$. No vertical asymptotes.



(b) No intercepts or extrema. Vertical asymptotes: $x = \pm 1$. Horizontal asymptotes: $y = \pm 2$



17. Vertical asymptotes where

$x^2 - 4 = 0 \Rightarrow x = \pm 2.$

18. Vertical asymptotes where

$x^2 - 9 = 0 \Rightarrow x = \pm 3.$

19. Vertical asymptotes where

$x^2 + 3x - 10 = 0$

$\Rightarrow (x + 5)(x - 2) = 0$

$\Rightarrow x = -5$ or $x = 2$

20. Vertical asymptotes where

$x^2 - 2x - 15 = (x - 5)(x + 3) = 0$

$\Rightarrow x = -3$ or $x = 5$

21. Vertical asymptotes where

$x^3 + 3x^2 + 2x = 0$

$\Rightarrow x(x^2 + 3x + 2) = 0$

$\Rightarrow x(x + 2)(x + 1) = 0$

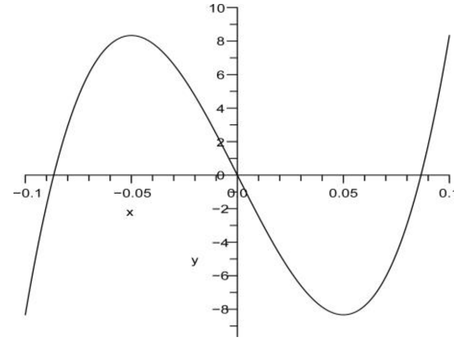
$\Rightarrow x = 0, -2$ or $x = -1$

Since none of these x values make the numerator zero, they are all vertical asymptotes.

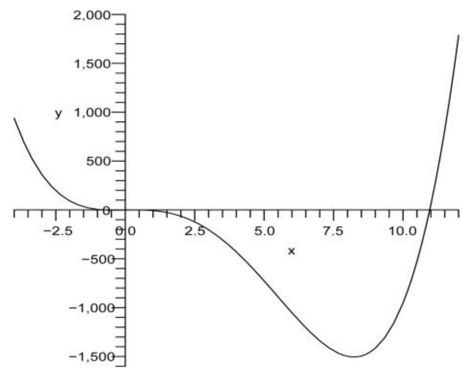
22. Vertical asymptotes where

$x^2 - 9 = 0 \Rightarrow x = \pm 3.$

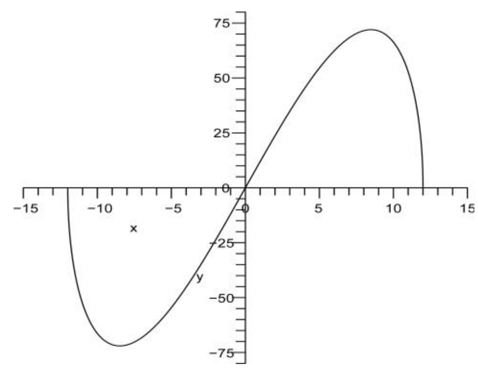
23. A window with $-0.1 \leq x \leq 0.1$ and $-0.0001 \leq y \leq 0.0001$ shows all details.



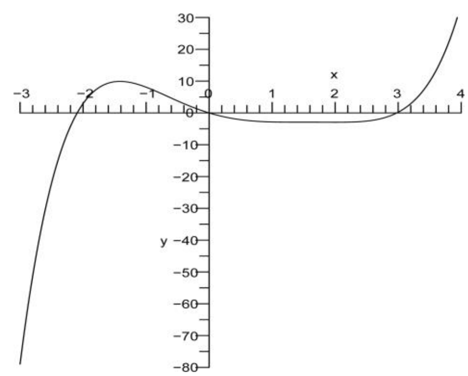
24. A window with $-4 \leq x \leq 12$ and $-1600 \leq y \leq 2000$ shows all details.



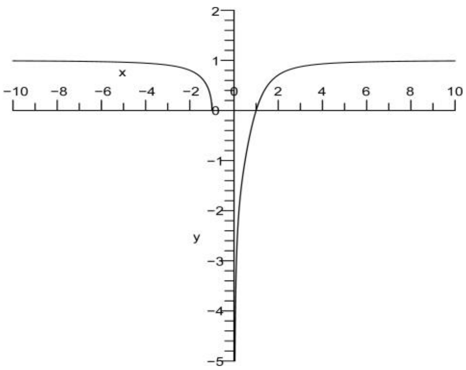
25. A window with $-15 \leq x \leq 15$ and $-80 \leq y \leq 80$ shows all details.



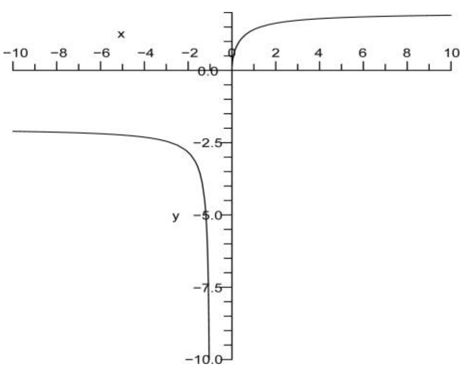
26. A window with $-3 \leq x \leq 4$ and $-80 \leq y \leq 30$ shows all details.



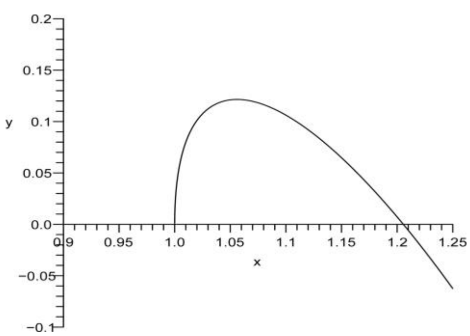
27. A window with $-10 \le x \le 10$ and $-5 \le y \le 2$ shows all details.



28. A window with $-10 \le x \le 10$ and $-11 \le y \le 2$ shows all details.



29. Graph of $y=\sqrt{(x-1)}-(x^2-1)$:



The blow-up makes it appear that there are two intersection points. Solving algebraically,

$\sqrt{x-1}=x^2-1$ (for $x \ge 1$) when

$x - 1 = (x^2 - 1)^2 = ((x - 1)(x + 1))^2$

$= (x - 1)^2(x + 1)^2$

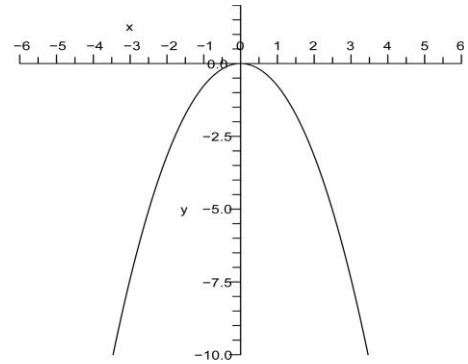
We see that $x = 1$ is one solution (obvious from the start), while for any other, we can cancel one factor of $x - 1$ and find

$1 = (x - 1)(x + 1)^2 = (x^2 - 1)(x + 1)$

$= x^3 + x^2 - x - 1$

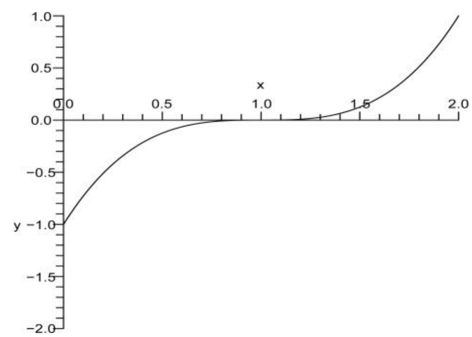
Hence $x^3 + x^2 - x - 2 = 0$. By solver or spreadsheet, this equation has only the one solution $x \approx 1.206$.

30. Graph of $y=\sqrt{(x^2+4)}-(x^2+2)$:



Graph shows one intersection at $x = 0$.

31. Graph of $y = (x^3 - 3x^2) - (1 - 3x)$:



The graph shows the only intersection near $x = 1$. Solving algebraically,

$x^3 - 3x^2 = 1 - 3x$

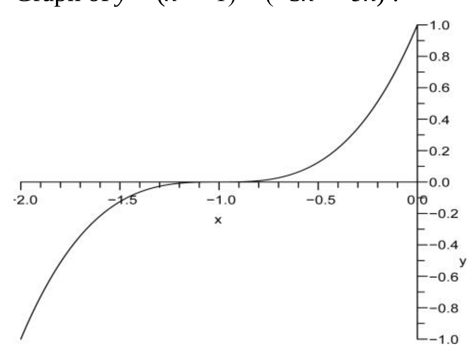
$\Rightarrow x^3 - 3x^2 + 3x - 1 = 0$

$\Rightarrow (x - 1)^3 = 0$

$\Rightarrow\ x = 1$

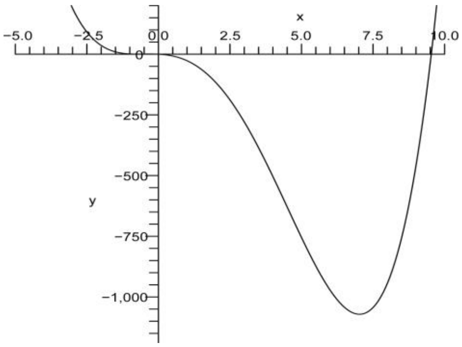
So there is only one solution: $x = 1$.

32. Graph of $y = (x^3 + 1) - (-3x^2 - 3x)$:



Graph shows only intersection near $x = -1$. Algebraically, $x^3 + 1 = -3x^2 - 3x$ when $x^3 + 3x^2 + 3x + 1 = (x + 1)^3 = 0$ and the only solution is $x = -1$.

33. Graph of $y = (x^2 - 1)^2 - (2x + 1)^3$:



After zooming out, the graph shows that there are two solutions: one near zero, and one around ten. Algebraically,

$(x^2 - 1)^{2/3} = 2x + 1$

$\Rightarrow\ (x^2 - 1)^2 = (2x + 1)^3$

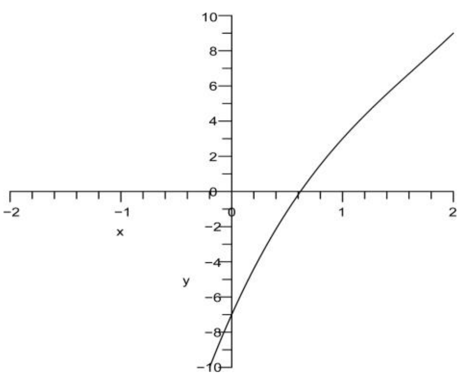
$\Rightarrow x^4 - 2x^2 + 1 = 8x^3 + 12x^2 + 6x + 1$

$\Rightarrow x^4 - 8x^3 - 14x^2 - 6x = 0$

$\Rightarrow\ x\,(x^3 - 8x^2 - 14x - 6) = 0$

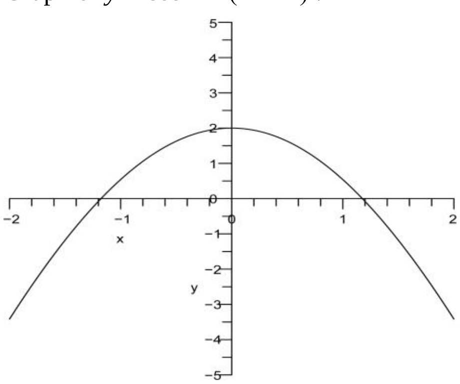
We thus confirm the obvious solution $x = 0$, and by solver or spreadsheet, find the second solution $x \approx 9.534$.

34. Graph of $y = (x + 1)^2 - (2 - x)^3$:



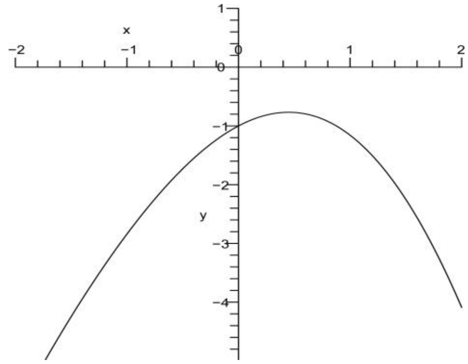
Graph shows one solution at approximately $x = 0.62$.

35. Graph of $y = \cos x - (x^2 - 1)$:



The graph shows that there are two solutions: $x \approx \pm 1.177$ by calculator or spreadsheet

36. Graph of $y = \sin x - (x^2 + 1)$:



Graph shows no intersections.

37. Calculator shows zeros at approximately −1.879, 0.347 and 1.532.

38. Calculator shows zeros at approximately 3.87, 0.79 and −0.66.

39. Calculator shows zeros at approximately .5637 and 3.0715.

40. Calculator shows zeros at approximately 1 and 0.54.

41. Calculator shows zeros at approximately −5.248 and 10.006.

42. Calculator shows zeros at approximately 2.02, −0.26, − 1.10 and −2.04.

43. The graph of $y = x^2$ on the window $-10 \le x \le 10$, $-10 \le y \le 10$ appears identical (except for labels) to the graph of $y = 2(x - 1)^2 + 3$ if the latter is drawn on a graphing window centered at the point (1, 3) with $1 - 5\sqrt{2} \le x \le 1 + 5\sqrt{2}$, $-7 \le y \le 13$.

44. The graph of $y = x^4$ is below the graph of $y = x^2$ when $-1 \le x \le 1$, and above it when $x > 1$. Both graphs have roughly the same upward parabola shape, but $y = x^4$ is flatter at the bottom.

45. $\sqrt{y^2}$ is the distance from (x, y) to the x-axis $\sqrt{x^2 + (y - 2)^2}$ is the distance from (x, y) to the point (0, 2). If we require that these be the same, and we square both quantities, we have

$y^2 = x^2 + (y - 2)^2$

$y^2 = x^2 + y^2 - 4y + 4$

$4y = x^2 + 4$

$$y = \frac{1}{4}x^2 + 1$$

In this relation, we see that y is a quadratic function of x. The graph is commonly known as a parabola.

46. The distance between (x, y) and the x-axis is $\sqrt{y^2}$. The distance between $(x, y)$and $(1,4)$ is $\sqrt{(x-1)^2+(y-4)^2}$. Setting these equal and squaring both sides yields $y^2 = (x - 1)^2 + (y - 4)^2$ which simplifies to $y = \frac{1}{8}(x-1)^2 + 16$ (a parabola).

0.3 Inverse Fuctions

1. $f(x) = x^5$ and $g(x) = x^{1/5}$

$f(g(x)) = f(x^{1/5}) = (x^{1/5})^5 = x$

$g(f(x)) = g(x^5) = (x^5)^{1/5} = x^{(5/5)} = x$

2. $f(x) = 4x^3$ and $g(x) = \left(\frac{1}{4}x\right)^{1/3}$

$$f(g(x)) = 4\left(\left(\frac{1}{4}x\right)^{1/3}\right)^3 = 4\left(\frac{1}{4}x\right) = x$$

$$g(f(x)) = \left(\frac{1}{4}4x^3\right)^{1/3} = x$$

3. $f(x) = 2x^3 + 1$ and $g(x) = \sqrt[3]{\frac{x-1}{2}}$

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x-1}{2}}\right)^3 + 1$$

$$= 2\left(\frac{x-1}{2}\right) + 1 = x$$

$$g(f(x)) = \sqrt[3]{\frac{f(x)-1}{2}}$$

$$= \sqrt[3]{\frac{2x^3+1-1}{2}} = \sqrt[3]{x^3} = x$$

4. $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1-2x}{x}$

$$f(g(x)) = \frac{1}{\frac{1-2x}{x}+2} = \frac{1}{\frac{1-2x}{x}+\frac{2x}{x}} = x$$

$$g(f(x)) = \frac{1-2(\frac{1}{x+2})}{\frac{1}{x+2}}$$

$$= \left(1-2\left(\frac{1}{x+2}\right)\right)(x+2)$$

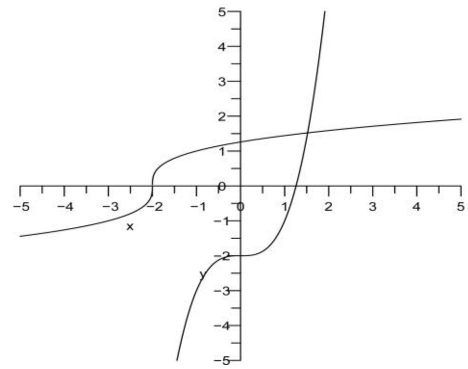
$= (x + 2) - 2 = x$

5. The function is one-to-one since $f(x) = x^3$ is one-to-one. To find the inverse function, write $y = x^3 - 2$

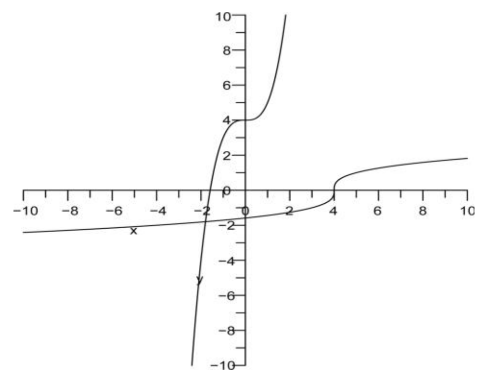
$y + 2 = x^3$

$\sqrt[3]{y+2} = x$

So $f^{-1}(x) = \sqrt[3]{x+2}$



6. The function is one-to-one with inverse $f^{-1}(x) = \sqrt[3]{x-4}$



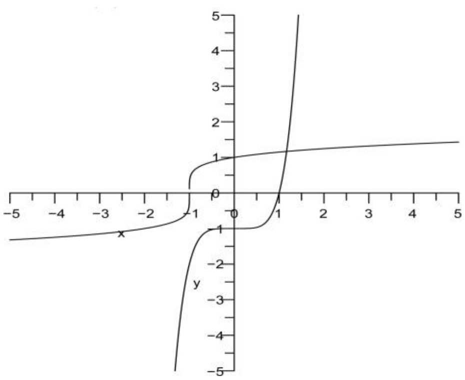
7. The graph of $y = x^5$ is one-to-one and hence so is $f(x) = x^5 - 1$. To find a formula for the inverse, write

$y = x^5 - 1$

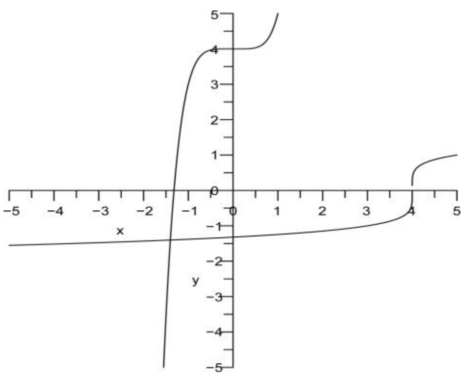
$y + 1 = x^5$

$\sqrt[5]{y+1} = x$

So $f^{-1}(x) = \sqrt[5]{x+1}$



8. The function is one-to-one with inverse $f^{-1}(x) = \sqrt[5]{x-4}$.



9. The function is not one-to-one since it is an even function ($f(-x) = f(x)$). In particular, $f(2) = 18 = f(-2)$.

10. Not one-to-one. Fails horizontal line test.

11. Here, the natural domain requires that the radicand (the object inside the radical) be nonnegative. Hence $x \geq -1$ is required, while all function values are non negative. Therefore the inverse, if defined at all, will be defined only for nonnegative numbers. Sometimes one can determine the existence of an inverse in the process of trying to find its formula. This is an example: Write

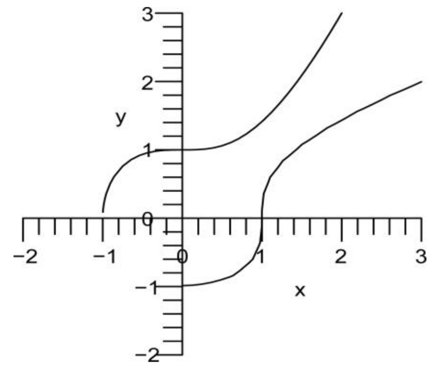
$$y = \sqrt{x^3 + 1}$$

$$y^2 = x^3 + 1$$

$$y^2 - 1 = x^3$$

$$\sqrt[3]{y^2 - 1} = x$$

The left side is a formula for $f^{-1}(y)$, good for $y \geq 0$. Therefore, $f^{-1}(x) = \sqrt[3]{x^2 - 1}$ whenever $x \geq 0$.



12. Not one-to-one. Fails horizontal line test.

13. (a) Since $f(0) = -1$, we know $f^{-1}(-1) = 0$

(b) Since $f(1) = 4$, we know $f^{-1}(4) = 1$

14. (a) Since $f(0) = 1$, we know $f^{-1}(1) = 0$.

(b) Since $f(2) = 13$, we know $f^{-1}(13) = 2$.

15. (a) Since $f(-1) = -5$, we know $f^{-1}(-5) = -1$.

(b) Since $f(1) = 5$, we know $f^{-1}(5) = 1$.

16. (a) Since $f(2) = 38$, we know $f^{-1}(38) = 2$

(b) Since $f(1) = 3$, we know $f^{-1}(3) = 1$.

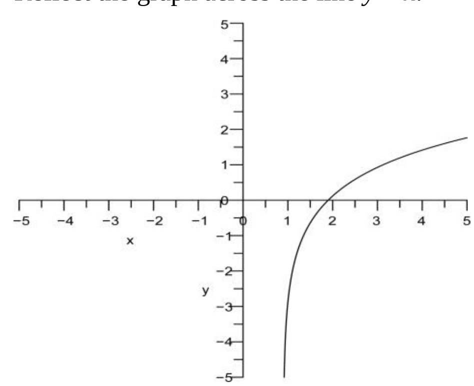
17. (a) Since $f(2) = 4$, we know $f^{-1}(4) = 2$.

(b) Since $f(0) = 2$, we know $f^{-1}(2) = 0$.

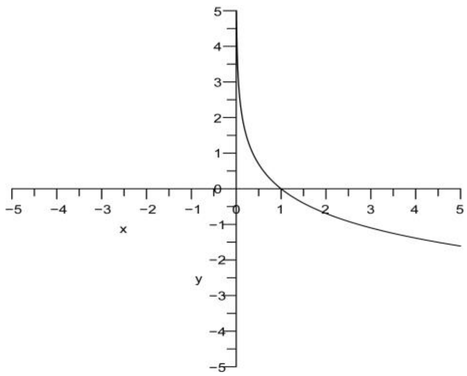
18. (a) Since $f(1) = 3$, we know $f^{-1}(3) = 1$

(b) Since $f(0) = 1$, we know $f^{-1}(1) = 0$.

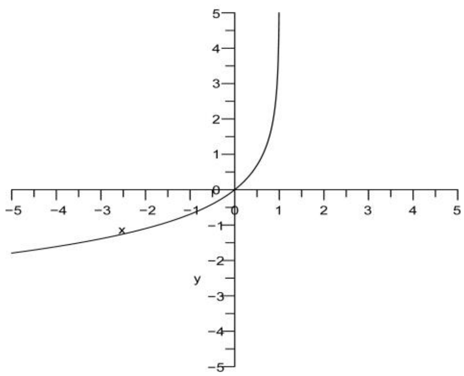
19. Reflect the graph across the line $y = x$.



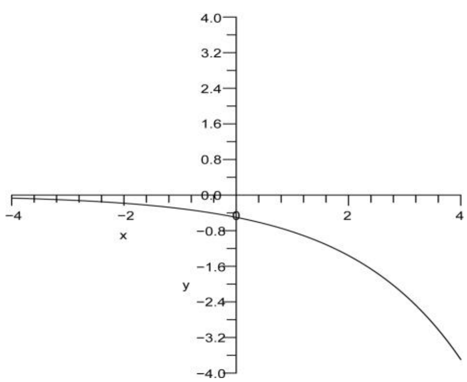
20. Reflect the graph across the line $y = x$.



21. Reflect the graph across the line $y = x$.



22. Reflect the graph across the line $y = x$.



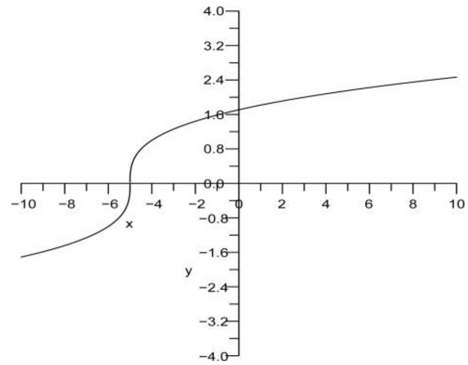
23. The range of function f is the domain of its inverse. Therefore, if the range of f is all $y > 0$, then the domain of the f^{-1} is $x > 0$.

24. If the graph of f includes (a, b), then $b = f(a)$, which implies $f^{-1}(b) = a$. Therefore, the graph of f^{-1} includes (b, a).

25. If the line $y = 3$ does not intersect the graph of f, there is no x such that $f(x) = 3$. Hence f^{-1} is not defined at $x = 3$.

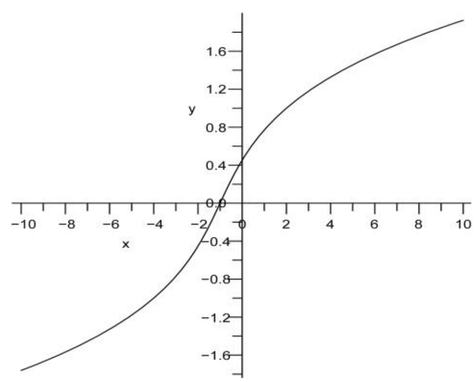
26. The range of function f^{-1} is the domain of the function f. Therefore, if the domain of f is all real numbers, the range of f^{-1} is all real numbers.

27. If $f(x) = x^3 - 5$, then the horizontal line test is passed, so $f(x)$ is one-to-one.



28. Not one-to-one. Fails horizontal line test.

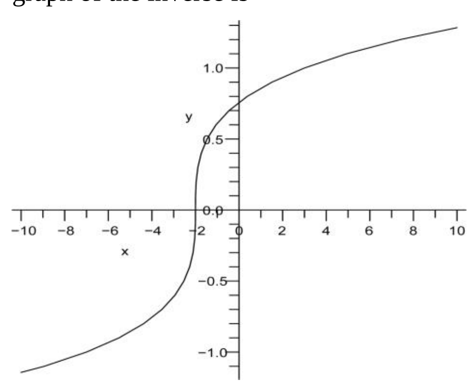
29. The function $f(x) = x^3 + 2x - 1$ easily passes the horizontal line test and is invertible.



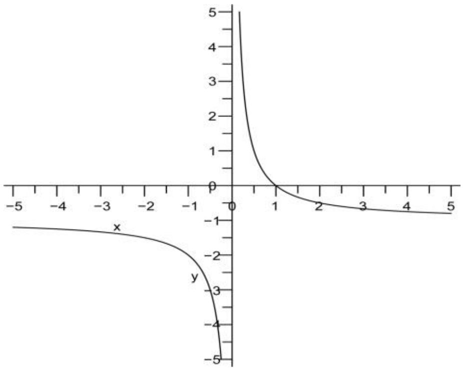
30. Not one-to-one. Fails horizontal line test.

31. Not one-to-one. Fails horizontal line test.

32. The function $f(x) = x^5 + 4x^3 - 2$ is one-to-one. The graph of the inverse is

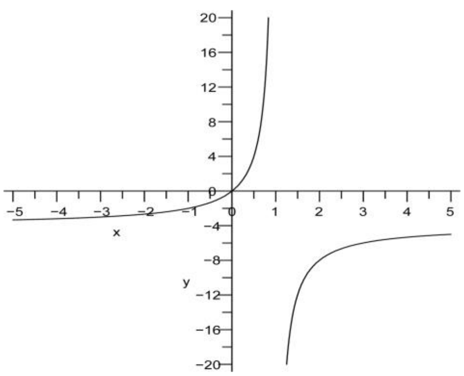


33. If $f(x)=\dfrac{1}{x+1}$, then the horizontal line test is passed, so $f(x)$ is one-to-one.

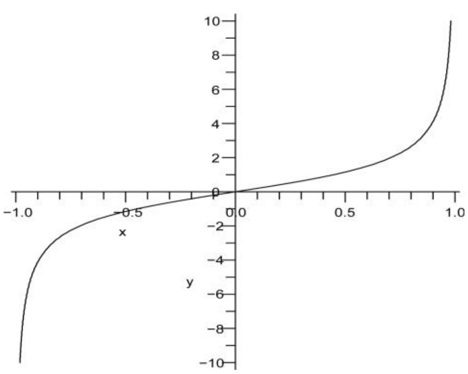


34. Not one-to-one. Fails horizontal line test.

35. If $f(x)=\dfrac{x}{x+4}$, then the horizontal line test is passed, so $f(x)$ is one-to-one.



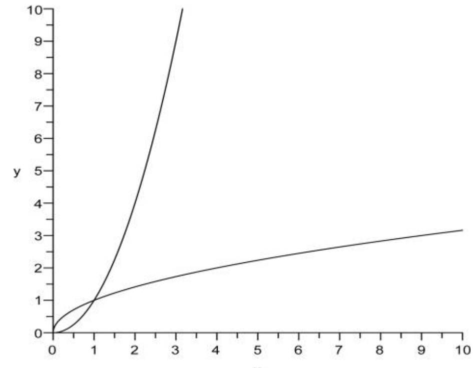
36. The function $f(x)=\dfrac{x}{\sqrt{x^2+4}}$ is one-to-one. The graph of the inverse is



37. $f(g(x))=(g(x))^2=(\sqrt{x})^2=x$

$g(f(x))=\sqrt{f(x)}=\sqrt{x^2}=|x|$

Because $x \geq 0$, the absolute value is the same as x. Thus these functions (both defined only when $x \geq 0$) are inverses.

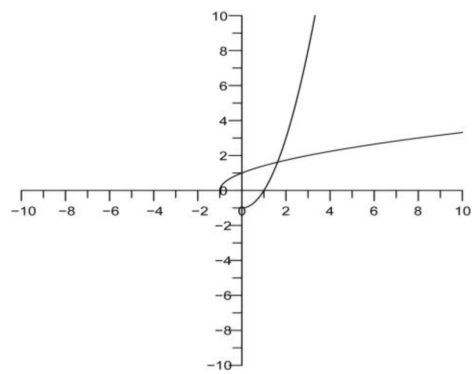


38. $f(x) = x^2 - 1 (x \geq 0)$ and $g(x)=\sqrt{x+1}(x \geq -1)$.

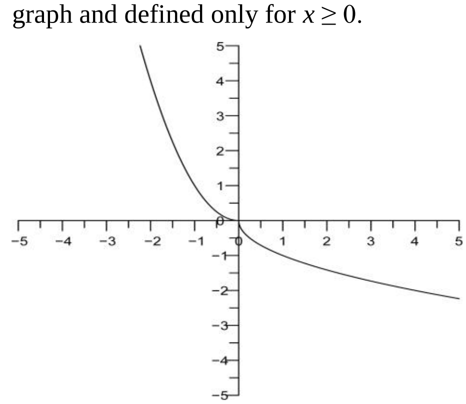
$f(g(x))=(\sqrt{x+1})^2-1=x$ and

$g(f(x))=\sqrt{(x^2-1)+1}=x$ (because $x \geq 0$),

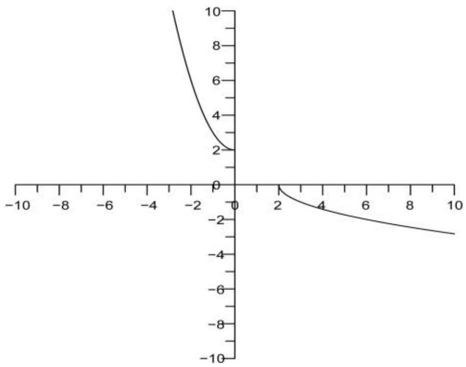
therefore f and g are inverse functions.



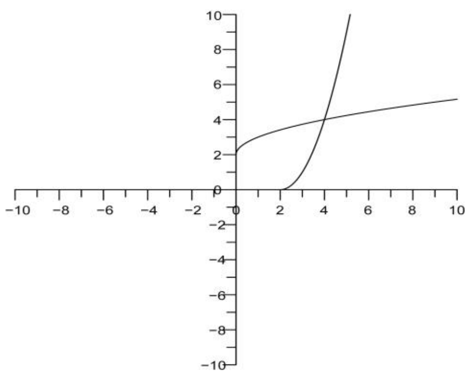
39. With $f(x) = x^2$ defined only for $x \leq 0$, (shown below as the upper left graph) the horizontal line test is easily passed. The formula for the inverse function g is $g(x)=-\sqrt{x}$ shown below as the lower right graph and defined only for $x \geq 0$.



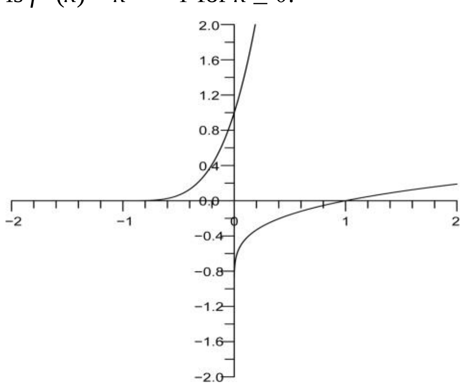
40. The inverse function is $f^{-1}(x) = -\sqrt{x-2}$.



41. The graph of $y = (x - 2)^2$ is a simple parabola with vertex at (2, 0). If we take only the right half $\{x \ge 2\}$ (shown below as the lower right graph) the horizontal line test is easily passed, and the formula for the inverse function g is $g(x) = 2 + \sqrt{x}$ defined only for $x \ge 0$ and shown below as the upper left graph.



42. $f(x) = (x + 1)^4$ is one-to-one for $x \ge -1$. The inverse is $f^{-1}(x) = x^{1/4} - 1$ for $x \ge 0$.



43. In the first place, for $f(x)$ to be defined, the radicand must be nonnegative, *i.e.*, $0 \le x^2 - 2x = x(x - 2)$ which entails either $x \le 0$ or $x \ge 2$. One can restrict the domain to either of these intervals and have an invertible function. Taking the latter for convenience, the inverse will be found as follows:

$y = \sqrt{x^2 - 2x}$

$y^2 = x^2 - 2x = (x - 1)^2 - 1$

$y^2 + 1 = (x - 1)^2$

$\sqrt{y^2 + 1} = \pm(x - 1)$

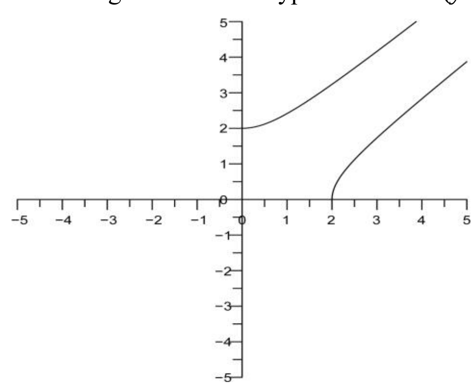
With $x \ge 2$ and the left side nonnegative, we must choose the plus sign. We can then write $x = 1 + \sqrt{y^2 + 1}$. The right side is now a formula for $f^{-1}(y)$ seemingly good for any y, but we recall from the original formula (as a radical) that y must be nonnegative. We summarize the conclusion:

$$f^{-1}(x) = 1 + \sqrt{x^2 + 1}, \quad (x \ge 0)$$

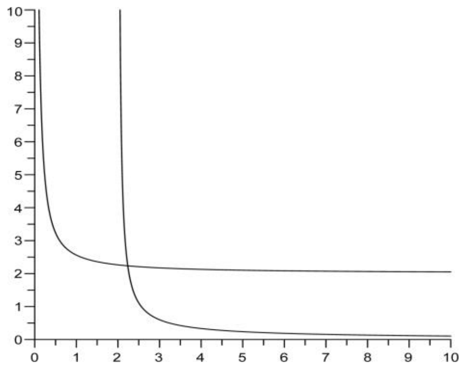
This is the upper graph below. The lower graph is the original $f(x) = \sqrt{x^2 - 2x}$. Had we chosen $\{x \le 0\}$, the "other half of the domain", and called the new function h, (same formula as f but a different domain, not shown) we would have come by choosing the minus sign, to the formula

$$h^{-1}(x) = 1 - \sqrt{x^2 + 1}, \quad (x \ge 0).$$

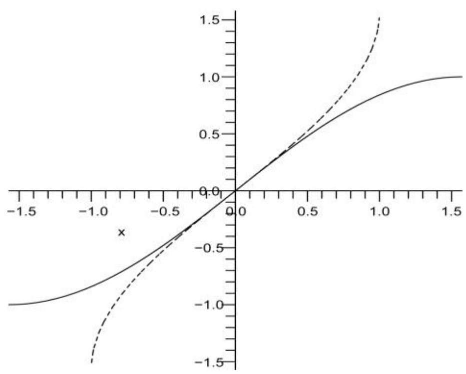
The two inverse formulae, if graphed together, fill out the right half of the hyperbola $-x^2 + (y - 1)^2 = 1$



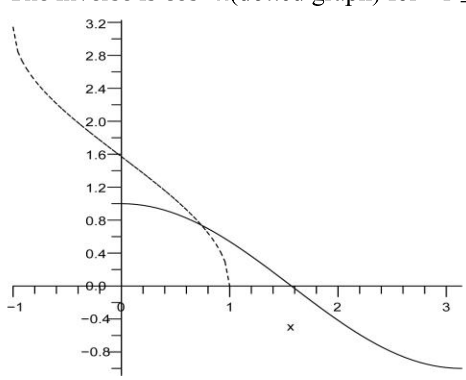
44. $y = \dfrac{x}{x^2 - 4}$ is one-to-one for $x > 2$. To solve for x, clear the denominator and use the quadratic formula $yx^2 - x - 4y = 0$ so $x = \dfrac{1 \pm \sqrt{1 + 16y^2}}{2y}$. Since $x > 2$, we use the plus sign. Switch x and y to get $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 16x^2}}{2x}$ for $x > 0$.



45. The function sin(x) (solid below) is increasing and one-to-one on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. One does not "find" the inverse in the sense of solving the equation $y = \sin(x)$ and obtaining a formula. It is done only in theory or as a graph. The name of the inverse is the "arcsin" function ($y = \arcsin(x)$ shown dotted), and some of its properties are developed in the next section.



46. $f(x) = \cos x$(solid graph) is one-to-one for $0 \le x \le \pi$. The inverse is $\cos^{-1}x$(dotted graph) for $-1 \le x \le 1$



47. A company's income is not in fact a function of time, but a function of a time interval (income is defined as the change in net worth). When income is viewed as a function of time, it is usually after picking a fixed time interval (week, month, quarter, or year) and assigning the income for the period in a consistent manner to either the beginning or the ending date as in "...income for the quarter beginning...". This much said, income more often than not rises and falls over time, so the function is unlikely to be one-to-one. In short, income functions usually do not have inverses.

48. Height of a person over time is not one-to-one since it stays fairly constant.

49. During an interval of free fall following a drop, the height is decreasing with time and (barring a powerful updraft, as with hail) an inverse exists. After impact, if there is a bounce then some of the heights are repeated and the function is no longer one-to-one on the expanded time interval.

50. Height of a ball thrown upward will be one-to-one until it reaches its apex, so on this domain it has an inverse.

51. Two three-dimensional shapes with congruent profiles will cast identical shadows if the congruent profiles face the light source. Such objects need not be fully identical in shape. (For an example, think of a sphere and a hemisphere with the flat side of the latter facing the light). The shadow as a function of shape is not one-to-one and does not have an inverse.

52. The number of calories burned increases as running speed increases. This is likely one-to-one and will have an inverse.

53. The usual meaning of a "ten percent cut in salary" is that the new salary is 90% of the old. Thus after a ten percent raise the salary is 1.1 times the original, and after a subsequent ten percent cut, the salary is 90% of the raised salary, or.9 times 1.1 times the original salary. The combined effect is 99% of the original, and therefore the ten percent raise and the ten percent cut are not inverse operations. The 10%-raise function is $y = f(x) = (1.1)\ x$, and the inverse relation is $x = y/1.1 = (0.90909...)\ y$. Thus $f^{-1}(x) = (0.90909)\ x$ and in the language of cuts, this is a pay cut of fractional value $1 - 0.90909... = 0.090909...$ or 9.0909...percent.

54. (a) If x is the original salary of the employee, then the new salary is $y = f(x) = 1.06x + 500$.

The inverse relation is $x = \dfrac{y-500}{1.06}$.

Therefore, $f^{-1}(x) = \dfrac{x-500}{1.06}$.

(b) If x is the original salary of the employee, then the new salary is $y = f(x) = 1.06(x + 500)$. The inverse relation is $x = \dfrac{y-530}{1.06}$.

Therefore, $f^{-1}(x) = \dfrac{x-530}{1.06}$.

0.4 Trigonometric and Inverse Trigonometric Functions

1. (a) $\left(\dfrac{\pi}{4}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) = 45^{\text{o}}$

(b) $\left(\dfrac{\pi}{3}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) = 60^{\text{o}}$

(c) $\left(\dfrac{\pi}{6}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) = 30^{\text{o}}$

(d) $\left(\dfrac{4\pi}{3}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) = 240^{\text{o}}$

2. (a) $\left(\dfrac{3\pi}{5}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) = 108^{\text{o}}$

(b) $\left(\dfrac{\pi}{7}\right)\left(\dfrac{180^{\text{o}}}{\pi}\right) \approx 25.71^{\text{o}}$

(c) $2\left(\dfrac{180^{\text{o}}}{\pi}\right) \approx 114.59^{\circ}$

(d) $3\left(\dfrac{180^{\text{o}}}{\pi}\right) \approx 171.89^{\text{o}}$

3. (a) $(180^{\text{o}})\left(\dfrac{\pi}{180^{\text{o}}}\right) = \pi$

(b) $(270^{\text{o}})\left(\dfrac{\pi}{180^{\text{o}}}\right) = \dfrac{3\pi}{2}$

(c) $(120^{\text{o}})\left(\dfrac{\pi}{180^{\text{o}}}\right) = \dfrac{2\pi}{3}$

(d) $(30^{\text{o}})\left(\dfrac{\pi}{180^{\text{o}}}\right) = \dfrac{\pi}{6}$

4. (a) $40^{\text{o}}\,\dfrac{\pi}{180^{\text{o}}} = \dfrac{2\pi}{9}$

(b) $80^{\text{o}}\,\dfrac{\pi}{180^{\text{o}}} = \dfrac{4\pi}{9}$

(c) $450^{0}\,\dfrac{\pi}{180^{\text{o}}} = \dfrac{5\pi}{2}$

(d) $390^{0}\,\dfrac{\pi}{180^{\text{o}}} = \dfrac{13\pi}{6}$

5. $2\cos(x) - 1 = 0$ when $\cos(x) = 1/2$. This occurs whenever $x = \frac{\pi}{3} + 2k\pi$ or $x = -\frac{\pi}{3} + 2k\pi$ for any integer k.

6. $2\sin x + 1 = 0$ when $\sin x = -\frac{1}{2}$. This occurs whenever $x = -\frac{\pi}{6} + 2k\pi$ or $x = -\frac{5\pi}{6} + 2k\pi$ for any integer k.

7. $\sqrt{2}\cos(x) - 1 = 0$ when $\cos(x) = 1/\sqrt{2}$. This occurs whenever $x = \frac{\pi}{4} + 2k\pi$ or $x = -\frac{\pi}{4} + 2k\pi$ for any integer k.

8. $2\sin x - \sqrt{3} = 0$ when $\sin x = \frac{\sqrt{3}}{2}$. This occurs whenever $x = \frac{\pi}{3} + 2k\pi$ or $x = \frac{2\pi}{3} + 2k\pi$ for any integer k.

9. $\sin^2 x - 4\sin x + 3 = (\sin x - 1)(\sin x - 3)$ when $\sin x = 1$ ($\sin x \neq 3$ for any x). This occurs whenever $x = \frac{\pi}{2} + 2k\pi$ for any integer k.

10. $\sin^2 x - 2\sin x - 3 = (\sin x - 3)(\sin x + 1)$ when $\sin x = -1$ ($\sin x \neq 3$ for any x). $\sin x = -1$ whenever $x = \frac{3\pi}{2} + 2k\pi$ for any integer k.

11. $\sin^2 x + \cos x - 1 = (1 - \cos^2 x) + \cos x - 1$

$= (\cos x)(\cos x - 1) = 0$

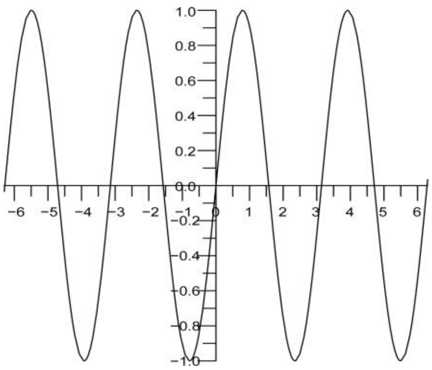
when $\cos x = 0$ or $\cos x = 1$. This occurs whenever $x = \frac{\pi}{2} + k\pi$ or $x = 2k\pi$ for any integer k.

12. Use the sine double angle formula to get $2\sin x\cos x - \cos x = (2\sin x - 1)\cos x = 0$ then $(2\sin x - 1) = 0$ whenever $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$ and $\cos x = 0$ whenever $x = \frac{\pi}{2} + k\pi$ for any integer k.

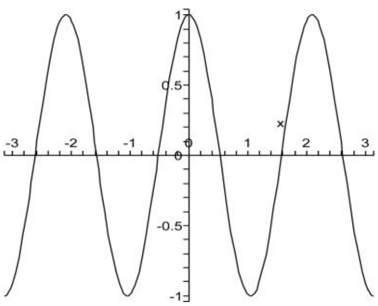
13. $\cos^2 x + \cos x = (\cos x)(\cos x + 1) = 0$ when $\cos x = 0$ or $\cos x = -1$ this occurs whenever $x = \frac{\pi}{2} + k\pi$ or $x = \pi + 2k\pi$ for any integer k.

14. $\sin^2 x - \sin x = \sin x(\sin x - 1) = 0$ whenever $x = k\pi$ or $x = \frac{\pi}{2} + 2k\pi$ for any integer k.

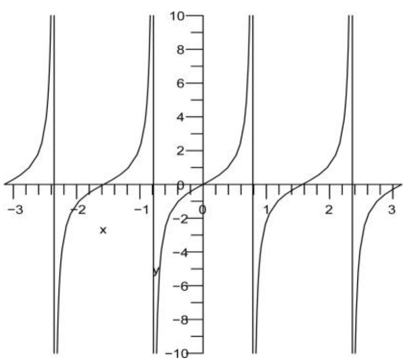
15. The graph of $f(x) = \sin 2x$.



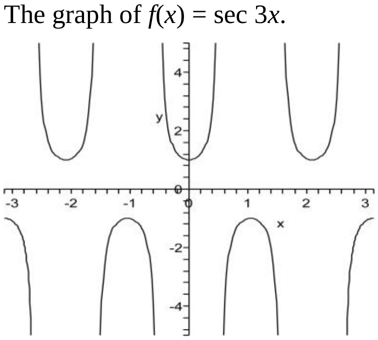
16. The graph of $f(x) = \cos 3x$.



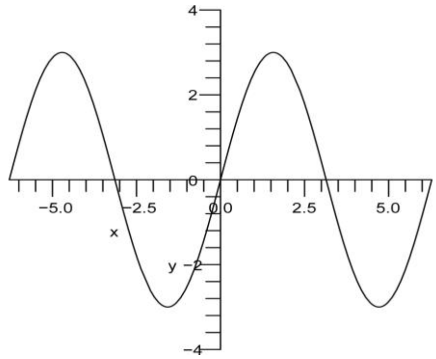
17. The graph of $f(x) = \tan 2x$.



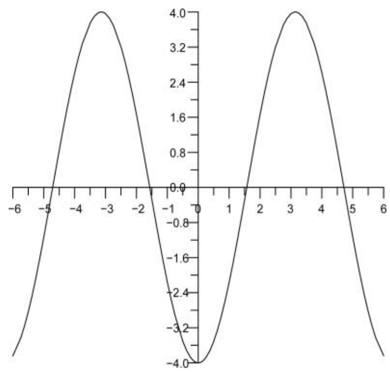
18. The graph of $f(x) = \sec 3x$.



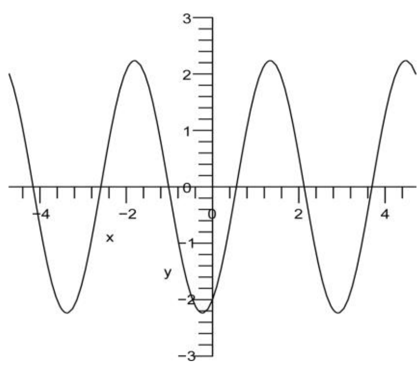
19. The graph of $f(x) = 3\cos(x - \pi/2)$.



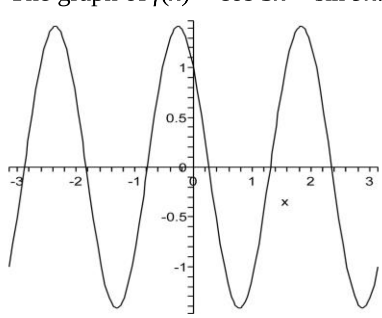
20. The graph of $f(x) = 4\cos(x + \pi)$.



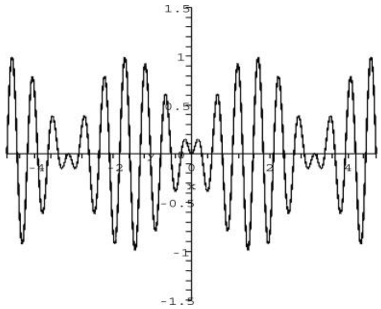
21. The graph of $f(x) = \sin 2x - 2\cos 2x$.



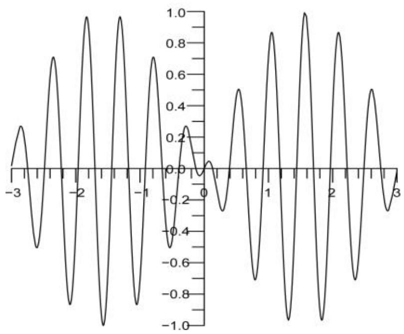
22. The graph of $f(x) = \cos 3x - \sin 3x$.



23. The graph of $f(x) = \sin x \sin 12x$.



24. The graph of $f(x) = \sin x \cos 12x$.



25. Amplitude is 3, period is $\frac{2\pi}{2} = \pi$, frequency is $\frac{1}{\pi}$.

26. Amplitude is 2, period is $\frac{2\pi}{3}$, frequency is $\frac{3}{2\pi}$.

27. Amplitude is 5, period is $\frac{2\pi}{3}$, frequency is $\frac{3}{2\pi}$.

28. Amplitude is 3, period is $\frac{2\pi}{5}$, frequency is $\frac{5}{2\pi}$.

29. Amplitude is 3, period is $\frac{2\pi}{2} = \pi$, frequency is $\frac{1}{\pi}$. We are completely ignoring the presence of $-\pi/2$. This has an influence on the so-called "phase shift" which will be studied in Chapter 6.

30. Amplitude is 4, period is $\frac{2\pi}{3}$, frequency is $\frac{3}{2\pi}$.

31. Amplitude is 4 (the graph oscillates between −4 and 4, so we may ignore the minus sign), period is 2π, frequency is $\frac{1}{2\pi}$.

32. Amplitude is 2, period is $\frac{2\pi}{3}$, frequency is $\frac{3}{2\pi}$.

33.
$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\sin(-\beta)\cos\alpha\\ &= \sin\alpha\cos\beta-\sin\beta\cos\alpha\end{aligned}$$

34.
$$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

35. (a)
$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)\\ &= \cos^2\theta-\sin^2\theta\\ &= \cos^2\theta-(1-\cos^2\theta)\\ &= 2\cos^2\theta-1\end{aligned}$$

(b) Just continue on, writing
$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta-1\\ &= 2(1-\sin^2\theta)-1\\ &= 1-2\sin^2\theta\end{aligned}$$

36. (a) Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ to get $\frac{\sin^2\theta}{\cos^2\theta}+1=\frac{1}{\cos^2\theta}$ or $\tan^2\theta + 1 = \sec^2\theta$.

(b) Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ yields $\cot^2\theta + 1 = \csc^2\theta$

37. $\cos^{-1}(0) = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}$

Any arbitrary point on the unit circle is $(\cos\theta, \sin\theta)$, therefore the ordered pair on the circle is (0, 1).

38. $\tan^{-1}(0) = 0 \Rightarrow \theta = 0$

The ordered pair on the circle is (1, 0).

39. $\sin^{-1}(-1) = -\frac{\pi}{2} \Rightarrow \theta = -\frac{\pi}{2}$

The ordered pair on the circle is (0, −1).

40. $\cos^{-1}(1) = 0 \Rightarrow \theta = 0$

The ordered pair on the circle is (1, 0).

41. $\sec^{-1}(1) = 0 \Rightarrow \theta = 0$

The ordered pair on the circle is (1, 0).

42. $\tan^{-1}(-1) = -\frac{\pi}{4} \Rightarrow \theta = -\frac{\pi}{4}$

The ordered pair on the circle is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

43. $\sec^{-1}(2) = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

The ordered pair on the circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

44. $\csc^{-1}(2) = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$

The ordered pair on the circle is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

45. $\cot^{-1}(1) = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

The ordered pair on the circle is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

46. $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

The ordered pair on the circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

47. Use the formula

$\cos(x + \beta) = \cos x \cos \beta - \sin \beta \sin x$. Now we see that $\cos \beta$ must equal 4/5 and $\sin \beta$ must equal 3/5. Since $(4/5)^2 + (3/5)^2 = 1$, this is possible. We see that $\beta = \sin^{-1}(3/5) \approx 0.6435$ radians, or 36.87°.

48. Use the formula

$\sin(x + \beta) = \sin x \cos \beta + \sin \beta \cos x$. Now we see that $\cos \beta$ must equal $2/\sqrt{5}$ and $\sin \beta$ must equal $1/\sqrt{5}$. Since $(2/\sqrt{5})^2 + (1/\sqrt{5})^2 = 1$, this is possible. We see that $\beta = \sin^{-1}(2/\sqrt{5}) \approx 0.4636$ radians, or 26.57°.

49. $\cos(2x)$ has period $\frac{2\pi}{2} = \pi$ and $\sin(\pi x)$ has period $\frac{2\pi}{\pi} = 2$. There are no common integer multiples of the periods, so the function $f(x) = \cos(2x) + 3\sin(\pi x)$ is not periodic.

50. $\sin x$ has period 2π and $\cos\sqrt{2}x$ has period $\sqrt{2}\pi$. There are no common integer multiples of the periods, so the function $f(x) = \sin x - \cos\sqrt{2}x$ is not periodic.

51. $\sin(2x)$ has period $\frac{2\pi}{2} = \pi$ and $\cos(5x)$ has period $\frac{2\pi}{5}$. The smallest integer multiple of both of these is the fundamental period, and it is 2π.

52. $\cos 3x$ has period $\frac{2\pi}{3}$ and $\sin 7x$ has period $\frac{2\pi}{7}$. The smallest integer multiple of both of these is the fundamental period, and it is 2π.

53. $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$ Because θ s in the first quadrant, its cosine is non-negative. Hence $\cos\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} = 0.9428$.

54. First quadrant, 3-4-5 right triangle, so $\sin\theta = \frac{3}{5}$.

55. Second quadrant, $1\text{-}\sqrt{3}\text{-}2$ right triangle, so

$\cos\theta = -\frac{\sqrt{3}}{2}$.

56. Second quadrant, $1\text{-}\sqrt{3}\text{-}2$ right triangle, so

$\tan\theta = -\frac{1}{\sqrt{3}}$.

57. Assume $0 < x < 1$ and give the temporary name θ to $\sin^{-1}(x)$. In a right triangle with hypotenuse 1 and one leg of length x, the angle θ will show up opposite the x-side, and the adjacent side will have length $\sqrt{1-x^2}$ Write $\cos(\sin^{-1}(x)) = \cos(\theta)$

$$= \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

The formula is numerically correct it he cases $x = 0$ and $x = 1$, and both sides are even functions of x, i.e. $f(-x) = f(x)$ so the formula is good for $-1 \le x \le 1$.

58. $\tan^{-1}x$ relates to a triangle in the first or fourth quadrant with opposite side x, adjacent side 1, and hypotenuse $\sqrt{x^2+1}$. Therefore, $\cos(\tan^{-1}x) = \frac{1}{\sqrt{x^2+1}}$. This is valid for all x.

59. Assume $1 < x$ and give the temporary name θ to $\sec^{-1}(x)$. In a right triangle with hypotenuse x and one leg of length 1, the angle θ will show up adjacent to the side of length 1, and the opposite side will have length $\sqrt{x^2-1}$. Write

$\tan(\sec^{-1}(x)) = \tan(\theta)$

$$= \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

The formula is numerically correct in the case $x \geq 1$. Dealing with negative x is trickier: assume $x > 1$ for the moment. The key identity is $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$. Taking tangents on both sides and applying the identity

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1+\tan(a)\tan(b)}$$

with $a = \pi$, $\tan(a) = 0$, $b = \sec^{-1}x$, we find

$$\tan(\sec^{-1}x) = \frac{0-\tan(\sec^{-1}x)}{1+0}$$
$$= -\sqrt{x^2-1} = -\sqrt{(-x)^2-1}$$

In this identity, $-x$ (on both sides) plays the role of an arbitrary number < -1. Consequently, the final formula is $\tan(\sec^{-1}x) = -\sqrt{x^2-1}$ whenever $x \leq -1$.

60. $\cos^{-1}x$ relates to a triangle in the first or second quadrant with adjacent side x, hypotenuse 1, and opposite side $\sqrt{1-x^2}$. Therefore
$\cot(\cos^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$ is valid for $-1 \leq x \leq 1$.

61. One can use the formula $\sin(\cos^{-1}x) = \sqrt{1-x^2}$ derived in the text:

$$\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$

62. $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$ and $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$.

63. $\cos^{-1}(\frac{3}{5})$ relates to a triangle in the first quadrant with adjacent side 3 and hypotenuse 5, so the opposite side must be 4 and then

$$\tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \frac{4}{3}.$$

64. $\sin^{-1}\left(\dfrac{2}{3}\right)$ relates to a triangle in the first quadrant with opposite side 2 and hypotenuse 3, so

$$\csc\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \frac{3}{2}.$$

65. From graph the three solutions are 0, 1.109, and 3.698.

66. From graph the three solutions are 0 and ±2.28

67. From graph the two solutions are ±1.455

68. From graph the two solutions are 0 and 0.88

Applications

1. Let h be the height of the rocket. Then

$$\frac{h}{2} = \tan 20° \Rightarrow h = 2\tan 20° \approx 0.73(\text{kilometers})$$

2. The person and the shadow form a right triangle similar to the triangle formed by the light-pole and the distance from the base of the pole to the tip of the shadow. If x represents the height of the pole, we have that

$\dfrac{x}{4+2} = \dfrac{6}{2}$ and therefore $x = 18$.

3. Let h be the height of the steeple. Then

$$\frac{h}{80+20} = \tan 50°$$

$\Rightarrow h = 100 \tan 50° \approx 119.2$ (meters).

4. If the steeple is 20cm inside the building, the height is 100 tan 50° ≈ 119.18 meters. If the steeple is 21cm inside the building, the height is 101 tan 50° ≈ 120.37 meters. The difference is 1.19 meters.

5. Using meters as the measuring standard, we find

$$\tan A = \frac{60/100}{x} = \frac{3}{5x} \Rightarrow A = \tan^{-1}\left(\frac{3}{5x}\right)$$

The graph of $y = A(x)$ (of course, one has to choose an appropriate range to make this a function):

6. From the center of the hole to the left (or right) edge is 54 millimeters. Consider the right triangle formed by the golfer, the center of the hole and the left edge. The angle at the golfer is $\tan^{-1}\left(\dfrac{54}{x}\right)$. The margin of error is then twice that, or $A = 2\tan^{-1}\left(\dfrac{54}{x}\right)$.

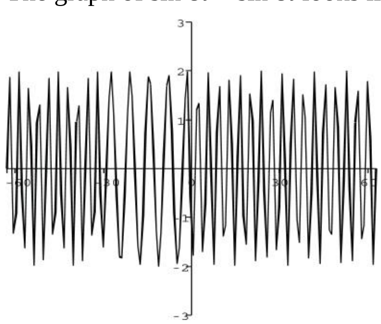
7. Presumably, the given amplitude (170) is the same as the "peak voltage" (v_p). Recalling an earlier discussion (#25 this section): the role of ω there is played by $2\pi f$ here, the frequency in cycles per second (Hz) was $\omega/2\pi$, which is now the f-parameter $(2\pi f/2\pi)$. The period was $2\pi/\omega$ (which is now $1/f$), given in this case to be $\pi/30$ (seconds). So, apparently, the frequency is $f = 30/\pi$ (cycles per second) and the meter voltage is $\frac{170}{\sqrt{2}} \approx 120.2$

8. Revolutions per minute measures frequency. The period is the reciprocal. The period of a $33\frac{1}{3}$ rpm record is $\frac{3}{100}$ minutes per revolution. Similarly, the period of a 45 rpm record is $\frac{1}{45}$ minutes per revolution.

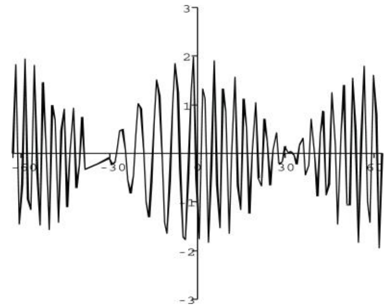
9. There seems to be a certain slowly increasing base for sales $(110 + 2t)$, and given that the sine function has period $\frac{2\pi}{\pi/6} = 12$ months, the sine term apparently represents some sort of seasonally cyclic pattern. If we assume that travel peaks at Thanksgiving, the effect is that time zero would correspond to a time one quarter period (3 months) prior to Thanksgiving, or very late August.

The annual increase for the year beginning at time t is given by $s(t + 12) - s(t)$ and automatically ignores both the seasonal factor and the basic 110, and indeed it is the constant $2 \times 12 = 24$ (in thousands of dollars per year and independent of the reference point t).

10. The graph of $\sin 8t + \sin 8t$ looks like



The graph of $\sin 8t + \sin 8.1t$ looks like



The difference in frequency produces clearly audible beats (to the trained ear).

0.5 Exponential and Logarithmic Functions

1. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

2. $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

3. $3^{1/2} = \sqrt{3}$

4. $6^{2/5} = \sqrt[5]{6^2} = \sqrt[5]{36}$

5. $5^{2/3} = \sqrt[3]{5^2} = \sqrt[3]{25}$

6. $4^{-2/3} = \frac{1}{\sqrt[3]{4^2}} = \frac{1}{\sqrt[3]{16}} = \frac{1}{\sqrt[3]{8 \cdot 2}} = \frac{1}{2\sqrt[3]{2}}$

7. $\frac{1}{x^2} = x^{-2}$

8. $\sqrt[3]{x^2} = x^{2/3}$

9. $\frac{2}{x^3} = 2x^{-3}$

10. $\frac{4}{x^2} = 4x^{-2}$

11. $\frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}} = \frac{1}{2}x^{-1/2}$

12. $\frac{3}{2\sqrt{x^3}} = \frac{3}{2x^{3/2}} = \frac{3}{2}x^{-3/2}$

13. $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

14. $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$

15. $\dfrac{\sqrt{8}}{2^{1/2}} = \dfrac{\sqrt{8}}{\sqrt{2}} = \sqrt{4} = 2$

16. $\dfrac{2}{(1/3)^2} = \dfrac{2}{(1/9)} = 18$

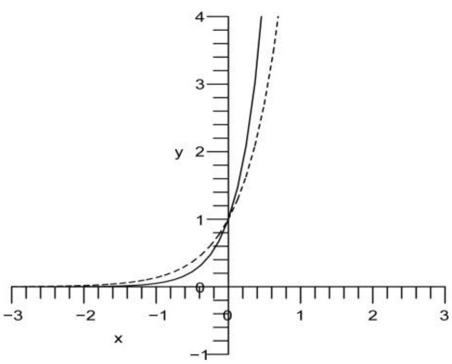
17. $2e^{-1/2} \approx 1.213$

18. $4e^{-2/3} \approx 2.05$

19. $\dfrac{12}{e} \approx 4.415$

20. $\dfrac{14}{\sqrt{e}} \approx 8.49$

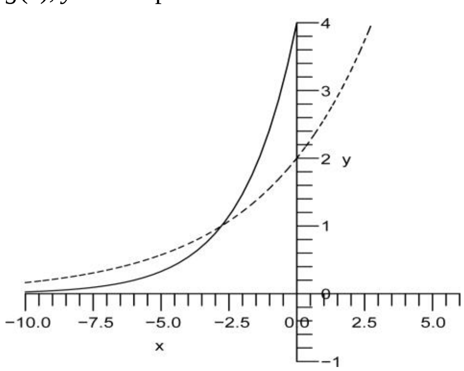
21. Both the graphs have same y-intercept.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

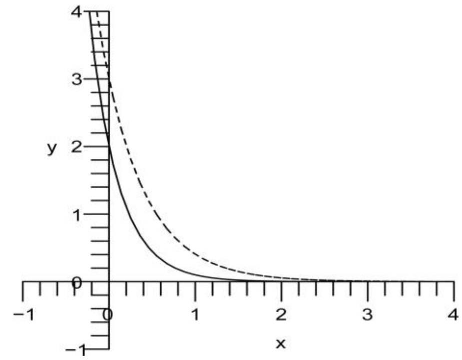
22. For the graph $f(x)$, y-intercept is 2 and for the graph $g(x)$, y-intercept is 4.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

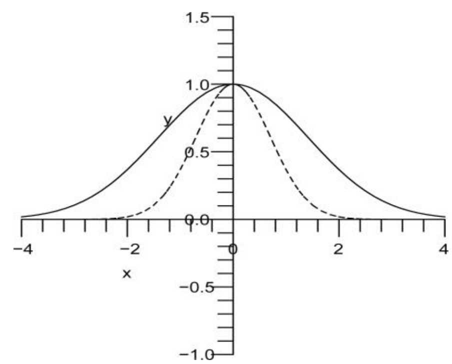
23. For the graph $f(x)$, y-intercept is 3 and for the graph $g(x)$, y-intercept is 2.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

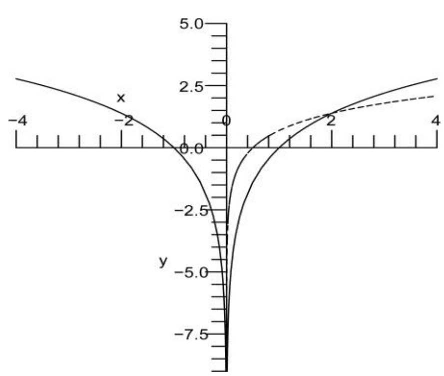
24. For both the graphs, y-intercept is 1. The graph of e^{-x^2} approaches the x-axis faster than the graph of $e^{-\frac{x^2}{4}}$.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

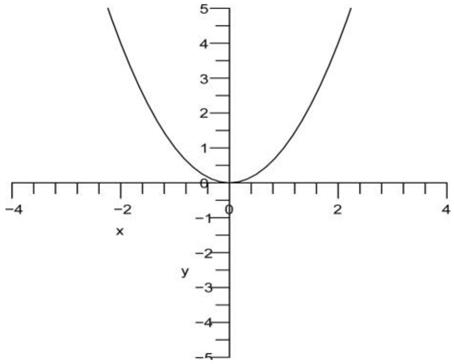
25. The graph $f(x)$ is defined for positive values of x only and the graph $g(x)$ is defined for all nonzero value of x.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

26. Both the graphs $f(x)$ and $g(x)$ are same.



Graph of $f(x)$: Dotted line.

Graph of $g(x)$: Solid line.

27. $e^{2x} = 2$

$\Rightarrow \ln e^{2x} = \ln 2$

$\Rightarrow 2x = \ln 2$

$$\Rightarrow x = \frac{\ln 2}{2} \approx 0.3466$$

28. $e^{4x} = 3$

$\Rightarrow 4x = \ln 3$

$$\Rightarrow x = \frac{\ln 3}{4} \approx 0.2747$$

29. $e^x (x^2 - 1) = 0 \Rightarrow x^2 - 1 = 0$(Since $e^x \neq 0$). Hence $x = 1$ or $x = -1$.

30. $$xe^{-2x} + 2e^{-2x} = 0 \Rightarrow \frac{x+2}{e^{2x}} = 0 \Rightarrow x = -2$$

31. $4 \ln x = -8$

$\Rightarrow \ln x = -2$

$$\Rightarrow x = e^{-2} = \frac{1}{e^2} \approx 0.13533$$

32. $x^2 \ln x - 9 \ln x = 0$

$\Rightarrow (x^2 - 9) \ln x = 0$

So either $\ln x = 0$ or $x^2 - 9 = 0$

$\Rightarrow x = 1, x = \pm 3$

33. $e^{2 \ln x} = 4$

$\Rightarrow 2 \ln x = \ln 4$

$\Rightarrow \ln x^2 = \ln 4$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

But in the original equation we had the expression $e^{2 \ln x}$ so $x \neq -2$ and thus the only solution is $x = 2$.

34. $\ln (e^{2x}) = 6 \Rightarrow 2x = 6 \Rightarrow x = 3$

35. $e^x = 1 + 6e^{-x}$

$\Rightarrow e^{2x} - e^x - 6 = 0$

$\Rightarrow (e^x - 3)(e^x + 2) = 0$

$\Rightarrow e^x - 3 = 0$ (Since $e^x + 2 \neq 0$)

$\Rightarrow x = \ln 3$

36. $\ln x + \ln(x - 1) = \ln 2$

Taking the exponential of both sides we get

$\Rightarrow x(x - 1) = 2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x - 2)(x + 1) = 0$

$\Rightarrow x = -1$ or $x = 2$

But $\ln x$ is not defined for $x = -1$. Hence $x = 2$ is the only solution.

37. (a) $\log_3 9 = \log_3 (3^2) = 2$

(b) $\log_4 64 = \log_4 (4^3) = 3$

(c) $\log_3 \frac{1}{27} = \log_3 (3^{-3}) = -3$

38. (a) $\log_4 \frac{1}{16} = \log_4 \frac{1}{4^2} = \log_4 4^{-2} = -2$

(b) $\log_4 2 = \log_4 4^{1/2} = \frac{1}{2}$

(c) $\log_9 3 = \log_9 9^{1/2} = \frac{1}{2}$

39. (a) $\log_3 7 = \frac{\ln 7}{\ln 3} \approx 1.771$

(b) $\log_4 60 = \frac{\ln 60}{\ln 4} \approx 2.953$

(c) $\log_3 \frac{1}{24} = \frac{\ln(1/24)}{\ln 3} \approx -2.893$

40. (a) $\log_3 \frac{1}{10} = -\frac{\ln 10}{\ln 3} \approx -1.66$

(b) $\log_4 3 = \frac{\ln 3}{\ln 4} \approx 0.79$

(c) $\log_9 8 = \dfrac{\ln 8}{\ln 9} \approx 0.95$

41. $\ln 3 - \ln 4 = \ln \frac{3}{4}$

42. $2\ln 4 - \ln 3 = \ln \dfrac{4^2}{3} = \ln \dfrac{16}{3}$

43. $\frac{1}{2}\ln 4 - \ln 2 = \frac{1}{2}\cdot 2\ln 2 - \ln 2 = 0$

44. $3\ln 2 - \ln \frac{1}{2} = \ln \frac{2^3}{1/2} = \ln 16$

45. $\ln \frac{3}{4} + 4\ln 2 = \ln \frac{3}{2^2} + \ln 2^4 = \ln\left(\frac{3}{2^2}\cdot 2^4\right) =$

$\ln(3\cdot 2^2) = \ln(12)$

46. $\ln 9 - 2\ln 3 = \ln \frac{9}{3^2} = \ln 1 = 0$

47. $f(0) = 2 \Rightarrow a \Rightarrow 2$.

Then $f(2) = 6$ gives $2e^{2b} = 6$, so $2b = \ln 3$ and $b = \dfrac{1}{2}\ln 3$. So $f(x) = 2e^{\left(\frac{1}{2}\ln 3\right)x} = 2\left[e^{\ln(3)}\right]^{x/2} = 2\cdot 3^{x/2}$

48. $f(0) = 3 \Rightarrow a = 3$.

Then $f(3) = 4$ gives $3e^{3b} = 4$, so $3b = \ln \frac{4}{3}$ and $b = \frac{1}{3}\ln \frac{4}{3}$. So $f(x) = 4e^{\left(\frac{1}{2}\ln \frac{1}{2}\right)x}$.

49. $f(0) = 4 \Rightarrow a = 4$.

Then $f(2) = 2$ gives $4e^{2b} = 2$, so $2b = \ln \frac{1}{2}$ and $b = \frac{1}{2}\ln \frac{1}{2}$. So $f(x) = 4e^{\left(\frac{1}{2}\ln \frac{1}{2}\right)x}$.

50. $f(0) = 5 \Rightarrow a = 5$.

Then $f(1) = 2$ gives $5e^{b} = 2$, so and $b = \ln \frac{2}{5}$.

So $f(x) = 5e^{\left(\ln \frac{2}{5}\right)x}$.

51. We know that $\cosh x = \dfrac{e^x + e^{-x}}{2}$. To show that $\cosh x \ge 1$ for all x is the same as showing that $\cosh x - 1 \ge 0$ for all x. So we ask when is the expression $\cosh x - 1 = \dfrac{e^x + e^{-x}}{2} - 1$ greater than or equal to 0?

We have:

$\dfrac{e^x + e^{-x}}{2} - 1 \ge 0$ if and only if

$\dfrac{e^x + e^{-x} - 2}{2} \ge 0$ if and only if

$e^x + e^{-x} - 2 \ge 0$ if and only if

$e^x + 1 - 2e^{-x} \ge 0$ if and only if*

$e^x - 2e^{-x} + 1 \ge 0$ if and only if

$(e^x - 1)^2 \ge 0$

But $(e^x - 1)^2$ is always greater than or equal to 0 since it is squared. It is actually equal to 0 at $x = 0$(i.e., $\cosh 0 = 1$), so the range of $y = \cosh x$ is $y \ge 1$.

* In the * step (above), we have multiplied on both sides by e^x, which we are allowed to do since $e^x > 0$ for all x. To show that the range of the hyperbolic sine is all real numbers, let a be any real number and solve the equation $\sinh(x) = a$. Let $u = e^x$. Then

$\dfrac{u - \frac{1}{u}}{2} = a$ if and only if

$u^2 - 1 = 2au$ if and only if

$u^2 - 2au - 1 = 0$ if and only if

$u = \dfrac{2a \pm \sqrt{4a^2 + 4}}{2} = a + \sqrt{a^2 + 1}$.

We simplified and chose the positive square root because $u > 0$. Because we found a unique solution no matter what a we had started with, we have shown that the range of $y = \sinh x$ is the whole real line.

52. $\cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

53. Since $\sinh^{-1}(0) = 0$, the equation is solved only by $x^2 - 1 = 0$, hence $x = 1$ or $x = -1$.

54. $\cosh(3x + 2) = 0$ has no solutions because $\cosh x \ge 1$ for all x.

Applications

1. $1 - \left(\dfrac{9}{10}\right)^{10} \approx 0.651$

2. The percentage decreases by almost 1%

3. We take on faith, whatever it may mean, that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Just to take a sample starting with n = 25, the numbers are

$\left(\frac{26}{25}\right)^{25}, \left(\frac{27}{26}\right)^{26}, \left(\frac{28}{27}\right)^{27}$ and so on. If we were to try taking a similar look at the numbers in

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$, the numbers starting at n = 26 would

be $\left(\frac{26}{25}\right)^{25}, \left(\frac{27}{26}\right)^{26}, \left(\frac{28}{27}\right)^{27}$ and so on.

We could rewrite these as

$$\left[\left(\frac{25}{26}\right)^{25}\right]^{\frac{26}{25}}, \left[\left(\frac{26}{27}\right)^{26}\right]^{\frac{27}{26}}, \left[\left(\frac{27}{28}\right)^{27}\right]^{\frac{28}{27}}$$

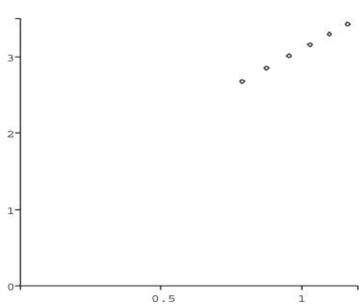
Here, the numbers inside the square brackets are the reciprocals of the numbers in the original list, which were all pretty close to e. Therefore these must all be pretty close to $1/e$. As to the external powers, they are all close to 1 and getting closer. This limit must be $1/e$. The expression in question must approach $1-\frac{1}{e}\approx 0.632$.

4. If $y = ax^m$ then $\ln y = \ln(ax^m) = \ln a + \ln x^m = \ln a + m \ln x$. Direct substitutions show that $v = mu + b$, and this is the equation of a line.

5.

$u = \ln x$	.78846	.87547	.95551
$v = \ln y$	2.6755	2.8495	3.0096

$u = \ln x$	1.0296	1.0986	1.1632
$v = \ln y$	3.1579	3.2958	3.4249



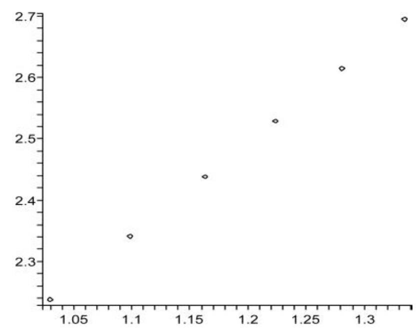
$$m=\frac{3.4249-2.6775}{1.1632-.78846}\approx 2.$$

Then we solve $2.6755 = 2\cdot(.78846) + b$ to find $b \approx 1.099$. Now $b = \ln a$, so $a = e^b \approx 3.001$, and the function is $y = 3.001x^2$.

6.

$u = \ln x$	1.0296	1.0986	1.1632
$v = \ln y$	2.2375	2.3408	2.4380

$u = \ln x$	1.2238	1.2809	1.3350
$v = \ln y$	2.5289	2.6145	2.6953



$$m=\frac{2.6953-2.2375}{1.3350-1.0296}=1.4990\approx\frac{3}{2}.$$

Then we solve $2.6953=\frac{3}{2}(1.3350)+b$ to find b = .6928. Now $b = \ln a$, so $a = e^b \approx 1.9993 \approx 2$, and the function is $y = 2x^{3/2}$.

7. We compute $u = \ln x$ and $v = \ln y$ for x values in number of decades since 1960 and y values in millions.

$u = \ln x$	0	0.693	1.099	1.386
$v = \ln y$	3.29	3.54	3.76	4.03

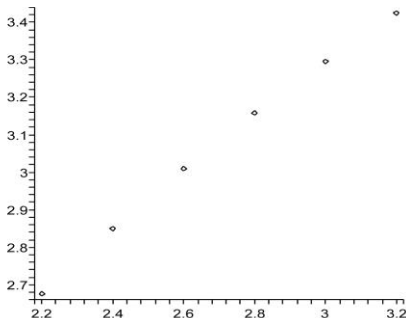
$u = \ln x$	1.609	1.792
$v = \ln y$	4.22	4.40

This plot does not look linear, which makes it clear that the population is not modeled by a power of x. The discussion in the Chapter has already strongly indicated that an exponential model is fairly good.

8.

x	2.2	2.4	2.6
ln y	2.6755	2.8495	3.0096

x	2.8	3.0	3.2
ln y	3.1579	3.2958	3.4249



This plot is slightly bowed concave down. The log-log plot looks more linear, and the function is modeled better by a power function.

9. (a) $7 = -\log[H^+] \Rightarrow [H^+] = 10^{-7}$

(b) $[H^+] = 10^{-8}$

(c) $[H^+] = 10^{-9}$

For each increase in pH of one, $[H^+]$ is reduced to one tenth of its previous value.

10. If the pH = 2.5 = $-\log[H^+]$, then $[H^+] = 10^{-2.5} \approx 3.16 \times 10^{-3}$. If the pH = 7.5 = $-\log[H^+]$, then $[H^+] = 10^{-7.5} \approx 3.16 \times 10^{-8}$. The concentration of hydrogen ions in blood is smaller by a factor of 10^5.

11. (a) $\log E = 4.4 + 1.5(4) = 10.4 \Rightarrow E = 10^{10.4}$

(b) $\log E = 4.4 + 1.5(5) = 11.9 \Rightarrow E = 10^{11.9}$

(c) $\log E = 4.4 + 1.5(6) = 13.4 \Rightarrow E = 10^{13.4}$

For each increase in M of one, E is increased by a factor of $10^{1.5} \approx 31.6$.

12. (a) $80 = 10\log\left(\dfrac{I}{10^{-12}}\right)$

$$\Rightarrow 8 = \log\left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow 10^8 = \left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow I = 10^8 10^{-12} = 10^{-4}$$

(b) $I = 10^{-3}$

(c) $I = 10^{-2}$

For each increase in dB of ten, I increases by a factor of 10.

13. From the graph, we see that it's pretty close, and these numbers would be considered equal according to the level of accuracy reported in the original measurements.

14. If $y = -c(x - 96)(x + 96)$, the x intercepts are at $x = \pm 96$ and are 192 meters apart as desired. The y intercept will be 192 provided that $c = \dfrac{1}{48}$. The parabola is narrower than the hyperbolic cosine.

15. $f = f(x) = 220e^{x \ln(2)}$

$$= 220e^{\ln(2^x)} = 220 \cdot 2^x$$

16. From problem 15, the frequency as a function of the number of octaves above the A below middle C is $f(x) = 220 \cdot 2^x$. We have then

$$f(\tfrac{1}{4}) = 220 \cdot 2^{1/4} \approx 261.6 \text{ Hz}.$$

0.6 Transformations of Functions

1. $(f \circ g)(x) = f(g(x)) = g(x) + 1 = \sqrt{x-3} + 1$ with domain $\{x|x \geq 3\}$.

$(g \circ f)(x) = g(f(x))$

$$= \sqrt{f(x) - 1}$$
$$= \sqrt{(x+1) - 3} = \sqrt{x-2}$$

with domain $\{x|x \geq 2\}$.

2. $f(g(x)) = \sqrt{x+1} - 2$

with doma $\{x|x \geq -1\}$.

$g(f(x)) = \sqrt{(x-2)+1} = \sqrt{x-1}$

with domain $\{x|x \geq 1\}$.

3. $(f \circ g)(x) = f(\ln x) = e^{\ln x} = x$

with domain $\{x|x > 0\}$.

$(g \circ f)(x) = g(e^x) = \ln e^x = x$

with domain $(-\infty, \infty)$ or all real numbers.

4. $f(g(x)) = \sqrt{1 - \ln x}$. For the domain, we need $(1 - \ln x) \geq 0$, so $0 \leq x \leq e$, but also the domain of $\ln x$ is $x > 0$ so the domain of $f(g(x))$ is $\{x \mid 0 < x \leq e\}$.

$g(f(x)) = \ln\sqrt{1-x}$ on $\{x|x < 1\}$.

5. $(f \circ g)(x) = f(\sin x) = \sin^2 x + 1$ with domain $(-\infty, \infty)$ or all real numbers.

$(g \circ f)(x) = g(x^2 + 1) = \sin(x^2 + 1)$ with domain $(-\infty, \infty)$ or all real numbers.

6.
$$f(g(x)) = \frac{1}{(x^2-2)^2 - 1} = \frac{1}{x^4 - 4x^2 + 3} = \frac{1}{(x^2-3)(x^2-1)}$$

This is valid if $x \neq \pm\sqrt{3}$ and $x \neq \pm 1$.

$g(f(x)) = \left(\frac{1}{x^2-1}\right)^2 - 2$. This is valid if $x \neq \pm 1$.

7. $\sqrt{x^4+1} = f(g(x))$ when $f(x) = \sqrt{x}$ and $g(x) = x^4 + 1$, for example.

8. $\sqrt[3]{x+3} = f(g(x))$ when $f(x) = \sqrt[3]{x}$ and $g(x) = x + 3$, for example.

9. $\frac{1}{x^2+1} = f(g(x))$ when $f(x) = 1/x$ and $g(x) = x^2 + 1$, for example.

10. $\frac{1}{x^2} + 1 = f(g(x))$ when $f(x) = x + 1$ and $g(x) = 1/x^2$, for example.

11. $(4x + 1)^2 + 3 = f(g(x))$ when $f(x) = x^2 + 3$ and $g(x) = 4x + 1$, for example.

12. $4(x + 1)^2 + 3 = f(g(x))$ when $f(x) = 4x^2 + 3$ and $g(x) = x + 1$, for example.

13. $\sin^3 x = f(g(x))$ when $f(x) = x^3$ and $g(x) = \sin x$, for example.

14. $\sin(x^3) = f(g(x))$ when $f(x) = \sin x$ and $g(x) = x^3$, for example.

15. $e^{x^2+1} = f(g(x))$ when $f(x) = e^x$ and $g(x) = x^2 + 1$, for example.

16. $e^{4x-2} = f(g(x))$ when $f(x) = e^x$ and $g(x) = 4x - 2$, for example.

17. $\frac{3}{\sqrt{\sin x + 2}} = f(g(h(x)))$ when $f(x) = 3/x$, $g(x) = \sqrt{x}$, and $h(x) = \sin x + 2$, for example.

18. $\sqrt{e^{4x}+1} = f(g(h(x)))$ when $f(x) = \sqrt{x}$, $g(x) = x + 1$, and $h(x) = e^{4x}$, for example.

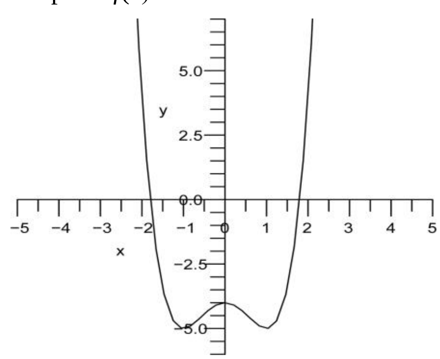
19. $\cos^3(4x - 2) = f(g(h(x)))$ when $f(x) = x^3$, $g(x) = \cos x$, and $h(x) = 4x - 2$, for example.

20. $\ln\sqrt{x^2+1} = f(g(h(x)))$ when $f(x) = \ln x$, $g(x) = \sqrt{x}$, and $h(x) = x^2 + 1$, for example.

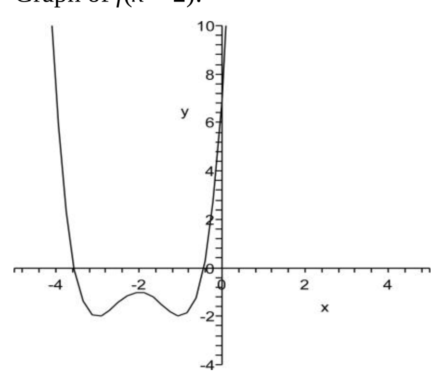
21. $4e^{x^2} - 5 = f(g(h(x)))$ when $f(x) = 4x - 5$, $g(x) = e^x$, and $h(x) = x^2$, for example.

22. $[\tan^{-1}(3x + 1)]^2 = f(g(h(x)))$ when $f(x) = x^2$, $g(x) = \tan^{-1} x$, and $h(x) = 3x + 1$, for example.

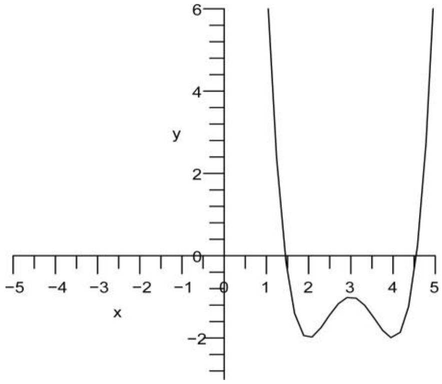
23. Graph of $f(x) - 3$:



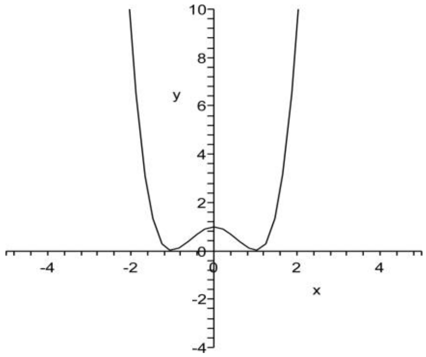
24. Graph of $f(x + 2)$:



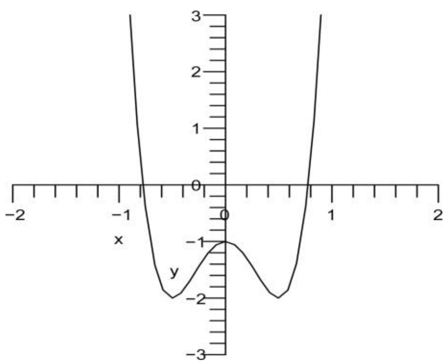
25. Graph of $f(x-3)$:



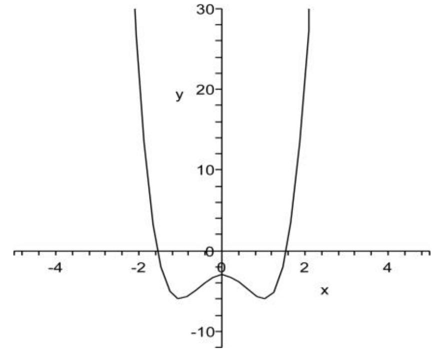
26. Graph of $f(x)+2$:



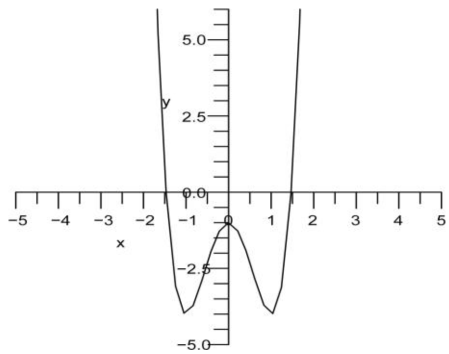
27. Graph of $f(2x)$:



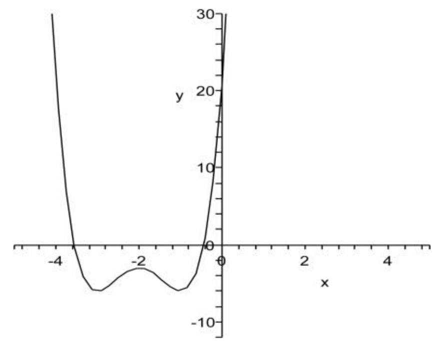
28. Graph of $3f(x)$:



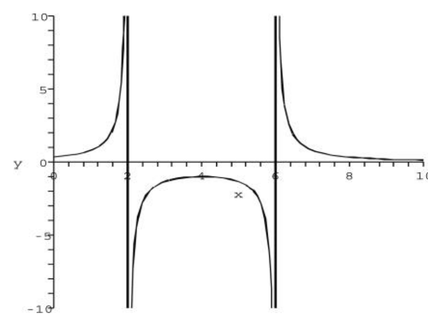
29. Graph of $-3f(x)+2$:



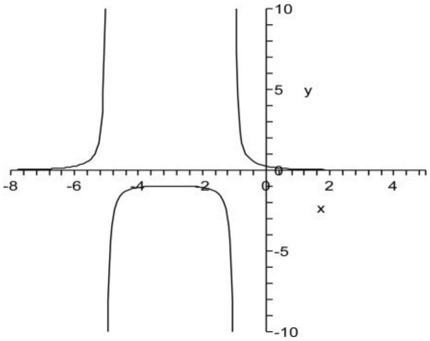
30. Graph of $3f(x+2)$:



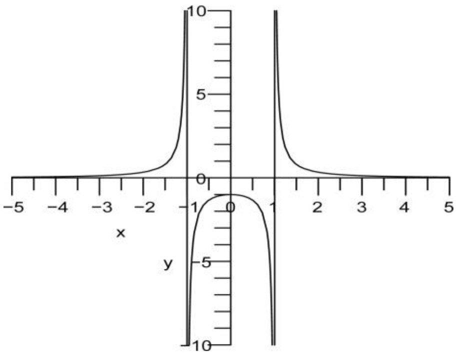
31. Graph of $f(x-4)$:



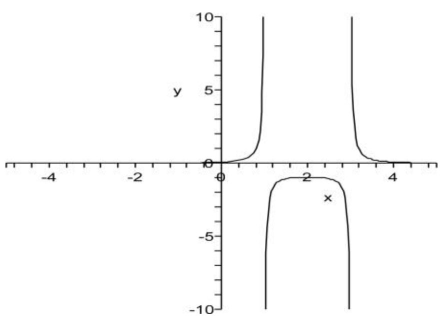
32. Graph of $f(x+3)$:



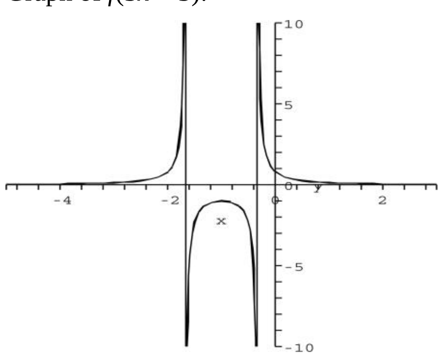
33. Graph of $f(2x)$:



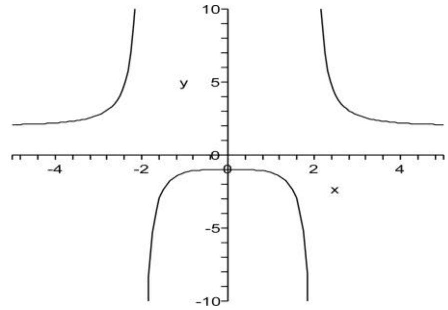
34. Graph of $f(2x-4)$:



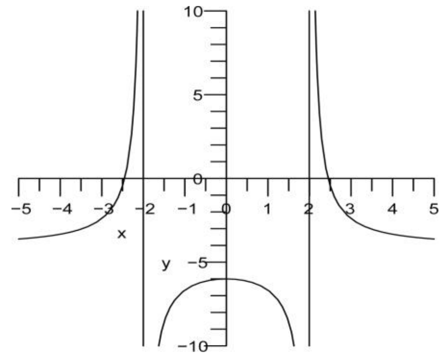
35. Graph of $f(3x+3)$:



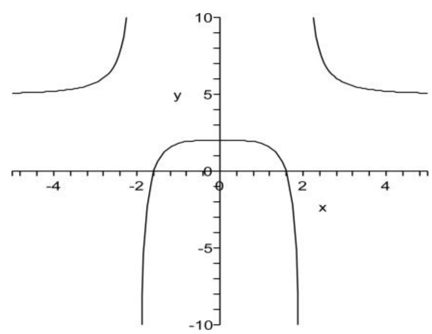
36. Graph of $3f(x)$:



37. Graph of $2f(x) - 4$:



38. Graph of $3f(x) + 3$:



39. $f(x) = x^2 + 2x + 1 = (x+1)^2$.
Shift $y = x^2$ to the left 1 unit.

40. $f(x) = x^2 - 4x + 4 = (x-2)^2$.
This is the graph of x^2 shifted 2 to the right.

41. $f(x) = x^2 + 2x + 4 = (x^2 + 2x + 1) + 4 - 1$
$= (x+1)^2 + 3$
Shift $y = x^2$ to the left 1 unit and up 3 units.

42. $f(x) = x^2 - 4x + 2 = x^2 - 4x + 4 - 2$
$= (x-2)^2 - 2$
This is the graph of x^2 shifted 2 to the right and 2 down.

43. $f(x) = 2x^2 + 4x + 4$

$= 2(x^2 + 2x + 1) + 4 - 2$

$= 2(x + 1)^2 + 2$

Shift $y = x^2$ to the left 1 unit, then multiply the scale on the y-axis by 2, then shift up 2 units.

44. $f(x) = 3x^2 - 6x + 2 = 3(x - 1)^2 - 1$.

This is the graph of x^2 with the y-scale multiplied by 3, shifted 1 to the right and 1 down.

45. Graph is reflected across the x-axis and the scale on the y-axis is multiplied by 2.

46. Graph is reflected across the x-axis, vertical scale tripled.

47. Graph is reflected across the x-axis, the scale on the y-axis is multiplied by 3, and the graph is shifted up 2 units.

48. Graph is reflected across the x-axis, vertical scale doubled, and shifted down 1 unit.

49. Graph is reflected across the y-axis.

50. Graph is reflected across the y-axis and then reflected across the x-axis, i.e. graph is rotated by an angle 2π about the origin.

51. $(-x + 1)^2 + 2(-x + 1) = (x - 1)^2 - 2(x - 1)$. Therefore graph is shifted 1 unit to the right.

52. Graph is reflected across the y-axis, horizontal scale tripled, and shifted down 3 units.

53. The graph is reflected across the x-axis and the scale on the y-axis is multiplied by $|c|$.

54. For $c < 0$, the graph of $f(cx)$ is the mirror image across the y-axis of $f(x)$ with the horizontal scale multiplied by $1/|c|$.

55. The graph of $y = |x|^3$ is identical to that of $y = x^3$ to the right of the y-axis because for $x > 0$ we have $|x|^3 = x^3$. For $y = |x|^3$ the graph to the left of the y-axis is the reflection through the y-axis of the graph to the right of the y-axis. In general to graph $y = f(|x|)$ based on the graph of $y = f(x)$, the procedure is to discard the part of the graph to the left of the y-axis, and replace it by a reflection in the y-axis of the part to the right of the y-axis.

56. If $f(x) = x^3$, then

$f(-x) = (-x)^3 = -x^3 = -f(x)$.

If in general you have the right half of a graph satisfying $f(-x) = -f(x)$, you can rotate 180° about the origin to see the left half.

57. The rest of the first 10 iterates of $f(x) = \cos x$ with $x_0 = 1$ are:

$x_4 = \cos .65 \approx .796$

$x_5 = \cos .796 \approx .70$

$x_6 = \cos .70 \approx .765$

$x_7 = \cos .765 \approx .721$

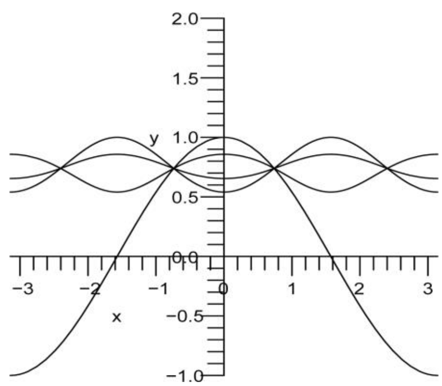
$x_8 = \cos .721 \approx .751$

$x_9 = \cos .751 \approx .731$

$x_{10} = \cos .731 \approx .744$

Continuing in this fashion and retaining more decimal places, one finds that x_{36} through x_{40} are all 0.739085. The same process is used with a different x_0.

58. We have $x_1 = f(x_0)$ so $x_2 = f(x_1) = f(f(x_0))$ and $x_3 = f(x_2) = f(f(f(x_0)))$ and so on. The graphs of cos x, cos cos x, cos cos cos x, and cos cos cos cos x:



The limiting line is $y = 0.739085$.

59. They converge to 0. One of the problems in Chapter 2 asks the student to prove that $|\sin(x)| < |x|$ for all but $x = 0$. This would show that 0 is the only solution to the equation $\sin(x) = x$ and offers a partial explanation (see the comments for #61) of the phenomena which the student observes..

60. If you start with a number x with $|x| < 1$, the iterations converge to 0. If you start with a number x with $|x| > 1$, the iterations diverge quickly. If you start with $x = \pm 1$, the iterations all equal 1.

61. If the iterates of a function f (starting from some point x_0) are going to go toward (and remain arbitrarily close to) a certain number L, this number

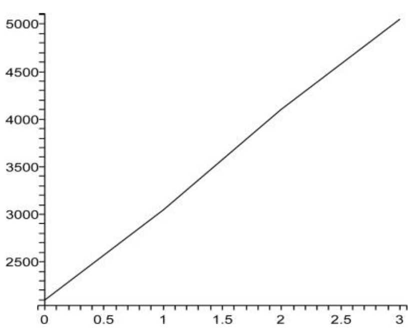
L must be a solution of the equation $f(x) = x$. For the list of iterates $x_0, x_1, x_2, x_3, \ldots$ is, apart from the first term, the same list as the list of numbers $f(x_0), f(x_1), f(x_2), f(x_3), \ldots$. (Remember that x_{n+1} is $f(x_n)$.) If any of the numbers in the first list are close to L, then the f-values (in the second list) are close to $f(L)$. But since the lists are *identical* (apart from the first term x_0 which is not in the second list), it must be true that L and $f(L)$ are the same number.

If conditions are right (and they are in the two cases $f(x) = \cos(x)$ (#57) and $f(x) = \sin(x)$ (#59), this "convergence" will indeed occur, and since there is in these cases only *one* solution x about 0.739085 in (#57) and $x = 0$ in (#59) it won't matter where you started.

62. The only fixed point is $x = 0$, since this is the only solution to $\sin x = x$. One can see that this is the only solution by graphing $y = \sin x$ and $y = x$ on the same axes and looking for intersection points.

Ch. 0 Review Exercises

1. $m = \dfrac{7-3}{0-2} = \dfrac{4}{-2} = -2$



7. The line apparently goes through (1, 1) and (3, 2). If so the slope would be $m = \frac{2-1}{3-1} = \frac{1}{2}$. The equation would be

$y = \frac{1}{2}(x-1)+1$ or $y = \frac{1}{2}x + \frac{1}{2}$.

Using the equation with $x = 4$, we find $y = \frac{1}{2}(4) + \frac{1}{2} = \frac{5}{2}$.

8. $f(0) = -4$, $f(2) = -6$, and $f(4) = 0$.

9. Using the point-slope method, we find $y = -\frac{1}{3}(x+1) - 1$

10. $y = \frac{1}{4}(x-0) - 2 = \frac{1}{4}x - 2$

2. $m = \dfrac{4-1}{1-3} = -\dfrac{3}{2}$

3. These lines both have slope 3. They are parallel unless they are coincident. But the first line includes the point (0, 1) which does not satisfy the equation of the second line. The lines are not coincident.

4. $m_1 = -1/m_2$, so the lines are perpendicular.

5. Let $P = (1, 2)$, $Q = (2, 4)$, $R = (0, 6)$.

Then PQ has slope $\dfrac{4-2}{2-1} = 2$

QR has slope $\dfrac{6-4}{0-2} = -1$

RP has slope $\dfrac{2-6}{1-0} = -4$

Since no two of these slopes are negative reciprocals, none of the angles are right angles. The triangle is not a right triangle.

6. The slopes between points seem to be alternating between 950 and 1050. If the pattern continues, the next points will be (4, 6100), (5, 7050), and (6, 8100).

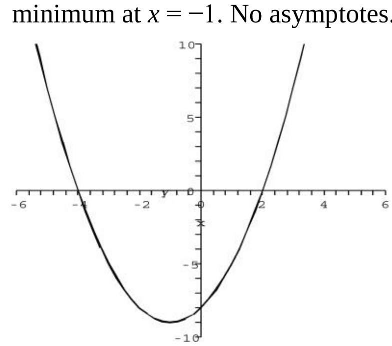
11. The graph passes the vertical line test, so it is a function.

12. Fails vertical line test: not a function.

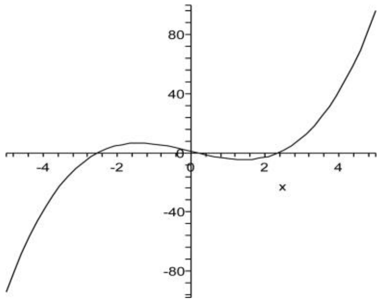
13. The radicand cannot be negative, hence we require $4 - x^2 \geq 0 \Rightarrow 4 \geq x^2$. Therefore the natural domain is $\{x \mid -2 \leq x \leq 2\}$ or, in "interval-language": $[-2, 2]$.

14. The function is not defined where the denominator is zero, so the domain for $f(x)$ is $\{x \mid x \neq \pm\sqrt{2}\}$.

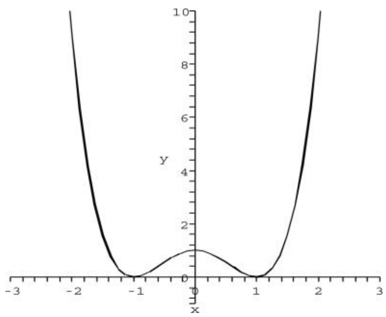
15. Intercepts at $x = -4$ and 2, and $y = -8$. Local minimum at $x = -1$. No asymptotes.



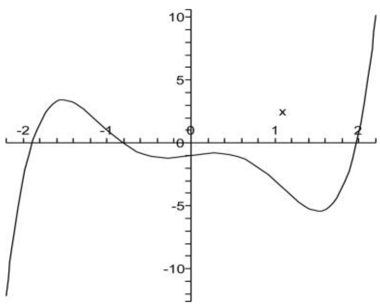
16. Intercepts at $x \approx 2.36$, 0.17 and -2.53, and $y = 1$. Local maximum at $x = -\sqrt{2}$. Local minimum at $x = \sqrt{2}$. No asymptotes.



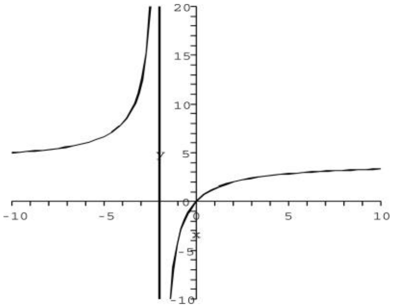
17. Intercepts at $x = -1$ and 1, and $y = 1$. Local minimum at $x = 1$ and at $x = -1$. Local maximum at $x = 0$. No asymptotes.



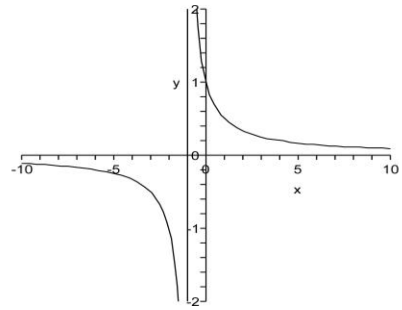
18. Intercepts at $x \approx 1.97$, -0.82, and -1.89, and $y = -1$. Local maximums at $x \approx -1.52$ and 0.29. Local minimums at $x \approx -0.29$ and 1.52. No asymptotes.



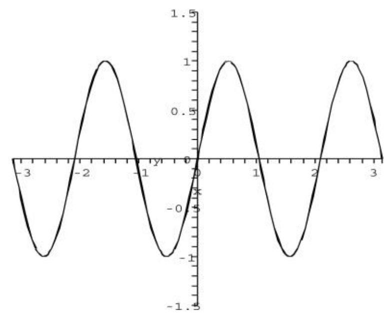
19. Intercept at $y = 0$ and at $x = 0$. No extrema. Horizontal asymptote $y = 4$. Vertical asymptote $x = -2$.



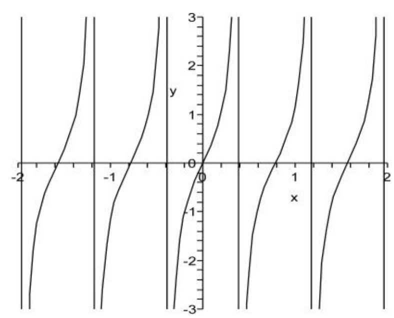
20. Intercept at $y = 1$. No x-intercept since the function is undefined at $x = 2$. No extrema. Horizontal asymptote $y = 0$. Vertical asymptote $x = -1$.



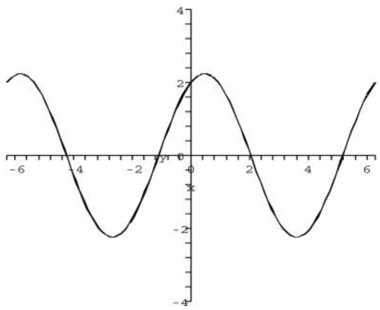
21. Intercept at $y = 0$ and $x = \frac{k\pi}{3}$ for integers k. Extrema: y takes maximum 1 and minimum -1 with great predictability and regularity. No asymptotes.



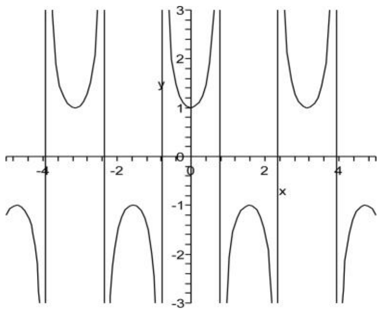
22. Intercept at $y = 0$ and $x = \frac{k\pi}{4}$ for integers k. No extrema. Vertical asymptotes at $x = \frac{(2k+1)\pi}{8}$ for integers k



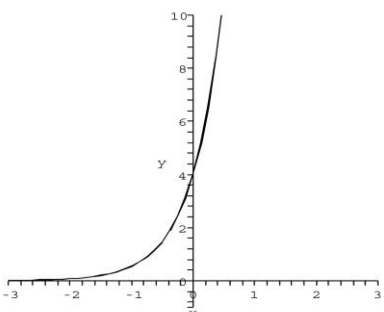
23. Intercept at $y = 2$ and from the amplitude/phase shift form $f(x) = \sqrt{5}\sin\left(x + \sin^{-1}(2/\sqrt{5})\right)$, we could write down all the intercepts only at considerable inconvenience. Extrema: y takes maximum $\sqrt{5}$ and minimum $-\sqrt{5}$ with great predictability and regularity. No asymptotes.



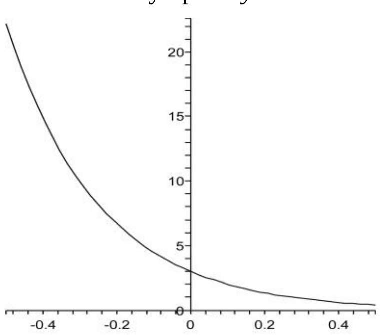
24. Intercept $y = 1$. Local maximums at $x = \frac{(2k+1)\pi}{2}$ for integers k. Local minimums at $x = k\pi$ for integers k. Vertical asymptotes at $x = \frac{(2k+1)\pi}{4}$ for integers k.



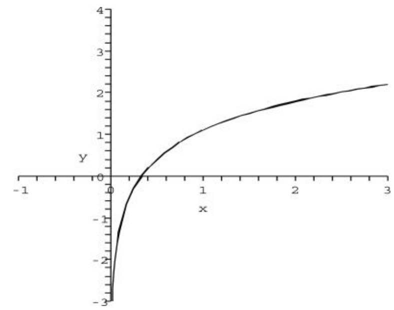
25. Intercept $y = 4$ (no x-intercepts). No extrema. Left horizontal asymptote $y = 0$.



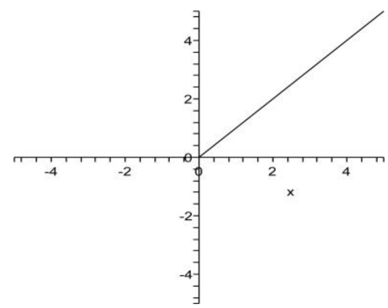
26. Intercept $y = 3$ (no x-intercepts). No extrema. Horizontal asymptote $y = 0$.



27. Intercept $x = 1/3$ (no y-intercepts). No extrema. Vertical asymptote $x = 0$.



28. No intercepts, extrema, or asymptotes. Function only defined for $x > 0$.



29. Intercepts at $x = -4$ and 2, and $y = -8$.

30. Intercepts $y = 1$, and $x = \pm 1$.

31. Vertical asymptote $x = -2$.

32. Vertical asymptote at $x = -1$. This is where the denominator is zero (and the numerator is not zero). Note that the function is not defined at $x = 2$.

33. $x^2 - 3x - 10 = (x-5)(x+2)$. The zeros are when $x = 5$ and $x = -2$.

34. $x^3 + 4x^2 + 3x = x(x+3)(x+1)$. Zeros are $x = 0, -1$ and -3.

35. Guess a root: $x = 1$. Factor the left side: $(x-1)(x^2 - 2x - 2)$. Solve the quadratic by formula:

$$x = \frac{2 \pm \sqrt{2^2 - 4(1)(-2)}}{2} = 1 \pm \sqrt{3}.$$

Complete list of three roots: $x = 1$, $1 - \sqrt{3} \approx -.732$, $x = 1 + \sqrt{3} \approx 2.732$.

36. Zeros are at $x \approx 1.618$, and -0.618. Exact values are $x = (1 \pm \sqrt{5})/2$.

37. There are 3 solutions, one at $x = 0$ and the other two negatives of one another. The value in question is.928632…, found using the function "Goal Seek" in Excel. The result can be checked, and a graphing calculator can find them by graphing $y = x^3$ and $y = \sin x$ on the same axes and finding the intersection points.

38. The graph shows two zeros. Squaring both sides gives $x^2 + 1 = x^4 - 2x^2 + 1$, or $0 = x^4 - 3x^2$. The solutions are $x = \pm\sqrt{3}$. ($x = 0$ is an extraneous solution.)

39. Let h be the height of the telephone pole. Then $\frac{h}{50} = \tan 34^{\text{o}} \Rightarrow h = 50\tan 34^{\text{o}} \approx 33.7$ meters .

40. The triangle in the first quadrant with adjacent side 1 and hypotenuse 5 has opposite side $\sqrt{24}$, so $\sin\theta = \frac{\sqrt{24}}{5}$.

41. (a) $5^{-1/2} = \dfrac{1}{5^{1/2}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

(b) $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

42. (a) $\dfrac{2}{\sqrt{x}} = \dfrac{2}{x^{1/2}} = 2x^{-1/2}$

(b) $\dfrac{3}{x^2} = 3x^{-2}$

43. $\ln 8 - 2\ln 2 = \ln 8 - \ln 2^2$

$= \ln 8 - \ln 4 = \ln\left(\frac{8}{4}\right) = \ln 2$

44. $e^{\ln 4x} = 8 \Rightarrow 4x = 8$ and $x = 2$.

45. $3e^{2x} = 8 \Rightarrow e^{2x} = \frac{8}{3}$

$\Rightarrow \ln e^{2x} = \ln\left(\frac{8}{3}\right)$

$\Rightarrow 2x = \ln\left(\frac{8}{3}\right)$

$\Rightarrow x = \frac{1}{2}\ln\frac{8}{3}$

46. $2\ln 3x = 5 \Rightarrow \ln 3x = \frac{5}{2}$

$\Rightarrow e^{5/2} = 3x$, so $x = \frac{1}{3}e^{5/2}$.

47. The natural domain for f is the full real line. The natural domain for g is $\{x|1 \le x\}$. Because f has a universal domain, the natural domain for $f \circ g$ is the same as the domain for g, namely $\{x|1 \le x\}$. Because g requires its inputs be not less than 1, the domain for $g \circ f$ is the set of x for which $1 \le f(x)$, i.e., $\{x|1 \le x^2\} = \{x|1 \le |x|\}$, or in interval language $(-\infty, -1] \cup [1, \infty)$.

The formulae are easier:

$$(f \circ g)(x) = f(\sqrt{x-1}) = (\sqrt{x-1})^2 = x - 1$$

$$(g \circ f)(x) = g(x^2) = \sqrt{x^2 - 1}$$

Caution: the *formula* for $f \circ g$ is defined for any x, but the *domain* for $f \circ g$ is restricted as stated earlier. The formula must be viewed as irrelevant outside the domain.

48. $(f \circ g)(x) = \left(\frac{1}{x^2-1}\right)^2$ and

$(g \circ f)(x) = \frac{1}{x^4-1}$

are both valid for $x \ne \pm 1$.

49. $e^{3x^2+2} = f(g(x))$ for $f(x) = e^x$ and $g(x) = 3x^2 + 2$.

50. $\sqrt{\sin x + 2} = f(g(x))$ for $f(x) = \sqrt{x}$ and $g(x) = \sin x + 2$.

51. $x^2 - 4x + 1 = x^2 - 4x + 4 - 4 + 1$, so $f(x) = (x-2)^2 - 3$. The graph of $f(x)$ is the graph of x^2 shifted two units to the right and three units down.

52. $x^2 + 4x + 6 = (x^2 + 4x + 4) + 2$, so $f(x) = (x+2)^2 + 2$. The graph of $f(x)$ is the graph of x^2 shifted two units to the left and two units up.

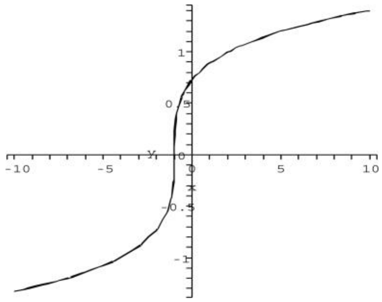
53. Like x^3, the function $f(x) = x^3 - 1$ passes the horizontal line test and is one-to-one. To find a formula for the inverse, solve for x to find $(y+1)^{1/3} = x$ then switch x and y to get $f^{-1}(x) = (x+1)^{1/3}$ for all x.

54. e^{-4x} is one-to-one, and its inverse is $-\frac{1}{4}\ln x$.

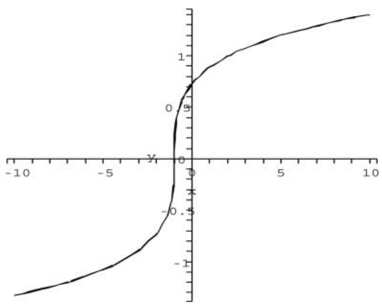
55. The function is *even* ($f(-x) = f(x)$). Every horizontal line (except $y = 0$) which meets the curve at all automatically meets it at least twice. The function is not one-to-one. There is no inverse.

56. $x^3 - 2x + 1$ is not one-to-one as it fails the horizontal line test.

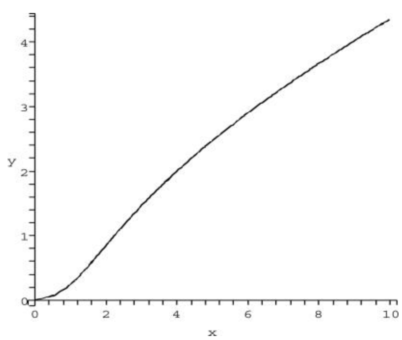
57. The inverse of $x^5 + 2x^3 - 1$:



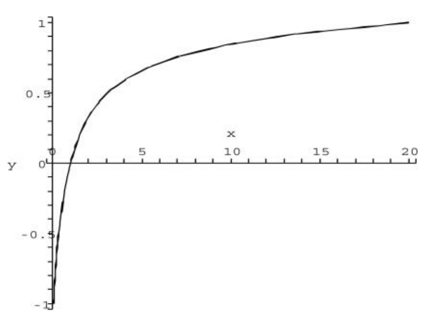
58. The inverse of $x^3 + 5x + 2$:



59. The inverse of $\sqrt{x^3 + 4x}$:



60. The inverse of e^{x^3+2x}:



61. On the unit circle, $y = \sin\theta = 1$ when $\theta = \frac{\pi}{2}$. Hence, $\sin^{-1} 1 = \frac{\pi}{2}$.

62. On the unit circle, $x = \cos\theta = -1/2$ when $y = \sin\theta = \pm\sqrt{3}/2$ in the second or third quadrant. This coincides with a 30°-60°-90° or $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ triangle, so $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$ or $\cos^{-1}(-\frac{1}{2}) = 4\pi/3$.

63. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we want $y = \cos\theta$ to be equal to $-x = -\sin\theta$ on the unit circle. This happens when $\theta = -\pi/4$ and $\theta = 3\pi/4$. Hence, $\tan^{-1}(-1) = -\frac{\pi}{4}$ or $\tan^{-1}(-1) = \frac{3\pi}{4}$.

64. We have that $\csc^{-1}(-2) = \sin^{-1}(-\frac{1}{2})$. On the unit circle, $y = \sin\theta = -1/2$ when $x = \cos\theta = \pm\sqrt{3}/2$ in the third or fourth quadrant. This coincides with a 30°-60°-90° or $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ triangle, so $\csc^{-1}(-2) = \sin^{-1}(-\frac{1}{2}) = -\pi/6$ or $\csc^{-1}(-2) = 7\pi/6$.

65. If an angle θ has $\sec(\theta) = 2$, then it has $\cos(\theta) = 1/2$. Its sine could be $\pm\frac{\sqrt{3}}{2}$. But if $\theta = \sec^{-1}(2)$, then in addition to all that has been stated, it is in the first quadrant, and the choice of sign (for its sine) is positive. In summary, $\sin(\sec^{-1} 2) = \sin\theta = \frac{\sqrt{3}}{2}$.

66. $\cos^{-1}(4/5)$ relates to a triangle in quadrant 1 with adjacent side 4 and hypotenuse 5, so the opposite side must be 3, and the tangent of this angle is $\frac{3}{4}$.

67. $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

68. $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ relates to a triangle in the second quadrant with angle $\frac{3\pi}{4}$.

69. $\sin 2x = 1 \Rightarrow$

$2x = \frac{\pi}{2} + 2k\pi$ for any integer k so

$x = \frac{\pi}{4} + k\pi$ for any integer k.

70. $\cos 3x = \frac{1}{2}$ whenever

$3x = \pm\frac{\pi}{3} + 2k\pi$ for any integer k, or

$x = \pm\frac{\pi}{9} + \frac{2k\pi}{3}$ for any integer k.