CHAPTER 1

Section 1.1

1.1.1: If $f(x) = \dfrac{1}{x}$, then:

(a) $f(-a) = \dfrac{1}{-a} = -\dfrac{1}{a}$;

(b) $f(a^{-1}) = \dfrac{1}{a^{-1}} = a$;

(c) $f\left(\sqrt{a}\,\right) = \dfrac{1}{\sqrt{a}} = \dfrac{1}{a^{1/2}} = a^{-1/2}$;

(d) $f(a^2) = \dfrac{1}{a^2} = a^{-2}$.

1.1.2: If $f(x) = x^2 + 5$, then:

(a) $f(-a) = (-a)^2 + 5 = a^2 + 5$;

(b) $f(a^{-1}) = (a^{-1})^2 + 5 = a^{-2} + 5 = \dfrac{1}{a^2} + 5 = \dfrac{1 + 5a^2}{a^2}$;

(c) $f\left(\sqrt{a}\,\right) = \left(\sqrt{a}\,\right)^2 + 5 = a + 5$;

(d) $f(a^2) = (a^2)^2 + 5 = a^4 + 5$.

1.1.3: If $f(x) = \dfrac{1}{x^2 + 5}$, then:

(a) $f(-a) = \dfrac{1}{(-a)^2 + 5} = \dfrac{1}{a^2 + 5}$;

(b) $f(a^{-1}) = \dfrac{1}{(a^{-1})^2 + 5} = \dfrac{1}{a^{-2} + 5} = \dfrac{1 \cdot a^2}{a^{-2} \cdot a^2 + 5 \cdot a^2} = \dfrac{a^2}{1 + 5a^2}$;

(c) $f\left(\sqrt{a}\,\right) = \dfrac{1}{\left(\sqrt{a}\,\right)^2 + 5} = \dfrac{1}{a + 5}$;

(d) $f(a^2) = \dfrac{1}{(a^2)^2 + 5} = \dfrac{1}{a^4 + 5}$.

1.1.4: If $f(x) = \sqrt{1 + x^2 + x^4}$, then:

(a) $f(-a) = \sqrt{1 + (-a)^2 + (-a)^4} = \sqrt{1 + a^2 + a^4}$;

(b) $f(a^{-1}) = \sqrt{1 + (a^{-1})^2 + (a^{-1})^4} = \sqrt{1 + a^{-2} + a^{-4}} = \sqrt{\dfrac{(a^4) \cdot (1 + a^{-2} + a^{-4})}{a^4}}$

$= \sqrt{\dfrac{a^4 + a^2 + 1}{a^4}} = \dfrac{\sqrt{a^4 + a^2 + 1}}{\sqrt{a^4}} = \dfrac{\sqrt{a^4 + a^2 + 1}}{a^2}$;

(c) $f\left(\sqrt{a}\,\right) = \sqrt{1 + \left(\sqrt{a}\,\right)^2 + \left(\sqrt{a}\,\right)^4} = \sqrt{1 + a + a^2}$;

(d) $f(a^2) = \sqrt{1 + (a^2)^2 + (a^4)^2} = \sqrt{1 + a^4 + a^8}$.

1.1.5: If $g(x) = 3x + 4$ and $g(a) = 5$, then $3a + 4 = 5$, so $3a = 1$; therefore $a = \frac{1}{3}$.

1.1.6: If $g(x) = \dfrac{1}{2x-1}$ and $g(a) = 5$, then:

$$\begin{aligned}
\frac{1}{2a-1} &= 5;\\
1 &= 5 \cdot (2a-1);\\
1 &= 10a - 5;\\
10a &= 6;\\
a &= \frac{3}{5}.
\end{aligned}$$

1.1.7: If $g(x) = \sqrt{x^2+16}$ and $g(a) = 5$, then:

$$\begin{aligned}
\sqrt{a^2+16} &= 5;\\
a^2 + 16 &= 25;\\
a^2 &= 9;\\
a &= 3 \text{ or } a = -3.
\end{aligned}$$

1.1.8: If $g(x) = x^3 - 3$ and $g(a) = 5$, then $a^3 - 3 = 5$, so $a^3 = 8$. Hence $a = 2$.

1.1.9: If $g(x) = \sqrt[3]{x+25} = (x+25)^{1/3}$ and $g(a) = 5$, then

$$\begin{aligned}
(a+25)^{1/3} &= 5;\\
a + 25 &= 5^3 = 125;\\
a &= 100.
\end{aligned}$$

1.1.10: If $g(x) = 2x^2 - x + 4$ and $g(a) = 5$, then:

$$\begin{aligned}
2a^2 - a + 4 &= 5;\\
2a^2 - a - 1 &= 0;\\
(2a+1)(a-1) &= 0;\\
2a + 1 = 0 &\text{ or } a - 1 = 0;\\
a = -\frac{1}{2} &\text{ or } a = 1.
\end{aligned}$$

1.1.11: If $f(x) = 3x - 2$, then

$$f(a+h) - f(a) = [3(a+h) - 2] - [3a - 2]$$

$$= 3a + 3h - 2 - 3a + 2 = 3h.$$

1.1.12: If $f(x) = 1 - 2x$, then

$$f(a+h) - f(a) = [1 - 2(a+h)] - [1 - 2a] = 1 - 2a - 2h - 1 + 2a = -2h.$$

1.1.13: If $f(x) = x^2$, then

$$f(a+h) - f(a) = (a+h)^2 - a^2$$

$$= a^2 + 2ah + h^2 - a^2 = 2ah + h^2 = h \cdot (2a + h).$$

1.1.14: If $f(x) = x^2 + 2x$, then

$$f(a+h) - f(a) = [(a+h)^2 + 2(a+h)] - [a^2 + 2a]$$

$$= a^2 + 2ah + h^2 + 2a + 2h - a^2 - 2a = 2ah + h^2 + 2h = h \cdot (2a + h + 2).$$

1.1.15: If $f(x) = \dfrac{1}{x}$, then

$$f(a+h) - f(a) = \frac{1}{a+h} - \frac{1}{a} = \frac{a}{a(a+h)} - \frac{a+h}{a(a+h)}$$

$$= \frac{a - (a+h)}{a(a+h)} = \frac{-h}{a(a+h)}.$$

1.1.16: If $f(x) = \dfrac{2}{x+1}$, then

$$f(a+h) - f(a) = \frac{2}{a+h+1} - \frac{2}{a+1} = \frac{2(a+1)}{(a+h+1)(a+1)} - \frac{2(a+h+1)}{(a+h+1)(a+1)}$$

$$= \frac{2a+2}{(a+h+1)(a+1)} - \frac{2a+2h+2}{(a+h+1)(a+1)} = \frac{(2a+2) - (2a+2h+2)}{(a+h+1)(a+1)}$$

$$= \frac{2a+2-2a-2h-2}{(a+h+1)(a+1)} = \frac{-2h}{(a+h+1)(a+1)}.$$

1.1.17: If $x > 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = 1.$$

If $x < 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1.$$

We are given $f(0) = 0$, so the range of f is $\{-1, 0, 1\}$. That is, the range of f is the set consisting of the three real numbers -1, 0, and 1.

1.1.18: Given $f(x) = [\![3x]\!]$, we see that

$$\begin{aligned} f(x) = 0 \quad & \text{if} \quad 0 \leqq x < \tfrac{1}{3}, \\ f(x) = 1 \quad & \text{if} \quad \tfrac{1}{3} \leqq x < \tfrac{2}{3}, \\ f(2) = 2 \quad & \text{if} \quad \tfrac{2}{3} \leqq x < 1; \end{aligned}$$

moreover,

$$\begin{aligned} f(x) = -3 \quad & \text{if} \quad -1 \leqq x < -\tfrac{2}{3}, \\ f(x) = -2 \quad & \text{if} \quad -\tfrac{2}{3} \leqq x < -\tfrac{1}{3}, \\ f(x) = -1 \quad & \text{if} \quad -\tfrac{1}{3} \leqq x < 0. \end{aligned}$$

In general, if m is any integer, then

$$\begin{aligned} f(x) = 3m \quad & \text{if} \quad m \leqq x < m + \tfrac{1}{3}, \\ f(x) = 3m + 1 \quad & \text{if} \quad m + \tfrac{1}{3} \leqq x < m + \tfrac{2}{3}, \\ f(x) = 3m + 2 \quad & \text{if} \quad m + \tfrac{2}{3} \leqq x < m + 1. \end{aligned}$$

Because every integer is equal to $3m$ or to $3m + 1$ or to $3m + 2$ for some integer m, we see that the range of f includes the set $\boldsymbol{Z}$ of all integers. Because f can assume no values other than integers, we can conclude that the range of f is exactly $\boldsymbol{Z}$.

1.1.19: Given $f(x) = (-1)^{[\![x]\!]}$, we first note that the values of the exponent $[\![x]\!]$ consist of all the integers and no other numbers. So all that matters about the exponent is whether it is an even integer or an odd integer, for if even then $f(x) = 1$ and if odd then $f(x) = -1$. No other values of $f(x)$ are possible, so the range of f is the set consisting of the two numbers -1 and 1.

1.1.20: If $0 < x \leqq 1$, then $f(x) = 39$. If $1 < x \leqq 2$ then $f(x) = 39 + 24 = 63$. If $2 < x \leqq 3$ then $f(x) = 39 + 2 \cdot 24 = 87$. We continue in this way and conclude with the observation that if $11 < x < 12$ then $f(x) = 39 + 11 \cdot 24 = 300$. So the range of f is the set

$$\{39, 63, 87, 111, 135, 159, 183, 207, 231, 255, 279, 303\}.$$

1.1.21: Given $f(x) = 10 - x^2$, note that for every real number x, x^2 is defined, and for every such real number x^2, $10 - x^2$ is also defined. Therefore the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.1.22: Given $f(x) = x^3 + 5$, we note that for each real number x, x^3 is defined; moreover, for each such real number x^3, $x^3 + 5$ is also defined. Thus the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.1.23: Given $f(t) = \sqrt{t^2}$, we observe that for every real number t, t^2 is defined and nonnegative, and hence that $\sqrt{t^2}$ is defined as well. Therefore the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.1.24: Given $g(t) = \left(\sqrt{t}\,\right)^2$, we observe that $\sqrt{t}$ is defined exactly when $t \geqq 0$. In this case, $\left(\sqrt{t}\,\right)^2$ is also defined, and hence the domain of g is the set $[0, +\infty)$ of all nonnegative real numbers.

1.1.25: Given $f(x) = \sqrt{3x - 5}$, we note that $3x - 5$ is defined for all real numbers x, but that its square root will be defined when and only when $3x - 5$ is nonnegative; that is, when $3x - 5 \geqq 0$, so that $x \geqq \frac{5}{3}$. So the domain of f consists of all those real numbers x in the interval $\left[\frac{5}{3}, +\infty\right)$.

1.1.26: Given $g(t) = \sqrt[3]{t+4} = (t+4)^{1/3}$, we note that $t + 4$ is defined for every real number t and the cube root of $t + 4$ is defined for every possible resulting value of $t + 4$. Therefore the domain of g is the set $\boldsymbol{R}$ of all real numbers.

1.1.27: Given $f(t) = \sqrt{1 - 2t}$, we observe that $1 - 2t$ is defined for every real number t, but that its square root is defined only when $1 - 2t$ is nonnegative. We solve the inequality $1 - 2t \geqq 0$ to find that $f(t)$ is defined exactly when $t \leqq \frac{1}{2}$. Hence the domain of f is the interval $\left(-\infty, \frac{1}{2}\right]$.

1.1.28: Given

$$g(x) = \frac{1}{(x+2)^2},$$

we see that $(x + 2)^2$ is defined for every real number x, but that $g(x)$, its reciprocal, will be defined only when $(x + 2)^2 \neq 0$; that is, when $x + 2 \neq 0$. So the domain of g consists of those real numbers $x \neq -2$.

1.1.29: Given

$$f(x) = \frac{2}{3 - x},$$

we see that $3 - x$ is defined for all real values of x, but that $f(x)$, double its reciprocal, is defined only when $3 - x \neq 0$. So the domain of f consists of those real numbers $x \neq 3$.

1.1.30: Given

$$g(t) = \sqrt{\frac{2}{3 - t}},$$

it is necessary that $3 - t$ be both nonzero (so that its reciprocal is defined) and nonnegative (so that the square root is defined). Thus $3 - t > 0$, and therefore the domain of g consists of those real numbers $t < 3$.

1.1.31: Given $f(x) = \sqrt{x^2 + 9}$, observe that for each real number x, $x^2 + 9$ is defined and, moreover, is positive. So its square root is defined for every real number x. Hence the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.1.32: Given

$$h(z) = \frac{1}{\sqrt{4 - z^2}},$$

we note that $4 - z^2$ is defined for every real number z, but that its square root will be defined only if $4 - z^2 \geqq 0$. Moreover, the square root cannot be zero, else its reciprocal will be undefined, so we need to solve the inequality $4 - z^2 > 0$; that is, $z^2 < 4$. The solution is $-2 < z < 2$, so the domain of h is the open interval $(-2, 2)$.

1.1.33: Given $f(x) = \sqrt{4 - \sqrt{x}}$, note first that we require $x \geqq 0$ in order that $\sqrt{x}$ be defined. In addition, we require $4 - \sqrt{x} \geqq 0$ so that *its* square root will be defined as well. So we solve [simultaneously] $x \geqq 0$ and $\sqrt{x} \leqq 4$ to find that $0 \leqq x \leqq 16$. So the domain of f is the closed interval $[0, 16]$.

1.1.34: Given

$$f(x) = \sqrt{\frac{x+1}{x-1}},$$

we require that $x \neq 1$ so that the fraction is defined. In addition we require that the fraction be nonnegative so that its square root will be defined. These conditions imply that both numerator and denominator be positive or that both be negative; moreover, the numerator may also be zero. But if the denominator is positive then the [larger] numerator will be positive as well; if the numerator is nonpositive then the [smaller] denominator will be negative. So the domain of f consists of those real numbers for which *either* $x - 1 > 0$ *or* $x + 1 \leqq 0$; that is, either $x > 1$ or $x \leqq -1$. So the domain of f is the union of the two intervals $(-\infty, -1]$ and $(1, +\infty)$. Alternatively, it consists of those real numbers x *not* in the interval $(-1, 1]$.

1.1.35: Given:

$$g(t) = \frac{t}{|t|}.$$

This fraction will be defined whenever its denominator is nonzero, thus for all real numbers $t \neq 0$. So the domain of g consists of the nonzero real numbers; that is, the union of the two intervals $(-\infty, 0)$ and $(0, +\infty)$.

1.1.36: If a square has edge length x, then its area A is given by $A = x^2$ and its perimeter P is given by $P = 4x$. To express A in terms of P:

$$x = \tfrac{1}{4}P;$$

$$A = x^2 = \left(\tfrac{1}{4}P\right)^2 = \tfrac{1}{16}P^2.$$

Thus to express A as a function of P, we write

$$A(P) = \tfrac{1}{16}P^2, \quad 0 \leqq P < +\infty.$$

(It will be convenient later in the course to allow the possibility that P, x, and A are zero. If this produces an answer that fails to meet real-world criteria for a solution, then that possibility can simply be eliminated when the answer to the problem is stated.)

1.1.37: If a circle has radius r, then its circumference C is given by $C = 2\pi r$ and its area A by $A = \pi r^2$. To express C in terms of A, we first express r in terms of A, then substitute in the formula for C:

$$A = \pi r^2; \qquad r = \sqrt{\frac{A}{\pi}}\,;$$

$$C = 2\pi r = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{\pi^2 A}{\pi}} = 2\sqrt{\pi A}.$$

Therefore to express C as a function of A, we write

$$C(A) = 2\sqrt{\pi A}, \qquad 0 \leqq A < +\infty.$$

It is also permissible simply to write $C(A) = 2\sqrt{\pi A}$ without mentioning the domain, because the "default" domain is correct. In the first displayed equation we do not write $r = \pm\sqrt{A/\pi}$ because we know that r is never negative.

1.1.38: If r denotes the radius of the sphere, then its volume is given by $V = \frac{4}{3}\pi r^3$ and its surface area by $S = 4\pi r^2$. Hence

$$r = \frac{1}{2}\sqrt{\frac{S}{\pi}}\,;$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot \frac{1}{8}\left(\frac{S}{\pi}\right)^{3/2} = \frac{1}{6}\pi\left(\frac{S}{\pi}\right)^{3/2}.$$

Answer: $V(S) = \dfrac{1}{6}\pi\left(\dfrac{S}{\pi}\right)^{3/2}, \quad 0 \leqq S < +\infty.$

1.1.39: To avoid decimals, we note that a change of 5°C is the same as a change of 9°F, so when the temperature is 10°C it is $32 + 18 = 50$°F; when the temperature is 20°C then it is $32 + 2 \cdot 18 = 68$°F. In general we get the Fahrenheit temperature F by adding 32 to the product of $\frac{1}{10}C$ and 18, where C is the Celsius temperature. That is,

$$F = 32 + \frac{9}{5}C,$$

and therefore $C = \frac{5}{9}(F - 32)$. Answer:

$$C(F) = \frac{5}{9}(F - 32), \quad F > -459.67.$$

1.1.40: Suppose that a rectangle has base length x and perimeter 100. Let h denote the height of such a rectangle. Then $2x + 2h = 100$, so that $h = 50 - x$. Because $x \geqq 0$ and $h \geqq 0$, we see that $0 \leqq x \leqq 50$. The area A of the rectangle is xh, so that

$$A(x) = x(50 - x), \quad 0 \leqq x \leqq 50.$$

1.1.41: Let y denote the height of such a rectangle. The rectangle is inscribed in a circle of diameter 4, so the bottom side x and the left side y are the two legs of a right triangle with hypotenuse 4. Consequently $x^2 + y^2 = 16$, so $y = \sqrt{16 - x^2}$ (not $-\sqrt{16 - x^2}$ because $y \geqq 0$). Because $x \geqq 0$ and $y \geqq 0$, we see that $0 \leqq x \leqq 4$. The rectangle has area $A = xy$, so

$$A(x) = x\sqrt{16 - x^2}, \quad 0 \leqq x \leqq 4.$$

1.1.42: We take the problem to mean that current production is 200 barrels per day per well, that if one new well is drilled then the 21 wells will produce 195 barrels per day per well; in general, that if x new wells are drilled then the $20 + x$ wells will produce $200 - 5x$ barrels per day per well. So total production would be $p = (20 + x)(200 - 5x)$ barrels per day. But because $200 - 5x \geqq 0$, we see that $x \leqq 40$. Because $x \geqq 0$ as well (you don't "undrill" wells), here's the answer:

$$p(x) \;=\; 4000 + 100x - 5x^2, \quad 0 \leqq x \leqq 40, \quad x \text{ an integer.}$$

1.1.43: The square base of the box measures x by x centimeters; let y denote its height (in centimeters). Because the volume of the box is 324 cm^3, we see that $x^2y = 324$. The base of the box costs $2x^2$ cents, each of its four sides costs xy cents, and its top costs x^2 cents. So the total cost of the box is

$$C = 2x^2 + 4xy + x^2 = 3x^2 + 4xy. \tag{1}$$

Because $x > 0$ and $y > 0$ (the box has positive volume), but because y can be arbitrarily close to zero (as well as x), we see also that $0 < x < +\infty$. We use the equation $x^2y = 324$ to eliminate y from Eq. (1) and thereby find that

$$C(x) = 3x^2 + \frac{1296}{x}, \quad 0 < x < +\infty.$$

1.1.44: If the rectangle is rotated around its side S of length x to produce a cylinder, then x will also be the height of the cylinder. Let y denote the length of the two sides perpendicular to S; then y will be the radius of the cylinder; moreover, the perimeter of the original rectangle is $2x + 2y = 36$. Hence $y = 18 - x$. Note also that $x \geqq 0$ and that $x \leqq 18$ (because $y \geqq 0$). The volume of the cylinder is $V = \pi y^2 x$, and so

$$V(x) = \pi x(18 - x)^2, \quad 0 \leqq x \leqq 18.$$

1.1.45: Let h denote the height of the cylinder. Its radius is r, so its volume is $\pi r^2 h = 1000$. The total surface area of the cylinder is

$$A = 2\pi r^2 + 2\pi r h \qquad \text{(look inside the front cover of the book);}$$
$$h = \frac{1000}{\pi r^2}, \qquad \text{so}$$
$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

Now r cannot be negative; r cannot be zero, else $\pi r^2 h \neq 1000$. But r can be arbitrarily small positive as well as arbitrarily large positive (by making h sufficiently close to zero). Answer:

$$A(r) = 2\pi r^2 + \frac{2000}{r}, \quad 0 < r < +\infty.$$

1.1.46: Let y denote the height of the box (in centimeters). Then

$$2x^2 + 4xy = 600, \qquad \text{so that} \qquad y = \frac{600 - 2x^2}{4x}. \tag{1}$$

The volume of the box is

$$V = x^2 y = \frac{(600 - 2x^2)\cdot x^2}{4x} = \frac{1}{4}(600x - 2x^3) = \frac{1}{2}(300x - x^3)$$

by Eq. (1). Also $x > 0$ by Eq. (1), but the maximum value of x is attained when Eq. (1) forces y to be zero, at which point $x = \sqrt{300} = 10\sqrt{3}$. Answer:

$$V(x) = \frac{300x - x^3}{2}, \quad 0 < x \leqq 10\sqrt{3}.$$

1.1.47: The base of the box will be a square measuring $50 - 2x$ in. on each side, so the open-topped box will have that square as its base and four rectangular sides each measuring $50 - 2x$ by x (the height of the box). Clearly $0 \leqq x$ and $2x \leqq 50$. So the volume of the box will be

$$V(x) = x(50 - 2x)^2, \quad 0 \leqq x \leqq 25.$$

1.1.48: Recall that $A(x) = x(50 - x)$, $0 \leqq x \leqq 50$. Here is a table of a few values of the function A at some special numbers in its domain:

x	0	5	10	15	20	25	30	35	40	45	50
A	0	225	400	525	600	625	600	525	400	225	0

It appears that when $x = 25$ (so the rectangle is a square), the rectangle has maximum area 625.

1.1.49: Recall that the total daily production of the oil field is $p(x) = (20 + x)(200 - 5x)$ if x new wells are drilled (where x is an integer satisfying $0 \leqq x \leqq 40$). Here is a table of *all* of the values of the production function p:

x	0	1	2	3	4	5	6	7
p	4000	4095	4180	4255	4320	4375	4420	4455
x	8	9	10	11	12	13	14	15
p	4480	4495	4500	4495	4480	4455	4420	4375
x	16	17	18	19	20	21	22	23
p	4320	4255	4180	4095	4000	3895	3780	3655
x	24	25	26	27	28	29	30	31
p	3520	3375	3220	3055	2880	2695	2500	2295
x	32	33	34	35	36	37	38	39
p	2080	1855	1620	1375	1120	855	580	295

and, finally, $p(40) = 0$. Answer: Drill ten new wells.

1.1.50: The surface area A of the box of Example 8 was

$$A(x) = 2x^2 + \frac{500}{x}, \qquad 0 < x < \infty.$$

The restrictions $x \geqq 1$ and $y \geqq 1$ imply that $1 \leqq x \leqq \sqrt{125}$. A small number of values of A, rounded to three places, are given in the following table.

x	1	2	3	4	5	6	7	8	9	10	11
A	502	258	185	157	150	155	169	191	218	250	287

It appears that A is minimized when $x = y = 5$.

1.1.51: If x is an integer, then $\text{CEILING}(x) = x$ and $-\text{FLOOR}(-x) = -(-x) = x$. If x is not an integer, then choose the integer n so that $n < x < n + 1$. Then $\text{CEILING}(x) = n + 1$, $-(n + 1) < -x < -n$, and

$$-\text{FLOOR}(-x) = -[-(n + 1)] = n + 1.$$

In both cases we see that $\text{CEILING}(x) = -\text{FLOOR}(-x)$.

1.1.52: The range of $\text{ROUND}(x)$ is the set $\boldsymbol{Z}$ of all integers. If k is a nonzero constant, then as x varies through all real number values, so does kx. Hence the range of $\text{ROUND}(kx)$ is $\boldsymbol{Z}$ if $k \neq 0$. If $k = 0$ then the range of $\text{ROUND}(kx)$ consists of the single number zero.

1.1.53: By the result of Problem 52, the range of $\text{ROUND}(10x)$ is the set of all integers, so the range of $g(x) = \frac{1}{10}\text{ROUND}(10x)$ is the set of all integral multiple of $\frac{1}{10}$.

1.1.54: What works for π will work for every real number; let $\text{ROUND2}(x) = \frac{1}{100}\text{ROUND}(100x)$. To be certain that this is correct (we will verify it only for positive numbers), write the [positive] real number x in the form

$$x = k + \frac{t}{10} + \frac{h}{100} + \frac{m}{1000} + r,$$

where k is a nonnnegative integer, t (the "tenths" digit) is a nonnegative integer between 0 and 9, h (the "hundredths" digit) is a nonnegative integer between 0 and 9, as is m, and $0 \leqq r < 0.001$. Then

$$\begin{aligned}\text{ROUND2}(x) &= \tfrac{1}{100}\text{FLOOR}(100x + 0.5)\\ &= \tfrac{1}{100}\text{FLOOR}(100k + 10t + h + \tfrac{1}{10}(m+5) + 100r).\end{aligned}$$

If $0 \leqq m \leqq 4$, the last expression becomes

$$\frac{1}{100}(100k + 10t + h) = k + \frac{t}{10} + \frac{h}{100},$$

which is the correct two-digit rounding of x. If $5 \leqq m \leqq 9$, it becomes

$$\frac{1}{100}(100k + 10t + h + 1) = k + \frac{t}{10} + \frac{h+1}{100},$$

also the correct two-digit rounding of x in this case.

1.1.55: Let $\text{ROUND4}(x) = \frac{1}{10000}\text{ROUND}(10000x)$. To verify that ROUND4 has the desired property for [say] positive values of x, write such a number x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + \frac{d_5}{100{,}000} + r,$$

where k is a nonnegative integer, each d_i is an integer between 0 and 9, and $0 \leqq r < 0.00001$. Application of ROUND4 to x then produces

$$\frac{1}{10000}\text{FLOOR}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + \tfrac{1}{10}(d_5 + 5) + 10000r).$$

Then consideration of the two cases $0 \leqq d_5 \leqq 4$ and $5 \leqq d_5 \leqq 9$ will show that ROUND4 produces the correct four-place rounding of x in both cases.

1.1.56: Let $\text{CHOP4}(x) = \frac{1}{10000}\text{FLOOR}(10000x)$. Suppose that $x > 0$. Write x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + r,$$

where k is a nonnegative integer, each of the d_i is an integer between 0 and 9, and $0 \leqq r < 0.0001$. Then $\textsc{Chop4}(x)$ produces

$$\frac{1}{10000}\textsc{Floor}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + 10000r)$$

$$= \frac{1}{10000}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4)$$

because $10000r < 1$. It follows that $\textsc{Chop4}$ has the desired effect.

1.1.57:

x	0.0	0.2	0.4	0.6	0.8	1.0
y	1.0	0.44	−0.04	−0.44	−0.76	−1.0

The sign change occurs between $x = 0.2$ and $x = 0.4$.

x	0.20	0.25	0.30	0.35	0.40
y	0.44	0.3125	0.19	0.0725	−0.04

The sign change occurs between $x = 0.35$ and $x = 0.40$.

x	0.35	0.36	0.37	0.38	0.39	0.40
y	0.0725	0.0496	0.0269	0.0044	−0.0179	−0.04

From this point on, the data for y will be rounded.

x	0.380	0.382	0.384	0.386	0.388	0.390
y	0.0044	−0.0001	−0.0045	−0.0090	−0.0135	−0.0179

Answer (rounded to two places): 0.38. The quadratic formula yields the two roots $\frac{1}{2}\left(3 \pm \sqrt{5}\,\right)$; the smaller of these is approximately 0.3819660112501051517795.

Problems 58 through 66 are worked in the same way as Problem 57.

1.1.58: The sign change intervals are [2, 3], [2.6, 2.8], [2.60, 2.64], and [2.616, 2.624]. Answer: $\frac{1}{2}\left(3 + \sqrt{5}\right) \approx 2.62$.

1.1.59: The sign change intervals are [1, 2], [1.2, 1.4], [1.20, 1.24], [1.232, 1.240], and [1.2352, 1.2368]. Answer: $-1 + \sqrt{5} \approx 1.24$.

1.1.60: The sign change intervals are $[-4, -3]$, $[-3.4, -3.2]$, $[-3.24, -3.20]$, $[-3.240, -3.232]$, and $[-3.2368, -3.2352]$. Answer: $-1 - \sqrt{5} \approx -3.24$.

1.1.61: The sign change intervals are [0, 1], [0.6, 0.8], [0.68, 0.72], [0.712, 0.720], and [0.7184, 0.7200]. Answer: $\frac{1}{4}\left(7 - \sqrt{17}\right) \approx 0.72$.

1.1.62: The sign change intervals are $[2, 3]$, $[2.6, 2.8]$, $[2.76, 2.80]$, $[2.776, 2.784]$, and $[2.7792, 2.7808]$. Answer: $\frac{1}{4}\left(7+\sqrt{17}\right) \approx 2.78$.

1.1.63: The sign change intervals are $[3, 4]$, $[3.2, 3.4]$, $[3.20, 3.24]$, $[3.208, 3.216]$, and $[3.2080, 3.2096]$. Answer: $\frac{1}{2}\left(11-\sqrt{21}\right) \approx 3.21$.

1.1.64: The sign change intervals are $[7, 8]$, $[7.6, 7.8]$, $[7.76, 7.80]$, $[7.784, 7.792]$, and $[7.7904, 7.7920]$. Answer: $\frac{1}{2}\left(11+\sqrt{21}\right) \approx 7.79$.

1.1.65: The sign change intervals are $[1, 2]$, $[1.6, 1.8]$, $[1.60, 1.64]$, $[1.608, 1.616]$, $[1.6144, 1.6160]$, and $[1.61568, 1.61600]$. Answer: $\frac{1}{6}\left(-23+\sqrt{1069}\right) \approx 1.62$.

1.1.66: The sign change intervals are $[-10, -9]$, $[-9.4, -9.2]$, $[-9.32, -9.28]$, $[-9.288, -9.280]$, and $[-9.2832, -9.2816]$. Answer: $\frac{1}{6}\left(-23-\sqrt{1069}\right) \approx -9.28$.

Section 1.2

1.2.1: The slope of L is $m = (3-0)/(2-0) = \frac{3}{2}$, so L has equation

$$y - 0 = \frac{3}{2}(x-0); \qquad \text{that is,} \qquad 2y = 3x.$$

1.2.2: Because L is vertical and $(7, 0)$ lies on L, every point on L has Cartesian coordinates $(7, y)$ for some number y (and every such point lies on L). Hence an equation of L is $x = 7$.

1.2.3: Because L is horizontal, it has slope zero. Hence an equation of L is

$$y - (-5) = 0 \cdot (x-3); \qquad \text{that is,} \qquad y = -5.$$

1.2.4: Because $(2, 0)$ and $(0, -3)$ lie on L, it has slope $(0+3)/(2-0) = \frac{3}{2}$. Hence an equation of L is

$$y - 0 = \tfrac{3}{2}(x-2); \qquad \text{that is,} \qquad y = \tfrac{3}{2}x - 3.$$

1.2.5: The slope of L is $(3-(-3))/(5-2) = 2$, so an equation of L is

$$y - 3 = 2(x-5); \qquad \text{that is,} \qquad y = 2x - 7.$$

1.2.6: An equation of L is $y - (-4) = \frac{1}{2}(x-(-1))$; that is, $2y + 7 = x$.

1.2.7: The slope of L is $\tan(135°) = -1$, so L has equation

$$y - 2 = -1 \cdot (x - 4); \qquad \text{that is,} \qquad x + y = 6.$$

1.2.8: Equation: $y - 7 = 6(x - 0)$; that is, $y = 6x + 7$.

1.2.9: The second line's equation can be written in the form $y = -2x + 10$ to show that it has slope -2. Because L is parallel to the second line, L also has slope -2 and thus equation $y - 5 = -2(x - 1)$.

1.2.10: The equation of the second line can be rewritten as $y = -\frac{1}{2}x + \frac{17}{2}$ to show that it has slope $-\frac{1}{2}$. Because L is perpendicular to the second line, L has slope 2 and thus equation $y - 4 = 2(x + 2)$.

1.2.11: $x^2 - 4x + 4 + y^2 = 4$: $(x-2)^2 + (y-0)^2 = 2^2$. Center $(2, 0)$, radius 2.

1.2.12: $x^2 + y^2 + 6y + 9 = 9$: $(x-0)^2 + (y+3)^2 = 3^2$. Center $(0, -3)$, radius 3.

1.2.13: $x^2 + 2x + 1 + y^2 + 2y + 1 = 4$: $(x+1)^2 + (y+1)^2 = 2^2$. Center $(-1, -1)$, radius 2.

1.2.14: $x^2 + 10x + 25 + y^2 - 20y + 100 = 25$: $(x+5)^2 + (y-10)^2 = 5^2$. Center $(-5, 10)$, radius 5.

1.2.15: $x^2 + y^2 + x - y = \frac{1}{2}$: $x^2 + x + \frac{1}{4} + y^2 - y + \frac{1}{4} = 1$; $(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = 1$. Center: $(-\frac{1}{2}, \frac{1}{2})$, radius 1.

1.2.16: $x^2 + y^2 - \frac{2}{3}x - \frac{4}{3}y = \frac{11}{9}$: $x^2 - \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{16}{9}$; $(x-\frac{1}{3})^2 + (y-\frac{2}{3})^2 = (\frac{4}{3})^2$. Center $(\frac{1}{3}, \frac{2}{3})$, radius $\frac{4}{3}$.

1.2.17: $y = (x-3)^2$: Opens upward, vertex at $(3, 0)$.

1.2.18: $y - 16 = -x^2$: Opens downward, vertex at $(0, 16)$.

1.2.19: $y - 3 = (x+1)^2$: Opens upward, vertex at $(-1, 3)$.

1.2.20: $2y = x^2 - 4x + 4 + 4$: $y - 2 = \frac{1}{2}(x-2)^2$. Opens upward, vertex at $(2, 2)$.

1.2.21: $y = 5(x^2 + 4x + 4) + 3 = 5(x+2)^2 + 3$: Opens upward, vertex at $(-2, 3)$.

1.2.22: $y = -(x^2 - x) = -(x^2 - x + \frac{1}{4}) + \frac{1}{4}$: $y - \frac{1}{4} = -(x - \frac{1}{2})^2$. Opens downward, vertex at $(\frac{1}{2}, \frac{1}{4})$.

1.2.23: $x^2 - 6x + 9 + y^2 + 8y + 16 = 25$: $(x-3)^2 + (y+4)^2 = 5^5$. Circle, center $(3, -4)$, radius 5.

1.2.24: $(x-1)^2 + (y+1)^2 = 0$: The graph consists of the single point $(1, -1)$.

1.2.25: $(x+1)^2 + (y+3)^2 = -10$: There are no points on the graph.

1.2.26: $x^2 + y^2 - x + 3y + 2.5 = 0$: $x^2 - x + 0.25 + y^2 + 3y + 2.25 = 0$: $(x - 0.5)^2 + (y + 1.5)^2 = 0$. The graph consists of the single point $(0.5, -1.5)$.

1.2.27: The graph is the straight line segment connecting the two points $(-1, 7)$ and $(1, -3)$ (including those two points).

1.2.28: The graph is the straight line segment connecting the two points $(0, 2)$ and $(2, -8)$, including the first of these two points but not the second.

1.2.29: The graph is the parabola that opens downward, symmetric around the y-axis, with vertex at $(0, 10)$ and x-intercepts $\pm\sqrt{10}$.

1.2.30: The graph of $y = 1 + 2x^2$ is a parabola that opens upwards, is symmetric around the y-axis, and has vertex at $(0, 1)$.

1.2.31: The graph of $y = x^3$ can be visualized by modifying the familiar graph of the parabola with equation $y = x^2$: The former may be obtained by multiplying the y-coordinate of the latter's point (x, x^2) by x. Thus both have flat spots at the origin. For $0 < x < 1$, the graph of $y = x^3$ is below that of $y = x^2$. They cross at $(1, 1)$, and for $x > 1$ the graph of $y = x^2$ is below that of $y = x^3$, with the difference becoming arbitrarily large as x increases without bound. If the graph of $y = x^3$ for $x \geqq 0$ is rotated $180°$ around the point $(0, 0)$, the graph of $y = x^3$ for $x < 0$ is the result.

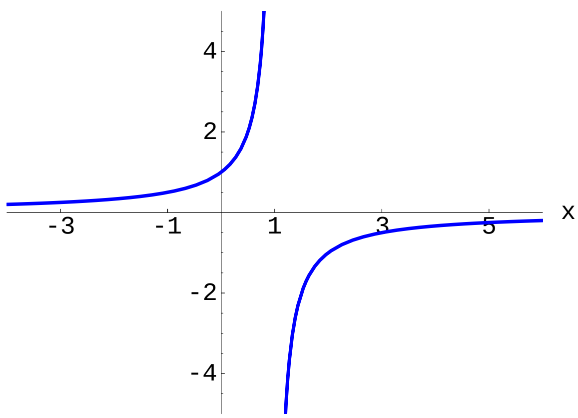
1.2.32: The graph of $f(x) = x^4$ can be visualized by first visualizing the graph of $y = x^2$. If the y-coordinate of each point on this graph is replaced with its square (x^4), the result is the graph of f. The effect on the graph of $y = x^2$ is to multiply the y-coordinate by x^2, which is between 0 and 1 for $0 < |x| < 1$ and which is larger than 1 for $|x| > 1$. Thus the graph of f superficially resembles that of $y = x^2$, but is much closer to the x-axis for $|x| < 1$ and much farther away for $|x| > 1$. The two graphs cross at $(0, 0)$ (where each has a flat spot) and at $(\pm 1, 1)$, but the graph of f is much steeper at the latter two points.

1.2.33: To graph $y = f(x) = \sqrt{4 - x^2}$, note that $y \geqq 0$ and that $y^2 = 4 - x^2$; that is, $x^2 + y^2 = 4$. Hence the graph of f is the *upper half* of the circle with center $(0, 0)$ and radius 2.

1.2.34: To graph $y = f(x) = -\sqrt{9 - x^2}$, note that $y \leqq 0$ and that $y^2 = 9 - x^2$; that is, that $x^2 + y^2 = 9$. Hence the graph of f is the *lower half* of the circle with center $(0, 0)$ and radius 3.

1.2.35: To graph $f(x) = \sqrt{x^2 - 9}$, note that there is no graph for $-3 < x < 3$, that $f(\pm 3) = 0$, and that $f(x) > 0$ for $x < -3$ and for $x > 3$. If x is large positive, then $\sqrt{x^2 - 9} \approx \sqrt{x^2} = x$, so the graph of f has x-intercept $(3, 0)$ and rises as x increases, nearly coinciding with the graph of $y = x$ for x large positive. The case $x < -3$ is trickier. In this case, if x is a large negative number, then $f(x) = \sqrt{x^2 - 9} \approx \sqrt{x^2} = -x$ (Note the minus sign!). So for $x \leqq -3$, the graph of f has x-intercept $(-3, 0)$ and, for x large negative, almost coincides with the graph of $y = -x$. Later we will see that the graph of f becomes arbitrarily steep as x gets closer and closer to ± 3.

1.2.36: As x increases without bound—either positively or negatively—$f(x)$ gets arbitrarily close to zero. Moreover, if x is large positive then $f(x)$ is negative and close to zero, so the graph of f lies just below the x-axis for such x. Similarly, the graph of f lies just above the x-axis for x large negative. If x is slightly less than 1 but very close to 1, then $f(x)$ is the reciprocal of a tiny positive number, hence is a large positive number. So the graph of f just to the left of the vertical line $x = 1$ almost coincides with the top half of that line. Similarly, just to the right of the line $x = 1$, then graph of f almost coincides with the bottom half of that line. There is no graph where $x = 1$, so the graph resembles the one in the next figure. The only intercept is the y-intercept $(0, 1)$. The graph correctly shows that the graph of f is increasing for $x < 1$ and for $x > 1$.



1.2.37: Note that $f(x)$ is positive and close to zero for x large positive, so that the graph of f is just above the x-axis—and nearly coincides with it—for such x. Similarly, the graph of f is just below the x-axis and nearly coincides with it for x large negative. There is no graph where $x = -2$, but if x is slightly greater than -2 then $f(x)$ is the reciprocal of a very small positive number, so $f(x)$ is large and nearly coincides with the upper half of the vertical line $x = -2$. Similarly, if x is slightly less than -2, then the graph of $f(x)$ is large negative and nearly coincides with the the lower half of the line $x = -3$. The graph of f is decreasing for $x < -2$ and for $x > -2$ and its only intercept is the y-intercept $\left(0, \frac{1}{2}\right)$.

1.2.38: Note that $f(x)$ is very small but positive if x is either large positive or large negative. There is no graph for $x = 0$, but if x is very close to zero, then $f(x)$ is the reciprocal of a very small positive number, and hence is large positive. So the graph of f is just above the x-axis and almost coincides with it if $|x|$ is large, whereas the graph of f almost coincides with the positive y-axis for x near zero. There are no intercepts; the graph of f is increasing for $x < 0$ and is decreasing for $x > 0$.

1.2.39: Note that $f(x) > 0$ for all x other than $x = 1$, where f is not defined. If $|x|$ is large, then $f(x)$ is near zero, so the graph of f almost coincides with the x-axis for such x. If x is very close to 1, then $f(x)$ is the reciprocal of a very small positive number, hence $f(x)$ is large positive. So for such x, the graph of $f(x)$ almost coincides with the upper half of the vertical line $x = 1$. The only intercept is $(0, 1)$.

1.2.40: Note first that $f(x)$ is undefined at $x = 0$. To handle the absolute value symbol, we look at two cases: If $x > 0$, then $f(x) = 1$; if $x < 0$, then $f(x) = -1$. So the graph of f consists of the part of the horizontal line $y = 1$ for which $x > 0$, together with the part of the horizontal line $y = -1$ for which $x < 0$.

1.2.41: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis. If x is large negative, then $f(x)$ is negative and close to zero, so the graph of f is slightly below the x-axis and almost coincides with the x-axis. If x is slightly greater than $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. If x is slightly less than $-\frac{3}{2}$ then $f(x)$ is very large negative, so the graph of f almost coincides with the lower half of that vertical line. The graph of f is decreasing for $x < -\frac{3}{2}$ and also decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{3}\right)$.

1.2.42: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive or large negative, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis for $|x|$ large. If x is close to $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. The graph of f is increasing for $x < -\frac{3}{2}$ and decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{9}\right)$.

1.2.43: Given $y = f(x) = \sqrt{1 - x}$, note that $y \geqq 0$ and that $y^2 = 1 - x$; that is, $x = 1 - y^2$. So the graph is the part of the parabola $x = 1 - y^2$ for which $y \geqq 0$. This parabola has horizontal axis of symmetry the y-axis, opens to the left (because the coefficient of y^2 is negative), and has vertex $(1, 0)$. Therefore the graph of f is the upper half of this parabola.

1.2.44: Note that the interval $x < 1$ is the domain of f, so there is no graph for $x \geqq 1$. If x is a large negative number, then the denominator is large positive, so that its reciprocal $f(x)$ is very small positive. As x gets closer and closer to 1 (while $x < 1$), the denominator approaches zero, so its reciprocal $f(x)$ takes on arbitrarily large positive values. So the graph of f is slightly above the x-axis and almost coincides with that axis for x large negative; the graph of f almost coincides with the upper half of the vertical line $x = 1$ for x near (and less than) 1. The graph of f is increasing for all $x < 1$ and $(0, 1)$ is the only intercept.

1.2.45: Note that $f(x)$ is defined only if $2x + 3 > 0$; that is, if $x > -\frac{3}{2}$. Note also that $f(x) > 0$ for all such x. If x is large positive, then $f(x)$ is positive but near zero, so the graph of f is just above the x-axis and almost coincides with it. If x is very close to $-\frac{3}{2}$ (but larger), then the denominator in $f(x)$ is very tiny positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$ for such x. The graph of f is decreasing for all $x > -\frac{3}{2}$.

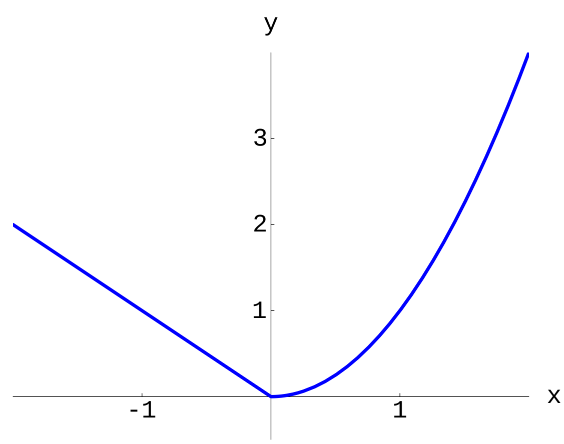
1.2.46: Given: $f(x) = |2x - 2|$. Case 1: $x \geqq 1$. Then $2x - 2 \geqq 0$, so that $f(x) = 2x - 2$. Because $f(1) = 0$, the graph of f for $x \geqq 1$ consists of the part of the straight line through $(1, 0)$ with slope 2. Case 2: $x < 1$. Then $2x - 2 < 0$, so that $f(x) = -2x + 2$. The line $y = -2x + 2$ passes through $(1, 0)$, so the graph of f for $x < 1$ consists of the part of the straight line through $(1, 0)$ with slope -2.

1.2.47: Given: $f(x) = |x| + x$. If $x \geqq 0$ then $f(x) = x + x = 2x$, so if $x \geqq 0$ then the graph of f is the part of the straight line through $(0, 0)$ with slope 2 for which $x \geqq 0$. If $x < 0$ then $f(x) = -x + x = 0$, so the rest of the graph of f coincides with the negative x-axis.

1.2.48: Given: $f(x) = |x - 3|$. If $x \geqq 0$ then $f(x) = x - 3$, so the graph of f consists of the straight line through $(3, 0)$ with slope 1 for $x \geqq 3$. If $x < 0$ then $f(x) = -x + 3$, so the graph of f consists of that part of the straight line with slope -3 and y-intercept $(0, 3)$. These two line segments fit together perfectly at the point $(3, 0)$; there is no break or gap or discontinuity in the graph of f.

1.2.49: Given: $f(x) = |2x + 5|$. The two cases are determined by the point where $2x + 5$ changes sign, which is where $x = -\frac{5}{2}$. If $x \geqq -\frac{5}{2}$, then $f(x) = 2x + 5$, so the graph of f consists of the part of the line with slope 2 and y-intercept 5 for which $x \geqq -\frac{5}{2}$. If $x < -\frac{5}{2}$, then the graph of f is the part of the straight line $y = -2x - 5$ for which $x < -\frac{5}{2}$.

1.2.50: The graph consists of the part of the line $y = -x$ for which $x < 0$ together with the part of the parabola $y = x^2$ for which $x \geqq 0$. These two parts of the graph fit together perfectly at the point $(0, 0)$; there is no break, gap, jump, or discontinuity there.



1.2.51: The graph consists of the horizontal line $y = 0$ for $x < 0$ together with the horizontal line $y = 1$ for $x \geqq 0$. As x moves from left to right through the value zero, there is an abrupt and unavoidable "jump" in the value of f from 0 to 1. That is, f is discontinuous at $x = 0$. To see part of the graph of f, enter the *Mathematica* commands

```
f[x_] := If[x < 0,  0,  1]
Plot[ f[x], {x, −3.5, 3.5 }, PlotRange −> { { −3.5, 3.5 },  {−1.5, 2.5 }} ];
```

1.2.52: The graph of f consists of the open intervals $\dots, (-2,-1), (-1,0), (0,1), (1,2), (2,3), \dots$ on the x-axis together with the isolated points $\dots, (-1,1), (0,1), (1,1), (2,1), (3,1), \dots$. There is a discontinuity at every integral value of x. A *Mathematica* plot of

```
f[ x_] := If[IntegerQ[x],  1,  0 ]
```

will produce a graph that's completely different because *Mathematica,* like most plotting programs, "connects the dots," in effect assuming that every function is continuous at every point in its domain.

1.2.53: Because the graph of the greatest integer function changes at each integral value of x, the graph of $f(x) = [\![2x]\!]$ changes twice as often—at each integral multiple of $\frac{1}{2}$. So as x moves from left to right through such points, the graph jumps upward one unit. Thus there is a discontinuity at each integral multiple of $\frac{1}{2}$. Because f is constant otherwise, these are the only discontinuities. To see something like the graph of f, enter the *Mathematica* commands

```
f[x_] := Floor[ 2*x ];
Plot[ f[x], {x, −3.5, 3.5 }, PlotRange −> { { −3.5, 3.5 },  {−4.5, 4.5 }} ];
```

Mathematica will draw vertical lines connecting points that it shouldn't, making the graph look like treads and risers of a staircase, whereas only the treads are on the graph.

1.2.54: The function f is undefined at $x = 1$. The graph consists of the horizontal line $y = 1$ for $x > 1$ together with the horizontal line $y = -1$ for $x < 1$. There is a discontinuity at $x = 1$.

1.2.55: Given: $f(x) = [\![x]\!]$. If n is an integer and $n \leqq x < n+1$, then express x as $x = n + (\!(x)\!)$ where $(\!(x)\!) = x - [\![x]\!]$ is the *fractional part* of x. Then $f(x) = n - x = n - [n + (\!(x)\!)] = -(\!(x)\!)$. So $f(x)$ is the negative of the fractional part of x. So as x ranges from n up to (but not including) $n+1$, $f(x)$ begins at 0 and drops linearly down not quite to -1. That is, on the interval $(n, n+1)$, the graph of f is the straight line segment connecting the two points $(n, 0)$ and $(n+1, -1)$ with the first of these points included and the second excluded. There is a discontinuity at each integral value of x.

1.2.56: Given: $f(x) = [\![x]\!] + [\![-x]\!] + 1$. If x is an integer, then $f(x) = x + (-x) + 1 = 1$. If x is not an integer, then choose the integer n such that $n < x < n+1$. Then $-(n+1) < -x < -n$, so

$$f(x) = [\![x]\!] + [\![-x]\!] + 1 = n - (n+1) + 1 = 0.$$

So f is the same function as the one defined in Problem 52 and has the same discontinuities: one at each integral value of x.

1.2.57: Because $y = 2x^2 - 6x + 7 = 2(x^2 - 3x + 3.5) = 2(x^2 - 3x + 2.25 + 1.25) = 2(x - 1.5)^2 + 2.5$, the vertex of the parabola is at $(1.5, 2.5)$.

1.2.58: Because $y = 2x^2 - 10x + 11 = 2(x^2 - 5x + 5.5) = 2(x^2 - 5x + 6.25 - 0.75) = 2(x - 2.5)^2 - 1.5$, the vertex of the parabola is at $(2.5, -1.5)$.

1.2.59: Because $y = 4x^2 - 18x + 22 = 4(x^2 - (4.5)x + 5.5) = 4(x^2 - (4.5)x + 5.0625 + 0.4375) = 4(x - 2.25)^2 + 1.75$, the vertex of the parabola is at $(2.25, 1.75)$.

1.2.60: Because $y = 5x^2 - 32x + 49 = 5(x^2 - (6.4)x + 9.8) = 5(x^2 - (6.4)x + 10.24 - 0.44) = 5(x - 3.2)^2 - 2.2$, the vertex of the parabola is at $(3.2, -2.2)$.

1.2.61: Because $y = -8x^2 + 36x - 32 = -8(x^2 - (4.5)x + 4) = -8(x^2 - (4.5)x + 5.0625 - 1.0625) = -8(x - 2.25)^2 + 8.5$, the vertex of the parabola is at $(2.25, 8.5)$.

1.2.62: Because $y = -5x^2 - 34x - 53 = -5(x^2 + (6.8)x + 10.6) = -5(x^2 + (6.8)x + 11.56 - 0.96) = -5(x + 3.4)^2 + 4.8$, the vertex of the parabola is at $(-3.4, 4.8)$.

1.2.63: Because $y = -3x^2 - 8x + 3 = -3\left(x^2 + \frac{8}{3}x - 1\right) = -3\left(x^2 + \frac{8}{3}x + \frac{16}{9} - \frac{25}{9}\right) = -3\left(x + \frac{4}{3}\right)^2 + \frac{25}{3}$, the vertex of the parabola is at $\left(-\frac{4}{3}, \frac{25}{3}\right)$.

1.2.64: Because $y = -9x^2 + 34x - 28 = -9\left(x^2 - \frac{34}{9}x + \frac{28}{9}\right) = -9\left(x^2 - \frac{34}{9}x + \frac{289}{81} - \frac{37}{81}\right) = -9\left(x - \frac{17}{9}\right)^2 + \frac{37}{9}$, the vertex of the parabola is at $\left(\frac{17}{9}, \frac{37}{9}\right)$.

1.2.65: To find the maximum height $y = -16t^2 + 96t$ of the ball, we find the vertex of the parabola: $y = -16(t^2 - 6t) = -16(t^2 - 6t + 9 - 9) = -16(t - 3)^2 + 144$. The vertex of the parabola is at $(3, 144)$ and therefore the maximum height of the ball is 144 ft.

1.2.66: Recall that the area of the rectangle is given by $y = A(x) = x(50 - x)$. To maximize $A(x)$ we find the vertex of the parabola: $y = 50x - x^2 = -(x^2 - 50x) = -(x^2 - 50x + 625 - 625) = -(x - 25)^2 + 625$. Because the vertex of the parabola is at $(25, 625)$ and $x = 25$ is in the domain of the function A, the maximum value of $A(x)$ occurs at $x = 25$ and is $A(25) = 625$ (ft^2).

1.2.67: If two positive numbers x and y have sum 50, then $y = 50 - x$ and $x < 50$ (because $y > 0$). To maximize their product $p(x)$ we find the vertex of the parabola

$$\begin{aligned} y = p(x) = x(50 - x) &= -(x^2 - 50x) \\ &= -(x^2 - 50x + 625 - 625) = -(x - 25)^2 + 625, \end{aligned}$$

which is at $(25, 625)$. Because $0 < 25 < 50$, $x = 25$ is in the domain of the product function $p(x) = x(50-x)$, and hence the maximum value of the product of x and y is $p(25) = 625$.

1.2.68: Recall that if x new wells are drilled, then the resulting total production p is given by $p(x) = 4000 + 100x - 5x^2$. To maximize $p(x)$ we find the vertex of the parabola

$$y = p(x) = -5x^2 + 100x + 4000 = -5(x^2 - 20x - 800)$$
$$= -5(x^2 - 20x + 100 - 900) = -5(x - 10)^2 + 4500.$$

The vertex of the parabola $y = p(x)$ is therefore at $(10, 4500)$. Because $x = 10$ is in the domain of p (it is an integer between 0 and 40) and because the parabola opens downward (the coefficient of x^2 is negative), $x = 10$ indeed maximizes $p(x)$.

1.2.69: The graph looks like the graph of $y = |x|$ because the slope of the left-hand part is -1 and that of the right-hand part is 1; but the vertex is shifted to $(-1, 0)$, so—using the translation principle—the graph in Fig. 1.2.29 must be the graph of $f(x) = |x + 1|$, $-2 \leqq x \leqq 2$.

1.2.70: Because the graph in Fig. 1.2.30 is composed of three straight-line segments, it can be described most easily using a "three-part" function:

$$f(x) = \begin{cases} 2x + 6 & \text{if } -3 \leqq x < -2; \\ 2 & \text{if } -2 \leqq x < 2; \\ \frac{1}{3}(10 - 2x) & \text{if } \quad 2 \leqq x \leqq 5. \end{cases}$$

1.2.71: The graph in Fig. 1.2.31 is much like the graph of the greatest integer function—it takes on only integral values—but the "jumps" occur twice as often, so this must be very like—indeed, it is exactly—the graph of $f(x) = [\![2x]\!]$, $-1 \leqq x < 2$.

1.2.72: The graph in Fig. 1.2.32 resembles the graph of the greatest integer function in that it takes on all integral values and only those, but it is decreasing rather than increasing and the "jumps" occur only at the even integers. Thus it must be the graph of something similar to $f(x) = -[\![\frac{1}{2}x]\!]$, $-4 \leqq x < 4$. Comparing values of f at $x = -4$, -3, -2.1, -2, -1, -0.1, 0, 1, 1.9, 2, 3, and 3.9 with points on the graph is sufficient evidence that the graph of f is indeed that shown in the figure.

1.2.73: Clearly $x(t) = 45t$ for the first hour; that is, for $0 \leqq t \leqq 1$. In the second hour the graph of $x(t)$ must be a straight line (because of constant speed) of slope 75, thus with equation $x(t) = 75t + C$ for some constant C. The constant C is determined by the fact that $45t$ and $75t + C$ must be equal at time $t = 1$, as the automobile cannot suddenly jump from one position to a completely different position in an instant. Hence $45 = 75 + C$, so that $C = -30$. Therefore

$$x(t) = \begin{cases} 45t & \text{if } 0 \leqq t \leqq 1; \\ 75t - 30 & \text{if } 1 < t \leqq 2. \end{cases}$$

To see the graph of $x(t)$, plot in *Mathematica*

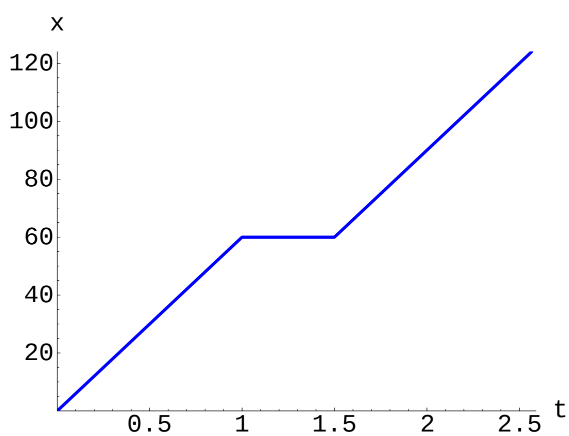
$$\text{x[t\_] := If[t < 1, 45*t, 75*t - 30]}$$

on the interval $0 \leqq t \leqq 2$.

1.2.74: The graph of $x(t)$ will consist of three straight-line segments (because of the constant speeds), the first of slope 60 for $0 \leqq t \leqq 1$, the second of slope zero for $1 \leqq t \leqq 1.5$, and the third of slope 60 for $1.5 \leqq t \leqq 2.5$. The first pair must coincide when $t = 1$ and the second pair must coincide when $t = 1.5$ because the graph of $x(t)$ can have no discontinuities. So if we write $x(t) = 60$ for $0 \leqq t \leqq 1$, we must have $x(t) = 60$ for $1 \leqq t \leqq 1.5$. Finally, $x(t) = 60t + C$ for some constant C if $1.5 \leqq t \leqq 2.5$, but the latter must equal 60 when $t = 1.5$, so that $C = -30$. Hence

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 60 & \text{if } 1 < t \leqq 1.5, \\ 60t - 30 & \text{if } 1.5 < t \leqq 2.5. \end{cases}$$

The graph of $x(t)$ is shown next.



1.2.75: The graph must consist of two straight-line segments (because of the constant speeds). The first must have slope 60, so we have $x(t) = 60t$ for $0 \leqq t \leqq 1$. The second must have slope -30, negative because you're driving in the reverse direction, so $x(t) = -30t + C$ for some constant C if $1 \leqq t \leqq 3$. The two segments must coincide when $t = 1$, so that $60 = -30 + C$. Thus $C = 90$ and thus a formula for $x(t)$ is

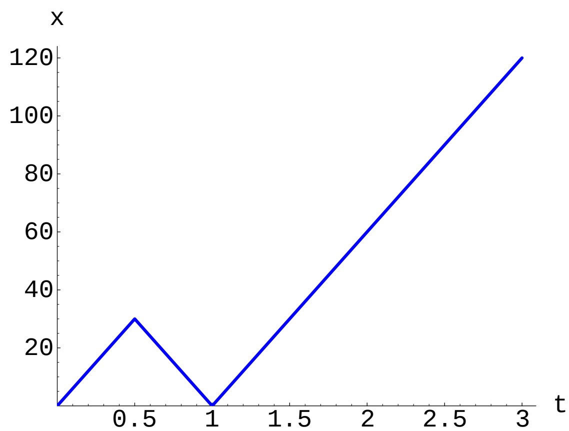
$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 90 - 30t & \text{if } 1 < t \leqq 3. \end{cases}$$

1.2.76: We need three straight line segments, the first of slope 60 for $0 \leqq t \leqq 0.5$, the second of slope -60 for $0.5 \leqq t \leqq 1$, and the third of slope 60 for $1 \leqq t \leqq 3$. Clearly the first must be $x(t) = 60t$ for $0 \leqq t \leqq 0.5$. The second must have the form $x(t) = -60t + C$ for some constant C, and the first and second must coincide when $t = 0.5$, so that $30 = -30 + C$, and thus $C = 60$. The third segment must have the form $x(t) = 60t + K$

for some constant K, and the second and third must coincide when $t = 1$, so that $0 = 60 + K$, and so $K = -60$. Therefore a formula for $x(t)$ is

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 0.5, \\ 60 - 60t & \text{if } 0.5 < t \leqq 1, \\ 60t - 60 & \text{if } 1 < t \leqq 3. \end{cases}$$

The graph of $x(t)$ is shown next.



1.2.77: Initially we work in units of pages and cents (to avoid decimals and fractions). The graph of C, as a function of p, must be a straight line segment, and its slope is (by information given)

$$\frac{C(79) - C(34)}{79 - 34} = \frac{305 - 170}{79 - 34} = \frac{135}{45} = 3.$$

Thus $C(p) = 3p + K$ for some constant K. So $3 \cdot 34 + K = 170$, and it follows that $K = 68$. So $C(p) = 3p + 68$, $1 \leqq p \leqq 100$, if C is to be expressed in cents. If C is to be expressed in dollars, we have

$$C(p) = (0.03)p + 0.68, \quad 1 \leqq p \leqq 100.$$

The "fixed cost" is incurred regardless of the number of pamphlets printed; it is \$0.68. The "marginal cost" of printing each additional page of the pamphlet is the coefficient \$0.03 of p.

1.2.78: We are given $C(x) = a + bx$ where a and b are constants; we are also given

$$99.45 = C(207) = a + 207b \qquad \text{and}$$

$$79.15 = C(149) = a + 149b.$$

Subtraction of the second equation from the first yields $20.3 = 58b$, so that $b = 0.35$. Substitution of this datum in the first of the preceding equations then yields

$$99.45 = a + 207 \cdot 0.35 = a + 72.45, \qquad \text{so that} \qquad a = 27.$$

Therefore $C(x) = 27 + (0.35)x,\ \ 0 \leqq x < +\infty$. Thus if you drive 175 miles on the third day, the cost for that day will be $C(175) = 88.25$ (in dollars). The slope $b = 0.35$ represents a cost of \$0.35 per mile. The C-intercept $a = 27$ represents the daily base cost of renting the car. In civil engineering and in some branches of applied mathematics, the intercept $a = 27$ is sometimes called the *offset*, representing the vertical amount by which $C(0)$ is "offset" from zero.

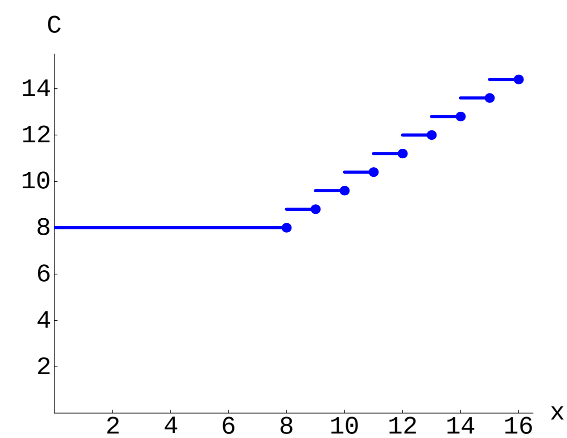
1.2.79: Suppose that the letter weighs x ounces, $0 < x \leqq 16$. If $x \leqq 8$, then the cost is simply 8 (dollars). If $8 < x \leqq 9$, add \$0.80; if $9 < x \leqq 10$, add \$1.60, and so on. Very roughly, one adds \$0.80 if $[\![x - 8]\!] = 1$, \$1.60 if $[\![x - 8]\!] = 2$, and so on. But this isn't quite right—we are using the FLOOR function of Section 1.1, whereas we should really be using the CEILING function. By the result of Problem 51 of that section, we see that instead of cost

$$C(x) = 8 + (0.8)[\![x - 8]\!]$$

for $8 < x \leqq 16$, we should instead write

$$C(x) = \begin{cases} 8 & \text{if } 0 < x \leqq 8, \\ 8 - (0.8)[\![-(x - 8)]\!] & \text{if } 8 < x \leqq 16. \end{cases}$$

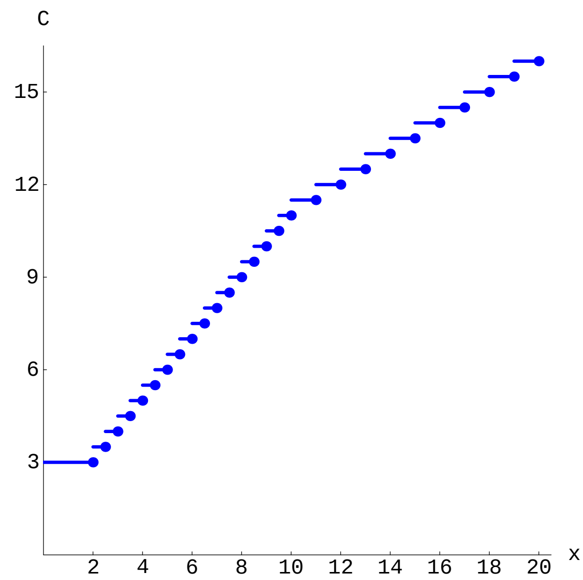
The graph of the cost function is shown next.



1.2.80: Solve this problem like Problem 79 (but it is more complicated). Result:

$$C(x) = \begin{cases} 3 & \text{if } 0 < x \leqq 2; \\ 3 - 0.5[\![-2(x - 2)]\!] & \text{if } 2 < x \leqq 10; \\ 11 - 0.5[\![-(x - 10)]\!] & \text{if } 10 < x \leqq 20. \end{cases}$$

The graph of C is shown below.



1.2.81: Boyle's law states that under conditions of constant temperature, the product of the pressure p and the volume V of a fixed mass of gas remains constant. If we assume that $pV = c$, a constant, for the given data, we find that the given five data points yield the values $c = 1.68$, 1.68, 1.675, 1.68, and 1.62. The average of these is 1.65 (to two places) and should be a good estimate of the true value of c. Alternatively, you can use a computer algebra program to find c; in *Mathematica*, for example, the command Fit will fit given data points to a sum of constant multiples of functions you specify. We used the commands

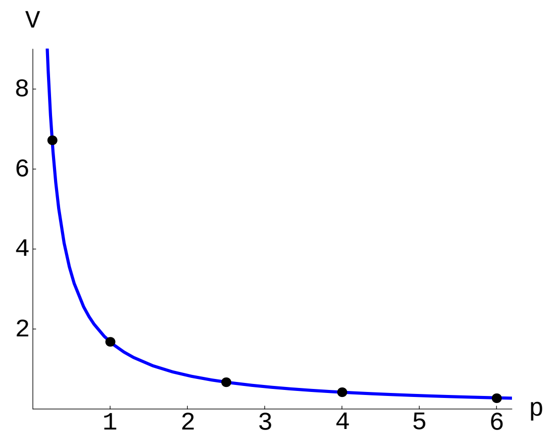
data = {{0.25, 6.72}, {1.0, 1.68}, {2.5, 0.67}, {4.0, 0.42}, {6.0, 0.27}};

Fit [data, { 1/p }, p]

to find that

$$V(p) = \frac{1.67986}{p}$$

yields the best *least-squares* fit of the given data to a function of the form $V(p) = c/p$. We rounded the numerator to 1.68 to find the estimates $V(0.5) \approx 3.36$ and $V(5) \approx 0.336$ (L). The graph of $V(p)$ together with the given data points is shown next.



1.2.82: It seems reasonable to assume that the maximum average temperature occurs on July 15 and the minimum on January 15, so that a multiple of a cosine function should fit the given data if we take $t = 0$ on July 15. So we assume a solution of the form

$$T(t) = c_1 + c_2 \cos\left(\frac{2\pi t}{365}\right).$$

Also assuming that the average year-round daily temperature is the average of the minimum and the maximum, we find that $c_1 = 61.25$, so we could find c_2 by the averaging method of Problem 81. Alternatively, we could use the Fit command in *Mathematica* to find both c_1 and c_2 simultaneously as follows:

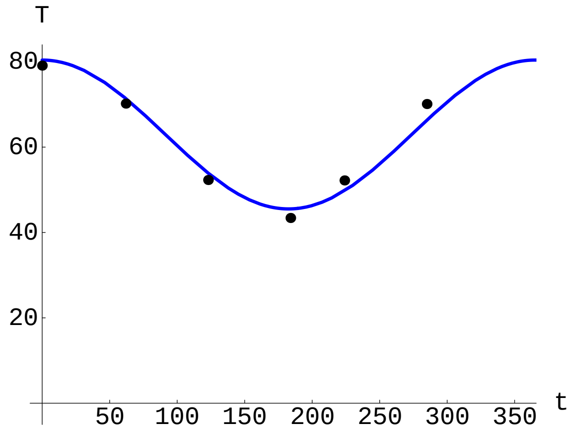
```
data = {{0, 79.1}, {62, 70.2}, {123, 52.3}, {184, 43.4}, {224, 52.2}, {285, 70.1}};

Fit [ data, { 1, Cos[2*Pi*t/365] }, t ]
```

The result is the formula

$$T(t) = 62.9602 + (17.437)\cos\left(\frac{2\pi t}{365}\right).$$

The values predicted by this function at the six dates in question are [approximately] 80.4, 71.4, 53.9, 45.5, 49.8, and 66.3. Not bad, considering we are dealing with weather, a most unpredictable phenomenon. The graph of $T(t)$ is shown next. Units on the horizontal axis are days, measured from July 15. Units on the vertical axis are degrees Fahrenheit. Remember that these are *average* daily temperatures; it is not uncommon for a winter low in Athens to be below 28°F and for a summer high to be as much as 92°F.



Section 1.3

1.3.1: The domain of f is $\boldsymbol{R}$, the set of all real numbers; so is the domain of g, but $g(x) = 0$ when $x = 1$ and when $x = -3$. So the domain of $f + g$ and $f \cdot g$ is the set $\boldsymbol{R}$ and the domain of f/g is the set of all real numbers other than 1 and -3. Their formulas are

$$(f+g)(x) = x^2 + 3x - 2,$$

$$(f \cdot g)(x) = (x+1)(x^2+2x-3) = x^3 + 3x^2 - x - 3, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{x+1}{x^2+2x-3}.$$

1.3.2: The domain of f consists of all real numbers other than 1 and the domain of g consists of all real numbers other than $-\frac{1}{2}$. Hence the domain of $f+g$, $f \cdot g$, and f/g consists of all real numbers other than $-\frac{1}{2}$ and 1. For such x,

$$(f+g)(x) = \frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{(x-1)(2x+1)},$$

$$(f \cdot g)(x) = \frac{1}{(x-1)(2x+1)}, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-1}.$$

Note that, in spite of the last equation, the domain of f/g does *not* include the number $-\frac{1}{2}$.

1.3.3: The domain of f is the interval $[0, +\infty)$ and the domain of g is the interval $[2, +\infty)$. Hence the domain of $f+g$ and $f \cdot g$ is the interval $[2, +\infty)$, but because $g(2) = 0$, the domain of f/g is the open interval $(2, +\infty)$. The formulas for these combinations are

$$(f+g)(x) = \sqrt{x} + \sqrt{x-2},$$

$$(f \cdot g)(x) = \sqrt{x}\,\sqrt{x-2} = \sqrt{x^2-2x}, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{x-2}} = \sqrt{\frac{x}{x-2}}.$$

1.3.4: The domain of f is the interval $[-1, +\infty)$ and the domain of g is the interval $(-\infty, 5]$. Hence the domain of $f+g$ and $f \cdot g$ is the closed interval $[-1, 5]$, but because $g(5) = 0$, the domain of f/g is the half-open interval $[-1, 5)$. Their formulas are

$$(f+g)(x) = \sqrt{x+1} + \sqrt{5-x}, \qquad (f \cdot g)(x) = \sqrt{x+1}\,\sqrt{5-x} = \sqrt{5+4x-x^2}, \quad \text{and}$$
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+1}}{\sqrt{5-x}} = \sqrt{\frac{x+1}{5-x}}.$$

1.3.5: The domain of f is the set $\boldsymbol{R}$ of all real numbers; the domain of g is the open interval $(-2, 2)$. Hence the domain of $f+g$ and $f \cdot g$ is the open interval $(-2, 2)$; because $g(x)$ is never zero, the domain of f/g is the same. Their formulas are

$$(f+g)(x) = \sqrt{x^2+1} + \frac{1}{\sqrt{4-x^2}}, \qquad (f \cdot g)(x) = \frac{\sqrt{x^2+1}}{\sqrt{4-x^2}}, \quad \text{and}$$
$$\left(\frac{f}{g}\right)(x) = \sqrt{x^2+1}\,\sqrt{4-x^2} = \sqrt{4+3x^2-x^4}.$$

1.3.6: The domain of f is the set of all real numbers other than 2 and the domain of g is the set of all real numbers other than -2. Hence the domain of $f+g$ and $f \cdot g$ is the set of all real numbers other than ± 2. But because $g(-1) = 0$, -1 does not belong to the domain of f/g, which therefore consists of all real numbers other than -2, -1, and 2. The formulas of these combinations are

$$(f+g)(x) = \frac{x-1}{x-2} + \frac{x+1}{x+2} = \frac{2x^2-4}{x^2-4}, \qquad (f \cdot g)(x) = \frac{x-1}{x-2} \cdot \frac{x+1}{x+2} = \frac{x^2-1}{x^2-4}, \quad \text{and}$$
$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{x-2} \cdot \frac{x+2}{x+1} = \frac{x^2+x-2}{x^2-x-2}.$$

1.3.7: $f(x) = x^3 - 3x + 1$ has 1, 2, or 3 zeros, approaches $+\infty$ as x does, and approaches $-\infty$ as x does. Because $f(0) \neq 0$, the graph does not match Fig. 1.3.26, so it must match Fig. 1.3.30.

1.3.8: $f(x) = 1 + 4x - x^3$ has one, two, or three zeros, approaches $-\infty$ as $x \to +\infty$ and approaches $+\infty$ as $x \to -\infty$. Hence its graph must be the one shown in Fig. 1.3.28.

1.3.9: $f(x) = x^4 - 5x^3 + 13x + 1$ has four or fewer zeros and approaches $+\infty$ as x approaches either $+\infty$ or $-\infty$. Hence its graph must be the one shown in Fig. 1.3.31.

1.3.10: $f(x) = 2x^5 - 10x^3 + 6x - 1$ has between one and five zeros, approaches $+\infty$ as x does, and approaches $-\infty$ as x does. So its graph might be the one shown in Fig. 1.3.26, the one in Fig. 1.3.29, or the one in Fig. 1.3.30. But $f(0) \neq 0$, so Fig. 1.3.26 is ruled out, and we have already found that the graph in Fig. 1.3.30 matches the function in Problem 7. Therefore the graph of f must be the one shown in Fig. 1.3.29.

Alternatively, the observation that $f(x)$ changes sign on the *five* intervals $[-3, -2]$, $[-1, 0]$, $[0, 0.5]$, $[0.5, 1]$, and $[2, 3]$ shows that $f(x)$ has five zeros; therefore the graph must be the one shown in Fig. 1.3.29.

1.3.11: $f(x) = 16 + 2x^2 - x^4$ approaches $-\infty$ as x approaches either $+\infty$ or $-\infty$, so its graph must be the one shown in Fig. 1.3.27.

1.3.12: $f(x) = x^5 + x$ approaches $+\infty$ as x does and approaches $-\infty$ as x does. Moreover, $f(x) > 0$ if $x > 0$ and $f(x) < 0$ if $x < 0$, which rules out every graph except for the one shown in Fig. 1.3.26.

1.3.13: The graph of f has vertical asymptotes at $x = -1$ and at $x = 2$, so its graph must be the one shown in Fig. 1.3.34.

1.3.14: The graph of $f(x)$ has vertical asymptotes at $x = \pm 3$, so its graph must be the one shown in Fig. 1.3.32.

1.3.15: The graph of f has no vertical asymptotes and has maximum value 3 when $x = 0$. Hence its graph must be the one shown in Fig. 1.3.33.

1.3.16: The denominator $x^3 - 1 = (x-1)(x^2+x+1)$ of $f(x)$ is zero only when $x = 1$ (because $x^2 + x + 1 > x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 \geqq 0$ for all x), so its graph must be the one shown in Fig. 1.3.35.

1.3.17: The domain of $f(x) = x\sqrt{x+2}$ is the interval $[-2, +\infty)$, so its graph must be the one shown in Fig. 1.3.38.

1.3.18: The domain of $f(x) = \sqrt{2x - x^2}$ consists of those numbers for which $2x - x^2 \geqq 0$; that is, $x(2-x) \geqq 0$. This occurs when x and $2 - x$ have the same sign and also when either is zero. If $x > 0$ and $2 - x > 0$, then $0 < x < 2$. If $x < 0$ and $2 - x < 0$, then $x < 0$ and $x > 2$, which is impossible. Hence the domain of f is the closed interval $[0, 2]$. So the graph of f must be the one shown in Fig. 1.3.36.

1.3.19: The domain of $f(x) = \sqrt{x^2 - 2x}$ consists of those numbers x for which $x^2 - 2x \geqq 0$; that is, $x(x-2) \geqq 0$. This occurs when x and $x - 2$ have the same sign and also when either is zero. If $x > 0$ and $x - 2 > 0$, then $x > 2$; if $x < 0$ and $x - 2 < 0$, then $x < 0$. So the domain of f is the union of the two intervals $(-\infty, 0]$ and $[2, +\infty)$. So the graph of f must be the one shown in Fig. 1.3.39.

1.3.20: The domain of $f(x) = 2(x^2 - 2x)^{1/3}$ is the set $\boldsymbol{R}$ of all real numbers because every real number has a [unique] cube root. By the analysis in the solution of Problem 19, $x^2 - 2x < 0$ if $0 < x < 2$ and $x^2 - 2x \geqq 0$ otherwise. Hence $f(x) < 0$ if $0 < x < 2$ and $f(x) \geqq 0$ otherwise. This makes it certain that the graph of f is the one shown in Fig. 1.3.37.

1.3.21: Good viewing window: $-2.5 \leqq x \leqq 2.5$. Three zeros, approximately -1.88, 0.35, and 1.53.

1.3.22: Good viewing window: $-3 \leqq x \leqq 3$. Two zeros: -2 and 1.

1.3.23: Good viewing window: $-3.5 \leqq x \leqq 2.5$. One zero, approximately -2.10.

1.3.24: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Four zeros, approximately -1.28, 0.61, 1.46, and 2.20.

1.3.25: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Three zeros: approximately -1.30, exactly 1, and approximately 2.30.

1.3.26: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Two zeros, approximately -1.33 and 2.37.

1.3.27: Good viewing window: $-7.5 \leqq x \leqq 8.5$. Three zeros: Approximately -5.70, -2.22, and 7.91.

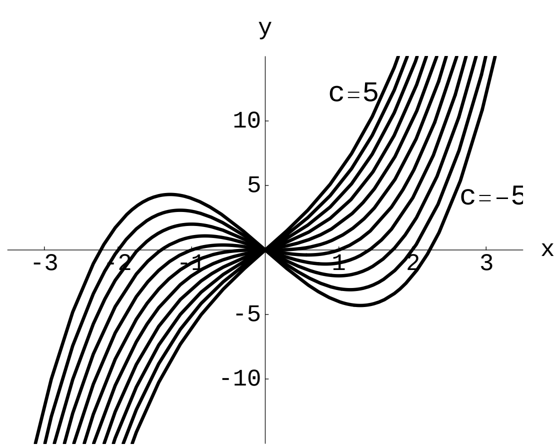
1.3.28: Good viewing window: None; it takes three: $-22 \leqq x \leqq 8$ shows that there is a zero near -20 and that the graph crosses the x-axis somewhere in the vicinity of $x = 0$. The window $-3 \leqq x \leqq 3$ shows that something interesting happens near $x = -1$ and that there is a zero near 1.8. The window $-1.4 \leqq x \leqq 0.4$ shows that there are zeros near -1.1 and -0.8. Closer approximations to these four zeros are -19.88, -1.09, -0.79, and 1.76.

1.3.29: The viewing window $-11 \leqq x \leqq 8$ shows that there are five zeros, although the two near 2.5 may be only one. The window $1.5 \leqq x \leqq 3.5$ shows that there are in fact two zeros near 2.5. Approximate values of the five zeros are -10.20, -7.31, 1.98, 3.25, and 7.28.

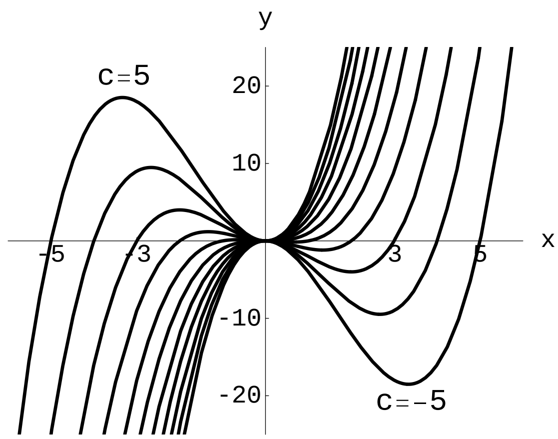
1.3.30: The viewing window $-16 \leqq x \leqq 16$ shows that there are zeros near ± 15 and perhaps a few more near $x = 0$. The window $-4 \leqq x \leqq 4$ shows that there are in fact four zeros near $x = 0$. Approximate values of the six are ± 15.48, ± 3.04, and ± 1.06.

1.3.31: Every time c increases by 1, the graph is raised 1 unit (in the positive y-direction), but there is no other change.

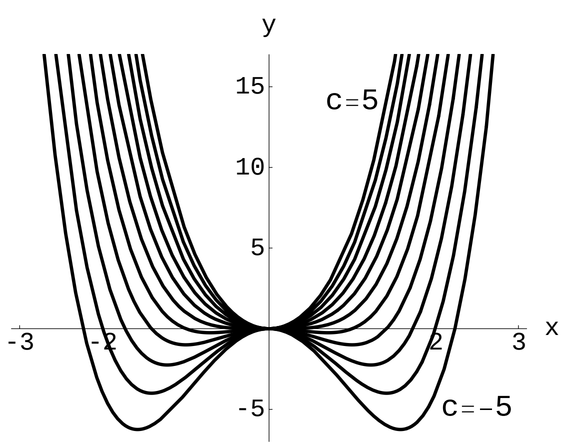
1.3.32: The graph starts with two "bends" when $c = -5$. As c increases the bends become narrower and narrower and disappear when $c = 0$. Then the graph gets steeper and steeper. See the following figure.



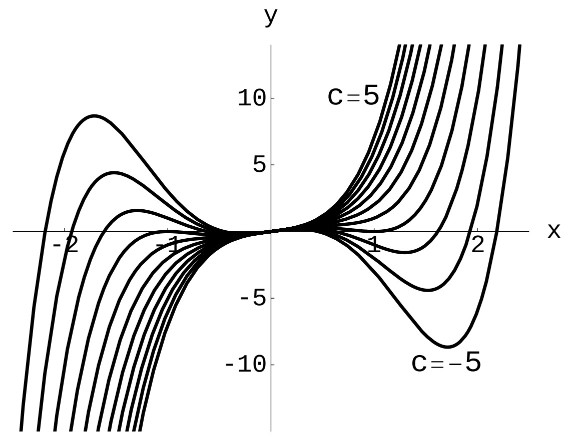
1.3.33: The graph always passes through (0, 0) and is tangent to the x-axis there. When $c = -5$ there is another zero at $x = 5$. As c increases this zero shifts to the left until it coincides with the one at $x = 0$ when $c = 0$. At this point the "bend" in the graph disappears. As c increases from 1 to 5, the bend reappears to the left of the x-axis and the second zero reappears at $-c$.



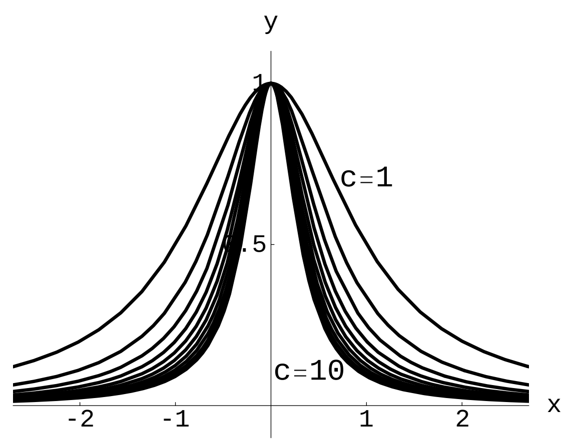
1.3.34: The graph is always tangent to the x-axis at $x = 0$ and is always symmetric around the y-axis. When $c = -5$ there is another pair of zeros near ± 2.2. As c increases these zeros move closer to $x = 0$ and the bends in the graph get smaller and smaller. They disappear when $c = 0$ and, at the same time, the zeros merge with the one at $x = 0$. Thereafter the graph simply becomes steeper and steeper. See the following figure.



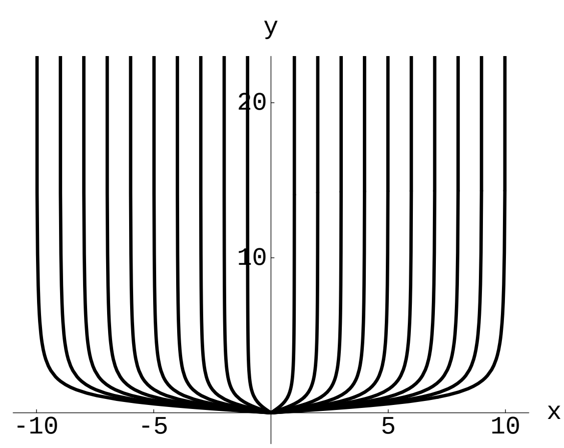
1.3.35: The graph is always symmetric around the origin (and, consequently, always passes through the origin). When $c = -5$ there is another pair of zeros near ± 2.2. As c increases the graph develops positive slope at $x = 0$, two more bends, and two more zeros on either side of the origin. They move outward and, when $c = -2$, they coincide with the outer pair of zeros, which have also been moving toward the origin. They reach the origin when $c = 0$ and thereafter the graph simply becomes steeper and steeper.



1.3.36: As c increases the "mountain" around the y-axis gets narrower and steeper. See the following figure.



1.3.37: As c increases the graph becomes wider; aside from the horizontal scale, its shape does not seem to change very much.



1.3.38: The length of the airfoil is approximately 1.0089 and its width is approximately 0.200057.

Section 1.4

1.4.1: Because $g(x) = 2^x$ increases—first slowly, then rapidly—on the set of all real numbers, with values in the range $(0, +\infty)$, the given function $f(x) = 2^x - 1$ must increase in the same way, but with values in the range $(-1, +\infty)$. Therefore its graph is the one shown in Fig. 1.4.29.

1.4.2: Given: $f(x) = 2 - 3^{-x}$. The graph of $g(x) = 3^x$ increases, first slowly, then rapidly, on its domain the set $\boldsymbol{R}$ of all real numbers. Hence $h(x) = 3^{-x}$ decreases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(0, +\infty)$. Hence $j(x) = -3^{-x}$ increases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(-\infty, 0)$. Therefore $f(x) = 2 - 3^{-x}$ increases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(-\infty, 2)$. Therefore its graph must be the one shown in Fig. 1.4.33.

1.4.3: The graph of $f(x) = 1 + \cos x$ is simply the graph of the ordinary cosine function raised 1 unit—moved upward 1 unit in the positive y-direction. Hence its graph is the one shown in Fig. 1.4.27.

1.4.4: The graph of $g(x) = 2\sin x$ resembles the graph of the ordinary sine function, but with values ranging from -2 to 2. The graph of $h(x) = -2\sin x$ is the same, but turned "upside down." Add 2 to get $f(x) = 2 - 2\sin x$ and the graph of h is raised 2 units, thus taking values in the range $[0, 4]$. So the graph of f is the one shown in Fig. 1.4.32.

1.4.5: The graph of $g(x) = 2\cos x$ resembles the graph of the cosine function, but with all values doubled, so that its range is the interval $[-2, 2]$. Add 1 to get $f(x) = 1 + 2\cos x$ and the range is now the interval $[-1, 3]$. So the graph of f is the one shown in Fig. 1.4.35.

1.4.6: Turn the graph of the sine function upside down, then add 2 to get $f(x) = 2 - \sin x$, with range the interval $[1, 3]$. Hence the graph of f is the one shown in Fig. 1.4.28.

1.4.7: The graph of $g(x) = 2^x$ increases, first slowly, then rapidly, on the set of all real numbers, with range the interval $(0, +\infty)$. So its reciprocal $h(x) = 2^{-x}$ decreases, first rapidly, then slowly, with the same domain and range. Multiply by x to obtain $f(x) = x \cdot 2^{-x}$. The effect of multiplication by x is to change large positive values into large negative values for $x < 0$, to cause $f(0)$ to be zero, and to multiply very small positive values (of 2^{-x}) by somewhat large positive values (of x) for $x > 0$, resulting in values that are still small and positive, even when x is quite large. So the graph of f must increase rapidly through negative values, pass through $(0, 0)$, rise to a maximum, then decrease rapidly through positive values toward zero. Hence the graph of f must be the one shown in Fig. 1.4.31.

1.4.8: The graph of $g(x) = \log x$ has domain the set $(0, +\infty)$ of all positive real numbers; it rises, first rapidly, then more slowly, with range the set of all real numbers, and its graph passes through the point

(1, 0). Division by $x > 0$ will have little effect if x is near zero, as this will merely multiply large negative values of $\log x$ by large positive numbers. But when x is large positive, it will be much larger than $\log x$, and thus the graph of $f(x)$ will rise to a maximum somewhere to the right of $x = 1$, then decreases fairly rapidly toward zero. So the graph of f is the one shown in Fig. 1.4.36.

1.4.9: The graph of $g(x) = 1 + \cos 6x$ will resemble the graph of the cosine function, but raised 1 unit (so that its range is the interval [0, 2]) and with much more "activity" on the x-axis (because of the factor 6). Division by $1 + x^2$ will have little effect until x is no longer close to zero, and then the effect will be to divide values of $g(x)$ by larger and larger positive numbers, so that the cosine oscillations have a much smaller range that $0 \leqq x \leqq 2$; they will range from 0 to smaller and smaller positive values as $|x|$ increases. So the graph of f is the one shown in Fig. 1.4.34.

1.4.10: The graph of $g(x) = \sin 10x$ resembles that of the sine function, but with much more "activity" because of the factor 10. Multiply by the rapidly decreasing positive numbers 2^{-x} and you will see the sine oscillations decreasing from the range $[-1, 1]$ when x is near zero to very small oscillations—near zero—as x increases. So the graph of f is the one shown in Fig. 1.4.30.

1.4.11: Given $f(x) = 1 - x^2$ and $g(x) = 2x + 3$,

$$f(g(x)) = 1 - (g(x))^2 = 1 - (2x+3)^2 = -4x^2 - 12x - 8 \quad \text{and}$$

$$g(f(x)) = 2f(x) + 3 = 2(1 - x^2) + 3 = -2x^2 + 5.$$

1.4.12: Given $f(x) = -17$ and $g(x) = |x|$,

$$f(g(x)) = -17 \quad \text{and}$$

$$g(f(x)) = |f(x)| = |-17| = 17.$$

The first result is a little puzzling until one realizes that to obtain $f(g(x))$, one substitutes $g(x)$ for x for every occurrence of x in the formula for f. No x there means there's no place to put $g(x)$. Indeed, $f(h(x)) = -17$ no matter what the formula of h.

1.4.13: If $f(x) = \sqrt{x^2 - 3}$ and $g(x) = x^2 + 3$, then

$$f(g(x)) = \sqrt{(g(x))^2 - 3} = \sqrt{(x^2+3)^2 - 3} = \sqrt{x^4 + 6x^2 + 6} \quad \text{and}$$

$$g(f(x)) = (f(x))^2 + 3 = \left(\sqrt{x^2 - 3}\right)^2 + 3 = x^2 - 3 + 3 = x^2.$$

The domain of $f(g)$ is the set $\boldsymbol{R}$ of all real numbers, but the domain of $g(f)$ is the same as the domain of f, the set of all real numbers x such that $x^2 \geqq 3$.

1.4.14: If $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x^2+1}$, then

$$f(g(x)) = (g(x))^2 + 1 = \frac{1}{(x^2+1)^2} + 1 = \frac{x^4 + 2x^2 + 2}{x^4 + 2x^2 + 1} \quad \text{and}$$

$$g(f(x)) = \frac{1}{(f(x))^2 + 1} = \frac{1}{(x^2+1)^2 + 1} = \frac{1}{x^4 + 2x^2 + 2}.$$

1.4.15: If $f(x) = x^3 - 4$ and $g(x) = (x+4)^{1/3}$, then

$$f(g(x)) = (g(x))^3 - 4 = \left((x+4)^{1/3}\right)^3 - 4 = x + 4 - 4 = x \quad \text{and}$$

$$g(f(x)) = (f(x) + 4)^{1/3} = \left(x^3 - 4 + 4\right)^{1/3} = \left(x^3\right)^{1/3} = x.$$

The domain of both $f(g)$ and $g(f)$ is the set $\boldsymbol{R}$ of all real numbers, so here is an example of the highly unusual case in which $f(g)$ and $g(f)$ are the same function.

1.4.16: If $f(x) = \sqrt{x}$ and $g(x) = \cos x$, then

$$f(g(x)) = f(\cos x) = \sqrt{\cos x} \quad \text{and}$$

$$g(f(x)) = g\left(\sqrt{x}\right) = \cos\left(\sqrt{x}\right).$$

1.4.17: If $f(x) = \sin x$ and $g(x) = x^3$, then

$$f(g(x)) = f\left(x^3\right) = \sin\left(x^3\right) = \sin x^3 \quad \text{and}$$

$$g(f(x)) = g(\sin x) = (\sin x)^3 = \sin^3 x.$$

We note in passing that $\sin x^3$ and $\sin^3 x$ don't mean the same thing!

1.4.18: If $f(x) = \sin x$ and $g(x) = \cos x$, then $f(g(x)) = f(\cos x) = \sin(\cos x)$ and $g(f(x)) = g(\sin x) = \cos(\sin x)$.

1.4.19: If $f(x) = 1 + x^2$ and $g(x) = \tan x$, then $f(g(x)) = f(\tan x) = 1 + (\tan x)^2 = 1 + \tan^2 x$ and $g(f(x)) = g(1 + x^2) = \tan(1 + x^2)$.

1.4.20: If $f(x) = 1 - x^2$ and $g(x) = \sin x$, then

$$f(g(x)) = f(\sin x) = 1 - (\sin x)^2 = 1 - \sin^2 x = \cos^2 x \quad \text{and}$$

$$g(f(x)) = g(1 - x^2) = \sin(1 - x^2).$$

Note: The answers to Problems 21 through 30 are not unique. We have generally chosen the simplest and most natural answer.

1.4.21: $h(x) = (2 + 3x)^2 = (g(x))^k = f(g(x))$ where $f(x) = x^k$, $k = 2$, and $g(x) = 2 + 3x$.

1.4.22: $h(x) = (4 - x)^3 = (g(x))^3 = f(g(x))$ where $f(x) = x^k$, $k = 3$, and $g(x) = 4 - x$.

1.4.23: $h(x) = (2x - x^2)^{1/2} = (g(x))^{1/2} = f(g(x))$ where $f(x) = x^k$, $k = \frac{1}{2}$, and $g(x) = 2x - x^2$.

1.4.24: $h(x) = (1 + x^4)^{17} = (g(x))^{17} = f(g(x))$ where $f(x) = x^k$, $k = 17$, and $g(x) = 1 + x^4$.

1.4.25: $h(x) = (5 - x^2)^{3/2} = (g(x))^{3/2} = f(g(x))$ where $f(x) = x^k$, $k = \frac{3}{2}$, and $g(x) = 5 - x^2$.

1.4.26: $h(x) = \left[(4x - 6)^{1/3}\right]^4 = (4x - 6)^{4/3} = (g(x))^{4/3} = f(g(x))$ where $f(x) = x^k$, $k = \frac{4}{3}$, and $g(x) = 4x - 6$. Alternatively, $h(x) = (g(x))^4 = f(g(x))$ where $f(x) = x^k$, $k = 4$, and $g(x) = (4x - 6)^{1/3}$.

1.4.27: $h(x) = (x + 1)^{-1} = (g(x))^{-1} = f(g(x))$ where $f(x) = x^k$, $k = -1$, and $g(x) = x + 1$.

1.4.28: $h(x) = (1 + x^2)^{-1} = (g(x))^{-1} = f(g(x))$ where $f(x) = x^k$, $k = -1$, and $g(x) = 1 + x^2$.

1.4.29: $h(x) = (x + 10)^{-1/2} = (g(x))^{-1/2} = f(g(x))$ where $f(x) = x^k$, $k = -\frac{1}{2}$, and $g(x) = x + 10$.

1.4.30: $h(x) = (1 + x + x^2)^{-3} = (g(x))^{-3} = f(g(x))$ where $f(x) = -x^k$, $k = -3$, and $g(x) = 1 + x + x^2$.

1.4.31: Recommended window: $-2 \leqq x \leqq 2$. The graph makes it evident that the equation has exactly one solution (approximately 0.641186).

1.4.32: Recommended window: $-5 \leqq x \leqq 5$. The graph makes it evident that the equation has exactly three solutions (approximately -3.63796, -1.86236, and 0.88947).

1.4.33: Recommended window: $-5 \leqq x \leqq 5$. The graph makes it evident that the equation has exactly one solution (approximately 1.42773).

1.4.34: Recommended window: $-6 \leqq x \leqq 6$. The graph makes it evident that the equation has exactly three solutions (approximately -3.83747, -1.97738, and 1.30644).

1.4.35: Recommended window: $-8 \leqq x \leqq 8$. The graph makes it evident that the equation has exactly five solutions (approximately -4.08863, -1.83622, 1.37333, 5.65222, and 6.61597).

1.4.36: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly one solution (approximately 1.32432).

1.4.37: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly three solutions (approximately 1.41841, 5.55211, and 6.86308).

1.4.38: Recommended window: $-4 \leqq x \leqq 4$. The graph makes it evident that the equation has exactly two solutions (approximately ± 1.37936).

1.4.39: Recommended window: $-11 \leqq x \leqq 11$. The graph makes it evident that the equation has exactly six solutions (approximately -5.92454, -3.24723, 3.04852, 6.75738, 8.59387, and [exactly] 0).

1.4.40: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly six solutions (approximately 0.372968, 1.68831, 4.29331, 8.05637, 11.1288, and 13.6582).

1.4.41: Graphical methods show that the solution of $10 \cdot 2^t = 100$ is slightly less than 3.322. We began with the viewing window $0 \leqq t \leqq 6$ and gradually narrowed it to $3.321 \leqq t \leqq 3.323$.

1.4.42: Under the assumption that the interest is compounded continuously at a rate of 7.696% (for an annual yield of 8%), we solved the equation $5000 \cdot (1.07696)^t = 15000$ for $t \approx 14.8176$. We began with the viewing window $10 \leqq t \leqq 20$ and gradually narrowed it to $14.81762 \leqq y \leqq 14.81763$. Under the assumption that the interest is compounded yearly at an annual rate of 8%, we solved the equation $A(t) = 5000 \cdot (1.08)^t = 15000$ by evaluating $A(14) \approx 14686$ and $A(15) \approx 15861$. Thus in this case you'd have to wait a full 15 years for your money to triple.

1.4.43: Graphical methods show that the solution of $(67.4) \cdot (1.026)^t = 134.8$ is approximately 27.0046. We began with the viewing window $20 \leqq t \leqq 30$ and gradually narrowed it to $27.0045 \leqq t \leqq 27.0047$.

1.4.44: Graphical methods show that the solution of $A(t) = (0.9975)^t = 0.5$ is approximately 276.912. We began with the viewing window $200 \leqq t \leqq 300$ and gradually narrowed it to $276.910 \leqq t \leqq 276.914$.

1.4.45: Graphical methods show that the solution of $A(t) = 12 \cdot (0.975)^t = 1$ is approximately 98.149. We began with the viewing window $50 \leqq t \leqq 250$ and gradually narrowed it to $98.148 \leqq t \leqq 98.150$.

1.4.46: Graphical methods show that the negative solution of $x^2 = 2^x$ is approximately -0.76666. We began with the viewing window $-1 \leqq x \leqq 0$ and gradually narrowed it to $-0.7667 \leqq t \leqq -0.7666$.

1.4.47: We plotted $y = \log_{10} x$ and $y = \frac{1}{2}x^{1/5}$ simultaneously. We began with the viewing window $1 \leqq x \leqq 10$ and gradually narrowed it to $4.84890 \leqq x \leqq 4.84892$. Answer: $x \approx 4.84891$.

1.4.48: We began with the viewing window $-2 \leqq x \leqq 2$, which showed the two smaller solutions but not the larger solution. We first narrowed this window to $-0.9054 \leqq x \leqq -0.9052$ to get the first solution, $x \approx -0.9053$. We returned to the original window and narrowed it to $1.1324 \leqq x \leqq 1.1326$ to get the second solution, $x \approx 1.1325$. We looked for a solution in the window $20 \leqq x \leqq 30$ but there was none. But the exponential graph was still below the polynomial graph, so we checked the window $30 \leqq x \leqq 32$. A solution was evident, and we gradually narrowed this window to $31.3636 \leqq x \leqq 31.3638$ to discover the third solution, $x \approx 31.3637$.

Chapter 1 Miscellaneous Problems

1.M.1: The domain of $f(x) = \sqrt{x-4}$ is the set of real numbers x for which $x - 4 \geqq 0$; that is, the interval $[4, +\infty)$.

1.M.2: The domain of f consists of those real numbers x for which $2 - x \neq 0$; that is, the set of all real numbers other than 2.

1.M.3: The domain of f consists of those real numbers for which the denominator is nonzero; that is, the set of all real numbers other than ± 3.

1.M.4: Because $x^2 + 1$ is never zero, the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.M.5: If $x \geqq 0$, then $\sqrt{x}$ exists; there is no obstruction to adding 1 to $\sqrt{x}$ nor to cubing the sum. Hence the domain of f is the set $[0, +\infty)$ of all nonnegative real numbers.

1.M.6: Given:

$$f(x) = \frac{x+1}{x^2 - 2x}.$$

The only obstruction to computing the number $f(x)$ is the possibility that the denominator is zero. Thus we must eliminate from the set of all real numbers those for which $x^2 - 2x = 0$; that is, $x(x-2) = 0$. Therefore the domain of f is the set of all real numbers other than 0 and 2.

1.M.7: The function $f(x) = \sqrt{2 - 3x}$ is defined whenever the radicand is nonnegative; that is, whenever

$$2 - 3x \geqq 0;$$

$$3x \leqq 2;$$

$$x \leqq \tfrac{2}{3}.$$

Hence the domain of f is the interval $\left(-\infty, \tfrac{2}{3}\right]$.

1.M.8: In order that the square root is defined, we require $9 - x^2 \geqq 0$; we also need the denominator in $f(x)$ to be nonzero, so we further require that $9 - x^2 \neq 0$. Hence $9 - x^2 > 0$; that is, $x^2 < 9$, so that $-3 < x < 3$. Hence the domain of f is the open interval $(-3, 3)$.

1.M.9: Regardless of the value of x, it's always possible to subtract 2 from x, to subtract x from 4, and to multiply the results. Hence the domain of f is the set $\boldsymbol{R}$ of all real numbers.

1.M.10: The domain of f consists of those real numbers x for which $(x-2)(4-x)$ is nonnegative. That is, $x-2$ and $4-x$ are both positive, or $x-2$ and $4-x$ are both negative, or either is zero. First case: $x-2>0$ and $4-x>0$. Then $2<x<4$, so the interval $(2, 4)$ is part of the domain of f. Second case: $x-2<0$ and $4-x<0$. These inequalities imply that $x<2$ and $4<x$. No real numbers satisfy both these inequalities. So the second case contributes no numbers to the domain of f. Third case: $x-2=0$ or $4-x=0$. That is, $x=2$ or $x=4$. Therefore the domain of f is the closed interval $[2, 4]$.

1.M.11: Because $100 \leqq V \leqq 200$ and $p>0$, it follows that $100p \leqq pV \leqq 200p$. Because $pV=800$, we see that $100p \leqq 800 \leqq 200p$, so that $p \leqq 8 \leqq 2p$. That is, $p \leqq 8$ and $4 \leqq p$, so that $4 \leqq p \leqq 8$. This is the range of possible values of p.

1.M.12: If $70 \leqq F \leqq 90$, then $70 \leqq 32+\frac{9}{5}C \leqq 90$. Hence

$$70-32 \leqq \tfrac{9}{5}C \leqq 90-32;$$

$$38 \leqq \tfrac{9}{5}C \leqq 58;$$

$$190 \leqq 9C \leqq 290;$$

$$\tfrac{190}{9} \leqq C \leqq \tfrac{290}{9}.$$

Answer: The Celsius temperature ranged from a low of about 21.1°C to a high of about 32.2°C.

1.M.13: Because $25<R<50$, $25I<IR<50I$, so that

$$25I<E<50I;$$

$$25I<100<50I;$$

$$I<4<2I;$$

$$I<4 \quad \text{and} \quad 2<I.$$

Therefore the current I lies in the range $2<I<4$.

1.M.14: Because $3<L<4$, we see that

$$\frac{3}{32}<\frac{L}{32}<\frac{4}{32};$$

$$\sqrt{\frac{3}{32}}<\sqrt{\frac{L}{32}}<\sqrt{\frac{1}{8}};$$

$$2\pi\sqrt{\frac{3}{32}}<2\pi\sqrt{\frac{L}{32}}<2\pi\sqrt{\frac{1}{8}};$$

$$\frac{\pi}{2}\sqrt{\frac{3}{2}}<T<\pi\sqrt{\frac{1}{2}}.$$

In approximate terms, $1.923825 < T < 2.221441$.

1.M.15: If a cube has edge length x, then its volume is $V = x^3$ and its total surface area is $S = 6x^2$ (because each of its six faces has area x^2). Hence $x = \sqrt{S/6}$, and therefore

$$V(S) = \left(\sqrt{\frac{S}{6}}\right)^3 = \left(\frac{S}{6}\right)^{3/2}, \qquad 0 < S < +\infty.$$

Under certain circumstances it would be both permissible and desirable to let the domain of V be the interval $[0, +\infty)$.

1.M.16: Let r denote the radius, and h the height, of the cylinder. Then its volume V and total surface area A are given by

$$V = \pi r^2 h \qquad \text{and} \qquad A = 2\pi rh + 2\pi r^2$$

(look inside the front cover of the textbook). In this problem we are given $h = r$, so that $V = \pi r^3$ and $A = 4\pi r^2$. Therefore

$$r = \left(\frac{V}{\pi}\right)^{1/3} \qquad \text{and so} \qquad A = 4\pi\left(\frac{V}{\pi}\right)^{2/3}.$$

$$\text{Answer:} \quad A(V) = 4\pi\left(\frac{V}{\pi}\right)^{2/3}, \quad 0 < V < +\infty.$$

It is permissible, and sometimes desirable, to use instead the domain $0 \leqq V < +\infty$.

1.M.17: Suppose the given equilateral triangle has sides of length $2x$ and height (perpendicular to its base) of length h. Then the height of the equilateral triangle divides it into two right triangles, each of which has base x, height h, and hypotenuse $2x$. Hence application of the Pythagorean theorem to one of these right triangles gives

$$x^2 + h^2 = (2x)^2, \qquad \text{so that} \qquad h = x\sqrt{3}.$$

Now the area of the original right triangle is $A = hx$ and its perimeter is $P = 6x$. So

$$A = x^2\sqrt{3} \qquad \text{and} \qquad x = \frac{P}{6}.$$

Therefore $A(P) = \dfrac{P^2\sqrt{3}}{36}, \quad 0 < P < \infty.$

1.M.18: The square has perimeter x and thus edge length $y = \frac{1}{4}x$. The circle has circumference $100 - x$. Thus if z is the radius of the circle, then $2\pi z = 100 - x$, so that $z = (100 - x)/(2\pi)$. The area of the square is y^2 and the area of the circle is πz^2, so that the sum of the areas of the square and the circle is given by

$$A(x) = \frac{x^2}{16} + \pi\left(\frac{100 - x}{2\pi}\right)^2, \quad 0 < x < 100.$$

Looking ahead to Chapter 3, it will be advantageous to use the *closed* interval $[0, 100]$ for the domain of the function A.

1.M.19: The slope of L is $\dfrac{13-5}{1-(-3)} = 2$, so an equation of L is

$$y - 5 = 2(x+3); \quad \text{that is,} \quad y = 2x + 11.$$

1.M.20: An equation of L is $y - (-1) = -3(x-4)$; that is, $3x + y = 11$.

1.M.21: The point $(0, -5)$ lies on L, so an equation of L is

$$y - (-5) = \frac{1}{2}(x-0); \quad \text{alternatively,} \quad 2y + 10 = x.$$

1.M.22: The equation $3x - 2y = 4$ of the other line may be written in the form $y = \frac{3}{2}x - 2$, revealing that it and L have slope $\frac{3}{2}$. Hence an equation of L is

$$y - (-3) = \frac{3}{2}(x-2); \quad \text{that is,} \quad y = \frac{3}{2}x - 6.$$

1.M.23: The equation $y - 2x = 10$ may be written in the form $y = 2x + 10$, showing that it has slope 2. Hence the perpendicular line L has slope $-\frac{1}{2}$. Therefore an equation of L is

$$y - 7 = -\frac{1}{2}(x-(-3)); \quad \text{that is,} \quad x + 2y = 11.$$

1.M.24: The segment S joining $(1, -5)$ and $(3, -1)$ has slope $(-1-(-5))/(3-1) = 2$ and midpoint $(2, -3)$, and hence L has slope $-\frac{1}{2}$ and passes through $(2, -3)$. So an equation of L is

$$y - (-3) = -\frac{1}{2}(x-2); \quad \text{that is,} \quad x + 2y = -4.$$

1.M.25: The graph of $y = f(x) = 2 - 2x - x^2$ is a parabola opening downward. The only such graph is shown in Fig. 1.MP.6.

1.M.26: Given: $f(x) = x^3 - 4x^2 + 5$. Because $f(-1) = 0$, $f(1) = 2 > 0 > -3 = f(2)$, and $f(3) = -4 < 0 < 5 = f(4)$, the graph of f crosses the x-axis at $x = -1$, between $x = 1$ and $x = 2$, and between $x = 3$ and $x = 4$. Hence the graph of f is the one shown in Fig. 1.MP.9.

1.M.27: Given: $f(x) = x^4 - 4x^3 + 5$. Because the graph of f has no vertical asymptotes and because $f(x)$ approaches $+\infty$ as x approaches either $+\infty$ or $-\infty$, the graph of f must be the one shown in Fig. 1.MP.4.

1.M.28: Given:

$$f(x) = \frac{5}{x^2 - x - 6} = \frac{5}{(x-3)(x+2)}.$$

The denominator in $f(x)$ is zero when $x = 3$ and when $x = -2$ (and the numerator is not zero), so the graph of $y = f(x)$ has vertical asymptotes at $x = -2$ and at $x = 3$. Also $f(x)$ approaches zero as x approaches either $+\infty$ or $-\infty$. Therefore the graph of $y = f(x)$ must be the one shown in Fig. 1.MP.11.

1.M.29: Given:

$$f(x) = \frac{5}{x^2 - x + 6} = \frac{20}{4x^2 - 4x + 1 + 23} = \frac{20}{(2x-1)^2 + 23}.$$

The algebra displayed here shows that the denominator in $f(x)$ is never zero, so there are no vertical asymptotes. It also shows that the maximum value of $f(x)$ occurs when the denominator is minimal; that is, when $x = \frac{1}{2}$. Finally, $f(x)$ approaches zero as x approaches either $+\infty$ or $-\infty$. So the graph of $y = f(x)$ must be the one shown in Fig. 1.MP.3.

1.M.30: If $y = f(x) = \sqrt{8 + 2x - x^2}$, then

$$\begin{aligned}
&y^2 = 8 + 2x - x^2;\\
&x^2 - 2x + 1 + y^2 = 9;\\
&(x-1)^2 + (y-0)^2 = 3^2.
\end{aligned}$$

The last is the equation of a circle with center $(1, 0)$ and radius 3. But $y \geqq 0$, so the graph of f is the upper half of that circle, and it is shown in Fig. 1.MP.10.

1.M.31: Given: $f(x) = 2^{-x} - 1$. The graph of $y = 2^x$ is an increasing exponential function, so the graph of $y = 2^{-x}$ is a decreasing exponential function, approaching 0 as x approaches $+\infty$. So the graph of f approaches -1 as x approaches $+\infty$. Moreover, $f(0) = 0$. Therefore the graph of f is the one shown in Fig. 1.MP.7.

1.M.32: The graph of $f(x) = \log_{10}(x+1)$ is obtained from the graph of $g(x) = \log_{10} x$ by translation one unit to the left; note also that $f(0) = 0$. Therefore the graph of f is the one shown in Fig. 1.MP.2.

1.M.33: The graph of $y = 3 \sin x$ oscillates between its minimum value -3 and its maximum value 3, so the graph of $f(x) = 1 + 3 \sin x$ oscillates between -2 and 4. This graph is shown in Fig. 1.MP.8.

1.M.34: The graph of $f(x) = x + 3 \sin x$ viewed at a great distance resembles the graph of $y = x$. A closer view shows oscillations, due to the sine function, superposed on the graph of $y = x$. Thus the graph of f is the one shown in Fig. 1.MP.5.

1.M.35: The graph of $2x - 5y = 7$ is the straight line with x-intercept $\frac{7}{2}$ and y-intercept $-\frac{7}{5}$.

1.M.36: If $|x-y|=1$, then $x-y=1$ or $x-y=-1$. The graph of the first of these is the straight line $y=x-1$ with slope 1 and y-intercept -1; the graph of the second is the straight line $y=x+1$ with slope 1 and y-intercept 1. So the graph of $|x-y|=1$ consists of these two parallel lines.

1.M.37: We complete the square: $x^2-2x+1+y^2=1$, so that $(x-1)^2+(y-0)^2=1^2$. Thus the graph of the given equation is the circle with center $(1,\,0)$ and radius 1.

1.M.38: We complete the square in x and in y to obtain

$$\begin{aligned}&x^2+6x+9+y^2-4y+4=16;\\&(x+3)^2+(y-2)^2=4^2.\end{aligned}$$

Therefore the graph of the given equation is the circle with center $(-3,\,2)$ and radius 4.

1.M.39: The graph is a parabola opening upward. To find its vertex, we complete the square:

$$\begin{aligned}y&=2\left(x^2-2x-\tfrac{1}{2}\right)\\&=2\left(x^2-2x+1-\tfrac{3}{2}\right)=2(x-1)^2-3.\end{aligned}$$

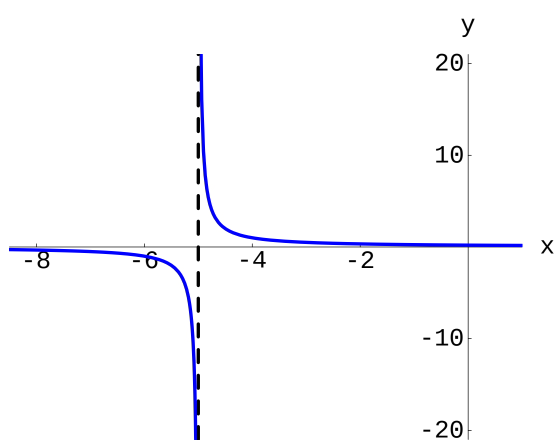
So the vertex of this parabola is at the point $(1,\,-3)$.

1.M.40: The graph is a parabola opening downward. To find its vertex, we complete the square:

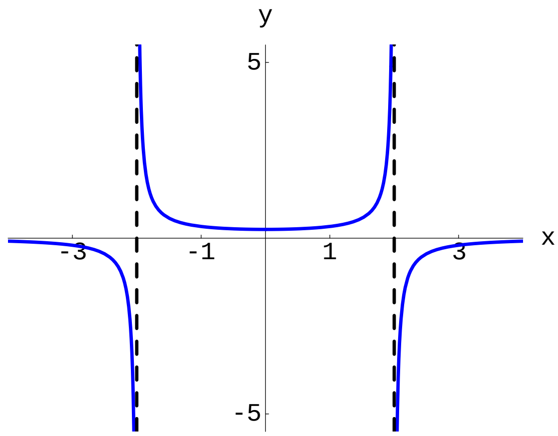
$$y=4x-x^2=-(x^2-4x)=-(x^2-4x+4-4)=4-(x-2)^2.$$

Thus the vertex of this parabola is at the point $(2,\,4)$.

1.M.41: The graph has a vertical asymptote at $x=-5$ and is shown next.

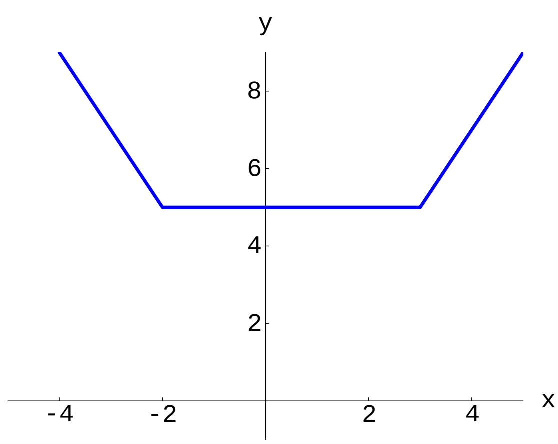


1.M.42: The graph has vertical asymptotes at $x=\pm 2$ and is shown next.



1.M.43: The graph of f is obtained by shifting the familiar absolute-value graph $y = |x|$ three units to the right, so that the graph of f has its "vertex" (or corner point) at the point $(3, 0)$.

1.M.44: Given: $f(x) = |x - 3| + |x + 2|$. If $x \geqq 3$ then $f(x) = x - 3 + x + 2 = 2x - 1$, so the graph is the unbounded line segment with slope 2 and endpoint $(3, 5)$ for $x \geqq 3$. If $-2 \leqq x \leqq 3$ then $f(x) = 3 - x + x + 2 = 5$, so another part of the graph is the horizontal line segment joining $(-2, 5)$ with $(3, 5)$. If $x \leqq -2$ then $f(x) = 3 - x - x - 2 = -2x + 1$, so the rest of the graph is the unbounded line segment with slope -2 and endpoint $(-2, 5)$ for $x \leqq -2$. The graph is shown next.



1.M.45: Suppose that a, b, and c are arbitrary real numbers. Then

$$|a + b + c| = |(a + b) + c| \leqq |a + b| + |c| \leqq |a| + |b| + |c| \,.$$

1.M.46: Suppose that a and b are arbitrary real numbers. Then $|a| = |(a - b) + b| \leqq |a - b| + |b|$. Therefore $|a| - |b| \leqq |a - b|$.

1.M.47: If $x - 3 > 0$ and $x + 2 > 0$, then $x > 3$ and $x > -2$, so $x > 3$. If $x - 3 < 0$ and $x + 2 < 0$, then $x < 3$ and $x < -2$, so $x < -2$. Answer: $(-\infty, -2) \cup (3, \infty)$.

1.M.48: $(x-1)(x-2)<0$: $x-1$ and $x-2$ have opposite signs, so either $x<1$ and $x>2$ (which leads to *no* values of x) or $x>1$ and $x<2$. Answer: $(1,\ 2)$.

1.M.49: $(x-4)(x+2)>0$: Either $x>4$ and $x>-2$ (so that $x>4$) or $x<4$ and $x<-2$ (so that $x<-2$). Answer: $(-\infty,\ -2)\cup(4,\ +\infty)$.

1.M.50: $2x \geqq 15-x^2$: $x^2+2x-15 \geqq 0$, so $(x-3)(x+5) \geqq 0$. Now $x+5>x-3$, so $x-3 \geqq 0$ or $x+5 \leqq 0$. Thus $x \geqq 3$ or $x \leqq -5$. Answer: $(-\infty,\ -5\,]\cup[\,3,\ +\infty)$.

1.M.51: The viewing window $-3 \leqq x \leqq 8$ shows a solution near -1 and another near 5. Gradual magnification of the region near -1 shows a solution between -1.1405 and -1.1395. Similarly, the other solution is between 6.1395 and 6.1405. So the solutions are approximately -1.140 and 6.140.

1.M.52: The viewing window $-2 \leqq x \leqq 5$ shows a solution near -1 and another near 4. To approximate the first more closely, we used the method of repeated tabulation on $[-1.0,\ -0.8]$, then on $[-0.88,\ -0,86]$, then on $[-0.872,\ -0.870]$. To approximate the second, we used the interval $[4.1,\ 4.3]$, then $[4.20,\ 4.22]$, then $[4.204,\ 4.206]$. To three places, the solutions are -0.872 and 4.205.

1.M.53: The viewing window $0.5 \leqq x \leqq 3$ shows one solution near 1.2 and another near 2.3. The method of repeated tabulation with successive intervals $[1.1,\ 1.3]$, $[1.18,\ 1.20]$, and $[1.190,\ 1.192]$ yields the approximation 1.191 to the first solution. The successive intervals $[2.2,\ 2.4]$, $[2.30,\ 2.32]$, and $[2.308,\ 2.310]$ yield the approximation 2.309 to the second solution.

1.M.54: The viewing window $-7 \leqq x \leqq 2$ shows one solution near -6 and another near 1. The method of repeated tabulation with successive intervals $[-6.1,\ -5.9]$, $[-5.98,\ -5.96]$, and $[-5.974,\ -5.970]$ yield the approximation -5.972 to the first solution. Similarly, we find the second solution to be approximately 1.172.

1.M.55: The viewing window $-6 \leqq x \leqq 2$ shows one solution near -5 and another near 1. The method of repeated tabulation with successive intervals $[-5.1,\ -4.9]$, $[-5.04,\ -5.02]$, and $[-5.022,\ -5.020]$, then with the intervals $[0.8,\ 1.0]$, $[0.88,\ 0.90]$, and $[0.896,\ 0.898]$, yields the two approximations -5.021 and 0.896 to the two solutions.

1.M.56: The viewing window $-11 \leqq x \leqq 3$ shows one solution near -10 and another near 1.7. The method of repeated tabulation, first with the intervals $[-10.0, -9.9]$, $[-9.97,\ -9.96]$, and $[-9.963,\ -9.962]$, then with $[1.7,\ 1.8]$, $[1.73,\ 1.74]$, and $[1.739,\ 1.741]$, yields the two approximations -9.962 and 1.740 to the two solutions.

1.M.57: The viewing window $2 \leqq x \leqq 3$ shows the low point with x-coordinate near 2.5. The method of repeated tabulation, using the successive intervals $[2.4,\ 2.6]$, $[2.48,\ 2.52]$, and $[2.496,\ 2.504]$, indicates that the low point is very close to $(2.5,\ 0.75)$.

1.M.58: The viewing window $-1 \leqq x \leqq 4$ shows the low point with x-coordinate near 1.7. The method of repeated tabulation, with the successive intervals [1.6, 1.8], [1.64, 1.68], and [1.664, 1.672], shows that the low point is quite close to (1.66, 2.67).

1.M.59: The viewing window $-0.5 \leqq x \leqq 4$ shows that the low point has x-coordinate near 1.8. The method of repeated tabulation, with the successive intervals [1.7, 1.9], [1.72, 1.78], and [1.744, 1.756], shows that the low point is very close to $(1.75, -1.25)$.

1.M.60: The viewing window $-5 \leqq x \leqq 1$ shows that the low point has x-coordinate near -2.5. The method of repeated tabulation indicates that the low point is very close to $(-2.4, 6.2)$.

1.M.61: The viewing window $-5 \leqq x \leqq 1$ show that the x-coordinate of the low point is close to -2. The method of repeated tabulation shows that the low point is very close to $(-2.0625, 0.96875)$.

1.M.62: The viewing window $-7 \leqq x \leqq 1$ shows that the x-coordinate of the low point is close to -4. The method of repeated tabulation indicates that the low point is very close to $(-4.111, 3.889)$.

1.M.63: The small rectangle has dimensions $10 - 4x$ by $7 - 2x$; $(7)(10) - (10 - 4x)(7 - 2x) = 20$, which leads to the quadratic equation $8x^2 - 48x + 20 = 0$. One solution of this equation is approximately 5.5495, which must be rejected; it is too large. The value of x is the other solution: $x \approx 0.4505$.

1.M.64: After shrinking, the tablecloth has dimensions $60 - x$ by $35 - x$. The area of this rectangle is 93% of the area of the original tablecloth, so $(60 - x)(35 - x) = (0.93)(35)(60)$. The larger solution of this quadratic equation is approximately 93.43, which we reject as too large. Answer: $x \approx 1.573$.

1.M.65: The viewing window $-4 \leqq x \leqq 4$ shows three solutions (and there can be no more).

1.M.66: The viewing window $-3 \leqq x \leqq 3$ shows two solutions, and there can be no more because $x^4 > |-3x^2 + 4x - 5|$ if $|x| > 3$.

1.M.67: We plotted $y = \sin x$ and $y = x^3 - 3x + 1$ simultaneously to see where they crossed. The viewing window $-2.2 \leqq x \leqq 2.2$ shows three solutions, and there can be no more because $|x^3 - 3x + 1| > 1$ if $|x| > 2.2$.

1.M.68: We plotted $y = \cos x$ and $y = x^4 - x$ simultaneously to see where they crossed. The viewing window $-2 \leqq x \leqq 2$ shows two solutions, and there can be no more because $x^4 - x > 1$ if $|x| > 2$.

1.M.69: We plotted $y = \cos x$ and $y = \log_{10} x$ simultaneously to see where they crossed. The viewing window $0.1 \leqq x \leqq 14$ shows three solutions, and there can be no more because $\log_{10} x < -1$ if $0 < x < 0.1$ and $\log_{10} x > 1$ if $x > 14$.

1.M.70: We plotted $y = 10^{-x}$ and $y = \log_{10} x$ simultaneously to see where they crossed. The viewing window $0.1 \leqq x \leqq 3$ shows one solution, and there can be no more because the exponential function is decreasing for all x and the logarithm function is increasing for all $x > 0$.