

# NOT FOR SALE

## CHAPTER 7 Applications of Integration

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# INSTRUCTOR USE ONLY

## CHAPTER 7

### Applications of Integration

#### Section 7.1 Area of a Region Between Two Curves

1.  $A = \int_0^6 [0 - (x^2 - 6x)] dx = -\int_0^6 (x^2 - 6x) dx$

2.  $A = \int_{-2}^2 [(2x + 5) - (x^2 + 2x + 1)] dx$   
 $= \int_{-2}^2 (-x^2 + 4) dx$

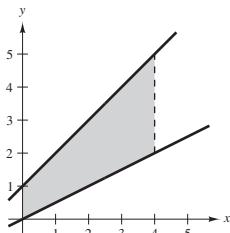
3.  $A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx$   
 $= \int_0^3 (-2x^2 + 6x) dx$

4.  $A = \int_0^1 (x^2 - x^3) dx$

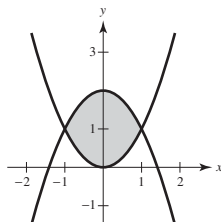
5.  $A = 2 \int_{-1}^0 3(x^3 - x) dx = 6 \int_{-1}^0 (x^3 - x) dx$   
 or  $-6 \int_0^1 (x^3 - x) dx$

6.  $A = 2 \int_0^1 [(x - 1)^3 - (x - 1)] dx$

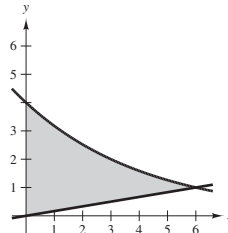
7.  $\int_0^4 \left[ (x + 1) - \frac{x}{2} \right] dx$



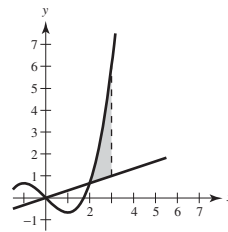
8.  $\int_{-1}^1 [(2 - x^2) - x^2] dx$



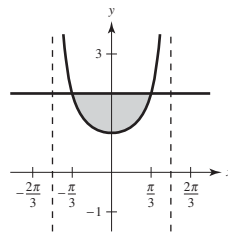
9.  $\int_0^6 \left[ 4(2^{-x/3}) - \frac{x}{6} \right] dx$



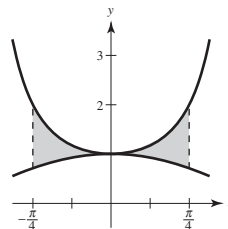
10.  $\int_2^3 \left[ \left( \frac{x^3}{3} - x \right) - \frac{x}{3} \right] dx$



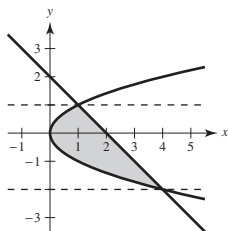
11.  $\int_{-\pi/3}^{\pi/3} (2 - \sec x) dx$



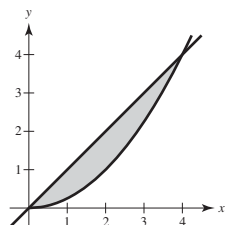
12.  $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x) dx$



13.  $\int_{-2}^1 [(2-y) - y^2] dy$

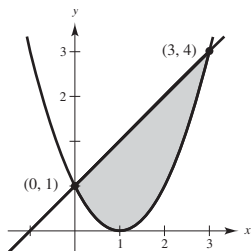


14.  $\int_0^4 (2\sqrt{y} - y) dy$



15.  $f(x) = x + 1$   
 $g(x) = (x - 1)^2$   
 $A \approx 4$

Matches (d)

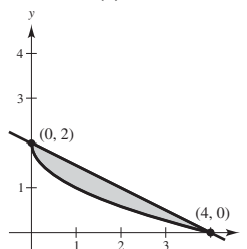


16.  $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

$A \approx 1$

Matches (a)



17. (a)  $x = 4 - y^2$

$x = y - 2$

$4 - y^2 = y - 2$

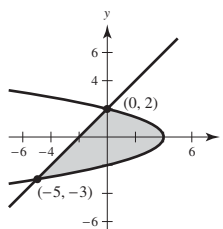
$y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

Intersection points:  $(0, 2)$  and  $(-5, -3)$

$$A = \int_{-5}^0 [(x + 2) + \sqrt{4 - x}] dx + \int_0^4 2\sqrt{4 - x} dx$$

$$= \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$



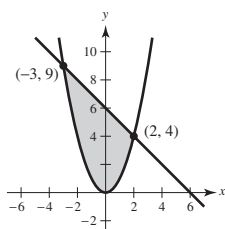
(b)  $A = \int_{-3}^2 [(4 - y^2) - (y - 2)] dy = \frac{125}{6}$

(c) The second method is simpler. Explanations will vary.

18. (a)  $y = x^2$  and  $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points:  $(2, 4)$  and  $(-3, 9)$

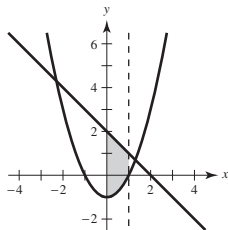


$A = \int_{-3}^2 [(6 - x) - x^2] dx = \frac{125}{6}$

(b)  $A = \int_0^4 2\sqrt{y} dy + \int_4^9 [(6 - y) + \sqrt{y}] dy = \frac{32}{3} + \frac{61}{6} = \frac{125}{6}$

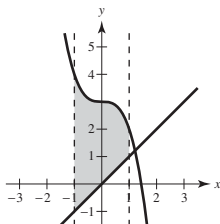
(c) The first method is simpler. Explanations will vary.

19.



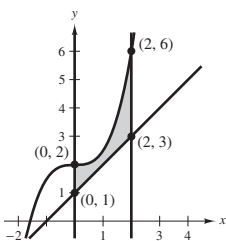
$$\begin{aligned}
 A &= \int_0^1 [(-x + 2) - (x^2 - 1)] dx \\
 &= \int_0^1 (-x^2 - x + 3) dx \\
 &= \left[ -\frac{x^3}{3} - \frac{x^2}{2} + 3x \right]_0^1 \\
 &= \left( -\frac{1}{3} - \frac{1}{2} + 3 \right) - 0 = \frac{13}{6}
 \end{aligned}$$

20.

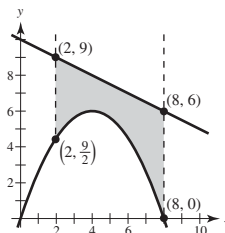


$$\begin{aligned}
 A &= \int_{-1}^1 [(-x^3 + 3) - x] dx \\
 &= \left[ -\frac{x^4}{4} + 3x - \frac{x^2}{2} \right]_{-1}^1 \\
 &= \left( -\frac{1}{4} + 3 - \frac{1}{2} \right) - \left( -\frac{1}{4} - 3 - \frac{1}{2} \right) = 6
 \end{aligned}$$

$$\begin{aligned}
 21. \quad A &= \int_0^2 \left[ \left( \frac{1}{2}x^3 + 2 \right) - (x + 1) \right] dx \\
 &= \int_0^2 \left( \frac{1}{2}x^3 - x + 1 \right) dx \\
 &= \left[ \frac{x^4}{8} - \frac{x^2}{2} + x \right]_0^2 \\
 &= \left( \frac{16}{8} - \frac{4}{2} + 2 \right) - 0 = 2
 \end{aligned}$$



$$\begin{aligned}
 22. \quad A &= \int_2^8 \left[ \left( 10 - \frac{1}{2}x \right) - \left( -\frac{3}{8}x(x - 8) \right) \right] dx \\
 &= \int_2^8 \left( \frac{3}{8}x^2 - \frac{7}{2}x + 10 \right) dx \\
 &= \left[ \frac{x^3}{8} - \frac{7x^2}{4} + 10x \right]_2^8 \\
 &= (64 - 112 + 80) - (1 - 7 + 20) = 18
 \end{aligned}$$

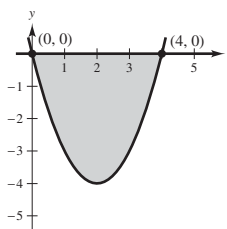


23. The points of intersection are given by:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0 \quad \text{when } x = 0, 4$$

$$\begin{aligned}
 A &= \int_0^4 [g(x) - f(x)] dx \\
 &= -\int_0^4 (x^2 - 4x) dx = -\left[ \frac{x^3}{3} - 2x^2 \right]_0^4 = \frac{32}{3}
 \end{aligned}$$



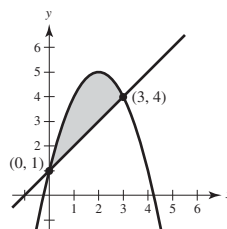
24. The points of intersection are given by:

$$-x^2 + 4x + 1 = x + 1$$

$$-x^2 + 3x = 0$$

$$x^2 = 3x \quad \text{when } x = 0, 3$$

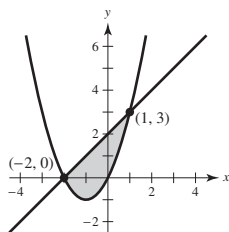
$$\begin{aligned}
 A &= \int_0^3 [(-x^2 + 4x + 1) - (x + 1)] dx \\
 &= \int_0^3 (-x^2 + 3x) dx \\
 &= \left[ -\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = -9 + \frac{27}{2} = \frac{9}{2}
 \end{aligned}$$





25. The points of intersection are given by:

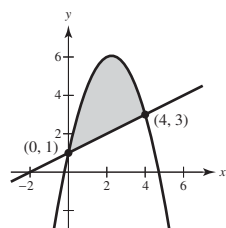
$$\begin{aligned}x^2 + 2x &= x + 2 \\x^2 + x - 2 &= 0 \\(x + 2)(x - 1) &= 0 \quad \text{when } x = -2, 1\end{aligned}$$



$$\begin{aligned}A &= \int_{-2}^1 [g(x) - f(x)] dx \\&= \int_{-2}^1 [(x + 2) - (x^2 + 2x)] dx \\&= \left[ \frac{-x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\&= \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) - \left( \frac{8}{3} - 2 - 4 \right) = \frac{9}{2}\end{aligned}$$

26. The points of intersection are given by:

$$\begin{aligned}-x^2 + \frac{9}{2}x + 1 &= \frac{1}{2}x + 1 \\x^2 - 4x &= 0 \quad \text{when } x = 0, 4\end{aligned}$$

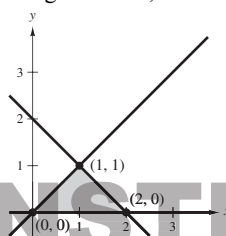


$$\begin{aligned}A &= \int_0^4 \left[ \left( -x^2 + \frac{9}{2}x + 1 \right) - \left( \frac{1}{2}x + 1 \right) \right] dx \\&= \int_0^4 (-x^2 + 4x) dx \\&= \left[ \frac{-x^3}{3} + 2x^2 \right]_0^4 = \frac{-64}{3} + 32 = \frac{32}{3}\end{aligned}$$

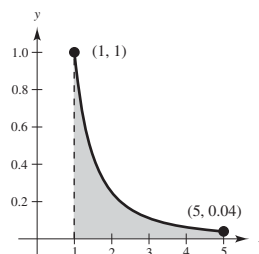
27. The points of intersection are given by:

$$\begin{aligned}x &= 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0 \\x &= 1 \quad \quad \quad x = 0 \quad \quad \quad x = 2 \\A &= \int_0^1 [(2 - y) - (y)] dy = [2y - y^2]_0^1 = 1\end{aligned}$$

Note that if you integrate with respect to  $x$ , you need two integrals. Also, note that the region is a triangle.



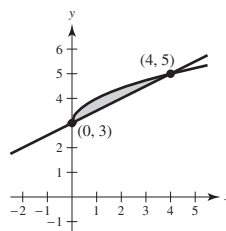
28.  $A = \int_1^5 \left( \frac{1}{x^2} - 0 \right) dx = \left[ -\frac{1}{x} \right]_1^5 = \frac{4}{5}$



29. The points of intersection are given by:

$$\begin{aligned}\sqrt{x} + 3 &= \frac{1}{2}x + 3 \\\sqrt{x} &= \frac{1}{2}x \\x &= \frac{x^2}{4} \quad \text{when } x = 0, 4\end{aligned}$$

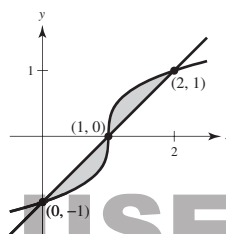
$$\begin{aligned}A &= \int_0^4 \left[ \left( \sqrt{x} + 3 \right) - \left( \frac{1}{2}x + 3 \right) \right] dx \\&= \left[ \frac{2}{3}x^{3/2} - \frac{x^2}{4} \right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}\end{aligned}$$



30. The points of intersection are given by:

$$\begin{aligned}\sqrt[3]{x-1} &= x-1 \\x-1 &= (x-1)^3 = x^3 - 3x^2 + 3x - 1 \\x^3 - 3x^2 + 2x &= 0 \\x(x^2 - 3x + 2) &= 0 \\x(x-2)(x-1) &= 0 \quad \text{when } x = 0, 1, 2\end{aligned}$$

$$\begin{aligned}A &= 2 \int_0^1 [(x-1) - \sqrt[3]{x-1}] dx \\&= 2 \left[ \frac{x^2}{2} - x - \frac{3}{4}(x-1)^{4/3} \right]_0^1 \\&= 2 \left[ \left( \frac{1}{2} - 1 - 0 \right) - \left( -\frac{3}{4} \right) \right] = \frac{1}{2}\end{aligned}$$



31. The points of intersection are given by:

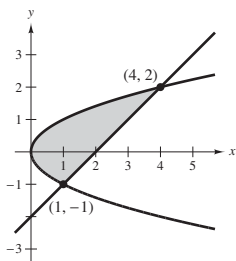
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$A = \int_{-1}^2 [g(y) - f(y)] dy$$

$$= \int_{-1}^2 [(y + 2) - y^2] dy$$

$$= \left[ 2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



32. The points of intersection are given by:

$$2y - y^2 = -y$$

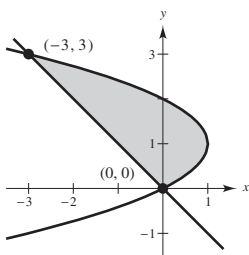
$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

$$A = \int_0^3 [f(y) - g(y)] dy$$

$$= \int_0^3 [(2y - y^2) - (-y)] dy$$

$$= \int_0^3 (3y - y^2) dy$$

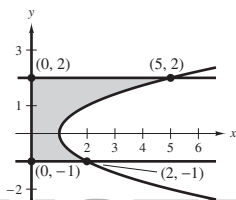
$$= \left[ \frac{3}{2}y^2 - \frac{1}{3}y^3 \right]_0^3 = \frac{9}{2}$$



$$33. A = \int_{-1}^2 [f(y) - g(y)] dy$$

$$= \int_{-1}^2 [(y^2 + 1) - 0] dy$$

$$= \left[ \frac{y^3}{3} + y \right]_{-1}^2 = 6$$

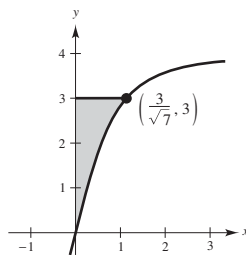


$$34. A = \int_0^3 [f(y) - g(y)] dy$$

$$= \int_0^3 \left[ \frac{y}{\sqrt{16 - y^2}} - 0 \right] dy$$

$$= -\frac{1}{2} \int_0^3 (16 - y^2)^{-1/2} (-2y) dy$$

$$= \left[ -\sqrt{16 - y^2} \right]_0^3 = 4 - \sqrt{7} \approx 1.354$$



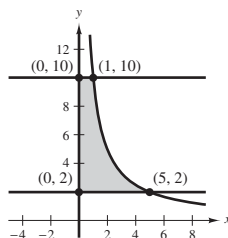
$$35. y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$$

$$A = \int_2^{10} \frac{10}{y} dy$$

$$= [10 \ln y]_2^{10}$$

$$= 10(\ln 10 - \ln 2)$$

$$= 10 \ln 5 \approx 16.0944$$

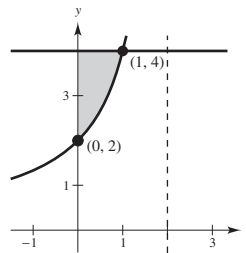


$$36. A = \int_0^1 \left( 4 - \frac{4}{2 - x} \right) dx$$

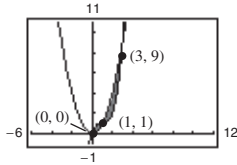
$$= [4x + 4 \ln |2 - x|]_0^1$$

$$= 4 - 4 \ln 2$$

$$\approx 1.227$$



37. (a)



(b) The points of intersection are given by:

$$x^3 - 3x^2 + 3x = x^2$$

$$x(x - 1)(x - 3) = 0 \quad \text{when } x = 0, 1, 3$$

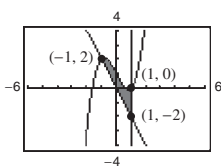
$$A = \int_0^1 [f(x) - g(x)] dx + \int_1^3 [g(x) - f(x)] dx$$

$$= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2] dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)] dx$$

$$= \int_0^1 (x^3 - 4x^2 + 3x) dx + \int_1^3 (-x^3 + 4x^2 - 3x) dx = \left[ \frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 + \left[ -\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2 \right]_1^3 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

(c) Numerical approximation:  $0.417 + 2.667 \approx 3.083$

38. (a)



(b) The point of intersection is given by:

$$x^3 - 2x + 1 = -2x$$

$$x^3 + 1 = 0 \quad \text{when } x = -1$$

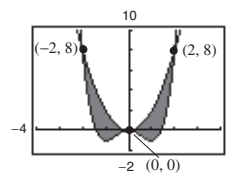
$$A = \int_{-1}^1 [f(x) - g(x)] dx$$

$$= \int_{-1}^1 [(x^3 - 2x + 1) - (-2x)] dx$$

$$= \int_{-1}^1 (x^3 + 1) dx = \left[ \frac{x^4}{4} + x \right]_{-1}^1 = 2$$

(c) Numerical approximation: 2.0

40. (a)



(b) The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$

$$x^2(x^2 - 4) = 0 \quad \text{when } x = 0, \pm 2$$

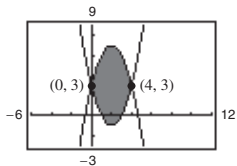
$$A = 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx$$

$$= 2 \int_0^2 (4x^2 - x^4) dx$$

$$= 2 \left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

39. (a)



(b) The points of intersection are given by:

$$x^2 - 4x + 3 = 3 + 4x - x^2$$

$$2x(x - 4) = 0 \quad \text{when } x = 0, 4$$

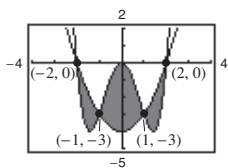
$$A = \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)] dx$$

$$= \int_0^4 (-2x^2 + 8x) dx$$

$$= \left[ -\frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{64}{3}$$

(c) Numerical approximation: 21.333

41. (a)  $f(x) = x^4 - 4x^2$ ,  $g(x) = x^2 - 4$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

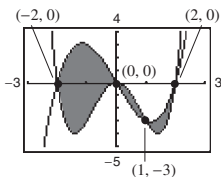
By symmetry:

$$\begin{aligned} A &= 2 \int_0^1 [(x^4 - 4x^2) - (x^2 - 4)] dx + 2 \int_1^2 [(x^2 - 4) - (x^4 - 4x^2)] dx \\ &= 2 \int_0^1 (x^4 - 5x^2 + 4) dx + 2 \int_1^2 (-x^4 + 5x^2 - 4) dx \\ &= 2 \left[ \frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 + 2 \left[ -\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2 \\ &= 2 \left[ \frac{1}{5} - \frac{5}{3} + 4 \right] + 2 \left[ \left( -\frac{32}{5} + \frac{40}{3} - 8 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) \right] = 8 \end{aligned}$$

(c) Numerical approximation:

$$5.067 + 2.933 = 8.0$$

42. (a)  $f(x) = x^4 - 4x^2$ ,  $g(x) = x^3 - 4x$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^3 - 4x$$

$$x^4 - x^3 - 4x^2 + 4x = 0$$

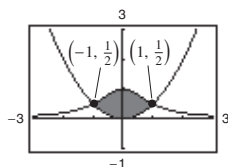
$$x(x-1)(x+2)(x-2) = 0 \quad \text{when } x = -2, 0, 1, 2$$

$$\begin{aligned} A &= \int_{-2}^0 [(x^3 - 4x) - (x^4 - 4x^2)] dx + \int_0^1 [(x^4 - 4x^2) - (x^3 - 4x)] dx + \int_1^2 [(x^3 - 4x) - (x^4 - 4x^2)] dx \\ &= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30} \end{aligned}$$

(c) Numerical approximation:

$$8.267 + 0.617 + 0.883 \approx 9.767$$

43. (a)

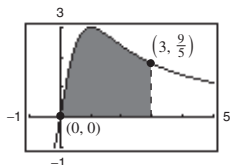


(b) The points of intersection are given by:

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{x^2}{2} \\ x^4 + x^2 - 2 &= 0 \\ (x^2 + 2)(x^2 - 1) &= 0 \\ x &= \pm 1 \\ A &= 2 \int_0^1 [f(x) - g(x)] dx \\ &= 2 \int_0^1 \left[ \frac{1}{1+x^2} - \frac{x^2}{2} \right] dx \\ &= 2 \left[ \arctan x - \frac{x^3}{6} \right]_0^1 \\ &= 2 \left( \frac{\pi}{4} - \frac{1}{6} \right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237\end{aligned}$$

(c) Numerical approximation: 1.237

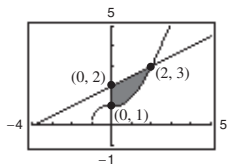
44. (a)



$$\begin{aligned}(b) \quad A &= \int_0^3 \left[ \frac{6x}{x^2 + 1} - 0 \right] dx \\ &= \left[ 3 \ln(x^2 + 1) \right]_0^3 \\ &= 3 \ln 10 \\ &\approx 6.908\end{aligned}$$

(c) Numerical approximation: 6.908

45. (a)

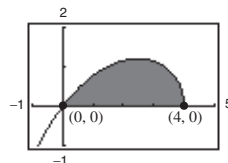


(b) and (c)  $\sqrt{1+x^3} \leq \frac{1}{2}x + 2$  on  $[0, 2]$

You must use numerical integration because  $y = \sqrt{1+x^3}$  does not have an elementary antiderivative.

$$A = \int_0^2 \left[ \frac{1}{2}x + 2 - \sqrt{1+x^3} \right] dx \approx 1.759$$

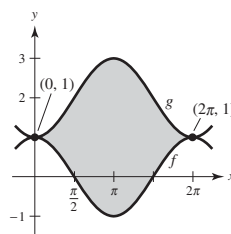
46. (a)



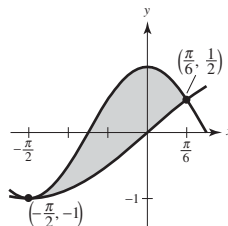
(b) and (c) You must use numerical integration:

$$A = \int_0^4 x \sqrt{\frac{4-x}{4+x}} dx \approx 3.434$$

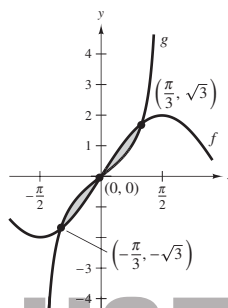
$$\begin{aligned}47. \quad A &= \int_0^{2\pi} [(2 - \cos x) - \cos x] dx \\ &= 2 \int_0^{2\pi} (1 - \cos x) dx \\ &= 2[x - \sin x]_0^{2\pi} = 4\pi \approx 12.566\end{aligned}$$



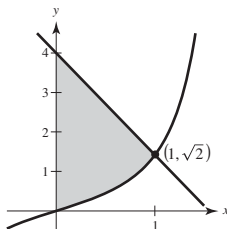
$$\begin{aligned}48. \quad A &= \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x) dx \\ &= \left[ \frac{1}{2} \sin 2x + \cos x \right]_{-\pi/2}^{\pi/6} \\ &= \left( \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} \right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299\end{aligned}$$



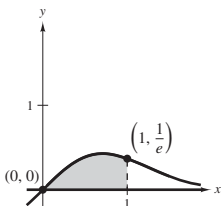
$$\begin{aligned}49. \quad A &= 2 \int_0^{\pi/3} [f(x) - g(x)] dx \\ &= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx \\ &= 2[-2 \cos x + \ln|\cos x|]_0^{\pi/3} = 2(1 - \ln 2) \approx 0.614\end{aligned}$$



$$\begin{aligned}
 50. \quad A &= \int_0^1 \left[ (\sqrt{2} - 4)x + 4 - \sec \frac{\pi x}{4} \tan \frac{\pi x}{4} \right] dx \\
 &= \left[ \frac{\sqrt{2} - 4}{2} x^2 + 4x - \frac{4}{\pi} \sec \frac{\pi x}{4} \right]_0^1 \\
 &= \left( \frac{\sqrt{2} - 4}{2} + 4 - \frac{4}{\pi} \sqrt{2} \right) - \left( -\frac{4}{\pi} \right) \\
 &= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi} (1 - \sqrt{2}) \approx 2.1797
 \end{aligned}$$

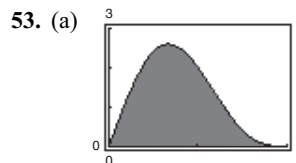
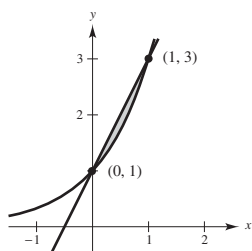


$$\begin{aligned}
 51. \quad A &= \int_0^1 [xe^{-x^2} - 0] dx \\
 &= \left[ -\frac{1}{2} e^{-x^2} \right]_0^1 = \frac{1}{2} \left( 1 - \frac{1}{e} \right) \approx 0.316
 \end{aligned}$$



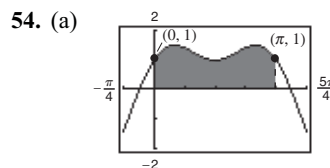
52. From the graph we see that  $f$  and  $g$  intersect twice at  $x = 0$  and  $x = 1$ .

$$\begin{aligned}
 A &= \int_0^1 [g(x) - f(x)] dx \\
 &= \int_0^1 [(2x + 1) - 3^x] dx \\
 &= \left[ x^2 + x - \frac{1}{\ln 3} (3^x) \right]_0^1 \\
 &= 2 \left( 1 - \frac{1}{\ln 3} \right) \approx 0.180
 \end{aligned}$$



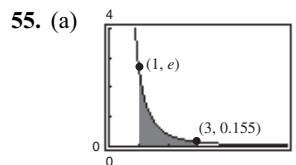
$$\begin{aligned}
 53. \quad (b) \quad A &= \int_0^\pi (2 \sin x + \sin 2x) dx \\
 &= \left[ -2 \cos x - \frac{1}{2} \cos 2x \right]_0^\pi \\
 &= \left( 2 - \frac{1}{2} \right) - \left( -2 - \frac{1}{2} \right) = 4
 \end{aligned}$$

(c) Numerical approximation: 4.0



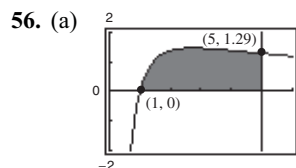
$$\begin{aligned}
 54. \quad (b) \quad A &= \int_0^\pi (2 \sin x + \cos 2x) dx \\
 &= \left[ -2 \cos x + \frac{1}{2} \sin 2x \right]_0^\pi = 4
 \end{aligned}$$

(c) Numerical approximation: 4



$$\begin{aligned}
 55. \quad (b) \quad A &= \int_1^3 \frac{1}{x^2} e^{1/x} dx \\
 &= \left[ -e^{-1/x} \right]_1^3 \\
 &= e - e^{1/3}
 \end{aligned}$$

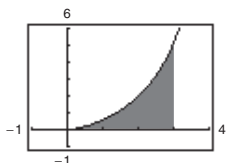
(c) Numerical approximation: 1.323



$$\begin{aligned}
 56. \quad (b) \quad A &= \int_1^5 \frac{4 \ln x}{x} dx \\
 &= \left[ 2(\ln x)^2 \right]_1^5 \\
 &= 2(\ln 5)^2
 \end{aligned}$$

(c) Numerical approximation: 5.181

57. (a)



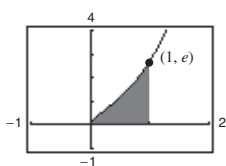
(b) The integral

$$A = \int_0^3 \sqrt{\frac{x^3}{4-x}} dx$$

does not have an elementary antiderivative.

(c)  $A \approx 4.7721$

58. (a)



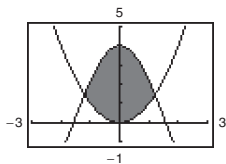
(b) The integral

$$A = \int_0^1 \sqrt{x} e^x dx$$

does not have an elementary antiderivative.

(c) 1.2556

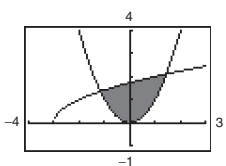
59. (a)



(b) The intersection points are difficult to determine by hand.

(c) Area =  $\int_{-c}^c [4 \cos x - x^2] dx \approx 6.3043$  where  $c \approx 1.201538$ .

60. (a)



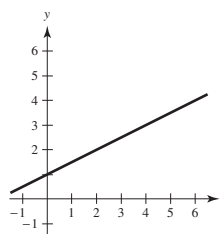
(b) The intersection points are difficult to determine.

(c) Intersection points:  $(-1.164035, 1.3549778)$  and  $(1.4526269, 2.1101248)$

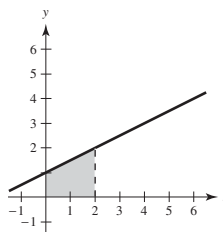
$$A = \int_{-1.164035}^{1.4526269} [\sqrt{3+x} - x^2] dx \approx 3.0578$$

$$61. F(x) = \int_0^x \left( \frac{1}{2}t + 1 \right) dt = \left[ \frac{t^2}{4} + t \right]_0^x = \frac{x^2}{4} + x$$

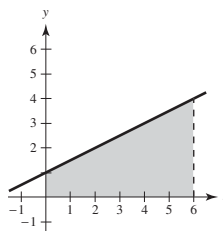
(a)  $F(0) = 0$



(b)  $F(2) = \frac{2^2}{4} + 2 = 3$

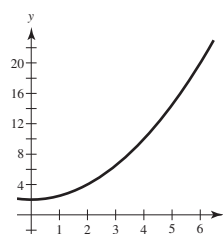


(c)  $F(6) = \frac{6^2}{4} + 6 = 15$

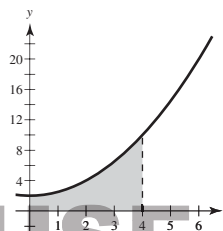


$$62. F(x) = \int_0^x \left( \frac{1}{2}t^2 + 2 \right) dt = \left[ \frac{1}{6}t^3 + 2t \right]_0^x = \frac{x^3}{6} + 2x$$

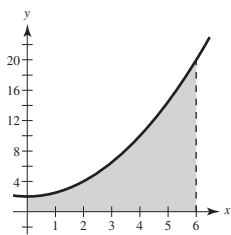
(a)  $F(0) = 0$



(b)  $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

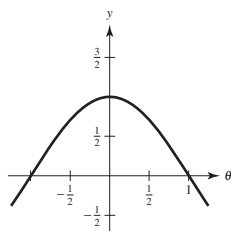


(c)  $F(6) = 36 + 12 = 48$

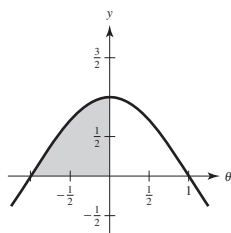


63.  $F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta = \left[ \frac{2}{\pi} \sin \frac{\pi\theta}{2} \right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$

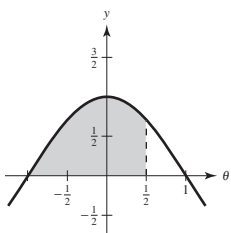
(a)  $F(-1) = 0$



(b)  $F(0) = \frac{2}{\pi} \approx 0.6366$

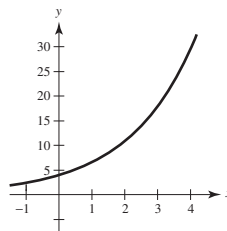


(c)  $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

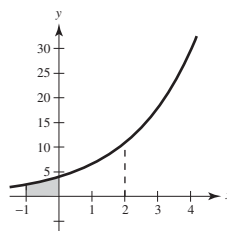


64.  $F(y) = \int_{-1}^y 4e^{x/2} dx = \left[ 8e^{x/2} \right]_{-1}^y = 8e^{y/2} - 8e^{-1/2}$

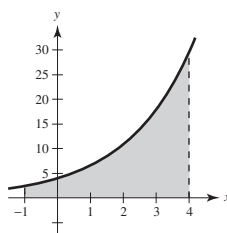
(a)  $F(-1) = 0$



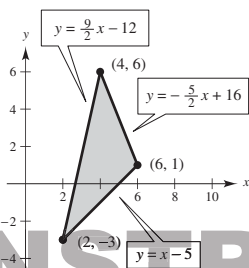
(b)  $F(0) = 8 - 8e^{-1/2} \approx 3.1478$



(c)  $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$

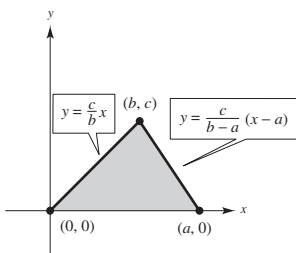


$$\begin{aligned}
 65. \quad A &= \int_2^4 \left[ \left( \frac{9}{2}x - 12 \right) - (x - 5) \right] dx + \int_4^6 \left[ \left( -\frac{5}{2}x + 16 \right) - (x - 5) \right] dx \\
 &= \int_2^4 \left( \frac{7}{2}x - 7 \right) dx + \int_4^6 \left( -\frac{7}{2}x + 21 \right) dx = \left[ \frac{7}{4}x^2 - 7x \right]_2^4 + \left[ -\frac{7}{4}x^2 + 21x \right]_4^6 = 7 + 7 = 14
 \end{aligned}$$

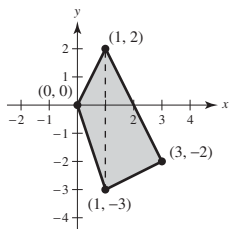




$$\begin{aligned}
 66. \quad A &= \int_0^c \left[ \left( \frac{b-a}{c}y + a \right) - \frac{b}{c}y \right] dy \\
 &= \int_0^c \left( -\frac{a}{c}y + a \right) dy \\
 &= \left[ -\frac{a}{2c}y^2 + ay \right]_0^c \\
 &= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left( = \frac{1}{2}(\text{base})(\text{height}) \right)
 \end{aligned}$$



$$\begin{aligned}
 68. \quad A &= \int_0^1 [2x - (-3x)] dx + \int_1^3 \left[ (-2x + 4) - \left( \frac{1}{2}x - \frac{7}{2} \right) \right] dx \\
 &= \int_0^1 5x dx + \int_1^3 \left( -\frac{5}{2}x + \frac{15}{2} \right) dx \\
 &= \left[ \frac{5x^2}{2} \right]_0^1 + \left[ -\frac{5x^2}{4} + \frac{15}{2}x \right]_1^3 \\
 &= \frac{5}{2} + \left( -\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2} \right) \\
 &= \frac{15}{2}
 \end{aligned}$$



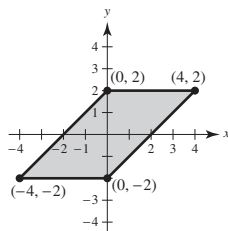
69. Answers will vary. *Sample answer:* If you let  $\Delta x = 6$  and  $n = 10$ ,  $b - a = 10(6) = 60$ .

$$\begin{aligned}
 \text{(a) Area} &\approx \frac{60}{2(10)} [0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0] = 3[322] = 966 \text{ ft}^2 \\
 \text{(b) Area} &\approx \frac{60}{3(10)} [0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0] = 2[502] = 1004 \text{ ft}^2
 \end{aligned}$$

70. Answers will vary. *Sample answer:*  $\Delta x = 4$ ,  $n = 8$ ,  $b - a = (8)(4) = 32$

$$\begin{aligned}
 \text{(a) Area} &\approx \frac{32}{2(8)} [0 + 2(11) + 2(13.5) + 2(14.2) + 2(14) + 2(14.2) + 2(15) + 2(13.5) + 0] \\
 &= 2[190.8] = 381.6 \text{ mi}^2 \\
 \text{(b) Area} &\approx \frac{32}{3(8)} [0 + 4(11) + 2(13.5) + 4(14.2) + 2(14) + 4(14.2) + 2(15) + 4(13.5) + 0] \\
 &= \frac{4}{3}[296.6] = 395.5 \text{ mi}^2
 \end{aligned}$$

67.



Left boundary line:  $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line:  $y = x - 2 \Leftrightarrow x = y + 2$

$$\begin{aligned}
 A &= \int_{-2}^2 [(y + 2) - (y - 2)] dy \\
 &= \int_{-2}^2 4 dy = [4y]_{-2}^2 = 8 - (-8) = 16
 \end{aligned}$$

71. On the interval  $\left[0, \frac{\pi}{4}\right]$ , the points of intersection of  $y_1 = \sin 2x$  and  $y_2 = \cos 4x$  are given by:

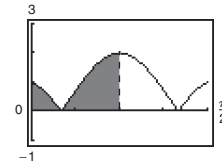
$$\sin 2x = \cos 4x = 1 - 2 \sin^2 2x$$

$$2 \sin^2 2x + \sin 2x - 1 = 0$$

$$(2 \sin 2x - 1)(\sin 2x + 1) = 0$$

$$\sin 2x = \frac{1}{2} \quad \text{when } x = \frac{\pi}{12}$$

$$\begin{aligned} \int_0^{\pi/4} |\sin 2x - \cos 4x| dx &= \int_0^{\pi/12} (\cos 4x - \sin 2x) dx + \int_{\pi/12}^{\pi/4} (\sin 2x - \cos 4x) dx \\ &= \left[ \frac{\sin 4x}{4} + \frac{\cos 2x}{2} \right]_0^{\pi/12} + \left[ -\frac{\cos 2x}{2} - \frac{\sin 4x}{4} \right]_{\pi/12}^{\pi/4} \\ &= \left[ \left( \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} \right) - \frac{1}{2} \right] + \left[ 0 + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} \right] = \frac{3\sqrt{3}}{4} - \frac{1}{2} \approx 0.7990 \end{aligned}$$



72. On the interval  $[0, 2]$ , the points of intersection of  $y_1 = \sqrt{x+3}$  and  $y_2 = 2x$  are given by:

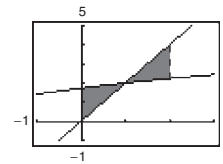
$$\sqrt{x+3} = 2x$$

$$x+3 = 4x^2$$

$$4x^2 - x - 3 = 0$$

$$(x-1)(4x+3) = 0 \quad \text{when } x = 1$$

$$\begin{aligned} \int_0^2 |\sqrt{x+3} - 2x| dx &= \int_0^1 (\sqrt{x+3} - 2x) dx + \int_1^2 (2x - \sqrt{x+3}) dx \\ &= \left[ \frac{2}{3}(x+3)^{3/2} - x^2 \right]_0^1 + \left[ x^2 - \frac{2}{3}(x+3)^{3/2} \right]_1^2 \\ &= \left[ \left( \frac{16}{3} - 1 \right) - \frac{2}{3}(3)^{3/2} \right] + \left[ \left( 4 - \frac{2}{3}(5)^{3/2} \right) - \left( 1 - \frac{16}{3} \right) \right] = \frac{38}{3} - 2\sqrt{3} - \frac{10}{3}\sqrt{5} \approx 1.749 \end{aligned}$$



73.  $f(x) = x^3$

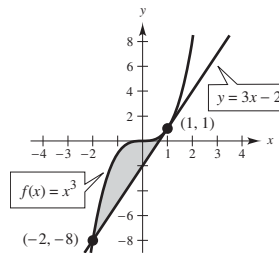
$$f'(x) = 3x^2$$

$$\text{At } (1, 1), f'(1) = 3.$$

$$\text{Tangent line: } y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The tangent line intersects  $f(x) = x^3$  at  $x = -2$ .

$$A = \int_{-2}^1 [x^3 - (3x - 2)] dx = \left[ \frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{27}{4}$$



74.  $y = x^3 - 2x, \quad (-1, 1)$

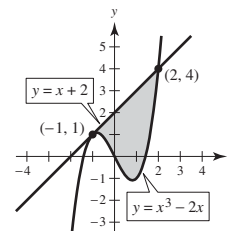
$$y' = 3x^2 - 2$$

$$y'(-1) = 3 - 2 = 1$$

$$\text{Tangent line: } y - 1 = 1(x + 1) \Rightarrow y = x + 2$$

Intersection points:  $(-1, 1)$  and  $(2, 4)$

$$\begin{aligned} A &= \int_{-1}^2 [(x+2) - (x^3 - 2x)] dx = \int_{-1}^2 (-x^3 + 3x + 2) dx \\ &= \left[ -\frac{x^4}{4} + \frac{3x^2}{2} + 2x \right]_{-1}^2 = \left[ (-4 + 6 + 4) - \left( -\frac{1}{4} + \frac{3}{2} - 2 \right) \right] = \frac{27}{4} \end{aligned}$$

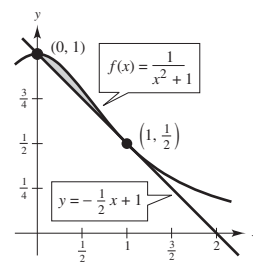


75.  $f(x) = \frac{1}{x^2 + 1}$   
 $f'(x) = -\frac{2x}{(x^2 + 1)^2}$

At  $\left(1, \frac{1}{2}\right)$ ,  $f'(1) = -\frac{1}{2}$ .

Tangent line:  $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$  or  $y = -\frac{1}{2}x + 1$

The tangent line intersects  $f(x) = \frac{1}{x^2 + 1}$  at  $x = 0$ .



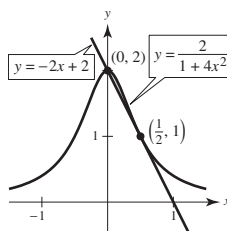
$$A = \int_0^1 \left[ \frac{1}{x^2 + 1} - \left( -\frac{1}{2}x + 1 \right) \right] dx = \left[ \arctan x + \frac{x^2}{4} - x \right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$

76.  $y = \frac{2}{1 + 4x^2}$ ,  $\left(\frac{1}{2}, 1\right)$

$$y' = \frac{-16x}{(1 + 4x^2)^2}$$

$$y'\left(\frac{1}{2}\right) = \frac{-8}{2^2} = -2$$

Tangent line:  $y - 1 = -2\left(x - \frac{1}{2}\right)$   
 $y = -2x + 2$

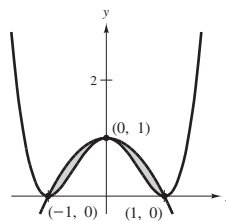


Intersection points:  $\left(\frac{1}{2}, 1\right)$ ,  $(0, 2)$

$$A = \int_0^{1/2} \left[ \frac{2}{1 + 4x^2} - (-2x + 2) \right] dx = \left[ \arctan(2x) + x^2 - 2x \right]_0^{1/2} = \arctan(1) + \frac{1}{4} - 1 = \frac{\pi}{4} - \frac{3}{4} \approx 0.0354$$

77.  $x^4 - 2x^2 + 1 \leq 1 - x^2$  on  $[-1, 1]$

$$\begin{aligned} A &= \int_{-1}^1 \left[ (1 - x^2) - (x^4 - 2x^2 + 1) \right] dx \\ &= \int_{-1}^1 (x^2 - x^4) dx \\ &= \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = \frac{4}{15} \end{aligned}$$



You can use a single integral because  $x^4 - 2x^2 + 1 \leq 1 - x^2$  on  $[-1, 1]$ .

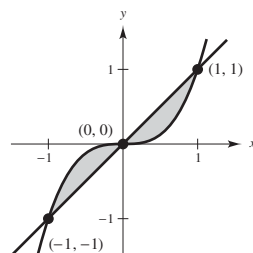
78.  $x^3 \geq x$  on  $[-1, 0]$ ,  $x^3 \leq x$  on  $[0, 1]$

Both functions symmetric to origin.

$$\int_{-1}^0 (x^3 - x) dx = -\int_0^1 (x^3 - x) dx$$

Thus,  $\int_{-1}^1 (x^3 - x) dx = 0$ .

$$A = 2 \int_0^1 (x - x^3) dx = 2 \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2}$$



79. Offer 2 is better because the accumulated salary (area under the curve) is larger.

80. Proposal 2 is better because the cumulative deficit (the area under the curve) is less.

81. (a)  $\int_0^5 [v_1(t) - v_2(t)] dt = 10$  means that Car 1 traveled 10 more meters than Car 2 on the interval  $0 \leq t \leq 5$ .

$\int_0^{10} [v_1(t) - v_2(t)] dt = 30$  means that Car 1 traveled 30 more meters than Car 2 on the interval  $0 \leq t \leq 10$ .

$\int_{20}^{30} [v_1(t) - v_2(t)] dt = -5$  means that Car 2 traveled 5 more meters than Car 1 on the interval  $20 \leq t \leq 30$ .

(b) No, it is not possible because you do not know the initial distance between the cars.

(c) At  $t = 10$ , Car 1 is ahead by 30 meters.

(d) At  $t = 20$ , Car 1 is ahead of Car 2 by 13 meters. From part (a), at  $t = 30$ , Car 1 is ahead by  $13 - 5 = 8$  meters.

82. The integral  $\int_a^b [f(x) - g(x)] dx$  gives the area of the region bounded by the graphs of  $f$  and  $g$  ( $g(x) \leq f(x)$  on  $[a, b]$ ) and the vertical lines  $x = a$  and  $x = b$ . No, the interpretation does not change.

83.  $A = \int_{-3}^3 (9 - x^2) dx = 36$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b] dx = 18$$

$$\int_0^{\sqrt{9-b}} [(9 - b) - x^2] dx = 9$$

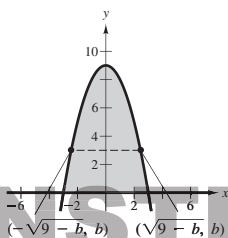
$$\left[ (9 - b)x - \frac{x^3}{3} \right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



84.  $A = 2 \int_0^9 (9 - x) dx = 2 \left[ 9x - \frac{x^2}{2} \right]_0^9 = 81$

$$2 \int_0^{9-b} [(9 - x) - b] dx = \frac{81}{2}$$

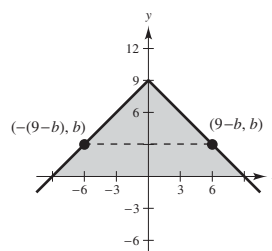
$$2 \int_0^{9-b} [(9 - b) - x] dx = \frac{81}{2}$$

$$2 \left[ (9 - b)x - \frac{x^2}{2} \right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



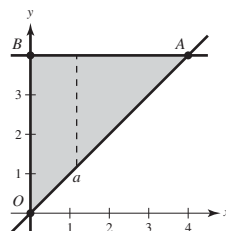
85. Area of triangle  $OAB$  is  $\frac{1}{2}(4)(4) = 8$ .

$$4 = \int_0^a (4 - x) dx = \left[ 4x - \frac{x^2}{2} \right]_0^a = 4a - \frac{a^2}{2}$$

$$a^2 - 8a + 8 = 0$$

$$a = 4 \pm 2\sqrt{2}$$

Because  $0 < a < 4$ , select  $a = 4 - 2\sqrt{2} \approx 1.172$ .



86. Total area =  $\int_{-2}^2 (4 - y^2) dy = 2 \int_0^2 (4 - y^2) dy$

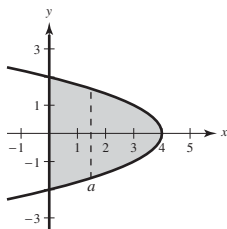
$$= 2 \left[ 4y - \frac{y^3}{3} \right]_0^2 = 2 \left[ 8 - \frac{8}{3} \right] = \frac{32}{3}$$

$$\frac{16}{3} = 2 \int_a^4 \sqrt{4-x} dx = -\frac{4}{3} (4-x)^{3/2} \Big|_a^4 = \frac{4}{3} (4-a)^{3/2}$$

$$4 = (4-a)^{3/2}$$

$$4^{2/3} = 4-a$$

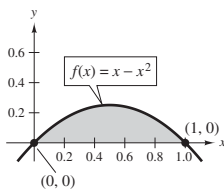
$$a = 4 - 4^{2/3} \approx 1.48$$



87.  $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$

where  $x_i = \frac{i}{n}$  and  $\Delta x = \frac{1}{n}$  is the same as

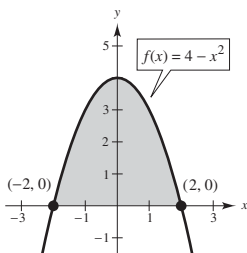
$$\int_0^1 (x - x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$



88.  $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (4 - x_i^2) \Delta x$

where  $x_i = -2 + \frac{4i}{n}$  and  $\Delta x = \frac{4}{n}$  is the same as

$$\int_{-2}^2 (4 - x^2) dx = \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$



89.  $f(x) = x^4 + 4x^3 + x + 7$

$$f'(x) = 4x^3 + 12x^2 + 1$$

$$f''(x) = 12x^2 + 24x = 12x(x + 2)$$

(a) Concavity changes when  $x = 0, -2$ .

Points of inflection:  $(0, 7), (-2, -11)$

(b) Slope =  $\frac{18}{2} = 9$

Tangent line:  $y - 7 = 9(x - 0)$

$$y = 9x + 7 \text{ or } g(x) = 9x + 7$$

(c) Intersection points:

$$x^4 + 4x^3 + x + 7 = 9x + 7$$

$$x^4 + 4x^3 - 8x = 0$$

$$x(x + 2)(x + \sqrt{5} + 1)(x - \sqrt{5} + 1) = 0$$

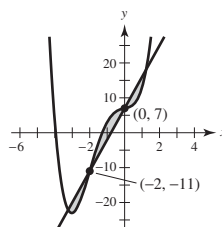
$$[x = -\sqrt{5} - 1, -2, 0, \sqrt{5} - 1]$$

$$\int_{-\sqrt{5}-1}^{-2} [g(x) - f(x)] dx = \frac{16}{5}$$

$$\int_{-2}^0 [f(x) - g(x)] dx = \frac{32}{5}$$

$$\int_0^{\sqrt{5}-1} [g(x) - f(x)] dx = \frac{16}{5}$$

The area between the two inflection points is the sum of the areas between the other two regions.



90.  $f(x) = 2x^4 - 12x^2 + 3x - 1$

$$f'(x) = 8x^3 - 24x + 3$$

$$f''(x) = 24x^2 - 24 = 24(x-1)(x+1)$$

(a) Concavity changes when  $x = -1, 1$ .

Points of inflection:  $(-1, -14), (1, -8)$

(b) Slope  $= \frac{-14 - (-8)}{-1 - 1} = 3$

Tangent line:

$$y + 8 = 3(x - 1)$$

$$y = 3x - 11 \quad \text{or} \quad g(x) = 3x - 11$$

(c) Intersection points:

$$2x^4 - 12x^2 + 3x - 1 = 3x - 11$$

$$2x^4 - 12x^2 + 10 = 0$$

$$x^4 - 6x^2 + 5 = 0$$

$$(x^2 - 1)(x^2 - 5) = 0$$

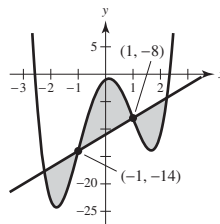
$$(x-1)(x+1)(x+\sqrt{5})(x-\sqrt{5}) = 0$$

$$x = -\sqrt{5}, -1, \sqrt{5}$$

$$\int_{-\sqrt{5}}^{-1} [g(x) - f(x)] dx = \frac{32}{5}$$

$$\int_{-1}^1 [f(x) - g(x)] dx = \frac{64}{5}$$

$$\int_1^{\sqrt{5}} [g(x) - f(x)] dx = \frac{32}{5}$$



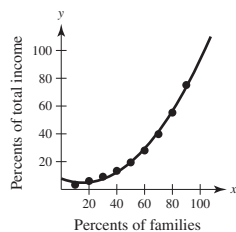
The area between the two inflection points is the sum of the areas between the other two regions.

91.  $\int_8^{13} [(7.21 + 0.58t) - (7.21 + 0.45t)] dt = \int_8^{13} 0.13t dt = \left[ \frac{0.13t^2}{2} \right]_8^{13} = \$6.825 \text{ billion}$

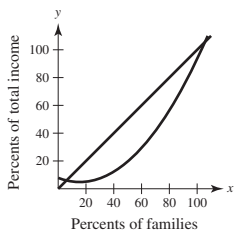
92.  $\int_8^{13} [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)] dt = \int_8^{13} (0.01t^2 + 0.16t) dt = \left[ \frac{0.01t^3}{3} + \frac{0.16t^2}{2} \right]_8^{13} \approx \$14.017 \text{ billion}$

93. (a)  $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) Income inequality  $= \int_0^{100} [x - y_1] dx \approx 2006.7$

94. 5%:  $P_1 = 15.9e^{0.05t}$  (in millions)

3.5%:  $P_2 = 15.9e^{0.035t}$  (in millions)

Difference in profits over 5 years:

$$\int_0^5 (P_1 - P_2) dt = \int_0^5 15.9(e^{0.05t} - e^{0.035t}) dt = 15.9 \left[ \frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035} \right]_0^5 \approx \$3.44 \text{ million}$$

95. The total area is 8 times the area of the shaded region to the right. A point  $(x, y)$  is on the upper boundary of the region if

$$\sqrt{x^2 + y^2} = 2 - y$$

$$x^2 + y^2 = 4 - 4y + y^2$$

$$x^2 = 4 - 4y$$

$$4y = 4 - x^2$$

$$y = 1 - \frac{x^2}{4}$$

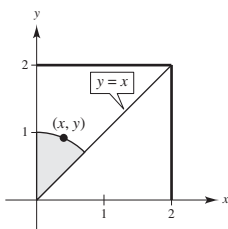
Now determine where this curve intersects the line  $y = x$ .

$$x = 1 - \frac{x^2}{4}$$

$$x^2 + 4x - 4 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \Rightarrow x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8 \int_0^{-2+2\sqrt{2}} \left( 1 - \frac{x^2}{4} - x \right) dx = 8 \left[ x - \frac{x^3}{12} - \frac{x^2}{2} \right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503$$



96. The curves intersect at the point where the slope of  $y_2$  equals that of  $y_1$ , 1.

$$y_2 = 0.08x^2 + k \Rightarrow y_2' = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

(a) The value of  $k$  is given by

$$y_1 = y_2$$

$$6.25 = (0.08)(6.25)^2 + k$$

$$k = 3.125.$$

(b)  $\text{Area} = 2 \int_0^{6.25} (y_2 - y_1) dx$

$$= 2 \int_0^{6.25} (0.08x^2 + 3.125 - x) dx$$

$$= 2 \left[ \frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2} \right]_0^{6.25}$$

$$= 2(6.510417) \approx 13.02083$$

$$\begin{aligned}
 97. (a) \quad A &= 2 \left[ \int_0^5 \left( 1 - \frac{1}{3} \sqrt{5-x} \right) dx + \int_5^{5.5} (1-0) dx \right] \\
 &= 2 \left( \left[ x + \frac{2}{9} (5-x)^{3/2} \right]_0^5 + [x]_5^{5.5} \right) \\
 &= 2 \left( 5 - \frac{10\sqrt{5}}{9} + 5.5 - 5 \right) \approx 6.031 \text{ m}^2
 \end{aligned}$$

$$(b) \quad V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$$

$$(c) \quad 5000V \approx 5000(12.062) = 60,310 \text{ pounds}$$

$$98. (a) \quad A \approx 6.031 - 2 \left[ \pi \left( \frac{1}{16} \right)^2 \right] - 2 \left[ \pi \left( \frac{1}{8} \right)^2 \right] \approx 5.908$$

$$(b) \quad V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$$

$$(c) \quad 5000V \approx 5000(11.816) = 59,082 \text{ pounds}$$

99. True. The region has been shifted  $C$  units upwards (if  $C > 0$ ), or  $C$  units downwards (if  $C < 0$ ).

100. True. This is a property of integrals.

101. False. Let  $f(x) = x$  and  $g(x) = 2x - x^2$ ,  $f$  and  $g$  intersect at  $(1, 1)$ , the midpoint of  $[0, 2]$ , but

$$\int_a^b [f(x) - g(x)] dx = \int_0^2 [x - (2x - x^2)] dx = \frac{2}{3} \neq 0.$$

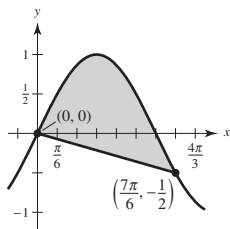
102. True. The area under  $f(x)$  between 0 and 1 is  $\frac{1}{6}$ . The

curves intersect at  $x = \frac{1}{2}^{1/3}$ , and the area between

$y = \left(1 - \frac{1}{2}\right)x$  and  $f$  on the interval  $\left[0, \frac{1}{2}^{1/3}\right]$  is  $\frac{1}{12}$ .

103. Line:  $y = \frac{-3}{7\pi}x$

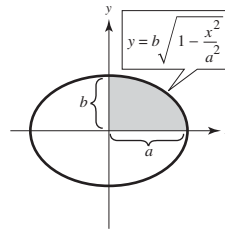
$$\begin{aligned}
 A &= \int_0^{7\pi/6} \left[ \sin x + \frac{3x}{7\pi} \right] dx \\
 &= \left[ -\cos x + \frac{3x^2}{14\pi} \right]_0^{7\pi/6} \\
 &= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1 \\
 &\approx 2.7823
 \end{aligned}$$



$$104. \quad A = 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx$$

$\int_0^a \sqrt{a^2 - x^2} dx$  is the area of  $\frac{1}{4}$  of a circle  $= \frac{\pi a^2}{4}$ .

$$\text{So, } A = \frac{4b}{a} \left( \frac{\pi a^2}{4} \right) = \pi ab.$$



105. You want to find  $c$  such that:

$$\int_0^b [(2x - 3x^3) - c] dx = 0$$

$$\left[ x^2 - \frac{3}{4}x^4 - cx \right]_0^b = 0$$

$$b^2 - \frac{3}{4}b^4 - cb = 0$$

But,  $c = 2b - 3b^3$  because  $(b, c)$  is on the graph.

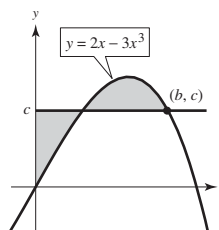
$$b^2 - \frac{3}{4}b^4 - (2b - 3b^3)b = 0$$

$$4 - 3b^2 - 8 + 12b^2 = 0$$

$$9b^2 = 4$$

$$b = \frac{2}{3}$$

$$c = \frac{4}{9}$$





## Section 7.2 Volume: The Disk Method

$$1. V = \pi \int_0^1 (-x + 1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx = \pi \left[ \frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{\pi}{3}$$

$$2. V = \pi \int_0^2 (4 - x^2)^2 dx = \pi \int_0^2 (x^4 - 8x^2 + 16) dx = \pi \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 = \frac{256\pi}{15}$$

$$3. V = \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx = \pi \left[ \frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$$

$$4. V = \pi \int_0^3 (\sqrt{9 - x^2})^2 dx = \pi \int_0^3 (9 - x^2) dx = \pi \left[ 9x - \frac{x^3}{3} \right]_0^3 = 18\pi$$

$$\begin{aligned} 5. V &= \pi \int_0^1 [(x^2)^2 - (x^5)^2] dx \\ &= \pi \int_0^1 (x^4 - x^{10}) dx \\ &= \pi \left[ \frac{x^5}{5} - \frac{x^{11}}{11} \right]_0^1 \\ &= \pi \left( \frac{1}{5} - \frac{1}{11} \right) = \frac{6\pi}{55} \end{aligned}$$

$$\begin{aligned} 6. \quad 2 &= 4 - \frac{x^2}{4} \\ 8 &= 16 - x^2 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} V &= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[ \left( 4 - \frac{x^2}{4} \right)^2 - (2)^2 \right] dx \\ &= 2\pi \int_0^{2\sqrt{2}} \left[ \frac{x^4}{16} - 2x^2 + 12 \right] dx \\ &= 2\pi \left[ \frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}} \\ &= 2\pi \left[ \frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right] \\ &= \frac{448\sqrt{2}}{15}\pi \approx 132.69 \end{aligned}$$

$$\begin{aligned} 7. \quad y &= x^2 \Rightarrow x = \sqrt{y} \\ V &= \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy \\ &= \pi \left[ \frac{y^2}{2} \right]_0^4 = 8\pi \end{aligned}$$

$$\begin{aligned} 8. \quad y &= \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2} \\ V &= \pi \int_0^4 (\sqrt{16 - y^2})^2 dy = \pi \int_0^4 (16 - y^2) dy \\ &= \pi \left[ 16y - \frac{y^3}{3} \right]_0^4 = \frac{128\pi}{3} \end{aligned}$$

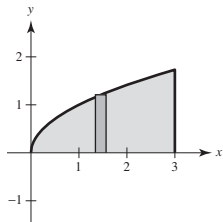
$$\begin{aligned} 9. \quad y &= x^{2/3} \Rightarrow x = y^{3/2} \\ V &= \pi \int_0^1 (y^{3/2})^2 dy = \pi \int_0^1 y^3 dy = \pi \left[ \frac{y^4}{4} \right]_0^1 = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} 10. \quad V &= \pi \int_1^4 (-y^2 + 4y)^2 dy = \pi \int_1^4 (y^4 - 8y^3 + 16y^2) dy \\ &= \pi \left[ \frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5} \end{aligned}$$

11.  $y = \sqrt{x}$ ,  $y = 0$ ,  $x = 3$

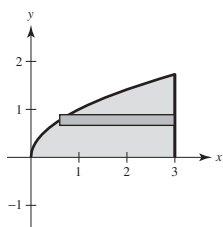
(a)  $R(x) = \sqrt{x}$ ,  $r(x) = 0$

$$V = \pi \int_0^3 (\sqrt{x})^2 dx = \pi \int_0^3 x dx = \pi \left[ \frac{x^2}{2} \right]_0^3 = \frac{9\pi}{2}$$



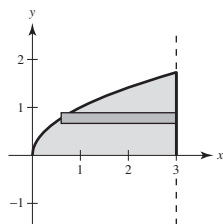
(b)  $R(y) = 3$ ,  $r(y) = y^2$

$$V = \pi \int_0^{\sqrt{3}} [3^2 - (y^2)^2] dy = \pi \int_0^{\sqrt{3}} (9 - y^4) dy = \pi \left[ 9y - \frac{y^5}{5} \right]_0^{\sqrt{3}} = \pi \left[ 9\sqrt{3} - \frac{9}{5}\sqrt{3} \right] = \frac{36\sqrt{3}\pi}{5}$$



(c)  $R(y) = 3 - y^2$ ,  $r(y) = 0$

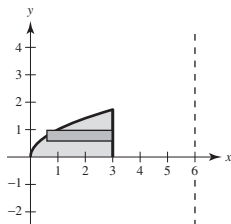
$$V = \pi \int_0^{\sqrt{3}} (3 - y^2)^2 dy = \pi \int_0^{\sqrt{3}} (9 - 6y^2 + y^4) dy = \pi \left[ 9y - 2y^3 + \frac{y^5}{5} \right]_0^{\sqrt{3}} = \pi \left[ 9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5} \right] = \frac{24\sqrt{3}\pi}{5}$$



(d)  $R(y) = 3 + (3 - y^2) = 6 - y^2$ ,  $r(y) = 3$

$$V = \pi \int_0^{\sqrt{3}} [(6 - y^2)^2 - 3^2] dy = \pi \int_0^{\sqrt{3}} (y^4 - 12y^2 + 27) dy$$

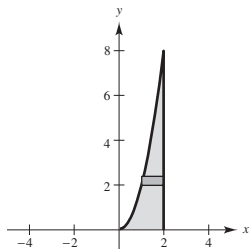
$$= \pi \left[ \frac{y^5}{5} - 4y^3 + 27y \right]_0^{\sqrt{3}} = \pi \left[ \frac{9\sqrt{3}}{5} - 12\sqrt{3} + 27\sqrt{3} \right] = \frac{84\sqrt{3}\pi}{5}$$



12.  $y = 2x^2$ ,  $y = 0$ ,  $x = 2$

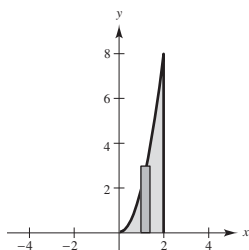
(a)  $R(y) = 2$ ,  $r(y) = \sqrt{y/2}$

$$V = \pi \int_0^8 \left( 4 - \frac{y}{2} \right) dy = \pi \left[ 4y - \frac{y^2}{4} \right]_0^8 = 16\pi$$



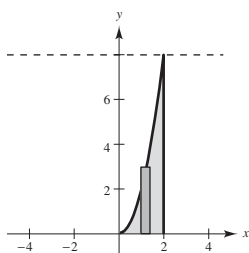
(b)  $R(x) = 2x^2$ ,  $r(x) = 0$

$$V = \pi \int_0^2 4x^4 dx = \pi \left[ \frac{4x^5}{5} \right]_0^2 = \frac{128\pi}{5}$$



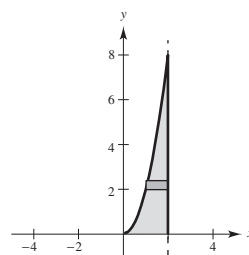
(c)  $R(x) = 8$ ,  $r(x) = 8 - 2x^2$

$$\begin{aligned} V &= \pi \int_0^2 [64 - (64 - 32x^2 + 4x^4)] dx \\ &= \pi \int_0^2 (32x^2 - 4x^4) dx = 4\pi \int_0^2 (8x^2 - x^4) dx \\ &= 4\pi \left[ \frac{8}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 \\ &= \frac{896\pi}{15} \end{aligned}$$



(d)  $R(y) = 2 - \sqrt{y/2}$ ,  $r(y) = 0$

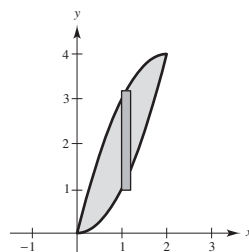
$$\begin{aligned} V &= \pi \int_0^8 \left( 2 - \sqrt{\frac{y}{2}} \right)^2 dy \\ &= \pi \int_0^8 \left( 4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2} \right) dy \\ &= \pi \left[ 4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4} \right]_0^8 = \frac{16\pi}{3} \end{aligned}$$



13.  $y = x^2$ ,  $y = 4x - x^2$  intersect at  $(0, 0)$  and  $(2, 4)$ .

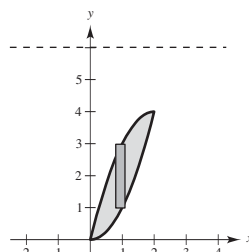
(a)  $R(x) = 4x - x^2$ ,  $r(x) = x^2$

$$\begin{aligned} V &= \pi \int_0^2 [(4x - x^2)^2 - x^4] dx \\ &= \pi \int_0^2 (16x^2 - 8x^3) dx \\ &= \pi \left[ \frac{16}{3}x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3} \end{aligned}$$



(b)  $R(x) = 6 - x^2$ ,  $r(x) = 6 - (4x - x^2)$

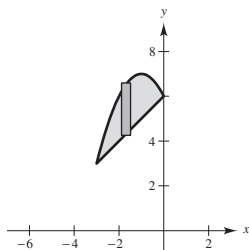
$$\begin{aligned} V &= \pi \int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2] dx \\ &= 8\pi \int_0^2 (x^3 - 5x^2 + 6x) dx \\ &= 8\pi \left[ \frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3} \end{aligned}$$



14.  $y = 6 - 2x - x^2$ ,  $y = x + 6$  intersect at  $(-3, 3)$  and  $(0, 6)$ .

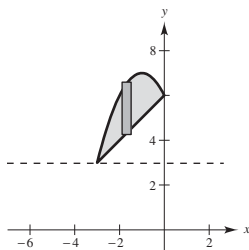
(a)  $R(x) = 6 - 2x - x^2$ ,  $r(x) = x + 6$

$$\begin{aligned} V &= \pi \int_{-3}^0 \left[ (6 - 2x - x^2)^2 - (x + 6)^2 \right] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 9x^2 - 36x) dx \\ &= \pi \left[ \frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2 \right]_{-3}^0 = \frac{243\pi}{5} \end{aligned}$$



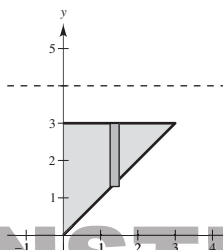
(b)  $R(x) = (6 - 2x - x^2) - 3$ ,  $r(x) = (x + 6) - 3$

$$\begin{aligned} V &= \pi \int_{-3}^0 \left[ (3 - 2x - x^2)^2 - (x + 3)^2 \right] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 3x^2 - 18x) dx \\ &= \pi \left[ \frac{1}{5}x^5 + x^4 - x^3 - 9x^2 \right]_{-3}^0 = \frac{108\pi}{5} \end{aligned}$$



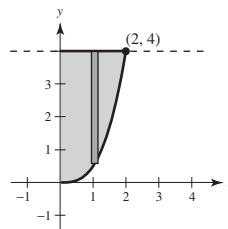
15.  $R(x) = 4 - x$ ,  $r(x) = 1$

$$\begin{aligned} V &= \pi \int_0^3 \left[ (4 - x)^2 - (1)^2 \right] dx \\ &= \pi \int_0^3 (x^2 - 8x + 15) dx \\ &= \pi \left[ \frac{x^3}{3} - 4x^2 + 15x \right]_0^3 \\ &= 18\pi \end{aligned}$$



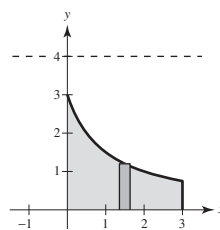
16.  $R(x) = 4 - \frac{x^3}{2}$ ,  $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^1 \left( 4 - \frac{x^3}{2} \right)^2 dx \\ &= \pi \int_0^1 \left[ 16 - 4x^3 + \frac{x^6}{4} \right] dx \\ &= \pi \left[ 16x - x^4 + \frac{x^7}{28} \right]_0^1 \\ &= \pi \left( 32 - 16 + \frac{128}{28} \right) \\ &= \frac{144}{7}\pi \end{aligned}$$



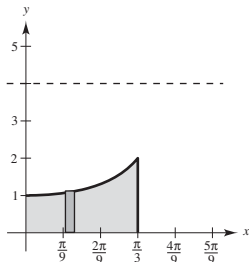
17.  $R(x) = 4$ ,  $r(x) = 4 - \frac{3}{1+x}$

$$\begin{aligned} V &= \pi \int_0^3 \left[ 4^2 - \left( 4 - \frac{3}{1+x} \right)^2 \right] dx \\ &= \pi \int_0^3 \left[ \frac{24}{1+x} - \frac{9}{(1+x)^2} \right] dx \\ &= \pi \left[ 24 \ln|1+x| + \frac{9}{1+x} \right]_0^3 \\ &= \pi \left[ \left( 24 \ln 4 + \frac{9}{4} \right) - 9 \right] \\ &= \left( 48 \ln 2 - \frac{27}{4} \right) \pi \approx 83.318 \end{aligned}$$



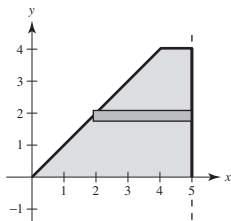
18.  $R(x) = 4, r(x) = 4 - \sec x$

$$\begin{aligned} V &= \pi \int_0^{\pi/3} \left[ (4)^2 - (4 - \sec x)^2 \right] dx \\ &= \pi \int_0^{\pi/3} (8 \sec x - \sec^2 x) dx \\ &= \pi \left[ 8 \ln |\sec x + \tan x| - \tan x \right]_0^{\pi/3} \\ &= \pi \left[ (8 \ln |2 + \sqrt{3}| - \sqrt{3}) - (8 \ln |1 + 0| - 0) \right] \\ &= \pi \left[ 8 \ln(2 + \sqrt{3}) - \sqrt{3} \right] \approx 27.66 \end{aligned}$$



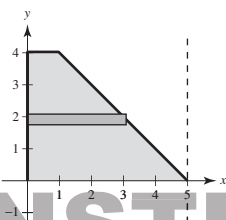
19.  $R(y) = 5 - y, r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (5 - y)^2 dy \\ &= \pi \int_0^4 (25 - 10y + y^2) dy \\ &= \pi \left[ 25y - 5y^2 + \frac{y^3}{3} \right]_0^4 \\ &= \pi \left[ 100 - 80 + \frac{64}{3} \right] = \frac{124\pi}{3} \end{aligned}$$



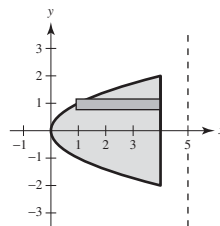
20.  $R(y) = 5, r(y) = 5 - (5 - y) = y$

$$\begin{aligned} V &= \pi \int_0^4 [5^2 - y^2] dy \\ &= \pi \left[ 25y - \frac{y^3}{3} \right]_0^4 \\ &= \pi \left[ 100 - \frac{64}{3} \right] = \frac{236\pi}{3} \end{aligned}$$



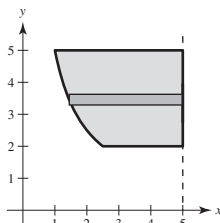
21.  $R(y) = 5 - y^2, r(y) = 1$

$$\begin{aligned} V &= \pi \int_{-2}^2 \left[ (5 - y^2)^2 - 1 \right] dy \\ &= 2\pi \int_0^2 [y^4 - 10y^2 + 24] dy \\ &= 2\pi \left[ \frac{y^5}{5} - \frac{10y^3}{3} + 24y \right]_0^2 \\ &= 2\pi \left[ \frac{32}{5} - \frac{80}{3} + 48 \right] = \frac{832\pi}{15} \end{aligned}$$



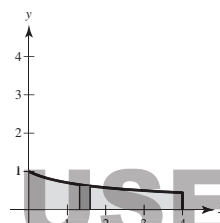
22.  $R(y) = 5 - \frac{5}{y}, r(y) = 0$

$$\begin{aligned} V &= \pi \int_2^5 \left( 5 - \frac{5}{y} \right)^2 dy \\ &= \pi \int_2^5 \left[ 25 - \frac{50}{y} + \frac{25}{y^2} \right] dy \\ &= 25\pi \left[ y - 2 \ln |y| - \frac{1}{y} \right]_2^5 \\ &= 25\pi \left[ \left( 5 - 2 \ln 5 - \frac{1}{5} \right) - \left( 2 - 2 \ln 2 - \frac{1}{2} \right) \right] \\ &= 25\pi \left[ \frac{33}{10} + 2 \ln \left( \frac{2}{5} \right) \right] \approx 115.251 \end{aligned}$$



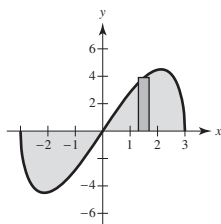
23.  $R(x) = \frac{1}{\sqrt{x+1}}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^4 \left( \frac{1}{\sqrt{x+1}} \right)^2 dx \\ &= \pi \int_0^4 \frac{1}{x+1} dx = \pi [\ln |x+1|]_0^4 = \pi \ln 5 \end{aligned}$$



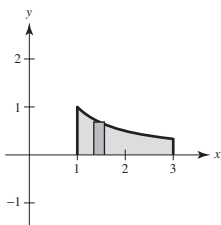
24.  $R(x) = x\sqrt{9 - x^2}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_{-3}^3 (x\sqrt{9 - x^2})^2 dx \\ &= 2\pi \int_0^3 x^2(9 - x^2) dx \\ &= 2\pi \left[ 3x^3 - \frac{x^5}{5} \right]_0^3 \\ &= 2\pi \left[ 81 - \frac{243}{5} \right] = \frac{324\pi}{5} \end{aligned}$$



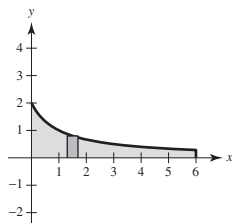
25.  $R(x) = \frac{1}{x}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^3 \left( \frac{1}{x} \right)^2 dx \\ &= \pi \left[ -\frac{1}{x} \right]_1^3 \\ &= \pi \left[ -\frac{1}{3} + 1 \right] = \frac{2}{3}\pi \end{aligned}$$



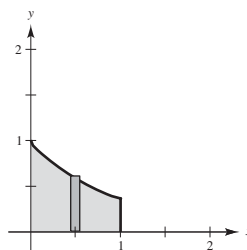
26.  $R(x) = \frac{2}{x+1}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^6 \left( \frac{2}{x+1} \right)^2 dx \\ &= 4\pi \int_0^6 (x+1)^{-2} dx \\ &= 4\pi \left[ \frac{-1}{x+1} \right]_0^6 \\ &= 4\pi \left[ -\frac{1}{7} + 1 \right] = \frac{24\pi}{7} \end{aligned}$$



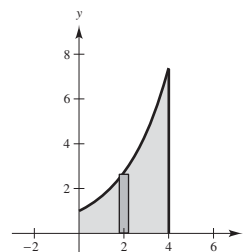
27.  $R(x) = e^{-x}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^1 (e^{-x})^2 dx \\ &= \pi \int_0^1 e^{-2x} dx \\ &= \left[ -\frac{\pi}{2} e^{-2x} \right]_0^1 \\ &= \frac{\pi}{2} (1 - e^{-2}) \approx 1.358 \end{aligned}$$



28.  $R(x) = e^{x/2}, r(x) = 0$

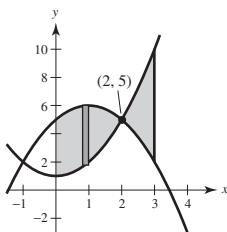
$$\begin{aligned} V &= \pi \int_0^4 (e^{x/2})^2 dx \\ &= \pi \int_0^4 e^x dx \\ &= \left[ \pi e^x \right]_0^4 \\ &= \pi (e^4 - 1) \approx 168.38 \end{aligned}$$



29.  $x^2 + 1 = -x^2 + 2x + 5$   
 $2x^2 - 2x - 4 = 0$   
 $x^2 - x - 2 = 0$   
 $(x - 2)(x + 1) = 0$

The curves intersect at  $(-1, 2)$  and  $(2, 5)$ .

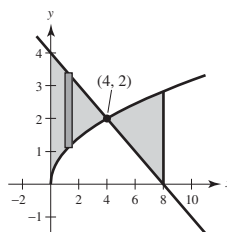
$$\begin{aligned} V &= \pi \int_0^2 \left[ (5 + 2x - x^2)^2 - (x^2 + 1)^2 \right] dx + \pi \int_2^3 \left[ (x^2 + 1)^2 - (5 + 2x - x^2)^2 \right] dx \\ &= \pi \int_0^2 (-4x^3 - 8x^2 + 20x + 24) dx + \pi \int_2^3 (4x^3 + 8x^2 - 20x - 24) dx \\ &= \pi \left[ -x^4 - \frac{8}{3}x^3 + 10x^2 + 24x \right]_0^2 + \pi \left[ x^4 + \frac{8}{3}x^3 - 10x^2 - 24x \right]_2^3 \\ &= \pi \frac{152}{3} + \pi \frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$



30.  $\sqrt{x} = -\frac{1}{2}x + 4$   
 $x = \frac{1}{4}x^2 - 4x + 16$   
 $0 = x^2 - 20x + 64$   
 $0 = (x - 4)(x - 16)$

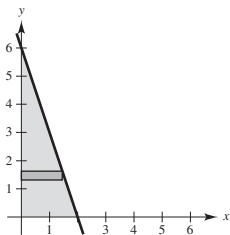
The curves intersect at  $(4, 2)$ . (Note  $x = 16$  is an extraneous root.)

$$\begin{aligned} V &= \pi \int_0^4 \left[ \left(4 - \frac{1}{2}x\right)^2 - (\sqrt{x})^2 \right] dx + \pi \int_4^8 \left[ (\sqrt{x})^2 - \left(4 - \frac{1}{2}x\right)^2 \right] dx \\ &= \pi \int_0^4 \left( \frac{x^2}{4} - 5x + 16 \right) dx + \pi \int_4^8 \left( -\frac{x^2}{4} + 5x - 16 \right) dx \\ &= \pi \left[ \frac{x^3}{12} - \frac{5x^2}{2} + 16x \right]_0^4 + \pi \left[ -\frac{x^3}{12} + \frac{5x^2}{2} - 16x \right]_4^8 \\ &= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi \end{aligned}$$



$$31. y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$$

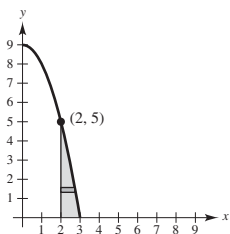
$$\begin{aligned} V &= \pi \int_0^6 \left[ \frac{1}{3}(6 - y) \right]^2 dy \\ &= \frac{\pi}{9} \int_0^6 [36 - 12y + y^2] dy \\ &= \frac{\pi}{9} \left[ 36y - 6y^2 + \frac{y^3}{3} \right]_0^6 \\ &= \frac{\pi}{9} \left[ 216 - 216 + \frac{216}{3} \right] \\ &= 8\pi = \frac{1}{3}\pi r^2 h, \text{ Volume of cone} \end{aligned}$$



$$32. y = 9 - x^2, y = 0, x = 2, x = 3$$

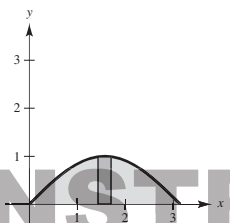
$$x = \sqrt{9 - y}$$

$$\begin{aligned} V &= \pi \int_0^5 (\sqrt{9 - y} - 2)^2 dy \\ &= \pi \int_0^5 (5 - y) dy \\ &= \pi \left[ 5y - \frac{y^2}{2} \right]_0^5 = \pi \left( 25 - \frac{25}{2} \right) = \frac{25\pi}{2} \end{aligned}$$



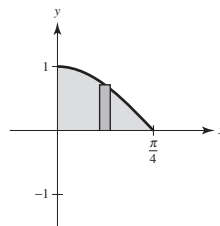
$$\begin{aligned} 33. V &= \pi \int_0^\pi (\sin x)^2 dx \\ &= \pi \int_0^\pi \frac{1 - \cos 2x}{2} dx \\ &= \frac{\pi}{2} \left[ x - \frac{1}{2} \sin 2x \right]_0^\pi = \frac{\pi}{2} [\pi] = \frac{\pi^2}{2} \end{aligned}$$

Numerical approximation: 4.9348



$$\begin{aligned} 34. V &= \pi \int_0^{\pi/4} \cos^2 2x dx \\ &= \pi \int_0^{\pi/4} \frac{1 + \cos 4x}{2} dx \\ &= \frac{\pi}{2} \left[ x + \frac{\sin 4x}{4} \right]_0^{\pi/4} \\ &= \frac{\pi}{2} \left[ \frac{\pi}{4} \right] = \frac{\pi^2}{8} \end{aligned}$$

Numerical approximation: 1.2337



$$\begin{aligned} 35. V &= \pi \int_1^2 (e^{x-1})^2 dx \\ &= \pi \int_1^2 e^{2x-2} dx \\ &= \frac{\pi}{2} e^{2x-2} \Big|_1^2 \\ &= \frac{\pi}{2} (e^2 - 1) \end{aligned}$$

Numerical approximation: 10.0359

$$\begin{aligned} 36. V &= \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 dx \\ &= \pi \int_{-1}^2 [e^x + e^{-x} + 2] dx \\ &= \pi [e^x - e^{-x} + 2x]_{-1}^2 \\ &= \pi [(e^2 - e^{-2} + 4) - (e^{-1} - e - 2)] \\ &= \pi (e^2 + e + 6 - e^{-2} - e^{-1}) \end{aligned}$$

Numerical approximation: 49.0218

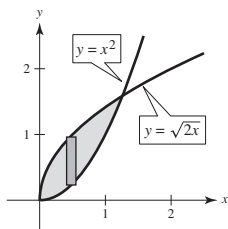
$$37. V = \pi \int_0^2 [e^{-x^2}]^2 dx \approx 1.9686$$

$$38. V = \pi \int_1^3 [\ln x]^2 dx \approx 3.2332$$

$$39. V = \pi \int_0^5 [2 \arctan(0.2x)]^2 dx \approx 15.4115$$



$$\begin{aligned}
 40. \quad x^2 &= \sqrt{2x} \\
 x^4 &= 2x \\
 x^3 &= 2 \\
 x &= 2^{1/3} \approx 1.2599 \\
 V &= \pi \int_0^{2^{1/3}} \left[ (\sqrt{2x})^2 - (x^2)^2 \right] dx \approx 2.9922
 \end{aligned}$$



$$41. V = \pi \int_0^1 y^2 dy = \pi \left[ \frac{y^3}{3} \right]_0^1 = \frac{\pi}{3}$$

$$\begin{aligned}
 42. \quad V &= \pi \int_0^1 [1^2 - (1-y)^2] dy \\
 &= \pi \int_0^1 [2y - y^2] dy \\
 &= \pi \left[ y^2 - \frac{y^3}{3} \right]_0^1 \\
 &= \pi \left( 1 - \frac{1}{3} \right) = \frac{2\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad V &= \pi \int_0^1 (x^2 - x^4) dx \\
 &= \pi \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
 &= \pi \left( \frac{1}{3} - \frac{1}{5} \right) \\
 &= \frac{2\pi}{15}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad V &= \pi \int_0^1 [(1-x^2)^2 - (1-x)^2] dx \\
 &= \pi \int_0^1 [1 - 2x^2 + x^4 - 1 + 2x - x^2] dx \\
 &= \pi \int_0^1 [2x - 3x^2 + x^4] dx \\
 &= \pi \left[ x^2 - x^3 + \frac{x^5}{5} \right]_0^1 \\
 &= \pi \left( \frac{1}{5} \right) = \frac{\pi}{5}
 \end{aligned}$$

$$\begin{aligned}
 45. \quad V &= \pi \int_0^1 (1-y) dy \\
 &= \pi \left[ y - \frac{y^2}{2} \right]_0^1 = \pi \left( 1 - \frac{1}{2} \right) = \frac{\pi}{2}
 \end{aligned}$$

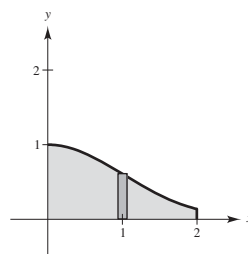
$$\begin{aligned}
 46. \quad V &= \pi \int_0^1 (1 - \sqrt{y})^2 dy \\
 &= \pi \int_0^1 (1 - 2\sqrt{y} + y) dy \\
 &= \pi \left[ y - \frac{4}{3}y^{3/2} + \frac{y^2}{2} \right]_0^1 \\
 &= \pi \left( 1 - \frac{4}{3} + \frac{1}{2} \right) \\
 &= \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 47. \quad V &= \pi \int_0^1 (y - y^2) dy \\
 &= \pi \left[ \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 48. \quad V &= \pi \int_0^1 [(1-y)^2 - (1-\sqrt{y})^2] dy \\
 &= \pi \int_0^1 [1 - 2y + y^2 - 1 + 2\sqrt{y} - y] dy \\
 &= \pi \int_0^1 [2\sqrt{y} - 3y + y^2] dy \\
 &= \pi \left[ \frac{4}{3}y^{3/2} - \frac{3y^2}{2} + \frac{y^3}{3} \right]_0^1 \\
 &= \pi \left( \frac{4}{3} - \frac{3}{2} + \frac{1}{3} \right) \\
 &= \frac{\pi}{6}
 \end{aligned}$$

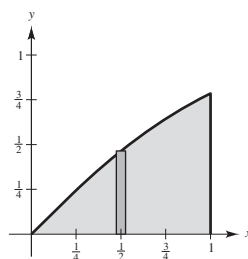
$$49. A \approx 3$$

Matches (a)

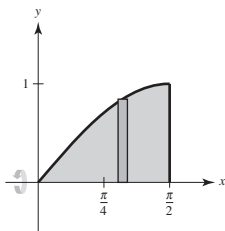


$$50. A \approx \frac{3}{4}$$

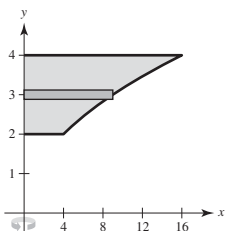
Matches (b)



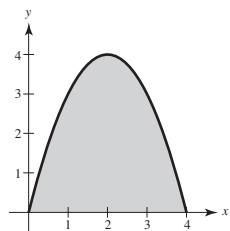
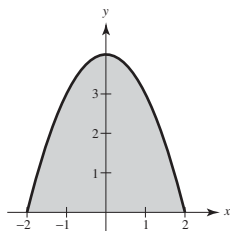
51.  $\pi \int_0^{\pi/2} \sin^2 x \, dx$  represents the volume of the solid generated by revolving the region bounded by  $y = \sin x$ ,  $y = 0$ ,  $x = 0$ ,  $x = \pi/2$  about the  $x$ -axis.



52.  $\pi \int_2^4 y^4 \, dy$  represents the volume of the solid generated by revolving the region bounded by  $x = y^2$ ,  $x = 0$ ,  $y = 2$ ,  $y = 4$  about the  $y$ -axis.

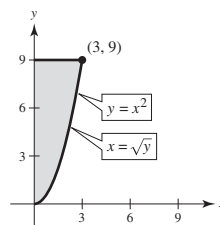


53.



The volumes are the same because the solid has been translated horizontally.  $(4x - x^2 = 4 - (x - 2)^2)$

54.



- (a) Around  $x$ -axis:

$$V = \pi \int_0^3 [9^2 - (x^2)^2] \, dx = \frac{972}{5} \pi = 194.4\pi$$

- (b) Around  $y$ -axis:

$$V = \pi \int_0^9 (\sqrt{y})^2 \, dy = \frac{81}{2} \pi = 40.5\pi$$

- (c) Around  $x = 3$ :

$$V = \pi(3^2)9 - \int_0^9 \pi(\sqrt{y} - 3)^2 \, dy = 81\pi - \frac{27}{2}\pi = \frac{135\pi}{2} \approx 67.5\pi$$

So,  $b < c < a$ .

55. (a) True. Answers will vary.

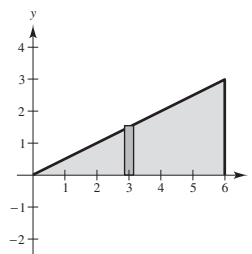
- (b) False. Answers will vary.

56. The answer is (b). See the formula for the Washer Method, page 461.

$$57. R(x) = \frac{1}{2}x, r(x) = 0$$

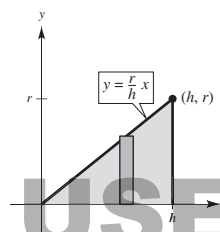
$$V = \pi \int_0^6 \frac{1}{4}x^2 \, dx = \left[ \frac{\pi}{12}x^3 \right]_0^6 = 18\pi$$

$$\text{Note: } V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3^2)6 = 18\pi$$



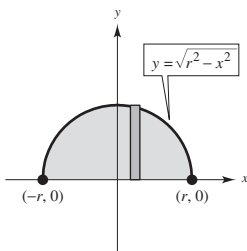
$$58. R(x) = \frac{r}{h}x, r(x) = 0$$

$$V = \pi \int_0^h \frac{r^2}{h^2}x^2 \, dx = \left[ \frac{r^2\pi}{3h^2}x^3 \right]_0^h = \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$



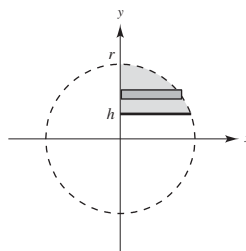
59.  $R(x) = \sqrt{r^2 - x^2}, r(x) = 0$

$$\begin{aligned} V &= \pi \int_{-r}^r (r^2 - x^2) dx \\ &= 2\pi \int_0^r (r^2 - x^2) dx \\ &= 2\pi \left[ r^2x - \frac{1}{3}x^3 \right]_0^r \\ &= 2\pi \left( r^3 - \frac{1}{3}r^3 \right) = \frac{4}{3}\pi r^3 \end{aligned}$$



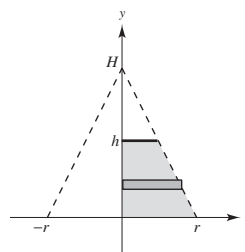
60.  $x = \sqrt{r^2 - y^2}, R(y) = \sqrt{r^2 - y^2}, r(y) = 0$

$$\begin{aligned} V &= \pi \int_h^r (\sqrt{r^2 - y^2})^2 dy \\ &= \pi \int_h^r (r^2 - y^2) dy \\ &= \pi \left[ r^2y - \frac{y^3}{3} \right]_h^r \\ &= \pi \left[ \left( r^3 - \frac{r^3}{3} \right) - \left( r^2h - \frac{h^3}{3} \right) \right] \\ &= \pi \left( \frac{2r^3}{3} - r^2h + \frac{h^3}{3} \right) = \frac{\pi}{3} (2r^3 - 3r^2h + h^3) \end{aligned}$$



61.  $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right), R(y) = r\left(1 - \frac{y}{H}\right), r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^h \left[ r\left(1 - \frac{y}{H}\right) \right]^2 dy = \pi r^2 \int_0^h \left( 1 - \frac{2}{H}y + \frac{1}{H^2}y^2 \right) dy \\ &= \pi r^2 \left[ y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3 \right]_0^h \\ &= \pi r^2 \left( h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right) = \pi r^2 h \left( 1 - \frac{h}{H} + \frac{h^2}{3H^2} \right) \end{aligned}$$



62. (a)  $V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \left[ \frac{\pi x^2}{2} \right]_0^4 = 8\pi$

Let  $0 < c < 4$  and set

$$\pi \int_0^c x dx = \left[ \frac{\pi x^2}{2} \right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

So, when  $x = 2\sqrt{2}$ , the solid is divided into two parts of equal volume.

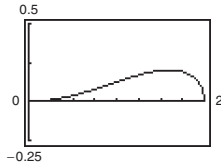
(b) Set  $\pi \int_0^c x dx = \frac{8\pi}{3}$  (one third of the volume). Then  $\frac{\pi c^2}{2} = \frac{8\pi}{3}, c^2 = \frac{16}{3}, c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$ .

To find the other value, set  $\pi \int_0^d x dx = \frac{16\pi}{3}$  (two thirds of the volume).

$$\text{Then } \frac{\pi d^2}{2} = \frac{16\pi}{3}, d^2 = \frac{32}{3}, d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}.$$

The  $x$ -values that divide the solid into three parts of equal volume are  $x = (4\sqrt{3})/3$  and  $x = (4\sqrt{6})/3$ .

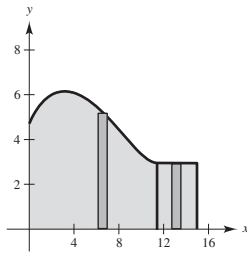
63.



$$V = \pi \int_0^2 \left( \frac{1}{8} x^2 \sqrt{2-x} \right)^2 dx = \frac{\pi}{64} \int_0^2 x^4 (2-x) dx = \frac{\pi}{64} \left[ \frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2 = \frac{\pi}{30} \text{ m}^3$$

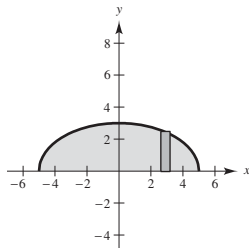
$$64. \quad y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \leq x \leq 11.5 \\ 2.95, & 11.5 < x \leq 15 \end{cases}$$

$$\begin{aligned} V &= \pi \int_0^{11.5} \left( \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2} \right)^2 dx + \pi \int_{11.5}^{15} 2.95^2 dx \\ &= \pi \left[ \frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x \right]_0^{11.5} + \pi [2.95^2 x]_{11.5}^{15} \\ &\approx 1031.9016 \text{ cubic centimeters} \end{aligned}$$



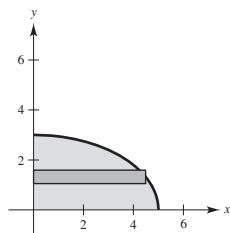
$$65. (a) \quad R(x) = \frac{3}{5} \sqrt{25-x^2}, r(x) = 0$$

$$V = \frac{9\pi}{25} \int_{-5}^5 (25-x^2) dx = \frac{18\pi}{25} \int_0^5 (25-x^2) dx = \frac{18\pi}{25} \left[ 25x - \frac{x^3}{3} \right]_0^5 = 60\pi$$



$$(b) \quad R(y) = \frac{5}{3} \sqrt{9-y^2}, r(y) = 0, x \geq 0$$

$$V = \frac{25\pi}{9} \int_0^3 (9-y^2) dy = \frac{25\pi}{9} \left[ 9y - \frac{y^3}{3} \right]_0^3 = 50\pi$$



66. Total volume:  $V = \frac{4\pi(50)^3}{3} = \frac{500,000\pi}{3} \text{ ft}^3$

Volume of water in the tank:

$$\pi \int_{-50}^{y_0} (\sqrt{2500 - y^2})^2 dy = \pi \int_{-50}^{y_0} (2500 - y^2) dy = \pi \left[ 2500y - \frac{y^3}{3} \right]_{-50}^{y_0} = \pi \left( 2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4} \left( \frac{500,000\pi}{3} \right) = \pi \left( 2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right)$$

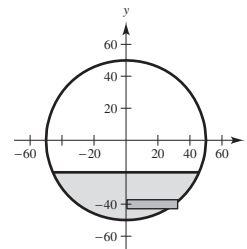
$$125,000 = 7500y_0 - y_0^3 + 250,000$$

$$y_0^3 - 7500y_0 - 125,000 = 0$$

$$y_0 \approx -17.36$$

Depth:  $-17.36 - (-50) = 32.64$  feet

When the tank is three-fourths of its capacity the depth is  $100 - 32.64 = 67.36$  feet.



67. (a) First find where  $y = b$  intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$

$$x^2 = 16 - 4b = 4(4 - b)$$

$$x = 2\sqrt{4 - b}$$

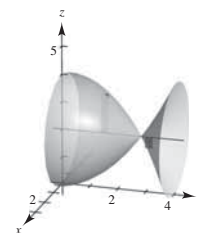
$$V = \int_0^{2\sqrt{4-b}} \pi \left[ 4 - \frac{x^2}{4} - b \right]^2 dx + \int_{2\sqrt{4-b}}^4 \pi \left[ b - 4 + \frac{x^2}{4} \right]^2 dx$$

$$= \int_0^4 \pi \left[ 4 - \frac{x^2}{4} - b \right]^2 dx$$

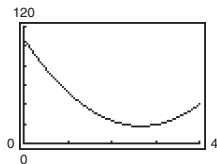
$$= \pi \int_0^4 \left[ \frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16 \right] dx$$

$$= \pi \left[ \frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x \right]_0^4$$

$$= \pi \left( \frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64 \right) = \pi \left( 4b^2 - \frac{64}{3}b + \frac{512}{15} \right)$$



(b) Graph of  $V(b) = \pi \left( 4b^2 - \frac{64}{3}b + \frac{512}{15} \right)$



Minimum volume is 17.87 for  $b = 2.67$ .

(c)  $V'(b) = \pi \left( 8b - \frac{64}{3} \right) = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$  is a relative minimum.

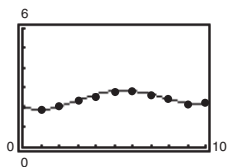
68. (a)  $V = \int_0^{10} \pi [f(x)]^2 dx$

Simpson's Rule:  $b - a = 10 - 0 = 10$ ,  $n = 10$

$$V \approx \frac{\pi}{3} [(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3} (178.405) \approx 186.83 \text{ cm}^3$$

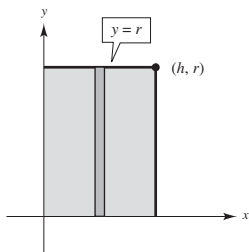
(b)  $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c)  $V \approx \int_0^{10} \pi f(x)^2 dx \approx 186.35 \text{ cm}^3$

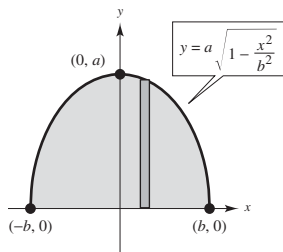
69. (a)  $\pi \int_0^h r^2 dx$  (ii)

is the volume of a right circular cylinder with radius  $r$  and height  $h$ .



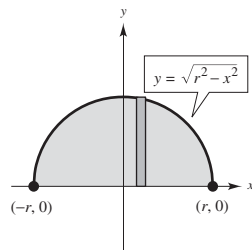
(b)  $\pi \int_{-b}^b \left( a \sqrt{1 - \frac{x^2}{b^2}} \right)^2 dx$  (iv)

is the volume of an ellipsoid with axes  $2a$  and  $2b$ .



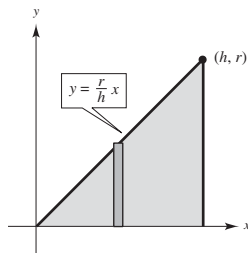
(c)  $\pi \int_{-r}^r \left( \sqrt{r^2 - x^2} \right)^2 dx$  (iii)

is the volume of a sphere with radius  $r$ .



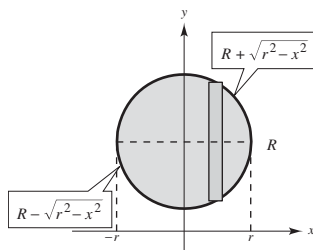
(d)  $\pi \int_0^h \left( \frac{rx}{h} \right)^2 dx$  (i)

is the volume of a right circular cone with the radius of the base as  $r$  and height  $h$ .



(e)  $\pi \int_{-r}^r \left[ \left( R + \sqrt{r^2 - x^2} \right)^2 - \left( R - \sqrt{r^2 - x^2} \right)^2 \right] dx$  (v)

is the volume of a torus with the radius of its circular cross section as  $r$  and the distance from the axis of the torus to the center of its cross section as  $R$ .

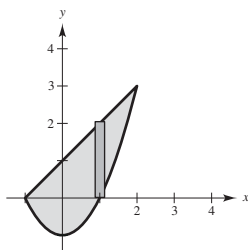


70. Let  $A_1(x)$  and  $A_2(x)$  equal the areas of the cross sections of the two solids for  $a \leq x \leq b$ . Because  $A_1(x) = A_2(x)$ , you have

$$V_1 = \int_a^b A_1(x) dx = \int_a^b A_2(x) dx = V_2.$$

So, the volumes are the same.

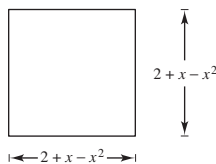
71.



$$\text{Base of cross section} = (x + 1) - (x^2 - 1) = 2 + x - x^2$$

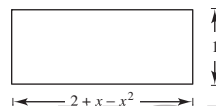
(a)  $A(x) = b^2 = (2 + x - x^2)^2 = 4 + 4x - 3x^2 - 2x^3 + x^4$

$$V = \int_{-1}^2 (4 + 4x - 3x^2 - 2x^3 + x^4) dx = \left[ 4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{81}{10}$$



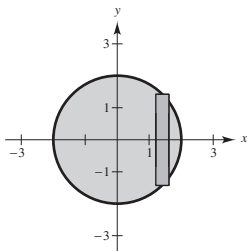
(b)  $A(x) = bh = (2 + x - x^2)1$

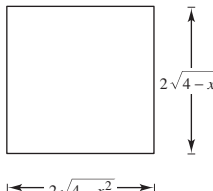
$$V = \int_{-1}^2 (2 + x - x^2) dx = \left[ 2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



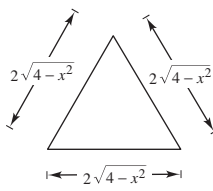
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72.

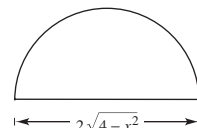
Base of cross section =  $2\sqrt{4 - x^2}$ 

$$\begin{aligned} \text{(a)} \quad A(x) &= b^2 = (2\sqrt{4 - x^2})^2 \\ V &= \int_{-2}^2 4(4 - x^2) dx \\ &= 4 \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{128}{3} \end{aligned}$$


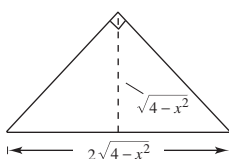
$$\begin{aligned} \text{(b)} \quad A(x) &= \frac{1}{2}bh = \frac{1}{2}(2\sqrt{4 - x^2})(\sqrt{3}\sqrt{4 - x^2}) \\ &= \sqrt{3}(4 - x^2) \end{aligned}$$

$$\begin{aligned} V &= \sqrt{3} \int_{-2}^2 (4 - x^2) dx \\ &= \sqrt{3} \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \frac{32\sqrt{3}}{3} \end{aligned}$$


$$\text{(c)} \quad A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}(\sqrt{4 - x^2})^2 = \frac{\pi}{2}(4 - x^2)$$

$$\begin{aligned} V &= \frac{\pi}{2} \int_{-2}^2 (4 - x^2) dx \\ &= \frac{\pi}{2} \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{16\pi}{3} \end{aligned}$$


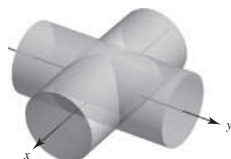
$$\text{(d)} \quad A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{4 - x^2})(\sqrt{4 - x^2}) = 4 - x^2$$

$$\begin{aligned} V &= \int_{-2}^2 (4 - x^2) dx \\ &= \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3} \end{aligned}$$


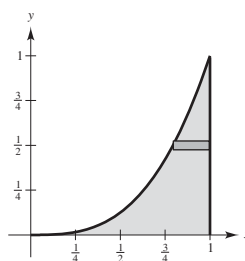
73. The cross sections are squares. By symmetry, you can set up an integral for an eighth of the volume and multiply by 8.

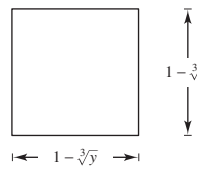
$$A(y) = b^2 = (\sqrt{r^2 - y^2})^2$$

$$\begin{aligned} V &= 8 \int_0^r (r^2 - y^2) dy \\ &= 8 \left[ r^2 y - \frac{1}{3} y^3 \right]_0^r \\ &= \frac{16}{3} r^3 \end{aligned}$$

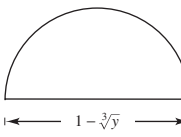


74.

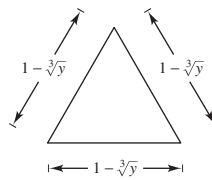
Base of cross section =  $1 - \sqrt[3]{y}$ 

$$\begin{aligned} \text{(a)} \quad A(y) &= b^2 = (1 - \sqrt[3]{y})^2 \\ V &= \int_0^1 (1 - \sqrt[3]{y})^2 dy \\ &= \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy \\ &= \left[ y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3} \right]_0^1 = \frac{1}{10} \end{aligned}$$


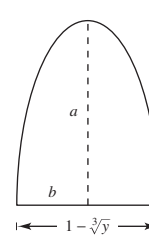
$$\text{(b)} \quad A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \left( \frac{1 - \sqrt[3]{y}}{2} \right)^2 = \frac{1}{8}\pi (1 - \sqrt[3]{y})^2$$

$$\begin{aligned} V &= \frac{1}{8}\pi \int_0^1 (1 - \sqrt[3]{y})^2 dy \\ &= \frac{\pi}{8} \left( \frac{1}{10} \right) = \frac{\pi}{80} \end{aligned}$$


$$\begin{aligned} \text{(c)} \quad A(y) &= \frac{1}{2}bh = \frac{1}{2}(1 - \sqrt[3]{y}) \left( \frac{\sqrt{3}}{2} (1 - \sqrt[3]{y}) \right) \\ &= \frac{\sqrt{3}}{4} (1 - \sqrt[3]{y})^2 \end{aligned}$$

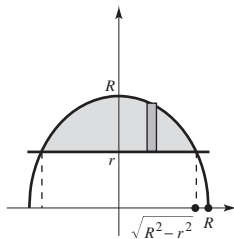
$$\begin{aligned} V &= \frac{\sqrt{3}}{4} \int_0^1 (1 - \sqrt[3]{y})^2 dy \\ &= \frac{\sqrt{3}}{4} \left( \frac{1}{10} \right) \\ &= \frac{\sqrt{3}}{40} \end{aligned}$$


$$\begin{aligned} \text{(d)} \quad A(y) &= \frac{1}{2}\pi ab \\ &= \frac{\pi}{2} (2) (1 - \sqrt[3]{y}) \frac{1 - \sqrt[3]{y}}{2} \\ &= \frac{\pi}{2} (1 - \sqrt[3]{y})^2 \end{aligned}$$

$$\begin{aligned} V &= \frac{\pi}{2} \int_0^1 (1 - \sqrt[3]{y})^2 dy \\ &= \frac{\pi}{2} \left( \frac{1}{10} \right) = \frac{\pi}{20} \end{aligned}$$




$$\begin{aligned}
 75. \quad V &= \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[ \left( \sqrt{R^2-x^2} \right)^2 - r^2 \right] dx \\
 &= 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - r^2 - x^2) dx \\
 &= 2\pi \left[ (R^2 - r^2)x - \frac{x^3}{3} \right]_0^{\sqrt{R^2-r^2}} \\
 &= 2\pi \left[ (R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3} \right] = \frac{4}{3}\pi(R^2 - r^2)^{3/2}
 \end{aligned}$$



76. Let  $R = 6$  in the previous Exercise.

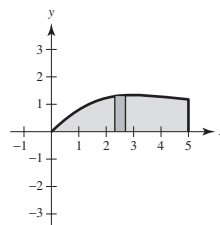
$$\begin{aligned}
 \frac{4}{3}\pi(36 - r^2)^{3/2} &= \frac{1}{2}\left(\frac{4}{3}\right)\pi(6)^3 \\
 (36 - r^2)^{3/2} &= 108 \\
 36 - r^2 &= (108)^{2/3} \\
 r^2 &= 36 - 108^{2/3} \\
 r &= \sqrt{36 - 108^{2/3}} \approx 3.65
 \end{aligned}$$

$$77. \quad y = \frac{8x}{9 + x^2}$$

$$V = \pi \int_0^5 \left( \frac{8x}{9 + x^2} \right)^2 dx$$

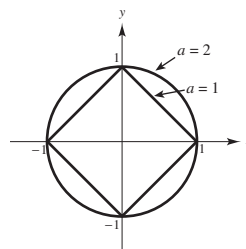
Using Simpson's Rule with  $n = 10$ , you obtain

$$V \approx 19.7444.$$



78. (a) When  $a = 1$ :  $|x| + |y| = 1$  represents a square.

When  $a = 2$ :  $|x|^2 + |y|^2 = 1$  represents a circle.



$$(b) \quad |y| = (1 - |x|^a)^{1/a}$$

$$A = 2 \int_{-1}^1 (1 - |x|^a)^{1/a} dx = 4 \int_0^1 (1 - x^a)^{1/a} dx$$

To approximate the volume of the solid, from  $n$  slices, each of whose area is approximated by the integral above. Then sum the volumes of these  $n$  slices.

79. (a) Because the cross sections are isosceles right triangles:

$$\begin{aligned}
 A(x) &= \frac{1}{2}bh = \frac{1}{2}(\sqrt{r^2 - y^2})(\sqrt{r^2 - y^2}) = \frac{1}{2}(r^2 - y^2) \\
 V &= \frac{1}{2} \int_{-r}^r (r^2 - y^2) dy = \int_0^r (r^2 - y^2) dy = \left[ r^2y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3
 \end{aligned}$$



$$(b) \quad A(x) = \frac{1}{2}bh = \frac{1}{2}\sqrt{r^2 - y^2}(\sqrt{r^2 - y^2} \tan \theta) = \frac{\tan \theta}{2}(r^2 - y^2)$$

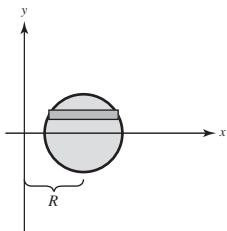
$$V = \frac{\tan \theta}{2} \int_{-r}^r (r^2 - y^2) dy = \tan \theta \int_0^r (r^2 - y^2) dy = \tan \theta \left[ r^2y - \frac{y^3}{3} \right]_0^r = \frac{2}{3}r^3 \tan \theta$$

As  $\theta \rightarrow 90^\circ$ ,  $V \rightarrow \infty$ .

80. (a)  $(x - R)^2 + y^2 = r^2$

$$x = R \pm \sqrt{r^2 - y^2}$$

$$V = 2\pi \int_0^r \left[ \left( R + \sqrt{r^2 - y^2} \right)^2 - \left( R - \sqrt{r^2 - y^2} \right)^2 \right] dy = 2\pi \int_0^r 4R\sqrt{r^2 - y^2} dy = 8\pi R \int_0^r \sqrt{r^2 - y^2} dy$$



(b)  $\int_0^r \sqrt{r^2 - y^2} dy$  is one-quarter of the area of a circle of radius  $r$ ,  $\frac{1}{4}\pi r^2$ .

$$V = 8\pi R \left( \frac{1}{4}\pi r^2 \right) = 2\pi^2 r^2 R$$

### Section 7.3 Volume: The Shell Method

1.  $p(x) = x, h(x) = x$

$$V = 2\pi \int_0^2 x(x) dx = \left[ \frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

2.  $p(x) = x, h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x) dx$$

$$= 2\pi \int_0^1 (x - x^2) dx = 2\pi \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{3}$$

3.  $p(x) = x, h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x} dx = 2\pi \int_0^4 x^{3/2} dx = \left[ \frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

4.  $p(x) = x, h(x) = 3 - \left( \frac{1}{2}x^2 + 1 \right) = 2 - \frac{1}{2}x^2$

$$V = 2\pi \int_0^2 x \left( 2 - \frac{1}{2}x^2 \right) dx$$

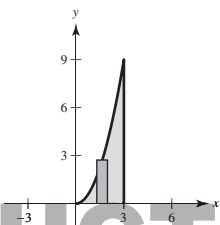
$$= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_0^2 = 2\pi(4 - 2) = 4\pi$$

5.  $p(x) = x, h(x) = x^2$

$$V = 2\pi \int_0^3 x(x^2) dx$$

$$= 2\pi \left[ \frac{x^4}{4} \right]_0^3$$

$$= 2\pi \left( \frac{81}{4} \right) = \frac{81\pi}{2}$$

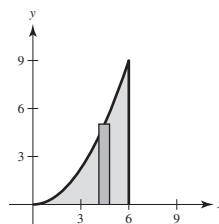


6.  $p(x) = x, h(x) = \frac{1}{4}x^2$

$$V = 2\pi \int_0^6 x \left( \frac{1}{4}x^2 \right) dx$$

$$= 2\pi \int_0^6 \frac{1}{4}x^3 dx$$

$$= 2\pi \left[ \frac{x^4}{16} \right]_0^6 = 162\pi$$

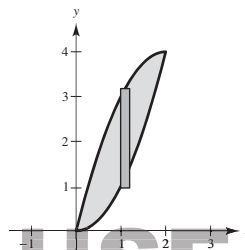


7.  $p(x) = x, h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x(4x - 2x^2) dx$$

$$= 4\pi \int_0^2 (2x^2 - x^3) dx$$

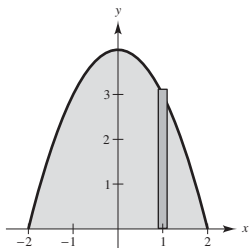
$$= 4\pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{16\pi}{3}$$



8.  $p(x) = x, h(x) = 4 - x^2$

$$V = 2\pi \int_0^2 (4x - x^3) dx$$

$$= 2\pi \left[ 2x^2 - \frac{1}{4}x^4 \right]_0^2 = 8\pi$$



9.  $p(x) = x$

$$h(x) = 4 - (4x - x^2)$$

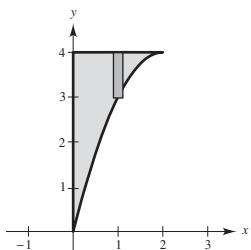
$$= x^2 - 4x + 4$$

$$V = 2\pi \int_0^2 x(x^2 - 4x + 4) dx$$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x) dx$$

$$= 2\pi \left[ \frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 \right]_0^2$$

$$= \frac{8\pi}{3}$$



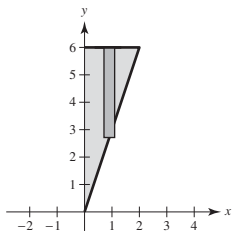
10.  $p(x) = x, h(x) = 6 - 3x$

$$V = 2\pi \int_0^2 x(6 - 3x) dx$$

$$= 2\pi \int_0^2 (6x - 3x^2) dx$$

$$= 2\pi \left[ 3x^2 - x^3 \right]_0^2$$

$$= 2\pi(12 - 8) = 8\pi$$



11.  $p(x) = x, h(x) = \sqrt{x - 2}$

$$V = 2\pi \int_2^4 x\sqrt{x - 2} dx$$

Let  $u = x - 2, x = u + 2, du = dx$ .

When  $x = 2, u = 0$ .

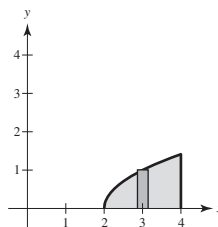
When  $x = 4, u = 2$ .

$$V = 2\pi \int_0^2 (u + 2)u^{1/2} du$$

$$= 2\pi \left[ \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} \right]_0^2$$

$$= 2\pi \left[ \frac{2}{5}(2)^{5/2} + \frac{4}{3}(2)^{3/2} \right]$$

$$= 2\pi\sqrt{2} \left[ \frac{2}{5}(4) + \frac{4}{3}(2) \right] = \frac{128\sqrt{2}\pi}{15}$$

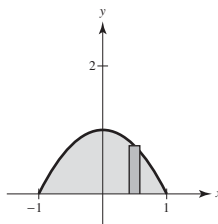


12.  $p(x) = x, h(x) = 1 - x^2$

$$V = 2\pi \int_0^1 x(1 - x^2) dx$$

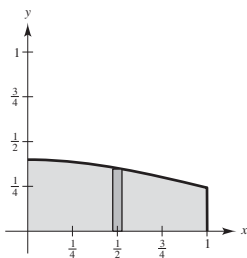
$$= 2\pi \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$$

$$= 2\pi \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}$$



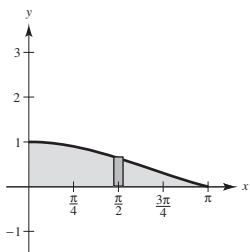
13.  $p(x) = x$ ,  $h(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$$\begin{aligned} V &= 2\pi \int_0^1 x \left( \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx \\ &= \sqrt{2\pi} \int_0^1 e^{-x^2/2} x dx \\ &= \left[ -\sqrt{2\pi} e^{-x^2/2} \right]_0^1 \\ &= \sqrt{2\pi} \left( 1 - \frac{1}{\sqrt{e}} \right) \\ &\approx 0.986 \end{aligned}$$



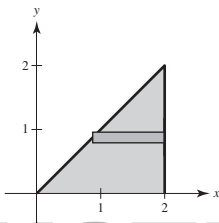
14.  $p(x) = x$ ,  $h(x) = \frac{\sin x}{x}$

$$\begin{aligned} V &= 2\pi \int_0^\pi x \left[ \frac{\sin x}{x} \right] dx \\ &= 2\pi \int_0^\pi \sin x dx \\ &= \left[ -2\pi \cos x \right]_0^\pi = 4\pi \end{aligned}$$



15.  $p(y) = y$ ,  $h(y) = 2 - y$

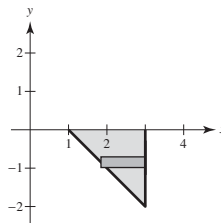
$$\begin{aligned} V &= 2\pi \int_0^2 y(2 - y) dy \\ &= 2\pi \int_0^2 (2y - y^2) dy \\ &= 2\pi \left[ y^2 - \frac{y^3}{3} \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



16.  $p(y) = -y$  (So,  $p(y) \geq 0$  on  $[-2, 0]$ )

$$h(y) = 3 - (1 - y) = 2 + y$$

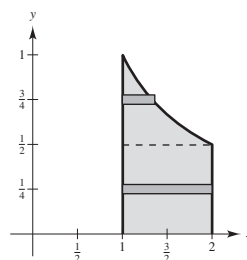
$$\begin{aligned} V &= 2\pi \int_{-2}^0 (-y)(2 + y) dy \\ &= 2\pi \int_{-2}^0 [-2y - y^2] dy \\ &= 2\pi \left[ -y^2 - \frac{y^3}{3} \right]_{-2}^0 \\ &= 2\pi \left[ 0 - \left( -4 + \frac{8}{3} \right) \right] = 2\pi \frac{4}{3} = \frac{8\pi}{3} \end{aligned}$$



17.  $p(y) = y$  and  $h(y) = 1$  if  $0 \leq y < \frac{1}{2}$ .

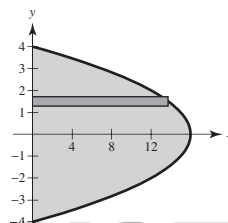
$$p(y) = y \text{ and } h(y) = \frac{1}{y} \text{ if } \frac{1}{2} \leq y \leq 1.$$

$$\begin{aligned} V &= 2\pi \int_0^{1/2} y dy + 2\pi \int_{1/2}^1 (1 - y) dy \\ &= 2\pi \left[ \frac{y^2}{2} \right]_0^{1/2} + 2\pi \left[ y - \frac{y^2}{2} \right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$



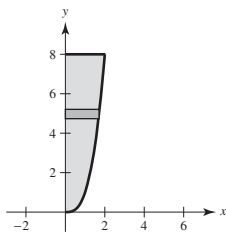
18.  $p(y) = y$ ,  $h(y) = 16 - y^2$

$$\begin{aligned} V &= 2\pi \int_0^4 y(16 - y^2) dy \\ &= 2\pi \int_0^4 (16y - y^3) dy \\ &= 2\pi \left[ 8y^2 - \frac{y^4}{4} \right]_0^4 = 2\pi(128 - 64) = 128\pi \end{aligned}$$



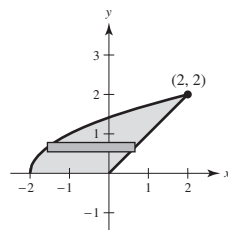
19.  $p(y) = y, h(y) = \sqrt[3]{y}$

$$\begin{aligned} V &= 2\pi \int_0^8 y \sqrt[3]{y} \, dy \\ &= 2\pi \int_0^8 y^{4/3} \, dy \\ &= \left[ 2\pi \left( \frac{3}{7} \right) y^{7/3} \right]_0^8 \\ &= \frac{6\pi}{7} (2^7) = \frac{768\pi}{7} \end{aligned}$$



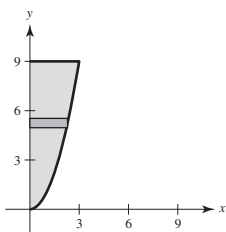
22.  $p(y) = y, h(y) = y - (y^2 - 2) = 2 + y - y^2$

$$\begin{aligned} V &= 2\pi \int_0^2 y(2 + y - y^2) \, dy \\ &= 2\pi \int_0^2 (2y + y^2 - y^3) \, dy \\ &= 2\pi \left[ y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_0^2 \\ &= 2\pi \left( 4 + \frac{8}{3} - 4 \right) = \frac{16\pi}{3} \end{aligned}$$



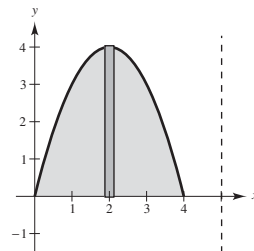
20.  $p(y) = y, h(y) = \sqrt{y}$

$$\begin{aligned} V &= 2\pi \int_0^9 y \sqrt{y} \, dy = 2\pi \int_0^9 y^{3/2} \, dy \\ &= 2\pi \left[ \frac{2}{5} y^{5/2} \right]_0^9 \\ &= \frac{4\pi}{5} (9)^{5/2} = \frac{4\pi}{5} (243) = \frac{972\pi}{5} \end{aligned}$$



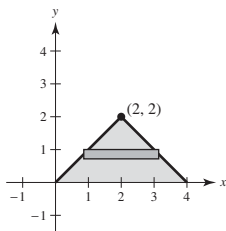
23.  $p(x) = 5 - x, h(x) = 4x - x^2$

$$\begin{aligned} V &= 2\pi \int_0^4 (5 - x)(4x - x^2) \, dx \\ &= 2\pi \int_0^4 (x^3 - 9x^2 + 20x) \, dx \\ &= 2\pi \left[ \frac{x^4}{4} - 3x^3 + 10x^2 \right]_0^4 = 64\pi \end{aligned}$$



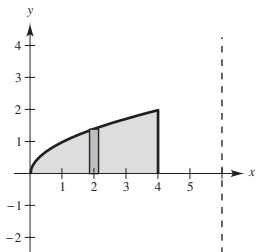
21.  $p(y) = y, h(y) = (4 - y) - (y) = 4 - 2y$

$$\begin{aligned} V &= 2\pi \int_0^2 y(4 - 2y) \, dy \\ &= 2\pi \int_0^2 (4y - 2y^2) \, dy \\ &= 2\pi \left[ 2y^2 - \frac{2}{3}y^3 \right]_0^2 \\ &= 2\pi \left( 8 - \frac{16}{3} \right) = \frac{16\pi}{3} \end{aligned}$$



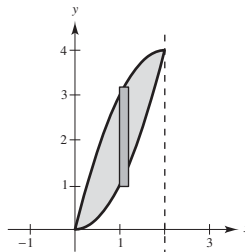
24.  $p(x) = 6 - x, h(x) = \sqrt{x}$

$$\begin{aligned} V &= 2\pi \int_0^4 (6 - x)\sqrt{x} \, dx \\ &= 2\pi \int_0^4 (6x^{1/2} - x^{3/2}) \, dx \\ &= 2\pi \left[ 4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^4 = \frac{192\pi}{5} \end{aligned}$$



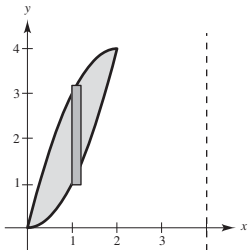
26.  $p(x) = 2 - x, h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$\begin{aligned} V &= 2\pi \int_0^2 (2 - x)(4x - 2x^2) \, dx \\ &= 2\pi \int_0^2 (8x - 8x^2 + 2x^3) \, dx \\ &= 2\pi \left[ 4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4 \right]_0^2 = \frac{16\pi}{3} \end{aligned}$$



25.  $p(x) = 4 - x, h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)(4x - 2x^2) \, dx \\ &= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x) \, dx \\ &= 4\pi \left[ \frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi \end{aligned}$$



27. The shell method would be easier:  $V = 2\pi \int_0^4 [4 - (y - 2)^2]y \, dy$

Using the disk method:  $V = \pi \int_0^4 \left[ (2 + \sqrt{4 - x})^2 - (2 - \sqrt{4 - x})^2 \right] dx$  [Note:  $V = \frac{128\pi}{3}$ ]

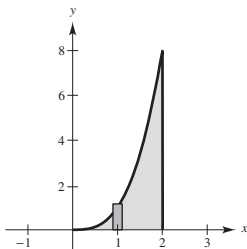
28. The shell method is easier:  $V = 2\pi \int_0^{\ln 4} x(4 - e^x) \, dx$

Using the disk method,  $x = \ln(4 - y)$  and  $V = \pi \int_0^3 (\ln(4 - y))^2 \, dy$ . [Note:  $V = \pi[8(\ln 2)^2 - 8 \ln 2 + 3]$ ]

29. (a) **Disk**

$$R(x) = x^3, r(x) = 0$$

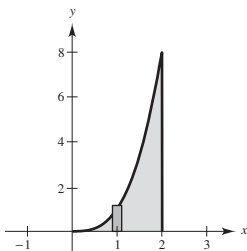
$$V = \pi \int_0^2 x^6 dx = \pi \left[ \frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$$



(b) **Shell**

$$p(x) = x, h(x) = x^3$$

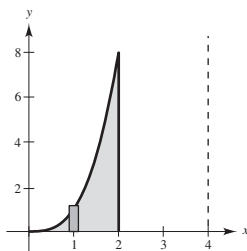
$$V = 2\pi \int_0^2 x^4 dx = 2\pi \left[ \frac{x^5}{5} \right]_0^2 = \frac{64\pi}{5}$$



(c) **Shell**

$$p(x) = 4 - x, h(x) = x^3$$

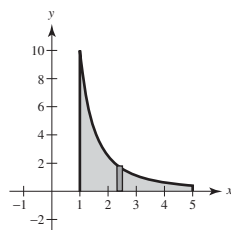
$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)x^3 dx \\ &= 2\pi \int_0^2 (4x^3 - x^4) dx \\ &= 2\pi \left[ x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{96\pi}{5} \end{aligned}$$



30. (a) **Disk**

$$R(x) = \frac{10}{x^2}, r(x) = 0$$

$$\begin{aligned} V &= \pi \int_1^5 \left( \frac{10}{x^2} \right)^2 dx \\ &= 100\pi \int_1^5 x^{-4} dx \\ &= 100\pi \left[ \frac{x^{-3}}{-3} \right]_1^5 \\ &= -\frac{100\pi}{3} \left( \frac{1}{125} - 1 \right) = \frac{496}{15}\pi \end{aligned}$$



(b) **Shell**

$$R(x) = x, r(x) = 0$$

$$\begin{aligned} V &= 2\pi \int_1^5 x \left( \frac{10}{x^2} \right) dx \\ &= 20\pi \int_1^5 \frac{1}{x} dx \\ &= 20\pi [\ln|x|]_1^5 = 20\pi \ln 5 \end{aligned}$$

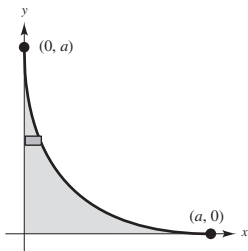
(c) **Disk**

$$R(x) = 10, r(x) = 10 - \frac{10}{x^2}$$

$$\begin{aligned} V &= \pi \int_1^5 \left[ 10^2 - \left( 10 - \frac{10}{x^2} \right)^2 \right] dx \\ &= \pi \left[ \frac{100}{3x^3} - \frac{200}{x} \right]_1^5 = \frac{1904}{15}\pi \end{aligned}$$

## 31. (a) Shell

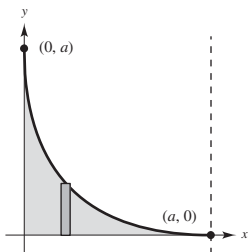
$$\begin{aligned}
 p(y) &= y, h(y) = (a^{1/2} - y^{1/2})^2 \\
 V &= 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y) dy \\
 &= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2) dy \\
 &= 2\pi \left[ \frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a \\
 &= 2\pi \left( \frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right) = \frac{\pi a^3}{15}
 \end{aligned}$$



(b) Same as part (a) by symmetry

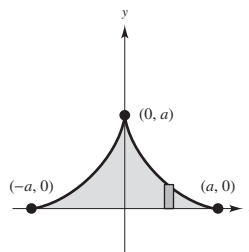
## (c) Shell

$$\begin{aligned}
 p(x) &= a - x, h(x) = (a^{1/2} - x^{1/2})^2 \\
 V &= 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2 dx \\
 &= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2) dx \\
 &= 2\pi \left[ a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a \\
 &= \frac{4\pi a^3}{15}
 \end{aligned}$$

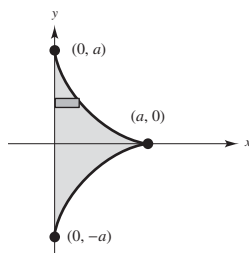


## 32. (a) Disk

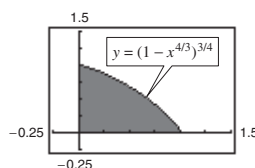
$$\begin{aligned}
 R(x) &= (a^{2/3} - x^{2/3})^{3/2}, r(x) = 0 \\
 V &= \pi \int_{-a}^a (a^{2/3} - x^{2/3})^3 dx \\
 &= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2) dx \\
 &= 2\pi \left[ a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3 \right]_0^a \\
 &= 2\pi \left( a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3 \right) = \frac{32\pi a^3}{105}
 \end{aligned}$$



(b) Same as part (a) by symmetry

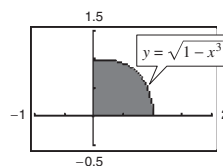


## 33. (a)

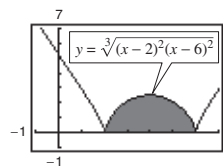
(b)  $x^{4/3} + y^{4/3} = 1, x = 0, y = 0$ 

$$\begin{aligned}
 y &= (1 - x^{4/3})^{3/4} \\
 V &= 2\pi \int_0^1 x(1 - x^{4/3})^{3/4} dx \approx 1.5056
 \end{aligned}$$

## 34. (a)

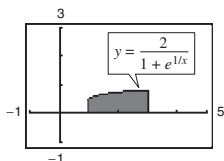
(b)  $V = 2\pi \int_0^1 x\sqrt{1 - x^3} dx \approx 2.3222$ 

## 35. (a)

(b)  $V = 2\pi \int_2^6 x\sqrt[3]{(x-2)^2(x-6)^2} dx \approx 187.249$



36. (a)

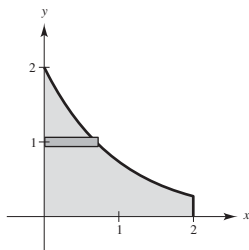


(b)  $V = 2\pi \int_1^5 \frac{2x}{1 + e^{1/x}} dx \approx 19.0162$

37.  $y = 2e^{-x}$ ,  $y = 0$ ,  $x = 0$ ,  $x = 2$

Volume  $\approx 7.5$

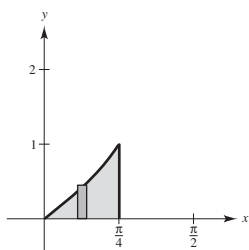
Matches (d)



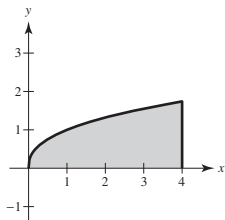
38.  $y = \tan x$ ,  $y = 0$ ,  $x = 0$ ,  $x = \frac{\pi}{4}$

Volume  $\approx 1$

Matches (e)



39.



(a) Around  $x$ -axis:  $V = \pi \int_0^4 (x^{2/5})^2 dx = \left[ \pi \frac{5}{9} x^{9/5} \right]_0^4$

$$= \frac{5}{9} \pi (4)^{9/5} \approx 6.7365\pi$$

(b) Around  $y$ -axis:  $V = 2\pi \int_0^4 x(x^{2/5}) dx$

$$= \left[ 2\pi \frac{5}{12} x^{12/5} \right]_0^4 \approx 23.2147\pi$$

(c) Around  $x = 4$ :

$$V = 2\pi \int_0^4 (4 - x)x^{2/5} dx \approx 16.5819\pi$$

So,  $a < c < b$ .

40. (a) The figure will be a circle of radius  $AB$  and center  $A$ .

(b) The figure will be a circular cylinder of radius  $AB$ .

(c) Disk method:  $V = \pi \int_0^3 [g(y)]^2 dy$

Shell method:  $V = 2\pi \int_0^{2.45} x f(x) dx$

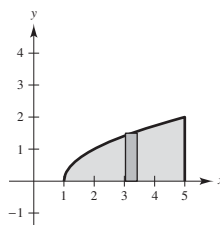
41.  $\pi \int_1^5 (x - 1) dx = \pi \int_1^5 (\sqrt{x - 1})^2 dx$

This integral represents the volume of the solid generated by revolving the region bounded by

$y = \sqrt{x - 1}$ ,  $y = 0$ , and  $x = 5$  about the  $x$ -axis by using the disk method.

$$2\pi \int_0^2 y[5 - (y^2 + 1)] dy$$

represents this same volume by using the shell method.



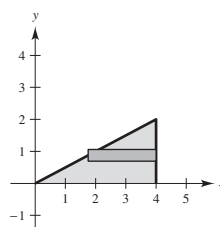
Disk method

42.  $2\pi \int_0^4 x\left(\frac{x}{2}\right) dx$

This integral represents the volume of the solid generated by revolving the region bounded by  $y = x/2$ ,  $y = 0$ , and  $x = 4$  about the  $y$ -axis by using the shell method.

$$\pi \int_0^2 [16 - (2y)^2] dy = \pi \int_0^2 [(4)^2 - (2y)^2] dy$$

represents this same volume by using the disk method.



Disk method

43. Answers will vary.

(a) The rectangles would be vertical.

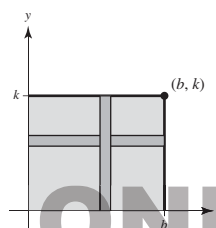
(b) The rectangles would be horizontal.

44. (a) radius =  $k$

height =  $b$

(b) radius =  $b$

height =  $k$



45.  $p(x) = x, h(x) = 2 - \frac{1}{2}x^2$

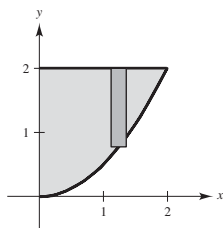
$$\begin{aligned} V &= 2\pi \int_0^2 x \left(2 - \frac{1}{2}x^2\right) dx \\ &= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx \\ &= 2\pi \left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \quad (\text{total volume}) \end{aligned}$$

Now find  $x_0$  such that:

$$\begin{aligned} \pi &= 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx \\ 1 &= 2 \left[x^2 - \frac{1}{8}x^4\right]_0^{x_0} \\ 1 &= 2x_0^2 - \frac{1}{4}x_0^4 \\ x_0^4 - 8x_0^2 + 4 &= 0 \\ x_0^2 &= 4 \pm 2\sqrt{3} \quad (\text{Quadratic Formula}) \end{aligned}$$

Take  $x_0 = \sqrt{4 - 2\sqrt{3}} \approx 0.73205$ , because the other root is too large.

Diameter:  $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$



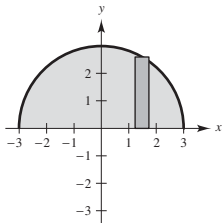
46. Total volume of the hemisphere is

$$\frac{1}{2} \left(\frac{4}{3}\right) \pi r^3 = \frac{2}{3} \pi (3)^3 = 18\pi. \text{ By the Shell Method,}$$

$p(x) = x, h(x) = \sqrt{9 - x^2}$ . Find  $x_0$  such that:

$$\begin{aligned} 6\pi &= 2\pi \int_0^{x_0} x \sqrt{9 - x^2} dx \\ 6 &= - \int_0^{x_0} (9 - x^2)^{1/2} (-2x) dx \\ &= \left[ -\frac{2}{3} (9 - x^2)^{3/2} \right]_0^{x_0} = 18 - \frac{2}{3} (9 - x_0^2)^{3/2} \\ (9 - x_0^2)^{3/2} &= 18 \\ x_0 &= \sqrt{9 - 18^{2/3}} \approx 1.460 \end{aligned}$$

Diameter:  $2\sqrt{9 - 18^{2/3}} \approx 2.920$



$$\begin{aligned} 47. V &= 4\pi \int_{-1}^1 (2 - x) \sqrt{1 - x^2} dx \\ &= 8\pi \int_{-1}^1 \sqrt{1 - x^2} dx - 4\pi \int_{-1}^1 x \sqrt{1 - x^2} dx \\ &= 8\pi \left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1 - x^2)^{1/2} (-2) dx \\ &= 4\pi^2 + \left[ 2\pi \left(\frac{2}{3}\right) (1 - x^2)^{3/2} \right]_{-1}^1 = 4\pi^2 \end{aligned}$$

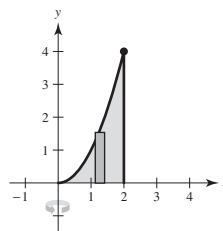
$$\begin{aligned} 48. V &= 4\pi \int_{-r}^r (R - x) \sqrt{r^2 - x^2} dx \\ &= 4\pi R \int_{-r}^r \sqrt{r^2 - x^2} dx - 4\pi \int_{-r}^r x \sqrt{r^2 - x^2} dx \\ &= 4\pi R \left(\frac{\pi r^2}{2}\right) + \left[ 2\pi \left(\frac{2}{3}\right) (r^2 - x^2)^{3/2} \right]_{-r}^r \\ &= 2\pi^2 r^2 R \end{aligned}$$

$$49. 2\pi \int_0^2 x^3 dx = 2\pi \int_0^2 x(x^2) dx$$

(a) Plane region bounded by

$$y = x^2, y = 0, x = 0, x = 2$$

(b) Revolved about the  $y$ -axis

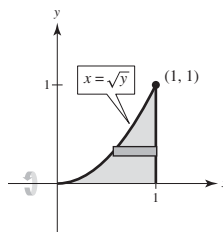


Other answers possible

$$50. 2\pi \int_0^1 (y - y^{3/2}) dy = 2\pi \int_0^1 y(1 - \sqrt{y}) dy$$

(a) Plane region bounded by  $x = \sqrt{y}, x = 1, y = 0$

(b) Revolved about the  $x$ -axis



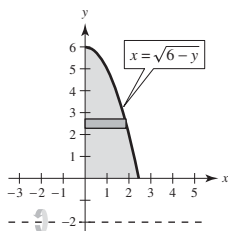
Other answers possible

51.  $2\pi \int_0^6 (y+2)\sqrt{6-y} \, dy$

(a) Plane region bounded by

$$x = \sqrt{6-y}, x = 0, y = 0$$

(b) Revolved around line  $y = -2$

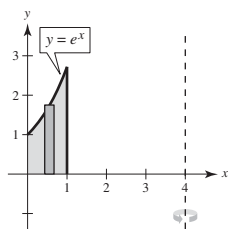


Other answers possible

52.  $2\pi \int_0^1 (4-x)e^x \, dx$

(a) Plane region bounded by  $y = e^x, y = 0, x = 0, x = 1$

(b) Revolved about the line  $x = 4$



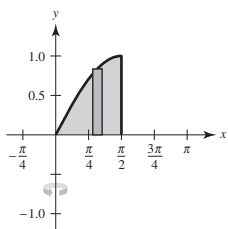
53. (a)  $\frac{d}{dx}[\sin x - x \cos x + C] = \cos x + x \sin x - \cos x$

$$= x \sin x$$

So,  $\int x \sin x \, dx = \sin x - x \cos x + C.$

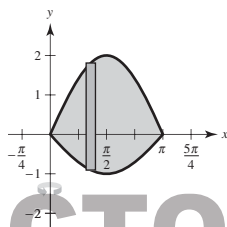
(b) (i)  $p(x) = x, h(x) = \sin x$

$$\begin{aligned} V &= 2\pi \int_0^{\pi/2} x \sin x \, dx \\ &= 2\pi [\sin x - x \cos x]_0^{\pi/2} \\ &= 2\pi [(1 - 0) - 0] = 2\pi \end{aligned}$$



(ii)  $p(x) = x, h(x) = 2 \sin x - (-\sin x) = 3 \sin x$

$$\begin{aligned} V &= 2\pi \int_0^{\pi} x(3 \sin x) \, dx \\ &= 6\pi \int_0^{\pi} x \sin x \, dx \\ &= 6\pi [\sin x - x \cos x]_0^{\pi} \\ &= 6\pi(\pi) = 6\pi^2 \end{aligned}$$

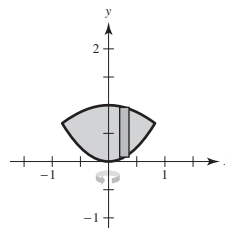


54. (a)  $\frac{d}{dx}[\cos x + x \sin x + C] = -\sin x + \sin x + x \cos x$   
 $= x \cos x$

Hence,  $\int x \cos x \, dx = \cos x + x \sin x + C.$

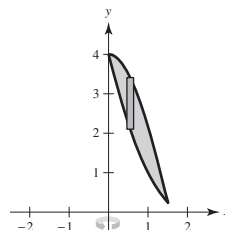
(b) (i)  $x^2 = \cos x \Rightarrow x \approx \pm 0.8241$

$$\begin{aligned} V &\approx 2(2\pi) \int_0^{0.8241} x[\cos x - x^2] \, dx \\ &= 4\pi \left[ \cos x + x \sin x - \frac{x^4}{4} \right]_0^{0.8241} \approx 2.1205 \end{aligned}$$



(ii)  $4 \cos x = (x-2)^2 \Rightarrow x = 0, 1.5110$

$$\begin{aligned} V &\approx 2\pi \int_0^{1.511} x[4 \cos x - (x-2)^2] \, dx \\ &= 2\pi \int_0^{1.511} \left[ 4 \cos x + 4x \sin x - \frac{(x-2)^3}{3} \right]_0^{1.511} \\ &= 6.2993 \end{aligned}$$

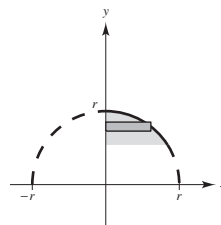


## 55. Disk Method

$$R(y) = \sqrt{r^2 - y^2}$$

$$r(y) = 0$$

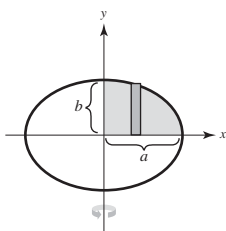
$$\begin{aligned} V &= \pi \int_{r-h}^r (r^2 - y^2) \, dy \\ &= \pi \left[ r^2 y - \frac{y^3}{3} \right]_{r-h}^r = \frac{1}{3} \pi h^2 (3r - h) \end{aligned}$$



56.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$



$$p(x) = x, h(x) = b \sqrt{1 - \frac{x^2}{a^2}}$$

$$V = 2(2\pi) \int_0^a x b \sqrt{1 - \frac{x^2}{a^2}} dx$$

$$= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - x^2} x dx$$

$$= \frac{4\pi b}{a} \left[ \frac{-(a^2 - x^2)^{3/2}}{3} \right]_0^a$$

$$= \frac{4\pi b}{3a} a^3 = \frac{4}{3} \pi a^2 b$$

If the region is revolved about the  $x$ -axis, then by symmetry the volume would be  $V = \frac{4}{3} \pi a b^2$ .

**Note:** If  $a = b$ , then volume is that of a sphere.

57. (a) Area of region =  $\int_0^b [ab^n - ax^n] dx$

$$= \left[ ab^n x - a \frac{x^{n+1}}{n+1} \right]_0^b$$

$$= ab^{n+1} - a \frac{b^{n+1}}{n+1}$$

$$= ab^{n+1} \left( 1 - \frac{1}{n+1} \right)$$

$$= ab^{n+1} \left( \frac{n}{n+1} \right)$$

$$R_1(n) = \frac{ab^{n+1} \left[ \frac{n}{n+1} \right]}{(ab^n)b} = \frac{n}{n+1}$$

(b)  $\lim_{n \rightarrow \infty} R_1(n) = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

$$\lim_{n \rightarrow \infty} (ab^n)b = \infty$$

(c) **Disk Method:**

$$V = 2\pi \int_0^b x(ab^n - ax^n) dx$$

$$= 2\pi a \int_0^b (xb^n - x^{n+1}) dx$$

$$= 2\pi a \left[ \frac{b^n}{2} x^2 - \frac{x^{n+2}}{n+2} \right]_0^b$$

$$= 2\pi a \left[ \frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2} \right] = \pi ab^{n+2} \left( \frac{n}{n+2} \right)$$

$$R_2(n) = \frac{\pi ab^{n+2} \left[ \frac{n}{n+2} \right]}{(\pi b^2)(ab^n)} = \left( \frac{n}{n+2} \right)$$

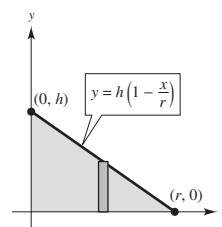
(d)  $\lim_{n \rightarrow \infty} R_2(n) = \lim_{n \rightarrow \infty} \left( \frac{n}{n+2} \right) = 1$

$$\lim_{n \rightarrow \infty} (\pi b^2)(ab^n) = \infty$$

(e) As  $n \rightarrow \infty$ , the graph approaches the line  $x = b$ .

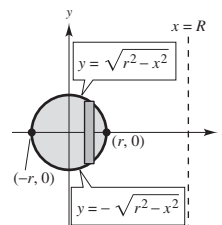
58. (a)  $2\pi \int_0^r hx \left( 1 - \frac{x}{r} \right) dx$  (ii)

is the volume of a right circular cone with the radius of the base as  $r$  and height  $h$ .



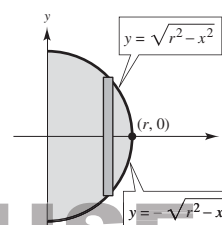
(b)  $2\pi \int_{-r}^r (R - x)(2\sqrt{r^2 - x^2}) dx$  (v)

is the volume of a torus with the radius of its circular cross section as  $r$  and the distance from the axis of the torus to the center of its cross section as  $R$ .



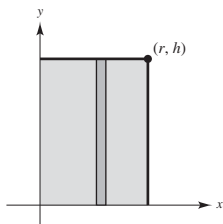
(c)  $2\pi \int_0^r 2x\sqrt{r^2 - x^2} dx$  (iii)

is the volume of a sphere with radius  $r$ .



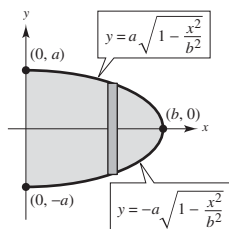
(d)  $2\pi \int_0^r hx \, dx$  (i)

is the volume of a right circular cylinder with a radius of  $r$  and a height of  $h$ .



(e)  $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)} \, dx$  (iv)

is the volume of an ellipsoid with axes  $2a$  and  $2b$ .



59. (a)  $V = 2\pi \int_0^4 xf(x) \, dx = \frac{2\pi(40)}{3(4)}[0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0] = \frac{20\pi}{3}(5800) \approx 121,475 \text{ ft}^3$

(b) Top line:  $y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x + 50$

Bottom line:  $y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \Rightarrow y = -2x + 80$

$$\begin{aligned} V &= 2\pi \int_0^{20} x\left(-\frac{1}{2}x + 50\right) dx + 2\pi \int_{20}^{40} x(-2x + 80) dx \\ &= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x\right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x) dx \\ &= 2\pi \left[-\frac{x^3}{6} + 25x^2\right]_0^{20} + 2\pi \left[-\frac{2x^3}{3} + 40x^2\right]_{20}^{40} = 2\pi \left(\frac{26,000}{3}\right) + 2\pi \left(\frac{32,000}{3}\right) \approx 121,475 \text{ ft}^3 \end{aligned}$$

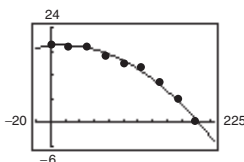
(Note that Simpson's Rule is exact for this problem.)

60. (a)  $V = 2\pi \int_0^{200} xf(x) \, dx$

$$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$$\approx 1,366,593 \text{ ft}^3$$

(b)  $d = -0.000561x^2 + 0.0189x + 19.39$



(c)  $V \approx 2\pi \int_0^{200} xd(x) \, dx \approx 2\pi(213,800) = 1,343,345 \text{ ft}^3$

(d) Number of gallons  $\approx V(7.48) = 10,048,221 \text{ gal}$

$$61. V_1 = \pi \int_{1/4}^c \frac{1}{x^2} dx = \pi \left[ -\frac{1}{x} \right]_{1/4}^c = \pi \left[ -\frac{1}{c} + 4 \right] = \frac{4c-1}{c} \pi$$

$$V_2 = \left[ 2\pi \int_{1/4}^c x \left( \frac{1}{x} \right) dx = 2\pi x \right]_{1/4}^c = 2\pi \left( c - \frac{1}{4} \right)$$

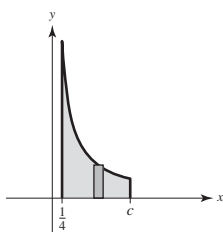
$$V_1 = V_2 \Rightarrow \frac{4c-1}{c} \pi = 2\pi \left( c - \frac{1}{4} \right)$$

$$4c-1 = 2c \left( c - \frac{1}{4} \right)$$

$$4c^2 - 9c + 2 = 0$$

$$(4c-1)(c-2) = 0$$

$$c = 2 \left( c = \frac{1}{4} \text{ yields no volume.} \right)$$



$$62. (a) p(x) = x, h(x) = r^2 - x^2$$

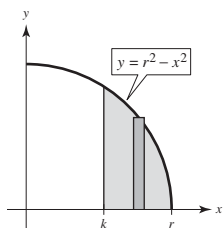
**Shell method:**

$$V = 2\pi \int_k^r x(r^2 - x^2) dx$$

$$= -\pi \int_k^r (r^2 - x^2)(-2x) dx$$

$$= -\pi \left[ \frac{(r^2 - x^2)^2}{2} \right]_k^r$$

$$= -\pi \left[ 0 - \frac{(r^2 - k^2)^2}{2} \right] = \frac{\pi}{2} (r^2 - k^2)^2$$



$$(b) y = r^2 - x^2$$

$$x = \sqrt{r^2 - y}$$

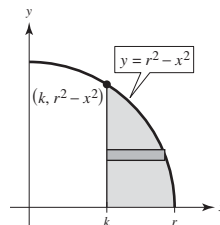
**Disk method:**

$$V = \pi \int_0^{r^2-k^2} \left[ (\sqrt{r^2 - y})^2 - k^2 \right] dy$$

$$= \pi \int_0^{r^2-k^2} [r^2 - y - k^2] dy$$

$$= \pi \left[ (r^2 - k^2)y - \frac{y^2}{2} \right]_0^{r^2-k^2}$$

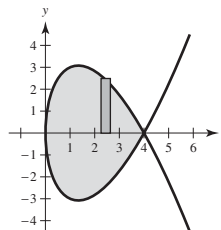
$$= \pi \left[ (r^2 - k^2)^2 - \frac{(r^2 - k^2)^2}{2} \right] = \frac{\pi}{2} (r^2 - k^2)^2$$



$$63. y^2 = x(4-x)^2, \quad 0 \leq x \leq 4$$

$$y_1 = \sqrt{x(4-x)^2} = (4-x)\sqrt{x}$$

$$y_2 = -\sqrt{x(4-x)^2} = -(4-x)\sqrt{x}$$



$$(a) V = \pi \int_0^4 x(4-x)^2 dx$$

$$= \pi \int_0^4 (x^3 - 8x^2 + 16x) dx$$

$$= \pi \left[ \frac{x^4}{4} - \frac{8x^3}{3} + 8x^2 \right]_0^4 = \frac{64\pi}{3}$$

$$(b) V = 4\pi \int_0^4 x(4-x)\sqrt{x} dx$$

$$= 4\pi \int_0^4 (4x^{3/2} - x^{5/2}) dx$$

$$= 4\pi \left[ \frac{8}{5}x^{5/2} - \frac{2}{7}x^{7/2} \right]_0^4 = \frac{2048\pi}{35}$$

$$(c) V = 4\pi \int_0^4 (4-x)(4-x)\sqrt{x} dx$$

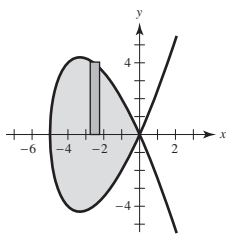
$$= 4\pi \int_0^4 (16\sqrt{x} - 8x^{3/2} + x^{5/2}) dx$$

$$= 4\pi \left[ \frac{32}{3}x^{3/2} - \frac{16}{5}x^{5/2} + \frac{2}{7}x^{7/2} \right]_0^4 = \frac{8192\pi}{105}$$

64.  $y^2 = x^2(x + 5), \quad -5 \leq x \leq 0$

$$y_1 = \sqrt{x^2(x + 5)} = x\sqrt{x + 5}$$

$$y_2 = -\sqrt{x^2(x + 5)} = -x\sqrt{x + 5}$$



(a)  $V = \pi \int_{-5}^0 x^2(x + 5) dx = \pi \left[ \frac{x^4}{4} + \frac{5x^3}{3} \right]_{-5}^0 = \frac{625\pi}{12}$

(b)  $V = 4\pi \int_{-5}^0 x(x\sqrt{x + 5}) dx$

Let  $u = x + 5, du = dx$ .

$$V = 4\pi \int_0^5 (u - 5)^2 \sqrt{u} du = 4\pi \int_0^5 (u^{5/2} - 10u^{3/2} + 25u^{1/2}) du = 4\pi \left[ \frac{2}{7}u^{7/2} - 4u^{5/2} + \frac{50}{3}u^{3/2} \right]_0^5 = \frac{1600\sqrt{5}\pi}{21}$$

(c)  $V = 4\pi \int_{-5}^0 (-5 - x)x\sqrt{x + 5} dx$

Let  $u = x + 5, du = dx$ .

$$V = 4\pi \int_0^5 (-u)(u - 5)\sqrt{u} du = 4\pi \int_0^5 (-u^{5/2} + 5u^{3/2}) du = 4\pi \left[ -\frac{2}{7}u^{7/2} + 2u^{5/2} \right]_0^5 = \frac{400\sqrt{5}\pi}{7}$$

## Section 7.4 Arc Length and Surfaces of Revolution

1.  $(0, 0), (8, 15)$

(a)  $d = \sqrt{(8 - 0)^2 + (15 - 0)^2}$   
 $= \sqrt{64 + 225}$   
 $= \sqrt{289} = 17$

(b)  $y = \frac{15}{8}x$

$$y' = \frac{15}{8}$$

$$s = \int_0^8 \sqrt{1 + \left(\frac{15}{8}\right)^2} dx = \int_0^8 \frac{17}{8} dx = \left[ \frac{17}{8}x \right]_0^8 = 17$$

2.  $(1, 2), (7, 10)$

(a)  $d = \sqrt{(7 - 1)^2 + (10 - 2)^2} = 10$

(b)  $y = \frac{4}{3}x + \frac{2}{3}$

$$y' = \frac{4}{3}$$

$$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2} dx = \left[ \frac{5}{3}x \right]_1^7 = 10$$

3.  $y = \frac{2}{3}(x^2 + 1)^{3/2}$

$$y' = (x^2 + 1)^{1/2}(2x), \quad 0 \leq x \leq 1$$

$$1 + (y')^2 = 1 + 4x^2(x^2 + 1)$$

$$= 4x^4 + 4x^2 + 1 = (2x^2 + 1)^2$$

$$s = \int_0^1 \sqrt{1 + (y')^2} dx$$

$$= \int_0^1 (2x^2 + 1) dx = \left[ \frac{2x^3}{3} + x \right]_0^1 = \frac{5}{3}$$

$$4. \quad y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2} \left( x^2 - \frac{1}{x^2} \right), \quad 1 \leq x \leq 2$$

$$1 + (y')^2 = 1 + \frac{1}{4} \left( x^4 - 2 + \frac{1}{x^4} \right)$$

$$= \frac{1}{4} \left( x^4 + 2 + \frac{1}{x^4} \right)$$

$$= \left[ \frac{1}{2} \left( x^2 + \frac{1}{x^2} \right) \right]^2$$

$$s = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \frac{1}{2} \left( x^2 + \frac{1}{x^2} \right) dx$$

$$= \frac{1}{2} \left[ \frac{x^3}{3} - \frac{1}{x} \right]_1^2$$

$$= \frac{1}{2} \left[ \left( \frac{8}{3} - \frac{1}{2} \right) - \left( \frac{1}{3} - 1 \right) \right]$$

$$= \frac{17}{12}$$

$$5. \quad y = \frac{2}{3}x^{3/2} + 1$$

$$y' = x^{1/2}, \quad 0 \leq x \leq 1$$

$$s = \int_0^1 \sqrt{1 + x} \, dx$$

$$= \left[ \frac{2}{3}(1 + x)^{3/2} \right]_0^1 = \frac{2}{3}(\sqrt{8} - 1) \approx 1.219$$

$$6. \quad y = 2x^{3/2} + 3$$

$$y' = 3x^{1/2}, \quad 0 \leq x \leq 9$$

$$s = \int_0^9 \sqrt{1 + 9x} \, dx$$

$$= \left[ \frac{2}{27}(1 + 9x)^{3/2} \right]_0^9 = \frac{2}{27}(82^{3/2} - 1) \approx 54.929$$

$$9. \quad y = \frac{x^5}{10} + \frac{1}{6x^3}, \quad 2 \leq x \leq 5$$

$$y' = \frac{x^4}{2} - \frac{1}{2x^4} = \frac{1}{2} \left( x^4 - \frac{1}{x^4} \right)$$

$$1 + (y')^2 = 1 + \frac{1}{4} \left( x^4 - \frac{1}{x^4} \right)^2 = 1 + \frac{1}{4} \left( x^8 - 2 + \frac{1}{x^8} \right)$$

$$= \frac{1}{4} \left( x^8 + 2 + \frac{1}{x^8} \right) = \frac{1}{4} \left( x^4 + \frac{1}{x^4} \right)^2$$

$$s = \int_2^5 \sqrt{1 + (y')^2} \, dx = \int_2^5 \frac{1}{2} \left( x^4 + \frac{1}{x^4} \right) dx$$

$$= \frac{1}{2} \left[ \frac{x^5}{5} - \frac{1}{3x^3} \right]_2^5 = \frac{1}{2} \left[ \left( 625 - \frac{1}{375} \right) - \left( \frac{32}{5} - \frac{1}{24} \right) \right]$$

$$= \frac{618639}{2000} \approx 309.320$$

$$7. \quad y = \frac{3}{2}x^{2/3}$$

$$y' = \frac{1}{x^{1/3}}, \quad 1 \leq x \leq 8$$

$$s = \int_1^8 \sqrt{1 + \left( \frac{1}{x^{1/3}} \right)^2} \, dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} \, dx$$

$$= \frac{3}{2} \int_1^8 \sqrt{x^{2/3} + 1} \left( \frac{2}{3x^{1/3}} \right) dx$$

$$= \frac{3}{2} \left[ \frac{2}{3} (x^{2/3} + 1)^{3/2} \right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

$$8. \quad y = \frac{x^4}{8} + \frac{1}{4x^2}$$

$$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, \quad 1 \leq x \leq 3$$

$$1 + (y')^2 = \left( \frac{1}{2}x^3 + \frac{1}{2x^3} \right)^2, \quad [1, 3]$$

$$s = \int_1^3 \sqrt{1 + (y')^2} \, dx$$

$$= \int_1^3 \left( \frac{1}{2}x^3 + \frac{1}{2x^3} \right) dx$$

$$= \left[ \frac{1}{8}x^4 - \frac{1}{4x^2} \right]_1^3$$

$$= \frac{92}{9} \approx 10.222$$



10.  $y = \frac{3}{2}x^{2/3} + 4$

$$y' = x^{-1/3}, \quad 1 \leq x \leq 27$$

$$\begin{aligned} s &= \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx \\ &= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx \\ &= \frac{3}{2} \int_1^{27} \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx \\ &= \left[ \frac{3}{2} \cdot \frac{2}{3} (x^{2/3} + 1)^{3/2} \right]_1^{27} \\ &= 10^{3/2} - 2^{3/2} \approx 28.794 \end{aligned}$$

11.  $y = \ln(\sin x), \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$y' = \frac{1}{\sin x} \cos x = \cot x$$

$$\begin{aligned} 1 + (y')^2 &= 1 + \cot^2 x = \csc^2 x \\ s &= \int_{\pi/4}^{3\pi/4} \csc x \, dx \\ &= \left[ \ln|\csc x - \cot x| \right]_{\pi/4}^{3\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \approx 1.763 \end{aligned}$$

12.  $y = \ln(\cos x), \quad 0 \leq x \leq \frac{\pi}{3}$

$$y' = \frac{-\sin x}{\cos x} = -\tan x$$

$$\begin{aligned} 1 + (y')^2 &= 1 + \tan^2 x = \sec^2 x \\ s &= \int_0^{\pi/3} \sqrt{\sec^2 x} \, dx \\ &= \int_0^{\pi/3} \sec x \, dx \\ &= \ln|\sec x + \tan x| \Big|_0^{\pi/3} \\ &= \ln(2 + \sqrt{3}) \approx 1.3170 \end{aligned}$$

13.  $y = \frac{1}{2}(e^x + e^{-x})$

$$y' = \frac{1}{2}(e^x - e^{-x}), \quad [0, 2]$$

$$\begin{aligned} 1 + (y')^2 &= \left[ \frac{1}{2}(e^x + e^{-x}) \right]^2, \quad [0, 2] \\ s &= \int_0^2 \sqrt{\left[ \frac{1}{2}(e^x + e^{-x}) \right]^2} dx \\ &= \frac{1}{2} \int_0^2 (e^x + e^{-x}) dx \\ &= \frac{1}{2} [e^x - e^{-x}]_0^2 = \frac{1}{2} \left( e^2 - \frac{1}{e^2} \right) \approx 3.627 \end{aligned}$$

14.  $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right) = \ln(e^x + 1) - \ln(e^x - 1)$

$$\frac{dy}{dx} = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1} = \frac{-2e^x}{e^{2x} - 1} = \frac{2e^x}{1 - e^{2x}}$$

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{4e^{2x}}{1 - 2e^{2x} + e^{4x}} \\ &= \frac{1 + 2e^{2x} + e^{4x}}{(1 - e^{2x})^2} = \left(\frac{1 + e^{2x}}{1 - e^{2x}}\right)^2 \\ s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\ln 2}^{\ln 3} \frac{1 + e^{2x}}{e^{2x} - 1} dx \\ &= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int_{\ln 2}^{\ln 3} \coth x \, dx \\ &= \ln(\sinh(x)) \Big|_{\ln 2}^{\ln 3} = \ln\left(\frac{4}{3}\right) - \ln\left(\frac{3}{4}\right) \\ &= \ln\left(\frac{4/3}{3/4}\right) = \ln \frac{16}{9} - 2 \ln\left(\frac{4}{3}\right) \approx 0.57536 \end{aligned}$$

15.  $x = \frac{1}{3}(y^2 + 2)^{3/2}, \quad 0 \leq y \leq 4$

$$\begin{aligned} \frac{dx}{dy} &= y(y^2 + 2)^{1/2} \\ s &= \int_0^4 \sqrt{1 + y^2(y^2 + 2)} \, dy \\ &= \int_0^4 \sqrt{y^4 + 2y^2 + 1} \, dy \\ &= \int_0^4 (y^2 + 1) \, dy \\ &= \left[ \frac{y^3}{3} + y \right]_0^4 = \frac{64}{3} + 4 = \frac{76}{3} \end{aligned}$$

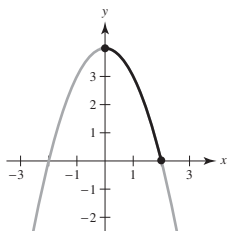
16.  $x = \frac{1}{3}\sqrt{y}(y - 3), \quad 1 \leq y \leq 4$

$$x = \frac{1}{3}(y^{3/2} - 3y^{1/2})$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$\begin{aligned} 1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \frac{1}{4}y + \frac{1}{4}y^{-1} - \frac{1}{2} \\ &= \frac{1}{4}(y + 2 + y^{-1}) = \frac{1}{4}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2 \\ s &= \int_1^4 \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy \\ &= \left[ \frac{1}{2}\left(\frac{3}{2}y^{3/2} + 2y^{1/2}\right) \right]_1^4 \\ &= \frac{1}{2}\left(\frac{16}{3} + 4\right) - \frac{1}{2}\left(\frac{2}{3} + 2\right) = \frac{10}{3} \end{aligned}$$

17. (a)  $y = 4 - x^2, \quad 0 \leq x \leq 2$



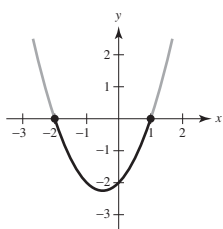
(b)  $y' = -2x$

$1 + (y')^2 = 1 + 4x^2$

$$L = \int_0^2 \sqrt{1 + 4x^2} \, dx$$

(c)  $L \approx 4.647$

18. (a)  $y = x^2 + x - 2, \quad -2 \leq x \leq 1$



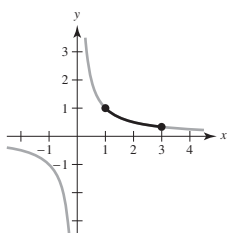
(b)  $y' = 2x + 1$

$1 + (y')^2 = 1 + 4x^2 + 4x + 1$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2} \, dx$$

(c)  $L \approx 5.653$

19. (a)  $y = \frac{1}{x}, \quad 1 \leq x \leq 3$



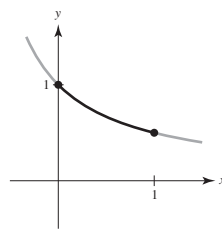
(b)  $y' = -\frac{1}{x^2}$

$1 + (y')^2 = 1 + \frac{1}{x^4}$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}} \, dx$$

(c)  $L \approx 2.147$

20. (a)  $y = \frac{1}{1+x}, \quad 0 \leq x \leq 1$



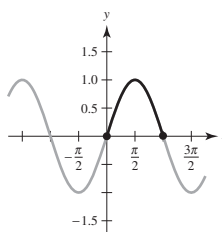
(b)  $y' = -\frac{1}{(1+x)^2}$

$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}} \, dx$$

(c)  $L \approx 1.132$

21. (a)  $y = \sin x, \quad 0 \leq x \leq \pi$



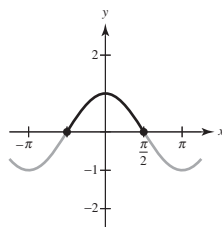
(b)  $y' = \cos x$

$1 + (y')^2 = 1 + \cos^2 x$

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx$$

(c)  $L \approx 3.820$

22. (a)  $y = \cos x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



(b)  $y' = -\sin x$

$1 + (y')^2 = 1 + \sin^2 x$

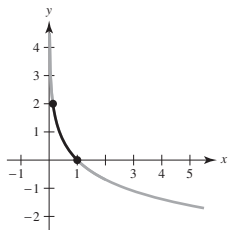
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} \, dx$$

(c) 3.820

23. (a)  $x = e^{-y}, \quad 0 \leq y \leq 2$

$y = -\ln x$

$1 \geq x \geq e^{-2} \approx 0.135$



(b)  $y' = -\frac{1}{x}$

$1 + (y')^2 = 1 + \frac{1}{x^2}$

$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}} dx$

(c)  $L \approx 2.221$

Alternatively, you can do all the computations with respect to  $y$ .

(a)  $x = e^{-y}, \quad 0 \leq y \leq 2$

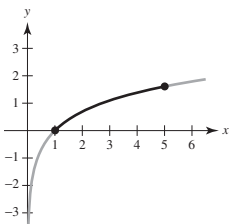
(b)  $\frac{dx}{dy} = -e^{-y}$

$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$

$L = \int_0^2 \sqrt{1 + e^{-2y}} dy$

(c)  $L \approx 2.221$

24. (a)  $y = \ln x, \quad 1 \leq x \leq 5$



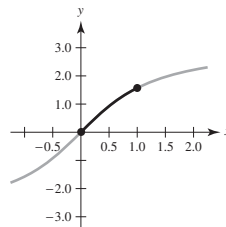
(b)  $y' = \frac{1}{x}$

$1 + (y')^2 = 1 + \frac{1}{x^2}$

$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}} dx$

(c)  $L \approx 4.367$

25. (a)  $y = 2 \arctan x, \quad 0 \leq x \leq 1$



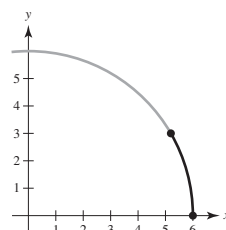
(b)  $y' = \frac{2}{1 + x^2}$

$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}} dx$

(c)  $L \approx 1.871$

26. (a)  $x = \sqrt{36 - y^2}, \quad 0 \leq y \leq 3$

$y = \sqrt{36 - x^2}, \quad 3\sqrt{3} \leq x \leq 6$



(b)  $\frac{dx}{dy} = \frac{1}{2}(36 - y^2)^{-1/2}(-2y)$

$= \frac{-y}{\sqrt{36 - y^2}}$

$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}} dy$

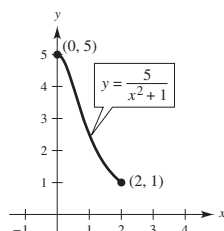
$= \int_0^3 \frac{6}{\sqrt{36 - y^2}} dy$

(c)  $L \approx 3.142 \quad (\pi)$

27.  $\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2} dx$

$s \approx 5$

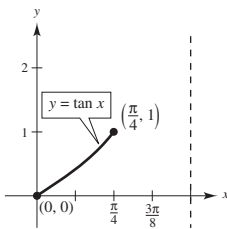
Matches (b)



$$28. \int_0^{\pi/4} \sqrt{1 + \left[ \frac{d}{dx}(\tan x) \right]^2} dx$$

$$s \approx 1$$

Matches (e)



$$29. y = x^3, \quad [0, 4]$$

$$(a) d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$$

$$(b) d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2} \\ \approx 64.525$$

$$(c) s = \int_0^4 \sqrt{1 + (3x^2)^2} dx = \int_0^4 \sqrt{1 + 9x^4} dx \approx 64.666 \quad (\text{Simpson's Rule, } n = 10)$$

$$(d) 64.672$$

$$30. f(x) = (x^2 - 4)^2, \quad [0, 4]$$

$$(a) d = \sqrt{(4-0)^2 + (144-16)^2} \approx 128.062$$

$$(b) d = \sqrt{(1-0)^2 + (9-16)^2} + \sqrt{(2-1)^2 + (0-9)^2} + \sqrt{(3-2)^2 + (25-0)^2} + \sqrt{(4-3)^2 + (144-25)^2} \\ \approx 160.151$$

$$(c) s = \int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2} dx \approx 159.087$$

$$(d) 160.287$$

$$31. y = 20 \cosh \frac{x}{20}, \quad -20 \leq x \leq 20$$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh \frac{x}{20} dx = 2 \int_0^{20} \cosh \frac{x}{20} dx = \left[ 2(20) \sinh \frac{x}{20} \right]_0^{20} = 40 \sinh(1) \approx 47.008 \text{ m}$$

$$32. y = 31 - 10(e^{x/20} + e^{-x/20})$$

$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10}) = \left[ \frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2$$

$$s = \int_{-20}^{20} \sqrt{\left[ \frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2} dx = \frac{1}{2} \int_{-20}^{20} (e^{x/20} + e^{-x/20}) dx = \left[ 10(e^{x/20} - e^{-x/20}) \right]_{-20}^{20} = 20 \left( e - \frac{1}{e} \right) \approx 47 \text{ ft}$$

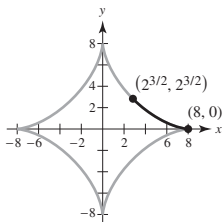
So, there are  $100(47) = 4700$  square feet of roofing on the barn.

33.  $y = 693.8597 - 68.7672 \cosh 0.0100333x$   
 $y' = -0.6899619478 \sinh 0.0100333x$   
 $s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2} dx \approx 1480$

(Use Simpson's Rule with  $n = 100$  or a graphing utility.)

34.  $x^{2/3} + y^{2/3} = 4$   
 $y^{2/3} = 4 - x^{2/3}$   
 $y = (4 - x^{2/3})^{3/2}$   
 $y' = \frac{3}{2}(4 - x^{2/3})^{1/2} \left( -\frac{2}{3}x^{-1/3} \right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$

In order to avoid division by 0, compute the arc length for  $2^{3/2} \leq x \leq 8$ , and multiply the answer by 8, as indicated in the figure.



$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}}, \quad 2^{3/2} \leq x \leq 8$$

$$= \frac{4}{x^{2/3}}$$

$$s = 8 \int_{2^{3/2}}^8 \sqrt{\frac{4}{x^{2/3}}} dx$$

$$= 16 \int_{2^{3/2}}^8 x^{-1/3} dx$$

$$= 16 \left[ \frac{3}{2} x^{2/3} \right]_{2^{3/2}}^8$$

$$= 24(4 - 2) = 48$$

35.  $y = \sqrt{9 - x^2}$   
 $y' = \frac{-x}{\sqrt{9 - x^2}}$   
 $1 + (y')^2 = \frac{9}{9 - x^2}$   
 $s = \int_0^2 \sqrt{\frac{9}{9 - x^2}} dx = \int_0^2 \frac{3}{\sqrt{9 - x^2}} dx$   
 $= \left[ 3 \arcsin \frac{x}{3} \right]_0^2 = 3 \left( \arcsin \frac{2}{3} - \arcsin 0 \right)$   
 $= 3 \arcsin \frac{2}{3} \approx 2.1892$

36.  $y = \sqrt{25 - x^2}$   
 $y' = \frac{-x}{\sqrt{25 - x^2}}$   
 $1 + (y')^2 = \frac{25}{25 - x^2}$   
 $s = \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}} dx = \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}} dx$   
 $= \left[ 5 \arcsin \frac{x}{5} \right]_{-3}^4 = 5 \left[ \arcsin \frac{4}{5} - \arcsin \left( -\frac{3}{5} \right) \right]$   
 $\approx 7.8540$

$$\frac{1}{4} [2\pi(5)] \approx 7.8540 = s$$

37.  $y = \frac{x^3}{3}$   
 $y' = x^2, \quad [0, 3]$   
 $S = 2\pi \int_0^3 \frac{x^3}{3} \sqrt{1 + x^4} dx$   
 $= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} (4x^3) dx$   
 $= \left[ \frac{\pi}{9} (1 + x^4)^{3/2} \right]_0^3$   
 $= \frac{\pi}{9} (82\sqrt{82} - 1) \approx 258.85$

38.  $y = 2\sqrt{x}$   
 $y' = \frac{1}{\sqrt{x}}, \quad [4, 9]$   
 $S = 2\pi \int_4^9 2\sqrt{x} \sqrt{1 + \frac{1}{x}} dx$   
 $= 4\pi \int_4^9 \sqrt{x+1} dx$   
 $= \left[ \frac{8}{3} \pi (x+1)^{3/2} \right]_4^9$   
 $= \frac{8\pi}{3} (10^{3/2} - 5^{3/2}) \approx 171.258$

$$39. \quad y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left( \frac{x^2}{2} + \frac{1}{2x^2} \right)^2, \quad [1, 2]$$

$$S = 2\pi \int_1^2 \left( \frac{x^3}{6} + \frac{1}{2x} \right) \left( \frac{x^2}{2} + \frac{1}{2x^2} \right) dx$$

$$= 2\pi \int_1^2 \left( \frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx$$

$$= 2\pi \left[ \frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2} \right]_1^2 = \frac{47\pi}{16}$$

$$40. \quad y = \frac{x}{2}$$

$$y' = \frac{1}{2}$$

$$1 + (y')^2 = \frac{5}{4}, \quad [0, 6]$$

$$S = 2\pi \int_0^6 \frac{x}{2} \sqrt{\frac{5}{4}} dx$$

$$= \left[ \frac{2\pi\sqrt{5}}{8} x^2 \right]_0^6 = 9\sqrt{5}\pi$$

$$41. \quad y = \sqrt{4 - x^2}$$

$$y' = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}, \quad -1 \leq x \leq 1$$

$$1 + (y')^2 = 1 + \frac{x^2}{4 - x^2} = \frac{4}{4 - x^2}$$

$$S = 2\pi \int_{-1}^1 \sqrt{4 - x^2} \cdot \sqrt{\frac{4}{4 - x^2}} dx$$

$$= 4\pi \int_{-1}^1 dx = 4\pi[x]_{-1}^1 = 8\pi$$

$$42. \quad y = \sqrt{9 - x^2}, \quad -2 \leq x \leq 2$$

$$y' = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{9 - x^2} = \frac{9}{9 - x^2}$$

$$S = 2\pi \int_{-2}^2 \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}} dx = 2\pi \int_{-2}^2 3 dx$$

$$= 2\pi[3x]_{-2}^2 = 24\pi$$

$$43. \quad y = \sqrt[3]{x} + 2$$

$$y' = \frac{1}{3x^{2/3}}, \quad [1, 8]$$

$$S = 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}} dx$$

$$= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3} + 1} dx$$

$$= \frac{\pi}{18} \int_1^8 (9x^{4/3} + 1)^{1/2} (12x^{1/3}) dx$$

$$= \left[ \frac{\pi}{27} (9x^{4/3} + 1)^{3/2} \right]_1^8$$

$$= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10}) \approx 199.48$$

$$44. \quad y = 9 - x^2, \quad [0, 3]$$

$$y' = -2x$$

$$S = 2\pi \int_0^3 x \sqrt{1 + 4x^2} dx$$

$$= \frac{\pi}{4} \int_0^3 (1 + 4x^2)^{1/2} (8x) dx$$

$$= \left[ \frac{\pi}{6} (1 + 4x^2)^{3/2} \right]_0^3 = \frac{\pi}{6} (37^{3/2} - 1) \approx 117.319$$

$$45. \quad y = 1 - \frac{x^2}{4}$$

$$y' = -\frac{x}{2}, \quad 0 \leq x \leq 2$$

$$1 + (y')^2 = 1 + \frac{x^2}{4} = \frac{4 + x^2}{4}$$

$$S = 2\pi \int_0^2 x \sqrt{\frac{4 + x^2}{4}} dx$$

$$= \pi \int_0^2 x \sqrt{4 + x^2} dx$$

$$= \frac{1}{2} \pi \int_0^2 (4 + x^2)^{1/2} (2x) dx$$

$$= \frac{1}{2} \pi \left[ \frac{2}{3} (4 + x^2)^{3/2} \right]_0^2$$

$$= \frac{\pi}{3} (8^{3/2} - 4^{3/2})$$

$$= \frac{\pi}{3} (16\sqrt{2} - 8) \approx 15.318$$

$$46. \quad y = 2x + 5$$

$$y' = 2, \quad 1 \leq x \leq 4$$

$$V = \int_1^4 2\pi x \sqrt{1 + 4} dx$$

$$= 2\pi\sqrt{5} \left[ \frac{x^2}{2} \right]_1^4$$

$$= 2\pi\sqrt{5} \left( 8 - \frac{1}{2} \right) = 15\sqrt{5}\pi$$

47.  $y = \sin x$   
 $y' = \cos x, \quad [0, \pi]$   
 $S = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} \, dx \approx 14.4236$

48.  $y = \ln x$   
 $y' = \frac{1}{x}$   
 $1 + (y')^2 = \frac{x^2 + 1}{x^2}, \quad [1, e]$   
 $S = 2\pi \int_1^e x \sqrt{\frac{x^2 + 1}{x^2}} \, dx = 2\pi \int_1^e \sqrt{x^2 + 1} \, dx$   
 $\approx 22.943$

49. A rectifiable curve is one that has a finite arc length.

50. The precalculus formula is the distance formula between two points. The representative element is

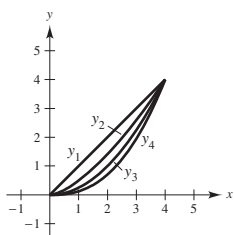
$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

51. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The formula is  $S = 2\pi rL$ , where  $r = \frac{1}{2}(r_1 + r_2)$ , which is the average radius of the frustum, and  $L$  is the length of a line segment on the frustum. The representative element is

$$2\pi f(d_i) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

52. The surface of revolution given by  $f_1$  will be larger.  $r(x)$  is larger for  $f_1$ .

53. (a)



(b)  $y_1, y_2, y_3, y_4$

(c)  $y'_1 = 1, \quad s_1 = \int_0^4 \sqrt{2} \, dx \approx 5.657$

$$y'_2 = \frac{3}{4}x^{1/2}, \quad s_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}} \, dx \approx 5.759$$

$$y'_3 = \frac{1}{2}x, \quad s_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}} \, dx \approx 5.916$$

$$y'_4 = \frac{5}{16}x^{3/2}, \quad s_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3} \, dx \approx 6.063$$

54. Let  $y = \ln x, 1 \leq x \leq e, y' = \frac{1}{x}$  and

$$s_1 = \int_1^e \sqrt{1 + \frac{1}{x^2}} \, dx.$$

Equivalently,  $x = e^y, 0 \leq y \leq 1, \frac{dx}{dy} = e^y$ , and

$$s_2 = \int_0^1 \sqrt{1 + e^{2y}} \, dy = \int_0^1 \sqrt{1 + e^{2x}} \, dx.$$

Numerically, both integrals yield  $s = 2.0035$ .

55.  $y = \frac{3x}{4}, \quad y' = \frac{3}{4}$

$$1 + (y')^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$S = 2\pi \int_0^4 x \sqrt{\frac{25}{16}} \, dx = \frac{5\pi}{2} \left[ \frac{x^2}{2} \right]_0^4 = 20\pi$$

56.  $y = \frac{hx}{r}$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x \sqrt{\frac{r^2 + h^2}{r^2}} \, dx$$

$$= \left[ \frac{2\pi \sqrt{r^2 + h^2}}{r} \left( \frac{x^2}{2} \right) \right]_0^r = \pi r \sqrt{r^2 + h^2}$$

57.  $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}} \, dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}} \, dx$$

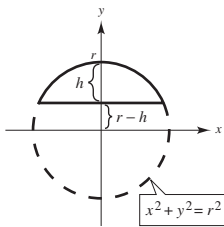
$$= \left[ -6\pi \sqrt{9 - x^2} \right]_0^2$$

$$= 6\pi(3 - \sqrt{5}) \approx 14.40$$

See figure in Exercise 58.

58. From Exercise 57 you have:

$$\begin{aligned}
 S &= 2\pi \int_0^a \frac{rx}{\sqrt{r^2 - x^2}} dx \\
 &= -r\pi \int_0^a \frac{-2x}{\sqrt{r^2 - x^2}} dx \\
 &= \left[ -2r\pi \sqrt{r^2 - x^2} \right]_0^a \\
 &= 2r^2\pi - 2r\pi \sqrt{r^2 - a^2} \\
 &= 2r\pi \left( r - \sqrt{r^2 - a^2} \right) \\
 &= 2\pi rh \text{ (where } h \text{ is the height of the zone)}
 \end{aligned}$$



59. (a) Approximate the volume by summing six disks of thickness 3 and circumference  $C_i$  equal to the average of the given circumferences:

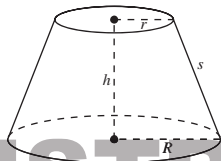
$$\begin{aligned}
 V &\approx \sum_{i=1}^6 \pi r_i^2 (3) = \sum_{i=1}^6 \pi \left( \frac{C_i}{2\pi} \right)^2 (3) = \frac{3}{4\pi} \sum_{i=1}^6 C_i^2 \\
 &= \frac{3}{4\pi} \left[ \left( \frac{50 + 65.5}{2} \right)^2 + \left( \frac{65.5 + 70}{2} \right)^2 + \left( \frac{70 + 66}{2} \right)^2 + \left( \frac{66 + 58}{2} \right)^2 + \left( \frac{58 + 51}{2} \right)^2 + \left( \frac{51 + 48}{2} \right)^2 \right] \\
 &= \frac{3}{4\pi} [57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2] = \frac{3}{4\pi} (21813.625) = 5207.62 \text{ in.}^3
 \end{aligned}$$

(b) The lateral surface area of a frustum of a right circular cone is  $\pi s(R + r)$ . For the first frustum:

$$\begin{aligned}
 S_1 &\approx \pi \left[ 3^2 + \left( \frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2} \left[ \frac{50}{2\pi} + \frac{65.5}{2\pi} \right] \\
 &= \left( \frac{50 + 65.5}{2} \right) \left[ 9 + \left( \frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2}.
 \end{aligned}$$

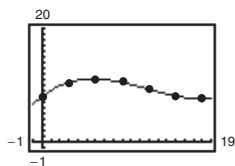
Adding the six frustums together:

$$\begin{aligned}
 S &\approx \left( \frac{50 + 65.5}{2} \right) \left[ 9 + \left( \frac{15.5}{2\pi} \right)^2 \right]^{1/2} + \left( \frac{65.5 + 70}{2} \right) \left[ 9 + \left( \frac{4.5}{2\pi} \right)^2 \right]^{1/2} \\
 &\quad + \left( \frac{70 + 66}{2} \right) \left[ 9 + \left( \frac{4}{2\pi} \right)^2 \right]^{1/2} + \left( \frac{66 + 58}{2} \right) \left[ 9 + \left( \frac{8}{2\pi} \right)^2 \right]^{1/2} \\
 &\quad + \left( \frac{58 + 51}{2} \right) \left[ 9 + \left( \frac{7}{2\pi} \right)^2 \right]^{1/2} + \left( \frac{51 + 48}{2} \right) \left[ 9 + \left( \frac{3}{2\pi} \right)^2 \right]^{1/2} \\
 &\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 = 1168.64
 \end{aligned}$$





(c)  $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



(d)  $V = \int_0^{18} \pi r^2 dy \approx 5275.9 \text{ in.}^3$

$S = \int_0^{18} 2\pi r(y) \sqrt{1 + r'(y)^2} dy \approx 1179.5 \text{ in.}^2$

60. (a)  $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

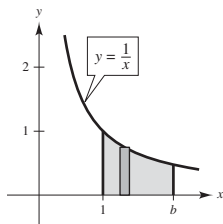
(b)  $\text{Area} = \int_0^{400} f(x) dx \approx 131,734.5 \text{ ft}^2 \approx 3.0 \text{ acres}$  (1 acre = 43,560 ft<sup>2</sup>)

(Answers will vary.)

(c)  $L = \int_0^{400} \sqrt{1 + f'(x)^2} dx \approx 794.9 \text{ ft}$

(Answers will vary.)

61. (a)  $V = \pi \int_1^b \frac{1}{x^2} dx = \left[ -\frac{\pi}{x} \right]_1^b = \pi \left( 1 - \frac{1}{b} \right)$



(b) 
$$\begin{aligned} S &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left( -\frac{1}{x^2} \right)^2} dx \\ &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \\ &= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx \end{aligned}$$

(c)  $\lim_{b \rightarrow \infty} V = \lim_{b \rightarrow \infty} \pi \left( 1 - \frac{1}{b} \right) = \pi$

(d) Because

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b],$$

you have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx > \int_1^b \frac{1}{x} dx = [\ln x]_1^b = \ln b$$

and  $\lim_{b \rightarrow \infty} \ln b \rightarrow \infty$ . So,

$$\lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx = \infty.$$

62. (a) Area of circle with radius  $L$ :  $A = \pi L^2$

Area of sector with central angle  $\theta$  (in radians):

$$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$$

(b) Let  $s$  be the arc length of the sector, which is the circumference of the base of the cone. Here,  $s = L\theta = 2\pi r$ , and you have

$$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left( \frac{s}{L} \right) = \frac{1}{2} Ls = \frac{1}{2} L(2\pi r) = \pi rL.$$

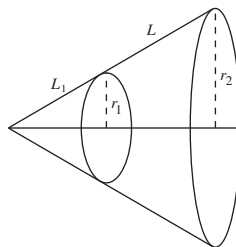
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$\begin{aligned} S &= \pi r_2(L + L_1) - \pi r_1 L_1 \\ &= \pi r_2 L + \pi L_1(r_2 - r_1) \end{aligned}$$

By similar triangles,

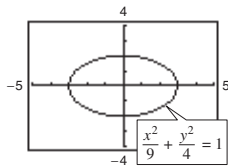
$$\frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow Lr_1 = L_1(r_2 - r_1). \text{ So,}$$

$$\begin{aligned} S &= \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi Lr_1 \\ &= \pi L(r_1 + r_2). \end{aligned}$$



63. (a)  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse:  $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$   
 $y_2 = -2\sqrt{1 - \frac{x^2}{9}}$



(b)  $y = 2\sqrt{1 - \frac{x^2}{9}}, \quad 0 \leq x \leq 3$

$$y' = 2\left(\frac{1}{2}\right)\left(1 - \frac{x^2}{9}\right)^{-1/2}\left(-\frac{2x}{9}\right)$$

$$= \frac{-2x}{9\sqrt{1 - (x^2/9)}} = \frac{-2x}{3\sqrt{9 - x^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}} dx$$

- (c) You cannot evaluate this definite integral, because the integrand is not defined at  $x = 3$ . Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

66.  $y = \frac{1}{3}x^{1/2} - x^{3/2}$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right) \sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2} dx = \frac{2\pi}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2}) dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2\right) dx = \frac{\pi}{3} \left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27} \text{ ft}^2 \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in.}^2$$

Amount of glass needed:  $V = \frac{\pi}{27} \left(\frac{0.015}{12}\right) \approx 0.00015 \text{ ft}^3 \approx 0.25 \text{ in.}^3$

67.  $x^{2/3} + y^{2/3} = 4$

$$y^{2/3} = 4 - x^{2/3}$$

$$y = (4 - x^{2/3})^{3/2}, \quad 0 \leq x \leq 8$$

$$y' = \frac{3}{2}(4 - x^{2/3})^{1/2} \left(-\frac{2}{3}x^{-1/3}\right) = \frac{-(4 - x^{2/3})^{1/2}}{x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}$$

$$S = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \sqrt{\frac{4}{x^{2/3}}} dx = 4\pi \int_0^8 \frac{(4 - x^{2/3})^{3/2}}{x^{1/3}} dx = \left[-\frac{12\pi}{5}(4 - x^{2/3})^{5/2}\right]_0^8 = \frac{384\pi}{5}$$

[Surface area of portion above the  $x$ -axis]

64. Essay

65.  $y = \frac{1}{3}(x^{3/2} - 3x^{1/2} + 2)$

When  $x = 0$ ,  $y = \frac{2}{3}$ . So, the fleeing object has traveled

$\frac{2}{3}$  unit when it is caught.

$$y' = \frac{1}{3}\left(\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}\right) = \left(\frac{1}{2}\right)\frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}} dx = \frac{1}{2} \int_0^1 (x^{1/2} + x^{-1/2}) dx$$

$$= \frac{1}{2} \left[ \frac{2}{3}x^{3/2} + 2x^{1/2} \right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

68.  $y^2 = \frac{1}{12}x(4-x)^2, \quad 0 \leq x \leq 4$

$$y = \frac{(4-x)\sqrt{x}}{\sqrt{12}}$$

$$y' = \frac{(4-3x)\sqrt{3}}{12\sqrt{x}}$$

$$\begin{aligned} 1 + (y')^2 &= 1 + \frac{(4-3x)^2}{48x} \\ &= \frac{48x + 16 - 24x + 9x^2}{48x} = \frac{(4+3x)^2}{48x}, \quad x \neq 0 \end{aligned}$$

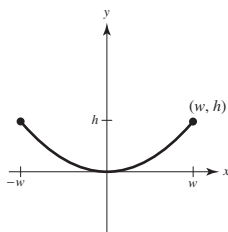
$$\begin{aligned} S &= 2\pi \int_0^4 \frac{(4-x)\sqrt{x}}{\sqrt{12}} \cdot \frac{(4+3x)}{\sqrt{48x}} dx \\ &= 2\pi \int_0^4 \frac{(4-x)(4+3x)}{24} dx \\ &= \frac{\pi}{12} \int_0^4 (16 + 8x - 3x^2) dx = \frac{\pi}{12} [16x + 4x^2 - x^3]_0^4 = \frac{\pi}{12} (64 + 64 - 64) = \frac{16\pi}{3} \end{aligned}$$

69.  $y = kx^2, y' = 2kx$

$$1 + (y')^2 = 1 + 4k^2x^2$$

$$h = kw^2 \Rightarrow k = \frac{h}{w^2} \Rightarrow 1 + (y')^2 = 1 + \frac{4h^2}{w^4}x^2$$

By symmetry,  $C = 2 \int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2} dx$ .



70.  $C = 2 \int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2} dx$

$$= 2 \int_0^{700} \sqrt{1 + \frac{4(155)^2}{700^4}x^2} dx = 1444.5 \text{ m}$$

71.  $y = f(x) = \cosh x$

$$y' = \sinh x$$

$$1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$$

$$\text{Area} = \int_0^t \cosh x dx = [\sinh x]_0^t = \sinh t$$

$$\text{Arc length} = \int_0^t \sqrt{1 + (y')^2} dx$$

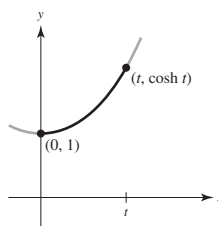
$$= \int_0^t \cosh x dx = \sinh x \Big|_0^t$$

$$= \sinh t.$$

Another curve with this property is  $g(x) = 1$ .

$$\text{Area} = \int_0^t dx = t$$

$$\text{Arc length} = t$$



72. Let  $(x_0, y_0)$  be the point on the graph of  $y^2 = x^3$  where the tangent line makes an angle of  $45^\circ$  with the  $x$ -axis.

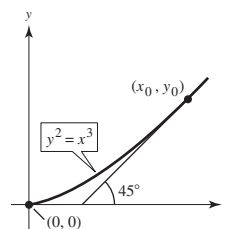
$$y = x^{3/2}$$

$$y' = \frac{3}{2}x^{1/2} = 1$$

$$x_0 = \frac{4}{9}$$

$$L = \int_0^{4/9} \sqrt{1 + \frac{9}{4}x} dx$$

$$= \frac{8}{27}(2\sqrt{2} - 1)$$



## Section 7.5 Work

$$1. W = Fd = (100)(20) = 2000 \text{ ft-lb}$$

$$2. W = Fd = (3500)(4) = 14,000 \text{ ft-lb}$$

$$3. W = Fd = (112)(8) = 896 \text{ joules (Newton-meters)}$$

$$4. W = Fd = [9(2000)]\left[\frac{1}{2}(5280)\right] = 47,520,000 \text{ ft-lb}$$

$$5. F(x) = kx$$

$$5 = k(3)$$

$$k = \frac{5}{3}$$

$$F(x) = \frac{5}{3}x$$

$$W = \int_0^7 F(x) dx = \int_0^7 \frac{5}{3}x dx = \left[\frac{5}{6}x^2\right]_0^7 = \frac{245}{6} \text{ in.-lb} \\ \approx 40.833 \text{ in.-lb} \approx 3.403 \text{ ft-lb}$$

$$6. \text{ From Exercise 5, } F(x) = \frac{5}{3}x.$$

$$W = \int_{15-10}^{15-6} F(x) dx = \int_5^9 \frac{5}{3}x dx = \left[\frac{5x^2}{6}\right]_5^9 = \frac{140}{3} \text{ in.-lb}$$

$$7. F(x) = kx$$

$$250 = k(30) \Rightarrow k = \frac{25}{3}$$

$$W = \int_{20}^{50} F(x) dx \\ = \int_{20}^{50} \frac{25}{3}x dx = \left[\frac{25x^2}{6}\right]_{20}^{50} \\ = 8750 \text{ n-cm} \\ = 87.5 \text{ joules or Nm}$$

$$8. F(x) = kx$$

$$800 = k(70) \Rightarrow k = \frac{80}{7}$$

$$W = \int_0^{70} F(x) dx \\ = \int_0^{70} \frac{80}{7}x dx = \left[\frac{40x^2}{7}\right]_0^{70} \\ = 28,000 \text{ n-cm} = 280 \text{ Nm}$$

$$9. F(x) = kx$$

$$20 = k(9)$$

$$k = \frac{20}{9}$$

$$W = \int_0^{12} \frac{20}{9}x dx = \left[\frac{10}{9}x^2\right]_0^{12} = 160 \text{ in.-lb} = \frac{40}{3} \text{ ft-lb}$$

$$10. F(x) = kx$$

$$15 = k(1) = k$$

$$W = 2 \int_0^4 15x dx = [15x^2]_0^4 = 240 \text{ ft-lb}$$

$$11. W = 18 = \int_0^{1/3} kx dx = \left[\frac{kx^2}{2}\right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$$

$$W = \int_{1/3}^{7/12} 324x dx = [162x^2]_{1/3}^{7/12} = 37.125 \text{ ft-lb}$$

$$\left[\text{Note: } 4 \text{ inches} = \frac{1}{3} \text{ foot}\right]$$

$$12. W = 7.5 = \int_0^{1/6} kx dx = \left[\frac{kx^2}{2}\right]_0^{1/6} = \frac{k}{72} \Rightarrow k = 540$$

$$W = \int_{1/6}^{5/24} 540x dx = [270x^2]_{1/6}^{5/24} = 4.21875 \text{ ft-lb}$$

$$13. \text{ Assume that Earth has a radius of 4000 miles.}$$

$$F(x) = \frac{k}{x^2}$$

$$5 = \frac{k}{(4000)^2}$$

$$k = 80,000,000$$

$$F(x) = \frac{80,000,000}{x^2}$$

$$(a) W = \int_{4000}^{4100} \frac{80,000,000}{x^2} dx = \left[\frac{-80,000,000}{x}\right]_{4000}^{4100} \\ \approx 487.8 \text{ mi-tons} \approx 5.15 \times 10^9 \text{ ft-lb}$$

$$(b) W = \int_{4000}^{4300} \frac{80,000,000}{x^2} dx \\ \approx 1395.3 \text{ mi-ton} \approx 1.47 \times 10^{10} \text{ ft-ton}$$

$$14. W = \int_{4000}^h \frac{80,000,000}{x^2} dx \\ = \left[\frac{-80,000,000}{x}\right]_{4000}^h \\ = \frac{-80,000,000}{h} + 20,000$$

$$\lim_{h \rightarrow \infty} W = 20,000 \text{ mi-ton} \approx 2.1 \times 10^{11} \text{ ft-lb}$$

15. Assume that Earth has a radius of 4000 miles.

$$F(x) = \frac{k}{x^2}$$

$$10 = \frac{k}{(4000)^2}$$

$$k = 160,000,000$$

$$F(x) = \frac{160,000,000}{x^2}$$

$$\begin{aligned} \text{(a) } W &= \int_{4000}^{15,000} \frac{160,000,000}{x^2} dx = \left[ -\frac{160,000,000}{x} \right]_{4000}^{15,000} \approx -10,666.667 + 40,000 \\ &= 29,333.333 \text{ mi-ton} \\ &\approx 2.93 \times 10^4 \text{ mi-ton} \\ &\approx 3.10 \times 10^{11} \text{ ft-lb} \end{aligned}$$

$$\begin{aligned} \text{(b) } W &= \int_{4000}^{26,000} \frac{160,000,000}{x^2} dx = \left[ -\frac{160,000,000}{x} \right]_{4000}^{26,000} \approx -6,153.846 + 40,000 \\ &= 33,846.154 \text{ mi-ton} \\ &\approx 3.38 \times 10^4 \text{ mi-ton} \\ &\approx 3.57 \times 10^{11} \text{ ft-lb} \end{aligned}$$

16. Weight on surface of moon:  $\frac{1}{6}(12) = 2$  tons

Weight varies inversely as the square of distance from the center of the moon. Therefore:

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

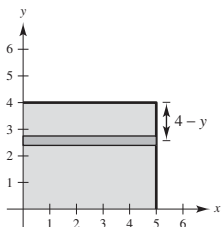
$$\begin{aligned} W &= \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2} dx = \left[ \frac{-2.42 \times 10^6}{x} \right]_{1100}^{1150} = 2.42 \times 10^6 \left( \frac{1}{1100} - \frac{1}{1150} \right) \\ &\approx 95.652 \text{ mi-ton} \approx 1.01 \times 10^9 \text{ ft-lb} \end{aligned}$$

17. Weight of each layer:  $62.4(20) \Delta y$

Distance:  $4 - y$

$$\text{(a) } W = \int_2^4 62.4(20)(4 - y) dy = [4992y - 624y^2]_2^4 = 2496 \text{ ft-lb}$$

$$\text{(b) } W = \int_0^4 62.4(20)(4 - y) dy = [4992y - 624y^2]_0^4 = 9984 \text{ ft-lb}$$



18. The bottom half had to be pumped a greater distance than the top half.

19. Volume of disk:  $\pi(2)^2 \Delta y = 4\pi \Delta y$

Weight of disk of water:  $9800(4\pi) \Delta y$

Distance the disk of water is moved:  $5 - y$

$$\begin{aligned} W &= \int_0^4 (5 - y)(9800)4\pi \, dy = 39,200\pi \int_0^4 (5 - y) \, dy \\ &= 39,200\pi \left[ 5y - \frac{y^2}{2} \right]_0^4 \\ &= 39,200\pi(12) = 470,400\pi \text{ newton-meters} \end{aligned}$$

20. Volume of disk:  $4\pi \Delta y$

Weight of disk:  $9800(4\pi) \Delta y$

Distance the disk of water is moved:  $y$

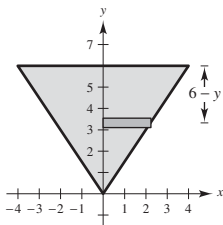
$$\begin{aligned} W &= \int_{10}^{12} y(9800)(4\pi) \, dy = 39,200\pi \left[ \frac{y^2}{2} \right]_{10}^{12} \\ &= 39,200\pi(22) \\ &= 862,400\pi \text{ newton-meters} \end{aligned}$$

21. Volume of disk:  $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk:  $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance:  $6 - y$

$$\begin{aligned} W &= \frac{4(62.4)\pi}{9} \int_0^6 (6 - y)y^2 \, dy \\ &= \frac{4}{9}(62.4)\pi \left[ 2y^3 - \frac{1}{4}y^4 \right]_0^6 \\ &= 2995.2\pi \text{ ft-lb} \end{aligned}$$



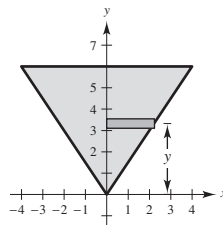
22. Volume of disk:  $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk:  $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance:  $y$

$$\begin{aligned} \text{(a) } W &= \frac{4}{9}(62.4)\pi \int_0^2 y^3 \, dy \\ &= \left[ \frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right) \right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb} \end{aligned}$$

$$\begin{aligned} \text{(b) } W &= \frac{4}{9}(62.4)\pi \int_4^6 y^3 \, dy \\ &= \left[ \frac{4}{9}(62.4)\pi\left(\frac{1}{4}y^4\right) \right]_4^6 \approx 7210.7\pi \text{ ft-lb} \end{aligned}$$

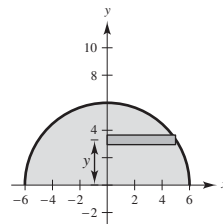


23. Volume of disk:  $\pi(\sqrt{36 - y^2})^2 \Delta y$

Weight of disk:  $62.4\pi(36 - y^2) \Delta y$

Distance:  $y$

$$\begin{aligned} W &= 62.4\pi \int_0^6 y(36 - y^2) \, dy \\ &= 62.4\pi \int_0^6 (36y - y^3) \, dy = 62.4\pi \left[ 18y^2 - \frac{1}{4}y^4 \right]_0^6 \\ &= 20,217.6\pi \text{ ft-lb} \end{aligned}$$

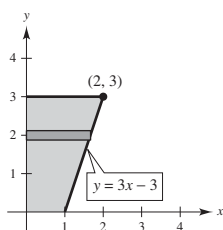


24. Volume of each layer:  $\frac{y+3}{3}(3) \Delta y = (y+3) \Delta y$

Weight of each layer:  $53.1(y+3) \Delta y$

Distance:  $6-y$

$$\begin{aligned} W &= \int_0^3 53.1(6-y)(y+3) dy \\ &= 53.1 \int_0^3 (18+3y-y^2) dy \\ &= 53.1 \left[ 18y + \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 \\ &= 53.1 \left( \frac{117}{2} \right) \\ &= 3106.35 \text{ ft-lb} \end{aligned}$$



25. Volume of layer:  $V = lwh = 4(2)\sqrt{(9/4) - y^2} \Delta y$

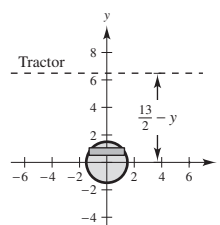
Weight of layer:  $W = 42(8)\sqrt{(9/4) - y^2} \Delta y$

Distance:  $\frac{13}{2} - y$

$$\begin{aligned} W &= \int_{-1.5}^{1.5} 42(8)\sqrt{\frac{9}{4} - y^2} \left( \frac{13}{2} - y \right) dy \\ &= 336 \left[ \frac{13}{2} \int_{-1.5}^{1.5} \sqrt{\frac{9}{4} - y^2} dy - \int_{-1.5}^{1.5} \sqrt{\frac{9}{4} - y^2} y dy \right] \end{aligned}$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius  $\frac{3}{2}$ . So, the work is

$$W = 336 \left( \frac{13}{2} \right) \pi \left( \frac{3}{2} \right)^2 \left( \frac{1}{2} \right) = 2457\pi \text{ ft-lb.}$$



26. Volume of layer:  $V = 12(2)\sqrt{(25/4) - y^2} \Delta y$

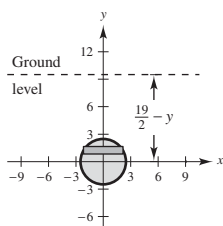
Weight of layer:  $W = 42(24)\sqrt{(25/4) - y^2} \Delta y$

Distance:  $\frac{19}{2} - y$

$$W = \int_{-2.5}^{2.5} 42(24)\sqrt{\frac{25}{4} - y^2} \left( \frac{19}{2} - y \right) dy = 1008 \left[ \frac{19}{2} \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} dy + \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} (-y) dy \right]$$

The second integral is zero because the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius  $\frac{5}{2}$ . So, the work is

$$W = 1008 \left( \frac{19}{2} \right) \pi \left( \frac{5}{2} \right)^2 \left( \frac{1}{2} \right) = 29,925\pi \text{ ft-lb} \approx 94,012.16 \text{ ft-lb.}$$



27. Weight of section of chain:  $3 \Delta y$

Distance:  $20-y$ .  $\Delta W = (\text{force increment})(\text{distance}) = (3 \Delta y)(20-y)$

$$W = \int_0^{20} (20-y)3 dy = 3 \left[ 20y - \frac{y^2}{2} \right]_0^{20} = 3 \left[ 400 - \frac{400}{2} \right] = 600 \text{ ft-lb}$$

28. The lower  $\frac{2}{3}(20)$  feet of fence are raised with a constant force

$$W_1 = 3\left(\frac{2}{3}(20)\right)\left(\frac{20}{3}\right) = \frac{800}{3} \text{ ft-lb}$$

The top  $\frac{1}{3}(20)$  feet are raised with a variable force.

Weight of section:  $3 \Delta y$

Distance:  $\frac{1}{3}(20) - y$

$$\begin{aligned} W_2 &= \int_0^{20/3} 3\left(\frac{20}{3} - y\right) dy = 3\left[\frac{20}{3}y - \frac{y^2}{2}\right]_0^{20/3} \\ &= \frac{200}{3} \text{ ft-lb} \end{aligned}$$

$$W = W_1 + W_2 = \frac{800}{3} + \frac{200}{3} = \frac{1000}{3} \text{ ft-lb}$$

29. The lower 10 feet of fence are raised 10 feet with a constant force.

$$W_1 = 3(10)(10) = 300 \text{ ft-lb}$$

The top 10 feet are raised with a variable force.

Weight of section:  $3 \Delta y$

Distance:  $10 - y$

$$W_2 = \int_0^{10} 3(10 - y) dy = 3\left[10y - \frac{y^2}{2}\right]_0^{10} = 150 \text{ ft-lb}$$

$$W = W_1 + W_2 = 300 + 150 = 450 \text{ ft-lb}$$

30. From Exercise 27, the work required to lift the chain is 600 ft-lb.

The work required to lift the 500-pound load is  $500(20) = 10,000$  ft-lb.

The total is  $600 + 10,000 = 10,600$  ft-lb.

31. Weight of section of chain:  $3 \Delta y$

Distance:  $15 - 2y$

$$\begin{aligned} W &= 3 \int_0^{7.5} (15 - 2y) dy = \left[-\frac{3}{4}(15 - 2y)^2\right]_0^{7.5} \\ &= \frac{3}{4}(15)^2 = 168.75 \text{ ft-lb} \end{aligned}$$

32.  $W = 3 \int_0^6 (12 - 2y) dy = \left[-\frac{3}{4}(12 - 2y)^2\right]_0^6$   
 $= \frac{3}{4}(12)^2 = 108 \text{ ft-lb}$

33. If an object is moved a distance  $D$  in the direction of an applied constant force  $F$ , then the work  $W$  done by the force is defined as force times distance,  $W = FD$ .

34.  $W = \int_a^b F(x) dx$  is the work done by a force  $F$  moving an object along a straight line from  $x = a$  to  $x = b$ .

35. (a) requires more work. In part (b) no work is done because the books are not moved:  
 $W = \text{force} \times \text{distance}$

36. Because the work equals the area under the force function, you have  $(c) < (d) < (a) < (b)$ .

37. (a)  $W = \int_0^9 6 dx = 54 \text{ ft-lb}$

$$\begin{aligned} \text{(b)} \quad W &= \int_0^7 20 dx + \int_7^9 (-10x + 90) dx = 140 + 20 \\ &= 160 \text{ ft-lb} \end{aligned}$$

$$\text{(c)} \quad W = \int_0^9 \frac{1}{27}x^2 dx = \frac{x^3}{81}\bigg|_0^9 = 9 \text{ ft-lb}$$

$$\text{(d)} \quad W = \int_0^9 \sqrt{x} dx = \frac{2}{3}x^{3/2}\bigg|_0^9 = \frac{2}{3}(27) = 18 \text{ ft-lb}$$

38. (a) Work to pull up the ball:

$$W_1 = 50(15) = 750 \text{ ft-lb}$$

Work to wind up the top 15 feet of cable:

Weight of section:  $2 \Delta y$

Distance:  $15 - y$

$$W_2 = \int_0^{15} 2(15 - y) dy = [30y - y^2]_0^{15} = 225 \text{ ft-lb}$$

Work to lift bottom 25 feet of cable

$$W_3 = 2(25)(15) = 750 \text{ ft-lb}$$

$$W = W_1 + W_2 + W_3 = 750 + 225 + 750 = 1725 \text{ ft-lb}$$

- (b) Work to pull up the ball:  $W_1 = 50(40) = 2000$  ft-lb

Work to wind up the cable.

$$W_2 = \int_0^{40} 2(40 - y) dy = [80y - y^2]_0^{40} = 1600 \text{ ft-lb}$$

$$W = W_1 + W_2 = 2000 + 1600 = 3600 \text{ ft-lb}$$

39.  $P = \frac{k}{V}$

$$1000 = \frac{k}{2}$$

$$k = 2000$$

$$W = \int_2^3 \frac{2000}{V} dV$$

$$= [2000 \ln|V|]_2^3 = 2000 \ln\left(\frac{3}{2}\right) \approx 810.93 \text{ ft-lb}$$



40.  $P = \frac{k}{V}$

$$2500 = \frac{k}{1} \Rightarrow k = 2500$$

$$W = \int_1^3 \frac{2500}{V} dV = [2500 \ln V]_1^3 = 2500 \ln 3 \approx 2746.53 \text{ ft-lb}$$

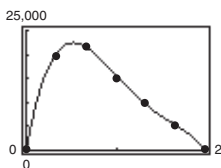
41.  $F(x) = \frac{k}{(2-x)^2}$

$$W = \int_{-2}^1 \frac{k}{(2-x)^2} dx = \left[ \frac{k}{2-x} \right]_{-2}^1 = k \left( 1 - \frac{1}{4} \right) = \frac{3k}{4} \text{ (units of work)}$$

42. (a)  $W = FD = (8000\pi)(2) = 16,000\pi \text{ ft} \cdot \text{lbs}$

(b)  $W \approx \frac{2-0}{3(6)} [0 + 4(20,000) + 2(22,000) + 4(15,000) + 2(10,000) + 4(5000) + 0] \approx 24,88.889 \text{ ft-lb}$

(c)  $F(x) = -16,261.36x^4 + 85,295.45x^3 - 157,738.64x^2 + 104,386.36x - 32.4675$



(d)  $F(x) = 0$  when  $x \approx 0.524$  feet.  $F(x)$  is a maximum when  $x \approx 0.524$  feet.

(e)  $W = \int_0^2 F(x) dx \approx 25,180.5 \text{ ft-lb}$

43.  $W = \int_0^5 1000[1.8 - \ln(x+1)] dx \approx 3249.44 \text{ ft-lb}$

45.  $W = \int_0^5 100x\sqrt{125-x^3} dx \approx 10,330.3 \text{ ft-lb}$

44.  $W = \int_0^4 \left( \frac{e^{x^2} - 1}{100} \right) dx \approx 11,494 \text{ ft-lb}$

46.  $W = \int_0^2 1000 \sinh x dx \approx 2762.2 \text{ ft-lb}$

## Section 7.6 Moments, Centers of Mass, and Centroids

1.  $\bar{x} = \frac{7(-5) + 3(0) + 5(3)}{7 + 3 + 5} = \frac{-20}{15} = -\frac{4}{3}$

3.  $\bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

2.  $\bar{x} = \frac{7(-3) + 4(-2) + 3(5) + 8(4)}{7 + 4 + 3 + 8} = \frac{9}{11}$

4.  $\bar{x} = \frac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$

5. (a) Add 4 to each  $x$ -value because each point is translated 4 units to the right.

$$\bar{x} = \frac{1(7+4) + 1(8+4) + 1(12+4) + 1(15+4) + 1(18+4)}{1 + 1 + 1 + 1 + 1} = \frac{80}{5} = 16$$

**Note:** From Exercise 3,  $12 + 4 = 16$ .

(b) Subtract 2 from each  $x$ -value because each point is translated 2 units to the left.

$$\bar{x} = \frac{12(-6-2) + 1(-4-2) + 6(-2-2) + 3(0-2) + 11(8-2)}{12 + 1 + 6 + 3 + 11} = \frac{-66}{33} = -2$$

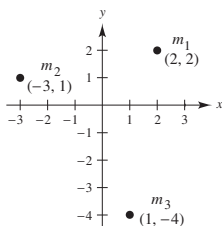
**Note:** From Exercise 4,  $0 - 2 = -2$ .

6. The center of mass is translated  $k$  units as well.

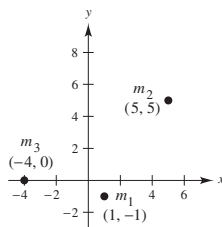
$$\begin{aligned}
 7. \quad 48x &= 72(L - x) = 72(10 - x) \\
 48x &= 720 - 72x \\
 120x &= 720 \\
 x &= 6 \text{ ft}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad 200x &= 600(5 - x) \quad (\text{person is on the left}) \\
 200x &= 3000 - 600x \\
 800x &= 3000 \\
 x &= \frac{15}{4} = 3\frac{3}{4} \text{ ft}
 \end{aligned}$$

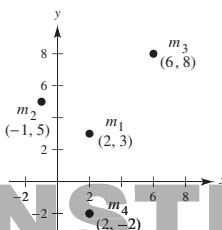
$$\begin{aligned}
 9. \quad \bar{x} &= \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9} \\
 \bar{y} &= \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9} \\
 (\bar{x}, \bar{y}) &= \left(\frac{10}{9}, -\frac{1}{9}\right)
 \end{aligned}$$



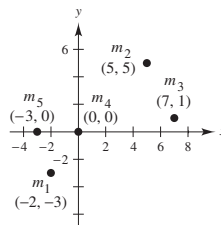
$$\begin{aligned}
 10. \quad \bar{x} &= \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0 \\
 \bar{y} &= \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0 \\
 (\bar{x}, \bar{y}) &= (0, 0)
 \end{aligned}$$



$$\begin{aligned}
 11. \quad \bar{x} &= \frac{12(2) + 6(-1) + (9/2)(6) + 15(2)}{12 + 6 + (9/2) + 15} = \frac{75}{37.5} = 2 \\
 \bar{y} &= \frac{12(3) + 6(5) + (9/2)(8) + 15(-2)}{12 + 6 + (9/2) + 15} = \frac{72}{37.5} = \frac{48}{25} \\
 (\bar{x}, \bar{y}) &= \left(2, \frac{48}{25}\right)
 \end{aligned}$$



$$\begin{aligned}
 12. \quad \bar{x} &= \frac{3(-2) + 4(5) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = \frac{5}{8} \\
 \bar{y} &= \frac{3(-3) + 4(5) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = \frac{13}{16} \\
 (\bar{x}, \bar{y}) &= \left(\frac{5}{8}, \frac{13}{16}\right)
 \end{aligned}$$



$$13. \quad m = \rho \int_0^2 \frac{x}{2} dx = \left[ \rho \frac{x^2}{4} \right]_0^2 = \rho$$

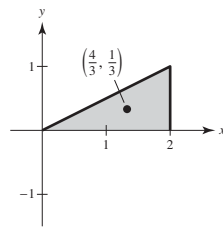
$$\begin{aligned}
 M_x &= \rho \int_0^2 \frac{1}{2} \left(\frac{x}{2}\right)^2 dx \\
 &= \frac{\rho}{8} \left[ \frac{x^3}{3} \right]_0^2 = \frac{\rho}{3}
 \end{aligned}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho/3}{\rho} = \frac{1}{3}$$

$$M_y = \rho \int_0^2 x \left(\frac{x}{2}\right) dx = \frac{\rho}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{4}{3} \rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{4/3 \rho}{\rho} = \frac{4}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{3}, \frac{1}{3}\right)$$



$$14. \quad m = \rho \int_0^3 (-x + 3) dx = \rho \left[ -\frac{x^2}{2} + 3x \right]_0^3 = \frac{9}{2}\rho$$

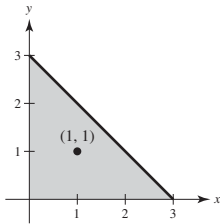
$$M_x = \rho \int_0^3 \frac{1}{2}(-x + 3)^2 dx = \frac{\rho}{2} \int_0^3 (x^2 - 6x + 9) dx = \frac{\rho}{2} \left[ \frac{x^3}{3} - 3x^2 + 9x \right]_0^3 = \frac{\rho}{2} [9 - 27 + 27] = \frac{9}{2}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/2\rho}{9/2\rho} = 1$$

$$M_y = \rho \int_0^3 x(-x + 3) dx = \rho \left[ -\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \rho \left[ -9 + \frac{27}{2} \right] = \frac{9}{2}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{9/2\rho}{9/2\rho} = 1$$

$$(\bar{x}, \bar{y}) = (1, 1)$$



$$15. \quad m = \rho \int_0^4 \sqrt{x} dx = \left[ \frac{2\rho}{3} x^{3/2} \right]_0^4 = \frac{16\rho}{3}$$

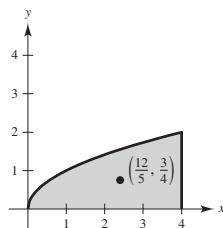
$$M_x = \rho \int_0^4 \frac{\sqrt{x}}{2} (\sqrt{x}) dx = \left[ \rho \frac{x^2}{4} \right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho \left( \frac{3}{16\rho} \right) = \frac{3}{4}$$

$$M_y = \rho \int_0^4 x\sqrt{x} dx = \left[ \rho \frac{2}{5} x^{5/2} \right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5} \left( \frac{3}{16\rho} \right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{12}{5}, \frac{3}{4} \right)$$



$$16. \quad m = \rho \int_0^3 \frac{1}{3} x^2 dx$$

$$= \rho \left[ \frac{x^3}{9} \right]_0^3 = 3\rho$$

$$M_x = \rho \int_0^3 \frac{1}{2} \left( \frac{1}{3} x^2 \right)^2 dx = \frac{\rho}{18} \int_0^3 x^4 dx$$

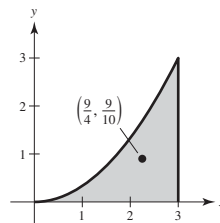
$$= \frac{\rho}{18} \left[ \frac{x^5}{5} \right]_0^3 = \frac{27}{10}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{27/10\rho}{3\rho} = \frac{9}{10}$$

$$M_y = \rho \int_0^3 x \left( \frac{1}{3} x^2 \right) dx = \frac{\rho}{3} \left[ \frac{x^4}{4} \right]_0^3 = \frac{27}{4}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{27/4\rho}{3\rho} = \frac{9}{4}$$

$$(\bar{x}, \bar{y}) = \left( \frac{9}{4}, \frac{9}{10} \right)$$



$$17. \quad m = \rho \int_0^1 (x^2 - x^3) dx = \rho \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\rho}{12}$$

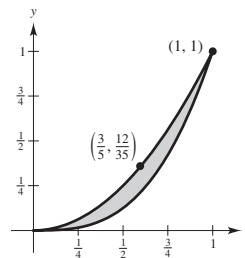
$$M_x = \rho \int_0^1 \frac{(x^2 + x^3)}{2} (x^2 - x^3) dx = \frac{\rho}{2} \int_0^1 (x^4 - x^6) dx = \frac{\rho}{2} \left[ \frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{\rho}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35} \left( \frac{12}{\rho} \right) = \frac{12}{35}$$

$$M_y = \rho \int_0^1 x(x^2 - x^3) dx = \rho \int_0^1 (x^3 - x^4) dx = \rho \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{\rho}{20}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20} \left( \frac{12}{\rho} \right) = \frac{3}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{3}{5}, \frac{12}{35} \right)$$



$$18. \quad m = \rho \int_0^4 \left( \sqrt{x} - \frac{x}{2} \right) dx = \rho \left[ \frac{2}{3} x^{3/2} - \frac{x^2}{4} \right]_0^4 = \rho \left[ \frac{16}{3} - 4 \right] = \frac{4}{3} \rho$$

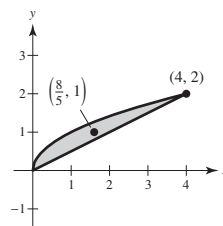
$$M_x = \rho \int_0^4 \frac{1}{2} \left( \sqrt{x} + \frac{x}{2} \right) \left( \sqrt{x} - \frac{x}{2} \right) dx = \frac{1}{2} \rho \int_0^4 \left( x - \frac{x^2}{4} \right) dx = \frac{\rho}{2} \left[ \frac{x^2}{2} - \frac{x^3}{12} \right]_0^4 = \frac{\rho}{2} \left[ 8 - \frac{16}{3} \right] = \frac{4}{3} \rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{4/3 \rho}{4/3 \rho} = 1$$

$$M_y = \rho \int_0^4 x \left( \sqrt{x} - \frac{x}{2} \right) dx = \rho \left[ \frac{2}{5} x^{5/2} - \frac{x^3}{6} \right]_0^4 = \rho \left[ \frac{64}{5} - \frac{32}{3} \right] = \frac{32}{15} \rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{32/15 \rho}{4/3 \rho} = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = (8/5, 1)$$



$$19. \quad m = \rho \int_0^3 \left[ (-x^2 + 4x + 2) - (x + 2) \right] dx = -\rho \left[ \frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{9\rho}{2}$$

$$M_x = \rho \int_0^3 \left[ \frac{(-x^2 + 4x + 2) + (x + 2)}{2} \right] \left[ (-x^2 + 4x + 2) - (x + 2) \right] dx$$

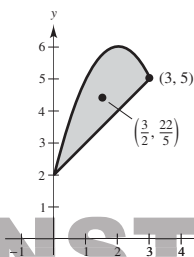
$$= \frac{\rho}{2} \int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x) dx = \frac{\rho}{2} \int_0^3 (x^4 - 8x^3 + 11x^2 + 12x) dx = \frac{\rho}{2} \left[ \frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2 \right]_0^3 = \frac{99\rho}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5} \left( \frac{2}{9\rho} \right) = \frac{22}{5}$$

$$M_y = \rho \int_0^3 x \left[ (-x^2 + 4x + 2) - (x + 2) \right] dx = \rho \int_0^3 (-x^3 + 3x^2) dx = \rho \left[ -\frac{x^4}{4} + x^3 \right]_0^3 = \frac{27\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4} \left( \frac{2}{9\rho} \right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left( \frac{3}{2}, \frac{22}{5} \right)$$



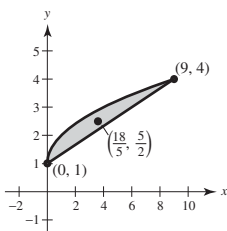
$$20. \quad m = \rho \int_0^9 \left[ (\sqrt{x} + 1) - \left( \frac{1}{3}x + 1 \right) \right] dx = \rho \int_0^9 \left( \sqrt{x} - \frac{1}{3}x \right) dx = \rho \left[ \frac{2}{3}x^{3/2} - \frac{x^2}{6} \right]_0^9 = \rho \left( 18 - \frac{27}{2} \right) = \frac{9}{2}\rho$$

$$\begin{aligned} M_x &= \rho \int_0^9 \frac{\sqrt{x} + 1 + (1/3)x + 1}{2} \left( \sqrt{x} + 1 - \frac{1}{3}x - 1 \right) dx = \frac{\rho}{2} \int_0^9 \left( \sqrt{x} + \frac{1}{3}x + 2 \right) \left( \sqrt{x} - \frac{1}{3}x \right) dx \\ &= \frac{\rho}{2} \int_0^9 \left( x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x \right) dx = \frac{\rho}{2} \int_0^9 \left( \frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x} \right) dx \\ &= \frac{\rho}{2} \left[ \frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2} \right]_0^9 = \frac{\rho}{2} \left[ \frac{27}{2} - 27 + 36 \right] = \frac{45}{4}\rho \end{aligned}$$

$$M_y = \rho \int_0^9 x \left[ \sqrt{x} + 1 - \frac{1}{3}x - 1 \right] dx = \rho \int_0^9 \left( x^{3/2} - \frac{1}{3}x^2 \right) dx = \rho \left[ \frac{2}{5}x^{5/2} - \frac{1}{9}x^3 \right]_0^9 = \rho \left[ \frac{486}{5} - 81 \right] = \frac{81}{5}\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{(81/5)\rho}{(9/2)\rho} = \frac{18}{5}; \quad \bar{y} = \frac{M_x}{m} = \frac{(45/4)\rho}{(9/2)\rho} = \frac{5}{2}$$

$$(\bar{x}, \bar{y}) = \left( \frac{18}{5}, \frac{5}{2} \right)$$



$$21. \quad m = \rho \int_0^8 x^{2/3} dx = \rho \left[ \frac{3}{5}x^{5/3} \right]_0^8 = \frac{96\rho}{5}$$

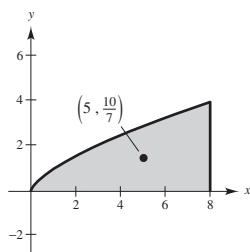
$$M_x = \rho \int_0^8 \frac{x^{2/3}}{2} (x^{2/3}) dx = \frac{\rho}{2} \left[ \frac{3}{7}x^{7/3} \right]_0^8 = \frac{192\rho}{7}$$

$$\bar{y} = \frac{M_x}{m} = \frac{192\rho}{7} \left( \frac{5}{96\rho} \right) = \frac{10}{7}$$

$$M_y = \rho \int_0^8 x(x^{2/3}) dx = \rho \left[ \frac{3}{8}x^{8/3} \right]_0^8 = 96\rho$$

$$\bar{x} = \frac{M_y}{m} = 96\rho \left( \frac{5}{96\rho} \right) = 5$$

$$(\bar{x}, \bar{y}) = \left( 5, \frac{10}{7} \right)$$



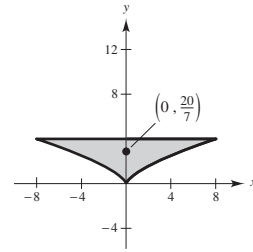
$$22. \quad m = 2\rho \int_0^8 (4 - x^{2/3}) dx = 2\rho \left[ 4x - \frac{3}{5}x^{5/3} \right]_0^8 = \frac{128\rho}{5}$$

By symmetry,  $M_y$  and  $\bar{x} = 0$ .

$$M_x = 2\rho \int_0^8 \left( \frac{4 + x^{2/3}}{2} \right) (4 - x^{2/3}) dx = \rho \left[ 16x - \frac{3}{7}x^{7/3} \right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7} \left( \frac{5}{128\rho} \right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{20}{7} \right)$$



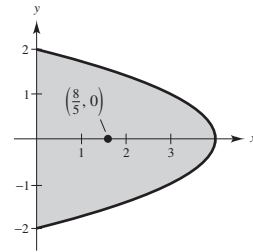
$$23. \quad m = 2\rho \int_0^2 (4 - y^2) dy = 2\rho \left[ 4y - \frac{y^3}{3} \right]_0^2 = \frac{32\rho}{3}$$

$$M_y = 2\rho \int_0^2 \left( \frac{4 - y^2}{2} \right) (4 - y^2) dy = \rho \left[ 16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_0^2 = \frac{256\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{256\rho}{15} \left( \frac{3}{32\rho} \right) = \frac{8}{5}$$

By symmetry,  $M_x$  and  $\bar{y} = 0$ .

$$(\bar{x}, \bar{y}) = \left( \frac{8}{5}, 0 \right)$$



$$24. \quad m = \rho \int_0^2 (2y - y^2) dy = \rho \left[ y^2 - \frac{y^3}{3} \right]_0^2 = \frac{4\rho}{3}$$

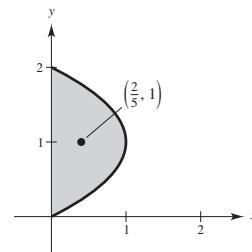
$$M_y = \rho \int_0^2 \left( \frac{2y - y^2}{2} \right) (2y - y^2) dy = \frac{\rho}{2} \left[ \frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right]_0^2 = \frac{8\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15} \left( \frac{3}{4\rho} \right) = \frac{2}{5}$$

$$M_x = \rho \int_0^2 y(2y - y^2) dy = \rho \left[ \frac{2y^3}{3} - \frac{y^4}{4} \right]_0^2 = \frac{4\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3} \left( \frac{3}{4\rho} \right) = 1$$

$$(\bar{x}, \bar{y}) = \left( \frac{2}{5}, 1 \right)$$



$$25. \quad m = \rho \int_0^3 [(2y - y^2) - (-y)] dy = \rho \left[ \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{9\rho}{2}$$

$$M_y = \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2} [(2y - y^2) - (-y)] dy = \frac{\rho}{2} \int_0^3 (y - y^2)(3y - y^2) dy$$

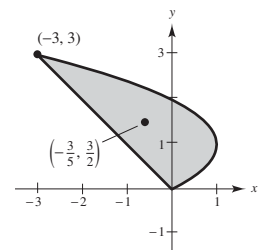
$$= \frac{\rho}{2} \int_0^3 (y^4 - 4y^3 + 3y^2) dy = \frac{\rho}{2} \left[ \frac{y^5}{5} - y^4 + y^3 \right]_0^3 = -\frac{27\rho}{10}$$

$$\bar{x} = \frac{M_y}{m} = -\frac{27\rho}{10} \left( \frac{2}{9\rho} \right) = -\frac{3}{5}$$

$$M_x = \rho \int_0^3 y[(2y - y^2) - (-y)] dy = \rho \int_0^3 (3y^2 - y^3) dy = \rho \left[ y^3 - \frac{y^4}{4} \right]_0^3 = \frac{27\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{27\rho}{4} \left( \frac{2}{9\rho} \right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left( -\frac{3}{5}, \frac{3}{2} \right)$$



$$26. \quad m = \rho \int_{-1}^2 [(y+2) - y^2] dy = \rho \left[ \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \frac{9\rho}{2}$$

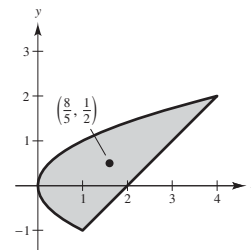
$$M_y = \rho \int_{-1}^2 \frac{[(y+2) + y^2]}{2} [(y+2) - y^2] dy = \frac{\rho}{2} \int_{-1}^2 [(y+2)^2 - y^4] dy = \frac{\rho}{2} \left[ \frac{(y+2)^3}{3} - \frac{y^5}{5} \right]_{-1}^2 = \frac{36\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5} \left( \frac{2}{9\rho} \right) = \frac{8}{5}$$

$$M_x = \rho \int_{-1}^2 y [(y+2) - y^2] dy = \rho \int_{-1}^2 (2y + y^2 - y^3) dy = \rho \left[ y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_{-1}^2 = \frac{9\rho}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4} \left( \frac{2}{9\rho} \right) = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left( \frac{8}{5}, \frac{1}{2} \right)$$



$$27. \quad A = \int_0^2 (2x - x^2) dx = \left[ x^2 - \frac{x^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$M_x = \frac{1}{2} \int_0^2 [(2x)^2 - (x^2)^2] dx = \frac{1}{2} \left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \frac{1}{2} \left[ \frac{32}{3} - \frac{32}{5} \right] = \frac{32}{15}$$

$$M_y = \int_0^2 x(2x - x^2) dx = \left[ \frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$$

$$28. \quad A = \int_1^4 \frac{1}{x} dx = [\ln|x|]_1^4 = \ln 4$$

$$M_x = \frac{1}{2} \int_1^4 \frac{1}{x^2} dx = \left[ \frac{1}{2} \left( -\frac{1}{x} \right) \right]_1^4 = \left( -\frac{1}{8} + \frac{1}{2} \right) = \frac{3}{8}$$

$$M_y = \int_1^4 x \left( \frac{1}{x} \right) dx = [x]_1^4 = 3$$

$$29. \quad A = \int_0^3 (2x + 4) dx = [x^2 + 4x]_0^3 = 9 + 12 = 21$$

$$M_x = \frac{1}{2} \int_0^3 (2x + 4)^2 dx = \int_0^3 (2x^2 + 8x + 8) dx = \left[ \frac{2x^3}{3} + 4x^2 + 8x \right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x) dx = \left[ \frac{2x^3}{3} + 2x^2 \right]_0^3 = 18 + 18 = 36$$

$$30. \quad A = \int_{-2}^2 -(x^2 - 4) dx = 2 \int_0^2 (4 - x^2) dx = \left[ 8x - \frac{2x^3}{3} \right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$$

$$M_x = \frac{1}{2} \int_{-2}^2 (x^2 - 4)(4 - x^2) dx = -\frac{1}{2} \int_{-2}^2 (x^4 - 8x^2 + 16) dx = -\frac{1}{2} \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^2 = -\left[ \frac{32}{5} - \frac{64}{3} + 32 \right] = -\frac{256}{15}$$

$$M_y = 0 \text{ by symmetry.}$$

$$31. \quad m = \rho \int_0^5 10x\sqrt{125 - x^3} \, dx \approx 1033.0\rho$$

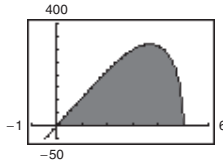
$$M_x = \rho \int_0^5 \left( \frac{10x\sqrt{125 - x^3}}{2} \right) (10x\sqrt{125 - x^3}) \, dx = 50\rho \int_0^5 x^2(125 - x^3) \, dx = \frac{3,124,375\rho}{24} \approx 130,208\rho$$

$$M_y = \rho \int_0^5 10x^2\sqrt{125 - x^3} \, dx = -\frac{10\rho}{3} \int_0^5 \sqrt{125 - x^3}(-3x^2) \, dx = \frac{12,500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is (3.0, 126.0).



$$32. \quad m = \rho \int_0^4 xe^{-x/2} \, dx \approx 2.3760\rho$$

$$M_x = \rho \int_0^4 \left( \frac{xe^{-x/2}}{2} \right) (xe^{-x/2}) \, dx$$

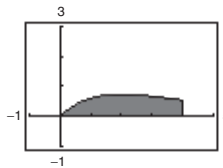
$$= \frac{\rho}{2} \int_0^4 x^2 e^{-x} \, dx \approx 0.7619\rho$$

$$M_y = \rho \int_0^4 x^2 e^{-x/2} \, dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is (2.2, 0.3).



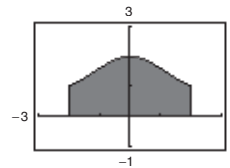
$$34. \quad m = \rho \int_{-2}^2 \frac{8}{x^2 + 4} \, dx \approx 6.2832\rho$$

$$M_x = \rho \int_{-2}^2 \frac{1}{2} \left( \frac{8}{x^2 + 4} \right) \left( \frac{8}{x^2 + 4} \right) \, dx$$

$$= 32\rho \int_{-2}^2 \frac{1}{(x^2 + 4)^2} \, dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$  by symmetry. Therefore, the centroid is (0, 0.8).



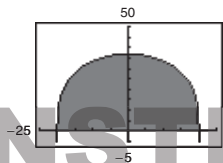
$$33. \quad m = \rho \int_{-20}^{20} 5\sqrt[3]{400 - x^2} \, dx \approx 1239.76\rho$$

$$M_x = \rho \int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2} (5\sqrt[3]{400 - x^2}) \, dx$$

$$= \frac{25\rho}{2} \int_{-20}^{20} (400 - x^2)^{2/3} \, dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$  by symmetry. Therefore, the centroid is (0, 16.2).





35.  $A = \frac{1}{2}(2a)c = ac$

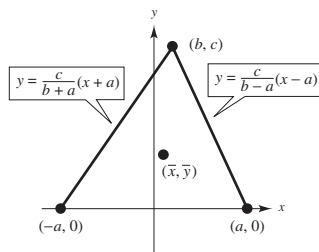
$$\frac{1}{A} = \frac{1}{ac}$$

$$\begin{aligned}\bar{x} &= \left(\frac{1}{ac}\right) \frac{1}{2} \int_0^c \left[ \left( \frac{b-a}{c}y + a \right)^2 - \left( \frac{b+a}{c}y - a \right)^2 \right] dy \\ &= \frac{1}{2ac} \int_0^c \left[ \frac{4ab}{c}y - \frac{4ab}{c^2}y^2 \right] dy = \frac{1}{2ac} \left[ \frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3 \right]_0^c = \frac{1}{2ac} \left( \frac{2}{3}abc \right) = \frac{b}{3}\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{1}{ac} \int_0^c y \left[ \left( \frac{b-a}{c}y + a \right) - \left( \frac{b+a}{c}y - a \right) \right] dy \\ &= \frac{1}{ac} \int_0^c y \left( -\frac{2a}{c}y + 2a \right) dy = \frac{2}{c} \int_0^c \left( y - \frac{y^2}{c} \right) dy = \frac{2}{c} \left[ \frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = \frac{c}{3}\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left( \frac{b}{3}, \frac{c}{3} \right)$$

From elementary geometry,  $(b/3, c/3)$  is the point of intersection of the medians.



36.  $A = bh = ac$

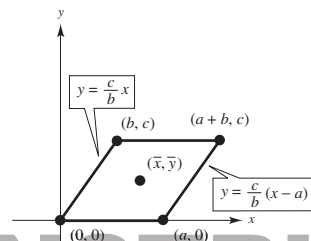
$$\frac{1}{A} = \frac{1}{ac}$$

$$\begin{aligned}\bar{x} &= \frac{1}{ac} \frac{1}{2} \int_0^c \left[ \left( \frac{b}{c}y + a \right)^2 - \left( \frac{b}{c}y \right)^2 \right] dy \\ &= \frac{1}{2ac} \int_0^c \left( \frac{2ab}{c}y + a^2 \right) dy \\ &= \frac{1}{2ac} \left[ \frac{ab}{c}y^2 + a^2y \right]_0^c \\ &= \frac{1}{2ac} [abc + a^2c] = \frac{1}{2}(b+a)\end{aligned}$$

$$\bar{y} = \frac{1}{ac} \int_0^c y \left[ \left( \frac{b}{c}y + a \right) - \left( \frac{b}{c}y \right) \right] dy = \left[ \frac{1}{c} \frac{y^2}{2} \right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left( \frac{b+a}{2}, \frac{c}{2} \right)$$

This is the point of intersection of the diagonals.



$$37. A = \frac{c}{2}(a+b)$$

$$\frac{1}{A} = \frac{2}{c(a+b)}$$

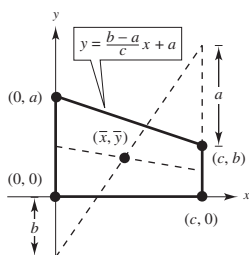
$$\begin{aligned}\bar{x} &= \frac{2}{c(a+b)} \int_0^c x \left( \frac{b-a}{c}x + a \right) dx = \frac{2}{c(a+b)} \int_0^c \left( \frac{b-a}{c}x^2 + ax \right) dx = \frac{2}{c(a+b)} \left[ \frac{b-a}{c} \frac{x^3}{3} + \frac{ax^2}{2} \right]_0^c \\ &= \frac{2}{c(a+b)} \left[ \frac{(b-a)c^2}{3} + \frac{ac^2}{2} \right] = \frac{2}{c(a+b)} \left[ \frac{2bc^2 - 2ac^2 + 3ac^2}{6} \right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{2}{c(a+b)} \frac{1}{2} \int_0^c \left( \frac{b-a}{c}x + a \right)^2 dx = \frac{1}{c(a+b)} \int_0^c \left[ \left( \frac{b-a}{c} \right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2 \right] dx \\ &= \frac{1}{c(a+b)} \left[ \left( \frac{b-a}{c} \right)^2 \frac{x^3}{3} + \frac{2a(b-a)}{c} \frac{x^2}{2} + a^2x \right]_0^c = \frac{1}{c(a+b)} \left[ \frac{(b-a)^2 c}{3} + ac(b-a) + a^2c \right] \\ &= \frac{1}{3c(a+b)} [(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c] \\ &= \frac{1}{3(a+b)} [b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}\end{aligned}$$

$$\text{So, } (\bar{x}, \bar{y}) = \left( \frac{(a+2b)c}{3(a+b)}, \frac{a^2 + ab + b^2}{3(a+b)} \right).$$

The one line passes through  $(0, a/2)$  and  $(c, b/2)$ . Its equation is  $y = \frac{b-a}{2c}x + \frac{a}{2}$ . The other line passes through  $(0, -b)$  and

$(c, a+b)$ . Its equation is  $y = \frac{a+2b}{c}x - b$ .  $(\bar{x}, \bar{y})$  is the point of intersection of these two lines.



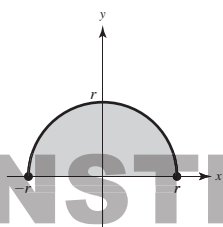
$$38. \bar{x} = 0 \text{ by symmetry.}$$

$$A = \frac{1}{2}\pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\begin{aligned}\bar{y} &= \frac{2}{\pi r^2} \frac{1}{2} \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx \\ &= \frac{1}{\pi r^2} \left[ r^2x - \frac{x^3}{3} \right]_{-r}^r = \frac{1}{\pi r^2} \left( \frac{4r^3}{3} \right) = \frac{4r}{3\pi}\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{4r}{3\pi} \right)$$



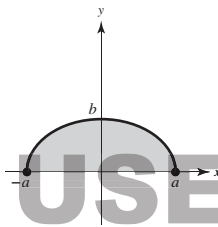
$$39. \bar{x} = 0 \text{ by symmetry.}$$

$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

$$\begin{aligned}\bar{y} &= \frac{2}{\pi ab} \frac{1}{2} \int_{-a}^a \left( \frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx \\ &= \frac{1}{\pi ab} \left( \frac{b^2}{a^2} \right) \left[ a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{b}{\pi a^3} \left( \frac{4a^3}{3} \right) = \frac{4b}{3\pi}\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{4b}{3\pi} \right)$$



40.  $A = \int_0^1 [1 - (2x - x^2)] dx = \frac{1}{3}$

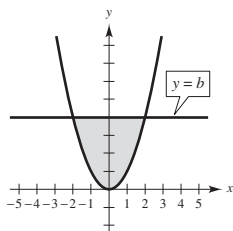
$$\frac{1}{A} = 3$$

$$\bar{x} = 3 \int_0^1 x [1 - (2x - x^2)] dx = 3 \int_0^1 [x - 2x^2 + x^3] dx = 3 \left[ \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\begin{aligned} \bar{y} &= 3 \int_0^1 \frac{[1 + (2x - x^2)]}{2} [1 - (2x - x^2)] dx = \frac{3}{2} \int_0^1 [1 - (2x - x^2)^2] dx \\ &= \frac{3}{2} \int_0^1 (1 - 4x^2 + 4x^3 - x^4) dx = \frac{3}{2} \left[ x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5} \right]_0^1 = \frac{7}{10} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left( \frac{1}{4}, \frac{7}{10} \right)$$

41. (a)



(b)  $\bar{x} = 0$  by symmetry.

(c)  $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2) dx = 0$  because  $bx - x^3$  is odd.

(d)  $\bar{y} > \frac{b}{2}$  because there is more area above  $y = \frac{b}{2}$  than below.

$$(e) \quad M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2} dx = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2} \left[ b^2x - \frac{x^5}{5} \right]_{-\sqrt{b}}^{\sqrt{b}} = b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2) dx = \left[ bx - \frac{x^3}{3} \right]_{-\sqrt{b}}^{\sqrt{b}} = \left( b\sqrt{b} - \frac{b\sqrt{b}}{3} \right) 2 = 4 \frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b$$

42. (a)  $M_y = 0$  by symmetry.

$$M_y = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} x(b - x^{2n}) dx = 0$$

because  $bx - x^{2n+1}$  is an odd function.

(b)  $\bar{y} > \frac{b}{2}$  because there is more area above  $y = \frac{b}{2}$  than below.

$$(c) \quad M_x = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2} dx = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{1}{2} (b^2 - x^{4n}) dx$$

$$= \frac{1}{2} \left( b^2x - \frac{x^{4n+1}}{4n+1} \right) \Big|_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} = b^2b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1} b^{(4n+1)/2n}$$

$$A = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} (b - x^{2n}) dx = 2 \left[ bx - \frac{x^{2n+1}}{2n+1} \right]_0^{2\sqrt[n]{b}} = 2 \left[ b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1} \right] = \frac{4n}{2n+1} b^{(2n+1)/2n}$$

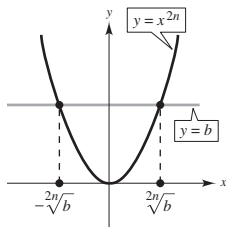
$$\bar{y} = \frac{M_x}{A} = \frac{4nb^{(4n+1)/2n}/(4n+1)}{4nb^{(2n+1)/2n}/(2n+1)} = \frac{2n+1}{4n+1} b$$

(d)

$n$	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

(e)  $\lim_{n \rightarrow \infty} \bar{y} = \lim_{n \rightarrow \infty} \frac{2n+1}{4n+1}b = \frac{1}{2}b$

(f) As  $n \rightarrow \infty$ , the figure gets narrower.



43. (a)  $\bar{x} = 0$  by symmetry.

$$A = 2 \int_0^{40} f(x) dx = \frac{2(40)}{3(4)} [30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3}(278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)} [30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3}(7216) = \frac{72,160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72,160/3}{5560/3} = \frac{72,160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

(b)  $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$  (Use nine data points.)

(c)  $\bar{y} = \frac{M_x}{A} \approx \frac{23,697.68}{1843.54} \approx 12.85$

$$(\bar{x}, \bar{y}) = (0, 12.85)$$

44. Let  $f(x)$  be the top curve, given by  $l + d$ . The bottom curve is  $d(x)$ .

$x$	0	0.5	1.0	1.5	2
$f$	2.0	1.93	1.73	1.32	0
$d$	0.50	0.48	0.43	0.33	0

(a) Area =  $2 \int_0^2 [f(x) - d(x)] dx$

$$\approx 2 \frac{2}{3(4)} [1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0] = \frac{1}{3}[13.86] = 4.62$$

$$M_x = \int_{-2}^2 \frac{f(x) + d(x)}{2} (f(x) - d(x)) dx$$

$$= \int_0^2 [f(x)^2 - d(x)^2] dx$$

$$= \frac{2}{3(4)} [3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0] = \frac{1}{6}[29.878] = 4.9797$$

$$\bar{y} = \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078$$

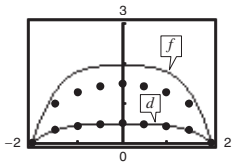
$$(\bar{x}, \bar{y}) = (0, 1.078)$$

(b)  $f(x) = -0.1061x^4 - 0.06126x^2 + 1.9527$

$d(x) = -0.02648x^4 - 0.01497x^2 + .4862$

(c)  $\bar{y} = \frac{M_x}{A} \approx \frac{4.9133}{4.59998} = 1.068$

$(\bar{x}, \bar{y}) = (0, 1.068)$



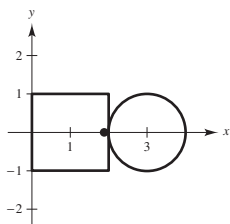
45. Centroids of the given regions:  $(1, 0)$  and  $(3, 0)$

Area:  $A = 4 + \pi$

$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$

$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$

$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0\right) \approx (1.88, 0)$



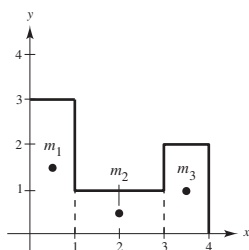
46. Centroids of the given regions:  $\left(\frac{1}{2}, \frac{3}{2}\right)$ ,  $\left(2, \frac{1}{2}\right)$ , and  $\left(\frac{7}{2}, 1\right)$

Area:  $A = 3 + 2 + 2 = 7$

$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$

$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$

$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$



47. Centroids of the given regions:  $\left(0, \frac{3}{2}\right)$ ,  $(0, 5)$ , and

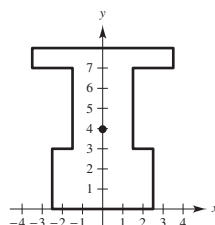
$\left(0, \frac{15}{2}\right)$

Area:  $A = 15 + 12 + 7 = 34$

$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$

$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$

$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right) \approx (0, 3.97)$



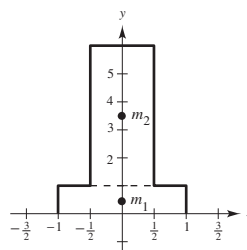
48.  $m_1 = \frac{7}{8}(2) = \frac{7}{4}$ ,  $P_1 = \left(0, \frac{7}{16}\right)$

$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}$ ,  $P_2 = \left(0, \frac{55}{16}\right)$

By symmetry,  $\bar{x} = 0$ .

$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16,569}{6384} = \frac{789}{304}$

$(\bar{x}, \bar{y}) = \left(0, \frac{789}{304}\right) \approx (0, 2.595)$



49. Centroids of the given regions:
- $(1, 0)$
- and
- $(3, 0)$

Mass:  $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left( \frac{2 + 3\pi}{2 + \pi}, 0 \right) \approx (2.22, 0)$$

50. Centroids of the given regions:
- $(3, 0)$
- and
- $(1, 0)$

Mass:  $8 + \pi$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left( \frac{8 + 3\pi}{8 + \pi}, 0 \right) \approx (1.56, 0)$$

- 51.
- $r = 5$
- is distance between center of circle and
- $y$
- axis.

$$A \approx \pi(4)^2 = 16\pi \text{ is area of circle. So,}$$

$$V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14.$$

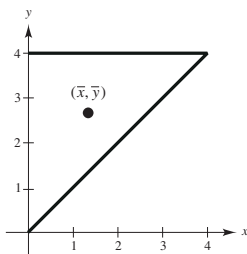
- 52.
- $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

- 53.
- $A = \frac{1}{2}(4)(4) = 8$

$$\bar{y} = \left( \frac{1}{8} \right) \frac{1}{2} \int_0^4 (4+x)(4-x) dx = \frac{1}{16} \left[ 16x - \frac{x^3}{3} \right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi rA = 2\pi \left( \frac{8}{3} \right) (8) = \frac{128\pi}{3} \approx 134.04$$



$$54. A = \int_2^6 2\sqrt{x-2} dx = \frac{4}{3}(x-2)^{3/2} \Big|_2^6 = \frac{32}{3}$$

$$M_y = \int_2^6 (x)2\sqrt{x-2} dx = 2 \int_2^6 x\sqrt{x-2} dx$$

Let  $u = x - 2$ ,  $x = u + 2$ ,  $du = dx$ :

$$M_y = 2 \int_0^4 (u+2)\sqrt{u} du$$

$$= 2 \int_0^4 (u^{3/2} + 2u^{1/2}) du$$

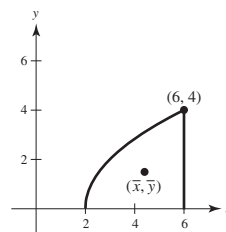
$$= 2 \left[ \frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} \right]_0^4$$

$$= 2 \left( \frac{64}{5} + \frac{32}{3} \right) = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi \left( \frac{22}{5} \right) \left( \frac{32}{3} \right) = \frac{1408\pi}{15} \approx 294.89$$



55. The center of mass
- $(\bar{x}, \bar{y})$
- is
- $\bar{x} = M_y/m$
- and

$$\bar{y} = M_x/m, \text{ where:}$$

1.  $m = m_1 + m_2 + \cdots + m_n$  is the total mass of the system.
2.  $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$  is the moment about the  $y$ -axis.
3.  $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$  is the moment about the  $x$ -axis.

56. A planar lamina is a thin flat plate of constant density. The center of mass
- $(\bar{x}, \bar{y})$
- is the balancing point on the lamina.

57. Let
- $R$
- be a region in a plane and let
- $L$
- be a line such that
- $L$
- does not intersect the interior of
- $R$
- . If
- $r$
- is the distance between the centroid of
- $R$
- and
- $L$
- , then the volume
- $V$
- of the solid of revolution formed by revolving
- $R$
- about
- $L$
- is
- $V = 2\pi rA$
- where
- $A$
- is the area of
- $R$
- .

58. (a) Yes.  $(\bar{x}, \bar{y}) = \left( \frac{5}{6}, \frac{5}{18} + 2 \right) = \left( \frac{5}{6}, \frac{41}{18} \right)$

(b) Yes.  $(\bar{x}, \bar{y}) = \left( \frac{5}{6} + 2, \frac{5}{18} \right) = \left( \frac{17}{6}, \frac{5}{18} \right)$

(c) Yes.  $(\bar{x}, \bar{y}) = \left( \frac{5}{6}, -\frac{5}{18} \right)$

(d) No

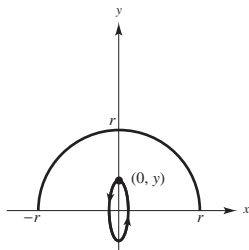
59. The surface area of the sphere is  $S = 4\pi r^2$ . The arc length of  $C$  is  $s = \pi r$ . The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

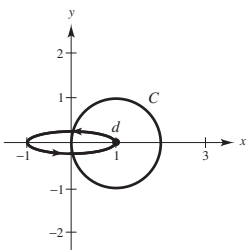
This distance is also the circumference of the circle of radius  $y$ .

$$d = 2\pi y$$

So,  $2\pi y = 4r$  and you have  $y = 2r/\pi$ . Therefore, the centroid of the semicircle  $y = \sqrt{r^2 - x^2}$  is  $(0, 2r/\pi)$ .



60. The centroid of the circle is  $(1, 0)$ . The distance traveled by the centroid is  $2\pi$ . The arc length of the circle is also  $2\pi$ . Therefore,  $S = (2\pi)(2\pi) = 4\pi^2$ .



$$61. \quad A = \int_0^1 x^n dx = \left[ \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2} \int_0^1 (x^n)^2 dx = \left[ \frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1} \right]_0^1 = \frac{\rho}{2(2n+1)}$$

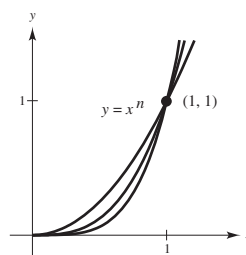
$$M_y = \rho \int_0^1 x(x^n) dx = \left[ \rho \cdot \frac{x^{n+2}}{n+2} \right]_0^1 = \frac{\rho}{n+2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

$$\text{Centroid: } \left( \frac{n+1}{n+2}, \frac{n+1}{4n+2} \right)$$

As  $n \rightarrow \infty$ ,  $(\bar{x}, \bar{y}) \rightarrow \left(1, \frac{1}{4}\right)$ . The graph approaches the  $x$ -axis and the line  $x = 1$  as  $n \rightarrow \infty$ .



62. Let  $T$  be the shaded triangle with vertices  $(-1, 4)$ ,  $(1, 4)$ , and  $(0, 3)$ . Let  $U$  be the large triangle with vertices  $(-4, 4)$ ,  $(4, 4)$ , and  $(0, 0)$ .  $V$  consists of the region  $U$  minus the region  $T$ .

$$\text{Centroid of } T: \left(0, \frac{11}{3}\right); \quad \text{Area} = 1$$

$$\text{Centroid of } U: \left(0, \frac{8}{3}\right); \quad \text{Area} = 16$$

$$\text{Area: } V = 16 - 1 = 15$$

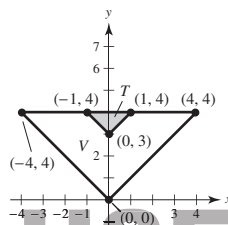
$$\bar{x} = 0 \text{ by symmetry.}$$

$$15\bar{y} + 1\left(\frac{11}{3}\right) = 16\left(\frac{8}{3}\right)$$

$$15\bar{y} = \frac{117}{3}$$

$$\bar{y} = \frac{13}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{13}{5}\right)$$



## Section 7.7 Fluid Pressure and Fluid Force

$$1. F = PA = [62.4(8)]3 = 1497.6 \text{ lb}$$

$$2. F = PA = [62.4(8)]16 = 7987.2 \text{ lb}$$

$$3. F = PA = [62.4(8)]10 = 4992 \text{ lb}$$

$$4. F = PA = [62.4(8)]22 = 10,982.4 \text{ lb}$$

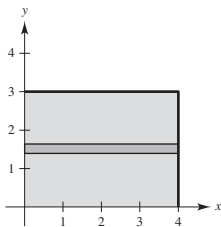
$$5. F = 62.4(h+2)(6) - (62.4)(h)(6) \\ = 62.4(2)(6) = 748.8 \text{ lb}$$

$$6. F = 62.4(h+4)(48) - (62.4)(h)(48) \\ = 62.4(4)(48) = 11,980.8 \text{ lb}$$

$$7. h(y) = 3 - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^3 (3 - y)(4) dy \\ = 249.6 \int_0^3 (3 - y) dy \\ = 249.6 \left[ 3y - \frac{y^2}{2} \right]_0^3 = 1123.2 \text{ lb}$$

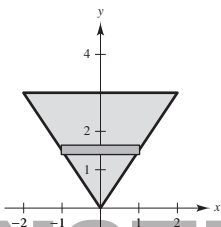


$$8. h(y) = 3 - y$$

$$L(y) = \frac{4}{3}y$$

$$F = 62.4 \int_0^3 (3 - y) \left( \frac{4}{3}y \right) dy \\ = \frac{4}{3}(62.4) \int_0^3 (3y - y^2) dy \\ = \frac{4}{3}(62.4) \left[ \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = 374.4 \text{ lb}$$

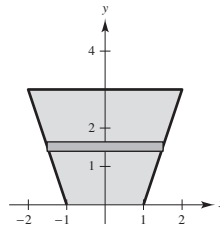
Force is one-third that of Exercise 7.



$$9. h(y) = 3 - y$$

$$L(y) = 2 \left( \frac{y}{3} + 1 \right)$$

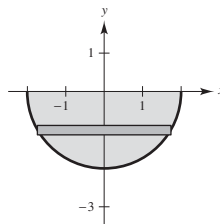
$$F = 2(62.4) \int_0^3 (3 - y) \left( \frac{y}{3} + 1 \right) dy \\ = 124.8 \int_0^3 \left( 3 - \frac{y^2}{3} \right) dy \\ = 124.8 \left[ 3y - \frac{y^3}{9} \right]_0^3 = 748.8 \text{ lb}$$



$$10. h(y) = -y$$

$$L(y) = 2\sqrt{4 - y^2}$$

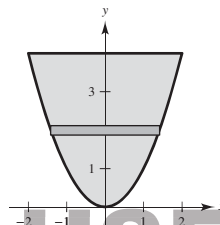
$$F = 62.4 \int_{-2}^0 (-y)(2)\sqrt{4 - y^2} dy \\ = \left[ 62.4 \left( \frac{2}{3} \right) (4 - y^2)^{3/2} \right]_{-2}^0 = 332.8 \text{ lb}$$



$$11. h(y) = 4 - y$$

$$L(y) = 2\sqrt{y}$$

$$F = 2(62.4) \int_0^4 (4 - y)\sqrt{y} dy \\ = 124.8 \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ = 124.8 \left[ \frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 1064.96 \text{ lb}$$

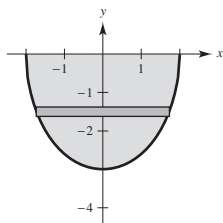




12.  $h(y) = -y$

$$L(y) = \frac{4}{3}\sqrt{9 - y^2}$$

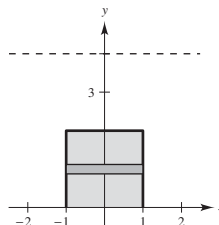
$$\begin{aligned} F &= 62.4 \int_{-3}^0 (-y) \frac{4}{3} \sqrt{9 - y^2} \, dy \\ &= 62.4 \left( \frac{2}{3} \right) \int_{-3}^0 (9 - y^2)^{1/2} (-2y) \, dy \\ &= \left[ 62.4 \left( \frac{4}{9} \right) (9 - y^2)^{3/2} \right]_{-3}^0 = 748.8 \text{ lb} \end{aligned}$$



13.  $h(y) = 4 - y$

$$L(y) = 2$$

$$\begin{aligned} F &= 9800 \int_0^2 2(4 - y) \, dy \\ &= 9800 \left[ 8y - y^2 \right]_0^2 = 117,600 \text{ newtons} \end{aligned}$$

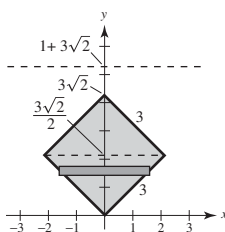


14.  $h(y) = (1 + 3\sqrt{2}) - y$

$$L_1(y) = 2y \quad (\text{lower part})$$

$$L_2(y) = 2(3\sqrt{2} - y) \quad (\text{upper part})$$

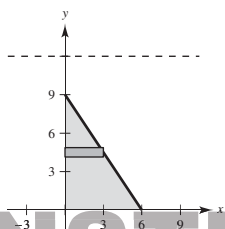
$$\begin{aligned} F &= 2(9800) \left[ \int_0^{3\sqrt{2}/2} (1 + 3\sqrt{2} - y)y \, dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} (1 + 3\sqrt{2} - y)(3\sqrt{2} - y) \, dy \right] \\ &= 19,600 \left[ \left[ \frac{y^2}{2} - 3\sqrt{2}y + \frac{y^3}{3} \right]_0^{3\sqrt{2}/2} + \left[ 3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y \right]_{3\sqrt{2}/2}^{3\sqrt{2}} \right] \\ &= 19,600 \left[ \frac{9(2\sqrt{2} + 1)}{4} + \frac{9(\sqrt{2} + 1)}{4} \right] = 44,100(3\sqrt{2} + 2) \text{ newtons} \end{aligned}$$



15.  $h(y) = 12 - y$

$$L(y) = 6 - \frac{2y}{3}$$

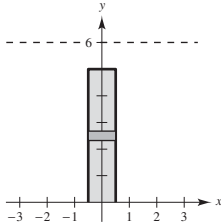
$$F = 9800 \int_0^9 (12 - y) \left( 6 - \frac{2y}{3} \right) dy = 9800 \left[ 72y - 7y^2 + \frac{2y^3}{9} \right]_0^9 = 2,381,400 \text{ newtons}$$



16.  $h(y) = 6 - y$

$L(y) = 1$

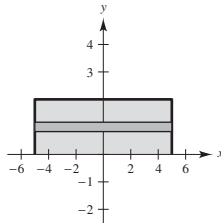
$$\begin{aligned}
 F &= 9800 \int_0^5 1(6 - y) \, dy \\
 &= 9800 \left[ 6y - \frac{y^2}{2} \right]_0^5 \\
 &= 171,500 \text{ newtons}
 \end{aligned}$$



17.  $h(y) = 2 - y$

$L(y) = 10$

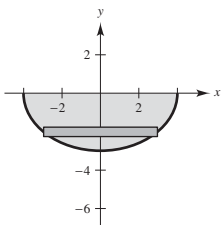
$$\begin{aligned}
 F &= 140.7 \int_0^2 (2 - y)(10) \, dy \\
 &= 1407 \int_0^2 (2 - y) \, dy \\
 &= 1407 \left[ 2y - \frac{y^2}{2} \right]_0^2 = 2814 \text{ lb}
 \end{aligned}$$



18.  $h(y) = -y$

$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

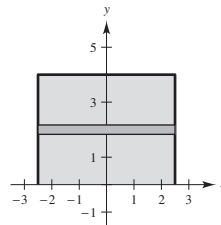
$$\begin{aligned}
 F &= 140.7 \int_{-3}^0 (-y)\left(2\left(\frac{4}{3}\sqrt{9 - y^2}\right)\right) \, dy \\
 &= \frac{(140.7)(4)}{3} \int_{-3}^0 \sqrt{9 - y^2}(-2y) \, dy \\
 &= \left[ \frac{(140.7)(4)}{3} \left(\frac{2}{3}\right)(9 - y^2)^{3/2} \right]_{-3}^0 \\
 &= 3376.8 \text{ lb}
 \end{aligned}$$



19.  $h(y) = 4 - y$

$L(y) = 6$

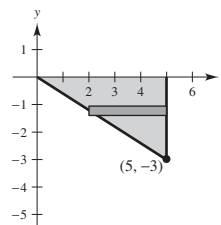
$$\begin{aligned}
 F &= 140.7 \int_0^4 (4 - y)(6) \, dy \\
 &= 844.2 \int_0^4 (4 - y) \, dy \\
 &= 844.2 \left[ 4y - \frac{y^2}{2} \right]_0^4 = 6753.6 \text{ lb}
 \end{aligned}$$



20.  $h(y) = -y$

$L(y) = 5 + \frac{5}{3}y$

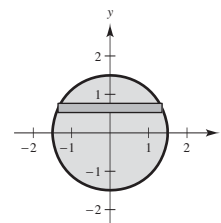
$$\begin{aligned}
 F &= 140.7 \int_{-3}^0 (-y)\left(5 + \frac{5}{3}y\right) \, dy \\
 &= 140.7 \int_{-3}^0 \left(-5y - \frac{5}{3}y^2\right) \, dy \\
 &= 140.7 \left[ -\frac{5}{2}y^2 - \frac{5}{9}y^3 \right]_{-3}^0 \\
 &= 140.7 \left[ \frac{45}{2} - 15 \right] \\
 &= 1055.25 \text{ lb}
 \end{aligned}$$



21.  $h(y) = -y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$\begin{aligned}
 F &= 42 \int_{-3/2}^0 (-y)\sqrt{9 - 4y^2} \, dy \\
 &= \frac{42}{8} \int_{-3/2}^0 (9 - 4y^2)^{1/2}(-8y) \, dy \\
 &= \left[ \left(\frac{21}{4}\right)\left(\frac{2}{3}\right)(9 - 4y^2)^{3/2} \right]_{-3/2}^0 = 94.5 \text{ lb}
 \end{aligned}$$



22.  $h(y) = \frac{3}{2} - y$

$$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$$

$$F = 42 \int_{-3/2}^{3/2} \left(\frac{3}{2} - y\right)\sqrt{9 - 4y^2} dy = 63 \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} dy + \frac{21}{4} \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} (-8y) dy$$

The second integral is zero because it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius  $\frac{3}{2}$ .

$$\left(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2}\right)$$

So, the force is  $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32$  lb.

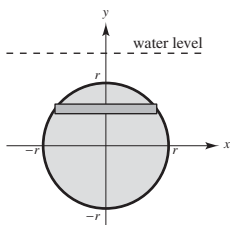
23.  $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$

$$F = w \int_{-r}^r (k - y)\sqrt{r^2 - y^2} (2) dy = w \left[ 2k \int_{-r}^r \sqrt{r^2 - y^2} dy + \int_{-r}^r \sqrt{r^2 - y^2} (-2y) dy \right]$$

The second integral is zero because its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius  $r$ .

$$F = w \left[ (2k) \frac{\pi r^2}{2} + 0 \right] = wk\pi r^2$$



24. (a)  $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi$  lb

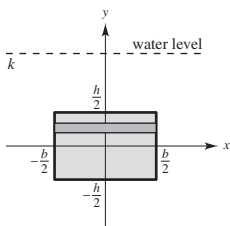
(b)  $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi$  lb

25.  $h(y) = k - y$

$$L(y) = b$$

$$F = w \int_{-h/2}^{h/2} (k - y)b dy$$

$$= wb \left[ ky - \frac{y^2}{2} \right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



26. (a)  $F = wkhb$

$$= (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148$$
 lb

(b)  $F = wkhb$

$$= (62.4)\left(\frac{17}{2}\right)(5)(10) = 26,520$$
 lb

27. From Exercise 25:

$$F = 64(15)(1)(1) = 960$$
 lb

28. From Exercise 23:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98$$
 lb

29.  $h(y) = 4 - y$

$$F = 62.4 \int_0^4 (4 - y)L(y) dy$$

Using Simpson's Rule with  $n = 8$  you have:

$$\begin{aligned} F &\approx 62.4 \left( \frac{4-0}{3(8)} \right) \left[ 0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0 \right] \\ &= 3010.8 \text{ lb} \end{aligned}$$

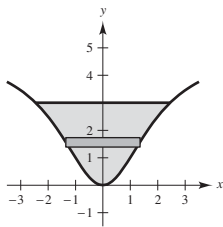
30.  $h(y) = 3 - y$

Solving  $y = 5x^2/(x^2 + 4)$  for  $x$ , you obtain

$$x = \sqrt{4y/(5 - y)}.$$

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

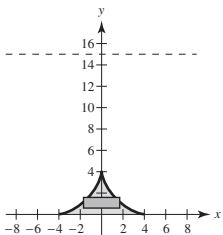
$$\begin{aligned} F &= 62.4(2) \int_0^3 (3 - y) \sqrt{\frac{4y}{5 - y}} dy \\ &= 2(124.8) \int_0^3 (3 - y) \sqrt{\frac{y}{5 - y}} dy \approx 546.265 \text{ lb} \end{aligned}$$



31.  $h(y) = 15 - y$

$$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$$

$$\begin{aligned} F &= 62.4 \int_0^4 2(15 - y)(4^{2/3} - y^{2/3})^{3/2} dy \\ &\approx 8213.04 \text{ lb} \end{aligned}$$



32.  $h(y) = 15 - y$

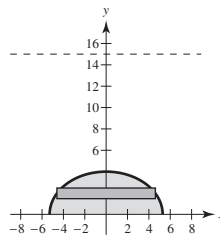
$$\frac{x^2}{28} = 1 - \frac{y_2}{16} = \frac{16 - y^2}{16}$$

$$x^2 = \frac{7}{4}(16 - y^2)$$

$$x = \frac{1}{2}\sqrt{7(16 - y^2)}$$

$$L(y) = \sqrt{7(16 - y^2)}$$

$$F = 62.4\sqrt{7} \int_0^4 (15 - y)\sqrt{16 - y^2} dy \approx 27,597.63 \text{ lb}$$



33. If the fluid force is one-half of 1123.2 lb, and the height of the water is  $b$ , then

$$h(y) = b - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^b (b - y)(4) dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y) dy = 2.25$$

$$\left[ by - \frac{y^2}{2} \right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

The pressure increases with increasing depth.

34. (a) Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

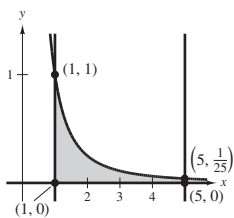
(b)  $F = Fw = w \int_c^d h(y)L(y) dy$ , see page 510.

35. You use horizontal representative rectangles because you are measuring total force against a region between two depths.

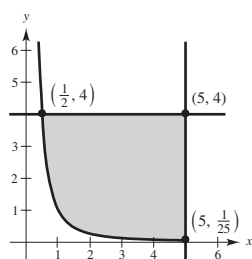
36. The left window experiences the greater fluid force because its centroid is lower.

## Review Exercises for Chapter 7

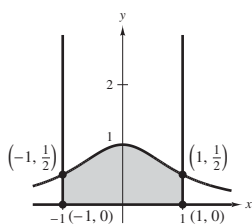
$$1. A = \int_1^5 \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^5 = \frac{4}{5}$$



$$2. A = \int_{1/2}^5 \left( 4 - \frac{1}{x^2} \right) dx = \left[ 4x + \frac{1}{x} \right]_{1/2}^5 = \frac{81}{5}$$



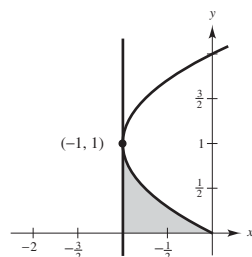
$$3. A = \int_{-1}^1 \frac{1}{x^2 + 1} dx = [\arctan x]_{-1}^1 = \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) = \frac{\pi}{2}$$



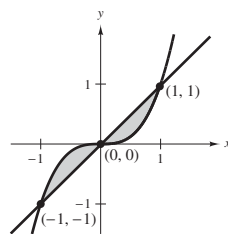
$$4. A = \int_0^1 [(y^2 - 2y) - (-1)] dy$$

$$= \int_0^1 (y^2 - 2y + 1) dy$$

$$= \int_0^1 (y - 1)^2 dy = \left[ \frac{(y - 1)^3}{3} \right]_0^1 = \frac{1}{3}$$

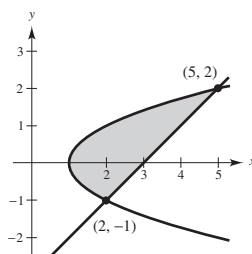


$$5. A = 2 \int_0^1 (x - x^3) dx = 2 \left[ \frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 = \frac{1}{2}$$

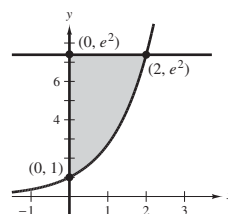


$$6. A = \int_{-1}^2 [(y + 3) - (y^2 + 1)] dy$$

$$= \int_{-1}^2 (2 + y - y^2) dy = \left[ 2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2 = \frac{9}{2}$$



$$7. A = \int_0^2 (e^2 - e^x) dx = [xe^2 - e^x]_0^2 = e^2 + 1$$

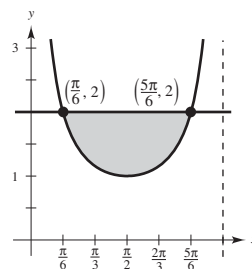


$$8. A = 2 \int_{\pi/6}^{\pi/2} (2 - \csc x) dx$$

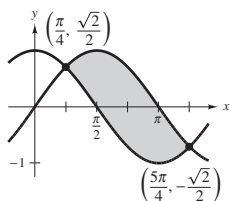
$$= 2 \left[ 2x - \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/2}$$

$$= 2 \left[ \left( \pi - 0 \right) - \left[ \frac{\pi}{3} - \ln(2 - \sqrt{3}) \right] \right]$$

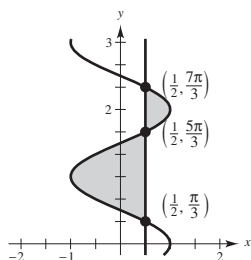
$$= 2 \left[ \frac{2\pi}{3} + \ln(2 - \sqrt{3}) \right] \approx 1.555$$



$$\begin{aligned}
 9. \quad A &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\
 &= [-\cos x - \sin x]_{\pi/4}^{5\pi/4} \\
 &= \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\
 &= \frac{4}{\sqrt{2}} = 2\sqrt{2}
 \end{aligned}$$



$$\begin{aligned}
 10. \quad A &= \int_{\pi/3}^{5\pi/3} \left( \frac{1}{2} - \cos y \right) dy + \int_{5\pi/3}^{7\pi/3} \left( \cos y - \frac{1}{2} \right) dy \\
 &= \left[ \frac{y}{2} - \sin y \right]_{\pi/3}^{5\pi/3} + \left[ \sin y - \frac{y}{2} \right]_{5\pi/3}^{7\pi/3} \\
 &= \frac{\pi}{3} + 2\sqrt{3}
 \end{aligned}$$

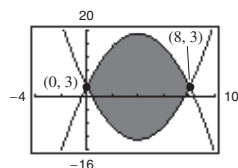


11. Points of intersection:

$$x^2 - 8x + 3 = 3 + 8x - x^2$$

$$2x^2 - 16x = 0 \quad \text{when} \quad x = 0, 8$$

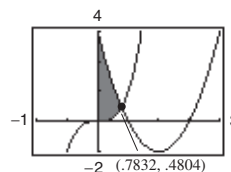
$$\begin{aligned}
 A &= \int_0^8 \left[ (3 + 8x - x^2) - (x^2 - 8x + 3) \right] dx \\
 &= \int_0^8 (16x - 2x^2) dx \\
 &= \left[ 8x^2 - \frac{2}{3}x^3 \right]_0^8 = \frac{512}{3} \approx 170.667
 \end{aligned}$$



12. Point of intersection:

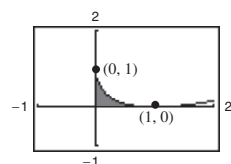
$$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$$

$$\begin{aligned}
 A &\approx \int_0^{0.783} (3 - 4x + x^2 - x^3) dx \\
 &= \left[ 3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^{0.783} \\
 &\approx 1.189
 \end{aligned}$$



$$13. \quad y = (1 - \sqrt{x})^2$$

$$\begin{aligned}
 A &= \int_0^1 (1 - \sqrt{x})^2 dx \\
 &= \int_0^1 (1 - 2x^{1/2} + x) dx \\
 &= \left[ x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{6} \approx 0.1667
 \end{aligned}$$

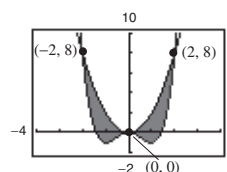


14. Points of intersection:

$$x^4 - 2x^2 = 2x^2$$

$$x^4 - 4x^2 = 0 \quad \text{when} \quad x = 0, \pm 2$$

$$\begin{aligned}
 A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\
 &= 2 \int_0^2 (4x^2 - x^4) dx \\
 &= 2 \left[ \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{128}{15} \approx 8.5333
 \end{aligned}$$



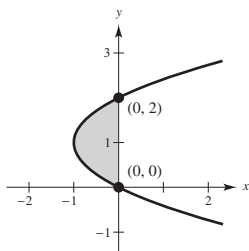
15.  $x = y^2 - 2y \Rightarrow x + 1 = (y - 1)^2 \Rightarrow y = 1 \pm \sqrt{x + 1}$

$$A = \int_{-1}^0 \left[ (1 + \sqrt{x + 1}) - (1 - \sqrt{x + 1}) \right] dx$$

$$= \int_{-1}^0 2\sqrt{x + 1} dx$$

$$A = \int_0^2 [0 - (y^2 - 2y)] dy$$

$$= \int_0^2 (2y - y^2) dy = \left[ y^2 - \frac{1}{3}y^3 \right]_0^2 = \frac{4}{3}$$



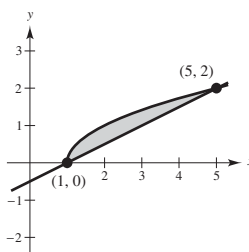
16.  $y = \sqrt{x - 1} \Rightarrow x = y^2 + 1$

$$y = \frac{x - 1}{2} \Rightarrow x = 2y + 1$$

$$A = \int_0^2 [(2y + 1) - (y^2 + 1)] dy$$

$$A = \int_1^5 \left[ \sqrt{x - 1} - \frac{x - 1}{2} \right] dx$$

$$= \left[ \frac{2}{3}(x - 1)^{3/2} - \frac{1}{4}(x - 1)^2 \right]_1^5 = \frac{4}{3}$$



17.  $A = \int_0^2 \left[ 1 - \left( 1 - \frac{x}{2} \right) \right] dx + \int_2^3 [1 - (x - 2)] dx$

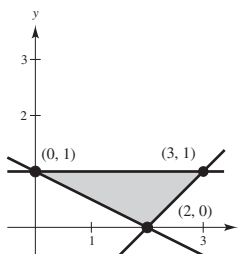
$$= \int_0^2 \frac{x}{2} dx + \int_2^3 (3 - x) dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2, y = 1$$

$$A = \int_0^1 [(y + 2) - (2 - 2y)] dy$$

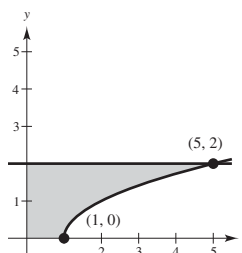
$$= \int_0^1 3y dy = \left[ \frac{3}{2}y^2 \right]_0^1 = \frac{3}{2}$$



18.  $A = \int_0^1 2 dx + \int_1^5 [2 - \sqrt{x - 1}] dx$

$$x = y^2 + 1$$

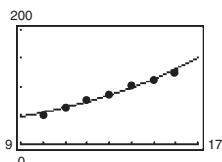
$$A = \int_0^2 (y^2 + 1) dy = \left[ \frac{1}{3}y^3 + y \right]_0^2 = \frac{14}{3}$$



19. (a) Trapezoidal: Area  $\approx \frac{160}{2(8)} [0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0] = 9920 \text{ ft}^2$

(b) Simpson's: Area  $\approx \frac{160}{3(8)} [0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0] = 10,413\frac{1}{3} \text{ ft}^2$

20. (a)  $y = 13.2945(1.1539)^t = 13.2945 e^{0.1432t}$

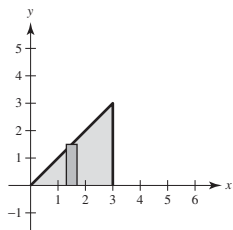


(b)  $R_2 = 6 + 13.9e^{0.14t}$

Difference:  $\int_{20}^{25} (y - R_2) dt \approx 17.7 \text{ billion dollars}$

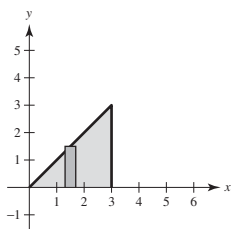
## 21. (a) Disk

$$V = \pi \int_0^3 x^2 dx = \left[ \frac{\pi x^3}{3} \right]_0^3 = 9\pi$$



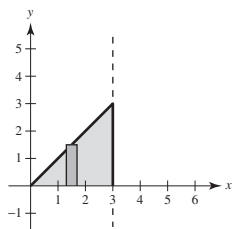
## (b) Shell

$$V = 2\pi \int_0^3 x(x) dx = 2\pi \left[ \frac{x^3}{3} \right]_0^3 = 18\pi$$



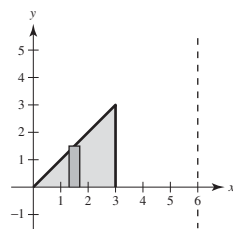
## (c) Shell

$$V = 2\pi \int_0^3 (3-x)x dx = 2\pi \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = 9\pi$$



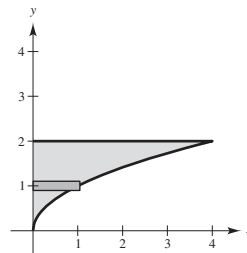
## (d) Shell

$$V = 2\pi \int_0^3 (6-x)x dx = 2\pi \left[ 3x^2 - \frac{x^3}{3} \right]_0^3 = 36\pi$$



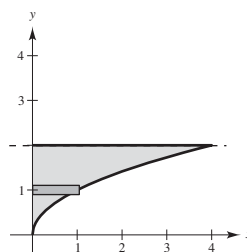
## 22. (a) Shell

$$V = 2\pi \int_0^2 y^3 dy = \left[ \frac{\pi}{2} y^4 \right]_0^2 = 8\pi$$



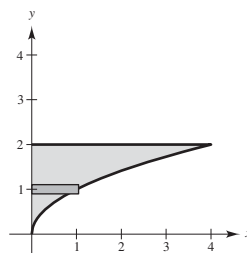
## (b) Shell

$$\begin{aligned} V &= 2\pi \int_0^2 (2-y)y^2 dy \\ &= 2\pi \int_0^2 (2y^2 - y^3) dy = 2\pi \left[ \frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



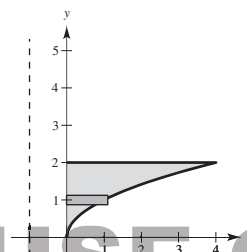
## (c) Disk

$$V = \pi \int_0^2 y^4 dy = \left[ \frac{\pi}{5} y^5 \right]_0^2 = \frac{32\pi}{5}$$



## (d) Disk

$$\begin{aligned} V &= \pi \int_0^2 \left[ (y^2 + 1)^2 - 1^2 \right] dy \\ &= \pi \int_0^2 (y^4 + 2y^2) dy = \pi \left[ \frac{1}{5}y^5 + \frac{2}{3}y^3 \right]_0^2 = \frac{176\pi}{15} \end{aligned}$$

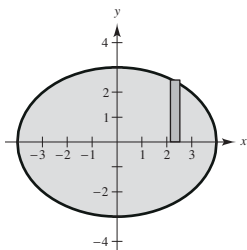




**23. (a) Shell**

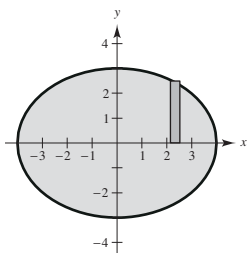
$$V = 4\pi \int_0^4 x \left(\frac{3}{4}\right) \sqrt{16 - x^2} dx$$

$$= \left[ 3\pi \left(-\frac{1}{2}\right) \left(\frac{2}{3}\right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi$$


**(b) Disk**

$$V = 2\pi \int_0^4 \left[ \frac{3}{4} \sqrt{16 - x^2} \right]^2 dx$$

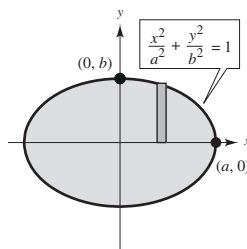
$$= \frac{9\pi}{8} \left[ 16x - \frac{x^3}{3} \right]_0^4 = 48\pi$$


**24. (a) Shell**

$$V = 4\pi \int_0^a x \frac{b}{a} \sqrt{a^2 - x^2} dx$$

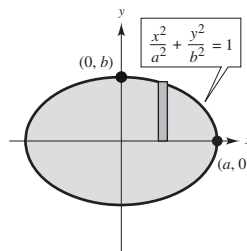
$$= \frac{-2\pi b}{a} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx$$

$$= \left[ \frac{-4\pi b}{3a} (a^2 - x^2)^{3/2} \right]_0^a = \frac{4}{3} \pi a^2 b$$

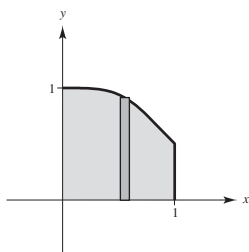

**(b) Disk**

$$V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx$$

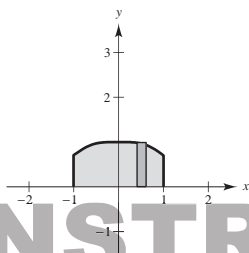
$$= \frac{2\pi b^2}{a^2} \left[ a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi a b^2$$


**25. Shell**

$$V = 2\pi \int_0^1 \frac{x}{x^4 + 1} dx = \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1} dx = \left[ \pi \arctan(x^2) \right]_0^1 = \pi \left( \frac{\pi}{4} - 0 \right) = \frac{\pi^2}{4}$$


**26. Disk**

$$V = 2\pi \int_0^1 \left[ \frac{1}{\sqrt{1+x^2}} \right]^2 dx = \left[ 2\pi \arctan x \right]_0^1 = 2\pi \left( \frac{\pi}{4} - 0 \right) = \frac{\pi^2}{2}$$



27. Shell:  $V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x-2}} dx$

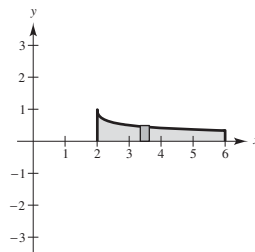
$$u = \sqrt{x-2}$$

$$x = u^2 + 2$$

$$dx = 2u du$$

$$V = 2\pi \int_2^6 \frac{x}{1 + \sqrt{x-2}} dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u} du$$

$$= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u} du = 4\pi \int_0^2 \left( u^2 - u + 3 - \frac{3}{1+u} \right) du = 4\pi \left[ \frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3 \ln(1+u) \right]_0^2 = \frac{4\pi}{3}(20 - 9 \ln 3) \approx 42.359$$



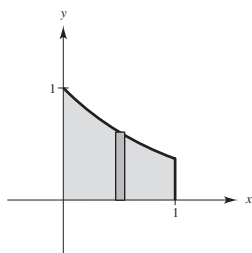
28. Disk

$$V = \pi \int_0^1 (e^{-x})^2 dx$$

$$= \pi \int_0^1 e^{-2x} dx$$

$$= \left[ -\frac{\pi}{2} e^{-2x} \right]_0^1$$

$$= \left( -\frac{\pi}{2e^2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \left( 1 - \frac{1}{e^2} \right)$$



29. (a) Because  $y \leq 0$ ,  $A = -\int_{-1}^0 x\sqrt{x+1} dx$ .

$$u = x + 1$$

$$x = u - 1$$

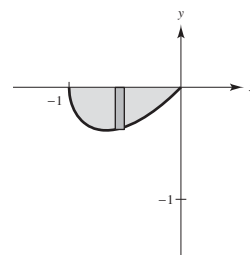
$$dx = du$$

$$A = -\int_0^1 (u-1)\sqrt{u} du$$

$$= -\int_0^1 (u^{3/2} - u^{1/2}) du$$

$$= -\left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1$$

$$= \frac{4}{15}$$



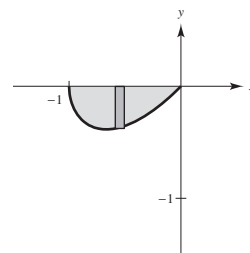
(b) Disk

$$V = \pi \int_{-1}^0 x^2(x+1) dx$$

$$= \pi \int_{-1}^0 (x^3 + x^2) dx$$

$$= \pi \left[ \frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0$$

$$= \frac{\pi}{12}$$



(c) Shell

$$u = \sqrt{x+1}$$

$$x = u^2 - 1$$

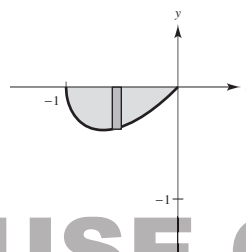
$$dx = 2u du$$

$$V = 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx$$

$$= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du$$

$$= 4\pi \int_0^1 (u^6 - 2u^4 + u^2) du$$

$$= 4\pi \left[ \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 \right]_0^1 = \frac{32\pi}{105}$$



30. (a) **Disk:**

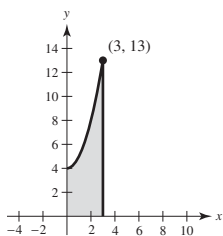
$$V = \pi \int_0^3 (x^2 + 4)^2 dx$$

(b) **Shell:**  $y = x^2 + 4 \Rightarrow x = \sqrt{4 - y}$

$$V = 2\pi \int_0^4 3y dy + 2\pi \int_4^{13} y(3 - \sqrt{y - 4}) dy$$

(c) No. The integral in (a) is respect to  $x$ , while those in (b) are with respect to  $y$ .

**Note:** The volume is  $843\pi/5$ .



31. From Exercise 23(a) you have:  $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

**Disk:**  $\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2) dy = 16\pi$

$$\frac{1}{9} \int_{-3}^{y_0} (9 - y^2) dy = 1$$

$$\left[ 9y - \frac{1}{3}y^3 \right]_{-3}^{y_0} = 9$$

$$\left( 9y_0 - \frac{1}{3}y_0^3 \right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method,  $y_0 \approx -1.042$  and the depth of the gasoline is  $3 - 1.042 = 1.958$  feet.

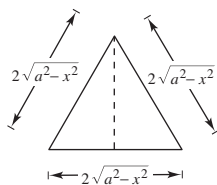
32.  $A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2})$   
 $= \sqrt{3}(a^2 - x^2)$

$$V = \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[ a^2x - \frac{x^3}{3} \right]_{-a}^a$$

$$= \sqrt{3} \left( \frac{4a^3}{3} \right)$$

Because  $(4\sqrt{3}a^3)/3 = 10$ , you have

$$a^3 = (5\sqrt{3})/2. \text{ So, } a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



33.  $f(x) = \frac{4}{5}x^{5/4}$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u - 1)^2$$

$$dx = 2(u - 1) du$$

$$s = \int_0^4 \sqrt{1 + \sqrt{x}} dx = 2 \int_1^3 \sqrt{u}(u - 1) du$$

$$= 2 \int_1^3 (u^{3/2} - u^{1/2}) du$$

$$= 2 \left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^3 = \frac{4}{15} [u^{3/2}(3u - 5)]_1^3$$

$$= \frac{8}{15} (1 + 6\sqrt{3}) \approx 6.076$$

34.  $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[ \frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

35.  $y = 300 \cosh\left(\frac{x}{2000}\right) - 280, -2000 \leq x \leq 2000$

$$y' = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[ \frac{3}{20} \sinh\left(\frac{x}{2000}\right) \right]^2} dx$$

$$= \frac{1}{20} \int_{-2000}^{2000} \sqrt{400 + 9 \sinh^2\left(\frac{x}{2000}\right)} dx$$

$$= 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

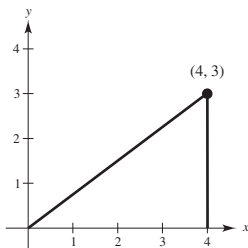
36. Because  $f(x) = \tan x$  has  $f'(x) = \sec^2 x$ , this integral represents the length of the graph of  $\tan x$  from  $x = 0$  to  $x = \pi/4$ . This length is a little over 1 unit.  
 Answer (b).

$$37. y = \frac{3}{4}x$$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right) \sqrt{\frac{25}{16}} dx = \left[\left(\frac{15\pi}{8}\right) \frac{x^2}{2}\right]_0^4 = 15\pi$$



$$38. y = 2\sqrt{x}, y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\begin{aligned} S &= 2\pi \int_3^8 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_3^8 \sqrt{x+1} dx \\ &= 4\pi \left[ \frac{2}{3}(x+1)^{3/2} \right]_3^8 = \frac{152\pi}{3} \end{aligned}$$

$$39. F = kx$$

$$5 = k(1)$$

$$F = 5x$$

$$W = \int_0^5 5x dx = \left[ \frac{5x^2}{2} \right]_0^5 = \frac{125}{2} \text{ in-lb} \approx 5.21 \text{ ft-lb}$$

$$40. F = kx$$

$$50 = k(1) \Rightarrow k = 50$$

$$W = \int_0^{10} 50x dx = \left[ 25x^2 \right]_0^{10} = 2500 \text{ in-lb} \approx 208.3 \text{ ft-lb}$$

$$41. \text{Volume of disk: } \pi \left(\frac{1}{3}\right)^2 \Delta y \quad \left[ \text{diameter} = \frac{2}{3} \text{ ft} \right]$$

$$\text{Weight of disk: } 62.4\pi \left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Distance: } 190 - y$$

$$\begin{aligned} W &= \frac{62.4\pi}{9} \int_0^{165} (190 - y) dy \\ &= \frac{62.4\pi}{9} \left[ 190y - \frac{y^2}{2} \right]_0^{165} \\ &= \frac{62.4\pi}{9} \left[ \frac{35,475}{2} \right] = 122,980\pi \text{ ft-lb} \end{aligned}$$

$$\approx 193.2 \text{ foot-tons}$$

42. You know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = \frac{-8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi \left(\frac{1}{9}\right) h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left( \frac{dV}{dt} \right) = \frac{9}{\pi} \left( \frac{-8}{7.481} \right) \approx -3.064 \text{ ft/min}$$

$$\text{Depth of water: } -3.064 t + 165$$

$$\text{Time to drain well: } t = \frac{165}{3.064} \approx 53.85 \text{ min}$$

$$(53.85)(12) \approx 646.2 \text{ gallons pumped}$$

Volume of water pumped in Exercise 41:

$$\left[ 165\pi \left(\frac{1}{9}\right) \text{ ft}^3 \right] \cdot [7.481 \text{ gal/ft}^3] = 430.87 \text{ gallons}$$

$$\frac{430.87}{386,353.1} = \frac{646.2}{x}$$

$$\Rightarrow x = \frac{646.2}{430.87} (386,353.1) \approx 579,435.5$$

$$\text{Work} \approx 579,435.5 \text{ ft-lb}$$

43. Weight of section of chain:  $4 \Delta x$

$$\text{Distance moved: } 10 - x$$

$$\begin{aligned} W &= 4 \int_0^{10} (10 - x) dx = 4 \left[ 10x - \frac{x^2}{2} \right]_0^{10} \\ &= 200 \text{ ft-lb} \end{aligned}$$

44. (a) Weight of section of cable:  $5 \Delta x$

$$\text{Distance: } 200 - x$$

$$\begin{aligned} W &= 5 \int_0^{200} (200 - x) dx \\ &= 5 \left[ 200x - \frac{x^2}{2} \right]_0^{200} \\ &= 100,000 \text{ ft-lb} \end{aligned}$$

(b) Work to move 300 pounds 200 feet vertically:

$$300(200) = 60,000 \text{ ft-lb.}$$

$$\text{Total work: } 100,000 + 60,000 = 160,000 \text{ ft-lb}$$

$$45. W = \int_a^b F(x) dx$$

$$80 = \int_0^4 ax^2 dx = \left[ \frac{ax^3}{3} \right]_0^4 = \frac{64}{3} a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

$$46. \quad W = \int_a^b F(x) \, dx$$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \leq x \leq 9 \\ -(4/3)x + 16, & 9 \leq x \leq 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\frac{2}{9}x + 6\right) dx + \int_9^{12} \left(-\frac{4}{3}x + 16\right) dx \\ &= \left[-\frac{1}{9}x^2 + 6x\right]_0^9 + \left[-\frac{2}{3}x^2 + 16x\right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) \\ &= 51 \text{ ft-lbs} \end{aligned}$$

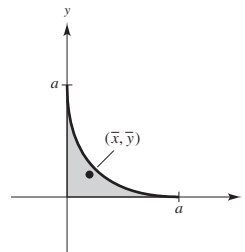
$$47. \quad A = \int_0^a (\sqrt{a} - \sqrt{x})^2 \, dx = \int_0^a (a - 2\sqrt{ax^{1/2}} + x) \, dx = \left[ax - \frac{4}{3}\sqrt{ax^{3/2}} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$$

$$\frac{1}{A} = \frac{6}{a^2}$$

$$\bar{x} = \frac{6}{a^2} \int_0^a x(\sqrt{a} - \sqrt{x})^2 \, dx = \frac{6}{a^2} \int_0^a (ax - 2\sqrt{ax^{3/2}} + x^2) \, dx = \frac{a}{5}$$

$$\begin{aligned} \bar{y} &= \left(\frac{6}{a^2}\right) \frac{1}{2} \int_0^a (\sqrt{a} - \sqrt{x})^4 \, dx \\ &= \frac{3}{a^2} \int_0^a (a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2) \, dx \\ &= \frac{3}{a^2} \left[ a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3 \right]_0^a = \frac{a}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



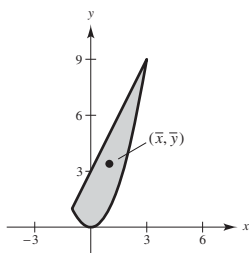
$$48. \quad A = \int_{-1}^3 [(2x + 3) - x^2] \, dx = \left[x^2 + 3x - \frac{1}{3}x^3\right]_{-1}^3 = \frac{32}{3}$$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32} \int_{-1}^3 x(2x + 3 - x^2) \, dx = \frac{3}{32} \int_{-1}^3 (3x + 2x^2 - x^3) \, dx = \frac{3}{32} \left[ \frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32}\right) \frac{1}{2} \int_{-1}^3 [(2x + 3)^2 - x^4] \, dx = \frac{3}{64} \int_{-1}^3 (9 + 12x + 4x^2 - x^4) \, dx \\ &= \frac{3}{64} \left[ 9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5}\right)$$



49. By symmetry,  $x = 0$ .

$$A = 2 \int_0^1 (a^2 - x^2) dx = 2 \left[ a^2 x - \frac{x^3}{3} \right]_0^1 = \frac{4a^3}{3}$$

$$\frac{1}{A} = \frac{3}{4a^3}$$

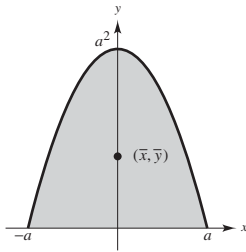
$$\bar{y} = \left( \frac{3}{4a^3} \right) \frac{1}{2} \int_{-a}^a (a^2 - x^2)^2 dx$$

$$= \frac{6}{8a^3} \int_0^a (a^4 - 2a^2 x^2 + x^4) dx$$

$$= \frac{6}{8a^3} \left[ a^4 x - \frac{2a^2}{3} x^3 + \frac{1}{5} x^5 \right]_0^a$$

$$= \frac{6}{8a^3} \left( a^5 - \frac{2}{3} a^5 + \frac{1}{5} a^5 \right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{2a^2}{5} \right)$$



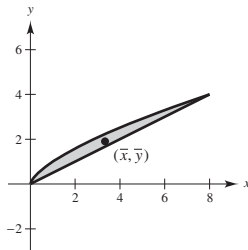
50.  $A = \int_0^8 \left( x^{2/3} - \frac{1}{2}x \right) dx = \left[ \frac{3}{5}x^{5/3} - \frac{1}{4}x^2 \right]_0^8 = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\bar{x} = \frac{5}{16} \int_0^8 x \left( x^{2/3} - \frac{1}{2}x \right) dx = \frac{5}{16} \left[ \frac{3}{8}x^{8/3} - \frac{1}{6}x^3 \right]_0^8 = \frac{10}{3}$$

$$\bar{y} = \left( \frac{5}{16} \right) \frac{1}{2} \int_0^8 \left( x^{4/3} - \frac{1}{4}x^2 \right) dx = \frac{1}{2} \left( \frac{5}{16} \right) \left[ \frac{3}{7}x^{7/3} - \frac{1}{12}x^3 \right]_0^8 = \frac{40}{21}$$

$$(\bar{x}, \bar{y}) = \left( \frac{10}{3}, \frac{40}{21} \right)$$



51.  $\bar{y} = 0$  by symmetry.

For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$M_y = \rho \int_0^6 x \left[ \left( \frac{1}{6}x + 1 \right) - \left( -\frac{1}{6}x - 1 \right) \right] dx = \rho \int_0^6 \left( \frac{1}{3}x^2 + 2x \right) dx = \rho \left[ \frac{x^3}{9} + x^2 \right]_0^6 = 60\rho$$

For the semicircle:

$$m = \left( \frac{1}{2} \right) (\pi)(2)^2 \rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x \left[ \sqrt{4 - (x-6)^2} - \left( -\sqrt{4 - (x-6)^2} \right) \right] dx = 2\rho \int_6^8 x \sqrt{4 - (x-6)^2} dx$$

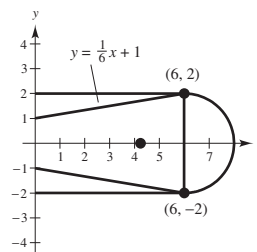
Let  $u = x - 6$ , then  $x = u + 6$  and  $dx = du$ . When  $x = 6$ ,  $u = 0$ . When  $x = 8$ ,  $u = 2$ .

$$\begin{aligned} M_y &= 2\rho \int_0^2 (u+6) \sqrt{4-u^2} du = 2\rho \int_0^2 u \sqrt{4-u^2} du + 12\rho \int_0^2 \sqrt{4-u^2} du \\ &= 2\rho \left[ -\frac{1}{2} \left( \frac{2}{3} \right) (4-u^2)^{3/2} \right]_0^2 + 12\rho \left[ \frac{\pi(2)^2}{4} \right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4+9\pi)}{3} \end{aligned}$$

So, you have:

$$\begin{aligned} \bar{x}(18\rho + 2\pi\rho) &= 60\rho + \frac{4\rho(4+9\pi)}{3} \\ \bar{x} &= \frac{180\rho + 4\rho(4+9\pi)}{3} \cdot \frac{1}{2\rho(9+\pi)} = \frac{2(9\pi+49)}{3(\pi+9)} \end{aligned}$$

The centroid of the blade is  $\left( \frac{2(9\pi+49)}{3(\pi+9)}, 0 \right)$ .



52. Wall at shallow end:

$$F = 62.4 \int_0^5 y(20) dy = \left[ (1248) \frac{y^2}{2} \right]_0^5 = 15,600 \text{ lb}$$

Wall at deep end:

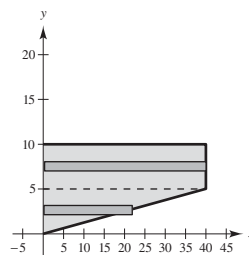
$$F = 62.4 \int_0^{10} y(20) dy = \left[ (624) y^2 \right]_0^{10} = 62,400 \text{ lb}$$

Side wall:

$$F_1 = 62.4 \int_0^5 y(40) dy = \left[ (1248) y^2 \right]_0^5 = 31,200 \text{ lb}$$

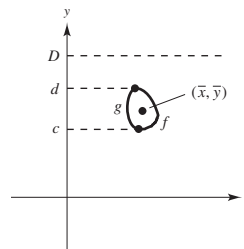
$$F_2 = 62.4 \int_0^5 (10-y)8y dy = 62.4 \int_0^5 (80y - 8y^2) dy$$

$$F = F_1 + F_2 = 72,800 \text{ lb}$$



53. Let  $D$  = surface of liquid;  $\rho$  = weight per cubic volume.

$$\begin{aligned} F &= \rho \int_c^d (D-y) [f(y) - g(y)] dy \\ &= \rho \left[ \int_c^d D[f(y) - g(y)] dy - \int_c^d y[f(y) - g(y)] dy \right] \\ &= \rho \left[ \int_c^d [f(y) - g(y)] dy \right] \left[ D - \frac{\int_c^d y[f(y) - g(y)] dy}{\int_c^d [f(y) - g(y)] dy} \right] \\ &= \rho(\text{Area})(D - \bar{y}) \\ &= \rho(\text{Area})(\text{depth of centroid}) \end{aligned}$$



54.  $F = 62.4(16\pi)10 = 9984\pi$  lb

### Problem Solving for Chapter 7

1.  $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2) dx = \left[ \frac{cx^2}{2} - \frac{x^3}{3} \right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c \rightarrow 0^+} \frac{T}{R} = \lim_{c \rightarrow 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

2. (a) By symmetry,  $M_x = 0$  for  $L$

(b) Because

$$(M_y \text{ for } L) + (M_y \text{ for } A) = (M_y \text{ for } B),$$

you have

$$(M_y \text{ for } L) = (M_y \text{ for } B) - (M_y \text{ for } A)$$

(c)  $M_y$  for  $B = 0$ , because  $B$  is a circle at the origin

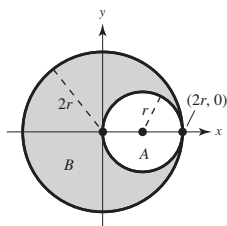
$$\text{For } A, \bar{x} = \frac{M_y}{\text{Area}} \Rightarrow M_y = r(\pi r^2) = \pi r^3$$

$$\text{So, } (M_y \text{ for } L) = 0 - \pi r^3 = -\pi r^3$$

(d)  $\bar{y} = 0$  by symmetry.

$$\bar{x} = \frac{M_y \text{ of } L}{\text{Area of } L} = \frac{-\pi r^3}{4\pi r^2 - \pi r^2} = -\frac{r}{3}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{r}{3}, 0\right)$$



4. (a)  $\frac{1}{2}V = \int_0^1 \left[ \pi(2 + \sqrt{1-y^2})^2 - \pi(2 - \sqrt{1-y^2})^2 \right] dy$

$$= \pi \int_0^1 \left[ (4 + 4\sqrt{1-y^2} + (1-y^2)) - (4 - 4\sqrt{1-y^2} + (1-y^2)) \right] dy$$

$$= 8\pi \int_0^1 \sqrt{1-y^2} dy \quad (\text{Integral represents } 1/4 \text{ (area of circle)})$$

$$= 8\pi \left( \frac{\pi}{4} \right) = 2\pi^2 \Rightarrow V = 4\pi^2$$

(b)  $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$$\frac{1}{2}V = \int_0^r \left[ \pi(R + \sqrt{r^2 - y^2})^2 - \pi(R - \sqrt{r^2 - y^2})^2 \right] dy = \pi \int_0^r 4R\sqrt{r^2 - y^2} dy = \pi(4R) \frac{1}{4} \pi r^2 = \pi^2 r^2 R$$

$$V = 2\pi^2 r^2 R$$

3.  $R = \int_0^1 x(1-x) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let  $(c, mc)$  be the intersection of the line and the parabola.

$$\text{Then, } mc = c(1-c) \Rightarrow m = 1-c \text{ or } c = 1-m.$$

$$\frac{1}{2} \left( \frac{1}{6} \right) = \int_0^{1-m} (x - x^2 - mx) dx$$

$$\frac{1}{12} = \left[ \frac{x^2}{2} - \frac{x^3}{3} - m \frac{x^2}{2} \right]_0^{1-m}$$

$$= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m \frac{(1-m)^2}{2}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

$$\frac{1}{2} = (1-m)^3$$

$$\left( \frac{1}{2} \right)^{1/3} = 1-m$$

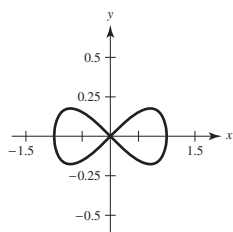
$$m = 1 - \left( \frac{1}{2} \right)^{1/3} \approx 0.2063$$

$$\text{So, } y = 0.2063x.$$



5.  $8y^2 = x^2(1 - x^2)$

$$y = \pm \frac{|x|\sqrt{1-x^2}}{2\sqrt{2}}$$



For  $x > 0$ ,  $y' = \frac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}}$

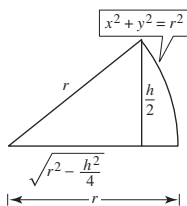
$$S = 2(2\pi) \int_0^1 x \sqrt{1 + \left( \frac{1-2x^2}{2\sqrt{2}\sqrt{1-x^2}} \right)^2} dx$$

$$= \frac{5\sqrt{2}\pi}{3}$$

6.  $V = 2(2\pi) \int_{\sqrt{r^2-(h^2/4)}}^r x \sqrt{r^2-x^2} dx$

$$= -2\pi \left[ \frac{2}{3} (r^2 - x^2)^{3/2} \right]_{\sqrt{r^2-(h^2/4)}}^r$$

$$= \frac{-4\pi}{3} \left[ -\frac{h^3}{8} \right] = \frac{\pi h^3}{6} \text{ which does not depend on } r$$



7. By the Theorem of Pappus,

$$V = 2\pi rA$$

$$= 2\pi \left[ d + \frac{1}{2}\sqrt{w^2 + l^2} \right] lw$$

8. (a) Tangent at A:  $y = x^3$ ,  $y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

To find point B:

$$x^3 = 3x - 2$$

$$x^3 - 3x + 2 = 0$$

$$(x-1)^2(x+2) = 0 \Rightarrow B = (-2, -8)$$

Tangent at B:  $y = x^3$ ,  $y' = 3x^2$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

To find point C:

$$x^3 = 12x + 16$$

$$x^3 - 12x - 16 = 0$$

$$(x+2)^2(x-4) = 0 \Rightarrow C = (4, 64)$$

$$\text{Area of } R = \int_{-2}^1 (x^3 - 3x + 2) dx = \frac{27}{4}$$

$$\text{Area of } S = \int_{-2}^4 (12x + 16 - x^3) dx = 108$$

$$\text{Area of } S = 16(\text{area of } R) \left[ \frac{\text{area } S}{\text{area } R} = 16 \right]$$

(b) Tangent at A( $a, a^3$ ):  $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

To find point B:  $x^3 - 3a^2x + 2a^3 = 0$

$$(x-a)^2(x+2a) = 0$$

$$\Rightarrow B = (-2a, -8a^3)$$

Tangent at B:  $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

To find point C:  $x^3 - 12a^2x - 16a^3 = 0$

$$(x+2a)^2(x-4a) = 0$$

$$\Rightarrow C = (4a, 64a^3)$$

$$\text{Area of } R = \int_{-2a}^a [x^3 - 3a^2x + 2a^3] dx = \frac{27}{4}a^4$$

$$\text{Area of } S = \int_{-2a}^{4a} [12a^2x + 16a^3 - x^3] dx = 108a^4$$

$$\text{Area of } S = 16(\text{area of } R)$$

9.  $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$

$$f(x) = 2e^{x/2} + C$$

$$f(0) = 0 \Rightarrow C = -2$$

$$f(x) = 2e^{x/2} - 2$$

$$10. s(x) = \int_a^x \sqrt{1 + f'(t)^2} dt$$

$$(a) s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$$

$$(b) ds = \sqrt{1 + f'(x)^2} dx$$

$$(ds)^2 = [1 + f'(x)^2](dx)^2$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 = (dx)^2 + (dy)^2$$

$$(c) s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2}\right)^2} dt = \int_1^x \sqrt{1 + \frac{9}{4}t} dt$$

$$(d) s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t} dt$$

$$= \left[ \frac{8}{27} \left(1 + \frac{9}{4}t\right)^{3/2} \right]_1^2$$

$$= \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

This is the length of the curve  $y = x^{3/2}$  from  $x = 1$  to  $x = 2$ .

12. (a)  $\bar{y} = 0$  by symmetry

$$M_y = \int_1^6 x \left( \frac{1}{x^3} - \left(-\frac{1}{x^3}\right) \right) dx = \int_1^6 \frac{2}{x^2} dx = \left[ -2\frac{1}{x} \right]_1^6 = \frac{5}{3}$$

$$m = 2 \int_1^6 \frac{1}{x^3} dx = \left[ -\frac{1}{x^2} \right]_1^6 = \frac{35}{36}$$

$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \quad (\bar{x}, \bar{y}) = \left( \frac{12}{7}, 0 \right)$$

$$(b) m = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$M_y = 2 \int_1^b \frac{1}{x^2} dx = \frac{2(b-1)}{b}$$

$$\bar{x} = \frac{2(b-1)/b}{(b^2-1)/b^2} = \frac{2b}{b+1} \quad (\bar{x}, \bar{y}) = \left( \frac{2b}{b+1}, 0 \right)$$

$$(c) \lim_{b \rightarrow \infty} \bar{x} = \lim_{b \rightarrow \infty} \frac{2b}{b+1} = 2 \quad (\bar{x}, \bar{y}) = (2, 0)$$

11. Let  $\rho_f$  be the density of the fluid and  $\rho_0$  the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^0 A(y) dy$$

where  $A(y)$  is a typical cross section and  $g$  is the acceleration due to gravity. The weight of the object is

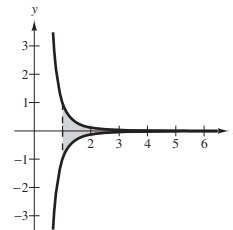
$$W = \rho_0 g \int_{-h}^{L-h} A(y) dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^0 A(y) dy = \rho_0 g \int_{-h}^{L-h} A(y) dy$$

$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}}$$

$$= \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

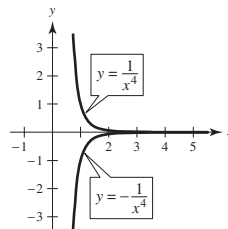


13. (a)  $\bar{y} = 0$  by symmetry

$$M_y = 2 \int_1^6 x \frac{1}{x^4} dx = 2 \int_1^6 \frac{1}{x^3} dx = \frac{35}{36}$$

$$m = 2 \int_1^6 \frac{1}{x^4} dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \quad (\bar{x}, \bar{y}) = \left( \frac{63}{43}, 0 \right)$$



$$(b) \quad M_y = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$m = 2 \int_1^b \frac{1}{x^4} dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \quad (\bar{x}, \bar{y}) = \left( \frac{3b(b + 1)}{2(b^2 + b + 1)}, 0 \right)$$

$$(c) \quad \lim_{b \rightarrow \infty} \bar{x} = \frac{3b(b + 1)}{2(b^2 + b + 1)} = \frac{3}{2} \quad (\bar{x}, \bar{y}) = \left( \frac{3}{2}, 0 \right)$$

14. (a)  $W = \text{area} = 2 + 4 + 6 = 12$

(b)  $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

15. Point of equilibrium:  $50 - 0.5x = 0.125x$

$$x = 80, p = 10$$

$$(P_0, x_0) = (10, 80)$$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10] dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} (10 - 0.125x) dx = 400$$

16. Point of equilibrium:  $1000 - 0.4x^2 = 42x$

$$x = 20, p = 840$$

$$(P_0, x_0) = (840, 20)$$

$$\begin{aligned} \text{Consumer surplus} &= \int_0^{20} [(1000 - 0.4x^2) - 840] dx \\ &= 2133.33 \end{aligned}$$

$$\text{Producer surplus} = \int_0^{20} (840 - 42x) dx = 8400$$

18. (a) Answers will vary.

$$f_1(x) = 6(x - x^2)$$

$$f_2(x) = \frac{\pi}{2} \sin(\pi x)$$

(b)  $f_1$  arc length  $\approx 3.2490$

$$f_2$$
 arc length  $\approx 3.3655$

(c) See the article by Professor Larson Riddle at <http://ecademy.agnesscott.edu/~Iriddle/arc/contest.htm>

One such function is

$$f_3(x) = \frac{8}{\pi} \sqrt{x - x^2} \quad (\text{arc length} \approx 2.9195)$$

17. Use Exercise 25, Section 7.7, which gives  $F = wkhb$  for a rectangle plate.

Wall at shallow end

$$\text{From Exercise 25: } F = 62.4(2)(4)(20) = 9984 \text{ lb}$$

Wall at deep end

$$\text{From Exercise 25: } F = 62.4(4)(8)(20) = 39,936 \text{ lb}$$

Side wall

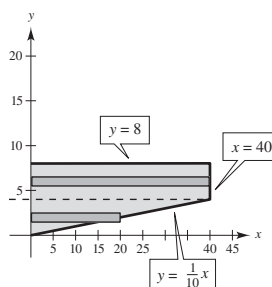
$$\text{From Exercise 25: } F_1 = 62.4(2)(4)(40) = 19,968 \text{ lb}$$

$$F_2 = 62.4 \int_0^4 (8 - y)(10y) dy$$

$$= 624 \int_0^4 (8y - y^2) dy = 624 \left[ 4y^2 - \frac{y^3}{3} \right]_0^4$$

$$= 26,624 \text{ lb}$$

$$\text{Total force: } F_1 + F_2 = 46,592 \text{ lb}$$



**NOT FOR SALE**

**C H A P T E R 8**

**Integration Techniques, L'Hôpital's Rule,  
and Improper Integrals**

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**INSTRUCTOR USE ONLY**

## CHAPTER 8

### Integration Techniques, L'Hôpital's Rule, and Improper Integrals

#### Section 8.1 Basic Integration Rules

$$1. (a) \frac{d}{dx} [2\sqrt{x^2 + 1} + C] = 2\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2 + 1}}$$

$$(b) \frac{d}{dx} [\sqrt{x^2 + 1} + C] = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$(c) \frac{d}{dx} \left[ \frac{1}{2}\sqrt{x^2 + 1} + C \right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2 + 1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2 + 1}}$$

$$(d) \frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$$

$$\int \frac{x}{\sqrt{x^2 + 1}} dx \text{ matches (b).}$$

$$2. (a) \frac{d}{dx} [\ln \sqrt{x^2 + 1} + C] = \frac{1}{2} \left( \frac{2x}{x^2 + 1} \right) = \frac{x}{x^2 + 1}$$

$$(b) \frac{d}{dx} \left[ \frac{2x}{(x^2 + 1)^2} + C \right] = \frac{(x^2 + 1)^2(2) - (2x)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$(c) \frac{d}{dx} [\arctan x + C] = \frac{1}{1 + x^2}$$

$$(d) \frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$$

$$\int \frac{x}{x^2 + 1} dx \text{ matches (a).}$$

$$3. (a) \frac{d}{dx} [\ln \sqrt{x^2 + 1} + C] = \frac{1}{2} \left( \frac{2x}{x^2 + 1} \right) = \frac{x}{x^2 + 1}$$

$$(b) \frac{d}{dx} \left[ \frac{2x}{(x^2 + 1)^2} + C \right] = \frac{(x^2 + 1)^2(2) - (2x)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$(c) \frac{d}{dx} [\arctan x + C] = \frac{1}{1 + x^2}$$

$$(d) \frac{d}{dx} [\ln(x^2 + 1) + C] = \frac{2x}{x^2 + 1}$$

$$\int \frac{1}{x^2 + 1} dx \text{ matches (c).}$$

$$4. (a) \frac{d}{dx} [2x \sin(x^2 + 1) + C] = 2x[\cos(x^2 + 1)(2x)] + 2 \sin(x^2 + 1) = 2[2x^2 \cos(x^2 + 1) + \sin(x^2 + 1)]$$

$$(b) \frac{d}{dx} \left[ -\frac{1}{2} \sin(x^2 + 1) + C \right] = -\frac{1}{2} \cos(x^2 + 1)(2x) = -x \cos(x^2 + 1)$$

$$(c) \frac{d}{dx} \left[ \frac{1}{2} \sin(x^2 + 1) + C \right] = \frac{1}{2} \cos(x^2 + 1)(2x) = x \cos(x^2 + 1)$$

$$(d) \frac{d}{dx} [-2x \sin(x^2 + 1) + C] = -2x[\cos(x^2 + 1)(2x)] - 2 \sin(x^2 + 1) = -2[2x^2 \cos(x^2 + 1) + \sin(x^2 + 1)]$$

$$\int x \cos(x^2 + 1) dx \text{ matches (c).}$$

5.  $\int (5x - 3)^4 dx$

$u = 5x - 3, du = 5 dx, n = 4$

Use  $\int u^n du$ .

6.  $\int \frac{2t + 1}{t^2 + t - 4} dt$

$u = t^2 + t - 4, du = (2t + 1) dt$

Use  $\int \frac{du}{u}$ .

7.  $\int \frac{1}{\sqrt{x}(1 - 2\sqrt{x})} dx$

$u = 1 - 2\sqrt{x}, du = -\frac{1}{\sqrt{x}} dx$

Use  $\int \frac{du}{u}$ .

8.  $\int \frac{2}{(2t - 1)^2 + 4} dt$

$u = 2t - 1, du = 2 dt, a = 2$

Use  $\int \frac{du}{u^2 + a^2}$ .

9.  $\int \frac{3}{\sqrt{1 - t^2}} dt$

$u = t, du = dt, a = 1$

Use  $\int \frac{du}{\sqrt{a^2 - u^2}}$ .

10.  $\int \frac{-2x}{\sqrt{x^2 - 4}} dx$

$u = x^2 - 4, du = 2x dx, n = -\frac{1}{2}$

Use  $\int u^n du$ .

11.  $\int t \sin t^2 dt$

$u = t^2, du = 2t dt$

Use  $\int \sin u du$ .

12.  $\int \sec 5x \tan 5x dx$

$u = 5x, du = 5 dx$

Use  $\int \sec u \tan u du$ .

13.  $\int (\cos x) e^{\sin x} dx$

$u = \sin x, du = \cos x dx$

Use  $\int e^u du$ .

14.  $\int \frac{1}{x\sqrt{x^2 - 4}} dx$

$u = x, du = dx, a = 2$

Use  $\int \frac{du}{u\sqrt{u^2 - a^2}}$ .

15. Let  $u = x - 5, du = dx$ .

$\int 14(x - 5)^6 dx = 14 \int (x - 5)^6 dx = 2(x - 5)^7 + C$

16. Let  $u = t - 8, du = dt$ .

$\int \frac{9}{(t - 8)^2} dt = 9 \int (t - 8)^{-2} dt = -\frac{9}{t - 8} + C$

17. Let  $u = z - 10, du = dz$ .

$\int \frac{7}{(z - 10)^7} dz = 7 \int (z - 10)^{-7} dz = -\frac{7}{6(z - 10)^6} + C$

18. Let  $u = t^3 - 1, du = 3t^2 dt$ .

$$\begin{aligned} \int t^2 \sqrt[3]{t^3 - 1} dt &= \frac{1}{3} \int (t^3 - 1)^{1/3} (3t^2) dt \\ &= \frac{1}{3} \frac{(t^3 - 1)^{4/3}}{4/3} + C \\ &= \frac{(t^3 - 1)^{4/3}}{4} + C \end{aligned}$$

19.  $\int \left[ v + \frac{1}{(3v - 1)^3} \right] dv = \int v dv + \frac{1}{3} \int (3v - 1)^{-3} (3) dv$

$$= \frac{1}{2} v^2 - \frac{1}{6(3v - 1)^2} + C$$

20.  $\int \left[ x - \frac{5}{(3x + 5)^2} \right] dx = \int x dx - 5 \int (3x + 5)^{-2} dx$

$$= \int x dx - \frac{5}{3} \int (3x + 5)^{-2} (3) dx$$

$$= \frac{x^2}{2} + \frac{5}{3(3x + 5)} + C$$

21. Let  $u = -t^3 + 9t + 1$ ,

$du = (-3t^2 + 9) dt = -3(t^2 - 3) dt$ .

$$\begin{aligned} \int \frac{t^2 - 3}{-t^3 + 9t + 1} dt &= -\frac{1}{3} \int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1} dt \\ &= -\frac{1}{3} \ln |-t^3 + 9t + 1| + C \end{aligned}$$

22. Let  $u = x^2 + 2x - 4$ ,  $du = 2(x + 1) dx$ .

$$\int \frac{x+1}{\sqrt{x^2+2x-4}} dx = \frac{1}{2} \int (x^2+2x-4)^{-1/2} (2)(x+1) dx$$

$$= \sqrt{x^2+2x-4} + C$$

23.  $\int \frac{x^2}{x-1} dx = \int (x+1) dx + \int \frac{1}{x-1} dx$

$$= \frac{1}{2}x^2 + x + \ln|x-1| + C$$

24.  $\int \frac{4x}{x-8} dx = \int \left(4 + \frac{32}{x-8}\right) dx$

$$= 4x + 32 \ln|x-8| + C$$

25. Let  $u = 1 + e^x$ ,  $du = e^x dx$ .

$$\int \frac{e^x}{1+e^x} dx = \ln(1+e^x) + C$$

26.  $\int \left[ \frac{1}{7x-2} - \frac{1}{7x+2} \right] dx$

$$= \frac{1}{7} \int \frac{1}{7x-2} (7) dx - \frac{1}{7} \int \frac{1}{7x+2} (7) dx$$

$$= \frac{1}{7} \ln|7x-2| - \frac{1}{7} \ln|7x+2| + C$$

$$= \frac{1}{7} \ln \left| \frac{7x-2}{7x+2} \right| + C$$

27.  $\int (5 + 4x^2)^2 dx = \int (25 + 40x^2 + 16x^4) dx$

$$= 25x + \frac{40}{3}x^3 + \frac{16}{5}x^5 + C$$

$$= \frac{x}{15} (48x^5 + 200x^3 + 375) + C$$

28.  $\int x \left(1 + \frac{1}{x}\right)^3 dx = \int x \left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3}\right) dx$

$$= \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{2}x^2 + 3x + 3 \ln|x| - \frac{1}{x} + C$$

29. Let  $u = 2\pi x^2$ ,  $du = 4\pi x dx$ .

$$\int x(\cos 2\pi x^2) dx = \frac{1}{4\pi} \int (\cos 2\pi x^2)(4\pi x) dx$$

$$= \frac{1}{4\pi} \sin 2\pi x^2 + C$$

30.  $\int \sec 4x dx = \frac{1}{4} \int \sec(4x)(4) dx$

$$= \frac{1}{4} \ln|\sec 4x + \tan 4x| + C$$

31. Let  $u = \pi x$ ,  $du = \pi dx$ .

$$\int \csc \pi x \cot \pi x dx = \frac{1}{\pi} \int (\csc \pi x)(\cot \pi x) \pi dx$$

$$= -\frac{1}{\pi} \csc \pi x + C$$

32. Let  $u = \cos x$ ,  $du = -\sin x dx$ .

$$\int \frac{\sin x}{\sqrt{\cos x}} dx = -\int (\cos x)^{-1/2} (-\sin x) dx$$

$$= -2\sqrt{\cos x} + C$$

33.  $\int e^{11x} dx = \frac{1}{11} \int e^{11x} (11) dx = \frac{1}{11} e^{11x} + C$

34. Let  $u = \cot x$ ,  $du = -\csc^2 x dx$ .

$$\int \csc^2 x e^{\cot x} dx = -\int e^{\cot x} (-\csc^2 x) dx = -e^{\cot x} + C$$

35. Let  $u = 1 + e^x$ ,  $du = e^x dx$ .

$$\int \frac{2}{e^{-x} + 1} dx = 2 \int \left( \frac{2}{e^{-x} + 1} \right) \left( \frac{e^x}{e^x} \right) dx$$

$$= 2 \int \frac{e^x}{1 + e^x} dx = 2 \ln(1 + e^x) + C$$

36.  $\int \frac{5}{3e^x - 2} dx = 5 \int \left( \frac{1}{3e^x - 2} \right) \left( \frac{e^{-x}}{e^{-x}} \right) dx$

$$= 5 \int \frac{e^{-x}}{3 - 2e^{-x}} dx$$

$$= \frac{5}{2} \int \frac{1}{3 - 2e^{-x}} (2e^{-x}) dx$$

$$= \frac{5}{2} \ln|3 - 2e^{-x}| + C$$

37.  $\int \frac{\ln x^2}{x} dx = 2 \int (\ln x) \frac{1}{x} dx$

$$= 2 \frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$$

38. Let  $u = \ln(\cos x)$ ,  $du = \frac{-\sin x}{\cos x} dx = -\tan x dx$ .

$$\int (\tan x)(\ln \cos x) dx = -\int (\ln \cos x)(-\tan x) dx$$

$$= \frac{-[\ln(\cos x)]^2}{2} + C$$

$$\begin{aligned}
 39. \int \frac{1 + \sin x}{\cos x} dx &= \int \frac{1 + \sin x}{\cos x} \cdot \frac{1 - \sin x}{1 - \sin x} dx \\
 &= \int \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} dx \\
 &= \int \frac{\cos^2 x}{\cos x(1 - \sin x)} dx \\
 &= -\int \frac{-\cos x}{1 - \sin x} dx \\
 &= -\ln(1 - \sin x) + C, \quad (u = 1 - \sin x)
 \end{aligned}$$

**Alternate Solution:**

$$\begin{aligned}
 \int \frac{1 + \sin x}{\cos x} dx &= \int (\sec x + \tan x) dx \\
 &= \ln|\sec x + \tan x| + \ln|\sec x| + C \\
 &= \ln|\sec x(\sec x + \tan x)| + C
 \end{aligned}$$

$$\begin{aligned}
 40. \int \frac{1 + \cos \alpha}{\sin \alpha} d\alpha &= \int \csc \alpha d\alpha + \int \cot \alpha d\alpha \\
 &= -\ln|\csc \alpha + \cot \alpha| + \ln|\sin \alpha| + C
 \end{aligned}$$

$$\begin{aligned}
 41. \frac{1}{\cos \theta - 1} &= \frac{1}{\cos \theta - 1} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1} \\
 &= \frac{\cos \theta + 1}{-\sin^2 \theta} = -\csc \theta \cdot \cot \theta - \csc^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{1}{\cos \theta - 1} d\theta &= \int (-\csc \theta \cot \theta - \csc^2 \theta) d\theta \\
 &= \csc \theta + \cot \theta + C \\
 &= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C \\
 &= \frac{1 + \cos \theta}{\sin \theta} + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{2}{3(\sec x - 1)} dx &= \frac{2}{3} \int \frac{1}{\sec x - 1} \cdot \left( \frac{\sec x + 1}{\sec x + 1} \right) dx \\
 &= \frac{2}{3} \int \frac{\sec x + 1}{\tan^2 x} dx \\
 &= \frac{2}{3} \int \frac{\sec x}{\tan^2 x} dx + \frac{2}{3} \int \cot^2 x dx \\
 &= \frac{2}{3} \int \frac{\cos x}{\sin^2 x} dx + \frac{2}{3} \int (\csc^2 x - 1) dx \\
 &= \frac{2}{3} \left( -\frac{1}{\sin x} \right) - \frac{2}{3} \cot x - \frac{2}{3} x + C \\
 &= -\frac{2}{3} (\csc x + \cot x + x) + C
 \end{aligned}$$

$$48. \int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}} dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}} dx = \operatorname{arcsec}|2(x-1)| + C$$

$$43. \text{ Let } u = 4t + 1, du = 4 dt.$$

$$\begin{aligned}
 \int \frac{-1}{\sqrt{1 - (4t + 1)^2}} dt &= -\frac{1}{4} \int \frac{4}{\sqrt{1 - (4t + 1)^2}} dt \\
 &= -\frac{1}{4} \arcsin(4t + 1) + C
 \end{aligned}$$

$$44. \text{ Let } u = \sqrt{5} x, du = \sqrt{5} dx, a = 3.$$

$$\begin{aligned}
 \int \frac{1}{9 + 5x^2} dx &= \frac{1}{\sqrt{5}} \int \frac{\sqrt{5} dx}{3^2 + (\sqrt{5}x)^2} \\
 &= \frac{1}{\sqrt{5}} \frac{1}{3} \arctan\left(\frac{\sqrt{5}x}{3}\right) + C \\
 &= \frac{\sqrt{5}}{15} \arctan\left(\frac{\sqrt{5}x}{3}\right) + C
 \end{aligned}$$

$$45. \text{ Let } u = \cos\left(\frac{2}{t}\right), du = \frac{2 \sin(2/t)}{t^2} dt.$$

$$\begin{aligned}
 \int \frac{\tan(2/t)}{t^2} dt &= \frac{1}{2} \int \frac{1}{\cos(2/t)} \left[ \frac{2 \sin(2/t)}{t^2} \right] dt \\
 &= \frac{1}{2} \ln \left| \cos\left(\frac{2}{t}\right) \right| + C
 \end{aligned}$$

$$46. \text{ Let } u = \frac{1}{t}, du = \frac{-1}{t^2} dt.$$

$$\int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left( \frac{-1}{t^2} \right) dt = -e^{1/t} + C$$

$$\begin{aligned}
 47. \text{ Note: } 10x - x^2 &= 25 - (25 - 10x + x^2) \\
 &= 25 - (5 - x)^2
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{6}{\sqrt{10x - x^2}} dx &= 6 \int \frac{1}{\sqrt{25 - (5 - x)^2}} dx \\
 &= 6 \int \frac{-1}{\sqrt{5^2 - (5 - x)^2}} dx \\
 &= -6 \arcsin \frac{(5 - x)}{5} + C \\
 &= 6 \arcsin \left( \frac{x - 5}{5} \right) + C
 \end{aligned}$$



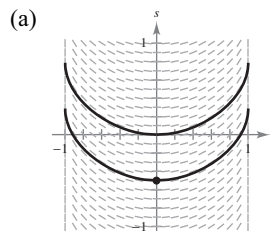
$$\begin{aligned}
 49. \int \frac{4}{4x^2 + 4x + 65} dx &= \int \frac{1}{[x + (1/2)]^2 + 16} dx \\
 &= \frac{1}{4} \arctan \left[ \frac{x + (1/2)}{4} \right] + C \\
 &= \frac{1}{4} \arctan \left( \frac{2x + 1}{8} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 50. \int \frac{1}{x^2 - 4x + 9} dx &= \int \frac{1}{x^2 - 4x + 4 + 5} dx \\
 &= \int \frac{1}{(x - 2)^2 + (\sqrt{5})^2} dx \\
 &= \frac{1}{\sqrt{5}} \arctan \left( \frac{x - 2}{\sqrt{5}} \right) + C \\
 &= \frac{\sqrt{5}}{5} \arctan \left( \frac{\sqrt{5}}{5} (x - 2) \right) + C
 \end{aligned}$$

$$\begin{aligned}
 51. \int \frac{1}{\sqrt{1 - 4x - x^2}} dx &= \int \frac{1}{\sqrt{5 - (x^2 + 4x + 4)}} dx \\
 &= \int \frac{1}{\sqrt{5 - (x + 2)^2}} dx \\
 &= \arcsin \left( \frac{x + 2}{\sqrt{5}} \right) + C, \quad (a = \sqrt{5})
 \end{aligned}$$

$$\begin{aligned}
 52. \int \frac{12}{\sqrt{3 - 8x - x^2}} dx &= \int \frac{12}{19 - (16 + 8x + x^2)} dx \\
 &= 12 \int \frac{1}{19 - (x + 4)^2} dx \\
 &= 12 \arcsin \left( \frac{x + 4}{\sqrt{19}} \right) + C
 \end{aligned}$$

$$53. \frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}, \quad \left(0, -\frac{1}{2}\right)$$

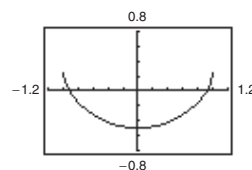


(b)  $u = t^2, du = 2t dt$

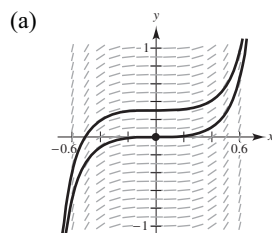
$$\begin{aligned}
 \int \frac{t}{\sqrt{1 - t^4}} dt &= \frac{1}{2} \int \frac{2t}{\sqrt{1 - (t^2)^2}} dt \\
 &= \frac{1}{2} \arcsin t^2 + C
 \end{aligned}$$

$$\left(0, -\frac{1}{2}\right): -\frac{1}{2} = \frac{1}{2} \arcsin 0 + C \Rightarrow C = -\frac{1}{2}$$

$$s = \frac{1}{2} \arcsin t^2 - \frac{1}{2}$$



$$54. \frac{dy}{dx} = \tan^2(2x), \quad (0, 0)$$

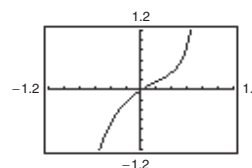


(b)  $\int \tan^2(2x) dx = \int (\sec^2(2x) - 1) dx$

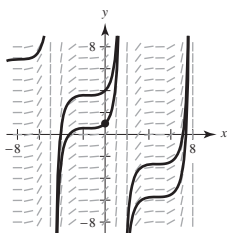
$$= \frac{1}{2} \tan(2x) - x + C$$

$$(0, 0): 0 = C$$

$$y = \frac{1}{2} \tan(2x) - x$$



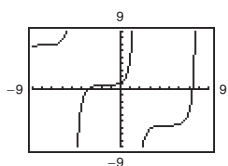
55. (a)  $\frac{dy}{dx} = (\sec x + \tan x), \quad (0, 1)$



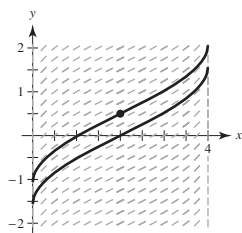
(b)  $y = \int (\sec x + \tan x)^2 dx$   
 $= \int (\sec^2 x + 2 \sec x \tan x + \tan^2 x) dx$   
 $= \int (\sec^2 x + 2 \sec x \tan x + (\sec^2 x - 1)) dx$   
 $= \int (2 \sec^2 x + 2 \sec x \tan x - 1) dx$   
 $= 2 \tan x + 2 \sec x - x + C$

$(0, 1): 1 = 0 + 2 - 0 + C \Rightarrow C = -1$

$y = 2 \tan x + 2 \sec x - x - 1$



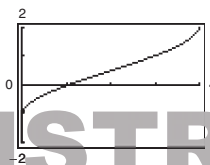
56. (a)  $\frac{dy}{dx} = \frac{1}{\sqrt{4x - x^2}}, \quad \left(2, \frac{1}{2}\right)$



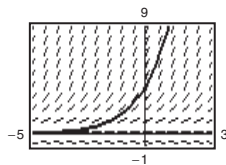
(b)  $y = \int \frac{1}{\sqrt{4x - x^2}} dx$   
 $= \int \frac{1}{\sqrt{4 - (x^2 - 4x + 4)}} dx$   
 $= \int \frac{1}{\sqrt{4 - (x - 2)^2}} dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

$\left(2, \frac{1}{2}\right): \frac{1}{2} = \arcsin(0) + C \Rightarrow C = \frac{1}{2}$

$y = \arcsin\left(\frac{x - 2}{2}\right) + \frac{1}{2}$

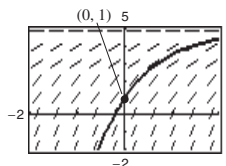


57.



$y = 4e^{0.8x}$

58.



$y = 5 - 4e^{-x}$

59.  $\frac{dy}{dx} = (e^x + 5)^2 = e^{2x} + 10e^x + 25$

$y = \int (e^{2x} + 10e^x + 25) dx$

$= \frac{1}{2}e^{2x} + 10e^x + 25x + C$

60.  $\frac{dy}{dx} = (3 - e^x)^2 = 9 - 6e^x + e^{2x}$

$y = \int (9 - 6e^x + e^{2x}) dx$

$= 9x - 6e^x + \frac{1}{2}e^{2x} + C$

61.  $\frac{dr}{dt} = \frac{10e^t}{\sqrt{1 - e^{2t}}}$

$r = \int \frac{10e^t}{1 - (e^t)^2} dt$

$= 10 \arcsin(e^t) + C$

62.  $\frac{dr}{dt} = \frac{(1 + e^t)^2}{e^t}$

$r = \int \frac{(1 + e^t)^2}{e^t} dt = \int \frac{1 + 2e^t + e^{2t}}{e^t} dt$

$= \int (e^{-t} + 2 + e^t) dt = -e^{-t} + 2t + e^t + C$

63.  $\frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$

Let  $u = \tan x, du = \sec^2 x dx$ .

$y = \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + C$

INSTRUCTOR USE ONLY

$$64. \frac{dy}{dx} = \frac{1}{x\sqrt{4x^2 - 1}}$$

$$\text{Let } u = 2x, du = 2 dx.$$

$$\begin{aligned} y &= \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx \\ &= \operatorname{arcsec}|2x| + C \end{aligned}$$

$$65. \text{ Let } u = 2x, du = 2 dx.$$

$$\begin{aligned} \int_0^{\pi/4} \cos 2x dx &= \frac{1}{2} \int_0^{\pi/4} \cos 2x(2) dx \\ &= \left[ \frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2} \end{aligned}$$

$$66. \text{ Let } u = \sin t, du = \cos t dt.$$

$$\int_0^{\pi} \sin^2 t \cos t dt = \left[ \frac{1}{3} \sin^3 t \right]_0^{\pi} = 0$$

$$67. \text{ Let } u = -x^2, du = -2x dx.$$

$$\begin{aligned} \int_0^1 x e^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx = \left[ -\frac{1}{2} e^{-x^2} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-1}) \approx 0.316 \end{aligned}$$

$$68. \text{ Let } u = 1 - \ln x, du = \frac{-1}{x} dx.$$

$$\begin{aligned} \int_1^e \frac{1 - \ln x}{x} dx &= -\int_1^e (1 - \ln x) \left( \frac{-1}{x} \right) dx \\ &= \left[ -\frac{1}{2} (1 - \ln x)^2 \right]_1^e = \frac{1}{2} \end{aligned}$$

$$69. \text{ Let } u = x^2 + 36, du = 2x dx.$$

$$\begin{aligned} \int_0^8 \frac{2x}{\sqrt{x^2 + 36}} dx &= \int_0^8 (x^2 + 36)^{-1/2} (2x) dx \\ &= 2 \left[ (x^2 + 36)^{1/2} \right]_0^8 = 8 \end{aligned}$$

$$\begin{aligned} 70. \int_1^2 \frac{x-2}{x} dx &= \int_1^2 \left( 1 - \frac{2}{x} \right) dx \\ &= [x - 2 \ln x]_1^2 = 1 - \ln 4 \approx -0.386 \end{aligned}$$

$$71. \text{ Let } u = 3x, du = 3 dx.$$

$$\begin{aligned} \int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx &= \frac{1}{3} \int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2} dx \\ &= \left[ \frac{1}{6} \arctan \left( \frac{3x}{2} \right) \right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175 \end{aligned}$$

$$72. \int_0^7 \frac{1}{\sqrt{100 - x^2}} dx = \left[ \arcsin \left( \frac{x}{10} \right) \right]_0^7 = \arcsin(7/10)$$

$$\begin{aligned} 73. A &= \int_0^{3/2} (-4x + 6)^{3/2} dx \\ &= -\frac{1}{4} \int_0^{3/2} (6 - 4x)^{3/2} (-4) dx \\ &= -\frac{1}{4} \left[ \frac{2}{5} (6 - 4x)^{5/2} \right]_0^{3/2} \\ &= -\frac{1}{10} (0 - 6^{5/2}) \\ &= \frac{18}{5} \sqrt{6} \approx 8.8182 \end{aligned}$$

$$\begin{aligned} 74. A &= \int_0^2 x \sqrt{8 - 2x^2} dx \\ &= -\frac{1}{4} \int_0^2 (8 - 2x^2)^{1/2} (-4x) dx \\ &= \left[ -\frac{1}{6} (8 - 2x^2)^{3/2} \right]_0^2 \\ &= 0 + \frac{1}{6} (8)^{3/2} \\ &= \frac{8\sqrt{2}}{3} \approx 3.7712 \end{aligned}$$

$$\begin{aligned} 75. A &= \int_0^5 \frac{3x + 2}{x^2 + 9} dx \\ &= \int_0^5 \frac{3x}{x^2 + 9} dx + \int_0^5 \frac{2}{x^2 + 9} dx \\ &= \left[ \frac{3}{2} \ln |x^2 + 9| + \frac{2}{3} \arctan \left( \frac{x}{3} \right) \right]_0^5 \\ &= \frac{3}{2} \ln(34) + \frac{2}{3} \arctan \left( \frac{5}{3} \right) - \frac{3}{2} \ln 9 \\ &= \frac{3}{2} \ln \left( \frac{34}{9} \right) + \frac{2}{3} \arctan \left( \frac{5}{3} \right) \\ &\approx 2.6806 \end{aligned}$$

$$\begin{aligned} 76. A &= \int_{-4}^4 \frac{5}{x^2 + 1} dx \\ &= 2(5) \int_0^4 \frac{1}{x^2 + 1} dx \\ &= 10 [\arctan x]_0^4 \\ &= 10 \arctan 4 \approx 13.2582 \end{aligned}$$

$$77. y^2 = x^2(1 - x^2)$$

$$y = \pm \sqrt{x^2(1 - x^2)}$$

$$A = 4 \int_0^1 x \sqrt{1 - x^2} \, dx$$

$$= -2 \int_0^1 (1 - x^2)^{1/2} (-2x) \, dx$$

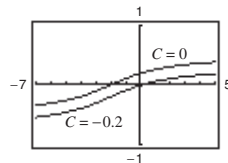
$$= -\frac{4}{3} \left[ (1 - x^2)^{3/2} \right]_0^1$$

$$= -\frac{4}{3} (0 - 1) = \frac{4}{3}$$

$$78. A = \int_0^{\pi/2} \sin 2x \, dx = -\frac{1}{2} [\cos 2x]_0^{\pi/2} = -\frac{1}{2} (-1 - 1) = 1$$

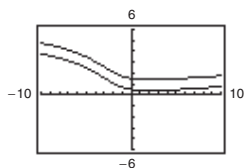
$$79. \int \frac{1}{x^2 + 4x + 13} \, dx = \frac{1}{3} \arctan\left(\frac{x + 2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



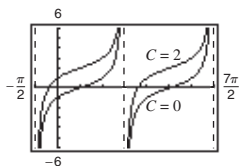
$$80. \int \frac{x - 2}{x^2 + 4x + 13} \, dx = \frac{1}{2} \ln(x^2 + 4x + 13) - \frac{4}{3} \arctan\left(\frac{x + 2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



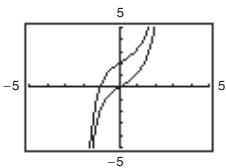
$$81. \int \frac{1}{1 + \sin \theta} \, d\theta = \tan \theta - \sec \theta + C \quad \left( \text{or } \frac{-2}{1 + \tan(\theta/2)} \right)$$

The antiderivatives are vertical translations of each other.



$$82. \int \left( \frac{e^x + e^{-x}}{2} \right)^3 \, dx = \frac{1}{24} [e^{3x} + 9e^x - 9e^{-x} - e^{-3x}] + C$$

The antiderivatives are vertical translations of each other.



$$83. \text{Power Rule: } \int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$u = x^2 + 1, n = 3$$

$$84. \int \sec u \tan u \, du = \sec u + C$$

$$85. \text{Log Rule: } \int \frac{du}{u} = \ln|u| + C, \quad u = x^2 + 1$$

$$86. \text{Arctan Rule: } \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

87.  $\sin x + \cos x = a \sin(x + b)$

$$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$$

$$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$$

Equate coefficients of like terms to obtain the following.

$$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$$

So,  $a = 1/\cos b$ . Now, substitute for  $a$  in  $1 = a \sin b$ .

$$1 = \left( \frac{1}{\cos b} \right) \sin b$$

$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

Because  $b = \frac{\pi}{4}$ ,  $a = \frac{1}{\cos(\pi/4)} = \sqrt{2}$ . So,

$$\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

$$\begin{aligned} \int \frac{dx}{\sin x + \cos x} &= \int \frac{dx}{\sqrt{2} \sin\left(x + \left(\pi/4\right)\right)} \\ &= \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx \\ &= -\frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) \right| + C \end{aligned}$$

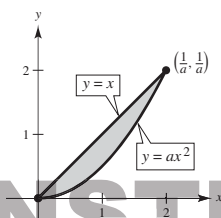
88. 
$$\begin{aligned} \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} &= \frac{\sin x(1 + \sin x) + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)} \\ &= \frac{\sin x + 1}{\cos x(1 + \sin x)} \\ &= \frac{1}{\cos x} = \sec x \end{aligned}$$

So,

$$\begin{aligned} \int \sec x \, dx &= \int \left[ \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x} \right] dx \\ &= -\ln|\cos x| + \ln|1 + \sin x| + C \\ &= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C \\ &= \ln|\sec x + \tan x| + C \end{aligned}$$

89.  $\int_0^{1/a} (x - ax^2) \, dx = \left[ \frac{1}{2}x^2 - \frac{a}{3}x^3 \right]_0^{1/a} = \frac{1}{6a^2}$

Let  $\frac{1}{6a^2} = \frac{2}{3}$ ,  $12a^2 = 3$ ,  $a = \frac{1}{2}$ .



90. (a) They are equivalent because

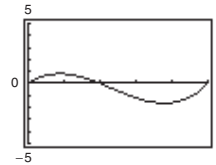
$$e^{x+C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}.$$

(b) They differ by a constant.

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$$

91.  $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$

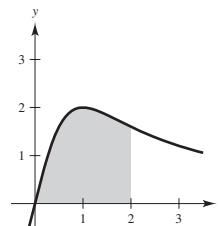
$\int_0^5 f(x) \, dx < 0$  because more area is below the  $x$ -axis than above.



92. No. When  $u = x^2$ , it does not follow that  $x = \sqrt{u}$  because  $x$  is negative on  $[-1, 0)$ .

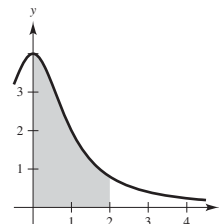
93.  $\int_0^2 \frac{4x}{x^2 + 1} \, dx \approx 3$

Matches (a).

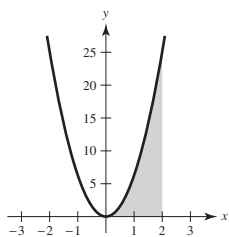


94.  $\int_0^2 \frac{4}{x^2 + 1} \, dx \approx 4$

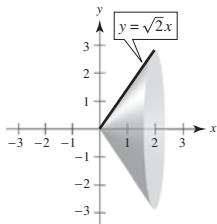
Matches (d).



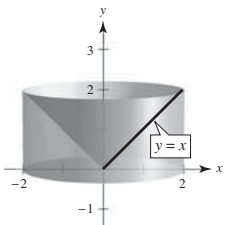
95. (a)  $y = 2\pi x^2, \quad 0 \leq x \leq 2$



(b)  $y = \sqrt{2}x, \quad 0 \leq x \leq 2$

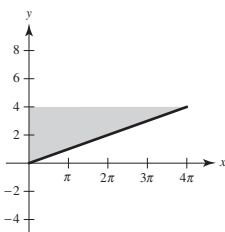


(c)  $y = x, \quad 0 \leq x \leq 2$



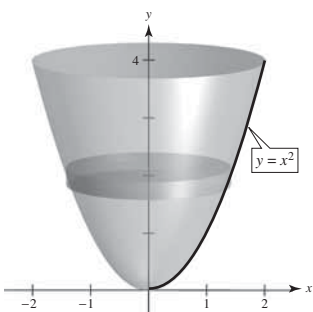
96. (a)  $x = \pi y, \quad 0 \leq y \leq 4$

$$y = \frac{1}{\pi}x, \quad 0 \leq x \leq 4\pi$$



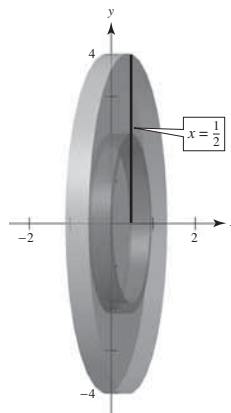
(b)  $x = \sqrt{y}, \quad 0 \leq y \leq 4$

$$y = x^2, \quad 0 \leq x \leq 2$$



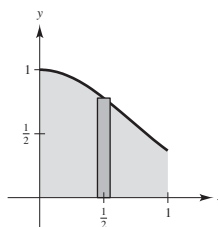
(c)  $x = \frac{1}{2}, \quad 0 \leq y \leq 4$

$$2\pi \int_0^4 y \left( \frac{1}{2} \right) dy$$

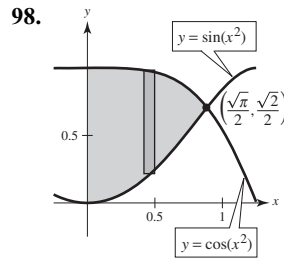

97. (a) **Shell Method:**

Let  $u = -x^2, du = -2x dx$ .

$$\begin{aligned} V &= 2\pi \int_0^1 x e^{-x^2} dx \\ &= -\pi \int_0^1 e^{-x^2} (-2x) dx \\ &= \left[ -\pi e^{-x^2} \right]_0^1 \\ &= \pi(1 - e^{-1}) \approx 1.986 \end{aligned}$$


(b) **Shell Method:**

$$\begin{aligned} V &= 2\pi \int_0^b x e^{-x^2} dx \\ &= \left[ -\pi e^{-x^2} \right]_0^b \\ &= \pi(1 - e^{-b^2}) = \frac{4}{3} \\ e^{-b^2} &= \frac{3\pi - 4}{3\pi} \\ b &= \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \approx 0.743 \end{aligned}$$



**Shell Method:**

$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{\pi/2}} x(\cos(x^2) - \sin(x^2)) dx \\ &= \pi \left[ \sin(x^2) + \cos(x^2) \right]_0^{\sqrt{\pi/2}} \\ &= \pi \left[ \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (0 + 1) \right] \\ &= \pi(\sqrt{2} - 1) \end{aligned}$$

99.  $y = f(x) = \ln(\sin x)$

$$\begin{aligned} f'(x) &= \frac{\cos x}{\sin x} \\ s &= \int_{\pi/4}^{\pi/2} \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} dx = \int_{\pi/4}^{\pi/2} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx \\ &= \int_{\pi/4}^{\pi/2} \frac{1}{\sin x} dx = \int_{\pi/4}^{\pi/2} \csc x dx \\ &= [-\ln|\csc x + \cot x|]_{\pi/4}^{\pi/2} \\ &= -\ln(1) + \ln(\sqrt{2} + 1) \\ &= \ln(\sqrt{2} + 1) \approx 0.8814 \end{aligned}$$

100.  $y = \ln(\cos x), \quad 0 \leq x \leq \pi/3$

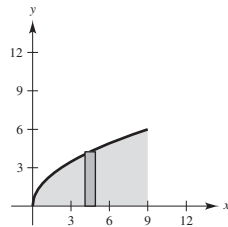
$$\begin{aligned} y' &= \frac{-\sin x}{\cos x} = -\tan x \\ 1 + (y')^2 &= 1 + \tan^2 x = \sec^2 x \\ s &= \int_0^{\pi/3} \sqrt{1 + (y')^2} dx = \int_0^{\pi/3} \sec x dx \\ &= [\ln|\sec x + \tan x|]_0^{\pi/3} \\ &= \ln(2 + \sqrt{3}) - \ln(1) = \ln(2 + \sqrt{3}) \approx 1.317 \end{aligned}$$

101.  $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

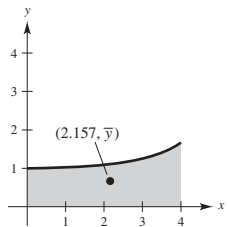
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$\begin{aligned} S &= 2\pi \int_0^9 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx \\ &= 2\pi \int_0^9 2\sqrt{x+1} dx \\ &= \left[ 4\pi \left( \frac{2}{3} \right) (x+1)^{3/2} \right]_0^9 = \frac{8\pi}{3} (10\sqrt{10} - 1) \approx 256.545 \end{aligned}$$



102.  $A = \int_0^4 \frac{5}{\sqrt{25-x^2}} dx = \left[ 5 \arcsin \frac{x}{5} \right]_0^4 = 5 \arcsin \frac{4}{5}$

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^4 x \left( \frac{5}{\sqrt{25-x^2}} \right) dx \\ &= \frac{1}{5 \arcsin(4/5)} \left( -\frac{5}{2} \right) \int_0^4 (25-x^2)^{-1/2} (-2x) dx \\ &= \frac{1}{5 \arcsin(4/5)} (-5) \left[ (25-x^2)^{1/2} \right]_0^4 \\ &= -\frac{1}{\arcsin(4/5)} [3 - 5] = \frac{2}{\arcsin(4/5)} \approx 2.157 \end{aligned}$$



103. Average value  $= \frac{1}{b-a} \int_a^b f(x) dx$

$$\begin{aligned} &= \frac{1}{3 - (-3)} \int_{-3}^3 \frac{1}{1+x^2} dx \\ &= \frac{1}{6} [\arctan(x)]_{-3}^3 \\ &= \frac{1}{6} [\arctan(3) - \arctan(-3)] \\ &= \frac{1}{3} \arctan(3) \approx 0.4163 \end{aligned}$$

$$\begin{aligned}
 104. \text{ Average value} &= \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{(\pi/n) - 0} \int_0^{\pi/n} \sin(nx) dx \\
 &= \frac{n}{\pi} \left[ -\frac{1}{n} \cos(nx) \right]_0^{\pi/n} \\
 &= -\frac{1}{\pi} [\cos(\pi) - \cos(0)] = \frac{2}{\pi}
 \end{aligned}$$

$$\begin{aligned}
 105. \quad y &= \tan(\pi x) \\
 y' &= \pi \sec^2(\pi x) \\
 1 + (y')^2 &= 1 + \pi^2 \sec^4(\pi x) \\
 s &= \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)} dx \approx 1.0320
 \end{aligned}$$

$$\begin{aligned}
 106. \quad y &= x^{2/3} \\
 y' &= \frac{2}{3x^{1/3}} \\
 1 + (y')^2 &= 1 + \frac{4}{9x^{2/3}} \\
 s &= \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}} dx \approx 7.6337
 \end{aligned}$$

$$107. (a) \int \cos^3 x dx = \int (1 - \sin^2 x) \cos x dx = \sin x - \frac{\sin^3 x}{3} + C = \frac{1}{3} \sin x (\cos^2 x + 2) + C$$

$$\begin{aligned}
 (b) \int \cos^5 x dx &= \int (1 - \sin^2 x)^2 \cos x dx = \int (1 - 2\sin^2 x + \sin^4 x) \cos x dx \\
 &= \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + C = \frac{1}{15} \sin x (3 \cos^4 x + 4 \cos^2 x + 8) + C
 \end{aligned}$$

$$\begin{aligned}
 (c) \int \cos^7 x dx &= \int (1 - \sin^2 x)^3 \cos x dx \\
 &= \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x dx \\
 &= \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C \\
 &= \frac{1}{35} \sin x (5 \cos^6 x + 6 \cos^4 x + 8 \cos^2 x + 16) + C
 \end{aligned}$$

$$(d) \int \cos^{15} x dx = \int (1 - \sin^2 x)^7 \cos x dx$$

You would expand  $(1 - \sin^2 x)^7$ .

$$\begin{aligned}
 108. (a) \int \tan^3 x dx &= \int (\sec^2 x - 1) \tan x dx \\
 &= \int \sec^2 x \tan x dx - \int \tan x dx \\
 &= \frac{\tan^2 x}{2} - \int \tan x dx
 \end{aligned}$$

$$\int \tan^3 x dx = \frac{\tan^2 x}{2} + \ln |\cos x| + C$$

$$\begin{aligned}
 (b) \int \tan^5 x dx &= \int (\sec^2 x - 1) \tan^3 x dx \\
 &= \frac{\tan^4 x}{4} - \int \tan^3 x dx
 \end{aligned}$$

$$\begin{aligned}
 (c) \int \tan^{2k+1} x dx &= \int (\sec^2 x - 1) \tan^{2k-1} x dx \\
 &= \frac{\tan^{2k} x}{2k} - \int \tan^{2k-1} x dx
 \end{aligned}$$

(d) You would use these formulas recursively.



109. Let  $f(x) = \frac{1}{2}(x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|) + C$ .

$$\begin{aligned} f'(x) &= \frac{1}{2} \left( x \frac{1}{2}(x^2 + 1)^{-1/2}(2x) + \sqrt{x^2 + 1} + \frac{1}{x + \sqrt{x^2 + 1}} \left( 1 + \frac{1}{2}(x^2 + 1)^{-1/2}(2x) \right) \right) \\ &= \frac{1}{2} \left( \frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1} + \frac{1}{x + \sqrt{x^2 + 1}} \left( 1 + \frac{x}{\sqrt{x^2 + 1}} \right) \right) \\ &= \frac{1}{2} \left( \frac{x^2 + (x^2 + 1)}{\sqrt{x^2 + 1}} + \frac{1}{x + \sqrt{x^2 + 1}} \left( \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) \right) \\ &= \frac{1}{2} \left( \frac{2x^2 + 1}{\sqrt{x^2 + 1}} + \frac{1}{\sqrt{x^2 + 1}} \right) = \frac{1}{2} \left( \frac{2(x^2 + 1)}{\sqrt{x^2 + 1}} \right) = \sqrt{x^2 + 1} \end{aligned}$$

So,  $\int \sqrt{x^2 + 1} \, dx = \frac{1}{2}(x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|) + C$ .

Let  $g(x) = \frac{1}{2}(x\sqrt{x^2 + 1} + \operatorname{arcsinh}(x))$ .

$$\begin{aligned} g'(x) &= \frac{1}{2} \left( x \frac{1}{2}(x^2 + 1)^{-1/2}(2x) + \sqrt{x^2 + 1} + \frac{1}{\sqrt{x^2 + 1}} \right) \\ &= \frac{1}{2} \left( \frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1} + \frac{1}{\sqrt{x^2 + 1}} \right) \\ &= \frac{1}{2} \left( \frac{x^2 + (x^2 + 1) + 1}{\sqrt{x^2 + 1}} \right) \\ &= \frac{1}{2} \left( \frac{2(x^2 + 1)}{\sqrt{x^2 + 1}} \right) = \sqrt{x^2 + 1} \end{aligned}$$

So,  $\int \sqrt{x^2 + 1} \, dx = \frac{1}{2}(x\sqrt{x^2 + 1} + \operatorname{arcsinh}(x)) + C$ .

110. Let  $I = \int_2^4 \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}} \, dx$ .

$I$  is defined and continuous on  $[2, 4]$ . Note the symmetry: as  $x$  goes from 2 to 4,  $9 - x$  goes from 7 to 5 and  $x + 3$  goes from 5 to 7. So, let  $y = 6 - x$ ,  $dy = -dx$ .

$$I = \int_2^4 \frac{\sqrt{\ln(3 + y)}}{\sqrt{\ln(3 + y)} + \sqrt{\ln(9 - y)}} (-dy) = \int_2^4 \frac{\sqrt{\ln(3 + y)}}{\sqrt{\ln(3 + y)} + \sqrt{\ln(9 - y)}} \, dy$$

Adding:

$$2I = \int_2^4 \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}} \, dx + \int_2^4 \frac{\sqrt{\ln(3 + x)}}{\sqrt{\ln(3 + x)} + \sqrt{\ln(9 - x)}} \, dx = \int_2^4 dx = 2 \Rightarrow I = 1$$

You can easily check this result numerically.

## Section 8.2 Integration by Parts

1.  $\int x e^{2x} dx$

$u = x, dv = e^{2x} dx$

2.  $\int x^2 e^{2x} dx$

$u = x^2, dv = e^{2x} dx$

3.  $\int (\ln x)^2 dx$

$u = (\ln x)^2, dv = dx$

4.  $\int \ln 4x dx$

$u = \ln 4x, dv = dx$

5.  $\int x \sec^2 x dx$

$u = x, dv = \sec^2 x dx$

6.  $\int x^2 \cos x dx$

$u = x^2, dv = \cos x dx$

7.  $dv = x^3 dx \Rightarrow v = \int x^3 dx = \frac{x^4}{4}$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$\int x^3 \ln x dx = uv - \int v du$

$= (\ln x) \frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} dx$

$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C$

$= \frac{1}{16} x^4 (4 \ln x - 1) + C$

8.  $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$u = 4x + 7 \Rightarrow du = 4 dx$

$\int (4x + 7) e^x dx = uv - \int v du$

$= (4x + 7) e^x - \int e^x 4 dx$

$= (4x + 7) e^x - 4e^x + C$

$= (4x + 3) e^x + C$

9.  $dv = \sin 3x dx \Rightarrow v = \int \sin 3x dx = -\frac{1}{3} \cos 3x$

$u = x \Rightarrow du = dx$

$\int x \sin 3x dx = uv - \int v du$

$= x \left( -\frac{1}{3} \cos 3x \right) - \int -\frac{1}{3} \cos 3x dx$

$= -\frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x + C$

10.  $dv = \cos 4x dx \Rightarrow v = \int \cos 4x dx = \frac{1}{4} \sin 4x$

$u = x \Rightarrow du = dx$

$\int x \cos 4x dx = uv - \int v du$

$= x \left( \frac{1}{4} \sin 4x \right) - \int \frac{1}{4} \sin 4x dx$

$= \frac{x}{4} \sin 4x + \frac{1}{16} \cos 4x + C$

11.  $dv = e^{-4x} dx \Rightarrow v = \int e^{-4x} dx = -\frac{1}{4} e^{-4x}$

$u = x \Rightarrow du = dx$

$\int x e^{-4x} dx = x \left( -\frac{1}{4} e^{-4x} \right) - \int -\frac{1}{4} e^{-4x} dx$

$= -\frac{x}{4} e^{-4x} - \frac{1}{16} e^{-4x} + C$

$= -\frac{1}{16 e^{4x}} (1 + 4x) + C$

12.  $dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$

$u = 4x \Rightarrow du = 4 dx$

$\int \frac{4x}{e^x} dx = \int 4x e^{-x} dx = 4x(-e^{-x}) - \int (-e^{-x}) 4 dx$

$= -4x e^{-x} - 4e^{-x} + C$

$= -4e^{-x}(x + 1) + C$

13. Use integration by parts three times.

$$(1) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^3 \Rightarrow du = 3x^2 dx$$

$$(2) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(3) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6 \int x e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x (x^3 - 3x^2 + 6x - 6) + C \end{aligned}$$

$$14. \quad \int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left( \frac{-1}{t^2} \right) dt = -e^{1/t} + C$$

$$15. \quad \int x^2 e^{x^3} dx = \frac{1}{3} \int e^{x^3} (3x^2) dx = \frac{1}{3} e^{x^3} + C$$

$$16. \quad dv = x^4 dx \Rightarrow v = \int x^4 dx = \frac{x^5}{5}$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$\begin{aligned} \int x^4 \ln x dx &= \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \left( \frac{1}{x} \right) dx \\ &= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 dx \\ &= \frac{x^5}{5} \ln x - \frac{1}{25} x^5 + C \\ &= \frac{x^5}{25} (5 \ln x - 1) + C \end{aligned}$$

$$17. \quad dv = t dt \Rightarrow v = \int t dt = \frac{t^2}{2}$$

$$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1} dt$$

$$\begin{aligned} \int t \ln(t+1) dt &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \frac{t^2}{t+1} dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \left( t - 1 + \frac{1}{t+1} \right) dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \left[ \frac{t^2}{2} - t + \ln(t+1) \right] + C \\ &= \frac{1}{4} [2(t^2 - 1) \ln|t+1| - t^2 + 2t] + C \end{aligned}$$

$$22. \quad dv = \frac{x}{(x^2+1)^2} dx \Rightarrow v = \int (x^2+1)^{-2} x dx = -\frac{1}{2(x^2+1)}$$

$$u = x^2 e^{x^2} \Rightarrow du = (2x^3 e^{x^2} + 2x e^{x^2}) dx = 2x e^{x^2} (x^2 + 1) dx$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2} dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int x e^{x^2} dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

$$18. \quad \text{Let } u = \ln x, du = \frac{1}{x} dx.$$

$$\int \frac{1}{x(\ln x)^3} dx = \int (\ln x)^{-3} \left( \frac{1}{x} \right) dx = \frac{-1}{2(\ln x)^2} + C$$

$$19. \quad \text{Let } u = \ln x, du = \frac{1}{x} dx.$$

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \left( \frac{1}{x} \right) dx = \frac{(\ln x)^3}{3} + C$$

$$20. \quad dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$21. \quad dv = \frac{1}{(2x+1)^2} dx \Rightarrow v = \int (2x+1)^{-2} dx$$

$$= -\frac{1}{2(2x+1)}$$

$$\begin{aligned} u &= x e^{2x} \Rightarrow du = (2x e^{2x} + e^{2x}) dx \\ &= e^{2x} (2x + 1) dx \end{aligned}$$

$$\begin{aligned} \int \frac{x e^{2x}}{(2x+1)^2} dx &= -\frac{x e^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2} dx \\ &= \frac{-x e^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C = \frac{e^{2x}}{4(2x+1)} + C \end{aligned}$$

23. Use integration by parts twice.

$$(1) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x \qquad (2) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^2 \Rightarrow du = 2x dx \qquad u = x \Rightarrow du = dx$$

$$\begin{aligned} \int (x^2 - 1)e^x dx &= \int x^2 e^x dx - \int e^x dx = x^2 e^x - 2 \int x e^x dx - e^x \\ &= x^2 e^x - 2(xe^x - \int e^x dx) - e^x = x^2 e^x - 2xe^x + e^x + C = (x - 1)^2 e^x + C \end{aligned}$$

$$24. \quad dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$u = \ln 2x \Rightarrow du = \frac{1}{x} dx$$

$$\int \frac{\ln(2x)}{x^2} dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2} dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x) + 1}{x} + C$$

$$25. \quad dv = \sqrt{x-5} dx \Rightarrow v = \int (x-5)^{1/2} dx = \frac{2}{3}(x-5)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{x-5} dx &= x \frac{2}{3}(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= \frac{2}{15}(x-5)^{3/2}(5x - 2(x-5)) + C \\ &= \frac{2}{15}(x-5)^{3/2}(3x+10) + C \end{aligned}$$

$$26. \quad dv = \frac{1}{\sqrt{5+4x}} dx \Rightarrow v = \int (5+4x)^{-1/2} dx = \frac{1}{2}(5+4x)^{1/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int \frac{x}{\sqrt{5+4x}} dx &= x \left( \frac{1}{2} \right) (5+4x)^{1/2} - \int \frac{1}{2} (5+4x)^{1/2} dx \\ &= \frac{x}{2} (5+4x)^{1/2} - \frac{1}{8} \frac{2}{3} (5+4x)^{3/2} + C \\ &= \frac{x}{2} (5+4x)^{1/2} - \frac{1}{12} (5+4x)^{3/2} + C \\ &= \frac{1}{12} (5+4x)^{1/2} (6x - (5+4x)) + C \\ &= \frac{1}{12} \sqrt{5+4x} (2x-5) + C \end{aligned}$$

$$27. \quad dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$$

$$u = x \Rightarrow du = dx$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

$$28. \quad dv = \sin x dx \Rightarrow v = \int \sin x dx = -\cos x$$

$$u = x \Rightarrow du = dx$$

$$\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + C$$

29. Use integration by parts three times.

$$(1) \quad u = x^3, du = 3x^2 dx, dv = \sin x dx, v = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3 \int x^2 \cos x dx$$

$$(2) \quad u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3(x^2 \sin x - 2 \int x \sin x dx) = -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx$$

$$(3) \quad u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3x^2 \sin x - 6(-x \cos x + \int \cos x dx) \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \\ &= (6x - x^3) \cos x + (3x^2 - 6) \sin x + C \end{aligned}$$

30. Use integration by parts twice.

$$(1) \quad u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

$$(2) \quad u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2(-x \cos x + \int \cos x dx) = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

31.  $u = t, du = dt, dv = \csc t \cot t dt, v = -\csc t$

$$\int t \csc t \cot t dt = -t \csc t + \int \csc t dt = -t \csc t - \ln|\csc t + \cot t| + C$$

32.  $dv = \sec \theta \tan \theta d\theta \Rightarrow v = \int \sec \theta \tan \theta d\theta = \sec \theta$

$$u = \theta \Rightarrow du = d\theta$$

$$\int \theta \sec \theta \tan \theta d\theta = \theta \sec \theta - \int \sec \theta d\theta = \theta \sec \theta - \ln|\sec \theta + \tan \theta| + C$$

33.  $dv = dx \Rightarrow v = \int dx = x$

$$u = \arctan x \Rightarrow du = \frac{1}{1+x^2} dx$$

$$\begin{aligned} \int \arctan x dx &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

34.  $dv = dx \Rightarrow v = \int dx = x$

$$u = \arccos x \Rightarrow du = -\frac{1}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} 4 \int \arccos x dx &= 4 \left( x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \right) \\ &= 4 \left( x \arccos x - \sqrt{1-x^2} \right) + C \end{aligned}$$

35. Use integration by parts twice.

$$(1) \quad dv = e^{2x} dx \Rightarrow v = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left( \frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right)$$

$$\frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x$$

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$$

$$(2) \quad dv = e^{2x} dx \Rightarrow v = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

36. Use integration by parts twice.

$$(1) \quad dv = e^{-3x} dx \Rightarrow v = \int e^{-3x} dx = -\frac{1}{3}e^{-3x}$$

$$u = \sin 5x \Rightarrow du = 5 \cos 5x dx$$

$$\int e^{-3x} \sin 5x dx = \sin 5x \left(-\frac{1}{3}e^{-3x}\right) - \int \left(-\frac{1}{3}e^{-3x}\right) 5 \cos 5x dx = -\frac{1}{3}e^{-3x} \sin 5x + \frac{5}{3} \int e^{-3x} \cos 5x dx$$

$$(2) \quad dv = e^{-3x} dx \Rightarrow v = \int e^{-3x} dx = -\frac{1}{3}e^{-3x}$$

$$u = \cos 5x \Rightarrow du = -5 \sin 5x dx$$

$$\begin{aligned} \int e^{-3x} \sin 5x dx &= -\frac{1}{3}e^{-3x} \sin 5x + \frac{5}{3} \left[ -\frac{1}{3}e^{-3x} \cos 5x - \int \left(-\frac{1}{3}e^{-3x}\right) (-5 \sin 5x) dx \right] \\ &= -\frac{1}{3}e^{-3x} \sin 5x - \frac{5}{9}e^{-3x} \cos 5x - \frac{25}{9} \int e^{-3x} \sin 5x dx \end{aligned}$$

$$\left(1 + \frac{25}{9}\right) \int e^{-3x} \sin 5x dx = -\frac{1}{3}e^{-3x} \sin 5x - \frac{5}{9}e^{-3x} \cos 5x$$

$$\int e^{-3x} \sin 5x dx = \frac{9}{34} \left(-\frac{1}{3}e^{-3x} \sin 5x - \frac{5}{9}e^{-3x} \cos 5x\right) + C = -\frac{3}{34}e^{-3x} \sin 5x - \frac{5}{34}e^{-3x} \cos 5x + C$$

37. Use integration by parts twice.

$$(1) \quad dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$$

$$u = \cos 2x \Rightarrow du = -2 \sin 2x dx$$

$$\int e^{-x} \cos 2x dx = \cos 2x (-e^{-x}) - \int (-e^{-x}) (-2 \sin 2x) dx = -e^{-x} \cos 2x - 2 \int e^{-x} \sin 2x dx$$

$$(2) \quad dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$$

$$u = \sin 2x \Rightarrow du = 2 \cos 2x dx$$

$$\begin{aligned} \int e^{-x} \cos 2x dx &= -e^{-x} \cos 2x - 2 \left[ \sin 2x (-e^{-x}) - \int (-e^{-x}) (2 \cos 2x) dx \right] \\ &= -e^{-x} \cos 2x + 2e^{-x} \sin 2x - 4 \int e^{-x} \cos 2x dx \end{aligned}$$

$$(4 + 1) \int e^{-x} \cos 2x dx = -e^{-x} \cos 2x + 2e^{-x} \sin 2x$$

$$\int e^{-x} \cos 2x dx = \frac{1}{5}e^{-x} (2 \sin 2x - \cos 2x) + C$$

38. Use integration by parts twice.

$$(1) \quad dv = e^{3x} dx \Rightarrow v = \int e^{3x} dx = \frac{1}{3}e^{3x}$$

$$u = \cos 4x \Rightarrow du = -4 \sin 4x dx$$

$$\int e^{3x} \cos 4x dx = \cos 4x \left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x} (-4 \sin 4x) dx = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3} \int e^{3x} \sin 4x dx$$

$$(2) \quad dv = e^{3x} dx \Rightarrow v = \int e^{3x} dx = \frac{1}{3}e^{3x}$$

$$u = \sin 4x \Rightarrow du = 4 \cos 4x dx$$

$$\begin{aligned} \int e^{3x} \cos 4x dx &= \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3} \left[ \sin 4x \left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x} (4 \cos 4x) dx \right] \\ &= \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x - \frac{16}{9} \int e^{3x} \cos 4x dx \end{aligned}$$

$$\left(1 + \frac{16}{9}\right) \int e^{3x} \cos 4x dx = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x$$

$$\int e^{3x} \cos 4x dx = \frac{9}{25} \left(\frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x\right) = \frac{3}{25}e^{3x} \cos 4x + \frac{4}{25}e^{3x} \sin 4x + C$$

39.  $y' = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

40.  $dv = dx \Rightarrow v = x$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$y' = \ln x$$

$$y = \int \ln x \, dx = x \ln x - \int x \left( \frac{1}{x} \right) dx = x \ln x - x + C = x(-1 + \ln x) + C$$

41. Use integration by parts twice.

(1)  $dv = \frac{1}{\sqrt{3+5t}} dt \Rightarrow v = \int (3+5t)^{-1/2} dt = \frac{2}{5}(3+5t)^{1/2}$

$$u = t^2 \Rightarrow du = 2t \, dt$$

$$\begin{aligned} \int \frac{t^2}{\sqrt{3+5t}} dt &= \frac{2t^2}{5}(3+5t)^{1/2} - \int \frac{2}{5}(3+5t)^{1/2} 2t \, dt \\ &= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{4}{5} \int t(3+5t)^{1/2} dt \end{aligned}$$

(2)  $dv = (3+5t)^{1/2} dt \Rightarrow v = \int (3+5t)^{1/2} dt = \frac{2}{15}(3+5t)^{3/2}$

$$u = t \Rightarrow du = dt$$

$$\begin{aligned} \int \frac{t^2}{\sqrt{3+5t}} dt &= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{4}{5} \left[ \frac{2t}{15}(3+5t)^{3/2} - \int \frac{2}{15}(3+5t)^{3/2} dt \right] \\ &= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{8t}{75}(3+5t)^{3/2} + \frac{8}{75} \int (3+5t)^{3/2} dt \\ &= \frac{2}{5}t^2(3+5t)^{1/2} - \frac{8t}{75}(3+5t)^{3/2} + \frac{16}{1875}(3+5t)^{5/2} + C \\ &= \frac{2}{1875} \sqrt{3+5t} (375t^2 - 100t(3+5t) + 8(3+5t)^2) + C \\ &= \frac{2}{625} \sqrt{3+5t} (25t^2 - 20t + 24) + C \end{aligned}$$

42. Use integration by parts twice.

(1)  $dv = \sqrt{x-3} \, dx \Rightarrow v = \int (x-3)^{1/2} dx = \frac{2}{3}(x-3)^{3/2}$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$\begin{aligned} \int x^2 \sqrt{x-3} \, dx &= \frac{2}{3}x^2(x-3)^{3/2} - \int \frac{2}{3}(x-3)^{3/2} 2x \, dx \\ &= \frac{2}{3}x^2(x-3)^{3/2} - \frac{4}{3} \int (x-3)^{3/2} x \, dx \end{aligned}$$

(2)  $dv = (x-3)^{3/2} dx \Rightarrow v = \int (x-3)^{3/2} dx = \frac{2}{5}(x-3)^{5/2}$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x^2 \sqrt{x-3} \, dx &= \frac{2}{3}x^2(x-3)^{3/2} - \frac{4}{3} \left[ \frac{2}{5}x(x-3)^{5/2} - \int \frac{2}{5}(x-3)^{5/2} dx \right] \\ &= \frac{2}{3}x^2(x-3)^{3/2} - \frac{8}{15}x(x-3)^{5/2} + \frac{8}{15} \frac{2}{7}(x-3)^{7/2} + C \\ &= \frac{2}{35}(x-3)^{3/2} (5x^2 + 12x + 24) + C \end{aligned}$$

43.  $(\cos y)y' = 2x$

$$\int \cos y \, dy = \int 2x \, dx$$

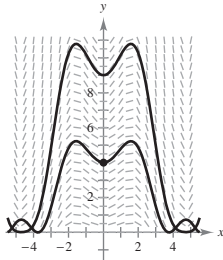
$$\sin y = x^2 + C$$

$$44. \, dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arctan \frac{x}{2} \quad \Rightarrow \quad du = \frac{1}{1 + (x/2)^2} \left( \frac{1}{2} \right) dx = \frac{2}{4 + x^2} dx$$

$$y = \int \arctan \frac{x}{2} dx = x \arctan \frac{x}{2} - \int \frac{2x}{4 + x^2} dx = x \arctan \frac{x}{2} - \ln(4 + x^2) + C$$

45. (a)



$$(b) \quad \frac{dy}{dx} = x\sqrt{y} \cos x, \quad (0, 4)$$

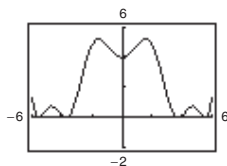
$$\int \frac{dy}{\sqrt{y}} = \int x \cos x \, dx$$

$$\int y^{-1/2} dy = \int x \cos x \, dx \quad (u = x, du = dx, dv = \cos x \, dx, v = \sin x)$$

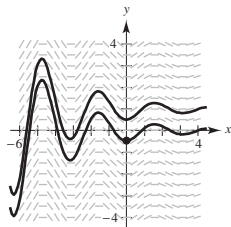
$$2y^{1/2} = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$$

$$(0, 4): 2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$$

$$2\sqrt{y} = x \sin x + \cos x + 3$$



46. (a)



$$(b) \quad \frac{dy}{dx} = e^{-x/3} \sin 2x, \quad \left( 0, -\frac{18}{37} \right)$$

$$y = \int e^{-x/3} \sin 2x \, dx$$

Use integration by parts twice.

$$(1) \quad u = \sin 2x, \, du = 2 \cos 2x$$

$$dv = e^{-x/3} \, dx, \, v = -3e^{-x/3}$$

$$\int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x + \int 6e^{-x/3} \cos 2x \, dx$$



(2)  $u = \cos 2x, du = -2 \sin 2x$

$dv = e^{-x/3} dx, v = -3e^{-x/3}$

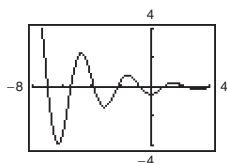
$$\int e^{-x/3} \sin 2x dx = -3e^{-x/3} \sin 2x + 6 \left( -3e^{-x/3} \cos 2x - \int 6e^{-x/3} \sin 2x dx \right) + C$$

$$37 \int e^{-x/3} \sin 2x dx = -3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x + C$$

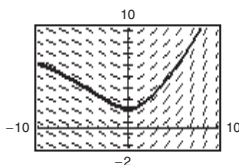
$$y = \int e^{-x/3} \sin 2x dx = \frac{1}{37} (-3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x) + C$$

$$\left(0, \frac{-18}{37}\right): \frac{-18}{37} = \frac{1}{37} [0 - 18] + C \Rightarrow C = 0$$

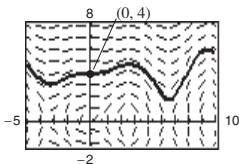
$$y = \frac{-1}{37} (3e^{-x/3} \sin 2x + 18e^{-x/3} \cos 2x)$$



47.  $\frac{dy}{dx} = \frac{x}{y} e^{x/8}, y(0) = 2$



48.  $\frac{dy}{dx} = \frac{x}{y} \sin x, y(0) = 4$



49.  $u = x, du = dx, dv = e^{x/2} dx, v = 2e^{x/2}$

$$\int x e^{x/2} dx = 2x e^{x/2} - \int 2e^{x/2} dx$$

$$= 2x e^{x/2} - 4e^{x/2} + C$$

So,

$$\int_0^3 x e^{x/2} dx = \left[ 2x e^{x/2} - 4e^{x/2} \right]_0^3$$

$$= (6e^{3/2} - 4e^{3/2}) - (-4)$$

$$= 4 + 2e^{3/2} \approx 12.963$$

50. Use integration by parts twice.

(1)  $u = x^2, du = 2x dx, dv = e^{-2x} dx,$

$$v = -\frac{1}{2} e^{-2x}$$

$$\int x^2 e^{-2x} dx = -\frac{1}{2} x^2 e^{-2x} - \int \left( -\frac{1}{2} e^{-2x} \right) 2x dx$$

$$= -\frac{1}{2} x^2 e^{-2x} + \int x e^{-2x} dx$$

(2)  $u = x, du = dx, dv = e^{-2x} dx, v = -\frac{1}{2} e^{-2x}$

$$\int x^2 e^{-2x} dx = -\frac{1}{2} x^2 e^{-2x} + \left( -\frac{1}{2} x e^{-2x} - \int -\frac{1}{2} e^{-2x} dx \right)$$

$$= -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C$$

$$= e^{-2x} \left( -\frac{1}{2} x^2 - \frac{1}{2} x - \frac{1}{4} \right)$$

So,

$$\int_0^2 x^2 e^{-2x} dx = \left[ e^{-2x} \left( -\frac{1}{2} x^2 - \frac{1}{2} x - \frac{1}{4} \right) \right]_0^2$$

$$= e^{-4} \left( -2 - 1 - \frac{1}{4} \right) - \left( -\frac{1}{4} \right)$$

$$= \frac{-13}{4e^4} + \frac{1}{4} \approx 0.190$$

$$51. \quad u = x, \quad du = dx, \quad dv = \cos 2x \, dx, \quad v = \frac{1}{2} \sin 2x$$

$$\begin{aligned} \int x \cos 2x \, dx &= \frac{1}{2}x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \\ &= \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x + C \end{aligned}$$

So,

$$\begin{aligned} \int_0^{\pi/4} x \cos 2x \, dx &= \left[ \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\pi/4} \\ &= \left( \frac{\pi}{8}(1) + 0 \right) - \left( 0 + \frac{1}{4} \right) \\ &= \frac{\pi}{8} - \frac{1}{4} \approx 0.143 \end{aligned}$$

$$52. \quad dv = \sin 2x \, dx \Rightarrow v = \int \sin 2x \, dx = -\frac{1}{2} \cos 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin 2x \, dx &= -\frac{1}{2}x \cos 2x + \frac{1}{2} \int \cos 2x \, dx \\ &= -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C \\ &= \frac{1}{4}(\sin 2x - 2x \cos 2x) + C \end{aligned}$$

So,

$$\int_0^{\pi} x \sin 2x \, dx = \left[ \frac{1}{4}(\sin 2x - 2x \cos 2x) \right]_0^{\pi} = -\frac{\pi}{2}.$$

$$54. \quad dv = x \, dx \Rightarrow v = \int x \, dx = \frac{x^2}{2}$$

$$u = \arcsin x^2 \Rightarrow du = \frac{2x}{\sqrt{1-x^4}} \, dx$$

$$\begin{aligned} \int x \arcsin x^2 \, dx &= \frac{x^2}{2} \arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}} \, dx \\ &= \frac{x^2}{2} \arcsin x^2 + \frac{1}{4}(2)(1-x^4)^{1/2} + C \\ &= \frac{1}{2}(x^2 \arcsin x^2 + \sqrt{1-x^4}) + C \end{aligned}$$

So,

$$\int_0^1 x \arcsin x^2 \, dx = \frac{1}{2} \left[ x^2 \arcsin x^2 + \sqrt{1-x^4} \right]_0^1 = \frac{1}{4}(\pi - 2).$$

$$53. \quad u = \arccos x, \quad du = -\frac{1}{\sqrt{1-x^2}} \, dx, \quad dv = dx, \quad v = x$$

$$\begin{aligned} \int \arccos x \, dx &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \arccos x - \sqrt{1-x^2} + C \end{aligned}$$

So,

$$\begin{aligned} \int_0^{1/2} \arccos x \, dx &= \left[ x \arccos x - \sqrt{1-x^2} \right]_0^{1/2} \\ &= \frac{1}{2} \arccos \left( \frac{1}{2} \right) - \sqrt{\frac{3}{4}} + 1 \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658. \end{aligned}$$

55. Use integration by parts twice.

$$(1) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (2) \quad dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin x \Rightarrow du = \cos x dx \quad u = \cos x \Rightarrow du = -\sin x dx$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$2 \int e^x \sin x dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\text{So, } \int_0^1 e^x \sin x dx = \left[ \frac{e^x}{2} (\sin x - \cos x) \right]_0^1 = \frac{e}{2} (\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$$

56. Use integration by parts twice.

$$(1) \quad dv = e^{-x}, v = -e^{-x}, u = \cos x, du = -\sin x dx$$

$$\int e^{-x} \cos x dx = e^{-x} \cos x - \int e^{-x} \sin x dx$$

$$(2) \quad dv = e^{-x} dx, v = -e^{-x}, u = \sin x, du = \cos x dx$$

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - (-e^{-x} \sin x + \int e^{-x} \cos x dx) \Rightarrow 2 \int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x$$

$$\text{So, } \int_0^2 e^{-x} \cos x dx = \left[ \frac{e^{-x} \sin x - e^{-x} \cos x}{2} \right]_0^2 = \frac{-e^{-2} (\sin 2 - \cos 2)}{2} + \frac{1}{2}.$$

57.  $u = \ln x, du = \frac{1}{x} dx, dv = \sqrt{x} dx, v = \frac{2}{3} x^{3/2}$

$$\int \sqrt{x} \ln x dx = \frac{2}{3} x^{3/2} \ln x - \int \frac{2}{3} x^{3/2} \frac{1}{x} dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + C$$

$$\text{So, } \int_1^2 \sqrt{x} \ln x dx = \left[ \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} \right]_1^2 = \left( \frac{4}{3} \sqrt{2} \ln 2 - \frac{4}{9} 2\sqrt{2} \right) - \left( 0 - \frac{4}{9} \right) = \frac{4}{3} \sqrt{2} \ln 2 - \frac{8}{9} \sqrt{2} + \frac{4}{9} \approx 0.494$$

58.  $u = \ln(4 + x^2), du = \frac{2x}{4 + x^2} dx, dv = dx, v = x$

$$\begin{aligned} \int \ln(4 + x^2) dx &= x \ln(4 + x^2) - \int \frac{2x^2}{4 + x^2} dx \\ &= x \ln(4 + x^2) - 2 \int \left( 1 - \frac{4}{4 + x^2} \right) dx \\ &= x \ln(4 + x^2) - 2 \left( x - \frac{4}{2} \arctan \frac{x}{2} \right) + C \\ &= x \ln(4 + x^2) - 2x + 4 \arctan \frac{x}{2} + C \end{aligned}$$

$$\text{So, } \int_0^1 \ln(4 + x^2) dx = \left[ x \ln(4 + x^2) - 2x + 4 \arctan \frac{x}{2} \right]_0^1 = \left( \ln 5 - 2 + 4 \arctan \left( \frac{1}{2} \right) \right) \approx 1.464$$

59.  $dv = x dx, v = \frac{x^2}{2}, u = \operatorname{arcsec} x, du = \frac{1}{x\sqrt{x^2 - 1}} dx$

$$\int x \operatorname{arcsec} x dx = \frac{x^2}{2} \operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2 - 1}} dx = \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{4} \int \frac{2x}{\sqrt{x^2 - 1}} dx = \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2 - 1} + C$$

So,

$$\int_2^4 x \operatorname{arcsec} x dx = \left[ \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2 - 1} \right]_2^4 = \left( 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} \right) - \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \approx 7.380.$$

# INSTRUCTOR USE ONLY

60.  $u = x, du = dx, dv = \sec^2 2x dx, v = \frac{1}{2} \tan 2x$

$$\int x \sec^2 2x dx = \frac{1}{2} x \tan 2x - \int \frac{1}{2} \tan 2x dx = \frac{1}{2} x \tan 2x + \frac{1}{4} \ln |\cos 2x| + C$$

So,

$$\int_0^{\pi/8} x \sec^2 2x dx = \left[ \frac{1}{2} x \tan 2x + \frac{1}{4} \ln |\cos 2x| \right]_0^{\pi/8} = \frac{\pi}{16} (1) + \frac{1}{4} \ln \left( \frac{\sqrt{2}}{2} \right) = \frac{\pi}{16} - \frac{1}{8} \ln(2) \approx 0.1097$$

61.  $\int x^2 e^{2x} dx = x^2 \left( \frac{1}{2} e^{2x} \right) - (2x) \left( \frac{1}{4} e^{2x} \right) + 2 \left( \frac{1}{8} e^{2x} \right) + C$

$$= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C$$

$$= \frac{1}{4} e^{2x} (2x^2 - 2x + 1) + C$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x^2$	$e^{2x}$
−	$2x$	$\frac{1}{2} e^{2x}$
+	$2$	$\frac{1}{4} e^{2x}$
−	$0$	$\frac{1}{8} e^{2x}$

62.  $\int x^3 e^{-2x} dx = x^3 \left( -\frac{1}{2} e^{-2x} \right) - 3x^2 \left( \frac{1}{4} e^{-2x} \right) + 6x \left( -\frac{1}{8} e^{-2x} \right) - 6 \left( \frac{1}{16} e^{-2x} \right) + C$

$$= -\frac{1}{8} e^{-2x} (4x^3 + 6x^2 + 6x + 3) + C$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x^3$	$e^{-2x}$
−	$3x^2$	$-\frac{1}{2} e^{-2x}$
+	$6x$	$\frac{1}{4} e^{-2x}$
−	$6$	$-\frac{1}{8} e^{-2x}$
+	$0$	$\frac{1}{16} e^{-2x}$

63.  $\int x^3 \sin x dx = x^3 (-\cos x) - 3x^2 (-\sin x) + 6x \cos x - 6 \sin x + C$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

$$= (3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x^3$	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	$6$	$\cos x$
+	$0$	$\sin x$

$$\begin{aligned}
 64. \int x^3 \cos 2x \, dx &= x^3 \left( \frac{1}{2} \sin 2x \right) - 3x^2 \left( -\frac{1}{4} \cos 2x \right) + 6x \left( -\frac{1}{8} \sin 2x \right) - 6 \left( \frac{1}{16} \cos 2x \right) + C \\
 &= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C \\
 &= \frac{1}{8} (4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3 \cos 2x) + C
 \end{aligned}$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x^3$	$\cos 2x$
−	$3x^2$	$\frac{1}{2} \sin 2x$
+	$6x$	$-\frac{1}{4} \cos 2x$
−	$6$	$-\frac{1}{8} \sin 2x$
+	$0$	$\frac{1}{16} \cos 2x$

$$65. \int x \sec^2 x \, dx = x \tan x + \ln |\cos x| + C$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x$	$\sec^2 x$
−	$1$	$\tan x$
+	$0$	$-\ln  \cos x $

$$66. \int x^2 (x-2)^{3/2} \, dx = \frac{2}{5} x^2 (x-2)^{5/2} - \frac{8}{35} x (x-2)^{7/2} + \frac{16}{315} (x-2)^{9/2} + C = \frac{2}{315} (x-2)^{5/2} (35x^2 + 40x + 32) + C$$

Alternate signs	$u$ and its derivatives	$v'$ and its antiderivatives
+	$x^2$	$(x-2)^{3/2}$
−	$2x$	$\frac{2}{5} (x-2)^{5/2}$
+	$2$	$\frac{4}{35} (x-2)^{7/2}$
−	$0$	$\frac{8}{315} (x-2)^{9/2}$

$$67. u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2u \, du = dx$$

$$\int \sin \sqrt{x} \, dx = \int \sin u (2u \, du) = 2 \int u \sin u \, du$$

Integration by parts:

$$w = u, dw = du, dv = \sin u \, du, v = -\cos u$$

$$\begin{aligned}
 2 \int u \sin u \, du &= 2(-u \cos u + \int \cos u \, du) \\
 &= 2(-u \cos u + \sin u) + C \\
 &= 2(-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}) + C
 \end{aligned}$$

$$68. u = \sqrt{x}, u^2 = x, 2u \, du = dx$$

$$\int \cos \sqrt{x} \, dx = \int \cos u (2u \, du) = 2 \int u \cos u \, du$$

Integration by parts:

$$w = u, dw = du, dv = \cos u \, du, v = \sin u$$

$$\begin{aligned}
 2 \int u \cos u \, du &= 2(u \sin u - \int \sin u \, du) \\
 &= 2(u \sin u + \cos u) + C \\
 &= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C
 \end{aligned}$$

$$69. \text{ Let } u = 4 - x, du = -dx, x = 4 - u.$$

$$\begin{aligned}
 \int_0^4 x \sqrt{4-x} \, dx &= \int_4^0 (4-u) u^{1/2} (-du) \\
 &= \int_0^4 (4u^{1/2} - u^{3/2}) \, du \\
 &= \left[ \frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_0^4 \\
 &= \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{128}{15}
 \end{aligned}$$

$$70. u = x^2, du = 2x \, dx$$

$$\int 2x^3 \cos(x^2) \, dx = \int x^2 \cos(x^2) (2x) \, dx = \int u \cos u \, du$$

Integration by parts:

$$w = u, dw = du, dv = \cos u \, du, v = \sin u$$

$$\begin{aligned}
 \int u \cos u \, du &= u \sin u - \int \sin u \, du \\
 &= u \sin u + \cos u + C \\
 &= x^2 \sin(x^2) + \cos(x^2) + C
 \end{aligned}$$

71.  $u = x^2, du = 2x \, dx$

$$\int x^5 e^{x^2} \, dx = \frac{1}{2} \int e^{x^2} x^4 \, 2x \, dx = \frac{1}{2} \int e^u u^2 \, du$$

Integration by parts twice.

(1)  $w = u^2, dw = 2u \, du, dv = e^u \, du, v = e^u$

$$\begin{aligned} \frac{1}{2} \int e^u u^2 \, du &= \frac{1}{2} \left[ u^2 e^u - \int 2u e^u \, du \right] \\ &= \frac{1}{2} u^2 e^u - \int u e^u \, du \end{aligned}$$

(2)  $w = u, dw = du, dv = e^u \, du, v = e^u$

$$\begin{aligned} \frac{1}{2} \int e^u u^2 \, du &= \frac{1}{2} u^2 e^u - (u e^u - \int e^u \, du) \\ &= \frac{1}{2} u^2 e^u - u e^u + e^u + C \\ &= \frac{1}{2} x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C \\ &= \frac{e^{x^2}}{2} (x^4 - 2x^2 + 2) + C \end{aligned}$$

73. Let  $w = \ln x, dw = \frac{1}{x} \, dx, x = e^w, dx = e^w \, dw$ .

$$\int \cos(\ln x) \, dx = \int \cos w (e^w \, dw)$$

Now use integration by parts twice.

$$\begin{aligned} \int (\cos w) e^w \, dw &= (\cos w) e^w + \int (\sin w) e^w \, dw & [u = \cos w, dv = e^w \, dw] \\ &= (\cos w) e^w + \left[ (\sin w) e^w - \int (\cos w) e^w \, dw \right] & [u = \sin w, dv = e^w \, dw] \end{aligned}$$

$$2 \int (\cos w) e^w \, dw = (\cos w) e^w + (\sin w) e^w$$

$$\int (\cos w) e^w \, dw = \frac{1}{2} e^w (\cos w + \sin w) + C$$

$$\int \cos(\ln x) \, dx = \frac{1}{2} x [\cos(\ln x) + \sin(\ln x)] + C$$

74. Let  $w = 1 + x^2, dw = 2x \, dx, x^2 = w - 1, x = \sqrt{w - 1}$ .

$$\int \ln(x^2 + 1) \, dx = \int \ln(w) \frac{dw}{2\sqrt{w-1}}$$

Integration by parts:  $u = \ln w, du = \frac{1}{w} \, dw, dv = \frac{1}{2\sqrt{w-1}} \, dw, v = \sqrt{w-1}$

$$\int \ln(x^2 + 1) \, dx = \ln(w) \sqrt{w-1} - \int \frac{\sqrt{w-1}}{2} \, dw$$

Substitution:  $z = \sqrt{w-1}, z^2 = w-1, 2z \, dz = dw$

$$\begin{aligned} \int \ln(x^2 + 1) \, dx &= \ln(w) \sqrt{w-1} - \int \frac{z}{z^2 + 1} (2z \, dz) \\ &= \ln(w) \sqrt{w-1} - 2 \int \left( 1 - \frac{1}{z^2 + 1} \right) dz \\ &= \ln(w) \sqrt{w-1} - 2z + 2 \arctan(z) + C \\ &= \ln(1 + x^2) x - 2x + 2 \arctan(x) + C \end{aligned}$$

72. Let  $u = \sqrt{2x}, u^2 = 2x, 2u \, du = 2 \, dx$ .

$$\begin{aligned} \int_0^2 e^{\sqrt{2x}} \, dx &= \int_0^2 e^u (u \, du) \\ &= \left[ u e^u - e^u \right]_0^2 & (\text{Integration by parts}) \\ &= (2e^2 - e^2) - (0 - 1) \\ &= e^2 + 1 \end{aligned}$$

75. Integration by parts is based on the Product Rule.

76. Answers will vary. *Sample answer:* You want  $dv$  to be the most complicated portion of the integrand.

77. In order for the integration by parts technique to be efficient, you want  $dv$  to be the most complicated portion of the integrand and you want  $u$  to be the portion of the integrand whose derivative is a function simpler than  $u$ . Suppose you let  $u = \sin x$  and  $dv = x \, dx$ . Then  $du = \cos x \, dx$  and  $v = x^2/2$ . So

$$\int x \sin x \, dx = uv - \int v \, du = \frac{x^2}{2} \sin x - \int \frac{x^2}{2} \cos x \, dx,$$

which is a more complicated integral than the original one.

78. (a) No

Substitution

(b) Yes

$$u = \ln x, \, dv = x \, dx$$

(c) Yes

$$u = x^2, \, dv = e^{-3x} \, dx$$

(d) No

Substitution

(e) Yes. Let  $u = x$  and

$$du = \frac{1}{\sqrt{x+1}} \, dx.$$

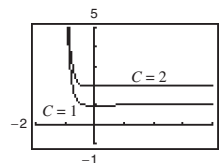
(Substitution also works. Let  $u = \sqrt{x+1}$ .)

(f) No

Substitution

79. (a)  $\int t^3 e^{-4t} \, dt = \frac{-e^{-4t}}{128} (32t^3 + 24t^2 + 12t + 3) + C$

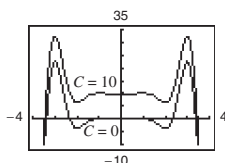
(b)



(c) The graphs are vertical translations of each other.

80. (a)  $\int \alpha^4 \sin(\pi\alpha) \, d\alpha = \frac{1}{\pi^5} [-(\alpha\pi)^4 \cos \pi\alpha + 4(\alpha\pi)^3 \sin \pi\alpha + 12(\alpha\pi)^2 \cos \pi\alpha - 24(\alpha\pi) \sin \pi\alpha - 24 \cos \pi\alpha] + C$

(b)

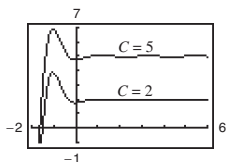


(c) The graphs are vertical translations of each other.

81. (a)  $\int e^{-2x} \sin 3x \, dx = \frac{e^{-2x}}{13} (-2 \sin 3x - 3 \cos 3x) + C$

$$\int_0^{\pi/2} e^{-2x} \sin 3x \, dx = \frac{1}{13} (2e^{-\pi} + 3) \approx 0.2374$$

(b)

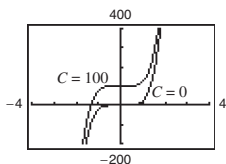


(c) The graphs are vertical translations of each other.

$$82. (a) \int x^4(25 - x^2)^{3/2} dx = \frac{1,171,875 \arcsin|x/5|}{128} - \frac{x(2x^2 + 25)(25 - x^2)^{5/2}}{16} + \frac{625x(25 - x^2)^{3/2}}{64} + \frac{46,875x\sqrt{25 - x^2}}{128} + C$$

$$\int_0^5 x^4(25 - x^2)^{2/3} dx = \frac{1,171,875}{256}\pi \approx 14,381.0699$$

(b)



(c) The graphs are vertical translations of each other.

$$83. (a) dv = \sqrt{2x - 3} dx \Rightarrow v = \int (2x - 3)^{1/2} dx = \frac{1}{3}(2x - 3)^{3/2}$$

$$u = 2x \Rightarrow du = 2 dx$$

$$\int 2x\sqrt{2x - 3} dx = \frac{2}{3}x(2x - 3)^{3/2} - \frac{2}{3} \int (2x - 3)^{3/2} dx$$

$$= \frac{2}{3}x(2x - 3)^{3/2} - \frac{2}{15}(2x - 3)^{5/2} + C$$

$$= \frac{2}{15}(2x - 3)^{3/2}(3x + 3) + C = \frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$$

$$(b) u = 2x - 3 \Rightarrow x = \frac{u + 3}{2} \text{ and } dx = \frac{1}{2} du$$

$$\int 2x\sqrt{2x - 3} dx = \int 2\left(\frac{u + 3}{2}\right)u^{1/2}\left(\frac{1}{2}\right) du = \frac{1}{2} \int (u^{3/2} + 3u^{1/2}) du$$

$$= \frac{1}{2} \left[ \frac{2}{5}u^{5/2} + 2u^{3/2} \right] + C$$

$$= \frac{1}{5}u^{3/2}(u + 5) + C$$

$$= \frac{1}{5}(2x - 3)^{3/2}[(2x - 3) + 5] + C$$

$$= \frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$$

$$84. (a) dv = \sqrt{9 + x} dx, v = \frac{2}{3}(9 + x)^{3/2}, u = x, du = dx$$

$$\int x\sqrt{9 + x} dx = \frac{2}{3}x(9 + x)^{3/2} - \int \frac{2}{3}(9 + x)^{3/2} dx$$

$$= \frac{2}{3}x(9 + x)^{3/2} - \frac{4}{15}(9 + x)^{5/2} + C$$

$$= \frac{2}{15}(9 + x)^{3/2}[5x - 2(9 + x)] + C$$

$$= \frac{2}{15}(9 + x)^{3/2}(3x - 18) + C = \frac{2}{5}(9 + x)^{3/2}(x - 6) + C$$

$$(b) u = 9 + x, du = dx, x = u - 9$$

$$\int x\sqrt{9 + x} dx = \int (u - 9)\sqrt{u} du = \int (u^{3/2} - 9u^{1/2}) du$$

$$= \frac{2}{5}u^{5/2} - 6u^{3/2} + C$$

$$= \frac{2}{5}(9 + x)^{5/2} - 6(9 + x)^{3/2} + C$$

$$= \frac{2}{15}(9 + x)^{3/2}[3(9 + x) - 45] + C$$

$$= \frac{2}{15}(9 + x)^{3/2}(3x - 18) + C$$

$$= \frac{2}{5}(9 + x)^{3/2}(x - 6) + C$$



$$85. (a) \quad dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \int (4+x^2)^{-1/2} x dx = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - 2 \int x \sqrt{4+x^2} dx \\ &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{1}{3} \sqrt{4+x^2} (x^2 - 8) + C \end{aligned}$$

$$(b) \quad u = 4+x^2 \Rightarrow x^2 = u-4 \text{ and } 2x dx = du \Rightarrow x dx = \frac{1}{2} du$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{x^2}{\sqrt{4+x^2}} x dx = \int \frac{u-4}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) du = \frac{1}{2} \left( \frac{2}{3} u^{3/2} - 8u^{1/2} \right) + C \\ &= \frac{1}{3} u^{1/2} (u - 12) + C \\ &= \frac{1}{3} \sqrt{4+x^2} [(4+x^2) - 12] + C = \frac{1}{3} \sqrt{4+x^2} (x^2 - 8) + C \end{aligned}$$

$$86. (a) \quad dv = \sqrt{4-x} dx \Rightarrow v = \int (4-x)^{1/2} dx = -\frac{2}{3} (4-x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sqrt{4-x} dx &= -\frac{2}{3} x (4-x)^{3/2} + \frac{2}{3} \int (4-x)^{3/2} dx \\ &= -\frac{2}{3} x (4-x)^{3/2} - \frac{4}{15} (4-x)^{5/2} + C \\ &= -\frac{2}{15} (4-x)^{3/2} [5x + 2(4-x)] + C = -\frac{2}{15} (4-x)^{3/2} (3x+8) + C \end{aligned}$$

$$(b) \quad u = 4-x \Rightarrow x = 4-u \text{ and } dx = -du$$

$$\begin{aligned} \int x \sqrt{4-x} dx &= -\int (4-u) \sqrt{u} du \\ &= -\int (4u^{1/2} - u^{3/2}) du \\ &= -\frac{8}{3} u^{3/2} + \frac{2}{5} u^{5/2} + C \\ &= -\frac{2}{15} u^{3/2} (20-3u) + C \\ &= -\frac{2}{15} (4-x)^{3/2} [20-3(4-x)] + C \\ &= -\frac{2}{15} (4-x)^{3/2} (3x+8) + C \end{aligned}$$

$$87. \quad n = 0: \int \ln x dx = x(\ln x - 1) + C$$

$$n = 1: \int x \ln x dx = \frac{x^2}{4} (2 \ln x - 1) + C$$

$$n = 2: \int x^2 \ln x dx = \frac{x^3}{9} (3 \ln x - 1) + C$$

$$n = 3: \int x^3 \ln x dx = \frac{x^4}{16} (4 \ln x - 1) + C$$

$$n = 4: \int x^4 \ln x dx = \frac{x^5}{25} (5 \ln x - 1) + C$$

$$\text{In general, } \int x^n \ln x dx = \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C.$$

$$88. n = 0: \int e^x dx = e^x + C$$

$$n = 1: \int x e^x dx = x e^x - e^x + C = x e^x - \int e^x dx$$

$$n = 2: \int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C = x^2 e^x - 2 \int x e^x dx$$

$$n = 3: \int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = x^3 e^x - 3 \int x^2 e^x dx$$

$$n = 4: \int x^4 e^x dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C = x^4 e^x - 4 \int x^3 e^x dx$$

$$\text{In general, } \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

$$89. dv = \sin x dx \Rightarrow v = -\cos x$$

$$u = x^n \Rightarrow du = n x^{n-1} dx$$

$$\int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx$$

$$90. dv = \cos x dx \Rightarrow v = \sin x$$

$$u = x^n \Rightarrow du = n x^{n-1} dx$$

$$\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$$

$$91. dv = x^n dx \Rightarrow v = \frac{x^{n+1}}{n+1}$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$\begin{aligned} \int x^n \ln x dx &= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} dx \\ &= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C \\ &= \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C \end{aligned}$$

$$92. dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = x^n \Rightarrow du = n x^{n-1} dx$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

93. Use integration by parts twice.

$$(1) dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx dx$$

$$(2) dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx dx$$

$$\begin{aligned} \int e^{ax} \sin bx dx &= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx dx \\ &= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left( \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx dx \right) = \frac{e^{ax} \sin bx}{a} - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx dx \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx dx &= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2} \\ \int e^{ax} \sin bx dx &= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C. \end{aligned}$$

94. Use integration by parts twice.

$$(1) dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx$$

$$(2) dv = e^{ax} dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx$$

$$\begin{aligned} \int e^{ax} \cos bx dx &= \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \left( \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx dx \right) \\ &= \frac{e^{ax} \cos bx}{a} + \frac{b e^{ax} \sin bx}{a^2} - \frac{b^2}{a^2} \int e^{ax} \cos bx dx \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \cos bx dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2} \\ \int e^{ax} \cos bx dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C. \end{aligned}$$

95.  $n = 5$  (Use formula in Exercise 91.)

$$\int x^5 \ln x \, dx = \frac{x^6}{6^2}(-1 + 6 \ln x) + C = \frac{x^6}{36}(-1 + 6 \ln x) + C$$

96.  $n = 2$ , (Use formula in Exercise 90.)

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2 \int x \sin x \, dx, \quad (\text{Use formula in Exercise 83.}) \quad (n = 1) \\ &= x^2 \sin x - 2(-x \cos x + \int \cos x \, dx) = x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

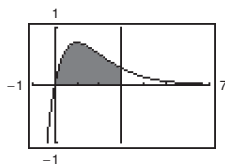
97.  $a = 2, b = 3$ , (Use formula in Exercise 94.)

$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2 \cos 3x + 3 \sin 3x)}{13} + C$$

98.  $n = 3, a = 2$ , (Use formula in Exercise 92 three times.)

$$\begin{aligned} \int x^3 e^{2x} \, dx &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \int x^2 e^{2x} \, dx, \quad (n = 3, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \left[ \frac{x^2 e^{2x}}{2} - \int x e^{2x} \, dx \right], \quad (n = 2, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3}{2} \left[ \frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx \right] \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3e^{2x}}{8} + C, \quad (n = 1, a = 2) \\ &= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C \end{aligned}$$

99.



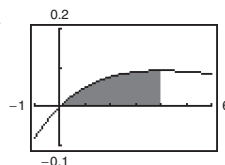
$$dv = e^{-x} \, dx \Rightarrow v = -e^{-x}$$

$$u = 2x \Rightarrow du = 2 \, dx$$

$$\begin{aligned} \int 2xe^{-x} \, dx &= 2x(-e^{-x}) - \int -2e^{-x} \, dx \\ &= -2xe^{-x} - 2e^{-x} + C \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 2xe^{-x} \, dx = [-2xe^{-x} - 2e^{-x}]_0^3 \\ &= (-6e^{-3} - 2e^{-3}) - (-2) \\ &= 2 - 8e^{-3} \approx 1.602 \end{aligned}$$

100.



$$dv = e^{-x/4} \, dx \Rightarrow v = -4e^{-x/4}$$

$$u = \frac{1}{16}x \Rightarrow du = \frac{1}{16} \, dx$$

$$\begin{aligned} \int \frac{1}{16} x e^{-x/4} \, dx &= \left( \frac{1}{16} x \right) (-4e^{-x/4}) - \int (-4e^{-x/4}) \frac{1}{16} \, dx \\ &= -\frac{1}{4} x e^{-x/4} - e^{-x/4} + C \end{aligned}$$

$$\begin{aligned} A &= \int_0^4 \frac{1}{16} x e^{-x/4} \, dx = \left[ -\frac{1}{4} x e^{-x/4} - e^{-x/4} \right]_0^4 \\ &= (-e^{-1} - e^{-1}) - (-1) \\ &= 1 - 2e^{-1} = 1 - \frac{2}{e} \approx 0.264 \end{aligned}$$

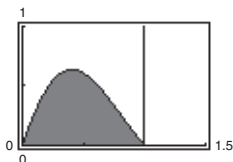
$$101. A = \int_0^1 e^{-x} \sin(\pi x) dx$$

$$= \left[ \frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2} \right]_0^1$$

$$= \frac{1}{1 + \pi^2} \left( \frac{\pi}{e} + \pi \right)$$

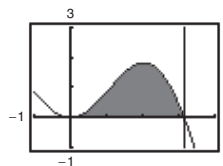
$$= \frac{\pi}{1 + \pi^2} \left( \frac{1}{e} + 1 \right)$$

$$\approx 0.395 \quad (\text{See Exercise 93.})$$



$$102. A = \int_0^\pi x \sin x dx = [-x \cos x + \sin x]_0^\pi$$

$$= \pi \quad (\text{See Exercise 89.})$$



$$103. (a) dv = dx \Rightarrow v = x$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

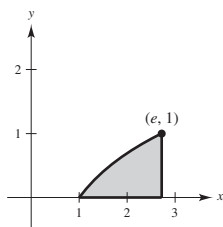
$$A = \int_1^e \ln x dx = [x \ln x - x]_1^e = 1 \quad (\text{Use integration by parts once.})$$

$$(b) R(x) = \ln x, r(x) = 0$$

$$V = \pi \int_1^e (\ln x)^2 dx$$

$$= \pi \left[ x(\ln x)^2 - 2x \ln x + 2x \right]_1^e \quad (\text{Use integration by parts twice, see Exercise 3.})$$

$$= \pi(e - 2) \approx 2.257$$



$$(c) p(x) = x, h(x) = \ln x$$

$$V = 2\pi \int_1^e x \ln x dx = 2\pi \left[ \frac{x^2}{4} (-1 + 2 \ln x) \right]_1^e$$

$$= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \quad (\text{See Exercise 91.})$$

$$(d) \bar{x} = \frac{\int_1^e x \ln x dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2} \int_1^e (\ln x)^2 dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left( \frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) \approx (2.097, 0.359)$$

104.  $y = x \sin x, \quad 0 \leq x \leq \pi$

$$(a) \quad V = \int_0^\pi \pi (x \sin x)^2 dx = \pi \int_0^\pi x^2 \sin^2 x dx$$

$$\text{Let } u = x^2, du = 2x dx, dv = \sin^2 x dx = \frac{1 - \cos 2x}{2} dx, v = \frac{1}{2}x - \frac{\sin 2x}{4}.$$

$$\begin{aligned} \int x^2 \sin^2 x dx &= x^2 \left( \frac{1}{2}x - \frac{\sin 2x}{4} \right) - \int \left( \frac{1}{2}x - \frac{\sin 2x}{4} \right) (2x dx) \\ &= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \int \left( x^2 - \frac{x \sin 2x}{2} \right) dx \\ &= \frac{1}{2}x^3 - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} + \int \frac{x \sin 2x}{2} dx \\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x) + C \quad (\text{Integration by Parts}) \end{aligned}$$

$$V = \pi \int_0^\pi x^2 \sin^2 x dx = \pi \left[ \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{8}(\sin 2x - 2x \cos 2x) \right]_0^\pi = \frac{1}{6}\pi^4 - \frac{1}{4}\pi^2$$

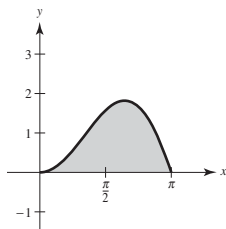
$$(b) \quad V = \int_0^\pi 2\pi x(x \sin x) dx = 2\pi \left[ 2 \cos x + 2x \sin x - x^2 \cos x \right]_0^\pi = 2\pi(\pi^2 - 4) = 2\pi^3 - 8\pi$$

$$(c) \quad m = \int_0^\pi x \sin(x) dx = [\sin x - x \cos x]_0^\pi = \pi$$

$$\begin{aligned} M_x &= \int_0^\pi \frac{1}{2}(x \sin x)^2 dx \\ &= \frac{1}{2} \left( \frac{1}{6}\pi^3 - \frac{1}{4}\pi \right) \quad (\text{See part (a).}) \\ &= \frac{1}{12}\pi^3 - \frac{1}{8}\pi \end{aligned}$$

$$M_y = \int_0^\pi x(x \sin x) dx = \pi^2 - 4 \quad (\text{See part (b).})$$

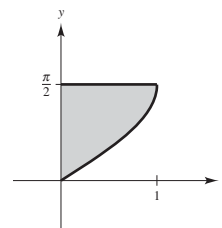
$$\bar{x} = \frac{M_y}{m} = \frac{\pi^2 - 4}{\pi} \approx 1.8684, \quad \bar{y} = \frac{M_x}{m} = \frac{(1/12)\pi^3 - (1/8)\pi}{\pi} = \frac{1}{2}\pi^2 - \frac{1}{8} \approx 0.6975$$



105. In Example 6, you showed that the centroid of an equivalent region was  $(1, \pi/8)$ . By symmetry, the centroid of this region is  $(\pi/8, 1)$ . You can also solve this problem directly.

$$\begin{aligned} A &= \int_0^1 \left( \frac{\pi}{2} - \arcsin x \right) dx = \left[ \frac{\pi}{2}x - x \arcsin x - \sqrt{1-x^2} \right]_0^1 \quad (\text{Example 3}) \\ &= \left( \frac{\pi}{2} - \frac{\pi}{2} - 0 \right) - (-1) = 1 \end{aligned}$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x \left( \frac{\pi}{2} - \arcsin x \right) dx = \frac{\pi}{8}, \quad \bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2} \left( \frac{\pi}{2} - \arcsin x \right) dx = 1$$

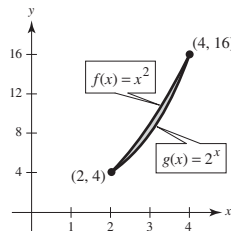


106.  $f(x) = x^2$ ,  $g(x) = 2^x$

$$f(2) = g(2) = 4, f(4) = g(4) = 16$$

$$m = \int_2^4 (x^2 - 2^x) dx = \left[ \frac{x^3}{3} - \frac{1}{\ln 2} 2^x \right]_2^4 = \left( \frac{64}{3} - \frac{16}{\ln 2} \right) - \left( \frac{8}{3} - \frac{4}{\ln 2} \right) = \frac{56}{3} - \frac{12}{\ln 2} \approx 1.3543$$

$$\begin{aligned} M_x &= \int_2^4 \frac{1}{2} (x^2 + 2^x) (x^2 - 2^x) dx \\ &= \frac{1}{2} \int_2^4 (x^4 - 2^{2x}) dx \\ &= \frac{1}{2} \left[ \frac{x^5}{5} - \frac{2^{2x}}{2 \ln 2} \right]_2^4 \\ &= \frac{1}{2} \left[ \left( \frac{1024}{5} - \frac{128}{\ln 2} \right) - \left( \frac{32}{5} - \frac{8}{\ln 2} \right) \right] \\ &= \frac{496}{5} - \frac{60}{\ln 2} \approx 12.6383 \end{aligned}$$



$$M_y = \int_2^4 x [x^2 - 2^x] dx = -\frac{56}{\ln 2} + \frac{12}{(\ln 2)^2} \approx 4.1855$$

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) \approx (3.0905, 9.3318)$$

107. Average value  $= \frac{1}{\pi} \int_0^\pi e^{-4t} (\cos 2t + 5 \sin 2t) dt$

$$\begin{aligned} &= \frac{1}{\pi} \left[ e^{-4t} \left( \frac{-4 \cos 2t + 2 \sin 2t}{20} \right) + 5e^{-4t} \left( \frac{-4 \sin 2t - 2 \cos 2t}{20} \right) \right]_0^\pi \quad (\text{From Exercises 93 and 94}) \\ &= \frac{7}{10\pi} (1 - e^{-4\pi}) \approx 0.223 \end{aligned}$$

108. (a) Average  $= \int_1^2 (1.6t \ln t + 1) dt = [0.8t^2 \ln t - 0.4t^2 + t]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average  $= \int_3^4 (1.6t \ln t + 1) dt = [0.8t^2 \ln t - 0.4t^2 + t]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

109.  $c(t) = 100,000 + 4000t$ ,  $r = 5\%$ ,  $t_1 = 10$

$$P = \int_0^{10} (100,000 + 4000t) e^{-0.05t} dt = 4000 \int_0^{10} (25 + t) e^{-0.05t} dt$$

Let  $u = 25 + t$ ,  $dv = e^{-0.05t} dt$ ,  $du = dt$ ,  $v = -\frac{100}{5} e^{-0.05t}$ .

$$P = 4000 \left\{ \left[ (25 + t) \left( -\frac{100}{5} e^{-0.05t} \right) \right]_0^{10} + \frac{100}{5} \int_0^{10} e^{-0.05t} dt \right\} = 4000 \left\{ \left[ (25 + t) \left( -\frac{100}{5} e^{-0.05t} \right) \right]_0^{10} - \left[ \frac{10,000}{25} e^{-0.05t} \right]_0^{10} \right\} \approx \$931,265$$

110.  $c(t) = 30,000 + 500t$ ,  $r = 7\%$ ,  $t_1 = 5$

$$P \int_0^5 (30,000 + 500t)e^{-0.07t} dt = 500 \int_0^5 (60 + t)e^{-0.07t} dt$$

Let  $u = 60 + t$ ,  $dv = e^{-0.07t} dt$ ,  $du = dt$ ,  $v = -\frac{100}{7}e^{-0.07t}$ .

$$P = 500 \left\{ \left[ (60 + t) \left( -\frac{100}{7} e^{-0.07t} \right) \right]_0^5 + \frac{100}{7} \int_0^5 e^{-0.07t} dt \right\} = 500 \left\{ \left[ (60 + t) \left( -\frac{100}{7} e^{-0.07t} \right) \right]_0^5 - \left[ \frac{10,000}{49} e^{-0.07t} \right]_0^5 \right\} \approx \$131,528.68$$

111.  $\int_{-\pi}^{\pi} x \sin nx dx = \left[ -\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_{-\pi}^{\pi} = -\frac{\pi}{n} \cos \pi n - \frac{\pi}{n} \cos(-\pi n) = -\frac{2\pi}{n} \cos \pi n = \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$

112.  $\int_{-\pi}^{\pi} x^2 \cos nx dx = \left[ \frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right]_{-\pi}^{\pi}$   
 $= \frac{2\pi}{n^2} \cos n\pi + \frac{2\pi}{n^2} \cos(-n\pi)$   
 $= \frac{4\pi}{n^2} \cos n\pi$   
 $= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$   
 $= \frac{(-1)^n 4\pi}{n^2}$

113. Let  $u = x$ ,  $dv = \sin\left(\frac{n\pi}{2}x\right) dx$ ,  $du = dx$ ,  $v = -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}x\right)$ .

$$I_1 = \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx = \left[ -\frac{2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos\left(\frac{n\pi}{2}x\right) dx$$

$$= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[ \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_0^1$$

$$= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

Let  $u = (-x + 2)$ ,  $dv = \sin\left(\frac{n\pi}{2}x\right) dx$ ,  $du = -dx$ ,  $v = -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}x\right)$ .

$$I_2 = \int_1^2 (-x + 2) \sin\left(\frac{n\pi}{2}x\right) dx = \left[ -\frac{2(-x + 2)}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx$$

$$= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \left[ \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_1^2$$

$$= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

$$h(I_1 + I_2) = b_n = h \left[ \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right] = \frac{8h}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

114. For any integrable function,  $\int f(x) dx = C + \int f(x) dx$ , but this cannot be used to imply that  $C = 0$ .

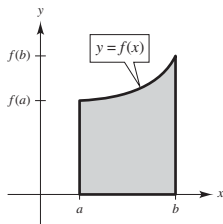
**115. Shell Method:**

$$V = 2\pi \int_a^b xf(x) dx$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

$$u = f(x) \Rightarrow du = f'(x) dx$$

$$\begin{aligned} V &= 2\pi \left[ \frac{x^2}{2} f(x) - \int_a^b \frac{x^2}{2} f'(x) dx \right]_a^b \\ &= \pi \left[ b^2 f(b) - a^2 f(a) - \int_a^b x^2 f'(x) dx \right] \end{aligned}$$

**Disk Method:**

$$\begin{aligned} V &= \pi \int_0^{f(a)} (b^2 - a^2) dy + \pi \int_{f(a)}^{f(b)} \left[ b^2 - [f^{-1}(y)]^2 \right] dy \\ &= \pi (b^2 - a^2) f(a) + \pi b^2 (f(b) - f(a)) \\ &\quad - \pi \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \\ &= \pi \left[ b^2 f(b) - a^2 f(a) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \right] \end{aligned}$$

Because  $x = f^{-1}(y)$ , you have  $f(x) = y$  and  $f'(x) dx = dy$ . When  $y = f(a)$ ,  $x = a$ . When  $y = f(b)$ ,  $x = b$ . So,

$\int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy = \int_a^b x^2 f'(x) dx$  and the volumes are the same.

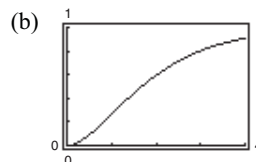
$$116. f'(x) = xe^{-x}$$

$$(a) f(x) = \int xe^{-x} dx = -xe^{-x} - e^{-x} + C$$

$$(\text{Parts: } u = x, dv = e^{-x} dx)$$

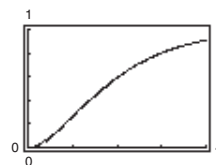
$$f(0) = 0 = -1 + C \Rightarrow C = 1$$

$$f(x) = -xe^{-x} - e^{-x} + 1$$



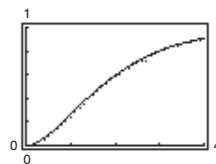
(c) Using  $h = 0.05$  you obtain the points:

$n$	$x_n$	$y_n$
0	0	0
1	0.05	0
2	0.10	$2.378 \times 10^{-3}$
3	0.15	0.0069
4	0.20	0.0134
$\vdots$	$\vdots$	$\vdots$
80	4.0	0.9064



(d) Using  $h = 0.1$  you obtain the points:

$n$	$x_n$	$y_n$
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
$\vdots$	$\vdots$	$\vdots$
40	4.0	0.9039



(e) The result in part (c) is better because  $h$  is smaller.



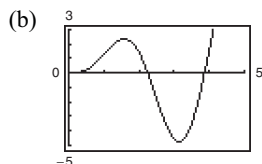
117.  $f'(x) = 3x \sin(2x)$ ,  $f(0) = 0$

(a)  $f(x) = \int 3x \sin 2x \, dx$   
 $= -\frac{3}{4}(2x \cos 2x - \sin 2x) + C$

(Parts:  $u = 3x$ ,  $dv = \sin 2x \, dx$ )

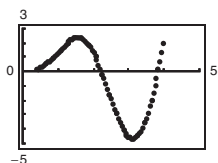
$f(0) = 0 = -\frac{3}{4}(0) + C \Rightarrow C = 0$

$f(x) = -\frac{3}{4}(2x \cos 2x - \sin 2x)$



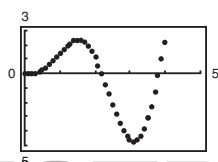
(c) Using  $h = 0.05$ , you obtain the points:

$n$	$x_n$	$y_n$
0	0	0
1	0.05	0
2	0.10	$7.4875 \times 10^{-4}$
3	0.15	0.0037
4	0.20	0.0104
$\vdots$	$\vdots$	$\vdots$
80	4.0	1.3181



(d) Using  $h = 0.1$ , you obtain the points:

$n$	$x_n$	$y_n$
0	0	0
1	0.1	0
2	0.2	0.0060
3	0.3	0.0293
4	0.4	0.0801
$\vdots$	$\vdots$	$\vdots$
40	4.0	1.0210



118.  $f'(x) = \cos \sqrt{x}$ ,  $f(0) = 1$

(a) Let  $w = \sqrt{x}$ ,  $w^2 = x$ ,  $2w \, dw = dx$ .

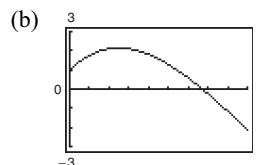
$\int \cos \sqrt{x} \, dx = \int \cos w (2w \, dw)$

Now use parts:  $u = 2w$ ,  $dv = \cos w \, dw$ .

$\int \cos \sqrt{x} \, dx = 2w \sin w + 2 \cos w + C$   
 $= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$

$f(0) = 1 = 2 + C \Rightarrow C = -1$

$f(x) = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} - 1$



(c) Using  $h = 0.05$ , you obtain the points:

$n$	$x_n$	$y_n$
0	0	1
1	0.05	1.05
2	0.1	1.0988
3	0.15	1.1463
4	0.2	1.1926
$\vdots$	$\vdots$	$\vdots$
80	4.0	1.8404

(d) Using  $h = 0.1$ , you obtain the points:

$n$	$x_n$	$y_n$
0	0	1
1	0.1	1.1
2	0.2	1.1950
3	0.3	1.2852
4	0.4	1.3706
$\vdots$	$\vdots$	$\vdots$
80	4.0	1.8759

119. On  $\left[0, \frac{\pi}{2}\right]$ ,  $\sin x \leq 1 \Rightarrow x \sin x \leq x \Rightarrow \int_0^{\pi/2} x \sin x \, dx \leq \int_0^{\pi/2} x \, dx$ .

120. (a)  $A = \int_0^{\pi} x \sin x \, dx = [\sin x - x \cos x]_0^{\pi} = \pi$

(b)  $\int_{\pi}^{2\pi} x \sin x \, dx = [\sin x - x \cos x]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$   
 $A = 3\pi$

(c)  $\int_{2\pi}^{3\pi} x \sin x \, dx = [\sin x - x \cos x]_{2\pi}^{3\pi} = 3\pi + 2\pi = 5\pi$   
 $A = 5\pi$

The area between  $y = x \sin x$  and  $y = 0$  on  $[n\pi, (n+1)\pi]$  is  $(2n+1)\pi$ :

$$\int_{n\pi}^{(n+1)\pi} x \sin x \, dx = [\sin x - x \cos x]_{n\pi}^{(n+1)\pi} = \pm(n+1)\pi \pm n\pi = \pm(2n+1)\pi$$

$$A = |\pm(2n+1)\pi| = (2n+1)\pi$$

### Section 8.3 Trigonometric Integrals

1.  $y = \sec x$

$$y' = \sec x \tan x = \sin x \sec^2 x$$

$$\int \sin x \sec^2 x \, dx = \sec x + C$$

Matches (c)

2.  $y = \cos x + \sec x$

$$y' = -\sin x + \sec x \tan x$$

$$= -\sin x + \sin x \sec^2 x$$

$$= -\sin x(1 - \sec^2 x)$$

$$= \sin x \tan^2 x$$

$$\int \sin x \tan^2 x \, dx = \cos x + \sec x + C$$

Matches (a)

4.  $y = 3x + 2 \sin x \cos^3 x + 3 \sin x \cos x$

$$y' = 3 + 2 \cos^4 x - 6 \sin^2 x \cos^2 x + 3 \cos^2 x - 3 \sin^2 x$$

$$= 3 + 2 \cos^4 x - 6 \cos^2 x(1 - \cos^2 x) + 3 \cos^2 x - 3(1 - \cos^2 x) = 8 \cos^4 x$$

$$\int 8 \cos^4 x \, dx = 3x + 2 \sin x \cos^3 x + 3 \sin x \cos x + C$$

Matches (b)

5. Let  $u = \cos x$ ,  $du = -\sin x \, dx$ .

$$\int \cos^5 x \sin x \, dx = -\int \cos^5 x (-\sin x) \, dx = -\frac{\cos^6 x}{6} + C$$

6.  $\int \cos^3 x \sin^4 x \, dx = \int \cos x(1 - \sin^2 x) \sin^4 x \, dx = \int (\sin^4 x - \sin^6 x) \cos x \, dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

3.  $y = x - \tan x + \frac{1}{3} \tan^3 x$

$$y' = 1 - \sec^2 x + \tan^2 x(\sec^2 x)$$

$$= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$$

$$= \tan^4 x$$

$$\int \tan^4 x \, dx = x - \tan x + \frac{1}{3} \tan^3 x + C$$

Matches (d)

7. Let  $u = \sin 2x$ ,  $du = 2 \cos 2x \, dx$ .

$$\begin{aligned}\int \sin^7 2x \cos 2x \, dx &= \frac{1}{2} \int \sin^6 2x (2 \cos 2x) \, dx \\ &= \frac{1}{2} \frac{\sin^8 2x}{8} + C \\ &= \frac{1}{16} \sin^8 2x + C\end{aligned}$$

8. Let  $u = \cos x$ ,  $du = -\sin x \, dx$ .

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin x (1 - \cos^2 x) \, dx \\ &= \int \cos^2 x (-\sin x) \, dx + \int \sin x \, dx \\ &= \frac{1}{3} \cos^3 x - \cos x + C\end{aligned}$$

$$\begin{aligned}9. \int \sin^3 x \cos^2 x \, dx &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\ &= \int (\cos^2 x - \cos^4 x) \sin x \, dx \\ &= -\int (\cos^2 x - \cos^4 x) (-\sin x) \, dx \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C\end{aligned}$$

10. Let  $u = \sin \frac{x}{3}$ ,  $du = \frac{1}{3} \cos \frac{x}{3} \, dx$ .

$$\begin{aligned}\int \cos^3 \frac{x}{3} \, dx &= \int \left( \cos \frac{x}{3} \right) \left( 1 - \sin^2 \frac{x}{3} \right) \, dx \\ &= 3 \int \left( 1 - \sin^2 \frac{x}{3} \right) \left( \frac{1}{3} \cos \frac{x}{3} \right) \, dx \\ &= 3 \left( \sin \frac{x}{3} - \frac{1}{3} \sin^3 \frac{x}{3} \right) + C \\ &= 3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C\end{aligned}$$

$$\begin{aligned}11. \int \sin^3 2\theta \sqrt{\cos 2\theta} \, d\theta &= \int (1 - \cos^2 2\theta) \sqrt{\cos 2\theta} \sin 2\theta \, d\theta \\ &= \int [(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}] \sin 2\theta \, d\theta \\ &= -\frac{1}{2} \int [(\cos 2\theta)^{1/2} - (\cos 2\theta)^{5/2}] (-2 \sin 2\theta) \, d\theta \\ &= -\frac{1}{2} \left[ \frac{2}{3} (\cos 2\theta)^{3/2} - \frac{2}{7} (\cos 2\theta)^{7/2} \right] + C \\ &= -\frac{1}{3} (\cos 2\theta)^{3/2} + \frac{1}{7} (\cos 2\theta)^{7/2} + C\end{aligned}$$

$$\begin{aligned}12. \int \frac{\cos^5 t}{\sqrt{\sin t}} \, dt &= \int \cos t (1 - \sin^2 t)^2 (\sin t)^{-1/2} \, dt \\ &= \int (1 - 2 \sin^2 t + \sin^4 t) (\sin t)^{-1/2} \cos t \, dt \\ &= \int [(\sin t)^{-1/2} - 2(\sin t)^{3/2} + (\sin t)^{7/2}] \cos t \, dt \\ &= 2\sqrt{\sin t} - \frac{4}{5} (\sin t)^{5/2} + \frac{2}{9} (\sin t)^{9/2} + C\end{aligned}$$

$$13. \int \cos^2 3x \, dx = \int \frac{1 + \cos 6x}{2} \, dx = \frac{1}{2} \left( x + \frac{1}{6} \sin 6x \right) + C = \frac{1}{12} (6x + \sin 6x) + C$$

$$14. \int \sin^2 5x \, dx = \int \frac{1 - \cos 10x}{2} \, dx = \frac{1}{2} \left( x - \frac{\sin 10x}{10} \right) + C = \frac{x}{2} - \frac{\sin 10x}{20}$$

$$\begin{aligned}15. \int \cos^4 3\alpha \, d\alpha &= \int \left( \frac{1 + \cos 6\alpha}{2} \right) \left( \frac{1 + \cos 6\alpha}{2} \right) \, d\alpha \\ &= \frac{1}{4} \int (1 + 2 \cos 6\alpha + \cos^2 6\alpha) \, d\alpha \\ &= \frac{1}{4} \int \left( 1 + 2 \cos 6\alpha + \frac{1 + \cos 12\alpha}{2} \right) \, d\alpha \\ &= \frac{1}{4} \left( \frac{3}{2} \alpha + \frac{\sin 6\alpha}{3} + \frac{\sin 12\alpha}{24} \right) + C \\ &= \frac{3}{8} \alpha + \frac{\sin 6\alpha}{12} + \frac{\sin 12\alpha}{96} + C\end{aligned}$$

$$\begin{aligned}
 16. \int \sin^4 6\theta \, d\theta &= \int \left( \frac{1 - \cos 12\theta}{2} \right) \left( \frac{1 - \cos 12\theta}{2} \right) d\theta \\
 &= \frac{1}{4} \int (1 - 2\cos 12\theta + \cos^2 12\theta) \, d\theta \\
 &= \frac{1}{4} \int \left( 1 - 2\cos 12\theta + \frac{1 + \cos 24\theta}{2} \right) d\theta \\
 &= \frac{1}{4} \int \left( \frac{3}{2} - 2\cos 12\theta + \frac{1}{2} \cos 24\theta \right) d\theta \\
 &= \frac{1}{4} \left( \frac{3}{2}\theta - \frac{1}{6} \sin 2\theta + \frac{1}{48} \sin 24\theta \right) + C = \frac{3}{8}\theta - \frac{1}{24} \sin 12\theta + \frac{1}{192} \sin 24\theta + C
 \end{aligned}$$

17. Integration by parts:

$$\begin{aligned}
 dv &= \sin^2 x \, dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x) \\
 u &= x \Rightarrow du = dx \\
 \int x \sin^2 x \, dx &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \int (2x - \sin 2x) \, dx \\
 &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \left( x^2 + \frac{1}{2} \cos 2x \right) + C = \frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C
 \end{aligned}$$

18. Use integration by parts twice.

$$\begin{aligned}
 dv &= \sin^2 x \, dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x) \\
 u &= x^2 \Rightarrow du = 2x \, dx \\
 dv &= \sin 2x \, dx \Rightarrow v = -\frac{1}{2} \cos 2x \\
 u &= x \Rightarrow du = dx \\
 \int x^2 \sin^2 x \, dx &= \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2} \int (2x^2 - x \sin 2x) \, dx \\
 &= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{3}x^3 + \frac{1}{2} \int x \sin 2x \, dx \\
 &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x + \frac{1}{2} \left( -\frac{1}{2}x \cos 2x + \frac{1}{2} \int \cos 2x \, dx \right) \\
 &= \frac{1}{6}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{1}{4}x \cos 2x + \frac{1}{8} \sin 2x + C \\
 &= \frac{1}{24}(4x^3 - 6x^2 \sin 2x - 6x \cos 2x + 3 \sin 2x) + C
 \end{aligned}$$

$$19. \int_0^{\pi/2} \cos^7 x \, dx = \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) \left( \frac{6}{7} \right) = \frac{16}{35}, (n = 7)$$

$$23. \int_0^{\pi/2} \sin^6 x \, dx = \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) \left( \frac{5}{6} \right) \frac{\pi}{2} = \frac{5\pi}{32}, (n = 6)$$

$$20. \int_0^{\pi/2} \cos^9 x \, dx = \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) \left( \frac{6}{7} \right) \left( \frac{8}{9} \right) = \frac{128}{315}, (n = 9)$$

$$24. \int_0^{\pi/2} \sin^8 x \, dx = \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) \left( \frac{5}{6} \right) \left( \frac{7}{8} \right) \left( \frac{\pi}{2} \right) = \frac{35\pi}{256}, (n = 8)$$

$$\begin{aligned}
 21. \int_0^{\pi/2} \cos^{10} x \, dx &= \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) \left( \frac{5}{6} \right) \left( \frac{7}{8} \right) \left( \frac{9}{10} \right) \left( \frac{\pi}{2} \right) \\
 &= \frac{63}{512} \pi, (n = 10)
 \end{aligned}$$

$$\begin{aligned}
 25. \int \sec 7x \, dx &= \frac{1}{7} \int \sec(7x) \, 7 \, dx \\
 &= \frac{1}{7} \ln |\sec 7x + \tan 7x| + C
 \end{aligned}$$

$$22. \int_0^{\pi/2} \sin^5 x \, dx = \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) = \frac{8}{15}, (n = 5)$$

$$26. \int \sec^2(2x - 1) \, dx = \frac{1}{2} \tan(2x - 1) + C$$

$$27. \int \sec^4 5x \, dx = \int (1 + \tan^2 5x) \sec^2 5x \, dx = \frac{1}{5} \left( \tan 5x + \frac{\tan^3 5x}{3} \right) + C = \frac{\tan 5x}{15} (3 + \tan^2 5x) + C$$

$$28. \int \sec^6 3x \, dx = \int (1 + \tan^2 3x)^2 \sec^2 3x \, dx = \int (1 + 2 \tan^2 3x + \tan^4 3x) \sec^2 3x \, dx = \frac{1}{3} \tan 3x + \frac{2}{9} \tan^3 3x + \frac{1}{15} \tan^5 3x + C$$

$$29. \, dv = \sec^2 \pi x \, dx \Rightarrow v = \frac{1}{\pi} \tan \pi x$$

$$u = \sec \pi x \Rightarrow du = \pi \sec \pi x \tan \pi x \, dx$$

$$\int \sec^3 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x (\sec^2 \pi x - 1) \, dx$$

$$2 \int \sec^3 \pi x \, dx = \frac{1}{\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x \, dx = \frac{1}{2\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C$$

$$\begin{aligned} 30. \int \tan^5 x \, dx &= \int (\sec^2 x - 1) \tan^3 x \, dx \\ &= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx \\ &= \frac{\tan^4 x}{4} - \int (\sec^2 x - 1) \tan x \, dx \\ &= \frac{\tan^4 x}{4} - \int \tan x \sec^2 x \, dx + \int \tan x \, dx \\ &= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln |\cos x| + C \end{aligned}$$

$$32. \int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx = \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$$

$$\begin{aligned} 33. \, u &= \tan x, \, du = \sec^2 x \, dx \\ \int \sec^2 x \tan x \, dx &= \frac{1}{2} \tan^2 x + C \\ \text{or, } u &= \sec x, \, du = \sec x \tan x \, dx, \\ \int \sec^2 x \tan x \, dx &= \frac{1}{2} \sec^2 x + C \end{aligned}$$

$$\begin{aligned} 31. \int \tan^5 \frac{x}{2} \, dx &= \int \left( \sec^2 \frac{x}{2} - 1 \right) \tan^3 \frac{x}{2} \, dx \\ &= \int \tan^3 \frac{x}{2} \sec^2 \frac{x}{2} \, dx - \int \tan^3 \frac{x}{2} \, dx \\ &= \frac{\tan^4 \frac{x}{2}}{2} - \int \left( \sec^2 \frac{x}{2} - 1 \right) \tan \frac{x}{2} \, dx \\ &= \frac{1}{2} \tan^4 \frac{x}{2} - \tan^2 \frac{x}{2} - 2 \ln \left| \cos \frac{x}{2} \right| + C \end{aligned}$$

$$34. \text{ Let } u = \sec 2t, \, du = 2 \sec 2t \tan 2t \, dt.$$

$$\begin{aligned} \int \tan^3 2t \cdot \sec^3 2t \, dt &= \int (\sec^2 2t - 1) \sec^3 2t \cdot \tan 2t \, dt \\ &= \int (\sec^4 2t - \sec^2 2t) (\sec 2t \tan 2t) \, dt = \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C \end{aligned}$$

$$\begin{aligned} 35. \int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx \\ &= \int \tan^4 x \sec^2 x \, dx + \int \tan^2 x \sec^2 x \, dx = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C \end{aligned}$$

$$\begin{aligned} 36. \int \tan^5 2x \sec^4 2x \, dx &= \int \tan^5 2x (\tan^2 2x + 1) \sec^2 2x \, dx \\ &= \int \tan^7 2x \sec^2 2x \, dx + \int \tan^5 2x \sec^2 2x \, dx \\ &= \frac{1}{2} \frac{\tan^8 2x}{8} + \frac{1}{2} \frac{\tan^6 2x}{6} + C \\ &= \frac{\tan^8 2x}{16} + \frac{\tan^6 2x}{12} + C \end{aligned}$$

$$\begin{aligned}
 37. \int \sec^6 4x \tan 4x \, dx &= \frac{1}{4} \int \sec^5 4x (4 \sec 4x \tan 4x) \, dx \\
 &= \frac{\sec^6 4x}{24} + C
 \end{aligned}$$

$$\begin{aligned}
 38. \int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx &= 2 \int \sec^2 \frac{x}{2} \left( \frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} \right) dx \\
 &= \sec^2 \frac{x}{2} + C \quad \text{or} \\
 \int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx &= 2 \int \tan \frac{x}{2} \left( \frac{1}{2} \sec^2 \frac{x}{2} \right) dx = \tan^2 \frac{x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 39. \int \sec^5 x \tan^3 x \, dx &= \int \sec^4 x \tan^2 x (\sec x \tan x) \, dx \\
 &= \int \sec^4 x (\sec^2 x - 1) (\sec x \tan x) \, dx \\
 &= \int (\sec^6 x - \sec^4 x) (\sec x \tan x) \, dx \\
 &= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C
 \end{aligned}$$

$$\begin{aligned}
 40. \int \tan^3 3x \, dx &= \int (\sec^2 3x - 1) \tan 3x \, dx \\
 &= \frac{1}{3} \int \tan 3x (3 \sec^2 3x) \, dx + \frac{1}{3} \int \frac{-3 \sin 3x}{\cos 3x} \, dx \\
 &= \frac{1}{6} \tan^2 3x + \frac{1}{3} \ln |\cos 3x| + C
 \end{aligned}$$

$$\begin{aligned}
 41. \int \frac{\tan^2 x}{\sec x} \, dx &= \int \frac{(\sec^2 x - 1)}{\sec x} \, dx \\
 &= \int (\sec x - \cos x) \, dx \\
 &= \ln |\sec x + \tan x| - \sin x + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{\tan^2 x}{\sec^5 x} \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x \, dx \\
 &= \int \sin^2 x \cdot \cos^3 x \, dx \\
 &= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx \\
 &= \int (\sin^2 x - \sin^4 x) \cos x \, dx \\
 &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C
 \end{aligned}$$

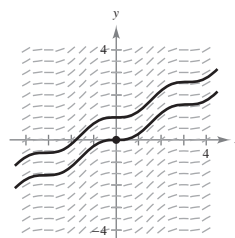
$$\begin{aligned}
 43. r &= \int \sin^4(\pi\theta) \, d\theta = \frac{1}{4} \int [1 - \cos(2\pi\theta)]^2 \, d\theta \\
 &= \frac{1}{4} \int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)] \, d\theta \\
 &= \frac{1}{4} \int \left[ 1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2} \right] \, d\theta \\
 &= \frac{1}{4} \left[ \theta - \frac{1}{\pi} \sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi} \sin(4\pi\theta) \right] + C \\
 &= \frac{1}{32\pi} [12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C
 \end{aligned}$$

$$\begin{aligned}
 44. s &= \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \, d\alpha \\
 &= \int \left( \frac{1 - \cos \alpha}{2} \right) \left( \frac{1 + \cos \alpha}{2} \right) \, d\alpha = \int \frac{1 - \cos^2 \alpha}{4} \, d\alpha \\
 &= \frac{1}{4} \int \sin^2 \alpha \, d\alpha = \frac{1}{8} \int (1 - \cos 2\alpha) \, d\alpha \\
 &= \frac{1}{8} \left( \theta - \frac{\sin 2\alpha}{2} \right) + C \\
 &= \frac{1}{16} (2\alpha - \sin 2\alpha) + C
 \end{aligned}$$

$$\begin{aligned}
 45. y &= \int \tan^3 3x \sec 3x \, dx \\
 &= \int (\sec^2 3x - 1) \sec 3x \tan 3x \, dx \\
 &= \frac{1}{3} \int \sec^2 3x (3 \sec 3x \tan 3x) \, dx - \frac{1}{3} \int 3 \sec 3x \tan 3x \, dx \\
 &= \frac{1}{9} \sec^3 3x - \frac{1}{3} \sec 3x + C
 \end{aligned}$$

$$\begin{aligned}
 46. y &= \int \sqrt{\tan x} \sec^4 x \, dx \\
 &= \int \tan^{1/2} x (\tan^2 x + 1) \sec^2 x \, dx \\
 &= \int (\tan^{5/2} x + \tan^{1/2} x) \sec^2 x \, dx \\
 &= \frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + C
 \end{aligned}$$

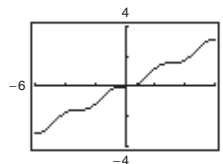
47. (a)



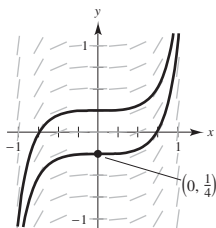
$$(b) \frac{dy}{dx} = \sin^2 x, \quad (0, 0)$$

$$\begin{aligned}
 y &= \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \\
 &= \frac{1}{2}x - \frac{\sin 2x}{4} + C
 \end{aligned}$$

$$(0, 0): 0 = C, \quad y = \frac{1}{2}x - \frac{\sin 2x}{4}$$



48. (a)

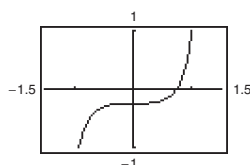


(b)  $\frac{dy}{dx} = \sec^2 x \tan^2 x, \quad \left(0, -\frac{1}{4}\right)$

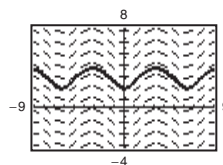
$$y = \int \sec^2 x \tan^2 x \, dx \quad u = \tan x, \, du = \sec^2 x \, dx$$

$$y = \frac{\tan^3 x}{3} + C$$

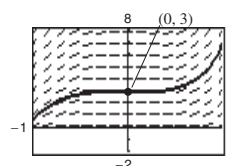
$$\left(0, -\frac{1}{4}\right): -\frac{1}{4} = C \Rightarrow y = \frac{1}{3} \tan^3 x - \frac{1}{4}$$



49.  $\frac{dy}{dx} = \frac{3 \sin x}{y}, \, y(0) = 2$



50.  $\frac{dy}{dx} = 3\sqrt{y} \tan^2 x, \, y(0) = 3$



$$\begin{aligned} 51. \int \cos 2x \cos 6x \, dx &= \frac{1}{2} \int [\cos((2-6)x) + \cos((2+6)x)] \, dx \\ &= \frac{1}{2} \int [\cos(-4x) + \cos 8x] \, dx \\ &= \frac{1}{2} \int (\cos 4x + \cos 8x) \, dx \\ &= \frac{1}{2} \left[ \frac{\sin 4x}{4} + \frac{\sin 8x}{8} \right] + C \\ &= \frac{\sin 4x}{8} + \frac{\sin 8x}{16} + C \\ &= \frac{1}{16} (2 \sin 4x + \sin 8x) + C \end{aligned}$$

$$\begin{aligned} 52. \int \cos 4\theta \cos(-3\theta) \, d\theta &= \int \cos 4\theta \cos 3\theta \, d\theta = \frac{1}{2} \int (\cos 7\theta + \cos \theta) \, d\theta \\ &= \frac{\sin 7\theta}{14} + \frac{\sin \theta}{2} + C = \frac{1}{14} (\sin 7\theta + 7 \sin \theta) + C \end{aligned}$$

$$\begin{aligned} 53. \int \sin 2x \cos 4x \, dx &= \frac{1}{2} \int [\sin((2-4)x) + \sin((2+4)x)] \, dx \\ &= \frac{1}{2} \int (\sin(-2x) + \sin 6x) \, dx \\ &= \frac{1}{2} \int (-\sin 2x + \sin 6x) \, dx \\ &= \frac{1}{2} \left[ \frac{\cos 2x}{2} - \frac{\cos 6x}{6} \right] + C \\ &= \frac{1}{4} \cos 2x - \frac{1}{12} \cos 6x + C \\ &= \frac{1}{12} (3 \cos 2x - \cos 6x) + C \end{aligned}$$

$$\begin{aligned}
 54. \int \sin(-4x) \cos 3x \, dx &= -\int \sin 4x \cos 3x \, dx \\
 &= -\frac{1}{2} \int (\sin x + \sin 7x) \, dx \\
 &= -\frac{1}{2} \left( -\cos x - \frac{1}{7} \cos 7x \right) + C \\
 &= \frac{1}{14} (7 \cos x + \cos 7x) + C
 \end{aligned}$$

$$\begin{aligned}
 55. \int \sin \theta \sin 3\theta \, d\theta &= \frac{1}{2} \int (\cos 2\theta - \cos 4\theta) \, d\theta \\
 &= \frac{1}{2} \left( \frac{1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right) + C \\
 &= \frac{1}{8} (2 \sin 2\theta - \sin 4\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 56. \int \sin 5x \sin 4x \, dx &= \frac{1}{2} \int (\cos x - \cos 9x) \, dx \\
 &= \frac{1}{2} \left( \sin x - \frac{\sin 9x}{9} \right) + C \\
 &= \frac{\sin x}{2} - \frac{\sin 9x}{18} + C \\
 &= \frac{1}{18} (9 \sin x - \sin 9x) + C
 \end{aligned}$$

$$\begin{aligned}
 57. \int \cot^3 2x \, dx &= \int (\csc^2 2x - 1) \cot 2x \, dx \\
 &= -\frac{1}{2} \int \cot 2x (-2 \csc^2 2x) \, dx - \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} \, dx \\
 &= -\frac{1}{4} \cot^2 2x - \frac{1}{2} \ln |\sin 2x| + C \\
 &= \frac{1}{4} (\ln |\csc^2 2x| - \cot^2 2x) + C
 \end{aligned}$$

$$58. \text{ Let } u = \tan \frac{x}{2}, \, du = \frac{1}{2} \sec^2 \frac{x}{2} \, dx.$$

$$\begin{aligned}
 \int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} \, dx &= \int \tan^4 \frac{x}{2} \left( \tan^2 \frac{x}{2} + 1 \right) \sec^2 \frac{x}{2} \, dx \\
 &= 2 \int \left( \tan^6 \frac{x}{2} + \tan^4 \frac{x}{2} \right) \left( \frac{1}{2} \sec^2 \frac{x}{2} \right) \, dx \\
 &= \frac{2}{7} \tan^7 \frac{x}{2} + \frac{2}{5} \tan^5 \frac{x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 59. \int \csc^4 2x \, dx &= \int \csc^2 2x (1 + \cot^2 2x) \, dx \\
 &= \int \csc^2 2x \, dx + \int \cot^2 2x \csc^2 2x \, dx \\
 &= -\frac{1}{2} \cot 2x - \frac{\cot^3 2x}{6} + C
 \end{aligned}$$

$$\begin{aligned}
 60. \int \cot^3 x \csc^3 x \, dx &= \int \cot^2 x \csc^2 x (\csc x \cot x) \, dx \\
 &= \int (\csc^2 x - 1) \csc^2 x (\csc x \cot x) \, dx \\
 &= \int (\csc^4 x - \csc^2 x) (\csc x \cot x) \, dx \\
 &= -\frac{\csc^5 x}{5} + \frac{\csc^3 x}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 61. \int \frac{\cot^2 t}{\csc t} \, dt &= \int \frac{\csc^2 t - 1}{\csc t} \, dt \\
 &= \int (\csc t - \sin t) \, dt \\
 &= \ln |\csc t - \cot t| + \cos t + C
 \end{aligned}$$

$$\begin{aligned}
 62. \int \frac{\cot^3 t}{\csc t} \, dt &= \int \frac{\cos^3 t}{\sin^2 t} \, dt = \int \frac{(1 - \sin^2 t) \cos t}{\sin^2 t} \, dt \\
 &= \int \frac{\cos t}{\sin^2 t} \, dt - \int \cos t \, dt \\
 &= \frac{-1}{\sin t} - \sin t + C = -\csc t - \sin t + C
 \end{aligned}$$



$$\begin{aligned} 63. \int \frac{1}{\sec x \tan x} dx &= \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx \\ &= \int (\csc x - \sin x) dx \\ &= \ln |\csc x - \cot x| + \cos x + C \end{aligned}$$

$$\begin{aligned} 64. \int \frac{\sin^2 x - \cos^2 x}{\cos x} dx &= \int \frac{1 - 2 \cos^2 x}{\cos x} dx \\ &= \int (\sec x - 2 \cos x) dx \\ &= \ln |\sec x + \tan x| - 2 \sin x + C \end{aligned}$$

$$\begin{aligned} 65. \int (\tan^4 t - \sec^4 t) dt &= \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t) dt, \quad (\tan^2 t - \sec^2 t = -1) \\ &= -\int (\tan^2 t + \sec^2 t) dt = -\int (2 \sec^2 t - 1) dt = -2 \tan t + t + C \end{aligned}$$

$$66. \int \frac{1 - \sec t}{\cos t - 1} dt = \int \frac{\cos t - 1}{(\cos t - 1) \cos t} dt = \int \sec t dt = \ln |\sec t + \tan t| + C$$

$$\begin{aligned} 67. \int_{-\pi}^{\pi} \sin^2 x dx &= 2 \int_0^{\pi} \frac{1 - \cos 2x}{2} dx \\ &= \left[ x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \pi \end{aligned}$$

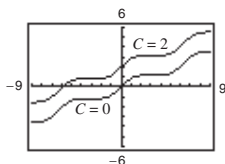
$$\begin{aligned} 68. \int_0^{\pi/3} \tan^2 x dx &= \int_0^{\pi/3} (\sec^2 x - 1) dx \\ &= [\tan x - x]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} 69. \int_0^{\pi/4} 6 \tan^3 x dx &= 6 \int_0^{\pi/4} (\sec^2 x - 1) \tan x dx \\ &= 6 \int_0^{\pi/4} [\tan x \sec^2 x - \tan x] dx \\ &= 6 \left[ \frac{\tan^2 x}{2} + \ln |\cos x| \right]_0^{\pi/4} \\ &= 6 \left[ \frac{1}{2} + \ln \left( \frac{\sqrt{2}}{2} \right) \right] = 6 \left( \frac{1}{2} - \ln \sqrt{2} \right) \\ &= 3(1 - \ln 2) \end{aligned}$$

70. Let  $u = \tan t$ ,  $du = \sec^2 t dt$ .

$$\int_0^{\pi/4} \sec^2 t \sqrt{\tan t} dt = \left[ \frac{2}{3} \tan^{3/2} t \right]_0^{\pi/4} = \frac{2}{3}$$

$$\begin{aligned} 75. \int \cos^4 \frac{x}{2} dx &= \frac{1}{16} (6x + 8 \sin x + \sin 2x) + C \\ &= \frac{1}{8} \left( 4 \sin \frac{x}{2} \cos^3 \frac{x}{2} + 6 \sin \frac{x}{2} \cos \frac{x}{2} + 3x \right) + C \end{aligned}$$



71. Let  $u = 1 + \sin t$ ,  $du = \cos t dt$ .

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t} dt = [\ln |1 + \sin t|]_0^{\pi/2} = \ln 2$$

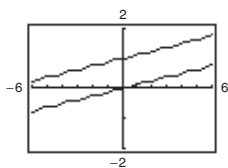
$$\begin{aligned} 72. \int_{-\pi}^{\pi} \sin 5x \cos 3x dx &= \frac{1}{2} \int_{-\pi}^{\pi} (\sin 2x + \sin 8x) dx \\ &= \frac{1}{2} \left[ -\frac{\cos 2x}{2} - \frac{\cos 8x}{8} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2} \left[ \left( -\frac{1}{2} - \frac{1}{8} \right) - \left( -\frac{1}{2} - \frac{1}{8} \right) \right] = 0 \end{aligned}$$

**Note:**  $f(x) = \sin 5x \cos 3x$  is odd, so  $\int_{-\pi}^{\pi} f(x) dx = 0$

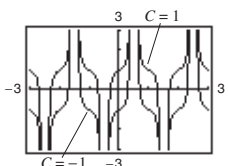
$$\begin{aligned} 73. \int_{-\pi/2}^{\pi/2} 3 \cos^3 x dx &= 3 \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) \cos x dx \\ &= 3 \left[ \sin x + \frac{\sin^3 x}{3} \right]_{-\pi/2}^{\pi/2} \\ &= 3 \left[ \left( 1 + \frac{1}{3} \right) - \left( -1 - \frac{1}{3} \right) \right] = 4 \end{aligned}$$

$$\begin{aligned} 74. \int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) dx &= \int_{-\pi/2}^{\pi/2} \left( \frac{1 - \cos 2x}{2} + 1 \right) dx \\ &= \int_{-\pi/2}^{\pi/2} \left( \frac{3}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left[ \frac{3}{2}x - \frac{1}{4} \sin 2x \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2} \end{aligned}$$

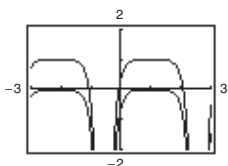
$$76. \int \sin^2 x \cos^2 x \, dx = \frac{1}{32}(4x - \sin 4x) + C$$



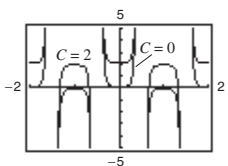
$$77. \int \sec^5 \pi x \, dx = \frac{1}{4\pi} \left\{ \sec^3 \pi x \tan \pi x + \frac{3}{2} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) \right\} + C$$



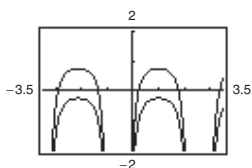
$$78. \int \tan^3(1-x) \, dx = -\frac{\tan^2(1-x)}{2} - \ln |\cos(1-x)| + C$$



$$79. \int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi} \sec^5 \pi x + C$$



$$80. \int \sec^4(1-x) \tan(1-x) \, dx = -\frac{\sec^4(1-x)}{4} + C$$



$$81. \int_0^{\pi/4} \sin 3\theta \sin 4\theta \, d\theta$$

$$= \left[ \frac{1}{2} \sin \theta - \frac{1}{14} \sin 7\theta \right]_0^{\pi/4} = \frac{2\sqrt{2}}{7}$$

$$82. \int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta = \left[ \frac{3}{2}\theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi}{4} - 2$$

$$83. \int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4} \left[ \frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right]_0^{\pi/2} = \frac{3\pi}{16}$$

$$84. \int_0^{\pi/2} \sin^{12} x \, dx = \frac{231\pi}{2048}$$

85. (a) Save one sine factor and convert the remaining factors to cosines. Then expand and integrate.  
 (b) Save one cosine factor and convert the remaining factors to sines. Then expand and integrate.  
 (c) Make repeated use of the power reducing formulas to convert the integrand to odd powers of the cosine. Then proceed as in part (b).

86. (a) Save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.  
 (b) Save a secant-squared factor and convert the remaining factors to secants. Then expand and integrate.  
 (c) Convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.  
 (d) Use integration by parts.

$$87. (a) \int \sin x \cos x \, dx = \frac{\sin^2 x}{2} + C$$

$$(b) -\int \cos x (-\sin x) \, dx = -\frac{\cos^2 x}{2} + C$$

$$(c) dv = \cos x \, dx \Rightarrow v = \sin x$$

$$u = \sin x \Rightarrow du = \cos x \, dx$$

$$\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx$$

$$2 \int \sin x \cos x \, dx = \sin^2 x$$

$$\int \sin x \cos x \, dx = \frac{\sin^2 x}{2} + C$$

(Answers will vary)

$$(d) \int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx = -\frac{1}{4} \cos 2x + C$$

The answers all differ by a constant.

88. (a) The second one is more difficult.

The first one is easy:

$$\int \sin^{372} x \cos x \, dx = \frac{\sin^{373} x}{373} + C$$

- (b) The second one is more difficult.

The first one is easy:

$$\int \tan^{400} x \sec^2 x \, dx = \frac{\tan^{401} x}{401} + C$$

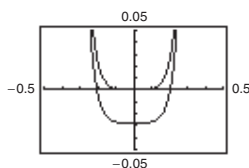
89. (a) Let  $u = \tan 3x$ ,  $du = 3 \sec^2 3x \, dx$ .

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx = \frac{1}{3} \int (\tan^2 3x + 1) \tan^3 3x (3 \sec^2 3x) \, dx \\ &= \frac{1}{3} \int (\tan^5 3x + \tan^3 3x) (3 \sec^2 3x) \, dx = \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1 \end{aligned}$$

Or let  $u = \sec 3x$ ,  $du = 3 \sec 3x \tan 3x \, dx$ .

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx \\ &= \frac{1}{3} \int \sec^3 3x (\sec^2 3x - 1) (3 \sec 3x \tan 3x) \, dx = \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C \end{aligned}$$

- (b)



$$\begin{aligned} \text{(c)} \quad \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C \\ &= \frac{1}{18} \tan^6 3x + \frac{1}{6} \tan^4 3x + \frac{1}{6} \tan^2 3x + \frac{1}{18} - \frac{1}{12} \tan^4 3x - \frac{1}{6} \tan^2 3x - \frac{1}{12} + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left( \frac{1}{18} - \frac{1}{12} \right) + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2 \end{aligned}$$

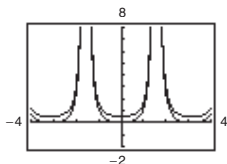
90. (a) Let  $u = \tan x$ ,  $du = \sec^2 x \, dx$ .

$$\int \sec^2 x \tan x \, dx = \frac{1}{2} \tan^2 x + C_1$$

Or let  $u = \sec x$ ,  $du = \sec x \tan x \, dx$ .

$$\int \sec x (\sec x \tan x) \, dx = \frac{1}{2} \sec^2 x + C$$

- (b)



$$\begin{aligned} \text{(c)} \quad \frac{1}{2} \sec^2 x + C &= \frac{1}{2} (\tan^2 x + 1) + C \\ &= \frac{1}{2} \tan^2 x + \left( \frac{1}{2} + C \right) \\ &= \frac{1}{2} \tan^2 x + C_2 \end{aligned}$$

$$\begin{aligned} \text{91. } A &= \int_0^{\pi/2} (\sin x - \sin^3 x) \, dx \\ &= \int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} \sin^3 x \, dx \\ &= [-\cos x]_0^{\pi/2} - \frac{2}{3} \quad (\text{Wallis's Formula}) \\ &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

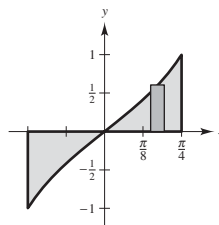
$$\begin{aligned} \text{92. } A &= \int_0^1 \sin^2(\pi x) \, dx \\ &= \int_0^1 \frac{1 - \cos(2\pi x)}{2} \, dx \\ &= \left[ \frac{1}{2}x - \frac{\sin 2\pi x}{4\pi} \right]_0^1 = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{93. } A &= \int_{-\pi/4}^{\pi/4} [\cos^2 x - \sin^2 x] \, dx \\ &= \int_{-\pi/4}^{\pi/4} \cos 2x \, dx \\ &= \left[ \frac{\sin 2x}{2} \right]_{-\pi/4}^{\pi/4} = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

$$\begin{aligned}
 94. \quad A &= \int_{-\pi/2}^{\pi/4} [\cos^2 x - \sin x \cos x] dx \\
 &= \int_{-\pi/2}^{\pi/4} \left[ \frac{1 + \cos 2x}{2} - \sin x \cos x \right] dx \\
 &= \left[ \frac{1}{2}x + \frac{\sin 2x}{4} - \frac{\sin^2 x}{2} \right]_{-\pi/2}^{\pi/4} \\
 &= \left( \frac{\pi}{8} + \frac{1}{4} - \frac{1}{4} \right) - \left( -\frac{\pi}{4} - \frac{1}{2} \right) \\
 &= \frac{3\pi}{8} + \frac{1}{2}
 \end{aligned}$$

**95. Disks**

$$\begin{aligned}
 R(x) &= \tan x, r(x) = 0 \\
 V &= 2\pi \int_0^{\pi/4} \tan^2 x \, dx \\
 &= 2\pi \int_0^{\pi/4} (\sec^2 x - 1) \, dx \\
 &= 2\pi [\tan x - x]_0^{\pi/4} \\
 &= 2\pi \left( 1 - \frac{\pi}{4} \right) \approx 1.348
 \end{aligned}$$



$$\begin{aligned}
 96. \quad V &= \pi \int_0^{\pi/2} \left[ \cos^2 \left( \frac{x}{2} \right) - \sin^2 \left( \frac{x}{2} \right) \right] dx \\
 &= \pi \int_0^{\pi/2} \cos x \, dx \\
 &= \pi [\sin x]_0^{\pi/2} = \pi
 \end{aligned}$$

$$97. (a) \quad V = \pi \int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx = \frac{\pi}{2} \left[ x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi^2}{2}$$

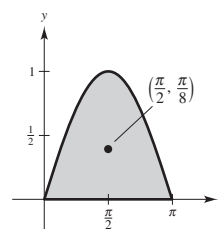
$$(b) \quad A = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 1 + 1 = 2$$

Let  $u = x$ ,  $dv = \sin x \, dx$ ,  $du = dx$ ,  $v = -\cos x$ .

$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx = \frac{1}{2} [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx = \frac{1}{2} [-x \cos x + \sin x]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx = \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx = \frac{1}{8} \left[ x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left( \frac{\pi}{2}, \frac{\pi}{8} \right)$$



$$98. (a) \quad V = \pi \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx = \frac{\pi}{2} \left[ x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi^2}{4}$$

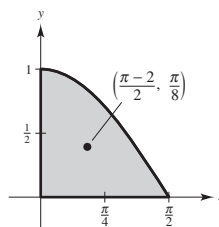
$$(b) \quad A = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$$

Let  $u = x$ ,  $dv = \cos x \, dx$ ,  $du = dx$ ,  $v = \sin x$ .

$$\bar{x} = \int_0^{\pi/2} x \cos x \, dx = [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = [x \sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\bar{y} = \frac{1}{2} \int_0^{\pi/2} \cos^2 x \, dx = \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx = \frac{1}{4} \left[ x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left( \frac{\pi - 2}{2}, \frac{\pi}{8} \right)$$



$$99. \quad dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \sin^{n-1} x \Rightarrow du = (n-1) \sin^{n-2} x \cos x \, dx$$

$$\begin{aligned}
 \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx
 \end{aligned}$$

$$\text{Therefore, } n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

100.  $dv = \cos x \, dx \Rightarrow v = \sin x$

$$u = \cos^{n-1} x \Rightarrow du = -(n-1) \cos^{n-2} x \sin x \, dx$$

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \end{aligned}$$

Therefore,  $n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

101. Let  $u = \sin^{n-1} x$ ,  $du = (n-1) \sin^{n-2} x \cos x \, dx$ ,  $dv = \cos^m x \sin x \, dx$ ,  $v = \frac{-\cos^{m+1} x}{m+1}$ .

$$\begin{aligned} \int \cos^m x \sin^n x \, dx &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^{m+2} x \, dx \\ &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x (1 - \sin^2 x) \, dx \\ &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx - \frac{n-1}{m+1} \int \sin^n x \cos^m x \, dx \\ \frac{m+n}{m+1} \int \cos^m x \sin^n x \, dx &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx \\ \int \cos^m x \sin^n x \, dx &= \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx \end{aligned}$$

102. Let  $u = \sec^{n-2} x$ ,  $du = (n-2) \sec^{n-2} x \tan x \, dx$ ,  $dv = \sec^2 x \, dx$ ,  $v = \tan x$ .

$$\begin{aligned} \int \sec^n x \, dx &= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \left[ \int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right] \\ (n-1) \int \sec^n x \, dx &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx \\ \int \sec^n x \, dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \end{aligned}$$

103.  $\int \sin^5 x \, dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \int \sin^3 x \, dx$

$$\begin{aligned} &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \left( -\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \int \sin x \, dx \right) \\ &= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C \\ &= -\frac{\cos x}{15} (3 \sin^4 x + 4 \sin^2 x + 8) + C \end{aligned}$$

104.  $\int \cos^4 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \int \cos^2 x \, dx$

$$\begin{aligned} &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left( \frac{\cos x \sin x}{2} + \frac{1}{2} \int dx \right) \\ &= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C \\ &= \frac{1}{8} (2 \cos^3 x \sin x + 3 \cos x \sin x + 3x) + C \end{aligned}$$

$$\begin{aligned}
 105. \int \sec^4 \frac{2\pi x}{5} dx &= \frac{5}{2\pi} \int \sec^4 \left( \frac{2\pi x}{5} \right) \frac{2\pi}{5} dx \\
 &= \frac{5}{2\pi} \left[ \frac{1}{3} \sec^2 \left( \frac{2\pi x}{5} \right) \tan \left( \frac{2\pi x}{5} \right) + \frac{2}{3} \int \sec^2 \left( \frac{2\pi x}{5} \right) \frac{2\pi}{5} dx \right] \\
 &= \frac{5}{6\pi} \left[ \sec^2 \left( \frac{2\pi x}{5} \right) \tan \left( \frac{2\pi x}{5} \right) + 2 \tan \left( \frac{2\pi x}{5} \right) \right] + C \\
 &= \frac{5}{6\pi} \tan \left( \frac{2\pi x}{5} \right) \left[ \sec^2 \left( \frac{2\pi x}{5} \right) + 2 \right] + C
 \end{aligned}$$

$$\begin{aligned}
 106. \int \sin^4 x \cos^2 x dx &= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \int \cos^2 x \sin^2 x dx \\
 &= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \left( -\frac{\cos^3 x \sin x}{4} + \frac{1}{4} \int \cos^2 x dx \right) \\
 &= -\frac{1}{6} \cos^3 x \sin^3 x - \frac{1}{8} \cos^3 x \sin x + \frac{1}{8} \left( \frac{\cos x \sin x}{2} + \frac{x}{2} \right) + C \\
 &= -\frac{1}{48} (8 \cos^3 x \sin^3 x + 6 \cos^3 x \sin x - 3 \cos x \sin x - 3x) + C
 \end{aligned}$$

$$107. f(t) = a_0 + a_1 \cos \frac{\pi t}{6} + b_1 \sin \frac{\pi t}{6}$$

$$a_0 = \frac{1}{12} \int_0^{12} f(t) dt, a_1 = \frac{1}{6} \int_0^{12} f(t) \cos \frac{\pi t}{6} dt, b_1 = \frac{1}{6} \int_0^{12} f(t) \sin \frac{\pi t}{6} dt$$

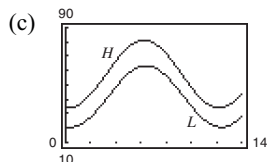
$$\begin{aligned}
 (a) \quad a_0 &\approx \frac{1}{12} \cdot \frac{(12-0)}{3(12)} [33.5 + 4(35.4) + 2(44.7) + 4(55.6) + 2(67.4) + 4(76.2) + 2(80.4) + 4(79.0) + 2(72.0) \\
 &\quad + 4(61.0) + 2(49.3) + 4(38.6) + 33.5] \\
 &\approx 57.72
 \end{aligned}$$

$$a_1 \approx -23.36$$

$$b_1 \approx -2.75 \quad (\text{Answers will vary.})$$

$$H(t) \approx 57.72 - 23.36 \cos \left( \frac{\pi t}{6} \right) - 2.75 \sin \left( \frac{\pi t}{6} \right)$$

$$(b) \quad L(t) \approx 42.04 - 20.91 \cos \left( \frac{\pi t}{6} \right) - 4.33 \sin \left( \frac{\pi t}{6} \right)$$



Temperature difference is greatest in the summer ( $t \approx 4.9$  or end of May).

108. (a)  $n$  is odd and  $n \geq 3$ .

$$\begin{aligned}
 \int_0^{\pi/2} \cos^n x \, dx &= \left[ \frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx \\
 &= \frac{n-1}{n} \left( \left[ \frac{\cos^{n-3} x \sin x}{n-2} \right]_0^{\pi/2} + \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x \, dx \right) \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \left( \left[ \frac{\cos^{n-5} x \sin x}{n-4} \right]_0^{\pi/2} + \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \right) \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos x \, dx \\
 &= \left[ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots (\sin x) \right]_0^{\pi/2} \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 1 \quad (\text{Reverse the order.}) \\
 &= (1) \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) \left( \frac{6}{7} \right) \cdots \left( \frac{n-1}{n} \right) = \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) \left( \frac{6}{7} \right) \cdots \left( \frac{n-1}{n} \right)
 \end{aligned}$$

(b)  $n$  is even and  $n \geq 2$ .

$$\begin{aligned}
 \int_0^{\pi/2} \cos^n x \, dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos^2 x \, dx \quad (\text{From part (a)}) \\
 &= \left[ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \left( \frac{x}{2} + \frac{1}{4} \sin 2x \right) \right]_0^{\pi/2} \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{\pi}{4} \quad (\text{Reverse the order.}) \\
 &= \left( \frac{\pi}{2} \cdot \frac{1}{2} \right) \left( \frac{3}{4} \right) \left( \frac{5}{6} \right) \cdots \left( \frac{n-1}{n} \right) = \left( \frac{1}{2} \right) \left( \frac{3}{4} \right) \left( \frac{5}{6} \right) \cdots \left( \frac{n-1}{n} \right) \left( \frac{\pi}{2} \right)
 \end{aligned}$$

109.  $\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \frac{1}{2} \left[ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$

$$\begin{aligned}
 \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx \\
 &= \frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n)
 \end{aligned}$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x] \, dx \\
 &= -\frac{1}{2} \left[ \frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right]_{-\pi}^{\pi}, \quad (m \neq n) \\
 &= -\frac{1}{2} \left[ \left( \frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} \right) - \left( \frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n} \right) \right] \\
 &= 0, \text{ because } \cos(-\theta) = \cos \theta.
 \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(mx) \, dx = \frac{1}{m} \left[ \frac{\sin^2(mx)}{2} \right]_{-\pi}^{\pi} = 0$$

110.  $f(x) = \sum_{i=1}^N a_i \sin(ix)$

(a)  $f(x) \sin(nx) = \left[ \sum_{i=1}^N a_i \sin(ix) \right] \sin(nx)$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin(nx) dx &= \int_{-\pi}^{\pi} \left[ \sum_{i=1}^N a_i \sin(ix) \right] \sin(nx) dx \\ &= \int_{-\pi}^{\pi} a_n \sin^2(nx) dx \quad (\text{by Exercise 109}) \\ &= \int_{-\pi}^{\pi} a_n \frac{1 - \cos(2nx)}{2} dx = \left[ \frac{a_n}{2} \left( x - \frac{\sin(2nx)}{2n} \right) \right]_{-\pi}^{\pi} = \frac{a_n}{2} (\pi + \pi) = a_n \pi \end{aligned}$$

So,  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$ .

(b)  $f(x) = x$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x dx = 2$$

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin 2x dx = -1$$

$$a_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin 3x dx = \frac{2}{3}$$

## Section 8.4 Trigonometric Substitution

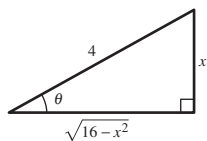
1. Use  $x = 3 \tan \theta$

2. Use  $x = 2 \sin \theta$

3. Use  $x = 4 \sin \theta$

4. Use  $x = 5 \sec \theta$

5. Let  $x = 4 \sin \theta$ ,  $dx = 4 \cos \theta d\theta$ ,  $\sqrt{16 - x^2} = 4 \cos \theta$ .



$$\int \frac{1}{(16 - x^2)^{3/2}} dx = \int \frac{4 \cos \theta}{(4 \cos \theta)^3} d\theta = \frac{1}{16} \int \sec^2 \theta d\theta = \frac{1}{16} \tan \theta + C = \frac{1}{16} \frac{x}{\sqrt{16 - x^2}} + C$$

6. Same substitution as in Exercise 5.

$$\int \frac{4}{x^2 \sqrt{16 - x^2}} dx = 4 \int \frac{4 \cos \theta}{(4 \sin \theta)^2 (4 \cos \theta)} d\theta = \frac{1}{4} \int \csc^2 \theta d\theta = -\frac{1}{4} \cot \theta + C = -\frac{1}{4} \frac{\sqrt{16 - x^2}}{x} + C = \frac{-\sqrt{16 - x^2}}{4x} + C$$



7. Same substitution as in Exercise 5

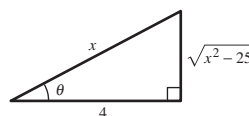
$$\begin{aligned}
 \int \frac{\sqrt{16-x^2}}{x} dx &= \int \frac{4 \cos \theta}{4 \sin \theta} 4 \cos \theta d\theta \\
 &= 4 \int \frac{\cos^2 \theta}{\sin \theta} d\theta \\
 &= 4 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta \\
 &= 4 \int (\csc \theta - \sin \theta) d\theta \\
 &= -4 \ln |\csc \theta + \cot \theta| + 4 \cos \theta + C \\
 &= -4 \ln \left| \frac{4}{x} + \frac{\sqrt{16-x^2}}{x} \right| + 4 \frac{\sqrt{16-x^2}}{4} + C \\
 &= -4 \ln \left| \frac{4 + \sqrt{16-x^2}}{x} \right| + \sqrt{16-x^2} + C \\
 &= 4 \ln \left| \frac{4 - \sqrt{16-x^2}}{x} \right| + \sqrt{16-x^2} + C
 \end{aligned}$$

8. Same substitution as in Exercise 5.

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{16-x^2}} dx &= \int \frac{(4 \sin \theta)^2}{4 \cos \theta} 4 \cos \theta d\theta \\
 &= 16 \int \sin^2 \theta d\theta \\
 &= 8 \int (1 - \cos 2\theta) d\theta \\
 &= 8 \left( \theta - \frac{\sin 2\theta}{2} \right) + C \\
 &= 8(\theta - \sin \theta \cos \theta) + C \\
 &= 8 \left( \arcsin \frac{x}{4} - \frac{x}{4} \frac{\sqrt{16-x^2}}{4} \right) + C \\
 &= 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C
 \end{aligned}$$

9. Let  $x = 5 \sec \theta$ ,  $dx = 5 \sec \theta \tan \theta d\theta$ ,

$$\sqrt{x^2 - 25} = 5 \tan \theta$$



$$\begin{aligned}
 \int \frac{1}{\sqrt{x^2 - 25}} dx &= \int \frac{5 \sec \theta \tan \theta}{5 \tan \theta} d\theta \\
 &= \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \frac{x}{5} + \frac{\sqrt{x^2 - 25}}{5} \right| + C \\
 &= \ln |x + \sqrt{x^2 - 25}| + C
 \end{aligned}$$

10. Same substitution as in Exercise 9

$$\begin{aligned}
 \int \frac{\sqrt{x^2 - 25}}{x} dx &= \int \frac{5 \tan \theta}{5 \sec \theta} 5 \sec \theta \tan \theta d\theta \\
 &= 5 \int \tan^2 \theta d\theta \\
 &= 5 \int (\sec^2 \theta - 1) d\theta \\
 &= 5(\tan \theta - \theta) + C \\
 &= 5 \left( \frac{\sqrt{x^2 - 25}}{5} - \operatorname{arcsec} \frac{x}{5} \right) + C \\
 &= \sqrt{x^2 - 25} - 5 \operatorname{arcsec} \frac{x}{5} + C
 \end{aligned}$$

$$\left[ \text{Note: } \operatorname{arcsec} \left( \frac{x}{5} \right) = \arctan \left( \frac{\sqrt{x^2 - 25}}{5} \right) \right]$$

11. Same substitution as in Exercise 9

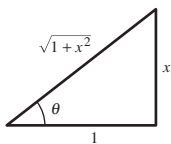
$$\begin{aligned}
\int x^3 \sqrt{x^2 - 25} \, dx &= \int (5 \sec \theta)^3 (5 \tan \theta) (5 \sec \theta \tan \theta) \, d\theta \\
&= 3125 \int \sec^4 \theta \tan^2 \theta \, d\theta \\
&= 3125 \int (1 + \tan^2 \theta) \tan^2 \theta \sec^2 \theta \, d\theta \\
&= 3125 \int (\tan^2 \theta + \tan^4 \theta) \sec^2 \theta \, d\theta \\
&= 3125 \left[ \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} \right] + C \\
&= 3125 \left[ \frac{(x^2 - 25)^{3/2}}{125(3)} + \frac{(x^2 - 25)^{5/2}}{5^5(5)} \right] + C \\
&= \frac{1}{15} (x^2 - 25)^{3/2} [125 + 3(x^2 - 25)] + C \\
&= \frac{1}{15} (x^2 - 25)^{3/2} (50 + 3x^2) + C
\end{aligned}$$

12. Same substitution as in Exercise 9

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2 - 25}} \, dx &= \int \frac{(5 \sec \theta)^3}{5 \tan \theta} 5 \sec \theta \tan \theta \, d\theta \\
&= 125 \int \sec^4 \theta \, d\theta \\
&= 125 \int (\tan^2 \theta + 1) \sec^2 \theta \, d\theta \\
&= 125 \left( \frac{\tan^3 \theta}{3} + \tan \theta \right) + C \\
&= \frac{125}{3} \frac{(x^2 - 25)^{3/2}}{125} + 125 \frac{\sqrt{x^2 - 25}}{5} + C \\
&= \frac{1}{3} (x^2 - 25)^{3/2} + 25 (x^2 - 25)^{1/2} + C \\
&= \frac{1}{3} \sqrt{x^2 - 25} (x^2 - 25 + 75) + C \\
&= \frac{1}{3} \sqrt{x^2 - 25} (50 + x^2) + C
\end{aligned}$$

13. Let  $x = \tan \theta$ ,  $dx = \sec^2 \theta \, d\theta$ ,  $\sqrt{1 + x^2} = \sec \theta$ .

$$\int x \sqrt{1 + x^2} \, dx = \int \tan \theta (\sec \theta) \sec^2 \theta \, d\theta = \frac{\sec^3 \theta}{3} + C = \frac{1}{3} (1 + x^2)^{3/2} + C$$

**Note:** This integral could have been evaluated with the Power Rule.

14. Same substitution as in Exercise 13

$$\begin{aligned}
\int \frac{9x^3}{\sqrt{1 + x^2}} \, dx &= 9 \int \frac{\tan^3 \theta}{\sec \theta} \sec^2 \theta \, d\theta = 9 \int (\sec^2 \theta - 1) \sec \theta \tan \theta \, d\theta = 9 \left[ \frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
&= 3 \sec \theta (\sec^2 \theta - 3) + C = 3 \sqrt{1 + x^2} [(1 + x^2) - 3] + C = 3 \sqrt{1 + x^2} (x^2 - 2) + C
\end{aligned}$$

15. Same substitution as in Exercise 13

$$\begin{aligned}
 \int \frac{1}{(1+x^2)^2} dx &= \int \frac{1}{(\sqrt{1+x^2})^4} dx = \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\
 &= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\
 &= \frac{1}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right] \\
 &= \frac{1}{2} [\theta + \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} \left[ \arctan x + \left( \frac{x}{\sqrt{1+x^2}} \right) \left( \frac{1}{\sqrt{1+x^2}} \right) \right] + C \\
 &= \frac{1}{2} \left( \arctan x + \frac{x}{1+x^2} \right) + C
 \end{aligned}$$

16. Same substitution as in Exercise 13

$$\begin{aligned}
 \int \frac{x^2}{(1+x^2)^2} dx &= \int \frac{x^2}{(\sqrt{1+x^2})^4} dx = \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^4 \theta} = \int \sin^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right] = \frac{1}{2} [\theta - \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} \left[ \arctan x - \left( \frac{x}{\sqrt{1+x^2}} \right) \left( \frac{1}{\sqrt{1+x^2}} \right) \right] + C = \frac{1}{2} \left( \arctan x - \frac{x}{1+x^2} \right) + C
 \end{aligned}$$

17. Let  $u = 4x$ ,  $a = 3$ ,  $du = 4 dx$ .

$$\begin{aligned}
 \int \sqrt{9+16x^2} dx &= \frac{1}{4} \int \sqrt{(4x)^2 + 3^2} (4) dx \\
 &= \frac{1}{4} \cdot \frac{1}{2} \left[ 4x \sqrt{16x^2 + 9} + 9 \ln |4x + \sqrt{16x^2 + 9}| \right] + C \\
 &= \frac{1}{2} x \sqrt{16x^2 + 9} + \frac{9}{8} \ln |4x + \sqrt{16x^2 + 9}| + C
 \end{aligned}$$

18. Let  $u = x$ ,  $a = 2$ ,  $du = dx$ .

$$\begin{aligned}
 \int \sqrt{4+x^2} dx &= \int \sqrt{x^2 + 2^2} dx \\
 &= \frac{1}{2} \left[ x \sqrt{x^2 + 4} + 4 \ln |x + \sqrt{x^2 + 4}| \right] + C \\
 &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C
 \end{aligned}$$

19.  $\int \sqrt{25-4x^2} dx = \int 2 \sqrt{\frac{25}{4} - x^2} dx, \quad a = \frac{5}{2}$

$$\begin{aligned}
 &= 2 \left[ \frac{25}{4} \arcsin \left( \frac{2x}{5} \right) + x \sqrt{\frac{25}{4} - x^2} \right] + C \\
 &= \frac{25}{4} \arcsin \left( \frac{2x}{5} \right) + \frac{x}{2} \sqrt{25 - 4x^2} + C
 \end{aligned}$$

20. Let  $u = \sqrt{5x}$ ,  $a = 1$ ,  $du = \sqrt{5} dx$ .

$$\begin{aligned}\int \sqrt{5x^2 - 1} dx &= \frac{1}{\sqrt{5}} \int \sqrt{(\sqrt{5x})^2 - 1} \sqrt{5} dx \\ &= \frac{1}{\sqrt{5}} \frac{1}{2} \left( \sqrt{5x} \sqrt{5x^2 - 1} - \ln \left| \sqrt{5x} + \sqrt{5x^2 - 1} \right| \right) + C \\ &= \frac{x}{2} \sqrt{5x^2 - 1} - \frac{\sqrt{5}}{10} \ln \left| \sqrt{5x} + \sqrt{5x^2 - 1} \right| + C\end{aligned}$$

21.  $\int \frac{x}{\sqrt{x^2 + 36}} dx = \frac{1}{2} \int (x^2 + 36)^{-1/2} (2x) dx$

$$\begin{aligned}&= \frac{1}{2} \frac{(x^2 + 36)^{1/2}}{1/2} + C \\ &= \sqrt{x^2 + 36} + C\end{aligned}$$

22.  $\int \frac{x}{\sqrt{36 - x^2}} dx = -\frac{1}{2} \int (36 - x^2)^{-1/2} (-2x) dx$

$$\begin{aligned}&= -\frac{1}{2} \frac{(36 - x^2)^{1/2}}{1/2} + C \\ &= -\sqrt{36 - x^2} + C\end{aligned}$$

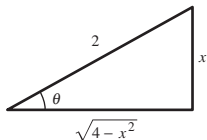
23.  $\int \frac{1}{\sqrt{16 - x^2}} dx = \arcsin\left(\frac{x}{4}\right) + C$

24.  $\int \frac{1}{\sqrt{49 - x^2}} dx = \arcsin\left(\frac{x}{7}\right) + C \quad (a = 7)$

25. Let  $x = 2 \sin \theta$ ,  $dx = 2 \cos \theta d\theta$ ,

$$\sqrt{4 - x^2} = 2 \cos \theta.$$

$$\begin{aligned}\int \sqrt{16 - 4x^2} dx &= 2 \int \sqrt{4 - x^2} dx \\ &= 2 \int 2 \cos \theta (2 \cos \theta d\theta) \\ &= 8 \int \cos^2 \theta d\theta \\ &= 4 \int (1 + \cos 2\theta) d\theta \\ &= 4 \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= 4\theta + 4 \sin \theta \cos \theta + C \\ &= 4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4 - x^2} + C\end{aligned}$$

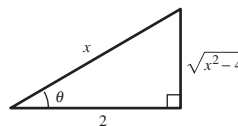


26. Let  $u = 16 - 4x^2$ ,  $du = -8x dx$ .

$$\begin{aligned}\int x \sqrt{16 - 4x^2} dx &= -\frac{1}{8} \int (16 - 4x^2)^{1/2} (-8x) dx \\ &= \left( -\frac{1}{12} (16 - 4x^2)^{3/2} \right) + C \\ &= -\frac{2}{3} (4 - x^2)^{3/2} + C\end{aligned}$$

27. Let  $x = 2 \sec \theta$ ,  $dx = 2 \sec \theta \tan \theta d\theta$ ,

$$\sqrt{x^2 - 4} = 2 \tan \theta.$$



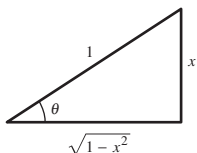
$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - 4}} dx &= \int \frac{2 \sec \theta \tan \theta}{2 \tan \theta} d\theta \\ &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C \\ &= \ln |x + \sqrt{x^2 - 4}| + C\end{aligned}$$

28. Let  $u = 1 - t^2$ ,  $du = -2t dt$ .

$$\begin{aligned}\int \frac{t}{(4 - t^2)^{3/2}} dt &= -\frac{1}{2} \int (4 - t^2)^{-3/2} (-2t) dt \\ &= -\frac{1}{2} \frac{(4 - t^2)^{-1/2}}{(-1/2)} + C \\ &= \frac{1}{\sqrt{4 - t^2}} + C\end{aligned}$$

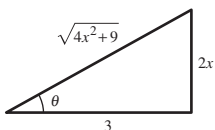
29. Let  $x = \sin \theta$ ,  $dx = \cos \theta d\theta$ ,  $\sqrt{1-x^2} = \cos \theta$ .

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^4} dx &= \int \frac{\cos \theta (\cos \theta d\theta)}{\sin^4 \theta} \\ &= \int \cot^2 \theta \csc^2 \theta d\theta \\ &= -\frac{1}{3} \cot^3 \theta + C \\ &= -\frac{(1-x^2)^{3/2}}{3x^3} + C\end{aligned}$$



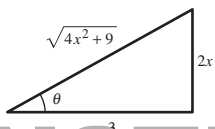
30. Let  $2x = 3 \tan \theta$ ,  $dx = \frac{3}{2} \sec^2 \theta d\theta$ ,

$$\begin{aligned}\sqrt{4x^2+9} &= 3 \sec \theta. \\ \int \frac{\sqrt{4x^2+9}}{x^4} dx &= \int \frac{3 \sec \theta [(3/2) \sec^2 \theta d\theta]}{(3/2)^4 \tan^4 \theta} \\ &= \frac{8}{9} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \\ &= \frac{-8}{27 \sin^3 \theta} + C \\ &= -\frac{8}{27} \csc^3 \theta + C \\ &= -\frac{(4x^2+9)^{3/2}}{27x^3} + C\end{aligned}$$



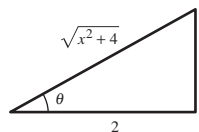
31. Same substitution as in Exercise 30

$$\begin{aligned}x &= \frac{3}{2} \tan \theta, dx = \frac{3}{2} \sec^2 \theta d\theta \\ \int \frac{1}{x\sqrt{4x^2+9}} dx &= \int \frac{(3/2) \sec^2 \theta d\theta}{(3/2) \tan \theta 3 \sec \theta} \\ &= \frac{1}{3} \int \csc \theta d\theta \\ &= -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C \\ &= -\frac{1}{3} \ln \left| \frac{\sqrt{4x^2+9} + 3}{2x} \right| + C\end{aligned}$$



32. Let  $2x = 4 \tan \theta$ ,  $dx = 2 \sec^2 \theta d\theta$ ,

$$\begin{aligned}\sqrt{4x^2+16} &= 4 \sec \theta. \\ \int \frac{1}{x\sqrt{4x^2+16}} dx &= \int \frac{2 \sec^2 \theta d\theta}{2 \tan \theta (4 \sec \theta)} \\ &= \frac{1}{4} \int \frac{\sec \theta}{\tan \theta} d\theta = \frac{1}{4} \int \csc \theta d\theta \\ &= -\frac{1}{4} \ln |\csc \theta + \cot \theta| + C \\ &= -\frac{1}{4} \ln \left| \frac{\sqrt{x^2+4} + 2}{x} \right| + C\end{aligned}$$

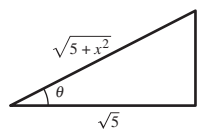


33. Let  $u = x^2 + 3$ ,  $du = 2x dx$ .

$$\begin{aligned}\int \frac{-3x}{(x^2+3)^{3/2}} dx &= -\frac{3}{2} \int (x^2+3)^{-3/2} (2x) dx \\ &= -\frac{3(x^2+3)^{-1/2}}{2(-1/2)} + C \\ &= \frac{3}{\sqrt{x^2+3}} + C\end{aligned}$$

34. Let  $x = \sqrt{5} \tan \theta$ ,  $dx = \sqrt{5} \sec^2 \theta d\theta$ ,

$$x^2 + 5 = 5 \sec^2 \theta.$$



$$\begin{aligned}\int \frac{1}{(x^2+5)^{3/2}} dx &= \int \frac{\sqrt{5} \sec^2 \theta d\theta}{(\sqrt{5} \sec \theta)^3} \\ &= \frac{1}{5} \int \cos \theta d\theta \\ &= \frac{1}{5} \sin \theta + C = \frac{x}{5\sqrt{5+x^2}} + C\end{aligned}$$

35. Let  $u = 1 + e^{2x}$ ,  $du = 2e^{2x} dx$ .

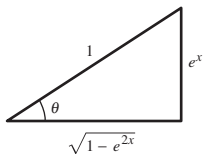
$$\begin{aligned}\int e^{2x} \sqrt{1+e^{2x}} dx &= \frac{1}{2} \int (1+e^{2x})^{1/2} (2e^{2x}) dx \\ &= \frac{1}{3} (1+e^{2x})^{3/2} + C\end{aligned}$$

36. Let  $u = x^2 + 2x + 2$ ,  $du = (2x+2) dx$ .

$$\begin{aligned}\int (x+1)\sqrt{x^2+2x+2} dx &= \frac{1}{2} \int (x^2+2x+2)^{1/2} (2x+2) dx \\ &= \frac{1}{3} (x^2+2x+2)^{3/2} + C\end{aligned}$$

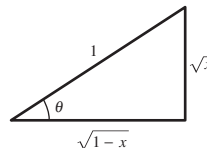
37. Let  $e^x = \sin \theta$ ,  $e^x dx = \cos \theta d\theta$ ,  $\sqrt{1 - e^{2x}} = \cos \theta$ .

$$\begin{aligned}\int e^x \sqrt{1 - e^{2x}} dx &= \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) \\ &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{2} (\arcsin e^x + e^x \sqrt{1 - e^{2x}}) + C\end{aligned}$$



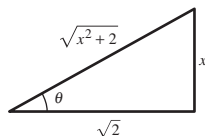
38. Let  $\sqrt{x} = \sin \theta$ ,  $x = \sin^2 \theta$ ,  $dx = 2 \sin \theta \cos \theta d\theta$ ,

$$\begin{aligned}\sqrt{1 - x} &= \cos \theta. \\ \int \frac{\sqrt{1 - x}}{\sqrt{x}} dx &= \int \frac{\cos \theta (2 \sin \theta \cos \theta d\theta)}{\sin \theta} \\ &= 2 \int \cos^2 \theta d\theta \\ &= \int (1 + \cos 2\theta) d\theta \\ &= (\theta + \sin \theta \cos \theta) + C \\ &= \arcsin \sqrt{x} + \sqrt{x} \sqrt{1 - x} + C\end{aligned}$$



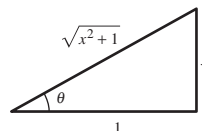
39. Let  $x = \sqrt{2} \tan \theta$ ,  $dx = \sqrt{2} \sec^2 \theta d\theta$ ,  $x^2 + 2 = 2 \sec^2 \theta$ .

$$\begin{aligned}\int \frac{1}{4 + 4x^2 + x^4} dx &= \int \frac{1}{(x^2 + 2)^2} dx = \int \frac{\sqrt{2} \sec^2 \theta d\theta}{4 \sec^4 \theta} \\ &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \left( \frac{1}{2} \right) \int (1 + \cos 2\theta) d\theta \\ &= \frac{\sqrt{2}}{8} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{\sqrt{2}}{8} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{\sqrt{2}}{8} \left( \arctan \frac{x}{\sqrt{2}} + \frac{x}{\sqrt{x^2 + 2}} \cdot \frac{\sqrt{2}}{\sqrt{x^2 + 2}} \right) = \frac{1}{4} \left( \frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} \right) + C\end{aligned}$$



40. Let  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ ,  $x^2 + 1 = \sec^2 \theta$ .

$$\begin{aligned}\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx &= \frac{1}{4} \int \frac{4x^3 + 4x}{x^4 + 2x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} dx \\ &= \frac{1}{4} \ln(x^4 + 2x^2 + 1) + \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{2} \left( \ln(x^2 + 1) + \arctan x + \frac{x}{x^2 + 1} \right) + C\end{aligned}$$



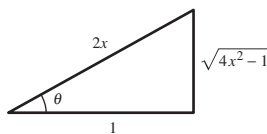
41. Use integration by parts. Because  $x > \frac{1}{2}$ ,

$$u = \operatorname{arcsec} 2x \Rightarrow du = \frac{1}{x\sqrt{4x^2 - 1}} dx, dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}} dx$$

$$2x = \sec \theta, dx = \frac{1}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2 - 1} = \tan \theta$$

$$\begin{aligned} \int \operatorname{arcsec} 2x dx &= x \operatorname{arcsec} 2x - \int \frac{(1/2) \sec \theta \tan \theta d\theta}{\tan \theta} = x \operatorname{arcsec} 2x - \frac{1}{2} \int \sec \theta d\theta \\ &= x \operatorname{arcsec} 2x - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = x \operatorname{arcsec} 2x - \frac{1}{2} \ln |2x + \sqrt{4x^2 - 1}| + C. \end{aligned}$$



42.  $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1 - x^2}} dx, dv = x dx \Rightarrow v = \frac{x^2}{2}$

$$\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1 - x^2}} dx$$

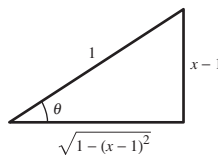
$$x = \sin \theta, dx = \cos \theta d\theta, \sqrt{1 - x^2} = \cos \theta$$

$$\begin{aligned} \int x \arcsin x dx &= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta \\ &= \frac{x^2}{2} \arcsin x - \frac{1}{4} \left( \theta - \frac{1}{2} \sin 2\theta \right) + C = \frac{x^2}{2} \arcsin x - \frac{1}{4} (\theta - \sin \theta \cos \theta) + C \\ &= \frac{x^2}{2} \arcsin x - \frac{1}{4} (\arcsin x - x\sqrt{1 - x^2}) + C = \frac{1}{4} [(2x^2 - 1) \arcsin x + x\sqrt{1 - x^2}] + C \end{aligned}$$

43.  $\int \frac{1}{\sqrt{4x - x^2}} dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}} dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

44. Let  $x - 1 = \sin \theta, dx = \cos \theta d\theta, \sqrt{1 - (x - 1)^2} = \sqrt{2x - x^2} = \cos \theta$ .

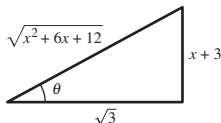
$$\begin{aligned} \int \frac{x^2}{\sqrt{2x - x^2}} dx &= \int \frac{x^2}{\sqrt{1 - (x - 1)^2}} dx \\ &= \int \frac{(1 + \sin \theta)^2 (\cos \theta d\theta)}{\cos \theta} \\ &= \int (1 + 2 \sin \theta + \sin^2 \theta) d\theta \\ &= \int \left( \frac{3}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta + C \\ &= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{2} \sin \theta \cos \theta + C \\ &= \frac{3}{2} \arcsin(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + C \\ &= \frac{3}{2} \arcsin(x - 1) - \frac{1}{2}\sqrt{2x - x^2}(x + 3) + C \end{aligned}$$



45.  $x^2 + 6x + 12 = x^2 + 6x + 9 + 3 = (x + 3)^2 + (\sqrt{3})^2$

Let  $x + 3 = \sqrt{3} \tan \theta$ ,  $dx = \sqrt{3} \sec^2 \theta d\theta$ .

$$\sqrt{x^2 + 6x + 12} = \sqrt{(x + 3)^2 + (\sqrt{3})^2} = \sqrt{3} \sec \theta$$



$$\begin{aligned} \int \frac{x}{\sqrt{x^2 + 6x + 12}} dx &= \int \frac{\sqrt{3} \tan \theta - 3}{\sqrt{3} \sec \theta} \sqrt{3} \sec^2 \theta d\theta \\ &= \int \sqrt{3} \sec \theta \tan \theta d\theta - 3 \int \sec \theta d\theta \\ &= \sqrt{3} \sec \theta - 3 \ln |\sec \theta + \tan \theta| + C \\ &= \sqrt{3} \left( \frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}} \right) - 3 \ln \left| \frac{\sqrt{x^2 + 6x + 12}}{\sqrt{3}} + \frac{x + 3}{\sqrt{3}} \right| + C \\ &= \sqrt{x^2 + 6x + 12} - 3 \ln |\sqrt{x^2 + 6x + 12} + (x + 3)| + C \end{aligned}$$

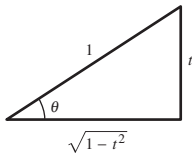
46. Let  $x - 3 = 2 \sec \theta$ ,  $dx = 2 \sec \theta \tan \theta d\theta$ ,  $\sqrt{(x - 3)^2 - 4} = 2 \tan \theta$ .

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 - 6x + 5}} dx &= \int \frac{x}{\sqrt{(x - 3)^2 - 4}} dx \\ &= \int \frac{(2 \sec \theta + 3)}{2 \tan \theta} (2 \sec \theta \tan \theta) d\theta \\ &= \int (2 \sec^2 \theta + 3 \sec \theta) d\theta \\ &= 2 \tan \theta + 3 \ln |\sec \theta + \tan \theta| + C_1 \\ &= 2 \left[ \frac{\sqrt{(x - 3)^2 - 4}}{2} \right] + 3 \ln \left| \frac{x - 3}{2} + \frac{\sqrt{(x - 3)^2 - 4}}{2} \right| + C_1 \\ &= \sqrt{x^2 - 6x + 5} + 3 \ln |(x - 3) + \sqrt{x^2 - 6x + 5}| + C \end{aligned}$$

47. Let  $t = \sin \theta$ ,  $dt = \cos \theta d\theta$ ,  $1 - t^2 = \cos^2 \theta$ .

(a)  $\int \frac{t^2}{(1 - t^2)^{3/2}} dt = \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^3 \theta} = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C = \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C$

So,  $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}} dt = \left[ \frac{t}{\sqrt{1 - t^2}} - \arcsin t \right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685$ .



(b) When  $t = 0$ ,  $\theta = 0$ . When  $t = \sqrt{3}/2$ ,  $\theta = \pi/3$ . So,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}} dt = [\tan \theta - \theta]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$



48. Same substitution as in Exercise 47

$$\begin{aligned} \text{(a)} \quad \int \frac{1}{(1-t^2)^{5/2}} dt &= \int \frac{\cos \theta d\theta}{\cos^5 \theta} = \int \sec^4 \theta d\theta = \int (\tan^2 \theta + 1) \sec^2 \theta d\theta \\ &= \frac{1}{3} \tan^3 \theta + \tan \theta + C = \frac{1}{3} \left( \frac{t}{\sqrt{1-t^2}} \right)^3 + \frac{t}{\sqrt{1-t^2}} + C \end{aligned}$$

$$\text{So, } \int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt = \left[ \frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}} \right]_0^{\sqrt{3}/2} = \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When  $t = 0$ ,  $\theta = 0$ . When  $t = \sqrt{3}/2$ ,  $\theta = \pi/3$ . So,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt = \left[ \frac{1}{3} \tan^3 \theta + \tan \theta \right]_0^{\pi/3} = \frac{1}{3} (\sqrt{3})^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

49. (a) Let  $x = 3 \tan \theta$ ,  $dx = 3 \sec^2 \theta d\theta$ ,  $\sqrt{x^2 + 9} = 3 \sec \theta$ .

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 9}} dx &= \int \frac{(27 \tan^3 \theta)(3 \sec^2 \theta d\theta)}{3 \sec \theta} \\ &= 27 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= 27 \left[ \frac{1}{3} \sec^3 \theta - \sec \theta \right] + C = 9 [\sec^3 \theta - 3 \sec \theta] + C \\ &= 9 \left[ \left( \frac{\sqrt{x^2 + 9}}{3} \right)^3 - 3 \left( \frac{\sqrt{x^2 + 9}}{3} \right) \right] + C = \frac{1}{3} (x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} + C \end{aligned}$$

$$\begin{aligned} \text{So, } \int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx &= \left[ \frac{1}{3} (x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} \right]_0^3 \\ &= \left( \frac{1}{3} (54\sqrt{2}) - 27\sqrt{2} \right) - (9 - 27) = 18 - 9\sqrt{2} = 9(2 - \sqrt{2}) \approx 5.272. \end{aligned}$$

(b) When  $x = 0$ ,  $\theta = 0$ . When  $x = 3$ ,  $\theta = \pi/4$ . So,

$$\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx = 9 [\sec^3 \theta - 3 \sec \theta]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1 - 3) = 9(2 - \sqrt{2}) \approx 5.272.$$

50. (a) Let  $5x = 3 \sin \theta$ ,  $dx = \frac{3}{5} \cos \theta d\theta$ ,  $\sqrt{9 - 25x^2} = 3 \cos \theta$ .

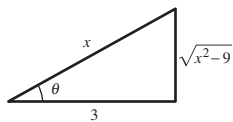
$$\begin{aligned} \int \sqrt{9 - 25x^2} dx &= \int (3 \cos \theta) \frac{3}{5} \cos \theta d\theta \\ &= \frac{9}{5} \int \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{9}{10} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{9}{10} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{9}{10} \left( \arcsin \frac{5x}{3} + \frac{5x}{3} \cdot \frac{\sqrt{9 - 25x^2}}{3} \right) + C \end{aligned}$$

$$\text{So, } \int_0^{3/5} \sqrt{9 - 25x^2} dx = \frac{9}{10} \left[ \arcsin \frac{5x}{3} + \frac{5x\sqrt{9 - 25x^2}}{9} \right]_0^{3/5} = \frac{9}{10} \left[ \frac{\pi}{2} \right] = \frac{9\pi}{20}.$$

(b) When  $x = 0$ ,  $\theta = 0$ . When  $x = \frac{3}{5}$ ,  $\theta = \frac{\pi}{2}$ .

$$\text{So, } \int_0^{3/5} \sqrt{9 - 25x^2} \, dx = \left[ \frac{9}{10} (\theta + \sin \theta \cos \theta) \right]_0^{\pi/2} = \frac{9}{10} \left( \frac{\pi}{2} \right) = \frac{9\pi}{20}.$$

51. (a) Let  $x = 3 \sec \theta$ ,  $dx = 3 \sec \theta \tan \theta \, d\theta$ ,  $\sqrt{x^2 - 9} = 3 \tan \theta$ .



$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 9}} \, dx &= \int \frac{9 \sec^2 \theta}{3 \tan \theta} 3 \sec \theta \tan \theta \, d\theta \\ &= 9 \int \sec^3 \theta \, d\theta \\ &= 9 \left( \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta \, d\theta \right) \quad (8.3 \text{ Exercise 102 or Example 5, Section 8.2}) \\ &= \frac{9}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \\ &= \frac{9}{2} \left( \frac{x}{3} \cdot \frac{\sqrt{x^2 - 9}}{3} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right) \end{aligned}$$

So,

$$\begin{aligned} \int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} \, dx &= \frac{9}{2} \left[ \frac{x\sqrt{x^2 - 9}}{9} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right]_4^6 \\ &= \frac{9}{2} \left[ \left( \frac{6\sqrt{27}}{9} + \ln \left| 2 + \frac{\sqrt{27}}{3} \right| \right) - \left( \frac{4\sqrt{7}}{9} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right) \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \left[ \ln \left( \frac{6 + \sqrt{27}}{3} \right) - \ln \left( \frac{4 + \sqrt{7}}{3} \right) \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left( \frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \approx 12.644 \end{aligned}$$

(b) When  $x = 4$ ,  $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$ . When  $x = 6$ ,  $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$ .

$$\begin{aligned} \int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} \, dx &= \frac{9}{2} \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\ &= \frac{9}{2} \left( 2 \cdot \sqrt{3} + \ln |2 + \sqrt{3}| \right) - \frac{9}{2} \left( \frac{4}{3} \frac{\sqrt{7}}{3} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left( \frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \approx 12.644 \end{aligned}$$

52. (a) Let  $x = 3 \sec \theta$ ,  $dx = 3 \sec \theta \tan \theta d\theta$ ,

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - 9}}{x^2} dx &= \int \frac{3 \tan \theta}{9 \sec^2 \theta} 3 \sec \theta \tan \theta d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\ &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\ &= \int (\sec \theta - \cos \theta) d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \\ &= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| - \frac{\sqrt{x^2 - 9}}{x} + C \end{aligned}$$

So,

$$\int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[ \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| - \frac{\sqrt{x^2 - 9}}{x} \right]_3^6 = \ln |2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

- (b) When  $x = 3$ ,  $\theta = 0$ ; when  $x = 6$ ,  $\theta = \frac{\pi}{3}$ . So,

$$\int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[ \ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\pi/3} = \ln |2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

53.  $x \frac{dy}{dx} = \sqrt{x^2 - 9}$ ,  $x \geq 3$ ,  $y(3) = 1$

$$y = \int \frac{\sqrt{x^2 - 9}}{x} dx$$

$$\text{Let } x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta.$$

$$\begin{aligned} y &= \int \frac{3 \tan \theta}{3 \sec \theta} 3 \sec \theta \tan \theta d\theta = 3 \int \tan^2 \theta d\theta \\ &= 3 \int (\sec^2 \theta - 1) d\theta = 3 [\tan \theta - \theta] + C \\ &= 3 \left[ \frac{\sqrt{x^2 - 9}}{3} - \arctan \left( \frac{\sqrt{x^2 - 9}}{3} \right) \right] + C \\ &= \sqrt{x^2 - 9} - 3 \arctan \left( \frac{\sqrt{x^2 - 9}}{3} \right) + C \end{aligned}$$

$$y(3) = 1: 1 = 0 - 3(0) + C \Rightarrow C = 1$$

$$y = \sqrt{x^2 - 9} - 3 \arctan \left( \frac{\sqrt{x^2 - 9}}{3} \right) + 1$$

54.  $\sqrt{x^2 + 4} \frac{dy}{dx} = 1, x \geq -2, y(0) = 4$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 4}}$$

$$y = \int \frac{1}{\sqrt{x^2 + 4}} dx$$

Let  $x = 2 \tan \theta, x^2 + 4 = 4 \sec^2 \theta, dx = 2 \sec^2 \theta d\theta$ .

$$y = \int \frac{1}{2 \sec \theta} 2 \sec^2 \theta d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{\sqrt{x^2 + 4}}{2} + \frac{x}{2} \right| + C = \ln |\sqrt{x^2 + 4} + x| + C_1$$

$$y(0) = 4 \Rightarrow 4 = \ln |2| + C_1 \Rightarrow C_1 = 4 - \ln 2$$

$$y = \ln |\sqrt{x^2 + 4} + x| + 4 - \ln 2$$

55.  $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}} dx = \frac{1}{2} \sqrt{x^2 + 10x + 9} (x - 15) + 33 \ln |(x + 5) + \sqrt{x^2 + 10x + 9}| + C$

56.  $\int (x^2 + 2x + 11)^{3/2} dx = \frac{1}{4} (x + 1) (x^2 + 2x + 26) \sqrt{x^2 + 2x + 11} + \frac{75}{2} \ln |\sqrt{x^2 + 2x + 11} + (x + 1)| + C$

57.  $\int \frac{x^2}{\sqrt{x^2 - 1}} dx = \frac{1}{2} (x \sqrt{x^2 - 1} + \ln |x + \sqrt{x^2 - 1}|) + C$

58.  $\int x^2 \sqrt{x^2 - 4} dx = \frac{1}{4} x^3 \sqrt{x^2 - 4} - \frac{1}{2} x \sqrt{x^2 - 4} - 2 \ln |x + \sqrt{x^2 - 4}| + C$

59. (a) Let  $u = a \sin \theta, \sqrt{a^2 - u^2} = a \cos \theta$ , where  $-\pi/2 \leq \theta \leq \pi/2$ .

(b) Let  $u = a \tan \theta, \sqrt{a^2 + u^2} = a \sec \theta$ , where  $-\pi/2 < \theta < \pi/2$ .

(c) Let  $u = a \sec \theta, \sqrt{u^2 - a^2} = \tan \theta$  if  $u > a$  and  $\sqrt{u^2 - a^2} = -\tan \theta$  if  $u < -a$ , where  $0 \leq \theta < \pi/2$  or  $\pi/2 < \theta \leq \pi$ .

60. Substitution:  $u = x^2 + 1, du = 2x dx$

61. Trigonometric substitution:  $x = \sec \theta$

62. (a)  $u = x^2 + 9, du = 2x dx$

$$\int \frac{x}{x^2 + 9} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln (x^2 + 9) + C$$

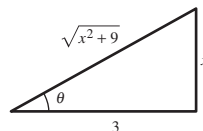
Let  $x = 3 \tan \theta, x^2 + 9 = 9 \sec^2 \theta, dx = 3 \sec^2 \theta d\theta$ .

$$\int \frac{x}{x^2 + 9} dx = \int \frac{3 \tan \theta}{9 \sec^2 \theta} 3 \sec^2 \theta d\theta = \int \tan \theta d\theta$$

$$= -\ln |\cos \theta| + C_1$$

$$= -\ln \left| \frac{3}{\sqrt{x^2 + 9}} \right| + C_1$$

$$= -\ln 3 + \ln \sqrt{x^2 + 9} + C_1 = \frac{1}{2} \ln (x^2 + 9) + C_2$$



The answers are equivalent.

$$(b) \int \frac{x^2}{x^2 + 9} dx = \int \frac{x^2 + 9 - 9}{x^2 + 9} dx = \int \left(1 - \frac{9}{x^2 + 9}\right) dx = x - 3 \arctan\left(\frac{x}{3}\right) + C$$

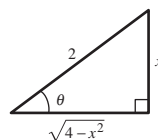
Let  $x = 3 \tan \theta$ ,  $x^2 + 9 = 9 \sec^2 \theta$ ,  $dx = 3 \sec^2 \theta d\theta$ .

$$\begin{aligned} \int \frac{x^2}{x^2 + 9} dx &= \int \frac{9 \tan^2 \theta}{9 \sec^2 \theta} 3 \sec^2 \theta d\theta \\ &= 3 \int \tan^2 \theta d\theta = 3 \int (\sec^2 \theta - 1) d\theta \\ &= 3 \tan \theta - 3\theta + C_1 \\ &= x - 3 \arctan\left(\frac{x}{3}\right) + C_1 \end{aligned}$$

The answers are equivalent.

$$(c) x = 2 \sin \theta, dx = 2 \cos \theta d\theta, 4 - x^2 = 4 \cos^2 \theta$$

$$\begin{aligned} \int \frac{4}{4 - x^2} dx &= \int \frac{4 \cdot 2 \cos \theta}{4 \cos^2 \theta} d\theta = 2 \int \sec \theta d\theta \\ &= 2 \ln |\sec \theta + \tan \theta| + C \\ &= 2 \ln \left| \frac{2}{\sqrt{4 - x^2}} + \frac{x}{\sqrt{4 - x^2}} \right| + C \\ &= \ln \left| \frac{2 + x}{\sqrt{(2 + x)(2 - x)}} \right|^2 + C = \ln \left| \frac{2 + x}{2 - x} \right| + C \end{aligned}$$



$$\int \frac{4}{4 - x^2} dx = \int \left( \frac{1}{x + 2} - \frac{1}{x - 2} \right) dx = \ln|x + 2| - \ln|x - 2| + C = \ln \left| \frac{x + 2}{x - 2} \right| + C$$

The answers are equivalent.

63. True

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta$$

64. False

$$\int \frac{\sqrt{x^2 - 1}}{x} dx = \int \frac{\tan \theta}{\sec \theta} (\sec \theta \tan \theta d\theta) = \int \tan^2 \theta d\theta$$

65. False

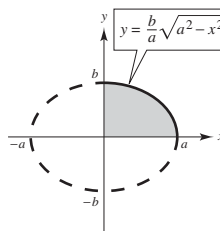
$$\int_0^{\sqrt{3}} \frac{dx}{(\sqrt{1 + x^2})^3} = \int_0^{\pi/3} \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int_0^{\pi/3} \cos \theta d\theta$$

66. True

$$\begin{aligned} \int_{-1}^1 x^2 \sqrt{1 - x^2} dx &= 2 \int_0^1 x^2 \sqrt{1 - x^2} dx \\ &= 2 \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta d\theta) \\ &= 2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \end{aligned}$$

$$\begin{aligned} 67. A &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= \left[ \frac{4b}{a} \left( \frac{1}{2} \right) \left( a^2 \arcsin \frac{x}{a} + x \sqrt{a^2 - x^2} \right) \right]_0^a \\ &= \frac{2b}{a} \left( a^2 \left( \frac{\pi}{2} \right) \right) = \pi ab \end{aligned}$$

**Note:** See Theorem 8.2 for  $\int \sqrt{a^2 - x^2} dx$ .



68.  $x^2 + y^2 = a^2$

$$x = \pm \sqrt{a^2 - y^2}$$

$$A = 2 \int_h^a \sqrt{a^2 - y^2} \, dy$$

$$= \left[ a^2 \arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2 - y^2} \right]_h^a \quad (\text{Theorem 8.2})$$

$$= \left( a^2 \frac{\pi}{2} \right) - \left( a^2 \arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2 - h^2} \right)$$

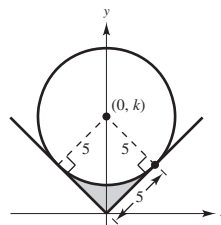
$$= \frac{a^2\pi}{2} - a^2 \arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2 - h^2}$$

69. (a)  $x^2 + (y - k)^2 = 25$

Radius of circle = 5

$$k^2 = 5^2 + 5^2 = 50$$

$$k = 5\sqrt{2}$$



(b) Area = square -  $\frac{1}{4}(\text{circle})$

$$= 25 - \frac{1}{4}\pi(5)^2 = 25\left(1 - \frac{\pi}{4}\right)$$

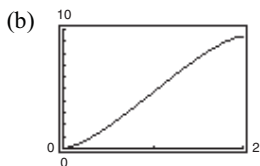
(c) Area =  $r^2 - \frac{1}{4}\pi r^2 = r^2\left(1 - \frac{\pi}{4}\right)$

70. (a) Place the center of the circle at  $(0, 1)$ ;  $x^2 + (y - 1)^2 = 1$ . The depth  $d$  satisfies  $0 \leq d \leq 2$ . The volume is

$$V = 3 \cdot 2 \int_0^d \sqrt{1 - (y - 1)^2} \, dy = 6 \cdot \frac{1}{2} \left[ \arcsin(y - 1) + (y - 1)\sqrt{1 - (y - 1)^2} \right]_0^d \quad (\text{Theorem 8.2 (1)})$$

$$= 3 \left[ \arcsin(d - 1) + (d - 1)\sqrt{1 - (d - 1)^2} - \arcsin(-1) \right]$$

$$= \frac{3\pi}{2} + 3 \arcsin(d - 1) + 3(d - 1)\sqrt{2d - d^2}.$$

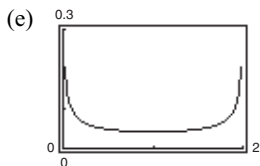
(c) The full tank holds  $3\pi \approx 9.4248$  cubic meters. The horizontal lines

$$y = \frac{3\pi}{4}, y = \frac{3\pi}{2}, y = \frac{9\pi}{4}$$

intersect the curve at  $d = 0.596, 1.0, 1.404$ . The dipstick would have these markings on it.

(d)  $V = 6 \int_0^d \sqrt{1 - (y - 1)^2} \, dy$

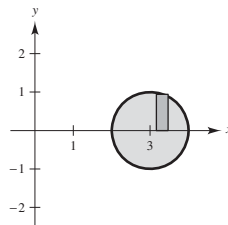
$$\frac{dV}{dt} = \frac{dV}{dd} \cdot \frac{dd}{dt} = 6\sqrt{1 - (d - 1)^2} \cdot d'(t) = \frac{1}{4} \Rightarrow d'(t) = \frac{1}{24\sqrt{1 - (d - 1)^2}}$$

The minimum occurs at  $d = 1$ , which is the widest part of the tank.

71. Let  $x - 3 = \sin \theta$ ,  $dx = \cos \theta d\theta$ ,  $\sqrt{1 - (x - 3)^2} = \cos \theta$ .

**Shell Method:**

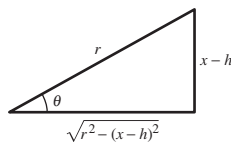
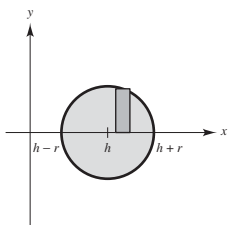
$$\begin{aligned} V &= 4\pi \int_2^4 x \sqrt{1 - (x - 3)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (3 + \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi \left[ \frac{3}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + \int_{-\pi/2}^{\pi/2} \cos^2 \theta \sin \theta d\theta \right] \\ &= 4\pi \left[ \frac{3}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) - \frac{1}{3} \cos^3 \theta \right]_{-\pi/2}^{\pi/2} = 6\pi^2 \end{aligned}$$



72. Let  $x - h = r \sin \theta$ ,  $dx = r \cos \theta d\theta$ ,  $\sqrt{r^2 - (x - h)^2} = r \cos \theta$ .

**Shell Method:**

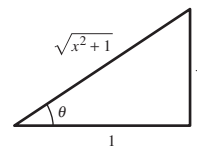
$$\begin{aligned} V &= 4\pi \int_{h-r}^{h+r} x \sqrt{r^2 - (x - h)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) r \cos \theta (r \cos \theta) d\theta = 4\pi r^2 \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi r^2 \left[ \frac{h}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + r \int_{-\pi/2}^{\pi/2} \sin \theta \cos^2 \theta d\theta \right] \\ &= 2\pi r^2 h \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} - \left[ 4\pi r^3 \left( \frac{\cos^3 \theta}{3} \right) \right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h \end{aligned}$$



73.  $y = \ln x$ ,  $y' = \frac{1}{x}$ ,  $1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

Let  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ ,  $\sqrt{x^2 + 1} = \sec \theta$ .

$$\begin{aligned} s &= \int_1^5 \frac{\sqrt{x^2 + 1}}{x^2} dx = \int_1^5 \frac{\sec \theta}{\tan^2 \theta} \sec^2 \theta d\theta = \int_a^b \frac{\sec \theta}{\tan^2 \theta} (1 + \tan^2 \theta) d\theta \\ &= \int_a^b (\csc \theta + \sec \theta \tan \theta) d\theta = [-\ln |\csc \theta + \cot \theta| + \sec \theta]_a^b \\ &= \left[ -\ln \left| \frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x} \right| + \sqrt{x^2 + 1} \right]_1^5 \\ &= \left[ -\ln \left( \frac{\sqrt{26} + 1}{5} \right) + \sqrt{26} \right] - \left[ -\ln(\sqrt{2} + 1) + \sqrt{2} \right] \\ &= \ln \left[ \frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1} \right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln \left[ \frac{\sqrt{26} - 1}{5(\sqrt{2} - 1)} \right] + \sqrt{26} - \sqrt{2} \end{aligned}$$



74.  $y = \frac{1}{2}x^2$ ,  $y' = x$ ,  $1 + (y')^2 = 1 + x^2$

$$s = \int_0^4 \sqrt{1 + x^2} \, dx = \left[ \frac{1}{2} \left( x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}| \right) \right]_0^4 \quad (\text{Theorem 8.2})$$

$$= \frac{1}{2} \left[ 4\sqrt{17} + \ln(4 + \sqrt{17}) \right] \approx 9.2936$$

75. Length of one arch of sine curve:  $y = \sin x$ ,  $y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx$$

Length of one arch of cosine curve:  $y = \cos x$ ,  $y' = -\sin x$

$$L_2 = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} \, dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 \left( x - \frac{\pi}{2} \right)} \, dx, \quad u = x - \frac{\pi}{2}, \, du = dx$$

$$= \int_{-\pi}^0 \sqrt{1 + \cos^2 u} \, du$$

$$= \int_0^\pi \sqrt{1 + \cos^2 u} \, du = L_1$$

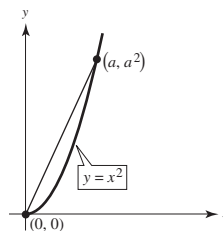
76. (a) Along line:  $d_1 = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$

Along parabola:  $y = x^2$ ,  $y' = 2x$

$$d_2 = \int_0^a \sqrt{1 + 4x^2} \, dx$$

$$= \frac{1}{4} \left[ 2x\sqrt{4x^2 + 1} + \ln|2x + \sqrt{4x^2 + 1}| \right]_0^a \quad (\text{Theorem 8.2})$$

$$= \frac{1}{4} \left[ 2a\sqrt{4a^2 + 1} + \ln(2a + \sqrt{4a^2 + 1}) \right]$$

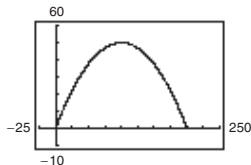


(b) For  $a = 1$ ,  $d_1 = \sqrt{2}$  and  $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}) \approx 1.4789$ .

For  $a = 10$ ,  $d_1 = 10\sqrt{101} \approx 100.4988$ ,  $d_2 \approx 101.0473$ .

(c) As  $a$  increases,  $d_2 - d_1 \rightarrow 0$ .

77. (a)



(b)  $y = 0$  for  $x = 200$  (range)

(c)  $y = x - 0.005x^2$ ,  $y' = 1 - 0.01x$ ,  $1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let  $u = 1 - 0.01x$ ,  $du = -0.01 \, dx$ ,  $a = 1$ . (See Theorem 8.2.)

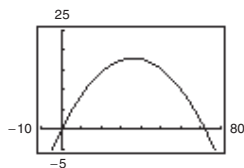
$$s = \int_0^{200} \sqrt{1 + (1 - 0.01x)^2} \, dx = -100 \int_0^{200} \sqrt{(1 - 0.01x)^2 + 1} (-0.01) \, dx$$

$$= -50 \left[ (1 - 0.01x) \sqrt{(1 - 0.01x)^2 + 1} + \ln \left| (1 - 0.01x) + \sqrt{(1 - 0.01x)^2 + 1} \right| \right]_0^{200}$$

$$= -50 \left[ \left( -\sqrt{2} + \ln|-1 + \sqrt{2}| \right) - \left( \sqrt{2} + \ln|1 + \sqrt{2}| \right) \right] = 100\sqrt{2} + 50 \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 229.559$$



78. (a)



(b)  $y = 0$  for  $x = 72$

$$(c) \quad y = x - \frac{x^2}{72}, \quad y' = 1 - \frac{x}{36}, \quad 1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$$

$$\begin{aligned} s &= \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \, dx = -36 \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \left(-\frac{1}{36}\right) dx \\ &= -\frac{36}{2} \left[ \left(1 - \frac{x}{36}\right) \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} + \ln \left| \left(1 - \frac{x}{36}\right) + \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \right| \right]_0^{72} \\ &= -18 \left[ \left(-\sqrt{2} + \ln |-1 + \sqrt{2}|\right) - \left(\sqrt{2} + \ln |1 + \sqrt{2}|\right) \right] = 36\sqrt{2} + 18 \ln \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 82.641 \end{aligned}$$

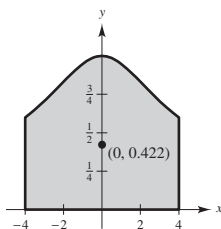
79. Let  $x = 3 \tan \theta$ ,  $dx = 3 \sec^2 \theta \, d\theta$ ,  $\sqrt{x^2 + 9} = 3 \sec \theta$ .

$$\begin{aligned} A &= 2 \int_0^4 \frac{3}{\sqrt{x^2 + 9}} \, dx = 6 \int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6 \int_a^b \frac{3 \sec^2 \theta \, d\theta}{3 \sec \theta} \\ &= 6 \int_a^b \sec \theta \, d\theta = \left[ 6 \ln |\sec \theta + \tan \theta| \right]_a^b = \left[ 6 \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| \right]_0^4 = 6 \ln 3 \end{aligned}$$

$\bar{x} = 0$  (by symmetry)

$$\bar{y} = \frac{1}{2} \left( \frac{1}{A} \right) \int_{-4}^4 \left( \frac{3}{\sqrt{x^2 + 9}} \right)^2 dx = \frac{9}{12 \ln 3} \int_{-4}^4 \frac{1}{x^2 + 9} dx = \frac{3}{4 \ln 3} \left[ \frac{1}{3} \arctan \frac{x}{3} \right]_{-4}^4 = \frac{2}{4 \ln 3} \arctan \frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left( 0, \frac{1}{2 \ln 3} \arctan \frac{4}{3} \right) \approx (0, 0.422)$$



80. First find where the curves intersect.

$$y^2 = 16 - (x - 4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x - 4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x - 4)(x^2 + 4x + 32) \Rightarrow x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2 dx + \frac{1}{4}\pi(4)^2 = \left[\frac{1}{12}x^3\right]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left(\frac{1}{4}x^2\right) dx + \int_4^8 x\sqrt{16 - (x - 4)^2} dx + \int_4^8 4\sqrt{16 - (x - 4)^2} dx$$

$$= \left[\frac{x^4}{16}\right]_0^4 + \int_4^8 (x - 4)\sqrt{16 - (x - 4)^2} dx + \int_4^8 4\sqrt{16 - (x - 4)^2} dx$$

$$= 16 + \left[\frac{-1}{3}(16 - (x - 4)^2)^{3/2}\right]_4^8 + 2\left[16 \arcsin \frac{x - 4}{4} + (x - 4)\sqrt{16 - (x - 4)^2}\right]_4^8$$

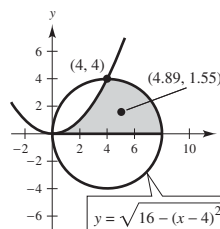
$$= 16 + \frac{1}{3}16^{3/2} + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$

$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}(16 - (x - 4)^2) dx = \left[\frac{1}{32} \cdot \frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x - 4)^3}{6}\right]_4^8 = \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$



81.  $y = x^2$ ,  $y' = 2x$ ,  $1 + (y')^2 = 1 + 4x^2$

$$2x = \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{1 + 4x^2} = \sec \theta$$

(For  $\int \sec^5 \theta d\theta$  and  $\int \sec^3 \theta d\theta$ , see Exercise 102 in Section 8.3.)

$$S = 2\pi \int_0^{\sqrt{2}} x^2 \sqrt{1 + 4x^2} dx = 2\pi \int_a^b \left(\frac{\tan \theta}{2}\right)^2 (\sec \theta) \left(\frac{1}{2} \sec^2 \theta\right) d\theta$$

$$= \frac{\pi}{4} \int_a^b \sec^3 \theta \tan^2 \theta d\theta = \frac{\pi}{4} \left[ \int_a^b \sec^5 \theta d\theta - \int_a^b \sec^3 \theta d\theta \right]$$

$$= \frac{\pi}{4} \left\{ \frac{1}{4} \left[ \sec^3 \theta \tan \theta + \frac{3}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right\} \Big|_a^b$$

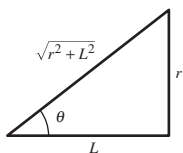
$$= \frac{\pi}{4} \left[ \frac{1}{4} \left[ (1 + 4x^2)^{3/2} (2x) \right] - \frac{1}{8} \left[ (1 + 4x^2)^{1/2} (2x) + \ln \left| \sqrt{1 + 4x^2} + 2x \right| \right] \right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{4} \left[ \frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{8} - \frac{1}{8} \ln(3 + 2\sqrt{2}) \right]$$

$$= \frac{\pi}{4} \left( \frac{51\sqrt{2}}{4} - \frac{\ln(3 + 2\sqrt{2})}{8} \right) = \frac{\pi}{32} [102\sqrt{2} - \ln(3 + 2\sqrt{2})] \approx 13.989$$

82. Let  $r = L \tan \theta$ ,  $dr = L \sec^2 \theta d\theta$ ,  $r^2 + L^2 = L^2 \sec^2 \theta$ .

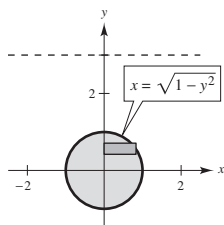
$$\frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr = \frac{2mL}{R} \int_a^b \frac{L \sec^2 \theta d\theta}{L^3 \sec^3 \theta} = \frac{2m}{RL} \int_a^b \cos \theta d\theta = \left[ \frac{2m}{RL} \sin \theta \right]_a^b = \left[ \frac{2m}{RL} \frac{r}{\sqrt{r^2 + L^2}} \right]_0^R = \frac{2m}{L\sqrt{R^2 + L^2}}$$



83. (a) Area of representative rectangle:  $2\sqrt{1 - y^2} \Delta y$

$$\text{Force: } 2(62.4)(3 - y)\sqrt{1 - y^2} \Delta y$$

$$\begin{aligned} F &= 124.8 \int_{-1}^1 (3 - y)\sqrt{1 - y^2} dy \\ &= 124.8 \left[ 3 \int_{-1}^1 \sqrt{1 - y^2} dy - \int_{-1}^1 y\sqrt{1 - y^2} dy \right] \\ &= 124.8 \left[ \frac{3}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{2} \left( \frac{2}{3} \right) (1 - y^2)^{3/2} \right]_{-1}^1 = (62.4)3 [\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb} \end{aligned}$$



$$\begin{aligned} \text{(b) } F &= 124.8 \int_{-1}^1 (d - y)\sqrt{1 - y^2} dy = 124.8d \int_{-1}^1 \sqrt{1 - y^2} dy - 124.8 \int_{-1}^1 y\sqrt{1 - y^2} dy \\ &= 124.8 \left( \frac{d}{2} \right) [\arcsin y + y\sqrt{1 - y^2}]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb} \end{aligned}$$

$$\begin{aligned} \text{84. (a) } F_{\text{inside}} &= 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2} dy \\ &= 96 \left[ 0.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy - \int_{-1}^{0.8} y\sqrt{1 - y^2} dy \right] \\ &= 96 \left[ \frac{0.8}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.8} \\ &\approx 96(1.263) \approx 121.3 \text{ lb} \end{aligned}$$

$$\begin{aligned} \text{(b) } F_{\text{outside}} &= 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2} dy \\ &= 128 \left[ 0.4 \int_{-1}^{0.4} \sqrt{1 - y^2} dy - \int_{-1}^{0.4} y\sqrt{1 - y^2} dy \right] \\ &= 128 \left[ \frac{0.4}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.4} \approx 92.98 \end{aligned}$$

85. Let  $u = a \sin \theta$ ,  $du = a \cos \theta d\theta$ ,  $\sqrt{a^2 - u^2} = a \cos \theta$ .

$$\begin{aligned}\int \sqrt{a^2 - u^2} du &= \int a^2 \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta \\&= \frac{a^2}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + C \\&= \frac{a^2}{2} \left[ \arcsin \frac{u}{a} + \left( \frac{u}{a} \right) \left( \frac{\sqrt{a^2 - u^2}}{a} \right) \right] + C = \frac{1}{2} \left( a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right) + C\end{aligned}$$

Let  $u = a \sec \theta$ ,  $du = a \sec \theta \tan \theta d\theta$ ,  $\sqrt{u^2 - a^2} = a \tan \theta$ .

$$\begin{aligned}\int \sqrt{u^2 - a^2} du &= \int a \tan \theta (a \sec \theta \tan \theta) d\theta = a^2 \int \tan^2 \theta \sec \theta d\theta \\&= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta = a^2 \int (\sec^3 \theta - \sec \theta) d\theta \\&= a^2 \left[ \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] - a^2 \int \sec \theta d\theta = a^2 \left[ \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] \\&= \frac{a^2}{2} \left[ \frac{u}{a} \cdot \frac{\sqrt{u^2 - a^2}}{a} - \ln \left| \frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a} \right| \right] + C_1 = \frac{1}{2} \left[ u \sqrt{u^2 - a^2} - a^2 \ln \left| u + \sqrt{u^2 - a^2} \right| \right] + C\end{aligned}$$

Let  $u = a \tan \theta$ ,  $du = a \sec^2 \theta d\theta$ ,  $\sqrt{u^2 + a^2} = a \sec \theta$ .

$$\begin{aligned}\int \sqrt{u^2 + a^2} du &= \int (a \sec \theta) (a \sec^2 \theta) d\theta \\&= a^2 \int \sec^3 \theta d\theta = a^2 \left[ \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C_1 \\&= \frac{a^2}{2} \left[ \frac{\sqrt{u^2 + a^2}}{a} \cdot \frac{u}{a} + \ln \left| \frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a} \right| \right] + C_1 = \frac{1}{2} \left[ u \sqrt{u^2 + a^2} + a^2 \ln \left| u + \sqrt{u^2 + a^2} \right| \right] + C\end{aligned}$$

86.  $y = \sin x$  on  $[0, 2]$

$$y' = \cos x$$

$$s_1 = 2 \int_0^\pi \sqrt{1 + \cos^2 x} dx \quad (\approx 3.820197789)$$

$$\text{Ellipse: } x^2 + 2y^2 = 2$$

$$\text{Upper half: } y = \sqrt{1 - \frac{1}{2}x^2}, \quad -\sqrt{2} \leq x \leq \sqrt{2}$$

$$y' = \frac{-x}{2\sqrt{1 - (1/2)x^2}}$$

$$s_2 = 2 \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \frac{x^2}{4(1 - (1/2)x^2)}} dx = 2 \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \frac{x^2}{4 - 2x^2}} dx$$

$$\text{Let } x = \sqrt{2} \sin \theta, dx = \sqrt{2} \cos \theta d\theta, x^2 = 2 \sin^2 \theta, 4 - 2x^2 = 4 - 4 \sin^2 \theta = 4 \cos^2 \theta.$$

$$\begin{aligned}s_2 &= 2 \int_{-\pi/2}^{\pi/2} \sqrt{1 + \frac{2 \sin^2 \theta}{4 \cos^2 \theta}} \sqrt{2} \cos \theta d\theta \\&= 2 \int_{-\pi/2}^{\pi/2} \frac{\sqrt{4 \cos^2 \theta + 2 \sin^2 \theta}}{2 \cos \theta} \sqrt{2} \cos \theta d\theta \\&= 2 \int_{-\pi/2}^{\pi/2} \frac{\sqrt{2 + 2 \cos^2 \theta}}{\sqrt{2}} d\theta = 2 \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 \theta} d\theta = 2 \int_0^\pi \sqrt{1 + \cos^2 \theta} d\theta = s_1\end{aligned}$$

87. Large circle:  $x^2 + y^2 = 25$

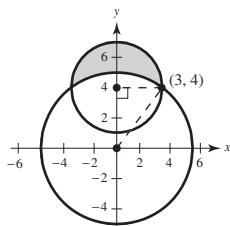
$$y = \sqrt{25 - x^2}, \text{ upper half}$$

From the right triangle, the center of the small circle is  $(0, 4)$ .

$$x^2 + (y - 4)^2 = 9$$

$$y = 4 + \sqrt{9 - x^2}, \text{ upper half}$$

$$\begin{aligned} A &= 2 \int_0^3 \left[ \left( 4 + \sqrt{9 - x^2} \right) - \sqrt{25 - x^2} \right] dx \\ &= 2 \left[ 4x + \frac{1}{2} \left[ 9 \arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2} \right] - \frac{1}{2} \left[ 25 \arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2} \right] \right]_0^3 \\ &= 2 \left[ 12 + \frac{9}{2} \arcsin(1) - \frac{25}{2} \arcsin \frac{3}{5} - 6 \right] \\ &= 12 + \frac{9\pi}{2} - 25 \arcsin \frac{3}{5} \approx 10.050 \end{aligned}$$

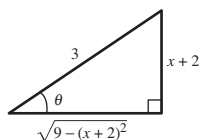


88. The left circle has equation  $(x + 2)^2 + y^2 = 9$ . The shaded area is four times the area in the first quadrant, under the curve

$$y = \sqrt{9 - (x + 2)^2}.$$

$$A = 4 \int_0^1 \sqrt{9 - (x + 2)^2} dx$$

$$\text{Let } x + 2 = 3 \sin \theta, dx = 3 \cos \theta d\theta, \sqrt{9 - (x + 2)^2} = 3 \cos \theta$$



$$\begin{aligned} \int \sqrt{9 - (x + 2)^2} dx &= \int 3 \cos \theta (3 \cos \theta) d\theta = 9 \int \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{9}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) + C = \frac{9}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{9}{2} \left( \arcsin\left(\frac{x+2}{3}\right) + \left(\frac{x+2}{3}\right) \left( \frac{\sqrt{9 - (x+2)^2}}{3} \right) \right) + C \end{aligned}$$

$$A = 4 \left[ \frac{9}{2} \left( \arcsin\left(\frac{x+2}{3}\right) + \left(\frac{x+2}{3}\right) \left( \frac{\sqrt{9 - (x+2)^2}}{3} \right) \right) \right]_0^1 = 18 \left[ \left( \frac{\pi}{2} \right) + 0 \right] - \left( \arcsin \frac{2}{3} + \frac{2\sqrt{5}}{3} \right) = 9\pi - 18 \arcsin \frac{2}{3} - 4\sqrt{5}$$

89. Let  $I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx$

Let  $x = \frac{1-u}{1+u}$ ,  $dx = \frac{-2}{(1+u)^2} du$

$x+1 = \frac{2}{1+u}$ ,  $x^2+1 = \frac{2+2u^2}{(1+u)^2}$

$$I = \int_1^0 \frac{\ln\left(\frac{2}{1+u}\right)}{\frac{2+2u^2}{(1+u)^2}} \left(\frac{-2}{(1+u)^2}\right) du$$

$$= \int_1^0 \frac{-\ln\left(\frac{2}{1+u}\right)}{1+u^2} du = \int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)}{1+u^2} du = \int_0^1 \frac{\ln 2}{1+u^2} - \int_0^1 \frac{\ln(1+u)}{1+u^2} du = (\ln 2)[\arctan u]_0^1 - I$$

$$\Rightarrow 2I = \ln 2 \left(\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{8} \ln 2 \approx 0.272198$$

## Section 8.5 Partial Fractions

1.  $\frac{4}{x^2-8x} = \frac{4}{x(x-8)} = \frac{A}{x} + \frac{B}{x-8}$

2.  $\frac{2x^2+1}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

3.  $\frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

4.  $\frac{x-4}{x^2+6x+5} = \frac{x-4}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$

5.  $\frac{x-9}{x^2-6x} = \frac{x-9}{x(x-6)} = \frac{A}{x} + \frac{B}{x-6}$

6.  $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

7.  $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x+3} + \frac{B}{x-3}$   
 $1 = A(x-3) + B(x+3)$

When  $x = 3$ ,  $1 = 6B \Rightarrow B = \frac{1}{6}$ .

When  $x = -3$ ,  $1 = -6A \Rightarrow A = -\frac{1}{6}$ .

$$\begin{aligned} \int \frac{1}{x^2-9} dx &= -\frac{1}{6} \int \frac{1}{x+3} dx + \frac{1}{6} \int \frac{1}{x-3} dx \\ &= -\frac{1}{6} \ln|x+3| + \frac{1}{6} \ln|x-3| + C \\ &= \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| + C \end{aligned}$$

8.  $\frac{1}{4x^2-1} = \frac{1}{(2x-1)(2x+1)} = \frac{A}{2x+1} + \frac{B}{2x-1}$   
 $1 = A(2x-1) + B(2x+1)$

When  $x = \frac{1}{2}$ ,  $1 = 2B \Rightarrow B = \frac{1}{2}$ .

When  $x = -\frac{1}{2}$ ,  $1 = -2A \Rightarrow A = -\frac{1}{2}$ .

$$\begin{aligned} \int \frac{1}{4x^2-1} dx &= -\frac{1}{2} \int \frac{1}{2x+1} dx + \frac{1}{2} \int \frac{1}{2x-1} dx \\ &= -\frac{1}{4} \ln|2x+1| + \frac{1}{4} \ln|2x-1| + C \\ &= \frac{1}{4} \ln \left| \frac{2x-1}{2x+1} \right| + C \end{aligned}$$

# INSTRUCTOR USE ONLY

$$9. \frac{5}{x^2 + 3x - 4} = \frac{5}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

$$5 = A(x-1) + B(x+4)$$

$$\text{When } x = 1, \quad 5 = 5B \Rightarrow B = 1.$$

$$\text{When } x = -4, \quad 5 = -5A \Rightarrow A = -1.$$

$$\int \frac{5}{x^2 + 3x - 4} dx = \int \frac{-1}{x+4} dx + \int \frac{1}{x-1} dx$$

$$= -\ln|x+4| + \ln|x-1| + C$$

$$= \ln \left| \frac{x-1}{x+4} \right| + C$$

$$10. \frac{x+2}{x^2 + 11x + 18} = \frac{x+2}{(x+9)(x+2)} = \frac{1}{x+9}$$

$$\int \frac{x+2}{x^2 + 11x + 18} dx = \int \frac{1}{x+9} dx = \ln|x+9| + C$$

$$13. \frac{x^2 + 12x + 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

$$\text{When } x = 0, 12 = -4A \Rightarrow A = -3.$$

$$\text{When } x = -2, -8 = 8B \Rightarrow B = -1.$$

$$\text{When } x = 2, 40 = 8C \Rightarrow C = 5.$$

$$\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx = 5 \int \frac{1}{x-2} dx - \int \frac{1}{x+2} dx - 3 \int \frac{1}{x} dx = 5 \ln|x-2| - \ln|x+2| - 3 \ln|x| + C$$

$$14. \frac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \frac{2x+1}{(x+2)(x-1)} = x - 1 + \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x+1 = A(x-1) + B(x+2)$$

$$\text{When } x = -2, -3 = -3A \Rightarrow A = 1.$$

$$\text{When } x = 1, 3 = 3B \Rightarrow B = 1.$$

$$\int \frac{x^3 - x + 3}{x^2 + x - 2} dx = \int \left( x - 1 + \frac{1}{x+2} + \frac{1}{x-1} \right) dx = \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2 + x - 2| + C$$

$$11. \frac{5-x}{2x^2 + x - 1} = \frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$5-x = A(x+1) + B(2x-1)$$

$$\text{When } x = \frac{1}{2}, \frac{9}{2} = \frac{3}{2}A \Rightarrow A = 3.$$

$$\text{When } x = -1, 6 = -3B \Rightarrow B = -2.$$

$$\int \frac{5-x}{2x^2 + x - 1} dx = 3 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x+1} dx$$

$$= \frac{3}{2} \ln|2x-1| - 2 \ln|x+1| + C$$

$$12. \frac{5x^2 - 12x - 12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$5x^2 - 12x - 12 = A(x^2 - 4) + Bx(x+2) + Cx(x-2)$$

$$\text{When } x = 0, -12 = -4A \Rightarrow A = 3.$$

$$\text{When } x = 2, -16 = 8B \Rightarrow B = -2.$$

$$\text{When } x = -2, 32 = 8C \Rightarrow C = 4.$$

$$\int \frac{5x^2 - 12x - 12}{x^3 - 4x} dx$$

$$= \int \frac{3}{x} dx + \int \frac{-2}{x-2} dx + \int \frac{4}{x+2} dx$$

$$= 3 \ln|x| - 2 \ln|x-2| + 4 \ln|x+2| + C$$

NOT FOR SALE

$$15. \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \frac{x + 5}{(x - 4)(x + 2)} = 2x + \frac{A}{x - 4} + \frac{B}{x + 2}$$

$$x + 5 = A(x + 2) + B(x - 4)$$

$$\text{When } x = 4, 9 = 6A \Rightarrow A = \frac{3}{2}.$$

$$\text{When } x = -2, 3 = -6B \Rightarrow B = -\frac{1}{2}.$$

$$\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx = \int \left( 2x + \frac{3/2}{x - 4} - \frac{1/2}{x + 2} \right) dx = x^2 + \frac{3}{2} \ln|x - 4| - \frac{1}{2} \ln|x + 2| + C$$

$$16. \frac{x + 2}{x(x - 4)} = \frac{A}{x - 4} + \frac{B}{x}$$

$$x + 2 = Ax + B(x - 4)$$

$$\text{When } x = 4, 6 = 4A \Rightarrow A = \frac{3}{2}.$$

$$\text{When } x = 0, 2 = -4B \Rightarrow B = -\frac{1}{2}.$$

$$\int \frac{x + 2}{x^2 - 4x} dx = \int \left( \frac{3/2}{x - 4} - \frac{1/2}{x} \right) dx$$

$$= \frac{3}{2} \ln|x - 4| - \frac{1}{2} \ln|x| + C$$

$$17. \frac{4x^2 + 2x - 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$

$$4x^2 + 2x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

$$\text{When } x = 0, B = -1.$$

$$\text{When } x = -1, C = 1.$$

$$\text{When } x = 1, A = 3.$$

$$\int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx = \int \left( \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x + 1} \right) dx$$

$$= 3 \ln|x| + \frac{1}{x} + \ln|x + 1| + C$$

$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

$$18. \frac{3x - 4}{(x - 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2}$$

$$3x - 4 = A(x - 1) + B$$

$$\text{When } x = 1, -1 = B.$$

$$\text{When } x = 0, -4 = -A - 1 \Rightarrow A = 3.$$

$$\int \frac{3x - 4}{(x - 1)^2} dx = \int \frac{3}{x - 1} dx - \int \frac{1}{(x - 1)^2} dx = 3 \ln|x - 1| + \frac{1}{x - 1} + C$$

$$19. \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} = \frac{x^2 + 3x - 4}{x(x - 2)^2} = \frac{A}{x} + \frac{B}{(x - 2)} + \frac{C}{(x - 2)^2}$$

$$x^2 + 3x - 4 = A(x - 2)^2 + Bx(x - 2) + Cx$$

$$\text{When } x = 0, -4 = 4A \Rightarrow A = -1.$$

$$\text{When } x = 2, 6 = 2C \Rightarrow C = 3.$$

$$\text{When } x = 1, 0 = -1 - B + 3 \Rightarrow B = 2.$$

$$\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} dx = \int \frac{-1}{x} dx + \int \frac{2}{(x - 2)} dx + \int \frac{3}{(x - 2)^2} dx = -\ln|x| + 2 \ln|x - 2| - \frac{3}{(x - 2)} + C$$

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$$20. \frac{4x^2}{x^3 + x^2 - x - 1} = \frac{4x^2}{x^2(x+1) - (x+1)} = \frac{4x^2}{(x^2-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\text{When } x = -1, 4 = -2C \Rightarrow C = -2.$$

$$\text{When } x = 1, 4 = 4A \Rightarrow A = 1.$$

$$\text{When } x = 0, 0 = 1 - B + 2 \Rightarrow B = 3.$$

$$\int \frac{4x^2}{x^3 + x^2 - x - 1} dx = \int \frac{1}{x-1} dx + \int \frac{3}{x+1} dx - \int \frac{2}{(x+1)^2} dx = \ln|x-1| + 3\ln|x+1| + \frac{2}{(x+1)} + C$$

$$21. \frac{x^2 - 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

$$\text{When } x = 0, A = -1.$$

$$\text{When } x = 1, 0 = -2 + B + C.$$

$$\text{When } x = -1, 0 = -2 + B - C.$$

$$\text{Solving these equations you have } A = -1, B = 2, C = 0.$$

$$\int \frac{x^2 - 1}{x^3 + x} dx = -\int \frac{1}{x} dx + \int \frac{2x}{x^2 + 1} dx = -\ln|x| + \ln|x^2 + 1| + C = \ln\left|\frac{x^2 + 1}{x}\right| + C$$

$$22. \frac{6x}{x^3 - 8} = \frac{6x}{(x-2)(x^2 + 2x + 4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 2x + 4}$$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x - 2)$$

$$\text{When } x = 2, 12 = 12A \Rightarrow A = 1.$$

$$\text{When } x = 0, 0 = 4 - 2C \Rightarrow C = 2.$$

$$\text{When } x = 1, 6 = 7 + (B + 2)(-1) \Rightarrow B = -1.$$

$$\begin{aligned} \int \frac{6x}{x^3 - 8} dx &= \int \frac{1}{x-2} dx + \int \frac{-x+2}{x^2+2x+4} dx \\ &= \int \frac{1}{x-2} dx + \int \frac{-x-1}{x^2+2x+4} dx + \int \frac{3}{(x^2+2x+1)+3} dx \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2+2x+4| + \frac{3}{\sqrt{3}} \arctan\left(\frac{x+1}{\sqrt{3}}\right) + C \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2+2x+4| + \sqrt{3} \arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C \end{aligned}$$

$$23. \frac{x^2}{x^4 - 2x^2 - 8} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2}$$

$$x^2 = A(x+2)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x+2)(x-2)$$

When  $x = 2, 4 = 24A$ .

When  $x = -2, 4 = -24B$ .

When  $x = 0, 0 = 4A - 4B - 4D$ .

When  $x = 1, 1 = 9A - 3B - 3C - 3D$ .

Solving these equations you have  $A = \frac{1}{6}, B = -\frac{1}{6}, C = 0, D = \frac{1}{3}$ .

$$\int \frac{x^2}{x^4 - 2x^2 - 8} dx = \frac{1}{6} \left( \int \frac{1}{x-2} dx - \int \frac{1}{x+2} dx + 2 \int \frac{1}{x^2+2} dx \right) = \frac{1}{6} \left( \ln \left| \frac{x-2}{x+2} \right| + \sqrt{2} \arctan \frac{x}{\sqrt{2}} \right) + C$$

$$24. \frac{x^2 - x + 9}{(x^2 + 9)^2} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2}$$

$$x^2 - x + 9 = (Ax+B)(x^2+9) + Cx+D$$

$$= Ax^3 + Bx^2 + (9A+C)x + (9B+D)$$

By equating coefficients of like terms, you have  $A = 0, B = 1, D = 0$ , and  $C = -1$ .

$$\int \frac{x^2 - x + 9}{(x^2 + 9)^2} dx = \int \frac{1}{x^2 + 9} dx - \int \frac{x}{(x^2 + 9)^2} dx = \frac{1}{3} \arctan \left( \frac{x}{3} \right) + \frac{1}{2(x^2 + 9)} + C$$

$$25. \frac{x}{(2x-1)(2x+1)(4x^2+1)} = \frac{A}{2x-1} + \frac{B}{2x+1} + \frac{Cx+D}{4x^2+1}$$

$$x = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

When  $x = \frac{1}{2}, \frac{1}{2} = 4A$ .

When  $x = -\frac{1}{2}, -\frac{1}{2} = -4B$ .

When  $x = 0, 0 = A - B - D$ .

When  $x = 1, 1 = 15A + 5B + 3C + 3D$ .

Solving these equations you have  $A = \frac{1}{8}, B = \frac{1}{8}, C = -\frac{1}{2}, D = 0$ .

$$\int \frac{x}{16x^4 - 1} dx = \frac{1}{8} \left( \int \frac{1}{2x-1} dx + \int \frac{1}{2x+1} dx - 4 \int \frac{x}{4x^2+1} dx \right) = \frac{1}{16} \ln \left| \frac{4x^2-1}{4x^2+1} \right| + C$$

$$26. \frac{x^2 - 4x + 7}{(x+1)(x^2 - 2x + 3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2 - 2x + 3}$$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx+C)(x+1)$$

When  $x = -1, 12 = 6A$ .

When  $x = 0, 7 = 3A + C$ .

When  $x = 1, 4 = 2A + 2B + 2C$ .

Solving these equations you have  $A = 2, B = -1, C = 1$ .

$$\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3} dx = 2 \int \frac{1}{x+1} dx + \int \frac{-x+1}{x^2 - 2x + 3} dx = 2 \ln|x+1| - \frac{1}{2} \ln|x^2 - 2x + 3| + C$$

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$$27. \frac{x^2 + 5}{(x+1)(x^2 - 2x + 3)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$\begin{aligned} x^2 + 5 &= A(x^2 - 2x + 3) + (Bx + C)(x + 1) \\ &= (A + B)x^2 + (-2A + B + C)x + (3A + C) \end{aligned}$$

When  $x = -1$ ,  $A = 1$ .

By equating coefficients of like terms, you have  $A + B = 1$ ,  $-2A + B + C = 0$ ,  $3A + C = 5$ .

Solving these equations you have  $A = 1$ ,  $B = 0$ ,  $C = 2$ .

$$\int \frac{x^2 + 5}{x^3 - x^2 + x + 3} dx = \int \frac{1}{x+1} dx + 2 \int \frac{1}{(x-1)^2 + 2} dx = \ln|x+1| + \sqrt{2} \arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

$$28. \frac{x^2 + x + 3}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

$$\begin{aligned} x^2 + x + 3 &= (Ax + B)(x^2 + 3) + Cx + D \\ &= Ax^3 + Bx^2 + (3A + C)x + (3B + D) \end{aligned}$$

By equating coefficients of like terms, you have

$$A = 0, B = 1, 3A + C = 1, 3B + D = 3.$$

Solving these equations you have

$$A = 0, B = 1, C = 1, D = 0.$$

$$\begin{aligned} \int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9} dx &= \int \left( \frac{1}{x^2 + 3} + \frac{x}{(x^2 + 3)^2} \right) dx \\ &= \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{1}{2(x^2 + 3)} + C \end{aligned}$$

$$29. \frac{3}{4x^2 + 5x + 1} = \frac{3}{(4x+1)(x+1)} = \frac{A}{4x+1} + \frac{B}{x+1}$$

$$3 = A(x+1) + B(4x+1)$$

When  $x = -1$ ,  $3 = -3B \Rightarrow B = -1$ .

When  $-\frac{1}{4}$ ,  $3 = \frac{3}{4}A \Rightarrow A = 4$ .

$$\begin{aligned} \int_0^2 \frac{3}{4x^2 + 5x + 1} dx &= \int_0^2 \frac{4}{4x+1} dx + \int_0^2 \frac{-1}{x+1} dx \\ &= [\ln|4x+1| - \ln|x+1|]_0^2 \\ &= \ln 9 - \ln 3 \\ &= 2 \ln 3 - \ln 3 = \ln 3 \end{aligned}$$

$$30. \frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$x-1 = Ax(x+1) + B(x+1) + Cx^2$$

When  $x = 0$ ,  $B = -1$ .

When  $x = -1$ ,  $C = -2$ .

When  $x = 1$ ,  $0 = 2A + 2B + C$ .

Solving these equations you have

$$A = 2, B = -1, C = -2.$$

$$\begin{aligned} \int_1^5 \frac{x-1}{x^2(x+1)} dx &= 2 \int_1^5 \frac{1}{x} dx - \int_1^5 \frac{1}{x^2} dx - 2 \int_1^5 \frac{1}{x+1} dx \\ &= \left[ 2 \ln|x| + \frac{1}{x} - 2 \ln|x+1| \right]_1^5 \\ &= \left[ 2 \ln \left| \frac{x}{x+1} \right| + \frac{1}{x} \right]_1^5 \\ &= 2 \ln \frac{5}{3} - \frac{4}{5} \end{aligned}$$

$$31. \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x+1 = A(x^2+1) + (Bx+C)x$$

When  $x = 0$ ,  $A = 1$ .

When  $x = 1$ ,  $2 = 2A + B + C$ .

When  $x = -1$ ,  $0 = 2A + B - C$ .

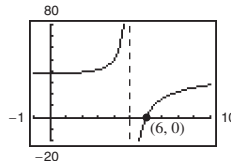
Solving these equations we have

$$A = 1, B = -1, C = 1.$$

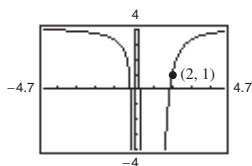
$$\begin{aligned} \int_1^2 \frac{x+1}{x(x^2+1)} dx &= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{x^2+1} dx + \int_1^2 \frac{1}{x^2+1} dx \\ &= \left[ \ln|x| - \frac{1}{2} \ln(x^2+1) + \arctan x \right]_1^2 \\ &= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2 \\ &\approx 0.557 \end{aligned}$$

$$\begin{aligned}
 32. \int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx &= \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1} dx \\
 &= \left[ x - \ln|x^2 + x + 1| \right]_0^1 \\
 &= 1 - \ln 3
 \end{aligned}$$

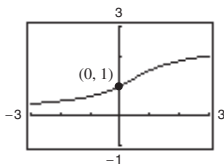
$$\begin{aligned}
 33. \int \frac{5x}{x^2 - 10x + 25} dx &= 5 \ln|x - 5| - \frac{5x}{x - 5} + C \\
 (6, 0): 5 \ln(1) - \frac{30}{1} + C &= 0 \Rightarrow C = 30 \\
 y &= 5 \ln|x - 5| - \frac{5x}{x - 5} + 30
 \end{aligned}$$



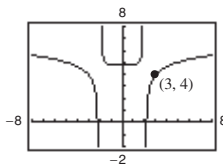
$$\begin{aligned}
 34. \int \frac{6x^2 + 1}{x^2(x - 1)^3} dx &= 3 \ln \left| \frac{x - 1}{x} \right| + \frac{1}{x} + \frac{2}{x - 1} - \frac{7}{2(x - 1)^2} + C \\
 (2, 1): 3 \ln \left| \frac{1}{2} \right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C &= 1 \Rightarrow C = 2 - 3 \ln \frac{1}{2}
 \end{aligned}$$



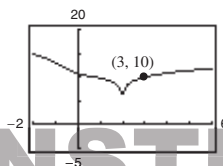
$$\begin{aligned}
 35. \int \frac{x^2 + x + 2}{(x^2 + 2)^2} dx &= \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} - \frac{1}{2(x^2 + 2)} + C \\
 (0, 1): 0 - \frac{1}{4} + C &= 1 \Rightarrow C = \frac{5}{4}
 \end{aligned}$$



$$\begin{aligned}
 36. \int \frac{x^3}{(x^2 - 4)^2} dx &= \frac{1}{2} \ln|x^2 - 4| - \frac{2}{x^2 - 4} + C \\
 (3, 4): \frac{1}{2} \ln 5 - \frac{2}{5} + C &= 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2} \ln 5
 \end{aligned}$$

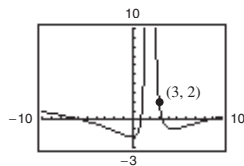


$$\begin{aligned}
 37. \int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2} dx &= \ln|x - 2| + \frac{1}{2} \ln|x^2 + x + 1| - \sqrt{3} \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) + C \\
 (3, 10): 0 + \frac{1}{2} \ln 13 - \sqrt{3} \arctan \frac{7}{\sqrt{3}} + C &= 10 \Rightarrow C = 10 - \frac{1}{2} \ln 13 + \sqrt{3} \arctan \frac{7}{\sqrt{3}}
 \end{aligned}$$



$$38. \int \frac{x(2x-9)}{x^3-6x^2+12x-8} dx = 2 \ln|x-2| + \frac{1}{x-2} + \frac{5}{(x-2)^2} + C$$

$$(3, 2): 0 + 1 + 5 + C = 2 \Rightarrow C = -4$$



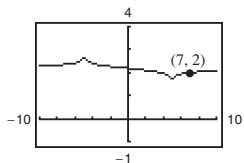
$$39. \int \frac{1}{x^2-25} dx = \frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| + C$$

$$(7, 2): \frac{1}{10} \ln \left| \frac{2}{12} \right| + C = 2$$

$$C = 2 - \frac{1}{10} \ln \left( \frac{1}{6} \right)$$

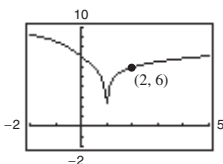
$$= 2 + \frac{1}{10} \ln 6$$

$$y = \frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| + 2 + \frac{1}{10} \ln 6$$



$$40. \int \frac{x^2-x+2}{x^3-x^2+x-1} dx = -\arctan x + \ln|x-1| + C$$

$$(2, 6): -\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$$



$$41. \text{ Let } u = \cos x, du = -\sin x dx.$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{When } u = 0, A = -1.$$

$$\text{When } u = 1, B = 1.$$

$$\begin{aligned} \int \frac{\sin x}{\cos x(\cos x-1)} dx &= -\int \frac{1}{u(u-1)} du \\ &= \int \frac{1}{u} du - \int \frac{1}{u-1} du \\ &= \ln|u| - \ln|u-1| + C \\ &= \ln \left| \frac{u}{u-1} \right| + C \\ &= \ln \left| \frac{\cos x}{\cos x-1} \right| + C \end{aligned}$$

$$42. \text{ Let } u = \cos x, du = -\sin x dx.$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

$$\text{When } u = 0, A = 1.$$

$$\text{When } u = -1, B = -1.$$

$$\begin{aligned} \int \frac{\sin x}{\cos x + \cos^2 x} dx &= -\int \frac{1}{u(u+1)} du \\ &= \int \frac{1}{u+1} du - \int \frac{1}{u} du \\ &= \ln|u+1| - \ln|u| + C \\ &= \ln \left| \frac{u+1}{u} \right| + C \\ &= \ln \left| \frac{\cos x + 1}{\cos x} \right| + C \\ &= \ln|1 + \sec x| + C \end{aligned}$$

43. Let
- $u = \sin x$
- ,
- $du = \cos x \, dx$

$$\frac{1}{u + u^2} = \frac{1}{u(1 + u)} = \frac{A}{u} + \frac{B}{u + 1}$$

$$1 = A(u + 1) + Bu$$

When  $u = 0$ ,  $1 = A$ .When  $u = -1$ ,  $1 = -B \Rightarrow B = -1$ .

$$\int \frac{\cos x}{\sin x + \sin^2 x} \, dx = \int \frac{1}{u + u^2} \, du$$

$$= \int \frac{1}{u} \, du - \int \frac{1}{u + 1} \, du$$

$$= \ln|u| - \ln|u + 1| + C$$

$$= \ln \left| \frac{\sin x}{\sin x + 1} \right| + C$$

$$44. \int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} \, dx = 5 \int \frac{1}{u^2 + 3u - 4} \, du$$

$$= \ln \left| \frac{u - 1}{u + 4} \right| + C$$

$$= \ln \left| \frac{-1 + \sin x}{4 + \sin x} \right| + C$$

(From Exercise 9 with  $u = \sin x$ ,  $du = \cos x \, dx$ )

45. Let
- $u = \tan x$
- ,
- $du = \sec^2 x \, dx$
- .

$$\frac{1}{u^2 + 5u + 6} = \frac{1}{(u + 3)(u + 2)} = \frac{A}{u + 3} + \frac{B}{u + 2}$$

$$1 = A(u + 2) + B(u + 3)$$

When  $u = -2$ ,  $1 = B$ .When  $u = -3$ ,  $1 = -A \Rightarrow A = -1$ .

$$\int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} \, dx = \int \frac{1}{u^2 + 5u + 6} \, du$$

$$= \int \frac{-1}{u + 3} \, du + \int \frac{1}{u + 2} \, du$$

$$= -\ln|u + 3| + \ln|u + 2| + C$$

$$= \ln \left| \frac{\tan x + 2}{\tan x + 3} \right| + C$$

$$46. \frac{1}{u(u + 1)} = \frac{A}{u} + \frac{B}{u + 1}, u = \tan x, du = \sec^2 x \, dx$$

$$1 = A(u + 1) + Bu$$

When  $u = 0$ ,  $A = 1$ .When  $u = -1$ ,  $1 = -B \Rightarrow B = -1$ .

$$\int \frac{\sec^2 x \, dx}{\tan x(\tan x + 1)} = \int \frac{1}{u(u + 1)} \, du$$

$$= \int \left( \frac{1}{u} - \frac{1}{u + 1} \right) \, du$$

$$= \ln|u| - \ln|u + 1| + C$$

$$= \ln \left| \frac{u}{u + 1} \right| + C$$

$$= \ln \left| \frac{\tan x}{\tan x + 1} \right| + C$$

47. Let
- $u = e^x$
- ,
- $du = e^x \, dx$
- .

$$\frac{1}{(u - 1)(u + 4)} = \frac{A}{u - 1} + \frac{B}{u + 4}$$

$$1 = A(u + 4) + B(u - 1)$$

When  $u = 1$ ,  $A = \frac{1}{5}$ .When  $u = -4$ ,  $B = -\frac{1}{5}$ .

$$\int \frac{e^x}{(e^x - 1)(e^x + 4)} \, dx = \int \frac{1}{(u - 1)(u + 4)} \, du$$

$$= \frac{1}{5} \left( \int \frac{1}{u - 1} \, du - \int \frac{1}{u + 4} \, du \right)$$

$$= \frac{1}{5} \ln \left| \frac{u - 1}{u + 4} \right| + C$$

$$= \frac{1}{5} \ln \left| \frac{e^x - 1}{e^x + 4} \right| + C$$

48. Let  $u = e^x$ ,  $du = e^x dx$ .

$$\frac{1}{(u^2 + 1)(u - 1)} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + 1}$$

$$1 = A(u^2 + 1) + (Bu + C)(u - 1)$$

When  $u = 1$ ,  $A = \frac{1}{2}$ .

When  $u = 0$ ,  $1 = A - C$ .

When  $u = -1$ ,  $1 = 2A + 2B - 2C$ .

Solving these equations you have  $A = \frac{1}{2}$ ,  $B = -\frac{1}{2}$ , and  $C = -\frac{1}{2}$ .

$$\begin{aligned} \int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx &= \int \frac{1}{(u^2 + 1)(u - 1)} du \\ &= \frac{1}{2} \left( \int \frac{1}{u - 1} du - \int \frac{u + 1}{u^2 + 1} du \right) \\ &= \frac{1}{2} \left( \ln|u - 1| - \frac{1}{2} \ln|u^2 + 1| - \arctan u \right) + C \\ &= \frac{1}{4} (2 \ln|e^x - 1| - \ln|e^{2x} + 1| - 2 \arctan e^x) + C \end{aligned}$$

49. Let  $u = \sqrt{x}$ ,  $u^2 = x$ ,  $2u du = dx$ .

$$\begin{aligned} \int \frac{\sqrt{x}}{x - 4} dx &= \int \frac{u(2u)du}{u^2 - 4} = \int \left( \frac{2u^2 - 8}{u^2 - 4} + \frac{8}{u^2 - 4} \right) du = \int \left( 2 + \frac{8}{u^2 - 4} \right) du \\ \frac{8}{u^2 - 4} &= \frac{8}{(u - 2)(u + 2)} = \frac{A}{u - 2} + \frac{B}{u + 2} \\ 8 &= A(u + 2) + B(u - 2) \end{aligned}$$

When  $u = -2$ ,  $8 = -4B \Rightarrow B = -2$ .

When  $u = 2$ ,  $8 = 4A \Rightarrow A = 2$ .

$$\begin{aligned} \int \left( 2 + \frac{8}{u^2 - 4} \right) du &= 2u + \int \left( \frac{2}{u - 2} - \frac{2}{u + 2} \right) du \\ &= 2u + 2 \ln|u - 2| - 2 \ln|u + 2| + C \\ &= 2\sqrt{x} + 2 \ln \left| \frac{\sqrt{x} - 2}{\sqrt{x} + 2} \right| + C \end{aligned}$$

50. Let  $u = x^{1/6}$ ,  $u^2 = x^{1/3}$ ,  $u^3 = x^{1/2}$ ,  $u^6 = x$ ,  $6u^5 du = dx$ .

$$\begin{aligned} \int \frac{1}{\sqrt{x} - \sqrt[3]{x}} dx &= \int \frac{6u^5 du}{u^3 - u^2} = 6 \int \frac{u^3 du}{u - 1} \\ &= 6 \int \left( u^2 + u + 1 + \frac{1}{u - 1} \right) du \quad (\text{long division}) \\ &= 6 \left( \frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u - 1| \right) + C \\ &= 2\sqrt{x} + 3x^{1/3} + 6x^{1/6} + 6 \ln|x^{1/6} - 1| + C \end{aligned}$$

$$51. \frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When  $x = 0, 1 = aA \Rightarrow A = 1/a$ .

When  $x = -a/b, 1 = -(a/b)B \Rightarrow B = -b/a$ .

$$\int \frac{1}{x(a+bx)} dx = \frac{1}{a} \int \left( \frac{1}{x} - \frac{b}{a+bx} \right) dx$$

$$= \frac{1}{a} (\ln|x| - \ln|a+bx|) + C$$

$$= \frac{1}{a} \ln \left| \frac{x}{a+bx} \right| + C$$

$$52. \frac{1}{a^2 - x^2} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$1 = A(a+x) + B(a-x)$$

When  $x = a, A = 1/2a$ .

When  $x = -a, B = 1/2a$ .

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \int \left( \frac{1}{a-x} + \frac{1}{a+x} \right) dx$$

$$= \frac{1}{2a} (-\ln|a-x| + \ln|a+x|) + C$$

$$= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

$$53. \frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$$

$$x = A(a+bx) + B$$

When  $x = -a/b, B = -a/b$ .

When  $x = 0, 0 = aA + B \Rightarrow A = 1/b$ .

$$\int \frac{x}{(a+bx)^2} dx = \int \left( \frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2} \right) dx$$

$$= \frac{1}{b} \int \frac{1}{a+bx} dx - \frac{a}{b} \int \frac{1}{(a+bx)^2} dx$$

$$= \frac{1}{b^2} \ln|a+bx| + \frac{a}{b^2} \left( \frac{1}{a+bx} \right) + C$$

$$= \frac{1}{b^2} \left( \frac{a}{a+bx} + \ln|a+bx| \right) + C$$

$$54. \frac{1}{x^2(a+bx)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{a+bx}$$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

When  $x = 0, 1 = Ba \Rightarrow B = 1/a$ . When  $x = -a/b$ ,

$1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$ . When  $x = 1$ ,

$1 = (a+b)A + (a+b)B + C \Rightarrow A = -b/a^2$ .

$$\int \frac{1}{x^2(a+bx)} dx = \int \left( \frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx} \right) dx$$

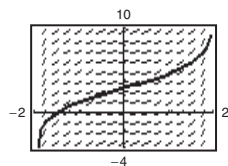
$$= -\frac{b}{a^2} \ln|x| - \frac{1}{ax} + \frac{b}{a^2} \ln|a+bx| + C$$

$$= -\frac{1}{ax} + \frac{b}{a^2} \ln \left| \frac{a+bx}{x} \right| + C$$

$$= -\frac{1}{ax} - \frac{b}{a^2} \ln \left| \frac{x}{a+bx} \right| + C$$

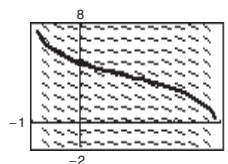
$$55. \frac{dy}{dx} = \frac{6}{4-x^2}, y(0) = 3$$

$$y = \frac{3}{2} \ln \left| \frac{2+x}{2-x} \right| + 3$$



$$56. \frac{dy}{dx} = \frac{4}{(x^2 - 2x - 3)}, y(0) = 5$$

$$y = \ln \left| \frac{x-3}{3(x+1)} \right| + 5$$



57. Dividing  $x^3$  by  $x - 5$

$$58. (a) \frac{N(x)}{D(x)} = \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

$$(b) \frac{N(x)}{D(x)} = \frac{A_1 + B_1x}{(ax^2+bx+c)} + \cdots + \frac{A_n + B_nx}{(ax^2+bx+c)^n}$$

# INSTRUCTOR USE ONLY



59.  $A = \int_0^1 \frac{12}{x^2 + 5x + 6} dx$

$$\frac{12}{x^2 + 5x + 6} = \frac{12}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = A(x+3) + B(x+2)$$

Let  $x = -3$ :  $12 = B(-1) \Rightarrow B = -12$

Let  $x = -2$ :  $12 = A(1) \Rightarrow A = 12$

$$A = \int_0^1 \left( \frac{12}{x+2} - \frac{12}{x+3} \right) dx$$

$$= [12 \ln|x+2| - 12 \ln|x+3|]_0^1$$

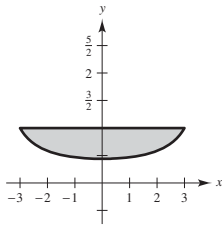
$$= 12(\ln 3 - \ln 4 - \ln 2 + \ln 3)$$

$$= 12 \ln\left(\frac{9}{8}\right) \approx 1.4134$$

61.  $A = 2 \int_0^3 \left( 1 - \frac{7}{16 - x^2} \right) dx = 2 \int_0^3 dx - 14 \int_0^3 \frac{1}{16 - x^2} dx$

$$= \left[ 2x - \frac{14}{8} \ln \left| \frac{4+x}{4-x} \right| \right]_0^3 \quad (\text{From Exercise 52})$$

$$= 6 - \frac{7}{4} \ln 7 \approx 2.595$$



62. (a) Substitution:  $u = x^2 + 2x - 8$   
 (b) Partial fractions  
 (c) Trigonometric substitution (tan) or inverse tangent rule

63. Average cost  $= \frac{1}{80 - 75} \int_{75}^{80} \frac{124p}{(10+p)(100-p)} dp$

$$= \frac{1}{5} \int_{75}^{80} \left( \frac{-124}{(10+p)11} + \frac{1240}{(100-p)11} \right) dp$$

$$= \frac{1}{5} \left[ \frac{-124}{11} \ln(10+p) - \frac{1240}{11} \ln(100-p) \right]_{75}^{80}$$

$$\approx \frac{1}{5} (24.51) = 4.9$$

Approximately \$490,000

60.  $\frac{15}{x^2 + 7x + 12} = \frac{15}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$

$$15 = A(x+4) + B(x+3)$$

When  $x = -4$ ,  $15 = -B \Rightarrow B = -15$ .

When  $x = -3$ ,  $15 = A$ .

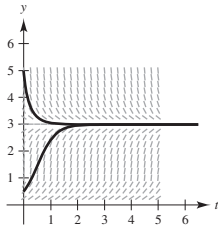
$$\text{Area} = \int_0^2 \frac{15}{x^2 + 7x + 12} dx = \int_0^2 \left( \frac{15}{x+3} - \frac{15}{x+4} \right) dx$$

$$= \left[ 15 \ln \left| \frac{x+3}{x+4} \right| \right]_0^2$$

$$= 15 \ln \left| \frac{5}{6} \right| - 15 \ln \left| \frac{3}{4} \right|$$

$$= 15 \ln \left| \frac{5/6}{3/4} \right| = 15 \ln \frac{10}{9}$$

64. (a)



(b) The slope is negative because the function is decreasing.

(c) For  $y > 0$ ,  $\lim_{t \rightarrow \infty} y(t) = 3$ .

$$(d) \quad \frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \Rightarrow A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k \, dt$$

$$\frac{1}{L} \left[ \int \frac{1}{y} \, dy + \int \frac{1}{L-y} \, dy \right] = \int k \, dt$$

$$\frac{1}{L} [\ln|y| - \ln|L-y|] = kt + C_1$$

$$\ln \left| \frac{y}{L-y} \right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

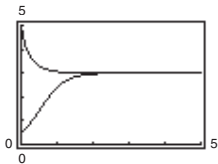
$$\text{When } t = 0, \frac{y_0}{L-y_0} = C_2 \Rightarrow \frac{y}{L-y} = \frac{y_0}{L-y_0} e^{kLt}.$$

$$\text{Solving for } y, \text{ you obtain } y = \frac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}.$$

(e)  $k = 1, L = 3$ 

$$(i) \quad y(0) = 5: \quad y = \frac{15}{5 - 2e^{-3t}}$$

$$(ii) \quad y(0) = \frac{1}{2}: \quad y = \frac{3/2}{(1/2) + (5/2)e^{-3t}} = \frac{3}{1 + 5e^{-3t}}$$



$$(f) \quad \frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2 y}{dt^2} = k \left[ y \left( \frac{-dy}{dt} \right) + (L-y) \frac{dy}{dt} \right] = 0$$

$$\Rightarrow y \frac{dy}{dt} = (L-y) \frac{dy}{dt}$$

$$\Rightarrow y = \frac{L}{2}$$

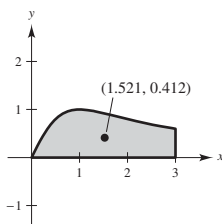
From the first derivative test, this is a maximum.

$$\begin{aligned}
 65. \quad V &= \pi \int_0^3 \left( \frac{2x}{x^2 + 1} \right)^2 dx = 4\pi \int_0^3 \frac{x^2}{(x^2 + 1)^2} dx \\
 &= 4\pi \int_0^3 \left( \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) dx && \text{(partial fractions)} \\
 &= 4\pi \left[ \arctan x - \frac{1}{2} \left( \arctan x + \frac{x}{x^2 + 1} \right) \right]_0^3 && \text{(trigonometric substitution)} \\
 &= 2\pi \left[ \arctan x - \frac{x}{x^2 + 1} \right]_0^3 = 2\pi \left( \arctan 3 - \frac{3}{10} \right) \approx 5.963
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_0^3 \frac{2x}{x^2 + 1} dx = \left[ \ln(x^2 + 1) \right]_0^3 = \ln 10 \\
 \bar{x} &= \frac{1}{A} \int_0^3 \frac{2x^2}{x^2 + 1} dx = \frac{1}{\ln 10} \int_0^3 \left( 2 - \frac{2}{x^2 + 1} \right) dx = \frac{1}{\ln 10} [2x - 2 \arctan x]_0^3 = \frac{2}{\ln 10} (3 - \arctan 3) \approx 1.521
 \end{aligned}$$

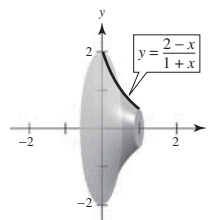
$$\begin{aligned}
 \bar{y} &= \frac{1}{A} \left( \frac{1}{2} \right) \int_0^3 \left( \frac{2x}{x^2 + 1} \right)^2 dx = \frac{2}{\ln 10} \int_0^3 \frac{x^2}{(x^2 + 1)^2} dx \\
 &= \frac{2}{\ln 10} \int_0^3 \left( \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) dx && \text{(partial fractions)} \\
 &= \frac{2}{\ln 10} \left[ \arctan x - \frac{1}{2} \left( \arctan x + \frac{x}{x^2 + 1} \right) \right]_0^3 && \text{(trigonometric substitution)} \\
 &= \frac{2}{\ln 10} \left[ \frac{1}{2} \arctan x - \frac{x}{2(x^2 + 1)} \right]_0^3 = \frac{1}{\ln 10} \left[ \arctan x - \frac{x}{x^2 + 1} \right]_0^3 = \frac{1}{\ln 10} \left( \arctan 3 - \frac{3}{10} \right) \approx 0.412
 \end{aligned}$$

$$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$$



$$66. \quad y^2 = \frac{(2-x)^2}{(1+x)^2}, \quad [0, 1]$$

$$\begin{aligned}
 V &= \int_0^1 \pi \frac{(2-x)^2}{(1+x)^2} dx \\
 &= \pi \left[ \int_0^1 \frac{4}{(1+x)^2} dx - \int_0^1 \frac{4x}{(1+x)^2} dx + \int_0^1 \frac{x^2}{(1+x)^2} dx \right] \\
 &= \pi \left[ 2 - (4 \ln 2 - 2) + \frac{3}{2} - 2 \ln 2 \right] \\
 &= \pi \left( \frac{11}{2} - 6 \ln 2 \right) = \frac{\pi}{2} (11 - 12 \ln 2)
 \end{aligned}$$



$$67. \quad \frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, \quad A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1} \int \left( \frac{1}{x+1} + \frac{1}{n-x} \right) dx = kt + C$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + C$$

$$\text{When } t = 0, x = 0, C = \frac{1}{n+1} \ln \frac{1}{n}.$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + \frac{1}{n+1} \ln \frac{1}{n}$$

$$\frac{1}{n+1} \left[ \ln \left| \frac{x+1}{n-x} \right| - \ln \frac{1}{n} \right] = kt$$

$$\ln \frac{nx+n}{n-x} = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$x = \frac{n \left[ e^{(n+1)kt} - 1 \right]}{n + e^{(n+1)kt}} \quad \text{Note: } \lim_{t \rightarrow \infty} x = n$$

$$68. (a) \quad \frac{1}{(y_0-x)(z_0-x)} = \frac{A}{y_0-x} + \frac{B}{z_0-x},$$

$$A = \frac{1}{z_0 - y_0}, \quad B = -\frac{1}{z_0 - y_0}, \quad (\text{Assume } y_0 \neq z_0.)$$

$$\frac{1}{z_0 - y_0} \int \left( \frac{1}{y_0 - x} - \frac{1}{z_0 - x} \right) dx = kt + C$$

$$\frac{1}{z_0 - y_0} \ln \left| \frac{z_0 - x}{y_0 - x} \right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0} \ln \frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0} \left[ \ln \left| \frac{z_0 - x}{y_0 - x} \right| - \ln \left( \frac{z_0}{y_0} \right) \right] = kt$$

$$\ln \left[ \frac{y_0(z_0 - x)}{z_0(y_0 - x)} \right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0 \left[ e^{(z_0 - y_0)kt} - 1 \right]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

(b) (1) If  $y_0 < z_0$ ,  $\lim_{t \rightarrow \infty} x = y_0$ .

(2) If  $y_0 > z_0$ ,  $\lim_{t \rightarrow \infty} x = z_0$ .

(3) If  $y_0 = z_0$ , then the original equation is:

$$\int \frac{1}{(y_0 - x)^2} dx = \int k dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kty_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kty_0 + 1}$$

$$x = y_0 - \frac{y_0}{kty_0 + 1}$$

As  $t \rightarrow \infty$ ,  $x \rightarrow y_0 = x_0$ .

69.  $\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$

$$x = (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)$$

$$= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)$$

$$0 = A+C \Rightarrow C = -A$$

$$0 = B+D-\sqrt{2}A+\sqrt{2}C \quad -2\sqrt{2}A = 0 \Rightarrow A = 0 \text{ and } C = 0$$

$$1 = A+C-\sqrt{2}B+\sqrt{2}D \quad -2\sqrt{2}B = 1 \Rightarrow B = -\frac{\sqrt{2}}{4} \text{ and } D = \frac{\sqrt{2}}{4}$$

$$0 = B+D \Rightarrow D = -B$$

So,

$$\begin{aligned} \int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 \left( \frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1} \right) dx \\ &= \frac{\sqrt{2}}{4} \int_0^1 \left[ \frac{-1}{\left[x + (\sqrt{2}/2)\right]^2 + (1/2)} + \frac{1}{\left[x - (\sqrt{2}/2)\right]^2 + (1/2)} \right] dx \\ &= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}} \left[ -\arctan\left(\frac{x + (\sqrt{2}/2)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x - (\sqrt{2}/2)}{1/\sqrt{2}}\right) \right]_0^1 \\ &= \frac{1}{2} \left[ -\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) \right]_0^1 \\ &= \frac{1}{2} \left[ \left( -\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1) \right) - \left( -\arctan 1 + \arctan(-1) \right) \right] \\ &= \frac{1}{2} \left[ \arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1) + \frac{\pi}{4} + \frac{\pi}{4} \right]. \end{aligned}$$

Because  $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$ , you have:

$$\int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \left[ \arctan\left(\frac{(\sqrt{2}-1) - (\sqrt{2}+1)}{1 + (\sqrt{2}-1)(\sqrt{2}+1)}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[ \arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left( -\frac{\pi}{4} + \frac{\pi}{2} \right) = \frac{\pi}{8}$$

70. The partial fraction decomposition is:

$$\begin{aligned}\frac{x^4(1-x)^4}{1+x^2} &= x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \\ \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx &= \left[ \frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4}{3}x^3 + 4x - 4 \arctan x \right]_0^1 \\ &= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right) \\ &= \frac{22}{7} - \pi\end{aligned}$$

**Note:** You can easily verify this calculation with a graphing utility.

## Section 8.6 Integration by Tables and Other Integration Techniques

1. By Formula 6: ( $a = 5$ ,  $b = 1$ )

$$\int \frac{x^2}{5+x} dx = \left[ -\frac{x}{2}(10-x) + 25 \ln|5+x| \right] + C$$

2. By Formula 13: ( $b = 2$ ,  $a = -5$ )

$$\begin{aligned}\frac{2}{3} \int \frac{1}{x^2(2x-5)^2} dx &= \frac{2}{3} \left( \frac{-1}{25} \right) \left[ \frac{-5+4x}{x(-5+2x)} + \frac{4}{-5} \ln \left| \frac{x}{2x-5} \right| \right] + C \\ &= \frac{8}{375} \ln \left| \frac{x}{2x-5} \right| - \frac{2}{75} \frac{(4x-5)}{x(2x-5)} + C\end{aligned}$$

3. By Formula 26:  $\int e^x \sqrt{1+e^{2x}} dx = \frac{1}{2} \left[ e^x \sqrt{e^{2x}+1} + \ln(e^x + \sqrt{e^{2x}+1}) \right] + C$   
 $u = e^x$ ,  $du = e^x dx$

4. By Formula 29: ( $a = 6$ )

$$\int \frac{\sqrt{x^2-36}}{6x} dx = \frac{1}{6} \left[ \sqrt{x^2-36} - 6 \operatorname{arcsec} \left| \frac{x}{6} \right| \right] + C$$

5. By Formula 44:  $\int \frac{1}{x^2 \sqrt{1-x^2}} dx = -\frac{\sqrt{1-x^2}}{x} + C$

6. By Formula 41: ( $u = x^2$ ,  $du = 2x dx$ ,  $a = 10$ )

$$\begin{aligned}\int \frac{x}{\sqrt{100-x^4}} dx &= \frac{1}{2} \int \frac{2x}{\sqrt{10^2-(x^2)^2}} dx \\ &= \frac{1}{2} \arcsin \left( \frac{x^2}{10} \right) + C\end{aligned}$$

7. By Formulas 51 and 49:

$$\begin{aligned}\int \cos^4 3x \, dx &= \frac{1}{3} \int \cos^4 3x (3) \, dx \\&= \frac{1}{3} \left[ \frac{\cos^3 3x \sin 3x}{4} + \frac{3}{4} \int \cos^2 3x \, dx \right] \\&= \frac{1}{12} \cos^3 3x \sin 3x + \frac{1}{4} \cdot \frac{1}{3} \int \cos^2 3x (3) \, dx \\&= \frac{1}{12} \cos^3 3x \sin 3x + \frac{1}{12} \cdot \frac{1}{2} (3x + \sin 3x \cos 3x) + C \\&= \frac{1}{24} (2 \cos^3 3x \sin 3x + 3x + \sin 3x \cos 3x) + C\end{aligned}$$

8. By Formulas 50 and 46:

$$\begin{aligned}\int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} \, dx &= 2 \int \sin^3 \sqrt{x} \left( \frac{1}{2\sqrt{x}} \right) dx \\&= 2 \left[ \frac{-\sin^2 \sqrt{x} \cos \sqrt{x}}{3} + \frac{2}{3} \int \sin \sqrt{x} \left( \frac{1}{2\sqrt{x}} \right) dx \right] \\&= -\frac{2}{3} \sin^2 \sqrt{x} \cos \sqrt{x} - \frac{4}{3} \cos \sqrt{x} + C \\u &= \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx\end{aligned}$$

9. By Formula 57:  $\int \frac{1}{\sqrt{x}(1 - \cos \sqrt{x})} \, dx = 2 \int \frac{1}{1 - \cos \sqrt{x}} \left( \frac{1}{2\sqrt{x}} \right) dx = -2(\cot \sqrt{x} + \csc \sqrt{x}) + C$

$$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$$

10. By Formula 71:

$$\begin{aligned}\int \frac{1}{1 - \tan 5x} \, dx &= \frac{1}{5} \int \frac{1}{1 - \tan 5x} (5) \, dx = \frac{1}{5} \left( \frac{1}{2} \right) (u - \ln |\cos u - \sin u|) + C = \frac{1}{10} (5x - \ln |\cos 5x - \sin 5x|) + C \\u &= 5x, du = 5 \, dx\end{aligned}$$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}} \, dx = x - \frac{1}{2} \ln(1 + e^{2x}) + C$$

12. By Formula 85:  $\left( a = -\frac{1}{2}, b = 2 \right)$

$$\int e^{-x/2} \sin 2x \, dx = \frac{e^{-x/2}}{(1/4) + 4} \left( -\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C = \frac{4}{17} e^{-x/2} \left( -\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C$$

13. By Formula 89: ( $n = 7$ )

$$\int x^7 \ln x \, dx = \frac{x^8}{64} [-1 + 8 \ln x] + C = \frac{1}{64} x^8 (8 \ln x - 1) + C$$

14. By Formulas 90 and 91:  $\int (\ln x)^3 \, dx = x(\ln x)^3 - 3 \int (\ln x)^2 \, dx$

$$= x(\ln x)^3 - 3x[2 - 2 \ln x + (\ln x)^2] + C$$

$$= x[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6] + C$$

15. (a) Let  $u = 3x$ ,  $x = \frac{u}{3}$ ,  $du = 3 dx$ .

$$\int x^2 e^{3x} dx = \int \left(\frac{u}{3}\right)^2 e^u \frac{1}{3} du = \frac{1}{27} \int u^2 e^u du$$

By Formulas 83 and 82:

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{1}{27} [u^2 e^u - 2 \int u e^u du] \\ &= \frac{1}{27} [u^2 e^u - 2((u-1)e^u)] + C \\ &= \frac{1}{27} e^{3x} (9x^2 - 6x + 2) + C \end{aligned}$$

- (b) Integration by parts:

$$u = x^2, du = 2x dx, dv = e^{3x} dx, v = \frac{1}{3} e^{3x}$$

$$\int x^2 e^{3x} dx = x^2 \frac{1}{3} e^{3x} - \int \frac{2}{3} x e^{3x} dx$$

$$\text{Parts again: } u = x, du = dx, dv = e^{3x}, v = \frac{1}{3} e^{3x}$$

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left[ \frac{x}{3} e^{3x} - \int \frac{1}{3} e^{3x} dx \right] \\ &= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C \\ &= \frac{1}{27} e^{3x} [9x^2 - 6x + 2] + C \end{aligned}$$

16. (a) By Formula 89: ( $n = 6$ )

$$\begin{aligned} \int x^6 \ln x dx &= \frac{x^7}{49} [-1 + 7 \ln x] + C \\ &= \frac{1}{7} x^7 \ln x - \frac{x^7}{49} + C \end{aligned}$$

- (b) Integration by parts:

$$u = \ln x, du = \frac{1}{x} dx, dv = x^6 dx, v = \frac{x^7}{7}$$

$$\begin{aligned} \int x^6 \ln x dx &= \frac{x^7}{7} \ln x - \int \frac{x^7}{7} \frac{1}{x} dx \\ &= \frac{x^7}{7} \ln x - \frac{x^7}{49} + C \end{aligned}$$

17. (a) By Formula 12: ( $a = b = 1, u = x$ )

$$\begin{aligned} \int \frac{1}{x^2(x+1)} dx &= \frac{-1}{1} \left( \frac{1}{x} + \frac{1}{1} \ln \left| \frac{x}{1+x} \right| \right) + C \\ &= \frac{-1}{x} - \ln \left| \frac{x}{1+x} \right| + C \\ &= \frac{-1}{x} + \ln \left| \frac{x+1}{x} \right| + C \end{aligned}$$

- (b) Partial fractions:

$$\begin{aligned} \frac{1}{x^2(x+1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \\ 1 &= Ax(x+1) + B(x+1) + Cx^2 \end{aligned}$$

$$x = 0: 1 = B$$

$$x = -1: 1 = C$$

$$x = 1: 1 = 2A + 2 + 1 \Rightarrow A = -1$$

$$\begin{aligned} \int \frac{1}{x^2(x+1)} dx &= \int \left[ \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx \\ &= -\ln|x| - \frac{1}{x} + \ln|x+1| + C \\ &= \frac{-1}{x} - \ln \left| \frac{x}{x+1} \right| + C \end{aligned}$$

18. (a) By Formula 24: ( $a = \sqrt{48} = 4\sqrt{3}$ )

$$\begin{aligned} \int \frac{1}{x^2 - 48} dx &= \frac{1}{2\sqrt{48}} \ln \left| \frac{x - \sqrt{48}}{x + \sqrt{48}} \right| + C \\ &= \frac{\sqrt{3}}{24} \ln \left| \frac{\sqrt{3}x - 12}{\sqrt{3}x + 12} \right| + C \end{aligned}$$

- (b) Partial fractions:

$$\begin{aligned} \frac{1}{x^2 - 48} &= \frac{A}{x + \sqrt{48}} + \frac{B}{x - \sqrt{48}} \\ 1 &= A(x - \sqrt{48}) + B(x + \sqrt{48}) \end{aligned}$$

When

$$x = \sqrt{48}, 1 = 2\sqrt{48} B = 8\sqrt{3} B \Rightarrow B = \frac{1}{8\sqrt{3}}$$

$$\text{When } x = -\sqrt{48}, 1 = -2\sqrt{48} A \Rightarrow A = \frac{-1}{8\sqrt{3}}$$

$$\begin{aligned} \int \frac{1}{x^2 - 48} dx &= \frac{-1}{8\sqrt{3}} \int \frac{dx}{x + \sqrt{48}} + \frac{1}{8\sqrt{3}} \int \frac{dx}{x - \sqrt{48}} \\ &= \frac{1}{8\sqrt{3}} \ln \left| \frac{x - \sqrt{48}}{x + \sqrt{48}} \right| + C \\ &= \frac{\sqrt{3}}{24} \ln \left| \frac{\sqrt{3}x - 12}{\sqrt{3}x + 12} \right| + C \end{aligned}$$



19. By Formula 80:

$$\begin{aligned}\int x \operatorname{arccsc}(x^2 + 1) dx &= \frac{1}{2} \int \operatorname{arccsc}(x^2 + 1)(2x) dx \\ &= \frac{1}{2} \left[ (x^2 + 1) \operatorname{arccsc}(x^2 + 1) + \ln \left| x^2 + 1 + \sqrt{(x^2 + 1)^2 - 1} \right| \right] + C \\ &= \frac{1}{2} (x^2 + 1) \operatorname{arccsc}(x^2 + 1) + \frac{1}{2} \ln (x^2 + 1 + \sqrt{x^4 + 2x^2}) + C\end{aligned}$$

20. By Formula 79:  $\int \operatorname{arcsec} 2x dx = \frac{1}{2} \left[ 2x \operatorname{arcsec} 2x - \ln |2x + \sqrt{4x^2 - 1}| \right] + C$   
 $u = 2x, du = 2 dx$

21. By Formula 35:  $\int \frac{1}{x^2 \sqrt{x^2 - 4}} dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

22. By Formula 14: ( $a = 8, b = 4, c = 1, b^2 < 4ac$ )

$$\begin{aligned}\int \frac{1}{x^2 + 4x + 8} dx &= \frac{2}{\sqrt{16}} \arctan \frac{2x + 4}{\sqrt{16}} + C \\ &= \frac{1}{2} \arctan \left( \frac{x + 2}{2} \right) + C\end{aligned}$$

23. By Formula 4: ( $a = 2, b = -5$ )

$$\begin{aligned}\int \frac{4x}{(2 - 5x)^2} dx &= 4 \left[ \frac{1}{25} \left( \frac{2}{2 - 5x} + \ln |2 - 5x| \right) \right] + C \\ &= \frac{4}{25} \left( \frac{2}{2 - 5x} + \ln |2 - 5x| \right) + C\end{aligned}$$

24. By Formula 56:

$$\begin{aligned}\int \frac{\theta^2}{1 - \sin \theta^3} d\theta &= \frac{1}{3} \int \frac{1}{1 - \sin \theta^3} 3\theta^2 d\theta \\ &= \frac{1}{3} (\tan \theta^3 + \sec \theta^3) + C\end{aligned}$$

28. By Formula 23:  $\int \frac{1}{t[1 + (\ln t)^2]} dt = \int \frac{1}{1 + (\ln t)^2} \left( \frac{1}{t} \right) dt = \arctan(\ln t) + C$   
 $u = \ln t, du = \frac{1}{t} dt$

29. By Formula 14:  $\int \frac{\cos \theta}{3 + 2 \sin \theta + \sin^2 \theta} d\theta = \frac{\sqrt{2}}{2} \arctan \left( \frac{1 + \sin \theta}{\sqrt{2}} \right) + C \quad (b^2 = 4 < 12 = 4ac)$   
 $u = \sin \theta, du = \cos \theta d\theta$

30. By Formula 27:  $\int x^2 \sqrt{2 + (3x)^2} dx = \frac{1}{27} \int (3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2} 3 dx$   
 $= \frac{1}{8(27)} \left[ 3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4 \ln |3x + \sqrt{2 + 9x^2}| \right] + C$

31. By Formula 35:  $\int \frac{1}{x^2 \sqrt{2 + 9x^2}} dx = 3 \int \frac{3}{(3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2}} dx = -\frac{3\sqrt{2 + 9x^2}}{6x} + C = -\frac{\sqrt{2 + 9x^2}}{2x} + C$

25. By Formula 76:

$$\begin{aligned}\int e^x \arccos e^x dx &= e^x \arccos e^x - \sqrt{1 - e^{2x}} + C \\ u &= e^x, du = e^x dx\end{aligned}$$

26. By Formula 71:

$$\begin{aligned}\int \frac{e^x}{1 - \tan e^x} dx &= \frac{1}{2} (e^x - \ln |\cos e^x - \sin e^x|) + C \\ u &= e^x, du = e^x dx\end{aligned}$$

27. By Formula 73:

$$\begin{aligned}\int \frac{x}{1 - \sec x^2} dx &= \frac{1}{2} \int \frac{2x}{1 - \sec x^2} dx \\ &= \frac{1}{2} (x^2 + \cot x^2 + \csc x^2) + C\end{aligned}$$

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32. By Formula 77:  $\int \sqrt{x} \arctan(x^{3/2}) dx = \frac{2}{3} \int \arctan(x^{3/2}) \left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3} \left[ x^{3/2} \arctan(x^{3/2}) - \ln \sqrt{1+x^3} \right] + C$

33. By Formula 3:  $\int \frac{\ln x}{x(3+2\ln x)} dx = \frac{1}{4} (2 \ln|x| - 3 \ln|3+2\ln|x||) + C$

$$u = \ln x, du = \frac{1}{x} dx$$

34. By Formula 45:  $\int \frac{e^x}{(1-e^{2x})^{3/2}} dx = \frac{e^x}{\sqrt{1-e^{2x}}} + C$

$$u = e^x, du = e^x dx$$

35. By Formulas 1, 25, and 33:  $\int \frac{x}{(x^2-6x+10)^2} dx = \frac{1}{2} \int \frac{2x-6+6}{(x^2-6x+10)^2} dx$

$$= \frac{1}{2} \int (x^2-6x+10)^{-2} (2x-6) dx + 3 \int \frac{1}{[(x-3)^2+1]^2} dx$$

$$= -\frac{1}{2(x^2-6x+10)} + \frac{3}{2} \left[ \frac{x-3}{x^2-6x+10} + \arctan(x-3) \right] + C$$

$$= \frac{3x-10}{2(x^2-6x+10)} + \frac{3}{2} \arctan(x-3) + C$$

36. By Formula 27:

$$\int (2x-3)^2 \sqrt{(2x-3)^2+4} dx = \frac{1}{2} \int (2x-3)^2 \sqrt{(2x-3)^2+4} (2) dx$$

$$= \frac{1}{8} (2x-3) \left[ (2x-3)^2+2 \right] \sqrt{(2x-3)^2+4} - \ln \left| 2x-3 + \sqrt{(2x-3)^2+4} \right| + C$$

$$u = 2x-3, du = 2 dx$$

37. By Formula 31:  $\int \frac{x}{\sqrt{x^4-6x^2+5}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{(x^2-3)^2-4}} dx = \frac{1}{2} \ln \left| x^2-3 + \sqrt{x^4-6x^2+5} \right| + C$

$$u = x^2-3, du = 2x dx$$

38. By Formula 31:  $\int \frac{\cos x}{\sqrt{\sin^2 x+1}} dx = \ln \left| \sin x + \sqrt{\sin^2 x+1} \right| + C$

$$u = \sin x, du = \cos x dx$$

39.  $\int \frac{x^3}{\sqrt{4-x^2}} dx = \int \frac{8 \sin^3 \theta (2 \cos \theta d\theta)}{2 \cos \theta}$

$$= 8 \int (1 - \cos^2 \theta) \sin \theta d\theta$$

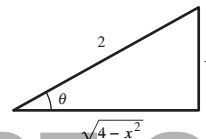
$$= 8 \int [\sin \theta - \cos^2 \theta (\sin \theta)] d\theta$$

$$= -8 \cos \theta + \frac{8 \cos^3 \theta}{3} + C$$

$$= -8 \frac{\sqrt{4-x^2}}{2} + \frac{8}{3} \left( \frac{\sqrt{4-x^2}}{2} \right)^3 + C$$

$$= \sqrt{4-x^2} \left[ -4 + \frac{1}{3}(4-x^2) \right] + C = \frac{-\sqrt{4-x^2}}{3} (x^2+8) + C$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$



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40. By Formula 41:

$$\begin{aligned}\int \sqrt{\frac{5-x}{5+x}} dx &= \int \frac{\sqrt{5-x}}{\sqrt{5+x}} \cdot \frac{\sqrt{5-x}}{\sqrt{5-x}} dx \\ &= \int \frac{5-x}{\sqrt{25-x^2}} dx \\ &= \int \frac{5 dx}{\sqrt{25-x^2}} - \int \frac{x}{\sqrt{25-x^2}} dx \\ &= 5 \arcsin\left(\frac{x}{5}\right) + \sqrt{25-x^2} + C\end{aligned}$$

41. By Formula 8:

$$\begin{aligned}\int \frac{e^{3x}}{(1+e^x)^3} dx &= \int \frac{(e^x)^2}{(1+e^x)^3} (e^x) dx \\ &= \frac{2}{1+e^x} - \frac{1}{2(1+e^x)^2} + \ln|1+e^x| + C\end{aligned}$$

$$u = e^x, du = e^x dx$$

42. By Formulas 68 and 64:

$$\begin{aligned}\int \cot^4 \theta d\theta &= -\frac{\cot^3 \theta}{3} - \int \cot^2 \theta d\theta \\ &= -\frac{\cot^3 \theta}{3} + \theta + \cot \theta + C\end{aligned}$$

43. By Formula 81:

$$\int_0^1 x e^{x^2} dx = \left[ \frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2}(e - 1) \approx 0.8591$$

49. By Formulas 54 and 55:

$$\begin{aligned}\int t^3 \cos t dt &= t^3 \sin t - 3 \int t^2 \sin t dt \\ &= t^3 \sin t - 3(-t^2 \cos t + 2 \int t \cos t dt) \\ &= t^3 \sin t + 3t^2 \cos t - 6(t \sin t - \int \sin t dt) \\ &= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi/2} t^3 \cos t dt &= \left[ t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t \right]_0^{\pi/2} \\ &= \left( \frac{\pi^3}{8} - 3\pi \right) + 6 = \frac{\pi^3}{8} + 6 - 3\pi \approx 0.4510\end{aligned}$$

44. By Formula 21: ( $a = 9, b = 1$ )

$$\begin{aligned}\int_0^7 \frac{x}{\sqrt{9+x}} dx &= \left[ \frac{-2(18-x)\sqrt{9+x}}{3} \right]_0^7 \\ &= -\frac{22}{3}\sqrt{16} + 12\sqrt{9} \\ &= -\frac{88}{3} + 36 = \frac{20}{3}\end{aligned}$$

45. By Formula 89: ( $n = 4$ )

$$\begin{aligned}\int_1^2 x^4 \ln x dx &= \frac{x^5}{25} [-1 + 5 \ln x]_1^2 \\ &= \frac{32}{25} [-1 + 5 \ln 2] - \frac{1}{25} [-1 + 0] \\ &= -\frac{31}{25} + \frac{32}{5} \ln 2 \approx 3.1961\end{aligned}$$

46. By Formula 53:

$$\begin{aligned}\int_0^{\pi/2} x \cos x dx &= [\cos x + x \sin x]_0^{\pi/2} \\ &= \frac{\pi}{2} - 1\end{aligned}$$

47. By Formula 23, and letting  $u = \sin x$ :

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx &= [\arctan(\sin x)]_{-\pi/2}^{\pi/2} \\ &= \arctan(1) - \arctan(-1) = \frac{\pi}{2}\end{aligned}$$

48. By Formula 7: ( $a = 7, b = 2$ )

$$\begin{aligned}\int_4^6 \frac{x^2}{(2x-7)^2} dx &= \left[ \frac{1}{8} \left( 2x - \frac{49}{2x-7} + 14 \ln|2x-7| \right) \right]_4^6 \\ &= \frac{1}{8} \left( 12 - \frac{49}{5} + 14 \ln 5 \right) - \frac{1}{8} (8 - 49) \\ &= \frac{27}{5} + \frac{7}{4} \ln 5 \approx 8.2165\end{aligned}$$

50. By Formula 26:

$$\int_0^1 \sqrt{3+x^2} \, dx = \left[ \frac{1}{2} \left( x\sqrt{x^2+3} + 3 \ln |x + \sqrt{x^2+3}| \right) \right]_0^1 = \frac{1}{2} \left[ (2) + 3 \ln 3 - 3 \ln \sqrt{3} \right] = 1 + \frac{3}{4} \ln 3$$

$$51. \quad \frac{u^2}{(a+bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a+bu)^2} = \frac{1}{b^2} + \frac{A}{a+bu} + \frac{B}{(a+bu)^2}$$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a+bu) + B = (aA+B) + bAu$$

Equating the coefficients of like terms you have  $aA + B = -a^2/b^2$  and  $bA = -2a/b$ . Solving these equations you have  $A = -2a/b^2$  and  $B = a^2/b^2$ .

$$\int \frac{u^2}{(a+bu)^2} \, du = \frac{1}{b^2} \int du - \frac{2a(1)}{b^2(b)} \int \frac{1}{a+bu} b \, du + \frac{a^2(1)}{b^2(b)} \int \frac{1}{(a+bu)^2} b \, du = \frac{1}{b^2} u - \frac{2a}{b^3} \ln |a+bu| - \frac{a^2}{b^3} \left( \frac{1}{a+bu} \right) + C$$

$$= \frac{1}{b^3} \left( bu - \frac{a^2}{a+bu} - 2a \ln |a+bu| \right) + C$$

52. Integration by parts:  $w = u^n$ ,  $dw = nu^{n-1} \, du$ ,  $dv = \frac{du}{\sqrt{a+bu}}$ ,  $v = \frac{2}{b} \sqrt{a+bu}$

$$\int \frac{u^n}{\sqrt{a+bu}} \, du = \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} \, du$$

$$= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} \cdot \frac{\sqrt{a+bu}}{\sqrt{a+bu}} \, du$$

$$= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int \frac{au^{n-1} + bu^n}{\sqrt{a+bu}} \, du$$

$$= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2na}{b} \int \frac{u^{n-1}}{\sqrt{a+bu}} \, du - 2n \int \frac{u^n}{\sqrt{a+bu}} \, du$$

$$\text{Therefore, } (2n+1) \int \frac{u^n}{\sqrt{a+bu}} \, du = \frac{2}{b} \left[ u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} \, du \right] \text{ and}$$

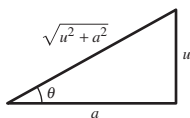
$$\int \frac{u^n}{\sqrt{a+bu}} \, du = \frac{2}{(2n+1)b} \left[ u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} \, du \right].$$

53. When you have  $u^2 + a^2$ :

$$u = a \tan \theta$$

$$du = a \sec^2 \theta \, d\theta$$

$$u^2 + a^2 = a^2 \sec^2 \theta$$



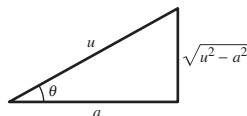
$$\int \frac{1}{(u^2 + a^2)^{3/2}} \, du = \int \frac{a \sec^2 \theta \, d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \int \cos \theta \, d\theta = \frac{1}{a^2} \sin \theta + C = \frac{u}{a^2 \sqrt{u^2 + a^2}} + C$$

When you have  $u^2 - a^2$ :

$$u = a \sec \theta$$

$$du = a \sec \theta \tan \theta \, d\theta$$

$$u^2 - a^2 = a^2 \tan^2 \theta$$



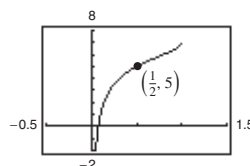
$$\int \frac{1}{(u^2 - a^2)^{3/2}} \, du = \int \frac{a \sec \theta \tan \theta \, d\theta}{a^3 \tan^3 \theta} = \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta = \frac{1}{a^2} \int \csc \theta \cot \theta \, d\theta = -\frac{1}{a^2} \csc \theta + C = \frac{-u}{a^2 \sqrt{u^2 - a^2}} + C$$

54.  $\int u^n (\cos u) du = u^n \sin u - n \int u^{n-1} (\sin u) du$   
 $w = u^n, dv = \cos u du, dw = nu^{n-1} du, v = \sin u$

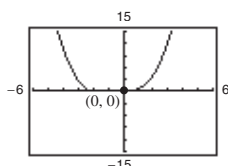
55.  $\int (\arctan u) du = u \arctan u - \frac{1}{2} \int \frac{2u}{1+u^2} du$   
 $= u \arctan u - \frac{1}{2} \ln(1+u^2) + C$   
 $= u \arctan u - \ln \sqrt{1+u^2} + C$   
 $w = \arctan u, dv = du, dw = \frac{du}{1+u^2}, v = u$

56.  $\int (\ln u)^n du = u(\ln u)^n - \int n(\ln u)^{n-1} \left(\frac{1}{u}\right) u du$   
 $= u(\ln u)^n - n \int (\ln u)^{n-1} du$   
 $w = (\ln u)^n, dv = du, dw = n(\ln u)^{n-1} \left(\frac{1}{u}\right) du,$   
 $v = u$

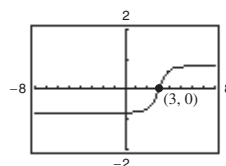
57.  $\int \frac{1}{x^{3/2} \sqrt{1-x}} dx = \frac{-2\sqrt{1-x}}{\sqrt{x}} + C$   
 $\left(\frac{1}{2}, 5\right): \frac{-2\sqrt{1/2}}{\sqrt{1/2}} + C = 5 \Rightarrow C = 7$   
 $y = \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7$



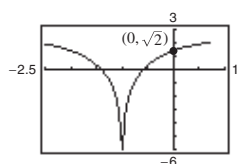
58.  $\int x\sqrt{x^2+2x} dx = \frac{1}{6} \left[ 2(x^2+2x)^{3/2} - 3(x+1)\sqrt{x^2+2x} + 3 \ln|x+1+\sqrt{x^2+2x}| \right] + C$   
 $(0, 0): \frac{1}{6} [3 \ln|1|] + C = 0 \Rightarrow C = 0$



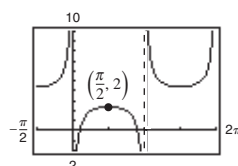
59.  $\int \frac{1}{(x^2-6x+10)^2} dx = \frac{1}{2} \left[ \arctan(x-3) + \frac{x-3}{x^2-6x+10} \right] + C$   
 $(3, 0): \frac{1}{2} \left[ 0 + \frac{0}{10} \right] + C = 0 \Rightarrow C = 0$   
 $y = \frac{1}{2} \left[ \arctan(x-3) + \frac{x-3}{x^2-6x+10} \right]$



60.  $\int \frac{\sqrt{2-2x-x^2}}{x+1} dx = \sqrt{2-2x-x^2} - \sqrt{3} \ln \left| \frac{\sqrt{3} + \sqrt{2-2x-x^2}}{x+1} \right| + C$   
 $(0, \sqrt{2}): \sqrt{2} - \sqrt{3} \ln(\sqrt{3} + \sqrt{2}) + C = \sqrt{2} \Rightarrow C = \sqrt{3} \ln(\sqrt{3} + \sqrt{2})$

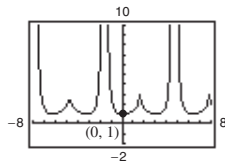


61.  $\int \frac{1}{\sin \theta \tan \theta} d\theta = -\csc \theta + C$   
 $\left(\frac{\pi}{4}, 2\right): -\frac{2}{\sqrt{2}} + C = 2 \Rightarrow C = 2 + \sqrt{2}$   
 $y = -\csc \theta + 2 + \sqrt{2}$



$$62. \int \frac{\sin \theta}{(\cos \theta)(1 + \sin \theta)} d\theta = \frac{1}{2} \left[ \frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + C$$

$$(0, 1): C = 1 \Rightarrow y = \frac{1}{2} \left[ \frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + 1$$



$$\begin{aligned} 63. \int \frac{1}{2 - 3 \sin \theta} d\theta &= \int \left[ \frac{\frac{2 du}{1 + u^2}}{2 - 3 \left( \frac{2u}{1 + u^2} \right)} \right], u = \tan \frac{\theta}{2} \\ &= \int \frac{2}{2(1 + u^2) - 6u} du \\ &= \int \frac{1}{u^2 - 3u + 1} du \\ &= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}} du \\ &= \frac{1}{\sqrt{5}} \ln \left| \frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}} \right| + C \\ &= \frac{1}{\sqrt{5}} \ln \left| \frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}} \right| + C \\ &= \frac{1}{\sqrt{5}} \ln \left| \frac{2 \tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2 \tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}} \right| + C \end{aligned}$$

$$\begin{aligned} 64. \int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta &= -\int \frac{-\sin \theta}{1 + (\cos \theta)^2} d\theta \\ &= -\arctan(\cos \theta) + C \end{aligned}$$

$$\begin{aligned} 65. \int_0^{\pi/2} \frac{1}{1 + \sin \theta + \cos \theta} d\theta &= \int_0^1 \left[ \frac{\frac{2 du}{1 + u^2}}{1 + \frac{2u}{1 + u^2} + \frac{1 - u^2}{1 + u^2}} \right] \\ &= \int_0^1 \frac{1}{1 + u} du \\ &= [\ln|1 + u|]_0^1 \\ &= \ln 2 \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned} 66. \int_0^{\pi/2} \frac{1}{3 - 2 \cos \theta} d\theta &= \int_0^1 \left[ \frac{\frac{2u}{1 + u^2}}{3 - \frac{2(1 - u^2)}{1 + u^2}} \right] \\ &= 2 \int_0^1 \frac{1}{5u^2 + 1} du \\ &= \left[ \frac{2}{\sqrt{5}} \arctan(\sqrt{5} u) \right]_0^1 \\ &= \frac{2}{\sqrt{5}} \arctan \sqrt{5} \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned} 67. \int \frac{\sin \theta}{3 - 2 \cos \theta} d\theta &= \frac{1}{2} \int \frac{2 \sin \theta}{3 - 2 \cos \theta} d\theta \\ &= \frac{1}{2} \ln|u| + C \\ &= \frac{1}{2} \ln(3 - 2 \cos \theta) + C \end{aligned}$$

$$u = 3 - 2 \cos \theta, du = 2 \sin \theta d\theta$$

$$\begin{aligned} 68. \int \frac{\cos \theta}{1 + \cos \theta} d\theta &= \int \frac{\cos \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} d\theta \\ &= \int \frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} d\theta \\ &= \int (\csc \theta \cot \theta - \cot^2 \theta) d\theta \\ &= \int (\csc \theta \cot \theta - (\csc^2 \theta - 1)) d\theta \\ &= -\csc \theta + \cot \theta + \theta + C \end{aligned}$$

$$\begin{aligned} 69. \int \frac{\sin \sqrt{\theta}}{\sqrt{\theta}} d\theta &= 2 \int \sin \sqrt{\theta} \left( \frac{1}{2\sqrt{\theta}} \right) d\theta \\ &= -2 \cos \sqrt{\theta} + C \end{aligned}$$

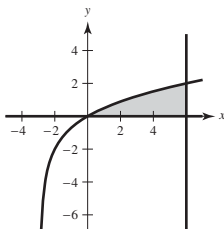
$$u = \sqrt{\theta}, du = \frac{1}{2\sqrt{\theta}} d\theta$$

$$\begin{aligned}
 70. \int \frac{4}{\csc \theta - \cot \theta} d\theta &= \int \frac{4}{\left(\frac{1}{\sin \theta}\right) - \left(\frac{\cos \theta}{\sin \theta}\right)} d\theta \\
 &= 4 \int \frac{\sin \theta}{1 - \cos \theta} d\theta \\
 &= 4 \ln|1 - \cos \theta| + C
 \end{aligned}$$

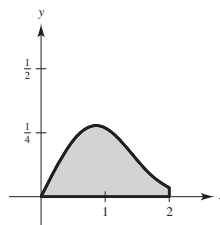
$$u = 1 - \cos \theta, du = \sin \theta d\theta$$

71. By Formula 21: ( $a = 3, b = 1$ )

$$\begin{aligned}
 A &= \int_0^6 \frac{x}{\sqrt{x+3}} dx = \left[ \frac{-2(6-x)}{3} \sqrt{x+3} \right]_0^6 \\
 &= 4\sqrt{3} \approx 6.928 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 72. A &= \int_0^2 \frac{x}{1 + e^{x^2}} dx \\
 &= \frac{1}{2} \int_0^2 \frac{2x dx}{1 + e^{x^2}} \\
 &= \frac{1}{2} \left[ x^2 - \ln(1 + e^{x^2}) \right]_0^2 \\
 &= \frac{1}{2} \left[ 4 - \ln(1 + e^4) \right] + \frac{1}{2} \ln 2 \\
 &\approx 0.337 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 73. (a) \quad n = 1: u &= \ln x, du = \frac{1}{x} dx, dv = x dx, v = \frac{x^2}{2} \\
 \int x \ln x dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C
 \end{aligned}$$

$$\begin{aligned}
 n = 2: u &= \ln x, du = \frac{1}{x} dx, dv = x^2 dx, v = \frac{x^3}{3} \\
 \int x^2 \ln x dx &= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \frac{1}{x} dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C
 \end{aligned}$$

$$\begin{aligned}
 n = 3: u &= \ln x, du = \frac{1}{x} dx, dv = x^3 dx, v = \frac{x^4}{4} \\
 \int x^3 \ln x dx &= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C
 \end{aligned}$$

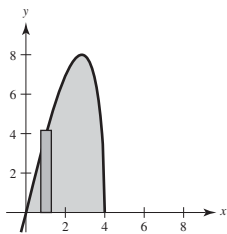
$$(b) \int x^n \ln x dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

74. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

75. False. You might need to convert your integral using substitution or algebra.

76. True.

77.



$$V = 2\pi \int_0^4 x(x\sqrt{16-x^2}) dx$$

$$= 2\pi \int_0^4 x^2\sqrt{16-x^2} dx$$

By Formula 38: ( $a = 4$ )

$$V = 2\pi \left[ \frac{1}{8} \left( x(2x^2 - 16)\sqrt{16-x^2} + 256 \arcsin\left(\frac{x}{4}\right) \right) \right]_0^4$$

$$= 2\pi \left[ 32 \left( \frac{\pi}{2} \right) \right] = 32\pi^2$$

78. (a) Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2 + 1} du, u = e^x$$

(b) Log Rule:  $\int \frac{1}{u} du, u = e^x + 1$ (c) Substitution:  $u = x^2, du = 2x dx$ 

Then Formula 81.

(d) Integration by parts

(e) Cannot be integrated.

(f) Formula 16 with  $u = e^{2x}$ 

$$79. W = \int_0^5 2000xe^{-x} dx$$

$$= -2000 \int_0^5 -xe^{-x} dx$$

$$= 2000 \int_0^5 (-x)e^{-x}(-1) dx$$

$$= 2000 \left[ (-x)e^{-x} - e^{-x} \right]_0^5$$

$$= 2000 \left( -\frac{6}{e^5} + 1 \right)$$

$$\approx 1919.145 \text{ ft-lb}$$

$$82. \frac{1}{2-0} \int_0^2 \frac{5000}{1+e^{4.8-1.9t}} dt = \frac{2500}{-1.9} \int_0^2 \frac{-1.9 dt}{1+e^{4.8-1.9t}}$$

$$= -\frac{2500}{1.9} \left[ (4.8-1.9t) - \ln(1+e^{4.8-1.9t}) \right]_0^2$$

$$= -\frac{2500}{1.9} \left[ (1 - \ln(1+e)) - (4.8 - \ln(1+e^{4.8})) \right]$$

$$= \frac{2500}{1.9} \left[ 3.8 + \ln\left(\frac{1+e}{1+e^{4.8}}\right) \right] \approx 401.4$$

$$80. W = \int_0^5 \frac{500x}{\sqrt{26-x^2}} dx$$

$$= -250 \int_0^5 (26-x^2)^{-1/2} (-2x) dx$$

$$= \left[ -500\sqrt{26-x^2} \right]_0^5$$

$$= 500(\sqrt{26}-1)$$

$$\approx 2049.51 \text{ ft-lb}$$

$$81. (a) V = 20(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy$$

$$= \left[ 80 \ln|y + \sqrt{1+y^2}| \right]_0^3$$

$$= 80 \ln(3 + \sqrt{10})$$

$$\approx 145.5 \text{ ft}^3$$

$$W = 148(80 \ln(3 + \sqrt{10}))$$

$$= 11,840 \ln(3 + \sqrt{10})$$

$$\approx 21,530.4 \text{ lb}$$

(b) By symmetry,  $\bar{x} = 0$ .

$$M = \rho(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy$$

$$= \left[ 4\rho \ln|y + \sqrt{1+y^2}| \right]_0^3$$

$$= 4\rho \ln(3 + \sqrt{10})$$

$$M_x = 2\rho \int_0^3 \frac{2y}{\sqrt{1+y^2}} dy$$

$$= \left[ 4\rho \sqrt{1+y^2} \right]_0^3$$

$$= 4\rho(\sqrt{10}-1)$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\rho(\sqrt{10}-1)}{4\rho \ln(3 + \sqrt{10})} \approx 1.19$$

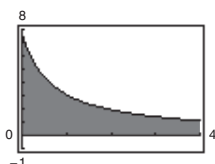
Centroid:  $(\bar{x}, \bar{y}) \approx (0, 1.19)$



83. (a)  $\int_0^4 \frac{k}{2+3x} dx = 10$

$$k = \frac{10}{\int_0^4 \frac{1}{2+3x} dx} = \frac{10}{\frac{1}{3} \ln 7} \approx \frac{10}{0.6486} = 15.417 \left( = \frac{30}{\ln 7} \right)$$

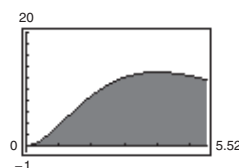
(b)  $\int_0^4 \frac{15.417}{2+3x} dx$



84. (a)  $\int_0^k 6x^2 e^{-x/2} dx = 50$

By trial and error,  $k = 5.51897$ .

(b)  $\int_0^{5.51897} 6x^2 e^{-x/2} dx$



85. Let  $I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$ .

For  $x = \frac{\pi}{2} - u$ ,  $dx = -du$ , and

$$I = \int_{\pi/2}^0 \frac{-du}{1 + (\tan(\pi/2 - u))^{\sqrt{2}}} = \int_0^{\pi/2} \frac{du}{1 + (\cot u)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}} + 1} du.$$

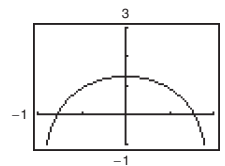
$$2I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{(\tan x)^{\sqrt{2}} + 1} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

So,  $I = \frac{\pi}{4}$ .

## Section 8.7 Indeterminate Forms and L'Hôpital's Rule

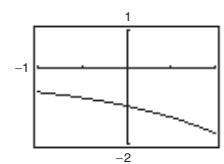
1.  $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 3x} \approx 1.3333$  (exact:  $\frac{4}{3}$ )

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177



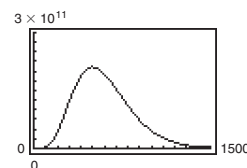
2.  $\lim_{x \rightarrow 0} \frac{1 - e^x}{x} \approx -1$

$x$	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



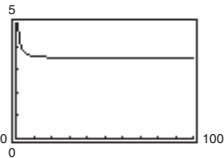
3.  $\lim_{x \rightarrow \infty} x^5 e^{-x/100} \approx 0$

$x$	1	10	$10^2$	$10^3$	$10^4$	$10^5$
$f(x)$	0.9900	90,484	$3.7 \times 10^9$	$4.5 \times 10^{10}$	0	0



$$4. \lim_{x \rightarrow \infty} \frac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \quad \left( \text{exact: } \frac{6}{\sqrt{3}} \right)$$

$x$	1	10	$10^2$	$10^3$	$10^4$	$10^5$
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



$$5. (a) \lim_{x \rightarrow 4} \frac{3(x-4)}{x^2-16} = \lim_{x \rightarrow 4} \frac{3(x-4)}{(x-4)(x+4)} = \lim_{x \rightarrow 4} \frac{3}{x+4} = \frac{3}{8}$$

$$(b) \lim_{x \rightarrow 4} \frac{3(x-4)}{x^2-16} = \lim_{x \rightarrow 4} \frac{d/dx[3(x-4)]}{d/dx[x^2-16]} = \lim_{x \rightarrow 4} \frac{3}{2x} = \frac{3}{8}$$

$$6. (a) \lim_{x \rightarrow -2} \frac{2x^2 + x - 6}{x+2} = \lim_{x \rightarrow -2} \frac{(2x-3)(x+2)}{x+2} = \lim_{x \rightarrow -2} (2x-3) = -7$$

$$(b) \lim_{x \rightarrow -2} \frac{2x^2 + x - 6}{x+2} = \lim_{x \rightarrow -2} \frac{d/dx[2x^2 + x - 6]}{d/dx[x+2]} = \lim_{x \rightarrow -2} \frac{4x+1}{1} = -7$$

$$7. (a) \lim_{x \rightarrow 6} \frac{\sqrt{x+10} - 4}{x-6} = \lim_{x \rightarrow 6} \frac{\sqrt{x+10} - 4}{x-6} \cdot \frac{\sqrt{x+10} + 4}{\sqrt{x+10} + 4} = \lim_{x \rightarrow 6} \frac{(x+10) - 16}{(x-6)(\sqrt{x+10} + 4)} = \lim_{x \rightarrow 6} \frac{1}{\sqrt{x+10} + 4} = \frac{1}{8}$$

$$(b) \lim_{x \rightarrow 6} \frac{\sqrt{x+10} - 4}{x-6} = \lim_{x \rightarrow 6} \frac{d/dx[\sqrt{x+10} - 4]}{d/dx[x-6]} = \lim_{x \rightarrow 6} \frac{\frac{1}{2}(x+10)^{-1/2}}{1} = 1/8$$

$$8. (a) \lim_{x \rightarrow 0} \frac{\sin 6x}{4x} = \lim_{x \rightarrow 0} \left( \frac{3 \sin 6x}{2 \cdot 6x} \right) = \frac{3}{2}(1) = \frac{3}{2}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 6x}{4x} = \lim_{x \rightarrow 0} \frac{d/dx[\sin 6x]}{d/dx[4x]} = \lim_{x \rightarrow 0} \frac{6 \cos 6x}{4} = \frac{3}{2}$$

$$9. (a) \lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x \rightarrow \infty} \frac{5 - (3/x) + (1/x^2)}{3 - (5/x^2)} = \frac{5}{3}$$

$$(b) \lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{3x^2 - 5} = \lim_{x \rightarrow \infty} \frac{(d/dx)[5x^2 - 3x + 1]}{(d/dx)[3x^2 - 5]} = \lim_{x \rightarrow \infty} \frac{10x - 3}{6x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[10x - 3]}{(d/dx)[6x]} = \lim_{x \rightarrow \infty} \frac{10}{6} = \frac{5}{3}$$

$$10. (a) \lim_{x \rightarrow \infty} \frac{2x+1}{4x^2+x} = \lim_{x \rightarrow \infty} \frac{(2/x) + (1/x^2)}{4 + (1/x)} = \frac{0}{4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{2x+1}{4x^2+x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[2x+1]}{(d/dx)[4x^2+x]} = \lim_{x \rightarrow \infty} \frac{2}{8x+1} = 0$$

$$11. \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{2x - 2}{1} = 4$$

$$13. \lim_{x \rightarrow 0} \frac{\sqrt{25-x^2} - 5}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}(25-x^2)^{-1/2}(-2x)}{1}$$

$$12. \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{x+1} = \lim_{x \rightarrow -1} \frac{2x-2}{1} = -4 \quad \quad \quad = \lim_{x \rightarrow 0} \frac{-x}{\sqrt{25-x^2}} = 0$$

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$$14. \lim_{x \rightarrow 5^-} \frac{\sqrt{25 - x^2}}{x - 5} = \lim_{x \rightarrow 5^-} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1} \\ = \lim_{x \rightarrow 5^-} \frac{-x}{\sqrt{25 - x^2}} = -\infty$$

$$15. \lim_{x \rightarrow 0} \frac{e^x - (1 - x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{1} = 2$$

$$16. \lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{2 \ln x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{2/x}{2x} = \lim_{x \rightarrow 1} \frac{1}{x^2} = 1$$

$$17. \lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{3x^2} = \lim_{x \rightarrow 0^+} \frac{e^x}{6x} = \infty$$

18. Case 1:  $n = 1$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{1} = 0$$

Case 2:  $n = 2$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^2} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3:  $n \geq 3$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^n} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{nx^{n-1}} \\ = \lim_{x \rightarrow 0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

$$19. \lim_{x \rightarrow 1} \frac{x^{11} - 1}{x^4 - 1} = \lim_{x \rightarrow 1} \frac{11x^{10}}{4x^3} = \frac{11}{4}$$

$$20. \lim_{x \rightarrow 1} \frac{x^a - 1}{x^b - 1} = \lim_{x \rightarrow 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

$$21. \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{3 \cos 3x}{5 \cos 5x} = \frac{3}{5}$$

$$22. \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$$

$$23. \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{1/\sqrt{1-x^2}}{1} = 1$$

$$24. \lim_{x \rightarrow 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x \rightarrow 1} \frac{1/(1+x^2)}{1} = \frac{1}{2}$$

$$25. \lim_{x \rightarrow \infty} \frac{5x^2 + 3x - 1}{4x^2 + 5} = \lim_{x \rightarrow \infty} \frac{10x + 3}{8x} = \lim_{x \rightarrow \infty} \frac{10}{8} = \frac{5}{4}$$

$$26. \lim_{x \rightarrow \infty} \frac{x - 6}{x^2 + 4x + 7} = \lim_{x \rightarrow \infty} \frac{1}{2x + 4} = 0$$

$$27. \lim_{x \rightarrow \infty} \frac{x^2 + 4x + 7}{x - 6} = \lim_{x \rightarrow \infty} \frac{2x + 4}{1} = \infty$$

$$28. \lim_{x \rightarrow \infty} \frac{x^3}{x + 1} = \lim_{x \rightarrow \infty} \frac{3x^2}{1} = \infty$$

$$29. \lim_{x \rightarrow \infty} \frac{x^3}{e^{x/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{(1/2)e^{x/2}} \\ = \lim_{x \rightarrow \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \rightarrow \infty} \frac{6}{(1/8)e^{x/2}} = 0$$

$$30. \lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2xe^{x^2}} \\ = \lim_{x \rightarrow \infty} \frac{6x}{(4x^2 + 2)e^{x^2}} \\ = \lim_{x \rightarrow \infty} \frac{6}{4x(2x^2 + 3)e^{x^2}} = 0$$

$$31. \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$$

**Note:** L'Hôpital's Rule does not work on this limit. See Exercise 89.

$$32. \lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{1 + (1/x^2)}} = \infty$$

$$33. \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by Squeeze Theorem}$$

$$\left( \frac{\cos x}{x} \leq \frac{1}{x}, \text{ for } x > 0 \right)$$

$$34. \lim_{x \rightarrow \infty} \frac{\sin x}{x - \pi} = 0$$

**Note:** Use the Squeeze Theorem for  $x > \pi$ .

$$-\frac{1}{x - \pi} \leq \frac{\sin x}{x - \pi} \leq \frac{1}{x - \pi}$$

$$35. \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0$$

$$36. \lim_{x \rightarrow \infty} \frac{\ln x^4}{x^3} = \lim_{x \rightarrow \infty} \frac{4 \ln x}{x^3} = \lim_{x \rightarrow \infty} \frac{4/x}{3x^2} = \lim_{x \rightarrow \infty} \frac{4}{3x^3} = 0$$

$$37. \lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \lim_{x \rightarrow \infty} \frac{e^x}{4x^3} \\ = \lim_{x \rightarrow \infty} \frac{e^x}{12x^2} \\ = \lim_{x \rightarrow \infty} \frac{e^x}{24x} \\ = \lim_{x \rightarrow \infty} \frac{e^x}{24} = \infty$$

$$38. \lim_{x \rightarrow \infty} \frac{e^{x/2}}{x} = \lim_{x \rightarrow \infty} \frac{(1/2)e^{x/2}}{1} = \infty$$

$$39. \lim_{x \rightarrow 0} \frac{\sin 5x}{\tan 9x} = \lim_{x \rightarrow 0} \frac{5 \cos 5x}{9 \sec^2 9x} = \frac{5}{9}$$

$$40. \lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x} = \lim_{x \rightarrow 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$$

$$41. \lim_{x \rightarrow 0} \frac{\arctan x}{\sin x} = \lim_{x \rightarrow 0} \frac{1/(1+x^2)}{\cos x} = 1$$

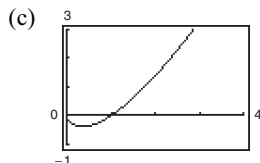
$$42. \lim_{x \rightarrow 0} \frac{x}{\arctan 2x} = \lim_{x \rightarrow 0} \frac{1}{2/(1+4x^2)} = 1/2$$

$$\begin{aligned} 43. \lim_{x \rightarrow \infty} \frac{\int_1^x \ln(e^{4t-1}) dt}{x} \\ &= \lim_{x \rightarrow \infty} \frac{\int_1^x (4t-1) dt}{x} \\ &= \lim_{x \rightarrow \infty} \frac{4x-1}{1} = \infty \end{aligned}$$

$$44. \lim_{x \rightarrow 1^+} \frac{\int_1^x \cos \theta d\theta}{x-1} = \lim_{x \rightarrow 1^+} \frac{\cos x}{1} = \cos(1)$$

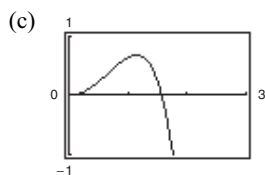
$$45. (a) \lim_{x \rightarrow \infty} x \ln x, \text{ not indeterminate}$$

$$(b) \lim_{x \rightarrow \infty} x \ln x = (\infty)(\infty) = \infty$$



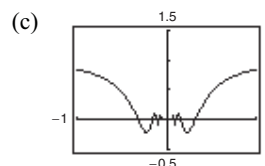
$$46. (a) \lim_{x \rightarrow 0^+} x^3 \cot x = (0)(\infty)$$

$$(b) \lim_{x \rightarrow 0^+} x^3 \cot x = \lim_{x \rightarrow 0^+} \frac{x^3}{\tan x} = \lim_{x \rightarrow 0^+} \frac{3x^2}{\sec^2 x} = 0$$



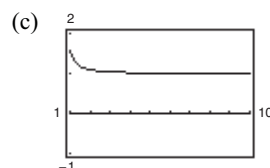
$$47. (a) \lim_{x \rightarrow \infty} \left( x \sin \frac{1}{x} \right) = (\infty)(0)$$

$$\begin{aligned} (b) \lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = 1 \end{aligned}$$



$$48. (a) \lim_{x \rightarrow \infty} \left( x \tan \frac{1}{x} \right) = (\infty)(0)$$

$$\begin{aligned} (b) \lim_{x \rightarrow \infty} x \tan \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{-(1/x^2) \sec^2(1/x)}{-(1/x^2)} \\ &= \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right) = 1 \end{aligned}$$



$$49. (a) \lim_{x \rightarrow 0^+} x^{1/x} = 0^\infty = 0, \text{ not indeterminate}$$

(See Exercise 116).

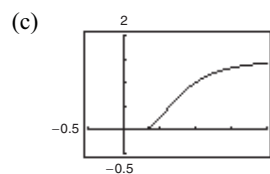
$$(b) \text{ Let } y = x^{1/x}$$

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

Because  $x \rightarrow 0^+$ ,  $\frac{1}{x} \ln x \rightarrow (\infty)(-\infty) = -\infty$ . So,

$$\ln y \rightarrow -\infty \Rightarrow y \rightarrow 0^+.$$

Therefore,  $\lim_{x \rightarrow 0^+} x^{1/x} = 0$ .

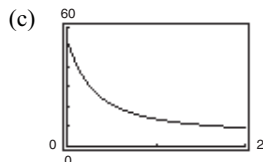


50. (a)  $\lim_{x \rightarrow 0^+} (e^x + x)^{2/x} = 1^\infty$

(b) Let  $y = \lim_{x \rightarrow 0^+} (e^x + x)^{2/x}$ .

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \frac{2 \ln(e^x + x)}{x} \\ &= \lim_{x \rightarrow 0^+} \frac{2(e^x + 1)/(e^x + x)}{1} = 4 \end{aligned}$$

So,  $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$ .



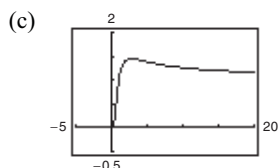
51. (a)  $\lim_{x \rightarrow \infty} x^{1/x} = \infty^0$

(b) Let  $y = \lim_{x \rightarrow \infty} x^{1/x}$ .

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \left( \frac{1/x}{1} \right) = 0$$

So,  $\ln y = 0 \Rightarrow y = e^0 = 1$ . Therefore,

$$\lim_{x \rightarrow \infty} x^{1/x} = 1.$$



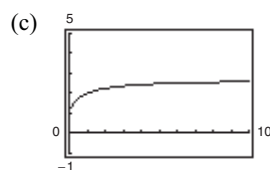
52. (a)  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 1^\infty$

(b) Let  $y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ .

$$\begin{aligned} \ln y &= \lim_{x \rightarrow \infty} \left[ x \ln \left(1 + \frac{1}{x}\right) \right] = \lim_{x \rightarrow \infty} \frac{\ln[1 + (1/x)]}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{(-1/x^2)}{1 + (1/x)}}{(-1/x^2)} = \lim_{x \rightarrow \infty} \frac{1}{1 + (1/x)} = 1 \end{aligned}$$

So,  $\ln y = 1 \Rightarrow y = e^1 = e$ . Therefore,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$



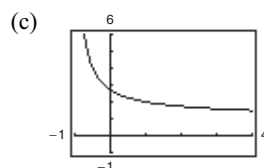
53. (a)  $\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = 1^\infty$

(b) Let  $y = \lim_{x \rightarrow 0^+} (1 + x)^{1/x}$ .

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1 + x)}{x} \\ &= \lim_{x \rightarrow 0^+} \left( \frac{1/(1 + x)}{1} \right) = 1 \end{aligned}$$

So,  $\ln y = 1 \Rightarrow y = e^1 = e$ .

Therefore,  $\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = e$ .



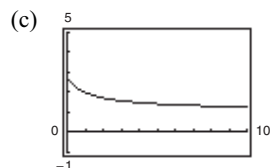
54. (a)  $\lim_{x \rightarrow \infty} (1 + x)^{1/x} = \infty^0$

(b) Let  $y = \lim_{x \rightarrow \infty} (1 + x)^{1/x}$ .

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln(1 + x)}{x} = \lim_{x \rightarrow \infty} \left( \frac{1/(1 + x)}{1} \right) = 0$$

So,  $\ln y = 0 \Rightarrow y = e^0 = 1$ .

Therefore,  $\lim_{x \rightarrow \infty} (1 + x)^{1/x} = 1$ .

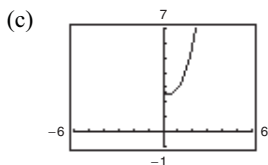


55. (a)  $\lim_{x \rightarrow 0^+} [3(x)^{x/2}] = 0^0$

(b) Let  $y = \lim_{x \rightarrow 0^+} 3(x)^{x/2}$ .

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 0^+} \left[ \ln 3 + \frac{x}{2} \ln x \right] \\ &= \lim_{x \rightarrow 0^+} \left[ \ln 3 + \frac{\ln x}{2/x} \right] \\ &= \lim_{x \rightarrow 0^+} \ln 3 + \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^2} \\ &= \lim_{x \rightarrow 0^+} \ln 3 - \lim_{x \rightarrow 0^+} \frac{x}{2} \\ &= \ln 3\end{aligned}$$

So,  $\lim_{x \rightarrow 0^+} 3(x)^{x/2} = 3$ .

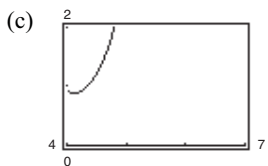


56. (a)  $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 0^0$

(b) Let  $y = \lim_{x \rightarrow 4^+} [3(x-4)]^{x-4}$ .

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 4^+} (x-4) \ln[3(x-4)] \\ &= \lim_{x \rightarrow 4^+} \frac{\ln[3(x-4)]}{1/(x-4)} \\ &= \lim_{x \rightarrow 4^+} \frac{1/(x-4)}{-1/(x-4)^2} \\ &= \lim_{x \rightarrow 4^+} [-(x-4)] = 0\end{aligned}$$

So,  $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 1$ .



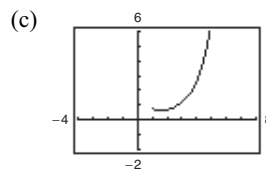
57. (a)  $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 0^0$

(b) Let  $y = (\ln x)^{x-1}$ .

$$\begin{aligned}\ln y &= \ln[(\ln x)^{x-1}] = (x-1) \ln(\ln x) \\ &= \frac{\ln(\ln x)}{(x-1)^{-1}}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 1^+} \ln y &= \lim_{x \rightarrow 1^+} \frac{\ln(\ln x)}{(x-1)^{-1}} \\ &= \lim_{x \rightarrow 1^+} \frac{1/(x \ln x)}{-(x-1)^{-2}} \\ &= \lim_{x \rightarrow 1^+} \frac{-(x-1)^2}{x \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{-2(x-1)}{1 + \ln x} = 0\end{aligned}$$

Because  $\lim_{x \rightarrow 1^+} \ln y = 0$ ,  $\lim_{x \rightarrow 1^+} y = 1$ .



58.  $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

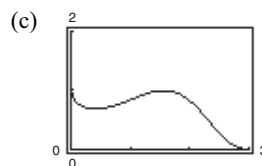
(a)  $\lim_{x \rightarrow 0^+} \left[ \cos\left(\frac{\pi}{2} - x\right) \right]^x = \lim_{x \rightarrow 0^+} [\sin x]^x = 0^0$

(b) Let  $y = (\sin x)^x$

$$\ln y = x \ln(\sin x) = \frac{\ln(\sin x)}{1/x}$$

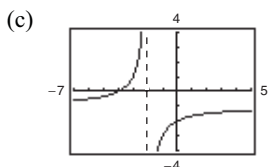
$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} &= \lim_{x \rightarrow 0^+} \frac{\cos x / \sin x}{-1/x^2} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^2 \cos x}{\sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \left( \frac{-x \cos x}{1} \right) \\ &= 0\end{aligned}$$

So,  $\lim_{x \rightarrow 0^+} \left[ \cos\left(\frac{\pi}{2} - x\right) \right]^x = 1$ .



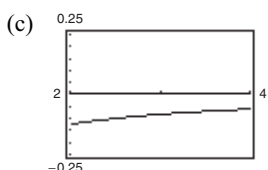
59. (a)  $\lim_{x \rightarrow 2^+} \left( \frac{8}{x^2 - 4} - \frac{x}{x - 2} \right) = \infty - \infty$

(b) 
$$\begin{aligned} \lim_{x \rightarrow 2^+} \left( \frac{8}{x^2 - 4} - \frac{x}{x - 2} \right) &= \lim_{x \rightarrow 2^+} \frac{8 - x(x + 2)}{x^2 - 4} \\ &= \lim_{x \rightarrow 2^+} \frac{(2 - x)(4 + x)}{(x + 2)(x - 2)} \\ &= \lim_{x \rightarrow 2^+} \frac{-(x + 4)}{x + 2} = \frac{-3}{2} \end{aligned}$$



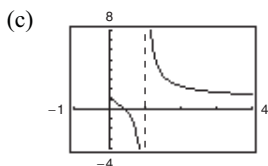
60. (a)  $\lim_{x \rightarrow 2^+} \left( \frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4} \right) = \infty - \infty$

(b) 
$$\begin{aligned} \lim_{x \rightarrow 2^+} \left( \frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4} \right) &= \lim_{x \rightarrow 2^+} \frac{1 - \sqrt{x - 1}}{x^2 - 4} \\ &= \lim_{x \rightarrow 2^+} \frac{-1/(2\sqrt{x - 1})}{2x} \\ &= \lim_{x \rightarrow 2^+} \frac{-1}{4x\sqrt{x - 1}} = \frac{-1}{8} \end{aligned}$$



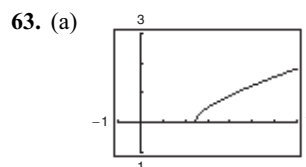
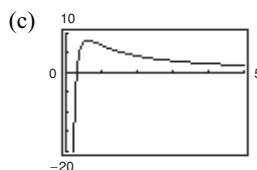
61. (a)  $\lim_{x \rightarrow 1^+} \left( \frac{3}{\ln x} - \frac{2}{x - 1} \right) = \infty - \infty$

(b) 
$$\begin{aligned} \lim_{x \rightarrow 1^+} \left( \frac{3}{\ln x} - \frac{2}{x - 1} \right) &= \lim_{x \rightarrow 1^+} \frac{3x - 3 - 2 \ln x}{(x - 1) \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{3 - (2/x)}{[(x - 1)/x] + \ln x} = \infty \end{aligned}$$

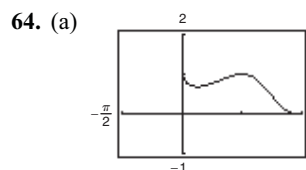


62. (a)  $\lim_{x \rightarrow 0^+} \left( \frac{10}{x} - \frac{3}{x^2} \right) = \infty - \infty$

(b)  $\lim_{x \rightarrow 0^+} \left( \frac{10}{x} - \frac{3}{x^2} \right) = \lim_{x \rightarrow 0^+} \left( \frac{10x - 3}{x^2} \right) = -\infty$



(b) 
$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x - 3}{\ln(2x - 5)} &= \lim_{x \rightarrow 3} \frac{1}{2/(2x - 5)} \\ &= \lim_{x \rightarrow 3} \frac{2x - 5}{2} = \frac{1}{2} \end{aligned}$$



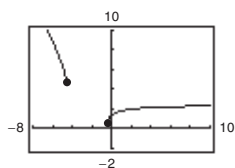
(b) Let  $y = (\sin x)^x$ , then  $\ln y = x \ln(\sin x)$ .

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} &= \lim_{x \rightarrow 0^+} \frac{\cos x / \sin x}{-1/x^2} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x} \\ &= \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x} = 0 \end{aligned}$$

Therefore, because  $\ln y = 0$ ,  $y = 1$  and

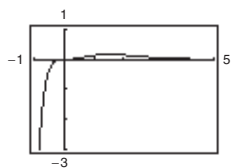
$$\lim_{x \rightarrow 0^+} (\sin x)^x = 1.$$

65. (a)



$$\begin{aligned}
 (b) \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) \frac{(\sqrt{x^2 + 5x + 2} + x)}{(\sqrt{x^2 + 5x + 2} + x)} \\
 &= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 2) - x^2}{\sqrt{x^2 + 5x + 2} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{5x + 2}{\sqrt{x^2 + 5x + 2} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{5 + (2/x)}{\sqrt{1 + (5/x) + (2/x^2)} + 1} = \frac{5}{2}
 \end{aligned}$$

66. (a)



$$(b) \lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

$$67. \frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty, \infty^0$$

68. See Theorem 8.4.

$$69. (a) \text{ Let } f(x) = x^2 - 25 \text{ and } g(x) = x - 5.$$

$$(b) \text{ Let } f(x) = (x - 5)^2 \text{ and } g(x) = x^2 - 25.$$

$$(c) \text{ Let } f(x) = x^2 - 25 \text{ and } g(x) = (x - 5)^3.$$

(Answers will vary.)

$$70. \text{ Let } f(x) = x + 25 \text{ and } g(x) = x.$$

(Answers will vary.)

71.

$x$	10	$10^2$	$10^4$	$10^6$	$10^8$	$10^{10}$
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

72.

$x$	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	$4.40 \times 10^5$	$2.30 \times 10^9$	$1.66 \times 10^{13}$	$2.69 \times 10^{33}$

$$73. \lim_{x \rightarrow \infty} \frac{x^2}{e^{5x}} = \lim_{x \rightarrow \infty} \frac{2x}{5e^{5x}} = \lim_{x \rightarrow \infty} \frac{2}{25e^{5x}} = 0$$



$$74. \lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

$$\begin{aligned} 75. \lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x} &= \lim_{x \rightarrow \infty} \frac{3(\ln x)^2(1/x)}{1} \\ &= \lim_{x \rightarrow \infty} \frac{3(\ln x)^2}{x} \\ &= \lim_{x \rightarrow \infty} \frac{6(\ln x)(1/x)}{1} \\ &= \lim_{x \rightarrow \infty} \frac{6(\ln x)}{x} = \lim_{x \rightarrow \infty} \frac{6}{x} = 0 \end{aligned}$$

$$\begin{aligned} 76. \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x^3} &= \lim_{x \rightarrow \infty} \frac{(2 \ln x)/x}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{2 \ln x}{3x^3} \\ &= \lim_{x \rightarrow \infty} \frac{2/x}{9x^2} = \lim_{x \rightarrow \infty} \frac{2}{9x^3} = 0 \end{aligned}$$

$$\begin{aligned} 77. \lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^m} &= \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}} \\ &= \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}}{mx^m} \\ &= \lim_{x \rightarrow \infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2 x^m} \\ &= \cdots = \lim_{x \rightarrow \infty} \frac{n!}{m^n x^m} = 0 \end{aligned}$$

$$\begin{aligned} 78. \lim_{x \rightarrow \infty} \frac{x^m}{e^{nx}} &= \lim_{x \rightarrow \infty} \frac{mx^{m-1}}{ne^{nx}} \\ &= \lim_{x \rightarrow \infty} \frac{m(m-1)x^{m-2}}{n^2 e^{nx}} \\ &= \cdots = \lim_{x \rightarrow \infty} \frac{m!}{n^m e^{nx}} = 0 \end{aligned}$$

$$79. y = x^{1/x}, x > 0$$

Horizontal asymptote:  $y = 1$  (See Exercise 51.)

$$\ln y = \frac{1}{x} \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left( \frac{1}{x} \right) + (\ln x) \left( -\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = x^{1/x} \left( \frac{1}{x^2} \right) (1 - \ln x) = x^{(1/x)-2} (1 - \ln x) = 0$$

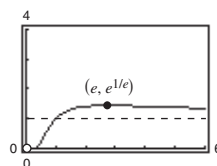
Critical number:  $x = e$

Intervals:  $(0, e)$   $(e, \infty)$

Sign of  $dy/dx$ :  $+$   $-$

$y = f(x)$ : Increasing Decreasing

Relative maximum:  $(e, e^{1/e})$



$$80. y = x^x, x > 0$$

$$\lim_{x \rightarrow \infty} x^x = \infty \text{ and } \lim_{x \rightarrow 0^+} x^x = 1$$

No horizontal asymptotes

$$\ln y = x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = x \left( \frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = x^x (1 + \ln x) = 0$$

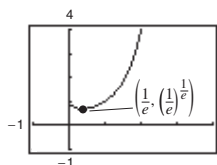
Critical number:  $x = e^{-1}$

Intervals:  $(0, e^{-1})$   $(e^{-1}, 0)$

Sign of  $dy/dx$ :  $-$   $+$

$y = f(x)$ : Decreasing Increasing

Relative maximum:  $\left( e^{-1}, (e^{-1})^{e^{-1}} \right) = \left( \frac{1}{e}, \left( \frac{1}{e} \right)^{1/e} \right)$

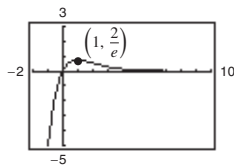


81.  $y = 2xe^{-x}$

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

Horizontal asymptote:  $y = 0$ 

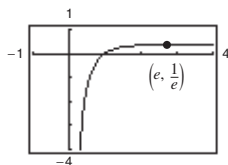
$$\begin{aligned} \frac{dy}{dx} &= 2x(-e^{-x}) + 2e^{-x} \\ &= 2e^{-x}(1 - x) = 0 \end{aligned}$$

Critical number:  $x = 1$ Intervals:  $(-\infty, 1)$   $(1, \infty)$ Sign of  $dy/dx$ :  $+$   $-$  $y = f(x)$ : Increasing DecreasingRelative maximum:  $\left(1, \frac{2}{e}\right)$ 

82.  $y = \frac{\ln x}{x}$

Horizontal asymptote:  $y = 0$  (See Exercise 31.)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number:  $x = e$ Intervals:  $(0, e)$   $(e, \infty)$ Sign of  $dy/dx$ :  $+$   $-$  $y = f(x)$ : Increasing DecreasingRelative maximum:  $\left(e, \frac{1}{e}\right)$ 

83.  $\lim_{x \rightarrow 2} (3x^2 + 4x + 1) = \infty$

Limit is not of the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ .

L'Hôpital's Rule does not apply.

84.  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form  $0/0$  or  $\infty/\infty$ .

L'Hôpital's Rule does not apply.

85.  $\lim_{x \rightarrow \infty} \frac{\sin \pi x - 1}{x} = 0$  (Numerator is bounded)

Limit is not of the form  $0/0$  or  $\infty/\infty$ .

L'Hôpital's Rule does not apply.

86.  $\lim_{x \rightarrow \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form  $0/0$  or  $\infty/\infty$ .

L'Hôpital's Rule does not apply.

87.  $\lim_{x \rightarrow \infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form  $0/0$  or  $\infty/\infty$ .

L'Hôpital's Rule does not apply.

88. (a) Yes:  $\frac{0}{0}$

(b) No:  $\frac{0}{-1}$

(c) Yes:  $\frac{\infty}{\infty}$

(d) Yes:  $\frac{0}{0}$

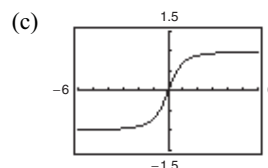
(e) No:  $\frac{-1}{0}$

(f) Yes:  $\frac{0}{0}$

89. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{1}{x/\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{x/\sqrt{x^2 + 1}}{1} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1}/x} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} = \frac{1}{\sqrt{1 + 0}} = 1 \end{aligned}$$

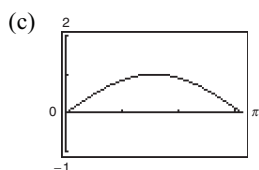


90. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x \rightarrow \pi/2^-} \frac{\tan x}{\sec x} \text{ is indeterminate: } \frac{\infty}{\infty}$$

$$\begin{aligned} \lim_{x \rightarrow \pi/2^-} \frac{\tan x}{\sec x} &= \lim_{x \rightarrow \pi/2^-} \frac{\sec^2 x}{\sec x \tan x} \\ &= \lim_{x \rightarrow \pi/2^-} \frac{\sec x}{\tan x} \left( \frac{\infty}{\infty} \right) \\ &= \lim_{x \rightarrow \pi/2^-} \frac{\sec x \tan x}{\sec^2 x} \\ &= \lim_{x \rightarrow \pi/2^-} \frac{\tan x}{\sec x} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow \pi/2^-} \frac{\tan x}{\sec x} &= \lim_{x \rightarrow \pi/2^-} \frac{\sin x}{\cos x} (\cos x) \\ &= \lim_{x \rightarrow \pi/2^-} \sin x = 1 \end{aligned}$$



91.  $f(x) = \sin(3x)$ ,  $g(x) = \sin(4x)$

$$f'(x) = 3 \cos(3x), \quad g'(x) = 4 \cos(4x)$$

$$y_1 = \frac{f(x)}{g(x)} = \frac{\sin 3x}{\sin 4x},$$

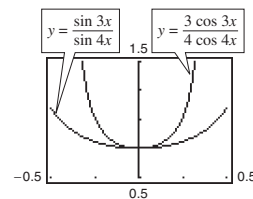
$$y_2 = \frac{f'(x)}{g'(x)} = \frac{3 \cos 3x}{4 \cos 4x}$$

As  $x \rightarrow 0$ ,  $y_1 \rightarrow 0.75$  and

$$y_2 \rightarrow 0.75$$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 4x} = \lim_{x \rightarrow 0} \frac{3 \cos 3x}{4 \cos 4x} = \frac{3}{4}$$



92.  $f(x) = e^{3x} - 1$ ,  $g(x) = x$

$$f'(x) = 3e^{3x}, \quad g'(x) = 1$$

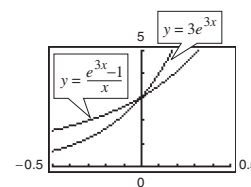
$$y_1 = \frac{f(x)}{g(x)} = \frac{e^{3x} - 1}{x},$$

$$y_2 = \frac{f'(x)}{g'(x)} = 3e^{3x}$$

As  $x \rightarrow 0$ ,  $y_1 \rightarrow 3$  and  $y_2 \rightarrow 3$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = \lim_{x \rightarrow 0} \frac{3e^{3x}}{1} = 3$$



93.  $\lim_{k \rightarrow 0} \frac{32(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32})}{k} = \lim_{k \rightarrow 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k \rightarrow 0} (v_0 e^{-kt}) = \lim_{k \rightarrow 0} \frac{32(0 + t e^{-kt})}{1} + \lim_{k \rightarrow 0} \left( \frac{v_0}{e^{kt}} \right) = 32t + v_0$

94.  $A = P \left( 1 + \frac{r}{n} \right)^{nt}$

$$\ln A = \ln P + nt \ln \left( 1 + \frac{r}{n} \right) = \ln P + \frac{\ln \left( 1 + \frac{r}{n} \right)}{\frac{1}{nt}}$$

$$\lim_{n \rightarrow \infty} \left[ \frac{\ln \left( 1 + \frac{r}{n} \right)}{\frac{1}{nt}} \right] = \lim_{n \rightarrow \infty} \left[ \frac{-\frac{r}{n^2} \left( \frac{1}{1 + (r/n)} \right)}{-\left( \frac{1}{n^2 t} \right)} \right] = \lim_{n \rightarrow \infty} \left[ rt \left( \frac{1}{1 + \frac{r}{n}} \right) \right] = rt$$

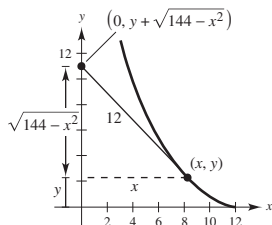
Because  $\lim_{n \rightarrow \infty} \ln A = \ln P + rt$ , you have  $\lim_{n \rightarrow \infty} A = e^{(\ln P + rt)} = e^{\ln P} e^{rt} = P e^{rt}$ . Alternatively,

$$\lim_{n \rightarrow \infty} A = \lim_{n \rightarrow \infty} P \left( 1 + \frac{r}{n} \right)^{nt} = \lim_{n \rightarrow \infty} P \left[ \left( 1 + \frac{r}{n} \right)^{n/r} \right]^{rt} = P e^{rt}.$$

95. Let  $N$  be a fixed value for  $n$ . Then

$$\lim_{x \rightarrow \infty} \frac{x^{N-1}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \cdots = \lim_{x \rightarrow \infty} \frac{(N-1)!}{e^x} = 0. \quad (\text{See Exercise 78.})$$

$$96. (a) \quad m = \frac{dy}{dx} = \frac{y - (y + \sqrt{144 - x^2})}{x - 0} = -\frac{\sqrt{144 - x^2}}{x}$$



$$(b) \quad y = -\int \frac{\sqrt{144 - x^2}}{x} dx$$

$$\text{Let } x = 12 \sin \theta, dx = 12 \cos \theta d\theta, \sqrt{144 - x^2} = 12 \cos \theta.$$

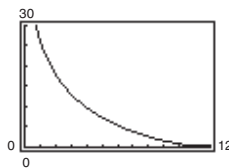
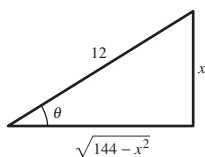
$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

$$= -12 \int (\csc \theta - \sin \theta) d\theta = -12 \ln |\csc \theta - \cot \theta| - 12 \cos \theta + C$$

$$= -12 \ln \left| \frac{12}{x} - \frac{\sqrt{144 - x^2}}{x} \right| - 12 \left( \frac{\sqrt{144 - x^2}}{12} \right) + C = -12 \ln \left| \frac{12 - \sqrt{144 - x^2}}{x} \right| - \sqrt{144 - x^2} + C$$

$$\text{When } x = 12, y = 0 \Rightarrow C = 0. \text{ So, } y = -12 \ln \left( \frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}.$$

$$\text{Note: } \frac{12 - \sqrt{144 - x^2}}{x} > 0 \text{ for } 0 < x \leq 12$$



(c) Vertical asymptote:  $x = 0$

$$(d) \quad y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$$

$$\text{So, } 12 - \sqrt{144 - x^2} = -12 \ln \left( \frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$$

$$-1 = \ln \left( \frac{12 - \sqrt{144 - x^2}}{x} \right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = (-\sqrt{144 - x^2})^2$$

$$x^2 e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

$$x[x(e^{-2} + 1) - 24e^{-1}] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

$$\text{Therefore, } s = \int_{7.77665}^{12} \sqrt{1 + \left( \frac{-\sqrt{144 - x^2}}{x} \right)^2} dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}} dx$$

$$= \int_{7.77665}^{12} \frac{12}{x} dx = [12 \ln |x|]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$

97.  $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{3c^2}{2c}$$

$$\frac{1}{1} = \frac{3c}{2}$$

$$c = \frac{2}{3}$$

98.  $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

99.  $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c$$

$$c = \frac{\pi}{4}$$

100.  $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

101. False. L'Hôpital's Rule does not apply because

$$\lim_{x \rightarrow 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \rightarrow 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \rightarrow 0^+} \left( x + 1 + \frac{1}{x} \right) = 1 + \infty = \infty$$

102. False. If  $y = e^x/x^2$ , then  $y' = \frac{x^2 e^x - 2x e^x}{x^4} = \frac{x e^x (x - 2)}{x^4} = \frac{e^x (x - 2)}{x^3}$ .

103. True

104. False. Let  $f(x) = x$  and  $g(x) = x + 1$ . Then

$$\lim_{x \rightarrow \infty} \frac{x}{x + 1} = 1, \text{ but } \lim_{x \rightarrow \infty} [x - (x + 1)] = -1.$$

105. Area of triangle:  $\frac{1}{2}(2x)(1 - \cos x) = x - x \cos x$

Shaded area: Area of rectangle - Area under curve

$$\begin{aligned} 2x(1 - \cos x) - 2 \int_0^x (1 - \cos t) dt &= 2x(1 - \cos x) - 2[t - \sin t]_0^x \\ &= 2x(1 - \cos x) - 2(x - \sin x) \\ &= 2 \sin x - 2x \cos x \end{aligned}$$

$$\text{Ratio: } \lim_{x \rightarrow 0} \frac{x - x \cos x}{2 \sin x - 2x \cos x} = \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2 \cos x + 2x \sin x - 2 \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2x \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x \cos x + \sin x + \sin x}{2x \cos x + 2 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{2x \cos x + 2 \sin x} \cdot \frac{1/\cos x}{1/\cos x} = \lim_{x \rightarrow 0} \frac{x + 2 \tan x}{2x + 2 \tan x} = \lim_{x \rightarrow 0} \frac{1 + 2 \sec^2 x}{2 + 2 \sec^2 x} = \frac{3}{4}$$

106. (a)  $\sin \theta = BD$

$$\cos \theta = DO \Rightarrow AD = 1 - \cos \theta$$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta)\sin \theta = \frac{1}{2}\sin \theta - \frac{1}{2}\sin \theta \cos \theta$$

(b) Area of sector:  $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2}\sin \theta \cos \theta$$

(c)  $R = \frac{(1/2)\sin \theta - (1/2)\sin \theta \cos \theta}{(1/2)\theta - (1/2)\sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$

(d)  $\lim_{\theta \rightarrow 0} R = \lim_{\theta \rightarrow 0} \frac{\sin \theta - (1/2)\sin 2\theta}{\theta - (1/2)\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta + 2\sin 2\theta}{2\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\cos \theta + 4\cos 2\theta}{4\cos 2\theta} = \frac{3}{4}$

107.  $\lim_{x \rightarrow 0} \frac{4x - 2\sin 2x}{2x^3} = \lim_{x \rightarrow 0} \frac{4 - 4\cos 2x}{6x^2} = \lim_{x \rightarrow 0} \frac{8\sin 2x}{12x} = \lim_{x \rightarrow 0} \frac{16\cos 2x}{12} = \frac{16}{12} = \frac{4}{3}$

Let  $c = \frac{4}{3}$ .

108. Let  $y = (e^x + x)^{1/x}$ .

$$\ln y = \frac{1}{x} \ln(e^x + x) = \frac{\ln(e^x + x)}{x}$$

$$\lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

So,  $\lim_{x \rightarrow 0} (e^x + x)^{1/x} = e^2$ .

Let  $c = e^2 \approx 7.389$ .

109.  $\lim_{x \rightarrow 0} \frac{a - \cos bx}{x^2} = 2$

Near  $x = 0$ ,  $\cos bx \approx 1$  and  $x^2 \approx 0 \Rightarrow a = 1$ .

Using L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{1 - \cos bx}{x^2} = \lim_{x \rightarrow 0} \frac{b \sin bx}{2x} = \lim_{x \rightarrow 0} \frac{b^2 \cos bx}{2} = 2.$$

So,  $b^2 = 4$  and  $b = \pm 2$ .

Answer:  $a = 1, b = \pm 2$

110. Use mathematical induction.

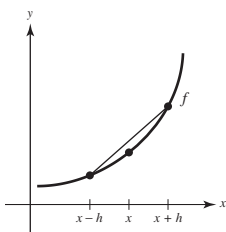
For  $n = 1$ ,  $\lim_{x \rightarrow \infty} \frac{x^1}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$ .

Assume that  $\lim_{x \rightarrow \infty} \frac{x^k}{e^x} = 0$ .

$$\begin{aligned} \text{Then, } \lim_{x \rightarrow \infty} \frac{x^{k+1}}{e^x} &= \lim_{x \rightarrow \infty} \frac{(k+1)x^k}{e^x} \\ &= (k+1) \lim_{x \rightarrow \infty} \frac{x^k}{e^x} \\ &= (k+1)(0) = 0. \end{aligned}$$

111. (a)  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h \rightarrow 0} \frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \lim_{h \rightarrow 0} \left[ \frac{f'(x+h) + f'(x-h)}{2} \right] = \frac{f'(x) + f'(x)}{2} = f'(x)$

(b)

Graphically, the slope of the line joining  $(x-h, f(x-h))$  and  $(x+h, f(x+h))$  is approximately  $f'(x)$ . And, as

$$h \rightarrow 0, \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

$$\begin{aligned}
 112. \quad \lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h)(1) + f'(x-h)(-1)}{2h} \\
 &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\
 &= \lim_{h \rightarrow 0} \frac{f''(x+h)(1) - f''(x-h)(-1)}{2} \\
 &= \lim_{h \rightarrow 0} \frac{f''(x+h) + f''(x-h)}{2} \\
 &= \frac{f''(x) + f''(x)}{2} = f''(x)
 \end{aligned}$$

$$113. \quad g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

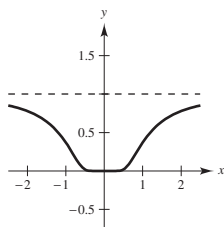
$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$$

Let  $y = \frac{e^{-1/x^2}}{x}$ , then  $\ln y = \ln\left(\frac{e^{-1/x^2}}{x}\right) = -\frac{1}{x^2} - \ln x = \frac{-1 - x^2 \ln x}{x^2}$ . Because

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x^2} = \lim_{x \rightarrow 0} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0} \left(-\frac{x^2}{2}\right) = 0$$

you have  $\lim_{x \rightarrow 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$ . So,  $\lim_{x \rightarrow 0} y = e^{-\infty} = 0 \Rightarrow g'(0) = 0$ .

**Note:** The graph appears to support this conclusion—the tangent line is horizontal at  $(0, 0)$ .



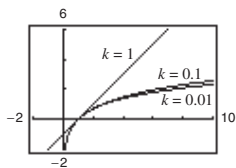
$$114. \quad f(x) = \frac{x^k - 1}{k}$$

$$k = 1, \quad f(x) = x - 1$$

$$k = 0.1, \quad f(x) = \frac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$$

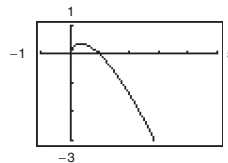
$$k = 0.01, \quad f(x) = \frac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$$

$$\lim_{k \rightarrow 0^+} \frac{x^k - 1}{k} = \lim_{k \rightarrow 0^+} \frac{x^k (\ln x)}{1} = \ln x$$



$$115. \quad (a) \quad \lim_{x \rightarrow 0^+} (-x \ln x) \text{ is the form } 0 \cdot \infty.$$

$$(b) \quad \lim_{x \rightarrow 0^+} \frac{-\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{-1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (x) = 0$$



$$116. \lim_{x \rightarrow a} f(x)^{g(x)}$$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$$

As  $x \rightarrow a$ ,  $\ln y \Rightarrow -\infty$ , and therefore  $y = 0$ . So,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = 0.$$

$$117. \lim_{x \rightarrow a} f(x)^{g(x)}$$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As  $x \rightarrow a$ ,  $\ln y \Rightarrow \infty$ , and therefore  $y = \infty$ . So,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \infty.$$

$$\begin{aligned} 118. f'(a)(b-a) - \int_a^b f''(t)(t-b) dt &= f'(a)(b-a) - \left\{ [f'(t)(t-b)]_a^b - \int_a^b f'(t) dt \right\} \\ &= f'(a)(b-a) + f'(a)(a-b) + [f(t)]_a^b = f(b) - f(a) \end{aligned}$$

$$dv = f''(t) dt \Rightarrow v = f'(t)$$

$$u = t - b \Rightarrow du = dt$$

$$119. (a) \lim_{x \rightarrow 0^+} x^{(\ln 2)/(1+\ln x)} \text{ is of form } 0^0.$$

$$\text{Let } y = x^{(\ln 2)/(1+\ln x)}$$

$$\ln y = \frac{\ln 2}{1+\ln x} \ln x$$

$$\lim_{x \rightarrow 0^+} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

$$\text{So, } \lim_{x \rightarrow 0^+} x^{(\ln 2)/(1+\ln x)} = 2.$$

$$(b) \lim_{x \rightarrow \infty} x^{(\ln 2)/(1+\ln x)} \text{ is of form } \infty^0.$$

$$\text{Let } y = x^{(\ln 2)/(1+\ln x)}$$

$$\ln y = \frac{\ln 2}{1+\ln x} \ln x$$

$$\lim_{x \rightarrow \infty} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

$$\text{So, } \lim_{x \rightarrow \infty} x^{(\ln 2)/(1+\ln x)} = 2.$$

$$(c) \lim_{x \rightarrow 0} (x+1)^{(\ln 2)/(x)} \text{ is of form } 1^\infty.$$

$$\text{Let } y = (x+1)^{(\ln 2)/(x)}$$

$$\ln y = \frac{\ln 2}{x} \ln(x+1)$$

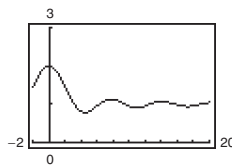
$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{(\ln 2)1/(x+1)}{1} = \ln 2.$$

$$\text{So, } \lim_{x \rightarrow 0} (x+1)^{(\ln 2)/(x)} = 2.$$

$$\begin{aligned} 120. \lim_{x \rightarrow a} \frac{\sqrt{2a^3x - x^4} - a^3\sqrt{a^2x}}{a - \sqrt[4]{ax^3}} \\ &= \lim_{x \rightarrow a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - \frac{a}{3}(a^2x)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}} \\ &= \frac{\frac{1}{2}(a^4)^{-1/2}(-2a^3) - \frac{a^3}{3}(a^3)^{-2/3}}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)} \\ &= \frac{a + \frac{a}{3}}{\frac{1}{4}(a^{-3})(3a^3)} \\ &= \frac{\frac{4}{3}a}{\frac{3}{4}} = \frac{16}{9}a \end{aligned}$$

$$121. (a) h(x) = \frac{x + \sin x}{x}$$

$$\lim_{x \rightarrow \infty} h(x) = 1$$



$$(b) h(x) = \frac{x + \sin x}{x} = \frac{x}{x} + \frac{\sin x}{x} = 1 + \frac{\sin x}{x}, x > 0$$

$$\text{So, } \lim_{x \rightarrow \infty} h(x) = \lim_{x \rightarrow \infty} \left[ 1 + \frac{\sin x}{x} \right] = 1 + 0 = 1.$$

(c) No.  $h(x)$  is not an indeterminate form.



122. (a)  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x + x \sin x}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{1 + \sin x}{x - 4/x} = 0$

(Because  $|1 + \sin x| \leq 1$  and  $x \rightarrow \infty$ .)

(b)  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} x(1 + \sin x) = \infty$

$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} (x^2 - 4) = \infty$

(c)  $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow \infty} \frac{1 + \sin x + x \cos x}{2x}$  undefined

(d) No. If  $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$  does not exist, then you cannot assume anything about  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ .

123. Let  $f(x) = \left[ \frac{1}{x} \cdot \frac{a^x - 1}{a - 1} \right]^{1/x}$ .

For  $a > 1$  and  $x > 0$ ,

$$\ln f(x) = \frac{1}{x} \left[ \ln \frac{1}{x} + \ln(a^x - 1) - \ln(a - 1) \right] = -\frac{\ln x}{x} + \frac{\ln(a^x - 1)}{x} - \frac{\ln(a - 1)}{x}.$$

As  $x \rightarrow \infty$ ,  $\frac{\ln x}{x} \rightarrow 0$ ,  $\frac{\ln(a - 1)}{x} \rightarrow 0$ , and  $\frac{\ln(a^x - 1)}{x} = \frac{\ln[(1 - a^{-x})a^x]}{x} = \frac{\ln(1 - a^{-x})}{x} + \ln a \rightarrow \ln a$ .

So,  $\ln f(x) \rightarrow \ln a$ .

For  $0 < a < 1$  and  $x > 0$ ,

$$\ln f(x) = \frac{-\ln x}{x} + \frac{\ln(1 - a^x)}{x} - \frac{\ln(1 - a)}{x} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Combining these results,  $\lim_{x \rightarrow \infty} f(x) = \begin{cases} a & \text{if } a > 1 \\ 1 & \text{if } 0 < a < 1 \end{cases}$ .

## Section 8.8 Improper Integrals

1.  $\int_0^1 \frac{dx}{5x - 3}$  is improper because  $5x - 3 = 0$  when  $x = \frac{3}{5}$ , and  $0 \leq \frac{3}{5} \leq 1$ .

2.  $\int_1^2 \frac{dx}{x^3}$  is not improper because  $f(x) = \frac{1}{x^3}$  is continuous on  $[1, 2]$ .

3.  $\int_0^1 \frac{2x - 5}{x^2 - 5x + 6} dx = \int_0^1 \frac{2x - 5}{(x - 2)(x - 3)} dx$  is not improper because  $\frac{2x - 5}{(x - 2)(x - 3)}$  is continuous on  $[0, 1]$ .

4.  $\int_1^\infty \ln(x^2) dx$  is improper because the upper limit of integration is  $\infty$ .

5.  $\int_0^2 e^{-x} dx$  is not improper because  $f(x) = e^{-x}$  is continuous on  $[0, 2]$ .

6.  $\int_0^\infty \cos x dx$  is improper because the upper limit of integration is  $\infty$ .

7.  $\int_{-\infty}^\infty \frac{\sin x}{4 + x^2} dx$  is improper because the limits of integration are  $-\infty$  and  $\infty$ .

8.  $\int_0^{\pi/4} \csc x dx$  is improper because  $f(x) = \csc x$  is undefined at  $x = 0$ .

9. Infinite discontinuity at
- $x = 0$
- .

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{x}} dx &= \lim_{b \rightarrow 0^+} \int_b^4 \frac{1}{\sqrt{x}} dx \\ &= \lim_{b \rightarrow 0^+} \left[ 2\sqrt{x} \right]_b^4 \\ &= \lim_{b \rightarrow 0^+} (4 - 2\sqrt{b}) = 4\end{aligned}$$

Converges

10. Infinite discontinuity at
- $x = 3$
- .

$$\begin{aligned}\int_3^4 \frac{1}{(x-3)^{3/2}} dx &= \lim_{b \rightarrow 3^+} \int_b^4 (x-3)^{-3/2} dx \\ &= \lim_{b \rightarrow 3^+} \left[ -2(x-3)^{-1/2} \right]_b^4 \\ &= \lim_{b \rightarrow 3^+} \left[ -2 + \frac{2}{\sqrt{b-3}} \right] = \infty\end{aligned}$$

Diverges

12. Infinite discontinuity at
- $x = 1$
- .

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^{2/3}} dx &= \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \left[ 3\sqrt[3]{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[ 3\sqrt[3]{x-1} \right]_c^2 = (0+3) + (3-0) = 6\end{aligned}$$

Converges

13. Infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-4x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-4x} dx \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{4}e^{-4x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{4}e^{-4b} + \frac{1}{4} \right] = \frac{1}{4}\end{aligned}$$

Converges

14. Infinite limit of integration.

$$\begin{aligned}\int_{-\infty}^0 e^{3x} dx &= \lim_{b \rightarrow -\infty} \int_b^0 e^{3x} dx \\ &= \lim_{b \rightarrow -\infty} \left[ \frac{1}{3}e^{3x} \right]_b^0 \\ &= \lim_{b \rightarrow -\infty} \left[ \frac{1}{3} - \frac{1}{3}e^{3b} \right] = \frac{1}{3}\end{aligned}$$

Converges

- 15.
- $\int_{-1}^1 \frac{1}{x^2} dx \neq -2$

because the integrand is not defined at  $x = 0$ .

The integral diverges.

11. Infinite discontinuity at
- $x = 1$
- .

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^2} dx &= \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left[ -\frac{1}{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[ -\frac{1}{x-1} \right]_c^2 \\ &= (\infty - 1) + (-1 + \infty)\end{aligned}$$

Diverges

- 16.
- $\int_{-2}^2 \frac{-2}{(x-1)^3} dx \neq \frac{8}{9}$

because the integral is not defined at  $x = 1$ . The integral diverges.

- 17.
- $\int_0^\infty e^{-x} dx \neq 0$
- . You need to evaluate the limit.

$$\begin{aligned}\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx &= \lim_{b \rightarrow \infty} \left[ -e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[ -e^{-b} + 1 \right] = 1\end{aligned}$$

- 18.
- $\int_0^\pi \sec x dx \neq 0$
- because
- $\sec x$
- is not defined at
- $x = \pi/2$
- .

The integral diverges.

- 19.
- $\int_1^\infty \frac{1}{x^3} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-3} dx$
- $$\begin{aligned}&= \lim_{b \rightarrow \infty} \left[ \frac{x^{-2}}{-2} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[ \frac{-1}{2b^2} + \frac{1}{2} \right] = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}
 20. \int_1^{\infty} \frac{3}{x^5} dx &= \lim_{b \rightarrow \infty} 3 \int_1^b x^{-5} dx \\
 &= \lim_{b \rightarrow \infty} 3 \left[ -\frac{x^{-4}}{4} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{-3}{4b^4} + \frac{3}{4} \right] = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 21. \int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-1/3} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{9}{2} x^{2/3} \right]_1^b = \infty
 \end{aligned}$$

Diverges

$$\begin{aligned}
 22. \int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 4x^{-1/4} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{16}{3} x^{3/4} \right]_1^b = \infty \quad \text{Diverges}
 \end{aligned}$$

$$\begin{aligned}
 23. \int_{-\infty}^0 xe^{-4x} dx &= \lim_{b \rightarrow -\infty} \int_b^0 xe^{-4x} dx \\
 &= \lim_{b \rightarrow -\infty} \left[ \left( -\frac{x}{4} - \frac{1}{16} \right) e^{-4x} \right]_b^0 \quad (\text{Integration by parts}) \\
 &= \lim_{b \rightarrow -\infty} \left[ -\frac{1}{16} + \frac{b}{4} + \frac{1}{16} e^{-4b} \right] = -\infty
 \end{aligned}$$

Diverges

$$\begin{aligned}
 24. \int_0^{\infty} xe^{-x/4} dx &= \lim_{b \rightarrow \infty} \int_0^b xe^{-x/4} dx \\
 &= \lim_{b \rightarrow \infty} \left[ (-4x - 16) e^{-x/4} \right]_0^b \quad (\text{Integration by parts}) \\
 &= \lim_{b \rightarrow \infty} \left[ (-4b - 16) e^{-b/4} + 16 \right] = 16
 \end{aligned}$$

$$\begin{aligned}
 25. \int_0^{\infty} x^2 e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -e^{-x} (x^2 + 2x + 2) \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{b^2 + 2b + 2}{e^b} + 2 \right) = 2
 \end{aligned}$$

Because  $\lim_{b \rightarrow \infty} \left( -\frac{b^2 + 2b + 2}{e^b} \right) = 0$  by L'Hôpital's Rule.

$$\begin{aligned}
 26. \int_0^{\infty} (x-1)e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b (x-1)e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -xe^{-x} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left( \frac{-b}{e^b} + 0 \right) = 0
 \end{aligned}$$

by L'Hôpital's Rule.

$$\begin{aligned}
 27. \int_0^{\infty} e^{-x} \cos x dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[ e^{-x} (-\cos x + \sin x) \right]_0^b \\
 &= \frac{1}{2} [0 - (-1)] = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 28. \int_0^{\infty} e^{-ax} \sin bx dx &= \lim_{c \rightarrow \infty} \left[ \frac{e^{-ax} (-a \sin bx - b \cos bx)}{a^2 + b^2} \right]_0^c \\
 &= 0 - \frac{-b}{a^2 + b^2} = \frac{b}{a^2 + b^2}
 \end{aligned}$$

$$\begin{aligned}
 29. \int_4^{\infty} \frac{1}{x(\ln x)^3} dx &= \lim_{b \rightarrow \infty} \int_4^b (\ln x)^{-3} \frac{1}{x} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{2} (\ln x)^{-2} \right]_4^b \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{2} (\ln b)^{-2} + \frac{1}{2} (\ln 4)^{-2} \right] \\
 &= \frac{1}{2} \frac{1}{(2 \ln 2)^2} = \frac{1}{2(\ln 4)^2}
 \end{aligned}$$

$$\begin{aligned}
 30. \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{(\ln x)^2}{2} \right]_1^b = \infty
 \end{aligned}$$

Diverges

$$\begin{aligned}
 31. \int_{-\infty}^{\infty} \frac{4}{16+x^2} dx &= \int_{-\infty}^0 \frac{4}{16+x^2} dx + \int_0^{\infty} \frac{4}{16+x^2} dx \\
 &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{4}{16+x^2} dx + \lim_{c \rightarrow \infty} \int_0^c \frac{4}{16+x^2} dx \\
 &= \lim_{b \rightarrow -\infty} \left[ \arctan\left(\frac{x}{4}\right) \right]_b^0 + \lim_{c \rightarrow \infty} \left[ \arctan\left(\frac{x}{4}\right) \right]_0^c \\
 &= \lim_{b \rightarrow -\infty} \left[ 0 - \arctan\left(\frac{b}{4}\right) \right] + \lim_{c \rightarrow \infty} \left[ \arctan\left(\frac{c}{4}\right) - 0 \right] \\
 &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi
 \end{aligned}$$

$$32. \int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{x}{x^2+1} dx - \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2+1)^2} dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{2} \ln(x^2+1) + \frac{1}{2(x^2+1)} \right]_0^b = \infty - \frac{1}{2}$$

Diverges

$$\begin{aligned}
 33. \int_0^{\infty} \frac{1}{e^x + e^{-x}} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1+e^{2x}} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \arctan(e^x) \right]_0^b \\
 &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

$$34. \int_0^{\infty} \frac{e^x}{1+e^x} dx = \lim_{b \rightarrow \infty} \left[ \ln(1+e^x) \right]_0^b = \infty - \ln 2$$

Diverges

$$35. \int_0^{\infty} \cos \pi x dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{\pi} \sin \pi x \right]_0^b$$

Diverges because  $\sin \pi b$  does not approach a limit as  $b \rightarrow \infty$ .

$$36. \int_0^{\infty} \sin \frac{x}{2} dx = \lim_{b \rightarrow \infty} \left[ -2 \cos \frac{x}{2} \right]_0^b$$

Diverges because  $\cos \frac{x}{2}$  does not approach a limit as  $x \rightarrow \infty$ .

$$37. \int_0^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \left[ -\frac{1}{x} \right]_b^{\infty} = \lim_{b \rightarrow 0^+} \left( -1 + \frac{1}{b} \right) = -1 + \infty$$

Diverges

$$\begin{aligned}
 38. \int_0^5 \frac{10}{x} dx &= \lim_{b \rightarrow 0^+} \int_b^5 \frac{10}{x} dx \\
 &= \lim_{b \rightarrow 0^+} [10 \ln x]_b^5 \\
 &= \lim_{b \rightarrow 0^+} (10 \ln 5 - 10 \ln b) = \infty
 \end{aligned}$$

Diverges

$$\begin{aligned}
 39. \int_0^8 \frac{1}{\sqrt[3]{8-x}} dx &= \lim_{b \rightarrow 8^-} \int_0^b \frac{1}{\sqrt[3]{8-x}} dx \\
 &= \lim_{b \rightarrow 8^-} \left[ \frac{-3}{2} (8-x)^{2/3} \right]_0^b = 6
 \end{aligned}$$

$$\begin{aligned}
 40. \int_0^{12} \frac{9}{\sqrt{12-x}} dx &= \lim_{b \rightarrow 12^-} \int_0^b 9(12-x)^{-1/2} dx \\
 &= \lim_{b \rightarrow 12^-} [-18\sqrt{12-x}]_0^b \\
 &= 18\sqrt{12} = 36\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 41. \int_0^1 x \ln x dx &= \lim_{b \rightarrow 0^+} \left[ \frac{x^2}{2} \ln|x| - \frac{x^2}{4} \right]_b^1 \\
 &= \lim_{b \rightarrow 0^+} \left( \frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4} \right) = \frac{-1}{4}
 \end{aligned}$$

because  $\lim_{b \rightarrow 0^+} (b^2 \ln b) = 0$  by L'Hôpital's Rule.

$$\begin{aligned}
 42. \int_0^e \ln x^2 dx &= \lim_{b \rightarrow 0^+} \int_0^e 2 \ln x dx \\
 &= \lim_{b \rightarrow 0^+} [2x \ln x - 2x]_b^e \\
 &= \lim_{b \rightarrow 0^+} [(2e - 2e) - (2b \ln b - 2b)] \\
 &= 0
 \end{aligned}$$

$$43. \int_0^{\pi/2} \tan \theta d\theta = \lim_{b \rightarrow (\pi/2)^-} [\ln|\sec \theta|]_0^b = \infty$$

Diverges

$$44. \int_0^{\pi/2} \sec \theta d\theta = \lim_{b \rightarrow (\pi/2)^-} [\ln|\sec \theta + \tan \theta|]_0^b = \infty$$

Diverges

$$\begin{aligned}
 45. \int_2^4 \frac{2}{x\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \int_b^4 \frac{2}{x\sqrt{x^2-4}} dx \\
 &= \lim_{b \rightarrow 2^+} \left[ \operatorname{arcsec} \left| \frac{x}{2} \right| \right]_b^4 \\
 &= \lim_{b \rightarrow 2^+} \left( \operatorname{arcsec} 2 - \operatorname{arcsec} \left( \frac{b}{2} \right) \right) \\
 &= \frac{\pi}{3} - 0 = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 46. \int_0^5 \frac{1}{\sqrt{25-x^2}} dx &= \lim_{b \rightarrow 5^-} \int_0^b \frac{1}{\sqrt{25-x^2}} dx \\
 &= \lim_{b \rightarrow 5^-} \left[ \arcsin \left( \frac{x}{5} \right) \right]_0^b \\
 &= \arcsin(1) - 0 = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 47. \int_2^4 \frac{1}{\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \left[ \ln |x + \sqrt{x^2-4}| \right]_b^4 \\
 &= \ln(4 + 2\sqrt{3}) - \ln 2 \\
 &= \ln(2 + \sqrt{3}) \approx 1.317
 \end{aligned}$$

$$\begin{aligned}
 48. \int_0^5 \frac{1}{25-x^2} dx &= \lim_{b \rightarrow 5^-} \int_0^b \frac{1}{25-x^2} dx \\
 &= \lim_{b \rightarrow 5^-} \int_0^b \frac{1}{10} \left( \frac{1}{x+5} - \frac{1}{x-5} \right) dx \quad (\text{partial fractions}) \\
 &= \lim_{b \rightarrow 5^-} \left[ \frac{1}{10} \ln \left| \frac{x+5}{x-5} \right| \right]_0^b \\
 &= \infty - 0 \quad \text{Diverges}
 \end{aligned}$$

$$49. \int_0^2 \frac{1}{\sqrt[3]{x-1}} dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}} dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}} dx = \lim_{b \rightarrow 1^-} \left[ \frac{3}{2} (x-1)^{2/3} \right]_0^b + \lim_{c \rightarrow 1^+} \left[ \frac{3}{2} (x-1)^{2/3} \right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0$$

$$\begin{aligned}
 50. \int_1^3 \frac{2}{(x-2)^{8/3}} dx &= \int_1^2 2(x-2)^{-8/3} dx + \int_2^3 2(x-2)^{-8/3} dx \\
 &= \lim_{b \rightarrow 2^-} \int_1^b 2(x-2)^{-8/3} dx + \lim_{c \rightarrow 2^+} \int_c^3 2(x-2)^{-8/3} dx = \lim_{b \rightarrow 2^-} \left[ -\frac{6}{5} (x-2)^{-5/3} \right]_1^b + \lim_{c \rightarrow 2^+} \left[ -\frac{6}{5} (x-2)^{-5/3} \right]_c^3 = \infty
 \end{aligned}$$

Diverges

$$\begin{aligned}
 51. \int_3^\infty \frac{1}{x\sqrt{x^2-9}} dx &= \lim_{b \rightarrow 3^+} \int_b^5 \frac{1}{x\sqrt{x^2-9}} dx + \lim_{c \rightarrow \infty} \int_5^c \frac{1}{x\sqrt{x^2-9}} dx \\
 &= \lim_{b \rightarrow 3^+} \left[ \frac{1}{3} \operatorname{arcsec} \frac{x}{3} \right]_b^5 + \lim_{c \rightarrow \infty} \left[ \frac{1}{3} \operatorname{arcsec} \left( \frac{x}{3} \right) \right]_5^c \\
 &= \lim_{b \rightarrow 3^+} \left[ \frac{1}{3} \operatorname{arcsec} \left( \frac{5}{3} \right) - \frac{1}{3} \operatorname{arcsec} \left( \frac{b}{3} \right) \right] + \lim_{c \rightarrow \infty} \left[ \frac{1}{3} \operatorname{arcsec} \left( \frac{c}{3} \right) - \frac{1}{3} \operatorname{arcsec} \left( \frac{5}{3} \right) \right] = -0 + \frac{1}{3} \left( \frac{\pi}{2} \right) = \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 52. \int_5^\infty \frac{1}{x\sqrt{x^2-25}} dx &= \lim_{b \rightarrow 5^+} \int_b^6 \frac{dx}{x\sqrt{x^2-25}} + \lim_{c \rightarrow \infty} \int_6^c \frac{dx}{x\sqrt{x^2-25}} \\
 &= \lim_{b \rightarrow 5^+} \left[ \frac{1}{5} \operatorname{arcsec} \left( \frac{x}{5} \right) \right]_b^6 + \lim_{c \rightarrow \infty} \left[ \frac{1}{5} \operatorname{arcsec} \left( \frac{x}{5} \right) \right]_6^c = \frac{1}{5} \operatorname{arcsec} \left( \frac{6}{5} \right) - \frac{1}{5} (0) + \frac{1}{5} \left( \frac{\pi}{2} \right) - \frac{1}{5} \operatorname{arcsec} \left( \frac{6}{5} \right) = \frac{\pi}{10}
 \end{aligned}$$

$$53. \int_0^{\infty} \frac{4}{\sqrt{x}(x+6)} dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)} dx + \int_1^{\infty} \frac{4}{\sqrt{x}(x+6)} dx$$

Let  $u = \sqrt{x}$ ,  $u^2 = x$ ,  $2u du = dx$ .

$$\int \frac{4}{\sqrt{x}(x+6)} dx = \int \frac{4(2u du)}{u(u^2+6)} = 8 \int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}} \arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

$$\begin{aligned} \text{So, } \int_0^{\infty} \frac{4}{\sqrt{x}(x+6)} dx &= \lim_{b \rightarrow 0^+} \left[ \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_b^1 + \lim_{c \rightarrow \infty} \left[ \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_1^c \\ &= \left[ \frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}(0) \right] + \left[ \frac{8}{\sqrt{6}} \frac{\pi}{2} - \frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) \right] = \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}. \end{aligned}$$

$$54. \int \frac{1}{x \ln x} dx = \ln|\ln|x|| + C$$

So,

$$\int_1^{\infty} \frac{1}{x \ln x} dx = \int_1^e \frac{1}{x \ln x} dx + \int_e^{\infty} \frac{1}{x \ln x} dx = \lim_{b \rightarrow 1^+} [\ln(\ln x)]_1^e + \lim_{c \rightarrow \infty} [\ln(\ln x)]_e^c.$$

Diverges

$$55. \text{ If } p = 1, \int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln x]_1^b = \lim_{b \rightarrow \infty} (\ln b) = \infty.$$

Diverges. For  $p \neq 1$ ,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \rightarrow \infty} \left( \frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right).$$

This converges to  $\frac{1}{p-1}$  if  $1-p < 0$  or  $p > 1$ .

$$56. \text{ If } p = 1, \int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} [\ln x]_a^1 = \lim_{a \rightarrow 0^+} -\ln a = \infty.$$

Diverges. If  $p \neq 1$ ,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left[ \frac{x^{1-p}}{1-p} \right]_a^1 = \lim_{a \rightarrow 0^+} \left( \frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right).$$

This converges to  $\frac{1}{1-p}$  if  $1-p > 0$  or  $p < 1$ .

57. For  $n = 1$ :

$$\begin{aligned} \int_0^{\infty} x e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx \\ &= \lim_{b \rightarrow \infty} [-e^{-x} x - e^{-x}]_0^b \quad (\text{Parts: } u = x, dv = e^{-x} dx) \\ &= \lim_{b \rightarrow \infty} (-e^{-b} b - e^{-b} + 1) \\ &= \lim_{b \rightarrow \infty} \left( \frac{-b}{e^b} - \frac{1}{e^b} + 1 \right) = 1 \quad (\text{L'Hôpital's Rule}) \end{aligned}$$

Assume that  $\int_0^{\infty} x^n e^{-x} dx$  converges. Then for  $n+1$  you have

$$\int x^{n+1} e^{-x} dx = -x^{n+1} e^{-x} + (n+1) \int x^n e^{-x} dx$$

by parts ( $u = x^{n+1}$ ,  $du = (n+1)x^n dx$ ,  $dv = e^{-x} dx$ ,  $v = -e^{-x}$ ).

So,

$$\int_0^{\infty} x^{n+1} e^{-x} dx = \lim_{b \rightarrow \infty} [-x^{n+1} e^{-x}]_0^b + (n+1) \int_0^{\infty} x^n e^{-x} dx = 0 + (n+1) \int_0^{\infty} x^n e^{-x} dx, \text{ which converges.}$$

58. (a)  $\int_1^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_1^b = 1$

Because  $e^{-x^2} \leq e^{-x}$  on  $[1, \infty)$   
and

$$\int_1^{\infty} e^{-x} dx$$

converges, then so does

$$\int_1^{\infty} e^{-x^2} dx.$$

(b)  $\int_1^{\infty} \frac{1}{x^5} dx$  converges (see Exercise 55).

Because  $\frac{1}{x^5 + 1} < \frac{1}{x^5}$  on  $[1, \infty)$ , then

$$\int_1^{\infty} \frac{1}{x^5 + 1} dx \text{ also converges.}$$

59.  $\int_0^1 \frac{1}{x^5} dx$  diverges by Exercise 56. ( $p = 5$ )

60.  $\int_0^1 \frac{1}{x^{1/5}} dx$  converges by Exercise 56. ( $p = \frac{1}{5}$ )

61.  $\int_1^{\infty} \frac{1}{x^5} dx$  converges by Exercise 55. ( $p = 5$ )

62.  $\int_0^{\infty} x^4 e^{-x} dx$  converges by Exercise 57. ( $n = 4$ )

63. Because  $\frac{1}{x^2 + 5} \leq \frac{1}{x^2}$  on  $[1, \infty)$  and

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ converges by Exercise 55,}$$

$$\int_1^{\infty} \frac{1}{x^2 + 5} dx \text{ converges.}$$

64. Because  $\frac{1}{\sqrt{x-1}} \geq \frac{1}{x}$  on  $[2, \infty)$  and  $\int_2^{\infty} \frac{1}{x} dx$  diverges

by Exercise 55,  $\int_2^{\infty} \frac{1}{\sqrt{x-1}} dx$  diverges.

65. Because  $\frac{1}{\sqrt[3]{x(x-1)}} \geq \frac{1}{\sqrt[3]{x^2}}$  on  $[2, \infty)$  and

$$\int_2^{\infty} \frac{1}{\sqrt[3]{x^2}} dx \text{ diverges by Exercise 55,}$$

$$\int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}} dx \text{ diverges.}$$

66. Because  $\frac{1}{\sqrt{x(1+x)}} \leq \frac{1}{x^{3/2}}$  on  $[1, \infty)$  and

$$\int_1^{\infty} \frac{1}{x^{3/2}} dx \text{ converges by Exercise 55,}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x(1+x)}} dx \text{ converges.}$$

67. Because  $\frac{2 + e^{-x}}{x} \geq \frac{2}{x}$  on  $[1, \infty)$  and  $\int_1^{\infty} \frac{2}{x} dx$  diverges

by Exercise 55,  $\int_1^{\infty} \frac{2 + e^{-x}}{x} dx$  diverges.

68.  $\int_0^{\infty} \frac{1}{e^x} dx = \int_0^{\infty} e^{-x} dx$  converges, and  $\frac{1}{e^x} \geq \frac{1}{e^x + x}$  on

$[0, \infty)$ , so  $\int_0^{\infty} \frac{1}{e^x + x} dx$  converges.

69.  $\int_1^{\infty} \frac{2}{x^2} dx$  converges, and  $\frac{1 - \sin x}{x^2} \leq \frac{2}{x^2}$  on  $[1, \infty)$ , so

$$\int_1^{\infty} \frac{1 - \sin x}{x^2} dx \text{ converges.}$$

70.  $\frac{1}{\sqrt{x} \ln x} \geq \frac{1}{x}$  because  $\sqrt{x} \ln x < x$  on  $[2, \infty)$ . Because

$$\int_2^{\infty} \frac{1}{x} dx \text{ diverges by Exercise 55,}$$

$$\int_2^{\infty} \frac{1}{\sqrt{x} \ln x} dx \text{ diverges.}$$

71. Answers will vary. *Sample answer:*

An integral with infinite integration limits or an integral with an infinite discontinuity at or between the integration limits

72. See the definitions, pages 580, 583.

73.  $\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$

These two integrals diverge by Exercise 56.

74.  $\frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx$$

75.  $A = \int_{-\infty}^1 e^x dx$

$$= \lim_{b \rightarrow -\infty} \int_b^1 e^x dx$$

$$= \lim_{b \rightarrow -\infty} [e^x]_b^1$$

$$= \lim_{b \rightarrow -\infty} (e - e^b) = e$$

$$76. A = \int_0^1 -\ln x \, dx$$

$$\begin{aligned} &= -\lim_{b \rightarrow 0^+} \int_b^1 \ln x \, dx \\ &= -\lim_{b \rightarrow 0^+} [x \ln x - x]_b^1 \\ &= -\lim_{b \rightarrow 0^+} [(0 - 1) - b \ln b + b] \\ &= 1 \end{aligned}$$

$$\text{Note: } \lim_{b \rightarrow 0^+} b \ln b = \lim_{b \rightarrow 0^+} \frac{\ln b}{1/b} = \lim_{b \rightarrow 0^+} \frac{1/b}{-1/b^2} = 0$$

$$77. A = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx$$

$$\begin{aligned} &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{1}{x^2 + 1} \, dx + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{x^2 + 1} \, dx \\ &= \lim_{b \rightarrow -\infty} [\arctan(x)]_b^0 + \lim_{b \rightarrow \infty} [\arctan(x)]_0^b \\ &= \lim_{b \rightarrow -\infty} [0 - \arctan(b)] + \lim_{b \rightarrow \infty} [\arctan(b) - 0] \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \end{aligned}$$

$$78. A = \int_{-\infty}^{\infty} \frac{8}{x^2 + 4} \, dx$$

$$\begin{aligned} &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{8}{x^2 + 4} \, dx + \lim_{b \rightarrow \infty} \int_0^b \frac{8}{x^2 + 4} \, dx \\ &= \lim_{b \rightarrow -\infty} \left[ 4 \arctan\left(\frac{x}{2}\right) \right]_b^0 + \lim_{b \rightarrow \infty} \left[ 4 \arctan\left(\frac{x}{2}\right) \right]_0^b \\ &= \lim_{b \rightarrow -\infty} \left[ 0 - 4 \arctan\left(\frac{b}{2}\right) \right] + \lim_{b \rightarrow \infty} \left[ 4 \arctan\left(\frac{b}{2}\right) - 0 \right] \\ &= -4\left(-\frac{\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) = 4\pi \end{aligned}$$

$$81. \quad x^{2/3} + y^{2/3} = 4$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}, \quad (x > 0)$$

$$s = 4 \int_0^8 \frac{2}{x^{1/3}} \, dx = \lim_{b \rightarrow 0^+} \left[ 8 \cdot \frac{3}{2} x^{2/3} \right]_b^8 = 48$$

$$82. \quad y = \sqrt{16 - x^2}, \quad 0 \leq x \leq 4$$

$$y' = \frac{-x}{\sqrt{16 - x^2}}$$

$$s = \int_0^4 \sqrt{1 + \frac{x^2}{16 - x^2}} \, dx = \int_0^4 \frac{4}{\sqrt{16 - x^2}} \, dx = \lim_{t \rightarrow 4^-} \int_0^t \frac{4}{\sqrt{16 - x^2}} \, dx = \lim_{t \rightarrow 4^-} \left[ 4 \arcsin\left(\frac{x}{4}\right) \right]_0^t = \lim_{t \rightarrow 4^-} 4 \arcsin\left(\frac{t}{4}\right) = 2\pi$$

$$79. (a) \quad A = \int_0^{\infty} e^{-x} \, dx$$

$$= \lim_{b \rightarrow \infty} [-e^{-x}]_0^b = 0 - (-1) = 1$$

(b) Disk:

$$\begin{aligned} V &= \pi \int_0^{\infty} (e^{-x})^2 \, dx \\ &= \lim_{b \rightarrow \infty} \pi \left[ -\frac{1}{2} e^{-2x} \right]_0^b = \frac{\pi}{2} \end{aligned}$$

(c) Shell:

$$\begin{aligned} V &= 2\pi \int_0^{\infty} x e^{-x} \, dx \\ &= \lim_{b \rightarrow \infty} 2\pi [-e^{-x}(x+1)]_0^b = 2\pi \end{aligned}$$

$$80. (a) \quad A = \int_1^{\infty} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_1^{\infty} = 1$$

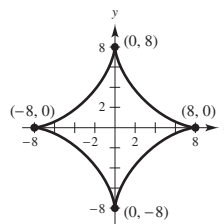
(b) Disk:

$$V = \pi \int_1^{\infty} \frac{1}{x^4} \, dx = \lim_{b \rightarrow \infty} \left[ -\frac{\pi}{3x^3} \right]_1^b = \frac{\pi}{3}$$

(c) Shell:

$$V = 2\pi \int_1^{\infty} x \left( \frac{1}{x^2} \right) \, dx = \lim_{b \rightarrow \infty} [2\pi(\ln x)]_1^b = \infty$$

Diverges





$$83. (x-2)^2 + y^2 = 1$$

$$2(x-2) + 2yy' = 0$$

$$y' = \frac{-(x-2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left[\frac{(x-2)^2}{y^2}\right]} = \frac{1}{y} \quad (\text{Assume } y > 0.)$$

$$\begin{aligned} S &= 4\pi \int_1^3 \frac{x}{y} dx = 4\pi \int_1^3 \frac{x}{\sqrt{1 - (x-2)^2}} dx = 4\pi \int_1^3 \left[ \frac{x-2}{\sqrt{1 - (x-2)^2}} + \frac{2}{\sqrt{1 - (x-2)^2}} \right] dx \\ &= \lim_{\substack{a \rightarrow 1^+ \\ b \rightarrow 3^-}} 4\pi \left[ -\sqrt{1 - (x-2)^2} + 2 \arcsin(x-2) \right]_a^b = 4\pi [0 + 2 \arcsin(1) - 2 \arcsin(-1)] = 8\pi^2 \end{aligned}$$

$$84. y = 2e^{-x}$$

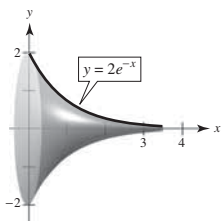
$$y' = -2e^{-x}$$

$$S = 2\pi \int_0^\infty (2e^{-x}) \sqrt{1 + 4e^{-2x}} dx$$

$$\text{Let } u = e^{-x}, du = -e^{-x} dx.$$

$$\begin{aligned} \int e^{-x} \sqrt{1 + 4e^{-2x}} dx &= -\int \sqrt{1 + 4u^2} du \\ &= -\frac{1}{4} \left[ 2u\sqrt{4u^2 + 1} + \ln \left| 2u + \sqrt{4u^2 + 1} \right| \right] + C \\ &= -\frac{1}{4} \left[ 2e^{-x}\sqrt{4e^{-2x} + 1} + \ln \left| 2e^{-x} + \sqrt{4e^{-2x} + 1} \right| \right] + C \end{aligned}$$

$$\begin{aligned} S &= 4\pi \lim_{b \rightarrow \infty} \int_0^b (e^{-x}) \sqrt{1 + 4e^{-2x}} dx \\ &= -\pi \lim_{b \rightarrow \infty} \left[ 2e^{-x}\sqrt{4e^{-2x} + 1} + \ln \left| 2e^{-x} + \sqrt{4e^{-2x} + 1} \right| \right]_0^b = \pi [2\sqrt{5} + \ln(2 + \sqrt{5})] \approx 18.5849 \end{aligned}$$



$$85. (a) F(x) = \frac{K}{x^2}, S = \frac{K}{(4000)^2}, K = 80,000,000$$

$$W = \int_{4000}^\infty \frac{80,000,000}{x^2} dx = \lim_{b \rightarrow \infty} \left[ \frac{-80,000,000}{x} \right]_{4000}^b = 20,000 \text{ mi-ton}$$

$$(b) \quad \frac{W}{2} = 10,000 = \left[ \frac{-80,000,000}{x} \right]_{4000}^b = \frac{-80,000,000}{b} + 20,000$$

$$\frac{80,000,000}{b} = 10,000$$

$$b = 8000$$

Therefore, 4000 miles above the earth's surface.

$$\begin{aligned}
 86. \text{ (a) } F(x) &= \frac{k}{x^2}, 10 = \frac{k}{4000^2}, k = 10(4000^2) \\
 W &= \int_{4000}^{\infty} \frac{10(4000^2)}{x^2} dx = \lim_{b \rightarrow \infty} \left[ \frac{-10(4000^2)}{x} \right]_{4000}^b \\
 &= \frac{10(4000^2)}{4000} = 40,000 \text{ mi-ton}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \frac{W}{2} &= 20,000 = \left[ \frac{-10(4000^2)}{x} \right]_{4000}^b \\
 &= \frac{-10(4000^2)}{b} + 40,000 \\
 \frac{10(4000^2)}{b} &= 20,000 \\
 b &= 8000
 \end{aligned}$$

Therefore, 4000 miles above the earth's surface.

$$89. \text{ (a) } C = 650,000 + \int_0^5 25,000 e^{-0.06t} dt = 650,000 - \left[ \frac{25,000}{0.06} e^{-0.06t} \right]_0^5 \approx \$757,992.41$$

$$\text{(b) } C = 650,000 + \int_0^{10} 25,000 e^{-0.06t} dt \approx \$837,995.15$$

$$\text{(c) } C = 650,000 + \int_0^{\infty} 25,000 e^{-0.06t} dt = 650,000 - \lim_{b \rightarrow \infty} \left[ \frac{25,000}{0.06} e^{-0.06t} \right]_0^b \approx \$1,066,666.67$$

$$\begin{aligned}
 90. \text{ (a) } C &= 650,000 + \int_0^5 25,000(1 + 0.08t)e^{-0.06t} dt \\
 &= 650,000 + 25,000 \left[ -\frac{1}{0.06} e^{-0.06t} - 0.08 \left( \frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^5 \approx \$778,512.58
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } C &= 650,000 + \int_0^{10} 25,000(1 + 0.08t)e^{-0.06t} dt \\
 &= 650,000 + 25,000 \left[ -\frac{1}{0.06} e^{-0.06t} - 0.08 \left( \frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^{10} \approx \$905,718.14
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } C &= 650,000 + \int_0^{\infty} 25,000(1 + 0.08t)e^{-0.06t} dt \\
 &= 650,000 + 25,000 \lim_{b \rightarrow \infty} \left[ -\frac{1}{0.06} e^{-0.06t} - 0.08 \left( \frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^b \approx \$1,622,222.22
 \end{aligned}$$

$$87. \text{ (a) } \int_{-\infty}^{\infty} \frac{1}{7} e^{-t/7} dt = \int_0^{\infty} \frac{1}{7} e^{-t/7} dt = \lim_{b \rightarrow \infty} \left[ -e^{-t/7} \right]_0^b = 1$$

$$\begin{aligned}
 \text{(b) } \int_0^4 \frac{1}{7} e^{-t/7} dt &= \left[ -e^{-t/7} \right]_0^4 = -e^{-4/7} + 1 \\
 &\approx 0.4353 = 43.53\%
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \int_0^{\infty} t \left[ \frac{1}{7} e^{-t/7} \right] dt &= \lim_{b \rightarrow \infty} \left[ -te^{-t/7} - 7e^{-t/7} \right]_0^b \\
 &= 0 + 7 = 7
 \end{aligned}$$

$$88. \text{ (a) } \int_{-\infty}^{\infty} \frac{2}{5} e^{-2t/5} dt = \int_0^{\infty} \frac{2}{5} e^{-2t/5} dt = \lim_{b \rightarrow \infty} \left[ -e^{-2t/5} \right]_0^b = 1$$

$$\begin{aligned}
 \text{(b) } \int_0^4 \frac{2}{5} e^{-2t/5} dt &= \left[ -e^{-2t/5} \right]_0^4 = -e^{-8/5} + 1 \\
 &\approx 0.7981 = 79.81\%
 \end{aligned}$$

$$\text{(c) } \int_0^{\infty} t \left[ \frac{2}{5} e^{-2t/5} \right] dt = \lim_{b \rightarrow \infty} \left[ -te^{2t/5} - \frac{5}{2} e^{-2t/5} \right]_0^b = \frac{5}{2}$$

91. Let  $K = \frac{2\pi N I r}{k}$ . Then

$$P = K \int_c^\infty \frac{1}{(r^2 + x^2)^{3/2}} dx.$$

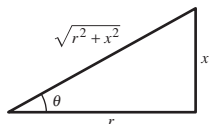
Let

$$x = r \tan \theta, dx = r \sec^2 \theta d\theta, \sqrt{r^2 + x^2} = r \sec \theta.$$

$$\begin{aligned} \int \frac{1}{(r^2 + x^2)^{3/2}} dx &= \int \frac{r \sec^2 \theta d\theta}{r^3 \sec^3 \theta} = \frac{1}{r^2} \int \cos \theta d\theta \\ &= \frac{1}{r^2} \sin \theta + C = \frac{1}{r^2} \frac{x}{\sqrt{r^2 + x^2}} + C \end{aligned}$$

So,

$$\begin{aligned} P &= K \frac{1}{r^2} \lim_{b \rightarrow \infty} \left[ \frac{x}{\sqrt{r^2 + x^2}} \right]_c^b \\ &= \frac{K}{r^2} \left[ 1 - \frac{c}{\sqrt{r^2 + c^2}} \right] \\ &= \frac{K(\sqrt{r^2 + c^2} - c)}{r^2 \sqrt{r^2 + c^2}} \\ &= \frac{2\pi N I (\sqrt{r^2 + c^2} - c)}{kr \sqrt{r^2 + c^2}}. \end{aligned}$$



92.  $F = \int_0^\infty \frac{GM\delta}{(a+x)^2} dx$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} \left[ \frac{-GM\delta}{a+x} \right]_0^b \\ &= \frac{GM\delta}{a} \end{aligned}$$

93. False.  $f(x) = 1/(x+1)$  is continuous on

$$[0, \infty), \lim_{x \rightarrow \infty} 1/(x+1) = 0, \text{ but}$$

$$\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} [\ln|x+1|]_0^b = \infty.$$

Diverges

94. False. This is equivalent to Exercise 93.

95. True

96. True

97. (a)  $\int_{-\infty}^\infty \sin x dx = \int_{-\infty}^0 \sin x dx + \int_0^\infty \sin x dx$

$$\begin{aligned} &= \lim_{b \rightarrow -\infty} \int_b^0 \sin x dx + \lim_{c \rightarrow \infty} \int_0^c \sin x dx \\ &= \lim_{b \rightarrow -\infty} [-\cos x]_b^0 + \lim_{c \rightarrow \infty} [-\cos x]_0^c \end{aligned}$$

Because  $\lim_{b \rightarrow -\infty} [-\cos b]$  diverges, as does  $\lim_{c \rightarrow \infty} [-\cos c]$ ,

$$\int_{-\infty}^\infty \sin x dx \text{ diverges.}$$

(b)  $\lim_{a \rightarrow \infty} \int_{-a}^a \sin x dx = \lim_{a \rightarrow \infty} [-\cos x]_{-a}^a$

$$= \lim_{a \rightarrow \infty} [-\cos(a) + \cos(-a)] = 0$$

(c) The definition of  $\int_{-\infty}^\infty f(x) dx$  is not

$$\lim_{a \rightarrow \infty} \int_{-a}^a f(x) dx.$$

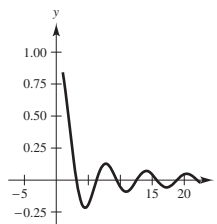
98. (a)  $b = 3$  (infinite discontinuity at 3)  
 (b)  $b = 4$  (infinite discontinuity at 4)  
 (c)  $b = 3$  (or  $b = 4$ ) (infinite discontinuity at 3)  
 (d)  $b = 0$  (infinite discontinuity at 0)  
 (e)  $b = \pi/4$  (infinite discontinuity at  $\pi/4$ )  
 (f)  $b = \pi/2$  (infinite discontinuity at  $\pi/2$ )

99. (a)  $\int_1^\infty \frac{1}{x} dx = \lim_{b \rightarrow \infty} [\ln|x|]_1^b = \infty$

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^b = 1$$

$\int_1^\infty \frac{1}{x^n} dx$  will converge if  $n > 1$  and will diverge if  $n \leq 1$ .

(b) It would appear to converge.



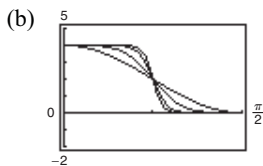
(c) Let  $dv = \sin x dx \Rightarrow v = -\cos x$

$$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx.$$

$$\begin{aligned} \int_1^\infty \frac{\sin x}{x} dx &= \lim_{b \rightarrow \infty} \left[ -\frac{\cos x}{x} \right]_1^b - \int_1^\infty \frac{\cos x}{x^2} dx \\ &= \cos 1 - \int_1^\infty \frac{\cos x}{x^2} dx \end{aligned}$$

Converges

100. (a) Yes, the integrand is not defined at  $x = \pi/2$ .



(c) As  $n \rightarrow \infty$ , the integral approaches  $4(\pi/4) = \pi$ .

$$(d) \quad I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n} dx$$

$$I_2 \approx 3.14159$$

$$I_4 \approx 3.14159$$

$$I_8 \approx 3.14159$$

$$I_{12} \approx 3.14159$$

101.  $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$$(a) \quad \Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_0^b = 1$$

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}(x+1)]_0^b = 1$$

$$\Gamma(3) = \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}]_0^b = 2$$

$$(b) \quad \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \lim_{b \rightarrow \infty} [-x^n e^{-x}]_0^b + \lim_{b \rightarrow \infty} n \int_0^b x^{n-1} e^{-x} dx = 0 + n\Gamma(n) \quad (u = x^n, dv = e^{-x} dx)$$

$$(c) \quad \Gamma(n) = (n-1)!$$

102. For  $n = 1$ ,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[ -\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For  $n > 1$ ,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[ \frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^{n+2}} \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$\left( \text{Parts: } u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}} \right)$$

$$(a) \quad \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[ -\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \quad \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left( \frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \quad \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left( \frac{1}{24} \right) = \frac{1}{60}$$

103.  $f(t) = 1$

$$F(s) = \int_0^{\infty} e^{-st} dt = \lim_{b \rightarrow \infty} \left[ -\frac{1}{s} e^{-st} \right]_0^b = \frac{1}{s}, s > 0$$

104.  $f(t) = t$

$$\begin{aligned} F(s) &= \int_0^{\infty} t e^{-st} dt = \lim_{b \rightarrow \infty} \left[ \frac{1}{s^2} (-st - 1) e^{-st} \right]_0^b \\ &= \frac{1}{s^2}, s > 0 \end{aligned}$$

105.  $f(t) = t^2$

$$\begin{aligned} F(s) &= \int_0^{\infty} t^2 e^{-st} dt = \lim_{b \rightarrow \infty} \left[ \frac{1}{s^3} (-s^2 t^2 - 2st - 2) e^{-st} \right]_0^b \\ &= \frac{2}{s^3}, s > 0 \end{aligned}$$

106.  $f(t) = e^{at}$

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{t(a-s)} dt \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{a-s} e^{t(a-s)} \right]_0^b \\ &= 0 - \frac{1}{a-s} = \frac{1}{s-a}, s > a \end{aligned}$$

109.  $f(t) = \cosh at$

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \cosh at dt = \int_0^{\infty} e^{-st} \left( \frac{e^{at} + e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} [e^{t(-s+a)} + e^{t(-s-a)}] dt \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[ \frac{1}{(-s+a)} e^{t(-s+a)} + \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[ \frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] \\ &= \frac{-1}{2} \left[ \frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] = \frac{s}{s^2 - a^2}, s > |a| \end{aligned}$$

110.  $f(t) = \sinh at$

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \sinh at dt = \int_0^{\infty} e^{-st} \left( \frac{e^{at} - e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} [e^{t(-s+a)} - e^{t(-s-a)}] dt \\ &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[ \frac{1}{(-s+a)} e^{t(-s+a)} - \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[ \frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] \\ &= \frac{-1}{2} \left[ \frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] = \frac{a}{s^2 - a^2}, s > |a| \end{aligned}$$

107.  $f(t) = \cos at$

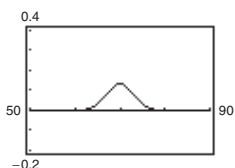
$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \cos at dt \\ &= \lim_{b \rightarrow \infty} \left[ \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^b \\ &= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2}, s > 0 \end{aligned}$$

108.  $f(t) = \sin at$

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \sin at dt \\ &= \lim_{b \rightarrow \infty} \left[ \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^b \\ &= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2}, s > 0 \end{aligned}$$

111. (a)  $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b)  $P(72 \leq x < \infty) \approx 0.2525$

(c)  $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because of symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

$$\begin{aligned} 113. \int_0^\infty \left( \frac{1}{\sqrt{x^2 + 1}} - \frac{c}{x + 1} \right) dx &= \lim_{b \rightarrow \infty} \int_0^b \left( \frac{1}{\sqrt{x^2 + 1}} - \frac{c}{x + 1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[ \ln |x + \sqrt{x^2 + 1}| - c \ln |x + 1| \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[ \ln(b + \sqrt{b^2 + 1}) - \ln(b + 1)^c \right] = \lim_{b \rightarrow \infty} \ln \left[ \frac{b + \sqrt{b^2 + 1}}{(b + 1)^c} \right] \end{aligned}$$

This limit exists for  $c = 1$ , and you have

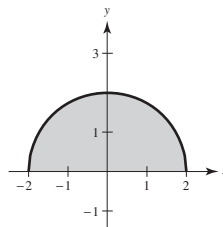
$$\lim_{b \rightarrow \infty} \ln \left[ \frac{b + \sqrt{b^2 + 1}}{(b + 1)} \right] = \ln 2.$$

$$\begin{aligned} 114. \int_1^\infty \left( \frac{cx}{x^2 + 2} - \frac{1}{3x} \right) dx &= \lim_{b \rightarrow \infty} \int_1^b \left( \frac{cx}{x^2 + 2} - \frac{1}{3x} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{c}{2} \ln(x^2 + 2) - \frac{1}{3} \ln |x| \right]_1^b \\ &= \lim_{b \rightarrow \infty} \ln \left[ \frac{(x^2 + 2)^{c/2}}{x^{1/3}} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[ \ln \frac{(b^2 + 2)^{c/2}}{b^{1/3}} - \ln 3^{c/2} \right] \end{aligned}$$

This limit exists if  $c = 1/3$ , and you have

$$\lim_{b \rightarrow \infty} \left[ \ln \frac{(b^2 + 2)^{1/6}}{b^{1/3}} - \ln 3^{1/6} \right] = -\ln 3^{1/6} = \frac{-\ln 3}{6}.$$

112. (a)



(b) Area =  $\frac{1}{2}\pi(2)^2 = 2\pi$

Arc length is also  $\frac{1}{2}(2\pi(2)) = 2\pi$ .

So, the corresponding integrals are equal.

Let  $y = \sqrt{4 - x^2}$ ,  $y' = \frac{-x}{\sqrt{4 - x^2}}$

$$1 + (y')^2 = \frac{4}{4 - x^2} \Rightarrow \sqrt{1 + (y')^2} = \frac{2}{\sqrt{4 - x^2}}.$$

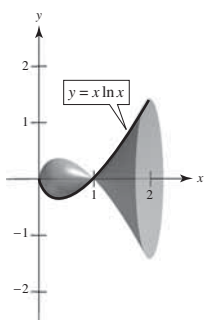
$$\begin{aligned} \text{So, } \int_{-2}^2 \sqrt{4 - x^2} dx &= \int_{-2}^2 \frac{2}{\sqrt{4 - x^2}} dx. \\ \text{(area)} &\qquad \qquad \text{(arc length)} \end{aligned}$$

115.  $f(x) = \begin{cases} x \ln x, & 0 < x \leq 2 \\ 0, & x = 0 \end{cases}$

$$V = \pi \int_0^2 (x \ln x)^2 dx$$

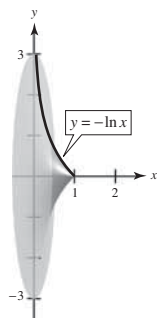
Let  $u = \ln x$ ,  $e^u = x$ ,  $e^u du = dx$ .

$$\begin{aligned} V &= \pi \int_{-\infty}^{\ln 2} e^{2u} u^2 (e^4 du) \\ &= \pi \int_{-\infty}^{\ln 2} e^{3u} u^2 du \\ &= \lim_{b \rightarrow -\infty} \left[ \pi \left[ \frac{u^2}{3} - \frac{2u}{9} + \frac{2}{27} \right] e^{3u} \right]_b^{\ln 2} \\ &= 8\pi \left[ \frac{(\ln 2)^2}{3} - \frac{2 \ln 2}{9} + \frac{2}{27} \right] \approx 2.0155 \end{aligned}$$



116.  $V = \pi \int_0^1 (-\ln x)^2 dx$

$$\begin{aligned} &= \lim_{b \rightarrow 0^+} \pi \int_b^1 (\ln x)^2 dx \\ &= \lim_{b \rightarrow 0^+} \pi x \left[ (\ln x)^2 - 2 \ln x + 2 \right]_b^1 \\ &= \lim_{b \rightarrow 0^+} \pi \left[ 2 - b(\ln b)^2 - 2b \ln b - 2b \right] \\ &= 2\pi \end{aligned}$$



117.  $u = \sqrt{x}$ ,  $u^2 = x$ ,  $2u du = dx$

$$\int_0^1 \frac{\sin x}{\sqrt{x}} dx = \int_0^1 \frac{\sin(u^2)}{u} (2u du) = \int_0^1 2 \sin(u^2) du$$

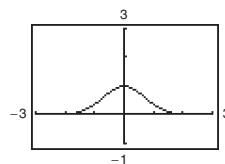
Trapezoidal Rule ( $n = 5$ ): 0.6278

118.  $u = \sqrt{1-x}$ ,  $1-x = u^2$ ,  $2u du = -dx$

$$\begin{aligned} \int_0^1 \frac{\cos x}{\sqrt{1-x}} dx &= \int_1^0 \frac{\cos(1-u^2)}{u} (-2u du) \\ &= \int_0^1 2 \cos(1-u^2) du \end{aligned}$$

Trapezoidal Rule ( $n = 5$ ): 1.4997

119. (a)



(b) Let  $y = e^{-x^2}$ ,  $0 \leq x < \infty$ .

$$\ln y = -x^2$$

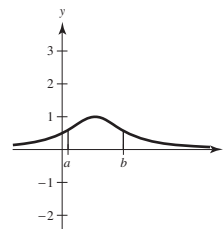
$$x = \sqrt{-\ln y} \text{ for } 0 < y \leq 1$$

The area bounded by  $y = e^{-x^2}$ ,  $x = 0$  and  $y = 0$  is

$$\int_0^\infty e^{-x^2} dx = \int_0^1 \sqrt{-\ln y} dy, \quad \left( = \frac{\sqrt{\pi}}{2} \right).$$

120. Assume  $a < b$ . The proof is similar if  $a > b$ .

$$\begin{aligned}
 \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx &= \lim_{c \rightarrow -\infty} \int_c^a f(x) dx + \lim_{d \rightarrow \infty} \int_a^d f(x) dx \\
 &= \lim_{c \rightarrow -\infty} \int_c^a f(x) dx + \lim_{d \rightarrow \infty} \left[ \int_a^b f(x) dx + \int_b^d f(x) dx \right] \\
 &= \lim_{c \rightarrow -\infty} \int_c^a f(x) dx + \int_a^b f(x) dx + \lim_{d \rightarrow \infty} \int_b^d f(x) dx \\
 &= \lim_{c \rightarrow -\infty} \left[ \int_c^a f(x) dx + \int_a^b f(x) dx \right] + \lim_{d \rightarrow \infty} \int_b^d f(x) dx \\
 &= \lim_{c \rightarrow -\infty} \int_c^b f(x) dx + \lim_{d \rightarrow \infty} \int_b^d f(x) dx \\
 &= \int_{-\infty}^b f(x) dx + \int_b^{\infty} f(x) dx
 \end{aligned}$$



## Review Exercises for Chapter 8

$$\begin{aligned}
 1. \int x\sqrt{x^2 - 36} dx &= \frac{1}{2} \int (x^2 - 36)^{1/2} (2x) dx \\
 &= \frac{1}{2} \frac{(x^2 - 36)^{3/2}}{3/2} + C \\
 &= \frac{1}{3} (x^3 - 36)^{3/2} + C
 \end{aligned}$$

$$2. \int x e^{x^2-1} dx = \frac{1}{2} \int e^{x^2-1} (2x) dx = \frac{1}{2} e^{x^2-1} + C$$

$$3. \int \frac{x}{x^2 - 49} dx = \frac{1}{2} \int \frac{2x}{x^2 - 49} dx = \frac{1}{2} \ln |x^2 - 49| + C$$

$$\begin{aligned}
 4. \int \frac{x}{\sqrt{25 - x^2}} dx &= -\frac{1}{2} \int (25 - x^2)^{-1/2} (-2x) dx \\
 &= -\frac{1}{2} \frac{(25 - x^2)^{1/2}}{1/2} + C \\
 &= -\sqrt{25 - x^2} + C
 \end{aligned}$$

$$5. \text{ Let } u = \ln(2x), du = \frac{1}{x} dx.$$

$$\begin{aligned}
 \int_1^e \frac{\ln(2x)}{x} dx &= \int_{\ln 2}^{1+\ln 2} u du \\
 &= \frac{u^2}{2} \Big|_{\ln 2}^{1+\ln 2} \\
 &= \frac{1}{2} [1 + 2 \ln 2 + (\ln 2)^2 - (\ln 2)^2] \\
 &= \frac{1}{2} + \ln 2 \approx 1.1931
 \end{aligned}$$

$$6. \text{ Let } u = 2x - 3, du = 2 dx, x = \frac{1}{2}(u + 3).$$

$$\begin{aligned}
 \int_{3/2}^2 2x\sqrt{2x-3} dx &= \int_0^1 (u+3)u^{1/2} \frac{1}{2} du \\
 &= \frac{1}{2} \int_0^1 (u^{3/2} + 3u^{1/2}) du \\
 &= \frac{1}{2} \left[ \frac{2}{5} u^{5/2} + 2u^{3/2} \right]_0^1 \\
 &= \frac{1}{2} \left( \frac{2}{5} + 2 \right) = \frac{6}{5}
 \end{aligned}$$

$$7. \int \frac{100}{\sqrt{100 - x^2}} dx = 100 \arcsin\left(\frac{x}{10}\right) + C$$

$$8. \frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} = 1 + \frac{x}{(x^2 + 1)^2}$$

$$\begin{aligned}
 \int \frac{x^4 + 2x^2 + x + 1}{(x^2 + 1)^2} dx &= \int dx + \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2} dx \\
 &= x - \frac{1}{2(x^2 + 1)} + C
 \end{aligned}$$

$$\begin{aligned}
 9. \int x e^{3x} dx &= \frac{x}{3} e^{3x} - \int \frac{1}{3} e^{3x} dx \\
 &= \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x} + C \\
 &= \frac{1}{9} e^{3x} (3x - 1) + C
 \end{aligned}$$

$$dv = e^{3x} dx \Rightarrow v = \frac{1}{3} e^{3x}$$

$$u = x \Rightarrow du = dx$$



$$\begin{aligned}
 10. \quad \int x^3 e^x dx &= x^3 e^x - \int 3x^2 e^x dx \\
 &= x^3 e^x - \left( 3x^2 e^x - \int 6xe^x dx \right) \\
 &= x^3 e^x - 3x^2 e^x + \int 6xe^x dx \\
 &= x^3 e^x - 3x^2 e^x + \left( 6xe^x - \int 6e^x dx \right) \\
 &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \\
 &= (x^3 - 3x^2 + 6x - 6)e^x + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad dv &= e^x dx \Rightarrow v = e^x \\
 u &= x^3 \Rightarrow du = 3x^2 dx
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad dv &= e^x dx \Rightarrow v = e^x \\
 u &= 3x^2 \Rightarrow du = 6x dx
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad dv &= e^x dx \Rightarrow v = e^x \\
 u &= 6x \Rightarrow du = 6 dx
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\
 &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left( \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \\
 \frac{13}{9} \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x \\
 \int e^{2x} \sin 3x dx &= \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad dv &= \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x \\
 u &= e^{2x} \Rightarrow du = 2e^{2x} dx
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad dv &= \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x \\
 u &= e^{2x} \Rightarrow du = 2e^{2x} dx
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \int (x^2 - 3x)e^x dx &= (x^2 - 3x)e^x - \int (2x - 3)e^x dx \\
 &= (x^2 - 3x)e^x - \left[ (2x - 3)e^x - \int 2e^x dx \right] \\
 &= (x^2 - 3x)e^x - (2x - 3)e^x + 2e^x + C \\
 &= (x^2 - 5x + 5)e^x + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad dv &= e^x dx \Rightarrow v = e^x \\
 u &= x^2 - 3x \Rightarrow du = (2x - 3) dx
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad dv &= e^x dx \Rightarrow v = e^x \\
 u &= 2x - 3 \Rightarrow du = 2 dx
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \int x\sqrt{x-1} dx &= \frac{2}{3}x(x-1)^{3/2} - \int \frac{2}{3}(x-1)^{3/2} dx \\
 &= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C \\
 &= \frac{2}{15}(x-1)^{3/2}(5x-2(x-1)) + C \\
 &= \frac{2}{15}(x-1)^{3/2}(3x+2) + C \\
 dv &= (x-1)^{1/2} dx \Rightarrow v = \frac{2}{3}(x-1)^{3/2} \\
 h &= x \Rightarrow du = dx
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int \arctan 2x dx &= x \arctan 2x - \int \frac{2x}{1+4x^2} dx \\
 &= x \arctan 2x - \frac{1}{4} \ln(1+4x^2) + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \arctan 2x \Rightarrow du = \frac{2}{1+4x^2} dx$$

NOT FOR SALE

$$\begin{aligned}
 15. \int x^2 \sin 2x \, dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x \, dx \\
 &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x \, dx \\
 &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \, dv &= \sin 2x \, dx \Rightarrow v = -\frac{1}{2} \cos 2x \\
 u &= x^2 \Rightarrow du = 2x \, dx
 \end{aligned}$$

$$\begin{aligned}
 (2) \, dv &= \cos 2x \, dx \Rightarrow v = \frac{1}{2} \sin 2x \\
 u &= x \Rightarrow du = dx
 \end{aligned}$$

$$\begin{aligned}
 17. \int x \arcsin 2x \, dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} \, dx \\
 &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} \, dx \\
 &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left( \frac{1}{2} \right) \left[ -(2x)\sqrt{1-4x^2} + \arcsin 2x \right] + C \quad (\text{by Formula 43 of Integration Tables}) \\
 &= \frac{1}{16} \left[ (8x^2 - 1) \arcsin 2x + 2x\sqrt{1-4x^2} \right] + C
 \end{aligned}$$

$$dv = x \, dx \Rightarrow v = \frac{x^2}{2}$$

$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} \, dx$$

$$\begin{aligned}
 18. \int e^x \arctan(e^x) \, dx &= e^x \arctan(e^x) - \int \frac{e^{2x}}{1+e^{2x}} \, dx \\
 &= e^x \arctan(e^x) - \frac{1}{2} \ln(1+e^{2x}) + C
 \end{aligned}$$

$$dv = e^x \, dx \Rightarrow v = e^x$$

$$u = \arctan e^x \Rightarrow du = \frac{e^x}{1+e^{2x}} \, dx$$

$$\begin{aligned}
 19. \int \cos^3(\pi x - 1) \, dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) \, dx \\
 &= \frac{1}{\pi} \left[ \sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\
 &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\
 &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\
 &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C
 \end{aligned}$$

$$20. \int \sin^2 \frac{\pi x}{2} \, dx = \int \frac{1}{2} (1 - \cos \pi x) \, dx = \frac{1}{2} \left( x - \frac{1}{\pi} \sin \pi x \right) + C = \frac{1}{2\pi} (\pi x - \sin \pi x) + C$$

$$\begin{aligned}
 16. \int \ln \sqrt{x^2 - 4} \, dx &= \frac{1}{2} \int \ln(x^2 - 4) \, dx \\
 &= \frac{1}{2} \left[ x \ln(x^2 - 4) - \int \frac{2x^2}{x^2 - 4} \, dx \right] \\
 &= \frac{1}{2} x \ln(x^2 - 4) - \int \left( 1 + \frac{4}{x^2 - 4} \right) dx \\
 &= \frac{1}{2} x \ln(x^2 - 4) - x - \ln \left| \frac{x-2}{x+2} \right| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 - 4) \Rightarrow du = \frac{2x}{x^2 - 4} \, dx$$

INSTRUCTOR USE ONLY

$$21. \int \sec^4\left(\frac{x}{2}\right) dx = \int \left[ \tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx$$

$$= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx$$

$$= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[ \tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C$$

$$22. \int \tan \theta \sec^4 \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta + \frac{1}{4} \tan^4 \theta + \frac{1}{2} \tan^2 \theta + C_1$$

or

$$\int \tan \theta \sec^4 \theta d\theta = \int \sec^3 \theta (\sec \theta \tan \theta) d\theta + \frac{1}{4} \sec^4 \theta + C_2$$

$$23. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1}{1 - \sin \theta} \cdot \frac{1 + \sin \theta}{1 + \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$24. \int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta = \int (\cos^2 \theta - \sin^2 \theta) (\sin \theta + \cos \theta)^2 d\theta$$

$$= \int (\sin \theta + \cos \theta)^3 (\cos \theta - \sin \theta) d\theta = \frac{1}{4} (\sin \theta + \cos \theta)^4 + C$$

$$25. A = \int_{\pi/4}^{3\pi/4} \sin^4 x dx. \text{ Using the Table of Integrals,}$$

$$\int \sin^4 x dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \left[ \frac{1}{2} (x - \sin x \cos x) \right] + C$$

$$\int_{\pi/4}^{3\pi/4} \sin^4 x dx = \left[ -\frac{\sin^3 x \cos x}{4} + \frac{3}{8} x - \frac{3}{8} \sin x \cos x \right]_{\pi/4}^{3\pi/4} = \left( \frac{1}{16} + \frac{9\pi}{32} + \frac{3}{16} \right) - \left( -\frac{1}{16} + \frac{3\pi}{32} - \frac{3}{16} \right) = \frac{3\pi}{16} + \frac{1}{2} \approx 1.0890$$

$$26. A = \int_0^{\pi/4} \sin 3x \cos 2x dx$$

$$= \frac{1}{2} \int_0^{\pi/4} [\sin x + \sin 5x] dx$$

$$= \frac{1}{2} \left[ -\cos x - \frac{1}{5} \cos 5x \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left[ -\frac{\sqrt{2}}{2} - \frac{1}{5} \left( -\frac{\sqrt{2}}{2} \right) + 1 + \frac{1}{5} \right]$$

$$= \frac{3}{5} - \frac{\sqrt{2}}{5} \approx 0.317$$

$$28. \int \frac{\sqrt{x^2 - 9}}{x} dx = \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta d\theta)$$

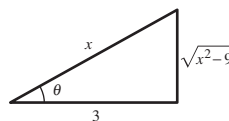
$$= 3 \int \tan^2 \theta d\theta$$

$$= 3 \int (\sec^2 \theta - 1) d\theta$$

$$= 3(\tan \theta - \theta) + C$$

$$= \sqrt{x^2 - 9} - 3 \operatorname{arcsec}\left(\frac{x}{3}\right) + C$$

$$x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta$$



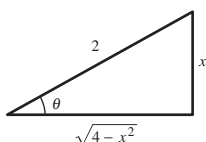
$$27. \int \frac{-12}{x^2 \sqrt{4 - x^2}} dx = \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)}$$

$$= -3 \int \csc^2 \theta d\theta$$

$$= 3 \cot \theta + C$$

$$= \frac{3\sqrt{4 - x^2}}{x} + C$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4 - x^2} = 2 \cos \theta$$

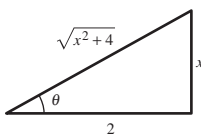


29.  $x = 2 \tan \theta$

$$dx = 2 \sec^2 \theta d\theta$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\ &= 8 \int \tan^3 \theta \sec \theta d\theta \\ &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\ &= 8 \left[ \frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\ &= 8 \left[ \frac{(x^2 + 4)^{3/2}}{24} - \frac{\sqrt{x^2 + 4}}{2} \right] + C \\ &= \sqrt{x^2 + 4} \left[ \frac{1}{3}(x^2 + 4) - 4 \right] + C \\ &= \frac{1}{3} x^2 \sqrt{x^2 + 4} - \frac{8}{3} \sqrt{x^2 + 4} + C \\ &= \frac{1}{3} (x^2 + 4)^{1/2} (x^2 - 8) + C \end{aligned}$$



30. 
$$\begin{aligned} \int \sqrt{25 - 9x^2} dx &= \frac{1}{3} \int \sqrt{5^2 - (3x)^2} (3) dx \\ &= \frac{1}{3} \frac{1}{2} \left[ 25 \arcsin\left(\frac{3x}{5}\right) + 3x \sqrt{25 - 9x^2} \right] + C = \frac{25}{6} \arcsin\left(\frac{3x}{5}\right) + \frac{x}{2} \sqrt{25 - 9x^2} + C \end{aligned}$$

(Theorem 8.2)

31. 
$$\int_{-2}^0 \sqrt{4 - x^2} dx = \frac{1}{2} \left[ 4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4 - x^2} \right]_{-2}^0 = \frac{1}{2} [0 - 4 \arcsin(-1)] = \frac{1}{2} \left[ -4 \left( \frac{-\pi}{2} \right) \right] = \pi$$

Note: The integral represents the area of a quarter circle of radius 2:  $A = \frac{1}{4}(\pi 2^2) = \pi$ .

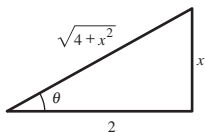
32. Let  $u = \cos \theta$ ,  $du = -\sin \theta d\theta$ .

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta &= \int_1^0 \frac{1}{1 + 2u^2} (-du) \\ &= \int_0^1 \frac{1}{1 + 2u^2} du \\ &= \frac{1}{2} \int_0^1 \frac{1}{(1/2) + u^2} du, \quad a = \frac{1}{\sqrt{2}} \\ &= \left[ \frac{1}{2} \sqrt{2} \arctan(\sqrt{2}u) \right]_0^1 \\ &= \frac{\sqrt{2}}{2} \arctan \sqrt{2} \end{aligned}$$

# INSTRUCTOR USE ONLY

33. (a) Let  $x = 2 \tan \theta$ ,  $dx = 2 \sec^2 \theta d\theta$ .

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\ &= 8 \int \tan^3 \theta \sec \theta d\theta \\ &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\ &= 8 \int (1 - \cos^2 \theta) \cos^{-4} \theta \sin \theta d\theta \\ &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\ &= 8 \left[ \frac{\cos^{-3} \theta}{-3} - \frac{\cos^{-1} \theta}{-1} \right] + C \\ &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\ &= \frac{8}{3} \frac{\sqrt{4+x^2}}{2} \left( \frac{4+x^2}{4} - 3 \right) + C \\ &= \frac{1}{3} \sqrt{4+x^2} (x^2 - 8) + C \end{aligned}$$



$$\begin{aligned} \text{(b)} \quad \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{x^2}{\sqrt{4+x^2}} x dx \\ &= \int \frac{(u^2 - 4)u du}{u} \\ &= \int (u^2 - 4) du \\ &= \frac{1}{3} u^3 - 4u + C \\ &= \frac{u}{3} (u^2 - 12) + C \\ &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned} \text{(c)} \quad \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - \int 2x \sqrt{4+x^2} dx \\ &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C \end{aligned}$$

$$\begin{aligned} dv &= \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \sqrt{4+x^2} \\ u &= x^2 \Rightarrow du = 2x dx \end{aligned}$$

NOT FOR SALE

$$\begin{aligned}
 34. (a) \quad \int x\sqrt{4+x} \, dx &= 64 \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= 64 \int (\sec^4 \theta - \sec^2 \theta) \sec \theta \tan \theta \, d\theta \\
 &= \frac{64 \sec^3 \theta}{15} (3 \sec^3 \theta - 5) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$x = 4 \tan^2 \theta, \, dx = 8 \tan \theta \sec^2 \theta \, d\theta,$$

$$\sqrt{4+x} = 2 \sec \theta$$

$$\begin{aligned}
 (b) \quad \int x\sqrt{4+x} \, dx &= 2 \int (u^4 - 4u^2) \, du \\
 &= \frac{2u^3}{15} (3u^2 - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u^2 = 4+x, \, dx = 2u \, du$$

$$\begin{aligned}
 (c) \quad \int x\sqrt{4+x} \, dx &= \int (u^{3/2} - 4u^{1/2}) \, du \\
 &= \frac{2u^{3/2}}{15} (3u - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u = 4+x, \, du = dx$$

$$\begin{aligned}
 (d) \quad \int x\sqrt{4+x} \, dx &= \frac{2x}{3} (4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} \, dx \\
 &= \frac{2x}{3} (4+x)^{3/2} - \frac{4}{15} (4+x)^{5/2} + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$dv = \sqrt{4+x} \, dx \Rightarrow v = \frac{2}{3} (4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}
 35. \quad \frac{x-39}{x^2-x-12} &= \frac{x-39}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3} \\
 x-39 &= A(x+3) + B(x-4)
 \end{aligned}$$

$$\text{Let } x = -3: -42 = -7B \Rightarrow B = 6$$

$$\text{Let } x = 4: -35 = 7A \Rightarrow A = -5$$

$$\begin{aligned}
 \int \frac{x-39}{x^2-x-12} \, dx &= \int \frac{-5}{x-4} \, dx + \int \frac{6}{x+3} \, dx \\
 &= -5 \ln|x-4| + 6 \ln|x+3| + C
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} &= 2x - 3 + \frac{4}{x} - \frac{3}{x-1} \\
 \int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx &= \int \left( 2x - 3 + \frac{4}{x} - \frac{3}{x-1} \right) \, dx = x^2 - 3x + 4 \ln|x| - 3 \ln|x-1| + C
 \end{aligned}$$

INSTRUCTOR USE ONLY

37.  $\frac{x^2 + 2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$x^2 + 2x = A(x^2 + 1) + (Bx + C)(x - 1)$$

Let  $x = 1$ :  $3 = 2A \Rightarrow A = \frac{3}{2}$     Let  $x = 0$ :  $0 = A - C \Rightarrow C = \frac{3}{2}$     Let  $x = 2$ :  $8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$

$$\begin{aligned} \int \frac{x^2 + 2x}{x^3 - x^2 + x - 1} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-3}{x^2+1} dx \\ &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{3}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{3}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{3}{2} \arctan x + C \\ &= \frac{1}{4} [6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C \end{aligned}$$

38.  $\frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

$$4x - 2 = 3A(x-1) + 3B$$

Let  $x = 1$ :  $2 = 3B \Rightarrow B = \frac{2}{3}$

Let  $x = 2$ :  $6 = 3A + 3B \Rightarrow A = \frac{4}{3}$

$$\int \frac{4x-2}{3(x-1)^2} dx = \frac{4}{3} \int \frac{1}{x-1} dx + \frac{2}{3} \int \frac{1}{(x-1)^2} dx = \frac{4}{3} \ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3} \left( 2 \ln|x-1| - \frac{1}{x-1} \right) + C$$

39.  $\frac{x^2}{x^2 + 5x - 24} = 1 - \frac{5x-24}{x^2 + 5x - 24} = 1 - \frac{5x-24}{(x+8)(x-3)}$

$$\frac{5x-24}{(x+8)(x-3)} = \frac{A}{x+8} + \frac{B}{x-3}$$

$$5x - 24 = A(x-3) + B(x+8)$$

Let  $x = 3$ :  $-9 = 11B \Rightarrow B = -9/11$

Let  $x = -8$ :  $-64 = -11A \Rightarrow A = 64/11$

$$\begin{aligned} \int \frac{x^2}{x^2 + 5x - 24} dx &= \int \left[ 1 - \frac{64/11}{x+8} + \frac{9/11}{x-3} \right] dx \\ &= x - \frac{64}{11} \ln|x+8| + \frac{9}{11} \ln|x-3| + C \end{aligned}$$

40.  $u = \tan \theta, du = \sec^2 \theta d\theta$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

Let  $u = 0$ :  $1 = -A \Rightarrow A = -1$

Let  $u = 1$ :  $1 = B$

$$\begin{aligned} \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} d\theta &= \int \frac{1}{u(u-1)} du = \int \frac{1}{u-1} du - \int \frac{1}{u} du \\ &= \ln|u-1| - \ln|u| + C = \ln \left| \frac{\tan \theta - 1}{\tan \theta} \right| + C = \ln|1 - \cot \theta| + C \end{aligned}$$

41. Using Formula 4: ( $a = 4, b = 5$ )

$$\int \frac{x}{(4 + 5x)^2} dx = \frac{1}{25} \left( \frac{4}{4 + 5x} + \ln|4 + 5x| \right) + C$$

42. Using Formula 21: ( $a = 4, b = 5$ )

$$\begin{aligned} \int \frac{x}{\sqrt{4 + 5x}} dx &= \frac{-2(8 - 5x)}{75} \sqrt{4 + 5x} + C \\ &= \frac{10x - 16}{75} \sqrt{4 + 5x} + C \end{aligned}$$

43. Let  $u = x^2, du = 2x dx$ .

$$\begin{aligned} \int_0^{\sqrt{\pi/2}} \frac{x}{1 + \sin x^2} dx &= \frac{1}{2} \int_0^{\pi/4} \frac{1}{1 + \sin u} du \\ &= \frac{1}{2} [\tan u - \sec u]_0^{\pi/4} \\ &= \frac{1}{2} [(1 - \sqrt{2}) - (0 - 1)] \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

44. Let  $u = x^2, du = 2x dx$ .

$$\begin{aligned} \int_0^1 \frac{x}{1 + e^{x^2}} dx &= \frac{1}{2} \int_0^1 \frac{1}{1 + e^u} du \\ &= \frac{1}{2} [u - \ln(1 + e^u)]_0^1 \\ &= \frac{1}{2} [(1 - \ln(1 + e)) + \ln 2] \\ &= \frac{1}{2} \left[ 1 + \ln \left( \frac{2}{1 + e} \right) \right] \end{aligned}$$

$$45. \int \frac{x}{x^2 + 4x + 8} dx = \frac{1}{2} \left[ \ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15})$$

$$\begin{aligned} &= \frac{1}{2} \left[ \ln|x^2 + 4x + 8| \right] - 2 \left[ \frac{2}{\sqrt{32 - 16}} \arctan \left( \frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14}) \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left( 1 + \frac{x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 46. \int \frac{3}{2x\sqrt{9x^2 - 1}} dx &= \frac{3}{2} \int \frac{1}{3x\sqrt{(3x)^2 - 1}} 3 dx \quad (u = 3x) \\ &= \frac{3}{2} \operatorname{arcsec}|3x| + C \quad (\text{Formula 33}) \end{aligned}$$

$$\begin{aligned} 47. \int \frac{1}{\sin \pi x \cos \pi x} dx &= \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58}) \end{aligned}$$

$$\begin{aligned} 48. \int \frac{1}{1 + \tan \pi x} dx &= \frac{1}{\pi} \int \frac{1}{1 + \tan \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \frac{1}{2} [\pi x + \ln|\cos \pi x + \sin \pi x|] + C \quad (\text{Formula 71}) \end{aligned}$$

$$49. dv = dx \quad \Rightarrow \quad v = x$$

$$u = (\ln x)^n \quad \Rightarrow \quad du = n(\ln x)^{n-1} \frac{1}{x} dx$$

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$



$$\begin{aligned}
 50. \int \tan^n x \, dx &= \int \tan^{n-2} x (\sec^2 x - 1) \, dx \\
 &= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx
 \end{aligned}$$

$$\begin{aligned}
 51. \int \theta \sin \theta \cos \theta \, d\theta &= \frac{1}{2} \int \theta \sin 2\theta \, d\theta \\
 &= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta \, d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 dv &= \sin 2\theta \, d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta \\
 u &= \theta \Rightarrow du = d\theta
 \end{aligned}$$

$$\begin{aligned}
 52. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} \, dx &= \sqrt{2} \int \csc \sqrt{2x} \left( \frac{1}{\sqrt{2x}} \right) dx = -\sqrt{2} \ln |\csc \sqrt{2x} + \cot \sqrt{2x}| + C \\
 u &= \sqrt{2x}, \, du = \frac{1}{\sqrt{2x}} dx
 \end{aligned}$$

$$\begin{aligned}
 53. \int \frac{x^{1/4}}{1+x^{1/2}} \, dx &= 4 \int \frac{u(u^3)}{1+u^2} \, du \\
 &= 4 \int \left( u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left( \frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$u = \sqrt[4]{x}, \, x = u^4, \, dx = 4u^3 \, du$$

$$\begin{aligned}
 54. \int \sqrt{1+\sqrt{x}} \, dx &= \int u(4u^3 - 4u) \, du = \int (4u^4 - 4u^2) \, du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15} (1+\sqrt{x})^{3/2} (3\sqrt{x} - 2) + C \\
 u &= \sqrt{1+\sqrt{x}}, \, x = u^4 - 2u^2 + 1, \, dx = (4u^3 - 4u) \, du
 \end{aligned}$$

$$\begin{aligned}
 55. \int \sqrt{1+\cos x} \, dx &= \int \frac{\sqrt{1+\cos x}}{1} \cdot \frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}} \, dx \\
 &= \int \frac{\sin x}{\sqrt{1-\cos x}} \, dx \\
 &= \int (1-\cos x)^{-1/2} (\sin x) \, dx \\
 &= 2\sqrt{1-\cos x} + C
 \end{aligned}$$

$$u = 1 - \cos x, \, du = \sin x \, dx$$

$$\begin{aligned}
 56. \frac{3x^3 + 4x}{(x^2 + 1)^2} &= \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \\
 3x^3 + 4x &= (Ax + B)(x^2 + 1) + Cx + D = Ax^3 + Bx^2 + (A + C)x + (B + D) \\
 A &= 3, \, B = 0, \, A + C = 4 \Rightarrow C = 1, \\
 B + D &= 0 \Rightarrow D = 0
 \end{aligned}$$

$$\int \frac{3x^3 + 4x}{(x^2 + 1)^2} \, dx = 3 \int \frac{x}{x^2 + 1} \, dx + \int \frac{x}{(x^2 + 1)^2} \, dx = \frac{3}{2} \ln(x^2 + 1) - \frac{1}{2(x^2 + 1)} + C$$

NOT FOR SALE

$$57. \int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \int \cos x dx = \sin x \ln(\sin x) - \sin x + C$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$58. \int (\sin \theta + \cos \theta)^2 d\theta = \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta$$

$$= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$$

$$59. y = \int \frac{25}{x^2 - 25} dx = 25 \left( \frac{1}{10} \right) \ln \left| \frac{x-5}{x+5} \right| + C$$

$$= \frac{5}{2} \ln \left| \frac{x-5}{x+5} \right| + C$$

(Formula 24)

$$60. y = \int \frac{\sqrt{4-x^2}}{2x} dx = \int \frac{2 \cos \theta (2 \cos \theta) d\theta}{4 \sin \theta}$$

$$= \int (\csc \theta - \sin \theta) d\theta$$

$$= [-\ln |\csc \theta + \cos \theta| + \cos \theta] + C$$

$$= -\ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + \frac{\sqrt{4-x^2}}{2} + C$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$61. y = \int \ln(x^2 + x) dx = x \ln |x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx$$

$$= x \ln |x^2 + x| - \int \frac{2x + 1}{x + 1} dx$$

$$= x \ln |x^2 + x| - \int 2 dx + \int \frac{1}{x+1} dx$$

$$= x \ln |x^2 + x| - 2x + \ln |x+1| + C$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x+1}{x^2+x} dx$$

$$62. y = \int \sqrt{1-\cos \theta} d\theta$$

$$= \int \frac{\sin \theta}{\sqrt{1+\cos \theta}} d\theta$$

$$= -\int (1+\cos \theta)^{-1/2} (-\sin \theta) d\theta$$

$$= -2\sqrt{1+\cos \theta} + C$$

$$u = 1 + \cos \theta, du = -\sin \theta d\theta$$

$$64. \int_0^1 \frac{x}{(x-2)(x-4)} dx = [2 \ln |x-4| - \ln |x-2|]_0^1$$

$$= 2 \ln 3 - 2 \ln 4 + \ln 2$$

$$= \ln \frac{9}{8} \approx 0.118$$

$$65. \int_1^4 \frac{\ln x}{x} dx = \left[ \frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 \approx 0.961$$

$$63. \int_2^{\sqrt{5}} x(x^2-4)^{3/2} dx = \left[ \frac{1}{5} (x^2-4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$66. \int_0^2 x e^{3x} dx = \left[ \frac{e^{3x}}{9} (3x-1) \right]_0^2 = \frac{1}{9} (5e^6 + 1) \approx 224.238$$

$$67. \int_0^\pi x \sin x dx = [-x \cos x + \sin x]_0^\pi = \pi$$

INSTRUCTOR USE ONLY

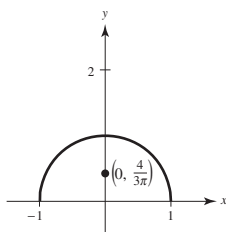
$$\begin{aligned}
 68. \int_0^5 \frac{x}{\sqrt{4+x}} dx &= \left[ \frac{2x-16}{3} \sqrt{4+x} \right]_0^5 \\
 &= -2(3) + \frac{16}{3}(2) = \frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 69. A &= \int_0^4 x\sqrt{4-x} dx = \int_2^0 (4-u^2) u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[ \frac{2}{5} \left( \frac{u^5}{5} - \frac{4u^3}{3} \right) \right]_2^0 = \frac{128}{15} \\
 u &= \sqrt{4-x}, x = 4-u^2, dx = -2u du
 \end{aligned}$$

$$\begin{aligned}
 70. A &= \int_0^4 \frac{1}{25-x^2} dx \\
 &= \left[ -\frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| \right]_0^4 = -\frac{1}{10} \ln \frac{1}{9} = \frac{1}{10} \ln 9 \approx 0.220
 \end{aligned}$$

$$71. \text{ By symmetry, } \bar{x} = 0, A = \frac{1}{2}\pi.$$

$$\begin{aligned}
 \bar{y} &= \frac{2}{\pi} \left( \frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[ x - \frac{1}{3}x^3 \right]_{-1}^1 = \frac{4}{3\pi} \\
 (\bar{x}, \bar{y}) &= \left( 0, \frac{4}{3\pi} \right)
 \end{aligned}$$



$$80. y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1^+} \left[ \frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[ \frac{\frac{1}{x-1}}{\left( \frac{1}{x} \right) \ln^2 x} \right] = \lim_{x \rightarrow 1^+} \left[ \frac{-\ln^2 x}{x-1} \right] = \lim_{x \rightarrow 1^+} \left[ \frac{-2 \left( \frac{1}{x} \right) (\ln x)}{\frac{1}{x^2}} \right] \\
 &= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0
 \end{aligned}$$

Because  $\ln y = 0$ ,  $y = 1$ .

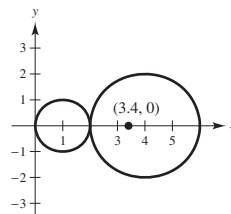
$$72. \text{ By symmetry, } \bar{y} = 0.$$

$$A = \pi + 4\pi = 5\pi$$

$$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$$

$$= \frac{17\pi}{5\pi} = 3.4$$

$$(\bar{x}, \bar{y}) = (3.4, 0)$$



$$73. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$74. s = \int_0^\pi \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$75. \lim_{x \rightarrow 1} \left[ \frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[ \frac{2(1/x) \ln x}{1} \right] = 0$$

$$76. \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 5\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{5\pi \cos \pi x} = \frac{1}{5}$$

$$77. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$78. \lim_{x \rightarrow \infty} x e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2x e^{x^2}} = 0$$

$$79. y = \lim_{x \rightarrow \infty} (\ln x)^{2/x}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[ \frac{2/(x \ln x)}{1} \right] = 0$$

Because  $\ln y = 0$ ,  $y = 1$ .

$$81. \lim_{n \rightarrow \infty} 1000 \left( 1 + \frac{0.09}{n} \right)^n = 1000 \lim_{n \rightarrow \infty} \left( 1 + \frac{0.09}{n} \right)^n$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \left( 1 + \frac{0.09}{n} \right)^n.$$

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left( 1 + \frac{0.09}{n} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left( 1 + \frac{0.09}{n} \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{0.09}{1 + \left( \frac{0.09}{n} \right)} = 0.09$$

$$\text{So, } \ln y = 0.09 \Rightarrow y = e^{0.09} \text{ and } \lim_{n \rightarrow \infty} 1000 \left( 1 + \frac{0.09}{n} \right)^n = 1000e^{0.09} \approx 1094.17.$$

$$\begin{aligned} 82. \lim_{x \rightarrow 1^+} \left( \frac{2}{\ln x} - \frac{2}{x-1} \right) &= \lim_{x \rightarrow 1^+} \left[ \frac{2x - 2 - 2 \ln x}{(\ln x)(x-1)} \right] \\ &= \lim_{x \rightarrow 1^+} \left[ \frac{2 - (2/x)}{(x-1)(1/x) + \ln x} \right] \\ &= \lim_{x \rightarrow 1^+} \frac{2x - 2}{(x-1) + x \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{2}{1 + 1 + \ln x} = 1 \end{aligned}$$

$$83. \int_0^{16} \frac{1}{\sqrt[4]{x}} dx = \lim_{b \rightarrow 0^+} \left[ \frac{4}{3} x^{3/4} \right]_b^{16} = \frac{32}{3}$$

$$84. \int_0^2 \frac{7}{x-2} dx = \lim_{b \rightarrow 2^-} [7 \ln |x-2|]_0^b = -\infty \quad \text{Diverges}$$

$$85. \int_1^\infty x^2 \ln x dx = \lim_{b \rightarrow \infty} \left[ \frac{x^3}{9} (-1 + 3 \ln x) \right]_1^b = \infty$$

Diverges

$$\begin{aligned} 89. \int_2^\infty \frac{1}{x\sqrt{x^2-4}} dx &= \int_2^3 \frac{1}{x\sqrt{x^2-4}} dx + \int_3^\infty \frac{1}{x\sqrt{x^2-4}} dx \\ &= \lim_{b \rightarrow 2^+} \left[ \frac{1}{2} \operatorname{arcsec} \left( \frac{x}{2} \right) \right]_b^3 + \lim_{c \rightarrow \infty} \left[ \frac{1}{2} \operatorname{arcsec} \left( \frac{x}{2} \right) \right]_3^c \\ &= \frac{1}{2} \operatorname{arcsec} \left( \frac{3}{2} \right) - \frac{1}{2} (0) + \frac{1}{2} \left( \frac{\pi}{2} \right) - \frac{1}{2} \operatorname{arcsec} \left( \frac{3}{2} \right) \\ &= \frac{\pi}{4} \end{aligned}$$

$$90. \text{ Let } u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2u du.$$

$$\begin{aligned} \int \frac{2}{\sqrt{x}(x+4)} dx &= \int \frac{2}{u(u^2+4)} 2u du = \int \frac{4}{u^2+4} du = 2 \arctan \left( \frac{u}{2} \right) + C = 2 \arctan \left( \frac{\sqrt{x}}{2} \right) + C \\ \int_0^\infty \frac{2}{\sqrt{x}(x+4)} dx &= \lim_{b \rightarrow 0^+} \left[ 2 \arctan \left( \frac{\sqrt{x}}{2} \right) \right]_b^1 + \lim_{c \rightarrow \infty} \left[ 2 \arctan \left( \frac{\sqrt{x}}{2} \right) \right]_1^c = \left( 2 \arctan \frac{1}{2} - 0 \right) + 2 \left( \frac{\pi}{2} \right) - 2 \arctan \frac{1}{2} = \pi \end{aligned}$$

$$86. \int_0^\infty \frac{e^{-1/x}}{x^2} dx = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} [e^{-1/x}]_a^b = 1 - 0 = 1$$

$$87. \text{ Let } u = \ln x, du = \frac{1}{x} dx, dv = x^{-2} dx, v = -x^{-1}.$$

$$\begin{aligned} \int \frac{\ln x}{x^2} dx &= \frac{-\ln x}{x} + \int \frac{1}{x^2} dx = \frac{-\ln x}{x} - \frac{1}{x} + C \\ \int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \left[ \frac{-\ln x}{x} - \frac{1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left( \frac{-\ln b}{b} - \frac{1}{b} \right) - (-1) \\ &= 0 + 1 = 1 \end{aligned}$$

$$\begin{aligned} 88. \int_1^\infty \frac{1}{\sqrt[4]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-1/4} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{4}{3} x^{3/4} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[ \frac{4}{3} b^{3/4} - \frac{4}{3} \right] \end{aligned}$$

Diverges

$$\begin{aligned} 91. \int_0^{t_0} 500,000 e^{-0.05t} dt &= \left[ \frac{500,000}{-0.05} e^{-0.05t} \right]_0^{t_0} \\ &= \frac{-500,000}{0.05} (e^{-0.05t_0} - 1) \\ &= 10,000,000(1 - e^{-0.05t_0}) \end{aligned}$$

$$(a) t_0 = 20: \$6,321,205.59$$

$$(b) t_0 \rightarrow \infty: \$10,000,000$$

$$93. (a) P(13 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.4581$$

$$(b) P(15 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.0135$$

## Problem Solving for Chapter 8

$$1. (a) \int_{-1}^1 (1 - x^2) dx = \left[ x - \frac{x^3}{3} \right]_{-1}^1 = 2 \left( 1 - \frac{1}{3} \right) = \frac{4}{3}$$

$$\int_{-1}^1 (1 - x^2)^2 dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = 2 \left( 1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15}$$

$$(b) \text{ Let } x = \sin u, dx = \cos u du, 1 - x^2 = 1 - \sin^2 u = \cos^2 u.$$

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (\cos^2 u)^n \cos u du \\ &= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u du \\ &= 2 \left[ \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{(2n)}{(2n+1)} \right] \quad (\text{Wallis's Formula}) \\ &= 2 \left[ \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)(2n+1)} \right] \\ &= \frac{2(2^{2n})(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

$$\begin{aligned} 2. (a) \int_0^1 \ln x dx &= \lim_{b \rightarrow 0^+} [x \ln x - x]_b^1 \\ &= (-1) - \lim_{b \rightarrow 0^+} (b \ln b - b) = -1 \end{aligned}$$

$$\text{Note: } \lim_{b \rightarrow 0^+} b \ln b = \lim_{b \rightarrow 0^+} \frac{\ln b}{b^{-1}} = \lim_{b \rightarrow 0^+} \frac{1/b}{-1/b^2} = 0$$

$$\begin{aligned} \int_0^1 (\ln x)^2 dx &= \lim_{b \rightarrow 0^+} [x(\ln x)^2 - 2x \ln x + 2x]_b^1 \\ &= 2 - \lim_{b \rightarrow 0^+} (b(\ln b)^2 - 2b \ln b + 2b) = 2 \end{aligned}$$

$$(b) \text{ Note first that } \lim_{b \rightarrow 0^+} b(\ln b)^n = 0 \text{ (Mathematical induction).}$$

$$\text{Also, } \int (\ln x)^{n+1} dx = x(\ln x)^{n+1} - (n+1) \int (\ln x)^n dx.$$

$$\text{Assume } \int_0^1 (\ln x)^n dx = (-1)^n n!.$$

$$\text{Then, } \int_0^1 (\ln x)^{n+1} dx = \lim_{b \rightarrow 0^+} \left[ x(\ln x)^{n+1} \right]_b^1 - (n+1) \int_0^1 (\ln x)^n dx = 0 - (n+1)(-1)^n n! = (-1)^{n+1} (n+1)!.$$

$$\begin{aligned}
 3. \quad \lim_{x \rightarrow \infty} \left( \frac{x+c}{x-c} \right)^x &= 9 \\
 \lim_{x \rightarrow \infty} x \ln \left( \frac{x+c}{x-c} \right) &= \ln 9 \\
 \lim_{x \rightarrow \infty} \frac{\ln(x+c) - \ln(x-c)}{1/x} &= \ln 9 \\
 \lim_{x \rightarrow \infty} \frac{\frac{1}{x+c} - \frac{1}{x-c}}{-\frac{1}{x^2}} &= \ln 9 \\
 \lim_{x \rightarrow \infty} \frac{-2c}{(x+c)(x-c)}(-x^2) &= \ln 9 \\
 \lim_{x \rightarrow \infty} \left( \frac{2cx^2}{x^2 - c^2} \right) &= \ln 9 \\
 2c &= \ln 9 \\
 2c &= 2 \ln 3 \\
 c &= \ln 3
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \lim_{x \rightarrow \infty} \left( \frac{x-c}{x+c} \right)^x &= \frac{1}{4} \\
 \lim_{x \rightarrow \infty} x \ln \left( \frac{x-c}{x+c} \right) &= \ln \frac{1}{4} \\
 \lim_{x \rightarrow \infty} \frac{\ln(x-c) - \ln(x+c)}{1/x} &= -\ln 4 \\
 \lim_{x \rightarrow \infty} \frac{\frac{1}{x-c} - \frac{1}{x+c}}{-\frac{1}{x^2}} &= -\ln 4 \\
 \lim_{x \rightarrow \infty} \frac{2c}{(x-c)(x+c)}(-x^2) &= -\ln 4 \\
 \lim_{x \rightarrow \infty} \frac{2cx^2}{x^2 - c^2} &= \ln 4 \\
 2c &= \ln 4 \\
 2x &= 2 \ln 2 \\
 c &= \ln 2
 \end{aligned}$$

$$6. \sin \theta = BD, \cos \theta = OD$$

$$\text{Area } \triangle DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1 - \cos \theta) \sin \theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2} \sin \theta$$

$$R = \frac{\Delta DAB}{\text{Shaded area}} = \frac{1/2(1 - \cos \theta) \sin \theta}{1/2(\theta - \sin \theta)}$$

$$\begin{aligned}
 \lim_{\theta \rightarrow 0^+} R &= \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \sin \theta}{\theta - \sin \theta} = \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \cos \theta + \sin^2 \theta}{1 - \cos \theta} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta)(-\sin \theta) + \cos \theta \sin \theta + 2 \sin \theta \cos \theta}{\sin \theta} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - 4 \cos \theta \sin \theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{4 \cos \theta - 1}{1} = 3
 \end{aligned}$$

$$5. \sin \theta = \frac{PB}{OP} = PB, \cos \theta = OB$$

$$AQ = \widehat{AP} = \theta$$

$$BR = OR + OB = OR + \cos \theta$$

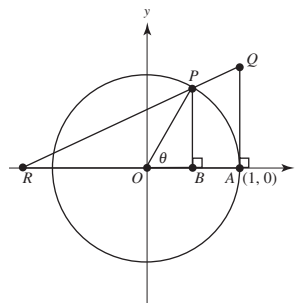
The triangles  $\triangle AQR$  and  $\triangle BPR$  are similar:

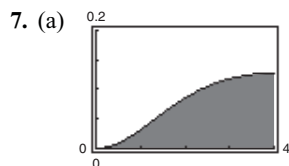
$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR+1}{\theta} = \frac{OR+\cos \theta}{\sin \theta}$$

$$\sin \theta(OR) + \sin \theta = (OR)\theta + \theta \cos \theta$$

$$OR = \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

$$\begin{aligned}
 \lim_{\theta \rightarrow 0^+} OR &= \lim_{\theta \rightarrow 0^+} \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta + \cos \theta - \cos \theta}{\cos \theta - 1} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta}{\cos \theta - 1} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - \theta \cos \theta}{-\sin \theta} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{\cos \theta + \cos \theta - \theta \sin \theta}{\cos \theta} \\
 &= 2
 \end{aligned}$$



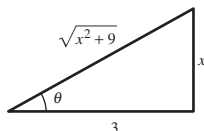


Area  $\approx 0.2986$

(b) Let  $x = 3 \tan \theta$ ,  $dx = 3 \sec^2 \theta d\theta$ ,  $x^2 + 9 = 9 \sec^2 \theta$ .

$$\begin{aligned} \int \frac{x^2}{(x^2 + 9)^{3/2}} dx &= \int \frac{9 \tan^2 \theta}{(9 \sec^2 \theta)^{3/2}} (3 \sec^2 \theta d\theta) \\ &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\ &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \left[ \ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\tan^{-1}(4/3)} \\ &= \left[ \ln \left( \frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right) - \frac{x}{\sqrt{x^2 + 9}} \right]_0^4 \\ &= \ln \left( \frac{5}{3} + \frac{4}{3} \right) - \frac{4}{5} = \ln 3 - \frac{4}{5} \end{aligned}$$



(c)  $x = 3 \sinh u$ ,  $dx = 3 \cosh u du$ ,  $x^2 + 9 = 9 \sinh^2 u + 9 = 9 \cosh^2 u$

$$\begin{aligned} A &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \int_0^{\sinh^{-1}(4/3)} \frac{9 \sinh^2 u}{(9 \cosh^2 u)^{3/2}} (3 \cosh u du) = \int_0^{\sinh^{-1}(4/3)} \tanh^2 u du \\ &= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u) du = \left[ u - \tanh u \right]_0^{\sinh^{-1}(4/3)} \\ &= \sinh^{-1} \left( \frac{4}{3} \right) - \tanh \left( \sinh^{-1} \left( \frac{4}{3} \right) \right) = \ln \left( \frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) - \tanh \left[ \ln \left( \frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) \right] \\ &= \ln \left( \frac{4}{3} + \frac{5}{3} \right) - \tanh \left( \ln \left( \frac{4}{3} + \frac{5}{3} \right) \right) = \ln 3 - \tanh(\ln 3) \\ &= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)} = \ln 3 - \frac{4}{5} \end{aligned}$$

8.  $u = \tan \frac{x}{2}, \cos x = \frac{1-u^2}{1+u^2},$

$$2 + \cos x = 2 + \frac{1-u^2}{1+u^2} = \frac{3+u^2}{1+u^2}$$

$$dx = \frac{2 du}{1+u^2}$$

$$\begin{aligned} \int_0^{\pi/2} \frac{1}{2 + \cos x} dx &= \int_0^1 \left( \frac{1+u^2}{3+u^2} \right) \left( \frac{2}{1+u^2} \right) du \\ &= \int_0^1 \frac{2}{3+u^2} du \\ &= \left[ 2 \frac{1}{\sqrt{3}} \arctan \left( \frac{u}{\sqrt{3}} \right) \right]_0^1 \\ &= \frac{2}{\sqrt{3}} \arctan \left( \frac{1}{\sqrt{3}} \right) \\ &= \frac{2}{\sqrt{3}} \frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \approx 0.6046 \end{aligned}$$

9.  $y = \ln(1-x^2), y' = \frac{-2x}{1-x^2}$

$$\begin{aligned} 1 + (y')^2 &= 1 + \frac{4x^2}{(1-x^2)^2} \\ &= \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} \\ &= \left( \frac{1+x^2}{1-x^2} \right)^2 \end{aligned}$$

$$\begin{aligned} \text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2} dx \\ &= \int_0^{1/2} \left( \frac{1+x^2}{1-x^2} \right) dx \\ &= \int_0^{1/2} \left( -1 + \frac{2}{1-x^2} \right) dx \\ &= \int_0^{1/2} \left( -1 + \frac{1}{x+1} + \frac{1}{1-x} \right) dx \\ &= [-x + \ln(1+x) - \ln(1-x)]_0^{1/2} \\ &= \left( -\frac{1}{2} + \ln \frac{3}{2} - \ln \frac{1}{2} \right) \\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\ &= \ln 3 - \frac{1}{2} \approx 0.5986 \end{aligned}$$

12. (a) Let  $y = f^{-1}(x), f(y) = x, dx = f'(y) dy$ .

$$\begin{aligned} \int f^{-1}(x) dx &= \int y f'(y) dy \\ &= y f(y) - \int f(y) dy \quad \left[ \begin{array}{l} u = y, du = dy \\ dv = f'(y) dy, v = f(y) \end{array} \right] \\ &= x f^{-1}(x) - \int f(y) dy \end{aligned}$$

10. Let  $u = cx, du = c dx$ .

$$\int_0^b e^{-c^2 x^2} dx = \int_0^{cb} e^{-u^2} \frac{du}{c} = \frac{1}{c} \int_0^{cb} e^{-u^2} du$$

As  $b \rightarrow \infty, cb \rightarrow \infty$ . So,  $\int_0^\infty e^{-c^2 x^2} dx = \frac{1}{c} \int_0^\infty e^{-u^2} du$ .

$\bar{x} = 0$  by symmetry.

$$\begin{aligned} \bar{y} = \frac{M_x}{m} &= \frac{2 \int_0^\infty \frac{e^{-c^2 x^2}}{2} dx}{2 \int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\int_0^\infty e^{-2c^2 x^2} dx}{\int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\frac{1}{\sqrt{2}c} \int_0^\infty e^{-x^2} dx}{\frac{1}{c} \int_0^\infty e^{-x^2} dx} \\ &= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \end{aligned}$$

So,  $(\bar{x}, \bar{y}) = \left( 0, \frac{\sqrt{2}}{4} \right)$ .

11. Consider  $\int \frac{1}{\ln x} dx$ .

Let  $u = \ln x, du = \frac{1}{x} dx, x = e^u$ . Then

$$\int \frac{1}{\ln x} dx = \int \frac{1}{u} e^u du = \int \frac{e^u}{u} du.$$

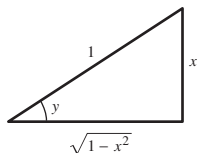
If  $\int \frac{1}{\ln x} dx$  were elementary, then  $\int \frac{e^u}{u} du$  would be too, which is false.

So,  $\int \frac{1}{\ln x} dx$  is not elementary.



$$(b) f^{-1}(x) = \arcsin x = y, f(x) = \sin x$$

$$\int \arcsin x \, dx = x \arcsin x - \int \sin y \, dy = x \arcsin x + \cos y + C = x \arcsin x + \sqrt{1-x^2} + C$$



$$(c) f(x) = e^x, f^{-1}(x) = \ln x = y \quad x = 1 \Leftrightarrow y = 0; x = e \Leftrightarrow y = 1$$

$$\int_1^e \ln x \, dx = [x \ln x]_1^e - \int_0^1 e^y \, dy = e - [e^y]_0^1 = e - (e - 1) = 1$$

$$13. \quad x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d) \\ = x^4 + (a+c)x^3 + (ac+b+d)x^2 + (ad+bc)x + bd$$

$$a = -c, b = d = 1, a = \sqrt{2}$$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\begin{aligned} \int_0^1 \frac{1}{x^4 + 1} \, dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1} \, dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1} \, dx \\ &= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} \, dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 - \sqrt{2}x + 1} \, dx \\ &= \frac{\sqrt{2}}{4} [\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1)]_0^1 + \frac{\sqrt{2}}{8} [\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1)]_0^1 \\ &= \frac{\sqrt{2}}{4} [\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)] + \frac{\sqrt{2}}{8} [\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})] - \frac{\sqrt{2}}{4} \left[ \frac{\pi}{4} - \frac{\pi}{4} \right] - \frac{\sqrt{2}}{8} [0] \\ &\approx 0.5554 + 0.3116 \\ &\approx 0.8670 \end{aligned}$$

$$14. (a) \text{ Let } x = \frac{\pi}{2} - u, dx = -du.$$

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} \, dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)} (-du) = \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} \, du$$

So,

$$2I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} \, dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

$$(b) \quad I = \int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)} (-du) = \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u} \, du$$

$$\text{So, } 2I = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

15. Using a graphing utility,

$$(a) \lim_{x \rightarrow 0^+} \left( \cot x + \frac{1}{x} \right) = \infty$$

$$(b) \lim_{x \rightarrow 0^+} \left( \cot x - \frac{1}{x} \right) = 0$$

$$(c) \lim_{x \rightarrow 0^+} \left( \cot x + \frac{1}{x} \right) \left( \cot x - \frac{1}{x} \right) \approx -\frac{2}{3}.$$

Analytically,

$$(a) \lim_{x \rightarrow 0^+} \left( \cot x + \frac{1}{x} \right) = \infty + \infty = \infty$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} \left( \cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x \cot x - 1}{x} = \lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{x \sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{-x \sin x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} = 0. \end{aligned}$$

$$\begin{aligned} (c) \left( \cot x + \frac{1}{x} \right) \left( \cot x - \frac{1}{x} \right) &= \cot^2 x - \frac{1}{x^2} \\ &= \frac{x^2 \cot^2 x - 1}{x^2} \\ \lim_{x \rightarrow 0^+} \frac{x^2 \cot^2 x - 1}{x^2} &= \lim_{x \rightarrow 0^+} \frac{2x \cot^2 x - 2x^2 \cot x \csc^2 x}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cot^2 x - x \cot x \csc^2 x}{1} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos^2 x \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{(1 - \sin^2 x) \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} - 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} &= \lim_{x \rightarrow 0^+} \frac{\cos x - \cos x + x \sin x}{3 \sin^2 x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{3 \sin x \cdot \cos x} \\ &= \lim_{x \rightarrow 0^+} \left( \frac{x}{\sin x} \right) \frac{1}{3 \cos x} = \frac{1}{3}. \end{aligned}$$

$$\text{So, } \lim_{x \rightarrow 0^+} \left( \cot x + \frac{1}{x} \right) \left( \cot x - \frac{1}{x} \right) = \frac{1}{3} - 1 = -\frac{2}{3}.$$

The form  $0 \cdot \infty$  is indeterminant.

$$16. \frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdots + \frac{P_n}{x - c_n}$$

$$N(x) = P_1(x - c_2)(x - c_3) \cdots (x - c_n) + P_2(x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + P_n(x - c_1)(x - c_2) \cdots (x - c_{n-1})$$

Let  $x = c_1$ :  $N(c_1) = P_1(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)}$$

Let  $x = c_2$ :  $N(c_2) = P_2(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)}$$

$$\vdots \qquad \qquad \qquad \vdots$$

Let  $x = c_n$ :  $N(c_n) = P_n(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})}$$

If  $D(x) = (x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n)$ , then by the Product Rule

$$D'(x) = (x - c_2)(x - c_3) \cdots (x - c_n) + (x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + (x - c_1)(x - c_2) \cdots (x - c_{n-1})$$

and

$$D'(c_1) = (c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$$

$$D'(c_2) = (c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$$

$$\vdots$$

$$D'(c_n) = (c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1}).$$

So,  $P_k = N(c_k)/D'(c_k)$  for  $k = 1, 2, \dots, n$ .

$$17. \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{P_1}{x} + \frac{P_2}{x - 1} + \frac{P_3}{x + 4} + \frac{P_4}{x - 3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$$

$$N(x) = x^3 - 3x^2 + 1$$

$$D'(x) = 4x^3 - 26x + 12$$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

$$\text{So, } \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{1/12}{x} + \frac{1/10}{x - 1} + \frac{111/140}{x + 4} + \frac{1/42}{x - 3}.$$

$$\begin{aligned}
 18. \quad s(t) &= \int \left[ -32t + 12,000 \ln \frac{50,000}{50,000 - 400t} \right] dt = -16t^2 + 12,000 \int [\ln 50,000 - \ln(50,000 - 400t)] dt \\
 &= 16t^2 + 12,000t \ln 50,000 - 12,000 \left[ t \ln(50,000 - 400t) - \int \frac{-400t}{50,000 - 400t} dt \right] \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t \int \left[ 1 - \frac{50,000}{50,000 - 400t} \right] dt \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t + 1,500,000 \ln(50,000 - 400t) + C
 \end{aligned}$$

$$s(0) = 1,500,000 \ln 50,000 + C = 0$$

$$C = -1,500,000 \ln 50,000$$

$$s(t) = -16t^2 + 12,000t \left[ 1 + \ln \frac{50,000}{50,000 - 400t} \right] + 1,500,000 \ln \frac{50,000 - 400t}{50,000}$$

When  $t = 100$ ,  $s(100) \approx 557,168.626$  feet.

19. By parts,

$$\begin{aligned}
 \int_a^b f(x)g''(x) dx &= [f(x)g'(x)]_a^b - \int_a^b f'(x)g'(x) dx \quad [u = f(x), dv = g''(x) dx] \\
 &= -\int_a^b f'(x)g'(x) dx \\
 &= [-f'(x)g(x)]_a^b + \int_a^b g(x)f''(x) dx \quad [u = f'(x), dv = g'(x) dx] \\
 &= \int_a^b f''(x)g(x) dx.
 \end{aligned}$$

20. Let  $u = (x - a)(x - b)$ ,  $du = [(x - a) + (x - b)] dx$ ,  $dv = f''(x) dx$ ,  $v = f'(x)$ .

$$\begin{aligned}
 \int_a^b (x - a)(x - b) f''(x) dx &= [(x - a)(x - b)f'(x)]_a^b - \int_a^b [(x - a) + (x - b)]f'(x) dx \\
 &= -\int_a^b (2x - a - b)f'(x) dx \quad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x) dx \end{pmatrix} \\
 &= [-(2x - a - b)f(x)]_a^b + \int_a^b 2f(x) dx = 2 \int_a^b f(x) dx
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int_2^\infty \left[ \frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} \right] dx &< \int_2^\infty \frac{1}{x^5 - 1} dx < \int_2^\infty \left[ \frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}} \right] dx \\
 \lim_{b \rightarrow \infty} \left[ -\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}} \right]_2^b &< \int_2^\infty \frac{1}{x^5 - 1} dx < \lim_{b \rightarrow \infty} \left[ -\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}} \right]_2^b \\
 0.015846 &< \int_2^\infty \frac{2}{x^5 - 1} dx < 0.015851
 \end{aligned}$$

$$22. \quad \frac{1}{2} V = \int_0^{\arcsin(c)} \pi(c - \sin x)^2 dx + \int_{\arcsin(c)}^{\pi/2} \pi(\sin x - c)^2 dx = \frac{2c^2\pi - 8c + \pi}{4} \pi = f(c)$$

$$f'(c) = \frac{4c\pi - 8}{4} \pi = 0 \Rightarrow c = \frac{2}{\pi}$$

$$\text{For } c = 0, \frac{1}{2} V = \frac{\pi^2}{4} \approx 2.4674$$

$$\text{For } c = 1, \frac{1}{2} V = \frac{\pi}{4}(3\pi - 8) \approx 1.1190$$

$$\text{For } c = \frac{2}{\pi}, \frac{1}{2} V = \frac{\pi^2 - 8}{4} \approx 0.4674$$

(a) Maximum:  $c = 0$

(b) Minimum:  $c = \frac{2}{\pi}$

**NOT FOR SALE**

**C H A P T E R 9**

**Infinite Series**

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**INSTRUCTOR USE ONLY**

## CHAPTER 9

### Infinite Series

#### Section 9.1 Sequences

1.  $a_n = 3^n$

$$a_1 = 3^1 = 3$$

$$a_2 = 3^2 = 9$$

$$a_3 = 3^3 = 27$$

$$a_4 = 3^4 = 81$$

$$a_5 = 3^5 = 243$$

2.  $a_n = \frac{3^n}{n!}$

$$a_1 = \frac{3}{1!} = 3$$

$$a_2 = \frac{3^2}{2!} = \frac{9}{2}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6} = \frac{9}{2}$$

$$a_4 = \frac{3^4}{4!} = \frac{81}{24} = \frac{27}{8}$$

$$a_5 = \frac{3^5}{5!} = \frac{243}{120} = \frac{81}{40}$$

3.  $a_n = \left(-\frac{1}{4}\right)^n$

$$a_1 = -\frac{1}{4}$$

$$a_2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$a_3 = \left(-\frac{1}{4}\right)^3 = -\frac{1}{64}$$

$$a_4 = \left(-\frac{1}{4}\right)^4 = \frac{1}{256}$$

$$a_5 = \left(-\frac{1}{4}\right)^5 = -\frac{1}{1024}$$

4.  $a_n = \left(-\frac{2}{3}\right)^n$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

$$a_4 = \frac{16}{81}$$

$$a_5 = -\frac{32}{243}$$

5.  $a_n = \sin \frac{n\pi}{2}$

$$a_1 = \sin \frac{\pi}{2} = 1$$

$$a_2 = \sin \pi = 0$$

$$a_3 = \sin \frac{3\pi}{2} = -1$$

$$a_4 = \sin 2\pi = 0$$

$$a_5 = \sin \frac{5\pi}{2} = 1$$

6.  $a_n = \frac{2n}{n+3}$

$$a_1 = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{4}{5}$$

$$a_3 = \frac{6}{6} = 1$$

$$a_4 = \frac{8}{7}$$

$$a_5 = \frac{10}{8} = \frac{5}{4}$$

7.  $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$$a_1 = \frac{(-1)^1}{1^2} = -1$$

$$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$$

$$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$$

$$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$$

$$8. a_n = (-1)^{n+1} \left( \frac{2}{n} \right)$$

$$a_1 = \frac{2}{1} = 2$$

$$a_2 = -\frac{2}{2} = -1$$

$$a_3 = \frac{2}{3}$$

$$a_4 = -\frac{2}{4} = -\frac{1}{2}$$

$$a_5 = \frac{2}{5}$$

$$9. a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$$

$$a_1 = 5 - 1 + 1 = 5$$

$$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$$

$$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$$

$$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$$

$$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$$

$$10. a_n = 10 + \frac{2}{n} + \frac{6}{n^2}$$

$$a_1 = 10 + 2 + 6 = 18$$

$$a_2 = 10 + 1 + \frac{3}{2} = \frac{25}{2}$$

$$a_3 = 10 + \frac{2}{3} + \frac{2}{9} = \frac{34}{3}$$

$$a_4 = 10 + \frac{1}{2} + \frac{3}{8} = \frac{87}{8}$$

$$a_5 = 10 + \frac{2}{5} + \frac{6}{25} = \frac{266}{25}$$

$$11. a_1 = 3, a_{k+1} = 2(a_k - 1)$$

$$a_2 = 2(a_1 - 1)$$

$$= 2(3 - 1) = 4$$

$$a_3 = 2(a_2 - 1)$$

$$= 2(4 - 1) = 6$$

$$a_4 = 2(a_3 - 1)$$

$$= 2(6 - 1) = 10$$

$$a_5 = 2(a_4 - 1)$$

$$= 2(10 - 1) = 18$$

$$12. a_1 = 4, a_{k+1} = \left( \frac{k+1}{2} \right) a_k$$

$$a_2 = \left( \frac{1+1}{2} \right) a_1 = 4$$

$$a_3 = \left( \frac{2+1}{2} \right) a_2 = 6$$

$$a_4 = \left( \frac{3+1}{2} \right) a_3 = 12$$

$$a_5 = \left( \frac{4+1}{2} \right) a_4 = 30$$

$$13. a_1 = 32, a_{k+1} = \frac{1}{2} a_k$$

$$a_2 = \frac{1}{2} a_1 = \frac{1}{2}(32) = 16$$

$$a_3 = \frac{1}{2} a_2 = \frac{1}{2}(16) = 8$$

$$a_4 = \frac{1}{2} a_3 = \frac{1}{2}(8) = 4$$

$$a_5 = \frac{1}{2} a_4 = \frac{1}{2}(4) = 2$$

$$14. a_1 = 6, a_{k+1} = \frac{1}{3} a_k^2$$

$$a_2 = \frac{1}{3} a_1^2 = \frac{1}{3}(6^2) = 12$$

$$a_3 = \frac{1}{3} a_2^2 = \frac{1}{3}(12^2) = 48$$

$$a_4 = \frac{1}{3} a_3^2 = \frac{1}{3}(48^2) = 768$$

$$a_5 = \frac{1}{3} a_4^2 = \frac{1}{3}(768^2) = 196,608$$

$$15. a_n = \frac{10}{n+1}, a_1 = \frac{10}{1+1} = 5, a_2 = \frac{10}{3}$$

Matches (c)

$$16. a_n = \frac{10n}{n+1}, a_1 = \frac{10}{2} = 5, a_2 = \frac{20}{3}$$

Matches (a)

$$17. a_n = (-1)^n, a_1 = -1, a_2 = 1, a_3 = -1, \dots$$

Matches (d)

$$18. a_n = \frac{(-1)^n}{n}, a_1 = \frac{-1}{1} = -1, a_2 = \frac{1}{2}$$

Matches (b)

$$19. -2, 0, \frac{2}{3}, 1, \dots \text{ matches (b)}$$

$$a_n = 2 - \frac{4}{n}$$

$$a_1 = 2 - \frac{4}{1} = -2, a_2 = 2 - \frac{4}{2} = 0, a_3 = 2 - \frac{4}{3} = \frac{2}{3}, \dots$$

20. 16, -8, 4, -2, ... matches (c)

$$a_n = 16(-0.5)^{n-1}$$

$$a_1 = 16(-0.5)^{1-1} = 16, a_2 = 16(-0.5)^{2-1} = -8, \dots$$

- 21.
- $\frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}, \dots$
- matches (a)

$$a_n = \frac{2}{3}n$$

$$a_1 = \frac{2}{3}, a_2 = \frac{4}{3}, a_3 = \frac{6}{3} = 2, \dots$$

- 22.
- $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \dots$
- matches (d)

$$a_n = \frac{2n}{n+1}$$

$$a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{6}{4} = \frac{3}{2}, \dots$$

- 23.
- $a_n = 3n - 1$

$$a_5 = 3(5) - 1 = 14$$

$$a_6 = 3(6) - 1 = 17$$

Add 3 to preceding term.

- 24.
- $a_n = \frac{n+6}{2}$

$$a_5 = \frac{5+6}{2} = \frac{11}{2}$$

$$a_6 = \frac{6+6}{2} = 6$$

Add  $\frac{1}{2}$  to preceding term.

- 25.
- $a_{n+1} = 2a_n, a_1 = 5$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

Multiply the preceding term by 2.

- 26.
- $a_n = -\frac{1}{2}a_{n-1}, a_1 = 1$

$$a_5 = \frac{1}{16}, a_6 = -\frac{1}{32}$$

Multiply the preceding term by  $-\frac{1}{2}$ .

- 27.
- $a_n = \frac{3}{(-2)^{n-1}}$

$$a_5 = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceding term by  $-\frac{1}{2}$ .

- 28.
- $a_n = -\frac{3}{2}a_{n-1}, a_1 = 1$

$$a_5 = \frac{81}{16}, a_6 = -\frac{243}{32}$$

Multiply the preceding term by  $-\frac{3}{2}$ .

- 29.
- $\frac{11!}{8!} = \frac{11(10)(9)8!}{8!} = 11(10)(9) = 990$

- 30.
- $\frac{25!}{20!} = \frac{25(24)(23)(22)(21)(20)!}{20!}$
- 
- $$= 25(24)(23)(22)(21) = 6,375,600$$

- 31.
- $\frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!} = n+1$

- 32.
- $\frac{(n+2)!}{n!} = \frac{n!(n+1)(n+2)}{n!} = (n+1)(n+2)$

- 33.
- $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)} = \frac{1}{2n(2n+1)}$

- 34.
- $\frac{(2n+2)!}{(2n)!} = \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$
- 
- $$= (2n+1)(2n+2)$$

- 35.
- $\lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + 2} = 5$

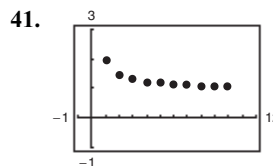
- 36.
- $\lim_{n \rightarrow \infty} \left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$

- 37.
- $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1 + (1/n^2)}} = \frac{2}{1} = 2$

- 38.
- $\lim_{n \rightarrow \infty} \frac{5n}{\sqrt{n^2 + 4}} = \lim_{n \rightarrow \infty} \frac{5}{\sqrt{1 + (4/n^2)}} = \frac{5}{1} = 5$

- 39.
- $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$

- 40.
- $\lim_{n \rightarrow \infty} \cos\left(\frac{2}{n}\right) = 1$

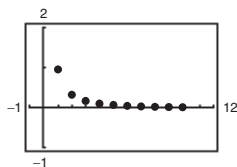


The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 = 1.$$



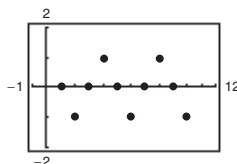
42.



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} = \lim_{x \rightarrow \infty} \frac{1}{x^{3/2}} = 0.$$

43.

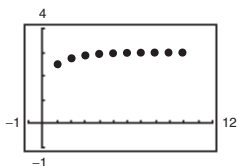


The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, \dots\}.$$

So,  $\lim_{n \rightarrow \infty} a_n$  does not exist.

44.



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

45.  $\lim_{n \rightarrow \infty} [(0.3)^n - 1] = 0 - 1 = -1, \text{ converges}$

46.  $\lim_{n \rightarrow \infty} \left[4 - \frac{3}{n}\right] = 4 - 0 = 4, \text{ converges}$

47.  $\lim_{n \rightarrow \infty} \frac{5}{n+2} = 0, \text{ converges}$

48.  $\lim_{n \rightarrow \infty} \frac{2}{n!} = 0, \text{ converges}$

49.  $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n+1}\right)$

does not exist (oscillates between  $-1$  and  $1$ ), diverges.

50.  $\lim_{n \rightarrow \infty} [1 + (-1)^n]$

does not exist, (alternates between  $0$  and  $2$ ), diverges.

51.  $\lim_{n \rightarrow \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}, \text{ converges}$

52.  $\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1, \text{ converges}$

53. 
$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$$

$$= \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} < \frac{1}{2n}$$

So,  $\lim_{n \rightarrow \infty} a_n = 0, \text{ converges.}$

54. The sequence diverges. To prove this analytically, you use mathematical induction to show that

$$a_n \geq \left(\frac{3}{2}\right)^{n-1}. \text{ Clearly, } a_1 = 1 \geq \left(\frac{3}{2}\right)^0 = 1. \text{ Assume that}$$

$$a_k \geq \left(\frac{3}{2}\right)^{k-1}. \text{ Then,}$$

$$a_{k+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{(k+1)!}$$

$$= a_k \frac{2k+1}{k+1} \geq \left(\frac{3}{2}\right)^{k-1} \left(\frac{2k+1}{k+1}\right).$$

Because

$$\frac{2k+1}{k+1} \geq \frac{3}{2},$$

you have  $a_{k+1} \geq \left(\frac{3}{2}\right)^k$ , which shows that

$$a_n \geq \left(\frac{3}{2}\right)^{n-1} \text{ for all } n.$$

55.  $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n} = 0, \text{ converges}$

56.  $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n^2} = 0, \text{ converges}$

57. 
$$\lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{2} \left(\frac{1}{n}\right) = 0, \text{ converges}$$

(L'Hôpital's Rule)

58. 
$$\lim_{n \rightarrow \infty} \frac{\ln \sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{(1/2) \ln n}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2n} = 0, \text{ converges}$$

(L'Hôpital's Rule)

$$59. \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0, \text{ converges}$$

$$60. \lim_{n \rightarrow \infty} (0.5)^n = 0, \text{ converges}$$

$$61. \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty, \text{ diverges}$$

$$62. \lim_{n \rightarrow \infty} \frac{(n-2)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} = 0, \text{ converges}$$

$$63. \lim_{n \rightarrow \infty} \left( \frac{n-1}{n} - \frac{n}{n-1} \right) = \lim_{n \rightarrow \infty} \frac{(n-1)^2 - n^2}{n(n-1)} \\ = \lim_{n \rightarrow \infty} \frac{1-2n}{n^2-n} = 0, \text{ converges}$$

$$64. \lim_{n \rightarrow \infty} \left( \frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right) = \lim_{n \rightarrow \infty} \frac{-2n^2}{4n^2-1} = -\frac{1}{2}, \text{ converges}$$

$$65. \lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0, \text{ converges} \\ (p > 0, n \geq 2)$$

$$66. a_n = n \sin \frac{1}{n}$$

$$\text{Let } f(x) = x \sin \frac{1}{x}.$$

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \\ = \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} \\ = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 \\ = 1 \text{ (L'Hôpital's Rule)}$$

or,

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \rightarrow 0^+} \frac{\sin(y)}{y} = 1. \text{ Therefore}$$

$$\lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1, \text{ converges.}$$

$$67. \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1, \text{ converges}$$

$$68. \lim_{n \rightarrow \infty} -3^{-n} = \lim_{n \rightarrow \infty} \frac{-1}{3^n} = 0, \text{ converges}$$

$$69. a_n = \left(1 + \frac{k}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u \rightarrow 0} \left[(1+u)^{1/u}\right]^k = e^k$$

where  $u = k/n$ , converges

$$70. a_n = \left(1 + \frac{1}{n^2}\right)^n = \left[\left(1 + \frac{1}{n^2}\right)^{n^2}\right]^{1/n} \\ \lim_{n \rightarrow \infty} a_n = e^{1/n} = 1, \text{ converges}$$

$$71. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} (\sin n) \frac{1}{n} = 0, \\ \text{converges (because } (\sin n) \text{ is bounded)}$$

$$72. \lim_{n \rightarrow \infty} \frac{\cos \pi n}{n^2} = 0, \text{ converges}$$

$$73. a_n = 3n - 2$$

$$74. a_n = 4n - 1$$

$$75. a_n = n^2 - 2$$

$$76. a_n = \frac{(-1)^{n-1}}{n^2}$$

$$77. a_n = \frac{n+1}{n+2}$$

$$78. a_n = \frac{(-1)^{n-1}}{2^{n-2}}$$

$$79. a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

$$80. a_n = 1 + \frac{2^n - 1}{2^n} \\ = \frac{2^{n+1} - 1}{2^n}$$

$$81. a_n = \frac{n}{(n+1)(n+2)}$$

$$82. a_n = \frac{1}{n!}$$

$$83. a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ = \frac{(-1)^{n-1} 2^n n!}{(2n)!}$$

$$84. a_n = \frac{x^{n-1}}{(n-1)!}$$

$$85. a_n = (2n)!, n = 1, 2, 3, \dots$$

$$86. a_n = (2n-1)!, n = 1, 2, 3, \dots$$

87.  $a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}$ ,

Monotonic;  $|a_n| < 4$ , bounded

88. Let  $f(x) = \frac{3x}{x+2}$ . Then  $f'(x) = \frac{6}{(x+2)^2}$ .

So,  $f$  is increasing which implies  $\{a_n\}$  is increasing.

$|a_n| < 3$ , bounded

89.  $\frac{n}{2^{n+2}} \stackrel{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$

$$2^{n+3}n \stackrel{?}{\geq} 2^{n+2}(n+1)$$

$$2n \stackrel{?}{\geq} n+1$$

$$n \geq 1$$

So,  $n \geq 1$

$$2n \geq n+1$$

$$2^{n+3}n \geq 2^{n+2}(n+1)$$

$$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$$

$$a_n \geq a_{n+1}.$$

Monotonic;  $|a_n| \leq \frac{1}{8}$ , bounded

90.  $a_n = ne^{-n/2}$

$$a_1 = 0.6065$$

$$a_2 = 0.7358$$

$$a_3 = 0.6694$$

Not monotonic;  $|a_n| \leq 0.7358$ , bounded

91.  $a_n = (-1)^n \left(\frac{1}{n}\right)$

$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

Not monotonic;  $|a_n| \leq 1$ , bounded

92.  $a_n = \left(-\frac{2}{3}\right)^n$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

Not monotonic;  $|a_n| \leq \frac{2}{3}$ , bounded

93.  $a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$

Monotonic;  $|a_n| \leq \frac{2}{3}$ , bounded

94.  $a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$

Monotonic;  $\lim_{n \rightarrow \infty} a_n = \infty$ , not bounded

95.  $a_n = \sin\left(\frac{n\pi}{6}\right)$

$$a_1 = 0.500$$

$$a_2 = 0.8660$$

$$a_3 = 1.000$$

$$a_4 = 0.8660$$

Not monotonic;  $|a_n| \leq 1$ , bounded

96.  $a_n = \cos\left(\frac{n\pi}{2}\right)$

$$a_1 = \cos \frac{\pi}{2} = 0$$

$$a_2 = \cos \pi = -1$$

$$a_3 = \cos\left(\frac{3\pi}{2}\right) = 0$$

Not monotonic;  $|a_n| \leq 1$ , bounded

97.  $a_n = \frac{\cos n}{n}$

$$a_1 = 0.5403$$

$$a_2 = -0.2081$$

$$a_3 = -0.3230$$

$$a_4 = -0.1634$$

Not monotonic;  $|a_n| \leq 1$ , bounded

98.  $a_n = \frac{\sin \sqrt{n}}{n}$

$$a_1 = \frac{\sin(1)}{1} \approx 0.8415$$

$$a_4 = \frac{\sin(4)}{4} \approx -0.1892$$

$$a_2 = \frac{\sin(2)}{2} \approx 0.4546$$

$$a_5 = \frac{\sin(5)}{5} \approx -0.1918$$

$$a_3 = \frac{\sin(3)}{3} \approx 0.0470$$

$$a_6 = \frac{\sin(6)}{6} \approx -0.0466$$

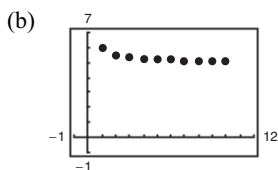
Not monotonic;  $|a_n| \leq 1$ , bounded

99. (a)  $a_n = 5 + \frac{1}{n}$

$$\left| 5 + \frac{1}{n} \right| \leq 6 \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1} \\ = a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



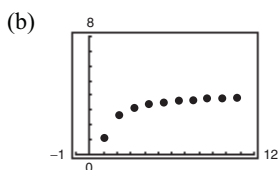
$$\lim_{n \rightarrow \infty} \left( 5 + \frac{1}{n} \right) = 5$$

100. (a)  $a_n = 4 - \frac{3}{n}$

$$\left| 4 - \frac{3}{n} \right| < 4 \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = 4 - \frac{3}{n} < 4 - \frac{3}{n+1} = a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



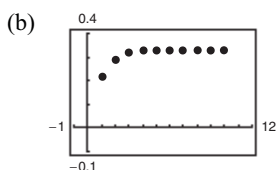
$$\lim_{n \rightarrow \infty} \left( 4 - \frac{3}{n} \right) = 4$$

101. (a)  $a_n = \frac{1}{3} \left( 1 - \frac{1}{3^n} \right)$

$$\left| \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) \right| < \frac{1}{3} \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) < \frac{1}{3} \left( 1 - \frac{1}{3^{n+1}} \right) \\ = a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



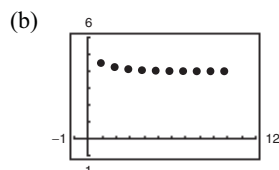
$$\lim_{n \rightarrow \infty} \left[ \frac{1}{3} \left( 1 - \frac{1}{3^n} \right) \right] = \frac{1}{3}$$

102. (a)  $a_n = 4 + \frac{1}{2^n}$

$$\left| 4 + \frac{1}{2^n} \right| \leq 4.5 \Rightarrow \{a_n\}, \text{ bounded}$$

$$a_n = 4 + \frac{1}{2^n} > 4 + \frac{1}{2^{n+1}} \\ = a_{n+1} \Rightarrow \{a_n\}, \text{ monotonic}$$

Therefore,  $\{a_n\}$  converges.



$$\lim_{n \rightarrow \infty} \left( 4 + \frac{1}{2^n} \right) = 4$$

103.  $\{a_n\}$  has a limit because it is a bounded, monotonic sequence. The limit is less than or equal to 4, and greater than or equal to 2.

$$2 \leq \lim_{n \rightarrow \infty} a_n \leq 4$$

104. The sequence  $\{a_n\}$  could converge or diverge. If  $\{a_n\}$  is increasing, then it converges to a limit less than or equal to 1. If  $\{a_n\}$  is decreasing, then it could converge (example:  $a_n = 1/n$ ) or diverge (example:  $a_n = -n$ ).

105.  $A_n = P \left( 1 + \frac{r}{12} \right)^n$

(a) Because  $P > 0$  and  $\left( 1 + \frac{r}{12} \right) > 1$ , the sequence diverges.  $\lim_{n \rightarrow \infty} A_n = \infty$

(b)  $P = 10,000, r = 0.055, A_n = 10,000 \left( 1 + \frac{0.055}{12} \right)^n$

$$A_0 = 10,000$$

$$A_1 = 10,045.83$$

$$A_2 = 10,091.88$$

$$A_3 = 10,138.13$$

$$A_4 = 10,184.60$$

$$A_5 = 10,231.28$$

$$A_6 = 10,278.17$$

$$A_7 = 10,325.28$$

$$A_8 = 10,372.60$$

$$A_9 = 10,420.14$$

$$A_{10} = 10,467.90$$

106. (a)  $A_n = 100(401)(1.0025^n - 1)$

$$A_0 = 0$$

$$A_1 = 100.25$$

$$A_2 = 200.75$$

$$A_3 = 301.50$$

$$A_4 = 402.51$$

$$A_5 = 503.76$$

$$A_6 = 605.27$$

(b)  $A_{60} = 6480.83$

(c)  $A_{240} = 32,912.28$

 107. No, it is not possible. See the "Definition of the Limit of a sequence". The number  $L$  is unique.

108. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit. See the definition.

(c) A sequence is monotonic if its terms are nondecreasing, or nonincreasing.

 (d) A sequence is bounded if it is bounded below ( $a_n \geq N$  for some  $N$ ) and bounded above ( $a_n \leq M$  for some  $M$ ).

 109. The graph on the left represents a sequence with alternating signs because the terms alternate from being above the  $x$ -axis to being below the  $x$ -axis.

110. (a)  $a_n = 10 - \frac{1}{n}$

(b) Impossible. The sequence converges by Theorem 9.5.

(c)  $a_n = \frac{3n}{4n+1}$

(d) Impossible. An unbounded sequence diverges.

111. (a)  $A_n = (0.8)^n 4,500,000,000$

(b)  $A_1 = \$3,600,000,000$

$$A_2 = \$2,880,000,000$$

$$A_3 = \$2,304,000,000$$

$$A_4 = \$1,843,200,000$$

(c)  $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} (0.8)^n (4.5) = 0$ , converges

112.  $P_n = 25,000(1.045)^n$

$$P_1 = \$26,125.00$$

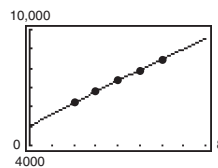
$$P_2 = \$27,300.63$$

$$P_3 = \$28,529.15$$

$$P_4 = \$29,812.97$$

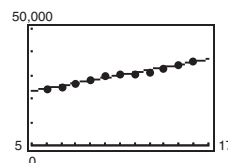
$$P_5 = \$31,154.55$$

113. (a)  $a_n = -5.364n^2 + 608.04n + 4998.3$



(b) For 2012,  $n = 12$ :  $a_{12} \approx \$11,522.4$  billion

114. (a)  $a_n = 1127.4n + 17684$



(b) For 2012,  $n = 22$ :  $a_{22} \approx \$42,487$

115.  $a_n = \frac{10^n}{n!}$

(a)  $a_9 = a_{10} = \frac{10^9}{9!} = \frac{1,000,000,000}{362,880} = \frac{1,562,500}{567}$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

116.  $a_n = \left(1 + \frac{1}{n}\right)^n$

$$a_1 = 2.0000$$

$$a_2 = 2.2500$$

$$a_3 \approx 2.3704$$

$$a_4 \approx 2.4414$$

$$a_5 \approx 2.4883$$

$$a_6 \approx 2.5216$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

117.  $a_n = \sqrt[n]{n} = n^{1/n}$

$$a_1 = 1^{1/1} = 1$$

$$a_2 = \sqrt{2} \approx 1.4142$$

$$a_3 = \sqrt[3]{3} \approx 1.4422$$

$$a_4 = \sqrt[4]{4} \approx 1.4142$$

$$a_5 = \sqrt[5]{5} \approx 1.3797$$

$$a_6 = \sqrt[6]{6} \approx 1.3480$$

Let  $y = \lim_{n \rightarrow \infty} n^{1/n}$ .

$$\ln y = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \ln n\right) = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0$$

 Because  $\ln y = 0$ , you have  $y = e^0 = 1$ . Therefore,

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

118. Because

$$\lim_{n \rightarrow \infty} s_n = L > 0,$$

there exists for each  $\varepsilon > 0$ ,an integer  $N$  such that  $|s_n - L| < \varepsilon$  for every  $n > N$ .Let  $\varepsilon = L > 0$  and you have,
 $|s_n - L| < L, -L < s_n - L < L$ , or  $0 < s_n < 2L$  for each  $n > N$ .

119. True

120. True

121. True

122. True

123. True

124. False. Let  $a_n = (-1)^n$  and  $b_n = (-1)^{n+1}$  then  $\{a_n\}$  and
 $\{b_n\}$  diverge. But  $\{a_n + b_n\} = \{(-1)^n + (-1)^{n+1}\}$ 

converges to 0.

125.  $a_{n+2} = a_n + a_{n+1}$ 

(a) $a_1 = 1$	$a_7 = 8 + 5 = 13$
$a_2 = 1$	$a_8 = 13 + 8 = 21$
$a_3 = 1 + 1 = 2$	$a_9 = 21 + 13 = 34$
$a_4 = 2 + 1 = 3$	$a_{10} = 34 + 21 = 55$
$a_5 = 3 + 2 = 5$	$a_{11} = 55 + 34 = 89$
$a_6 = 5 + 3 = 8$	$a_{12} = 89 + 55 = 144$

(b)  $b_n = \frac{a_{n+1}}{a_n}, n \geq 1$

$$b_1 = \frac{1}{1} = 1 \qquad b_6 = \frac{13}{8} = 1.625$$

$$b_2 = \frac{2}{1} = 2 \qquad b_7 = \frac{21}{13} \approx 1.6154$$

$$b_3 = \frac{3}{2} = 1.5 \qquad b_8 = \frac{34}{21} \approx 1.6190$$

$$b_4 = \frac{5}{3} \approx 1.6667 \qquad b_9 = \frac{55}{34} \approx 1.6176$$

$$b_5 = \frac{8}{5} = 1.6 \qquad b_{10} = \frac{89}{55} \approx 1.6182$$

$$\begin{aligned} \text{(c) } 1 + \frac{1}{b_{n-1}} &= 1 + \frac{1}{a_n/a_{n-1}} \\ &= 1 + \frac{a_{n-1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n \end{aligned}$$

$$\text{(d) If } \lim_{n \rightarrow \infty} b_n = \rho, \text{ then } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_{n-1}}\right) = \rho.$$

Because  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_{n-1}$ , you have

$$1 + (1/\rho) = \rho.$$

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Because  $a_n$ , and therefore  $b_n$ , is positive,

$$\rho = \frac{1 + \sqrt{5}}{2} \approx 1.6180.$$

$$126. x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \dots$$

$$x_1 = 1.5 \qquad x_6 = 1.414214$$

$$x_2 = 1.41667 \qquad x_7 = 1.414214$$

$$x_3 = 1.414216 \qquad x_8 = 1.414114$$

$$x_4 = 1.414214 \qquad x_9 = 1.414214$$

$$x_5 = 1.414214 \qquad x_{10} = 1.414214$$

The limit of the sequence appears to be  $\sqrt{2}$ . In fact, this sequence is Newton's Method applied to

$$f(x) = x^2 - 2.$$

127. (a)  $a_1 = \sqrt{2} \approx 1.4142$

$$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$$

(b)  $a_n = \sqrt{2 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{2}$

(c) First use mathematical induction to show that  $a_n \leq 2$ ; clearly  $a_1 \leq 2$ . So assume  $a_k \leq 2$ . Then

$$a_k + 2 \leq 4$$

$$\sqrt{a_k + 2} \leq 2$$

$$a_{k+1} \leq 2.$$

Now show that  $\{a_n\}$  is an increasing sequence. Because  $a_n \geq 0$  and  $a_n \leq 2$ ,

$$(a_n - 2)(a_n + 1) \leq 0$$

$$a_n^2 - a_n - 2 \leq 0$$

$$a_n^2 \leq a_n + 2$$

$$a_n \leq \sqrt{a_n + 2}$$

$$a_n \leq a_{n+1}.$$

Because  $\{a_n\}$  is a bounding increasing sequence, it converges to some number  $L$ , by Theorem 9.5.

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow \sqrt{2 + L} = L \Rightarrow 2 + L = L^2 \Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow (L - 2)(L + 1) = 0 \Rightarrow L = 2 \quad (L \neq -1)$$

128. (a)  $a_1 = \sqrt{6} \approx 2.4495$

$$a_2 = \sqrt{6 + \sqrt{6}} \approx 2.9068$$

$$a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}} \approx 2.9844$$

$$a_4 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \approx 2.9974$$

$$a_5 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}} \approx 2.9996$$

(b)  $a_n = \sqrt{6 + a_{n-1}}, \quad n \geq 2, a_1 = \sqrt{6}$

(c) First use mathematical induction to show that  $a_n \leq 3$ ; clearly  $a_1 \leq 3$ . So assume  $a_k \leq 3$ . Then

$$6 + a_k \leq 9$$

$$\sqrt{6 + a_k} \leq 3$$

$$a_{k+1} \leq 3.$$

Now show that  $\{a_n\}$  is an increasing sequence. Because  $a_n \geq 0$  and  $a_n \leq 3$ ,

$$(a_n - 3)(a_n + 2) \leq 0$$

$$a_n^2 - a_n - 6 \leq 0$$

$$a_n^2 \leq a_n + 6$$

$$a_n \leq \sqrt{a_n + 6}$$

$$a_n \leq a_{n+1}.$$

Because  $\{a_n\}$  is a bounded increasing sequence, it converges to some number  $L$ :  $\lim_{n \rightarrow \infty} a_n = L$ . So,

$$\sqrt{6 + L} = L \Rightarrow 6 + L = L^2 \Rightarrow L^2 - L - 6 = 0$$

$$\Rightarrow (L - 3)(L + 2) = 0 \Rightarrow L = 3 \quad (L \neq -2)$$

129. (a) Use mathematical induction to show that

$$a_n \leq \frac{1 + \sqrt{1 + 4k}}{2}.$$

[Note that if  $k = 2$ , and  $a_n \leq 3$ , and if  $k = 6$ , then  $a_n \leq 3$ .] Clearly,

$$a_1 = \sqrt{k} \leq \frac{\sqrt{1 + 4k}}{2} \leq \frac{1 + \sqrt{1 + 4k}}{2}.$$

Before proceeding to the induction step, note that

$$2 + 2\sqrt{1 + 4k} + 4k = 2 + 2\sqrt{1 + 4k} + 4k$$

$$\frac{1 + \sqrt{1 + 4k}}{2} + k = \frac{1 + 2\sqrt{1 + 4k} + 1 + 4k}{4}$$

$$\frac{1 + \sqrt{1 + 4k}}{2} + k = \left[ \frac{1 + \sqrt{1 + 4k}}{2} \right]^2$$

$$\sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k} = \frac{1 + \sqrt{1 + 4k}}{2}.$$

So assume  $a_n \leq \frac{1 + \sqrt{1 + 4k}}{2}$ . Then

$$a_n + k \leq \frac{1 + \sqrt{1 + 4k}}{2} + k$$

$$\sqrt{a_n + k} \leq \sqrt{\frac{1 + \sqrt{1 + 4k}}{2} + k}$$

$$a_{n+1} \leq \frac{1 + \sqrt{1 + 4k}}{2}.$$

$\{a_n\}$  is increasing because

$$\left( a_n - \frac{1 + \sqrt{1 + 4k}}{2} \right) \left( a_n - \frac{1 - \sqrt{1 + 4k}}{2} \right) \leq 0$$

$$a_n^2 - a_n - k \leq 0$$

$$a_n^2 \leq a_n + k$$

$$a_n \leq \sqrt{a_n + k}$$

$$a_n \leq a_{n+1}.$$

(b) Because  $\{a_n\}$  is bounded and increasing, it has a limit  $L$ .

(c)  $\lim_{n \rightarrow \infty} a_n = L$  implies that

$$L = \sqrt{k + L} \Rightarrow L^2 = k + L$$

$$\Rightarrow L^2 - L - k = 0$$

$$\Rightarrow L = \frac{1 \pm \sqrt{1 + 4k}}{2}.$$

$$\text{Because } L > 0, L = \frac{1 + \sqrt{1 + 4k}}{2}.$$



130. (a)  $a_0 = 10, b_0 = 3$

$$a_1 = \frac{a_0 + b_0}{2} = \frac{10 + 3}{2} = 6.5 \quad b_1 = \sqrt{a_0 b_0} = \sqrt{10(3)} \approx 5.4772$$

$$a_2 = \frac{a_1 + b_1}{2} \approx 5.9886 \quad b_2 = \sqrt{a_1 b_1} \approx 5.9667$$

$$a_3 = \frac{a_2 + b_2}{2} \approx 5.9777 \quad b_3 = \sqrt{a_2 b_2} \approx 5.9777$$

$$a_4 = \frac{a_3 + b_3}{2} \approx 5.9777 \quad b_4 = \sqrt{a_3 b_3} \approx 5.9777$$

$$a_5 = \frac{a_4 + b_4}{2} \approx 5.9777 \quad b_5 = \sqrt{a_4 b_4} \approx 5.9777$$

The terms of  $\{a_n\}$  are decreasing and those of  $\{b_n\}$  are increasing. They both seem to approach the same limit.

(b) For  $n = 0$ , you need to show  $a_0 > a_1 > b_1 > b_0$ . Because  $a_0 > b_0$ ,  $2a_0 > a_0 + b_0 \Rightarrow a_0 > \frac{a_0 + b_0}{2} = a_1$ .

Because  $(a_0 - b_0)^2 > 0$ ,

$$\begin{aligned} a_0^2 - 2a_0b_0 + b_0^2 > 0 &\Rightarrow a_0^2 + 2a_0b_0 + b_0^2 > 4a_0b_0 \\ &\Rightarrow (a_0 + b_0)^2 > 4a_0b_0 \Rightarrow a_0 + b_0 > 2\sqrt{a_0b_0} \Rightarrow a_1 > b_1. \end{aligned}$$

Because  $a_0 > b_0$ ,  $a_0b_0 > b_0^2 \Rightarrow \sqrt{a_0b_0} > b_0 \Rightarrow b_1 > b_0$ . So, you have shown that  $a_0 > a_1 > b_1 > b_0$ .

Now assume  $a_k > a_{k+1} > b_{k+1} > b_k$ . Because  $a_{k+1} > b_{k+1}$ ,

$$2a_{k+1} > a_{k+1} + b_{k+1} \Rightarrow a_{k+1} > \frac{a_{k+1} + b_{k+1}}{2} = a_{k+2}.$$

Because  $(a_{k+1} - b_{k+1})^2 > 0$ ,

$$\begin{aligned} a_{k+1}^2 - 2a_{k+1}b_{k+1} + b_{k+1}^2 > 0 &\Rightarrow a_{k+1}^2 + 2a_{k+1}b_{k+1} + b_{k+1}^2 > 4a_{k+1}b_{k+1} \\ &\Rightarrow (a_{k+1} + b_{k+1})^2 > 4a_{k+1}b_{k+1} \\ &\Rightarrow a_{k+1} + b_{k+1} > 2\sqrt{a_{k+1}b_{k+1}} \\ &\Rightarrow a_{k+2} > b_{k+2}. \end{aligned}$$

Because  $a_{k+1} > b_{k+1}$ ,  $a_{k+1}b_{k+1} > b_{k+1}^2 \Rightarrow \sqrt{a_{k+1}b_{k+1}} > b_{k+1} \Rightarrow b_{k+2} > b_{k+1}$ .

So, you have shown that  $a_k > a_{k+1} > b_{k+1} > b_k$ .

(c)  $\{a_n\}$  converges because it is decreasing and bounded below by 0.  $\{b_n\}$  converges because it is increasing and bounded above by  $a_0$ .

(d) Let  $\lim_{n \rightarrow \infty} a_n = A$  and  $\lim_{n \rightarrow \infty} b_n = B$ . Then  $A = \frac{A+B}{2} \Rightarrow A = B$ .

131. (a)  $f(x) = \sin x, a_n = n \sin \frac{1}{n}$

$$f'(x) = \cos x, f'(0) = 1$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{(1/n)} = 1 = f'(0)$$

$$(b) f'(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(h)}{h} = \lim_{n \rightarrow \infty} \frac{f(1/n)}{(1/n)} = \lim_{n \rightarrow \infty} n f\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} a_n$$

132.  $a_n = nr^n$

(a)  $r = \frac{1}{2}$ :  $a_n = n\left(\frac{1}{2}\right)^n = \frac{n}{2^n} \rightarrow 0$

(b)  $r = 1$ :  $a_n = n$ , diverges

(c)  $r = \frac{3}{2}$ :  $a_n = n\left(\frac{3}{2}\right)^n$ , diverges

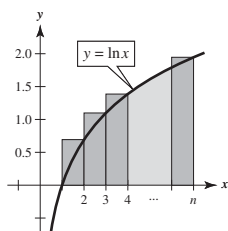
(d) If  $|r| \geq 1$ , the sequence diverges. If  $|r| < 1$ ,

$$nr^n = \frac{n}{r^{-n}} \quad \frac{\infty}{\infty}$$

$$\rightarrow \frac{1}{-r^{-n} \ln(r)} = \frac{-r^n}{\ln(r)} \rightarrow 0.$$

The sequence converges for  $|r| < 1$ .

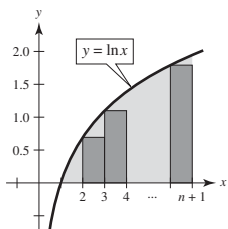
133. (a)



$$\int_1^n \ln x \, dx < \ln 2 + \ln 3 + \cdots + \ln n$$

$$= \ln(1 \cdot 2 \cdot 3 \cdots n) = \ln(n!)$$

(b)



$$\int_1^{n+1} \ln x \, dx > \ln 2 + \ln 3 + \cdots + \ln n = \ln(n!)$$

(c)  $\int \ln x \, dx = x \ln x - x + C$

$$\int_1^n \ln x \, dx = n \ln n - n + 1 = \ln n^n - n + 1$$

From part (a):  $\ln n^n - n + 1 < \ln(n!)$

$$e^{\ln n^n - n + 1} < n!$$

$$\frac{n^n}{e^{n-1}} < n!$$

$$\int_1^{n+1} \ln x \, dx = (n+1) \ln(n+1) - (n+1) + 1$$

$$= \ln(n+1)^{n+1} - n$$

From part (b):  $\ln(n+1)^{n+1} - n > \ln(n!)$

$$e^{\ln(n+1)^{n+1} - n} > n!$$

$$\frac{(n+1)^{n+1}}{e^n} > n!$$

(d)  $\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$

$$\frac{n}{e^{1-(1/n)}} < \sqrt[n]{n!} < \frac{(n+1)^{(n+1)/n}}{e}$$

$$\frac{1}{e^{1-(1/n)}} < \frac{\sqrt[n]{n!}}{n} < \frac{(n+1)^{1+(1/n)}}{ne}$$

$$\lim_{n \rightarrow \infty} \frac{1}{e^{1-(1/n)}} = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{1+(1/n)}}{ne} = \lim_{n \rightarrow \infty} \frac{(n+1)}{n} \frac{(n+1)^{1/n}}{e}$$

$$= (1) \frac{1}{e}$$

$$= \frac{1}{e}$$

By the Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ .

(e)  $n = 20$ :  $\frac{\sqrt[20]{20!}}{20} \approx 0.4152$

$$n = 50$$
:  $\frac{\sqrt[50]{50!}}{50} \approx 0.3897$

$$n = 100$$
:  $\frac{\sqrt[100]{100!}}{100} \approx 0.3799$

$$\frac{1}{e} \approx 0.3679$$

$$134. a_n = \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)}$$

$$(a) a_1 = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$a_2 = \frac{1}{2} \left[ \frac{1}{1 + (1/2)} + \frac{1}{1+1} \right] = \frac{1}{2} \left[ \frac{2}{3} + \frac{1}{2} \right] = \frac{7}{12} \approx 0.5833$$

$$a_3 = \frac{1}{3} \left[ \frac{1}{1 + (1/3)} + \frac{1}{1 + (2/3)} + \frac{1}{1+1} \right] = \frac{37}{60} \approx 0.6167$$

$$a_4 = \frac{1}{4} \left[ \frac{1}{1 + (1/4)} + \frac{1}{1 + (2/4)} + \frac{1}{1 + (3/4)} + \frac{1}{1+1} \right] = \frac{533}{840} \approx 0.6345$$

$$a_5 = \frac{1}{5} \left[ \frac{1}{1 + (1/5)} + \frac{1}{1 + (2/5)} + \frac{1}{1 + (3/5)} + \frac{1}{1 + (4/5)} + \frac{1}{1+1} \right] = \frac{1627}{2520} \approx 0.6456$$

$$(b) \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)} = \int_0^1 \frac{1}{1+x} dx = [\ln|1+x|]_0^1 = \ln 2$$

135. For a given  $\varepsilon > 0$ , you must find  $M > 0$  such that

$$|a_n - L| = \left| \frac{1}{n^3} \right| < \varepsilon$$

whenever  $n > M$ . That is,

$$n^3 > \frac{1}{\varepsilon} \text{ or } n > \left( \frac{1}{\varepsilon} \right)^{1/3}.$$

So, let  $\varepsilon > 0$  be given. Let  $M$  be an integer satisfying

$M > (1/\varepsilon)^{1/3}$ . For  $n > M$ , you have

$$n > \left( \frac{1}{\varepsilon} \right)^{1/3}$$

$$n^3 > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{n^3} \Rightarrow \left| \frac{1}{n^3} - 0 \right| < \varepsilon.$$

$$\text{So, } \lim_{n \rightarrow \infty} \frac{1}{n^3} = 0.$$

136. For a given  $\varepsilon > 0$ , you must find  $M > 0$  such that

$$|a_n - L| = |r^n| \varepsilon \text{ whenever } n > M. \text{ That is,}$$

$$n \ln|r| < \ln(\varepsilon) \text{ or}$$

$$n > \frac{\ln(\varepsilon)}{\ln|r|} \text{ (because } \ln|r| < 0 \text{ for } |r| < 1).$$

So, let  $\varepsilon > 0$  be given. Let  $M$  be an integer satisfying

$$M > \frac{\ln(\varepsilon)}{\ln|r|}.$$

For  $n > M$ , you have

$$n > \frac{\ln(\varepsilon)}{\ln|r|}$$

$$n \ln|r| < \ln(\varepsilon)$$

$$\ln|r|^n < \ln(\varepsilon)$$

$$|r|^n < \varepsilon$$

$$|r^n - 0| < \varepsilon.$$

So,  $\lim_{n \rightarrow \infty} r^n = 0$  for  $-1 < r < 1$ .

137. Answers will vary. *Sample answer:*

$$\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \dots\} \text{ diverges}$$

$$\{a_{2n}\} = \{(-1)^{2n}\} = \{1, 1, 1, 1, \dots\} \text{ converges}$$

138. Let  $f(x) = \sin(\pi x)$

$\lim_{x \rightarrow \infty} \sin(\pi x)$  does not exist.

$$a_n = f(n) = \sin(\pi n) = 0 \text{ for all } n$$

$$\lim_{n \rightarrow \infty} a_n = 0, \text{ converges}$$

139. If  $\{a_n\}$  is bounded, monotonic and nonincreasing, then  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$ . Then

$-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$  is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Because  $\{-a_n\}$  converges, then so does  $\{a_n\}$ .

140. Define  $a_n = \frac{x_{n+1} + x_{n-1}}{x_n}$ ,  $n \geq 1$ .

$$\begin{aligned} x_{n+1}^2 - x_n x_{n+2} &= 1 = x_n^2 - x_{n-1} x_{n+1} \Rightarrow \\ x_{n+1}(x_{n+1} + x_{n-1}) &= x_n(x_n + x_{n+2}) \\ \frac{x_{n+1} + x_{n-1}}{x_n} &= \frac{x_{n+2} + x_n}{x_{n+1}} \\ a_n &= a_{n+1} \end{aligned}$$

Therefore,  $a_1 = a_2 = \cdots = a$ . So,

$$x_{n+1} = a_n x_n - x_{n-1} = ax_n - x_{n-1}.$$

141.  $T_n = n! + 2^n$

Use mathematical induction to verify the formula.

$$T_0 = 1 + 1 = 2$$

$$T_1 = 1 + 2 = 3$$

$$T_2 = 2 + 4 = 6$$

Assume  $T_k = k! + 2^k$ . Then

$$\begin{aligned} T_{k+1} &= (k+1+4)T_k - 4(k+1)T_{k-1} + (4(k+1)-8)T_{k-2} \\ &= (k+5)[k! + 2^k] - 4(k+1)((k-1)! + 2^{k-1}) + (4k-4)((k-2)! + 2^{k-2}) \\ &= [(k+5)(k)(k-1) - 4(k+1)(k-1) + 4(k-1)](k-2)! + [(k+5)4 - 8(k+1) + 4(k-1)]2^{k-2} \\ &= [k^2 + 5k - 4k - 4 + 4](k-1)! + 8 \cdot 2^{k-2} \\ &= (k+1)! + 2^{k+1}. \end{aligned}$$

By mathematical induction, the formula is valid for all  $n$ .

## Section 9.2 Series and Convergence

1.  $S_1 = 1$

$$S_2 = 1 + \frac{1}{4} = 1.2500$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$$

2.  $S_1 = \frac{1}{6} \approx 0.1667$

$$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$$

$$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$$

$$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$$

$$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$$

3.  $S_1 = 3$

$$S_2 = 3 - \frac{9}{2} = -1.5$$

$$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$$

$$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$$

$$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$$

4.  $S_1 = 1$

$$S_2 = 1 + \frac{1}{3} \approx 1.3333$$

$$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$$

$$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \approx 1.6762$$

$$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \approx 1.7873$$

5.  $S_1 = 3$

$$S_2 = 3 + \frac{3}{2} = 4.5$$

$$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$$

$$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$$

$$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$$

6.  $S_1 = 1$

$$S_2 = 1 - \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

7.  $a_n = \frac{n+1}{n}$

$$\{a_n\} = \left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots\right\} \text{ converges to } 1$$

$$\sum_{n=1}^{\infty} a_n = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots \text{ diverges}$$

8.  $a_n = 3\left(\frac{4}{5}\right)^n$

$$\{a_n\} = \left\{3\left(\frac{4}{5}\right), 3\left(\frac{4}{5}\right)^2, 3\left(\frac{4}{5}\right)^3, \dots\right\} \text{ converges to } 0$$

$$\sum_{n=1}^{\infty} a_n = 3\left(\frac{4}{5}\right) + 3\left(\frac{4}{5}\right)^2 + 3\left(\frac{4}{5}\right)^3 + \dots \text{ converges}$$

$$\left(\text{Geometric series, } |r| = \frac{4}{5} < 1\right)$$

9.  $\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$

Geometric series

$$r = \frac{7}{6} > 1$$

Diverges by Theorem 9.6

10.  $\sum_{n=0}^{\infty} 5\left(\frac{11}{10}\right)^n$

Geometric series

$$r = \frac{11}{10} > 1$$

Diverges by Theorem 9.6

11.  $\sum_{n=0}^{\infty} 1000(1.055)^n$

Geometric series

$$r = 1.055 > 1$$

Diverges by Theorem 9.6

12.  $\sum_{n=0}^{\infty} 2(-1.03)^n$

Geometric series

$$|r| = 1.03 > 1$$

Diverges by Theorem 9.6

13.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Diverges by Theorem 9.9

14.  $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$$

Diverges by Theorem 9.9

15.  $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

Diverges by Theorem 9.9

16.  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 9.9

17.  $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \frac{2^n+1}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$$

Diverges by Theorem 9.9

18.  $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$$

Diverges by Theorem 9.9

19.  $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \dots\right]$

$$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \dots$$

Matches graph (c). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

$$20. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \dots$$

Matches graph (b). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - 2/3} = \frac{1}{1/3} = 3$$

$$21. \sum_{n=0}^{\infty} \frac{15}{4} \left(-\frac{1}{4}\right)^n = \frac{15}{4} \left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$$

$$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \dots$$

Matches graph (a). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4} \left(-\frac{1}{4}\right)^n = \frac{15/4}{1 - (-1/4)} = \frac{15/4}{5/4} = 3$$

$$22. \sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17}{3} \left[1 - \frac{8}{9} + \frac{64}{81} - \cdots\right]$$

$$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \dots$$

Matches (d). Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17/3}{1 - (-8/9)} = \frac{17/3}{17/9} = 3$$

$$23. \sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{1}{2}\right)^n = \frac{17}{3} \left[1 - \frac{1}{2} + \frac{1}{4} - \cdots\right]$$

$$S_0 = \frac{17}{3}, S_1 = \frac{17}{6}, \dots$$

Matches (f). The series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{1}{2}\right)^n = \frac{17}{3} \frac{1}{1 - (-1/2)} = \frac{17}{3} \frac{2}{3} = \frac{34}{9}$$

$$29. \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots, \quad S_n = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

$$30. \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)}\right) = \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$31. (a) \sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right) = 2 \left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right]$$

$$\left(S_n = 2 \left[1 + \frac{1}{2} + \frac{1}{3} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)\right]\right) = 2 \left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{3} \approx 3.667$$

(b)

$n$	5	10	20	50	100
$S_n$	2.7976	3.1643	3.3936	3.5513	3.6078

$$24. \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 1 + \frac{2}{5} + \frac{4}{25} + \cdots$$

$$S_0 = 1, S_1 = \frac{7}{5}, \dots$$

Matches (e). The series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1 - (2/5)} = \frac{5}{3}$$

$$25. \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$$

Geometric series with  $r = \frac{5}{6} < 1$

Converges by Theorem 9.6

$$26. \sum_{n=0}^{\infty} 2 \left(-\frac{1}{2}\right)^n$$

Geometric series with  $|r| = \left|-\frac{1}{2}\right| < 1$

Converges by Theorem 9.6

$$27. \sum_{n=0}^{\infty} (0.9)^n$$

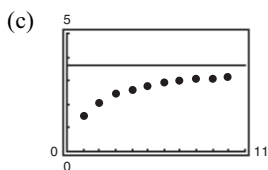
Geometric series with  $r = 0.9 < 1$

Converges by Theorem 9.6

$$28. \sum_{n=0}^{\infty} (-0.6)^n$$

Geometric series with  $|r| = |-0.6| < 1$

Converges by Theorem 9.6



(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

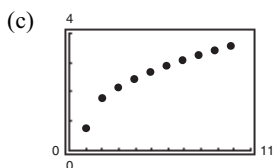
32. (a) 
$$\sum_{n=1}^{\infty} \frac{4}{n(n+4)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+4} \right)$$

$$= \left( 1 - \frac{1}{5} \right) + \left( \frac{1}{2} - \frac{1}{6} \right) + \left( \frac{1}{3} - \frac{1}{7} \right) + \left( \frac{1}{4} - \frac{1}{8} \right) + \left( \frac{1}{5} - \frac{1}{9} \right) + \left( \frac{1}{6} - \frac{1}{10} \right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833$$

(b)

$n$	5	10	20	50	100
$S_n$	1.5377	1.7607	1.9051	2.0071	2.0443

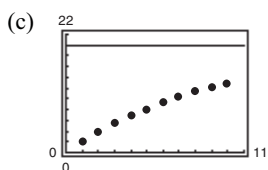


(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

33. (a) 
$$\sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$$

(b)

$n$	5	10	20	50	100
$S_n$	8.1902	13.0264	17.5685	19.8969	19.9995

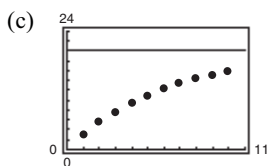


(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

34. (a) 
$$\sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1-0.85} = 20 \quad (\text{Geometric series})$$

(b)

$n$	5	10	20	50	100
$S_n$	11.1259	16.0625	19.2248	19.9941	19.999998

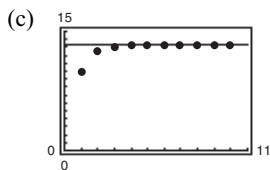


(d) The terms of the series decrease in magnitude slowly. So, the sequence of partial sums approaches the sum slowly.

35. (a)  $\sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$

(b)

$n$	5	10	20	50	100
$S_n$	13.3203	13.3333	13.3333	13.3333	13.3333

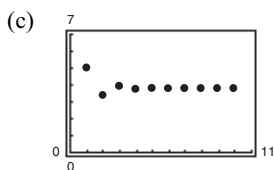


(d) The terms of the series decrease in magnitude rapidly. So, the sequence of partial sums approaches the sum rapidly.

36. (a)  $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1-(-1/3)} = \frac{15}{4} = 3.75$

(b)

$n$	5	10	20	50	100
$S_n$	3.7654	3.7499	3.7500	3.7500	3.7500



(d) The terms of the series decrease in magnitude rapidly. So, the sequence of partial sums approaches the sum rapidly.

37.  $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-(1/2)} = 2$

38.  $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1-(4/5)} = 30$

(Geometric)

39.  $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{1-(1/3)} = \frac{3}{4}$

40.  $\sum_{n=0}^{\infty} 3\left(-\frac{6}{7}\right)^n = \frac{3}{1-(-6/7)} = \frac{21}{13}$

41.  $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \sum_{n=2}^{\infty} \left( \frac{1/2}{n-1} - \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right)$

$$S_n = \frac{1}{2} \left[ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \right] = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{4}$$



$$42. \sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right)$$

$$S_n = 2 \left[ \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n+1} \right) + \left( \frac{1}{n} - \frac{1}{n+2} \right) \right] = 2 \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 2 \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = 3$$

$$43. \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8 \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$S_n = 8 \left[ \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \right] = 8 \left( \frac{1}{2} - \frac{1}{n+2} \right)$$

$$\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 8 \left( \frac{1}{2} - \frac{1}{n+2} \right) = 4$$

$$44. \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{2n+1} - \frac{1}{2n+3} \right)$$

$$S_n = \frac{1}{2} \left[ \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) + \left( \frac{1}{7} - \frac{1}{9} \right) + \cdots + \left( \frac{1}{2n+1} - \frac{1}{2n+3} \right) \right] = \frac{1}{2} \left( \frac{1}{3} - \frac{1}{2n+3} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{3} - \frac{1}{2n+3} \right) = \frac{1}{6}$$

$$45. \sum_{n=0}^{\infty} \left( \frac{1}{10} \right)^n = \frac{1}{1 - (1/10)} = \frac{10}{9}$$

$$46. \sum_{n=0}^{\infty} 8 \left( \frac{3}{4} \right)^n = \frac{8}{1 - (3/4)} = 32$$

$$47. \sum_{n=0}^{\infty} 3 \left( -\frac{1}{3} \right)^n = \frac{3}{1 - (-1/3)} = \frac{9}{4}$$

$$48. \sum_{n=0}^{\infty} 4 \left( -\frac{1}{2} \right)^n = \frac{4}{1 - (-1/2)} = \frac{8}{3}$$

$$49. \sum_{n=0}^{\infty} \left( \frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n$$

$$= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

$$52. S_n = \sum_{k=1}^n \frac{1}{9k^2 + 3k - 2} = \sum_{k=1}^n \frac{1}{(3k-1)(3k+2)}$$

$$= \sum_{k=1}^n \left[ \frac{1}{9k-3} - \frac{1}{9k+6} \right] = \frac{1}{3} \sum_{k=1}^n \left[ \frac{1}{3k-1} - \frac{1}{3k+2} \right]$$

$$= \frac{1}{3} \left[ \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{8} \right) + \left( \frac{1}{8} - \frac{1}{11} \right) + \cdots + \left( \frac{1}{3n-1} - \frac{1}{3n+2} \right) \right] = \frac{1}{3} \left( \frac{1}{2} - \frac{1}{3n+2} \right)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{1}{2} - \frac{1}{3n+2} \right) = \frac{1}{6}$$

$$50. \sum_{n=1}^{\infty} \left[ (0.7)^n + (0.9)^n \right] = \sum_{n=0}^{\infty} \left( \frac{7}{10} \right)^n + \sum_{n=0}^{\infty} \left( \frac{9}{10} \right)^n - 2$$

$$= \frac{1}{1 - (7/10)} + \frac{1}{1 - (9/10)} - 2$$

$$= \frac{10}{3} + 10 - 2 = \frac{34}{3}$$

51. Note that  $\sin(1) \approx 0.8415 < 1$ . The series  $\sum_{n=1}^{\infty} [\sin(1)]^n$  is geometric with  $r = \sin(1) < 1$ . So,

$$\sum_{n=1}^{\infty} [\sin(1)]^n = \sin(1) \sum_{n=0}^{\infty} [\sin(1)]^n = \frac{\sin(1)}{1 - \sin(1)} \approx 5.3080.$$

$$53. (a) 0.\bar{4} = \sum_{n=0}^{\infty} \frac{4}{10} \left( \frac{1}{10} \right)^n$$

$$(b) \text{ Geometric series with } a = \frac{4}{10} \text{ and } r = \frac{1}{10}$$

$$S = \frac{a}{1-r} = \frac{4/10}{1-(1/10)} = \frac{4}{9}$$

$$54. (a) 0.\bar{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \sum_{n=1}^{\infty} 9 \left( \frac{1}{10} \right)^n = \frac{9}{10} \sum_{n=0}^{\infty} \left( \frac{1}{10} \right)^n$$

$$(b) 0.\bar{9} = \frac{9}{101 - (1/10)} = 1$$

$$55. (a) 0.\bar{81} = \sum_{n=0}^{\infty} \frac{81}{100} \left( \frac{1}{100} \right)^n$$

$$(b) \text{ Geometric series with } a = \frac{81}{100} \text{ and } r = \frac{1}{100}$$

$$S = \frac{a}{1-r} = \frac{81/100}{1-(1/100)} = \frac{81}{99} = \frac{9}{11}$$

$$56. (a) 0.\bar{01} = \sum_{n=1}^{\infty} \left( \frac{1}{100} \right)^n = \frac{1}{100} \sum_{n=0}^{\infty} \left( \frac{1}{100} \right)^n$$

$$(b) 0.\bar{01} = \frac{1}{100} \cdot \frac{1}{1 - (1/100)} = \frac{1}{100} \cdot \frac{100}{99} = \frac{1}{99}$$

$$57. (a) 0.0\bar{75} = \sum_{n=0}^{\infty} \frac{3}{40} \left( \frac{1}{100} \right)^n$$

$$(b) \text{ Geometric series with } a = \frac{3}{40} \text{ and } r = \frac{1}{100}$$

$$S = \frac{a}{1-r} = \frac{3/40}{99/100} = \frac{5}{66}$$

$$63. \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right)$$

$$S_n = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n+1} \right) + \left( \frac{1}{n} - \frac{1}{n+2} \right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right) = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{2}, \text{ converges}$$

$$64. \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$S_n = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{n+2} \right) = \frac{1}{2}, \text{ converges}$$

$$58. (a) 0.2\bar{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200} \left( \frac{1}{100} \right)^n$$

$$(b) \text{ Geometric series with } a = \frac{3}{200} \text{ and } r = \frac{1}{100}$$

$$S = \frac{1}{5} + \frac{a}{1-r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

$$59. \sum_{n=0}^{\infty} (1.075)^n$$

Geometric series with  $r = 1.075$

Diverges by Theorem 9.6

$$60. \sum_{n=1}^{\infty} \frac{3^n}{1000}$$

Geometric series with  $r = 3 > 1$ .

Diverges by Theorem 9.6

$$61. \sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

$$\lim_{n \rightarrow \infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

Diverges by Theorem 9.9

$$62. \sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$$

$$\lim_{n \rightarrow \infty} \frac{4n+1}{3n-1} = \frac{4}{3} \neq 0$$

Diverges by Theorem 9.9

$$65. \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+3} \right)$$

$$S_n = \frac{1}{3} \left[ \left( \frac{1}{1} - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) + \left( \frac{1}{3} - \frac{1}{6} \right) + \left( \frac{1}{4} - \frac{1}{7} \right) + \cdots + \left( \frac{1}{n-2} - \frac{1}{n+1} \right) - \left( \frac{1}{n-1} - \frac{1}{n+2} \right) - \left( \frac{1}{n} - \frac{1}{n+3} \right) \right]$$

$$= \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{3} \left( 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{3} \left( \frac{11}{6} \right) = \frac{11}{8}, \text{ converges}$$

$$66. \sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

$$S_n = \frac{1}{2} \left[ \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) \right] = \frac{1}{2} \left( 1 - \frac{1}{n+1} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{2n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left( 1 - \frac{1}{n+1} \right) = \frac{1}{2}, \text{ converges}$$

$$67. \sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$$

$$\lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

Diverges by Theorem 9.9

$$68. \sum_{n=1}^{\infty} \frac{3^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n^3} = \lim_{n \rightarrow \infty} \frac{(\ln 2)3^n}{3n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule); diverges by Theorem 9.9

$$69. \sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n$$

Geometric series with  $r = \frac{1}{2}$

Converges by Theorem 9.6

$$70. \sum_{n=0}^{\infty} \frac{3}{5^n} = 3 \sum_{n=0}^{\infty} \left( \frac{1}{5} \right)^n, \text{ convergent}$$

Geometric series with  $r = \frac{1}{5}$

71. Because  $n > \ln(n)$ , the terms  $a_n = \frac{n}{\ln(n)}$  do not approach 0 as  $n \rightarrow \infty$ . So, the series  $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$  diverges.

$$72. S_n = \sum_{k=1}^n \ln\left(\frac{1}{k}\right) = \sum_{k=1}^n -\ln(k)$$

$$= 0 - \ln 2 - \ln 3 - \cdots - \ln(n)$$

Because  $\lim_{n \rightarrow \infty} S_n$  diverges,  $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$  diverges.

73. For  $k \neq 0$ ,

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{k}{n} \right)^n = \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{k}{n} \right)^{n/k} \right]^k$$

$$= e^k \neq 0.$$

For  $k = 0$ ,  $\lim_{n \rightarrow \infty} (1 + 0)^n = 1 \neq 0$ .

So,  $\sum_{n=1}^{\infty} \left[ 1 + \frac{k}{n} \right]^n$  diverges.

$$74. \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left( \frac{1}{e} \right)^n \text{ converges because it is geometric}$$

with

$$|r| = \frac{1}{e} < 1.$$

$$75. \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$$

So,  $\sum_{n=1}^{\infty} \arctan n$  diverges.

$$\begin{aligned}
 76. S_n &= \sum_{k=1}^n \ln\left(\frac{k+1}{k}\right) \\
 &= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right) \\
 &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) \\
 &= \ln(n+1) - \ln(1) = \ln(n+1)
 \end{aligned}$$

Diverges

77. See definitions on page 608.

78.  $\lim_{n \rightarrow \infty} a_n = 5$  means that the limit of the sequence  $\{a_n\}$  is 5.

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$  means that the limit of the partial sums is 5.

79. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

is a geometric series with ratio  $r$ . When  $0 < |r| < 1$ , the series converges to  $a/(1-r)$ . The series diverges if  $|r| \geq 1$ .

80. If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

81. (a)  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

(b)  $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$

These are the same. The third series is different, unless  $a_1 = a_2 = \cdots = a$  is constant.

(c)  $\sum_{n=1}^{\infty} a_k = a_k + a_k + \cdots$

82. (a) Yes, the new series will still diverge.

(b) Yes, the new series will converge.

$$83. \sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

Geometric series: converges for  $\left|\frac{x}{2}\right| < 1$  or  $|x| < 2$

$$\begin{aligned}
 f(x) &= \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n \\
 &= \frac{x}{2} \frac{1}{1 - (x/2)} = \frac{x}{2} \frac{2}{2 - x} = \frac{x}{2 - x}, \quad |x| < 2
 \end{aligned}$$

$$84. \sum_{n=1}^{\infty} (3x)^n = (3x) \sum_{n=0}^{\infty} (3x)^n$$

Geometric series: converges for  $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

$$f(x) = (3x) \sum_{n=0}^{\infty} (3x)^n = (3x) \frac{1}{1 - 3x} = \frac{3x}{1 - 3x}, \quad |x| < \frac{1}{3}$$

$$85. \sum_{n=1}^{\infty} (x-1)^n = (x-1) \sum_{n=0}^{\infty} (x-1)^n$$

Geometric series: converges for  $|x-1| < 1 \Rightarrow 0 < x < 2$

$$\begin{aligned}
 f(x) &= (x-1) \sum_{n=0}^{\infty} (x-1)^n \\
 &= (x-1) \frac{1}{1 - (x-1)} = \frac{x-1}{2-x}, \quad 0 < x < 2
 \end{aligned}$$

$$86. \sum_{n=0}^{\infty} 4 \left(\frac{x-3}{4}\right)^n$$

Geometric series: converges for

$$\left|\frac{x-3}{4}\right| < 1 \Rightarrow |x-3| < 4 \Rightarrow -1 < x < 7$$

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} 4 \left(\frac{x-3}{4}\right)^n = \frac{4}{1 - [(x-3)/4]} \\
 &= \frac{4}{(4-x+3)/4} = \frac{16}{7-x}, \quad -1 < x < 7
 \end{aligned}$$

$$87. \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$$

Geometric series: converges for

$$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}, \quad -1 < x < 1$$

$$88. \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n$$

Geometric series: converges for

$$|-x^2| < 1 \Rightarrow -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1+x^2}, \quad -1 < x < 1$$

$$89. \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$$

Geometric series: converges if  $\left|\frac{1}{x}\right| < 1$

$$\Rightarrow |x| > 1 \Rightarrow x < -1 \text{ or } x > 1$$

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n = \frac{1}{1 - (1/x)} = \frac{x}{x-1}, \quad x > 1 \text{ or } x < -1$$

90.  $\sum_{n=1}^{\infty} \left( \frac{x^2}{x^2 + 4} \right)^n = \frac{x^2}{x^2 + 4} \sum_{n=0}^{\infty} \left( \frac{x^2}{x^2 + 4} \right)^n$

Geometric series: converges for  $\left| \frac{x^2}{x^2 + 4} \right| < 1$

$\Rightarrow x^2 < x^2 + 4 \Rightarrow$  converges for all  $x$

$$f(x) = \frac{x^2}{x^2 + 4} \cdot \frac{1}{1 - \frac{x^2}{x^2 + 4}} = \frac{x^2}{x^2 + 4} \cdot \frac{x^2 + 4}{4} = \frac{x^2}{4}$$

91.  $\sum_{n=2}^{\infty} (1 + c)^{-n} = \sum_{n=0}^{\infty} (1 + c)^{-n-2} = \frac{1}{(1 + c)^2} \sum_{n=0}^{\infty} \left( \frac{1}{1 + c} \right)^n$

For convergence,  $\left| \frac{1}{1 + c} \right| < 1 \Rightarrow 1 < |1 + c|$

$$\Rightarrow c > 0 \text{ or } c < -2$$

$$2 = \frac{1}{(1 + c)^2} \sum_{n=0}^{\infty} \left( \frac{1}{1 + c} \right)^n$$

$$2(1 + c)^2 = \frac{1}{1 - \left( \frac{1}{1 + c} \right)} = \frac{1 + c}{c}$$

$$2(1 + c) = \frac{1}{c}$$

$$2c^2 + 2c - 1 = 0$$

$$c = \frac{-2 \pm \sqrt{4 + 8}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Because  $-2 < \frac{-1 - \sqrt{3}}{2} < 0$ , the answer is

$$c = \frac{-1 + \sqrt{3}}{2}$$

92.  $\sum_{n=0}^{\infty} e^{cn} = \sum_{n=0}^{\infty} (e^c)^n = 5$

$$\frac{1}{1 - e^c} = 5$$

$$1 - e^c = \frac{1}{5}$$

$$e^c = \frac{4}{5}$$

$$c = \ln\left(\frac{4}{5}\right)$$

93. Neither statement is true. The formula

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots$$

holds for  $-1 < x < 1$ .

94. (a) False. The fact that  $\frac{1}{n^4} \rightarrow 0$  is irrelevant to the convergence of  $\sum_{n=1}^{\infty} \frac{1}{n^4}$ . Furthermore,  $\sum_{n=1}^{\infty} \frac{1}{n^4} \neq 0$ .

(b) False. The fact that  $\frac{1}{\sqrt[4]{n}} \rightarrow 0$  is irrelevant to the convergence of  $\sum \frac{1}{\sqrt[4]{n}}$ . In fact,  $\sum \frac{1}{\sqrt[4]{n}}$  diverges.

95. (a)  $x$  is the common ratio.

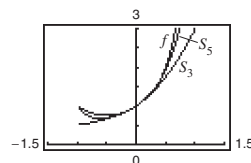
(b)  $1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \quad |x| < 1$

(c)  $y_1 = \frac{1}{1 - x}$

$$y_2 = S_3 = 1 + x + x^2$$

$$y_3 = S_5 = 1 + x + x^2 + x^3 + x^4$$

Answers will vary.



96. (a)  $\left(-\frac{x}{2}\right)$  is the common ratio.

(b)  $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$

$$= \frac{1}{1 - (-x/2)}$$

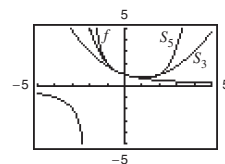
$$= \frac{2}{2 + x}, \quad |x| < 2$$

(c)  $y_1 = \frac{2}{2 + x}$

$$y_2 = S_3 = 1 - \frac{x}{2} + \frac{x^2}{4}$$

$$y_3 = S_5 = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}$$

Answers will vary.



$$97. f(x) = 3 \left[ \frac{1 - 0.5^x}{1 - 0.5} \right]$$

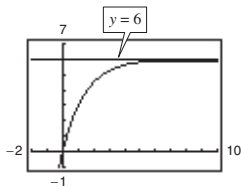
Horizontal asymptote:  $y = 6$

$$\sum_{n=0}^{\infty} 3 \left( \frac{1}{2} \right)^n$$

$$S = \frac{3}{1 - (1/2)} = 6$$

The horizontal asymptote is the sum of the series.

$f(n)$  is the  $n$ th partial sum.



$$98. f(x) = 2 \left[ \frac{1 - 0.8^x}{1 - 0.8} \right]$$

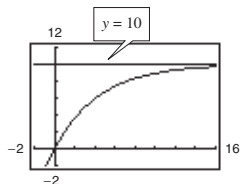
Horizontal asymptote:  $y = 10$

$$\sum_{n=0}^{\infty} 2 \left( \frac{4}{5} \right)^n$$

$$S = \frac{2}{1 - (4/5)} = 10$$

The horizontal asymptote is the sum of the series.

$f(n)$  is the  $n$ th partial sum.



$$99. \frac{1}{n(n+1)} < 0.0001$$

$$10,000 < n^2 + n$$

$$0 < n^2 + n - 10,000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10,000)}}{2}$$

Choosing the positive value for  $n$  you have  
 $n \approx 99.5012$ . The first term that is less than 0.0001 is  
 $n = 100$ .

$$\left( \frac{1}{8} \right)^n < 0.0001$$

$$10,000 < 8^n$$

This inequality is true when  $n = 5$ . This series converges at a faster rate.

$$100. \frac{1}{2^n} < 0.0001$$

$$10,000 < 2^n$$

This inequality is true when  $n = 14$ .

$$(0.01)^n < 0.0001$$

$$10,000 < 10^n$$

This inequality is true when  $n = 5$ . This series converges at a faster rate.

$$101. \sum_{i=0}^{n-1} 8000(0.95)^i = \frac{8000[1 - 0.95^n]}{1 - 0.95}$$

$$= 160,000[1 - 0.95^n], \quad n > 0$$

$$102. V(t) = 475,000(1 - 0.3)^n = 475,000(0.7)^n$$

$$V(5) = 475,000(0.7)^5 = \$79,833.25$$

$$103. \sum_{i=0}^{\infty} 200(0.75)^i = 800 \text{ million dollars}$$

$$104. \sum_{i=0}^{\infty} 200(0.60)^i = 500 \text{ million dollars}$$

$$105. D_1 = 16$$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$$\vdots$$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \dots$$

$$= -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81}$$

$$\approx 152.42 \text{ feet}$$

106. The ball in Exercise 105 takes the following times for each fall.

$$s_1 = -16t^2 + 16$$

$$s_1 = 0 \text{ if } t = 1$$

$$s_2 = -16t^2 + 16(0.81)$$

$$s_2 = 0 \text{ if } t = 0.9$$

$$s_3 = -16t^2 + 16(0.81)^2$$

$$s_3 = 0 \text{ if } t = (0.9)^2$$

$$\vdots$$

$$\vdots$$

$$s_n = -16t^2 + 16(0.81)^{n-1}$$

$$s_n = 0 \text{ if } t = (0.9)^{n-1}$$

Beginning with  $s_2$ , the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2 \sum_{n=1}^{\infty} (0.9)^n = -1 + 2 \sum_{n=0}^{\infty} (0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

$$107. P(n) = \frac{1}{2} \left( \frac{1}{2} \right)^n$$

$$P(2) = \frac{1}{2} \left( \frac{1}{2} \right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^n = \frac{1/2}{1 - (1/2)} = 1$$

$$108. P(n) = \frac{1}{3} \left( \frac{2}{3} \right)^n$$

$$P(2) = \frac{1}{3} \left( \frac{2}{3} \right)^2 = \frac{4}{27}$$

$$\sum_{n=0}^{\infty} \frac{1}{3} \left( \frac{2}{3} \right)^n = \frac{1/3}{1 - (2/3)} = 1$$

$$109. (a) \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^n = \frac{1}{2} \frac{1}{1 - (1/2)} = 1$$

(b) No, the series is not geometric.

$$(c) \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n = 2$$

$$110. \text{ Person 1: } \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{2} \frac{1}{1 - (1/8)} = \frac{4}{7}$$

$$\text{Person 2: } \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots = \frac{1}{4} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{4} \frac{1}{1 - (1/8)} = \frac{2}{7}$$

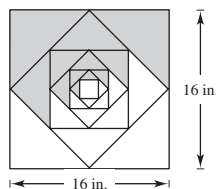
$$\text{Person 3: } \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots = \frac{1}{8} \sum_{n=0}^{\infty} \left( \frac{1}{8} \right)^n = \frac{1}{8} \frac{1}{1 - (1/8)} = \frac{1}{7}$$

$$\text{Sum: } \frac{4}{7} + \frac{2}{7} + \frac{1}{7} = 1$$

$$111. (a) 64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$$

$$(b) \sum_{n=0}^{\infty} 64 \left( \frac{1}{2} \right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$$

**Note:** This is one-half of the area of the original square



$$112. (a) \sin \theta = \frac{|Y_{y_1}|}{z} \Rightarrow |Y_{y_1}| = z \sin \theta$$

$$\sin \theta = \frac{|x_1 y_1|}{|Y_{y_1}|} \Rightarrow |x_1 y_1| = |Y_{y_1}| \sin \theta = z \sin^2 \theta$$

$$\sin \theta = \frac{|x_1 y_2|}{|x_1 y_1|} \Rightarrow |x_1 y_2| = |x_1 y_1| \sin \theta = z \sin^3 \theta$$

$$\text{Total: } z \sin \theta + z \sin^2 \theta + z \sin^3 \theta + \cdots = z \frac{\sin \theta}{1 - \sin \theta}$$

$$(b) \text{ If } z = 1 \text{ and } \theta = \frac{\pi}{6}, \text{ then total} = \frac{1/2}{1 - (1/2)} = 1.$$

$$113. \sum_{n=1}^{20} 100,000 \left( \frac{1}{1.06} \right)^n = \frac{100,000}{1.06} \sum_{i=0}^{19} \left( \frac{1}{1.06} \right)^i = \frac{100,000}{1.06} \left[ \frac{1 - 1.06^{-20}}{1 - 1.06^{-1}} \right] \quad (n = 20, r = 1.06^{-1}) \approx \$1,146,992.12$$

The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

$$114. \text{ Surface area} = 4\pi(1)^2 + 9 \left( 4\pi \left( \frac{1}{3} \right)^2 \right) + 9^2 \cdot 4\pi \left( \frac{1}{9} \right)^2 + \cdots = 4(\pi + \pi + \cdots) = \infty$$

# INSTRUCTOR USE ONLY

$$115. w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1-2^n)}{1-2} = 0.01(2^n - 1)$$

- (a) When  $n = 29$ :  $w = \$5,368,709.11$   
 (b) When  $n = 30$ :  $w = \$10,737,418.23$   
 (c) When  $n = 31$ :  $w = \$21,474,836.47$

$$116. \sum_{n=0}^{12t-1} P\left(1 + \frac{r}{12}\right)^n = \frac{P\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)}$$

$$= P\left(-\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

$$117. P = 45, \quad r = 0.03, \quad t = 20$$

$$(a) A = 45\left(\frac{12}{0.03}\right)\left[\left(1 + \frac{0.03}{12}\right)^{12(20)} - 1\right] \approx \$14,773.59$$

$$(b) A = \frac{45(e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$14,779.65$$

$$118. P = 75, \quad r = 0.055, \quad t = 25$$

$$(a) A = 75\left(\frac{12}{0.055}\right)\left[\left(1 + \frac{0.055}{12}\right)^{12(25)} - 1\right] \approx \$48,152.81$$

$$(b) A = \frac{75(e^{0.055(25)} - 1)}{e^{0.055/12} - 1} \approx \$48,245.07$$

$$119. P = 100, \quad r = 0.04, \quad t = 35$$

$$(a) A = 100\left(\frac{12}{0.04}\right)\left[\left(1 + \frac{0.04}{12}\right)^{12(35)} - 1\right] \approx \$91,373.09$$

$$(b) A = \frac{100(e^{0.04(35)} - 1)}{e^{0.04/12} - 1} \approx \$91,503.32$$

$$120. P = 30, \quad r = 0.06, \quad t = 50$$

$$(a) A = 30\left(\frac{12}{0.06}\right)\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$113,615.73$$

$$(b) A = \frac{30(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$114,227.18$$

$$121. T = 50,000 + 50,000(1.04) + \cdots + 50,000(1.04)^{39}$$

$$= \sum_{n=0}^{39} 50,000(1.04)^n$$

$$= 50,000\left[\frac{1 - 1.04^{40}}{1 - 1.04}\right] \approx \$4,751,275.79$$

$$122. T = \sum_{n=0}^{39} 50,000(1.045)^n \approx \$5,351,516.15$$

$$123. \text{False. } \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ but } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

124. True

$$125. \text{False; } \sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r}\right) - a$$

The formula requires that the geometric series begins with  $n = 0$ .

126. True

$$\lim_{n \rightarrow \infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$$

127. True

$$0.74999 \dots = 0.74 + \frac{9}{10^3} + \frac{9}{10^4} + \cdots$$

$$= 0.74 + \frac{9}{10^3} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$$

$$= 0.74 + \frac{9}{10^3} \cdot \frac{1}{1 - (1/10)}$$

$$= 0.74 + \frac{9}{10^3} \cdot \frac{10}{9}$$

$$= 0.74 + \frac{1}{100} = 0.75$$

128. True



129. By letting  $S_0 = 0$ , you have

$$a_n = \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}. \text{ So,}$$

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (S_n - S_{n-1}) \\ &= \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) \\ &= \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]. \end{aligned}$$

130. Let  $\{S_n\}$  be the sequence of partial sums for the convergent series

$$\sum_{n=1}^{\infty} a_n = L. \text{ Then } \lim_{n \rightarrow \infty} S_n = L \text{ and because}$$

$$R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

you have

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} (L - S_n) = \lim_{n \rightarrow \infty} L - \lim_{n \rightarrow \infty} S_n = L - L = 0.$$

131. Let  $\sum a_n = \sum_{n=0}^{\infty} 1$  and  $\sum b_n = \sum_{n=0}^{\infty} (-1)$ .

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

132. If  $\sum (a_n + b_n)$  converged, then

$\sum (a_n + b_n) - \sum a_n = \sum b_n$  would converge, which is a contradiction. So,  $\sum (a_n + b_n)$  diverges.

133. Suppose, on the contrary, that  $\sum ca_n$  converges. Because  $c \neq 0$ ,

$$\sum \left( \frac{1}{c} \right) ca_n = \sum a_n$$

converges. This is a contradiction since  $\sum ca_n$  diverged. So,  $\sum ca_n$  diverges.

134. If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ . So,  $\lim_{n \rightarrow \infty} \frac{1}{a_n} \neq 0$ ,

which implies that  $\sum \frac{1}{a_n}$  diverges.

135. (a)  $\frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} = \frac{a_{n+3} - a_{n+1}}{a_{n+1}a_{n+2}a_{n+3}} = \frac{a_{n+2}}{a_{n+1}a_{n+2}a_{n+3}} = \frac{1}{a_{n+1}a_{n+3}}$

(b) 
$$\begin{aligned} S_n &= \sum_{k=0}^n \frac{1}{a_{k+1}a_{k+3}} \\ &= \sum_{k=0}^n \left[ \frac{1}{a_{k+1}a_{k+2}} - \frac{1}{a_{k+2}a_{k+3}} \right] \\ &= \left[ \frac{1}{a_1a_2} - \frac{1}{a_2a_3} \right] + \left[ \frac{1}{a_2a_3} - \frac{1}{a_3a_4} \right] + \cdots + \left[ \frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}} \right] = \frac{1}{a_1a_2} - \frac{1}{a_{n+2}a_{n+3}} = 1 - \frac{1}{a_{n+2}a_{n+3}} \\ \sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{a_{n+2}a_{n+3}} \right] = 1 \end{aligned}$$

136.  $S_{2n} = 1 + 2x + x^2 + 2x^3 + \cdots + x^{2n} + 2x^{2n+1}$

$$= (1 + x^2 + x^4 + \cdots + x^{2n}) + (2x + 2x^3 + \cdots + 2x^{2n+1}) = \sum_{k=0}^n (x^2)^k + 2x \sum_{n=0}^n (x^2)^k$$

As  $n \rightarrow \infty$ , the two geometric series converge for  $|x| < 1$ :

$$\lim_{n \rightarrow \infty} S_{2n} = \frac{1}{1-x^2} + 2x \left( \frac{1}{1-x^2} \right) = \frac{1+2x}{1-x^2}, \quad |x| < 1$$

137.  $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r} \left( \frac{1}{r} \right)^n = \frac{1/r}{1-(1/r)} = \frac{1}{r-1} \quad \left( \text{since } \left| \frac{1}{r} \right| < 1 \right)$

This is a geometric series which converges if

$$\left| \frac{1}{r} \right| < 1 \Leftrightarrow |r| > 1.$$

$$138. \frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$$1 = A(n+1)(n+2) + B_n(n+2) + C_n(n+1)$$

When  $n = 0$ ,  $1 = 2A \Rightarrow A = 1/2$

When  $n = -1$ ,  $1 = -B \Rightarrow B = -1$

When  $n = -2$ ,  $1 = 2C \Rightarrow C = 1/2$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1/2}{n} + \frac{-1}{n+1} + \frac{1/2}{n+2}$$

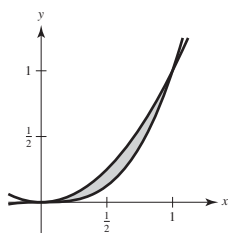
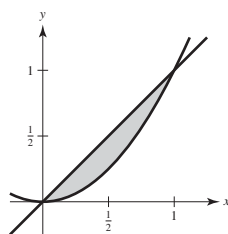
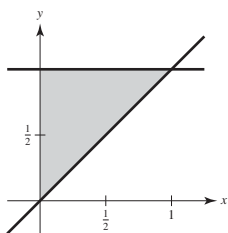
$$S_n = \left( \frac{1/2}{1} - \frac{1}{2} + \frac{1/2}{3} \right) + \left( \frac{1/2}{2} - \frac{1}{3} + \frac{1/2}{4} \right) + \left( \frac{1/2}{3} - \frac{1}{4} + \frac{1/2}{5} \right) + \cdots + \left( \frac{1/2}{n} - \frac{1}{n+1} + \frac{1/2}{n+2} \right)$$

$$= \left( \frac{1/2}{1} - \frac{1}{2} + \frac{1/2}{2} \right) + \left( \frac{1}{3} - \frac{1}{3} \right) + \left( \frac{1}{4} - \frac{1}{4} \right) + \cdots + \left( \frac{1/2}{n} - \frac{1}{n+1} + \frac{1/2}{n+2} \right)$$

As  $n \rightarrow \infty$ ,  $S_n \rightarrow 1/4$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

139. (a)



(b)  $\int_0^1 (1-x) dx = \left[ x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}$

$$\int_0^1 (x-x^2) dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\int_0^1 (x^2-x^3) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

(c)  $a_n = \int_0^1 (x^{n-1} - x^n) dx = \left[ \frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} a_n = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots = 1$$

Note that the square has area = 1.

140. The entire rectangle has area 2 because the height is 1

and the base is  $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$ . The squares all lie inside the rectangle, and the sum of their areas is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

So,  $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$ .

141. Let  $H$  represent the half-life of the drug. If a patient receives  $n$  equal doses of  $P$  units each of this drug, administered at equal time interval of length  $t$ , the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt} \text{ where}$$

$k = -(\ln 2)/H$ . One time interval after the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}$$

Two time intervals after the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt} \text{ and so on. Because } k < 0, T_{n+s} \rightarrow 0 \text{ as } s \rightarrow \infty, \text{ where } s \text{ is an integer.}$$

142. The series is telescoping:

$$S_n = \sum_{k=1}^n \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

$$= \sum_{k=1}^n \left[ \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \right]$$

$$= 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}}$$

$$\lim_{n \rightarrow \infty} S_n = 3 - 1 = 2$$

143.  $f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 4, \dots$

In general:  $f(n) = \begin{cases} n^2/4, & n \text{ even} \\ (n^2 - 1)/4, & n \text{ odd.} \end{cases}$

(See below for a proof of this.)

$x + y$  and  $x - y$  are either both odd or both even. If both even, then

$$f(x + y) - f(x - y) = \frac{(x + y)^2}{4} - \frac{(x - y)^2}{4} = xy.$$

If both odd,

$$f(x + y) - f(x - y) = \frac{(x + y)^2 - 1}{4} - \frac{(x - y)^2 - 1}{4} = xy.$$

Proof by induction that the formula for  $f(n)$  is correct. It is true for  $n = 1$ . Assume that the formula is valid for  $k$ . If  $k$  is even, then  $f(k) = k^2/4$  and

$$f(k + 1) = f(k) + \frac{k}{2} = \frac{k^2}{4} + \frac{k}{2} = \frac{k^2 + 2k}{4} = \frac{(k + 1)^2 - 1}{4}.$$

The argument is similar if  $k$  is odd.

## Section 9.3 The Integral Test and $p$ -Series

1.  $\sum_{n=1}^{\infty} \frac{1}{n+3}$

Let  $f(x) = \frac{1}{x+3}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x+3} dx = [\ln(x+3)]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

2.  $\sum_{n=1}^{\infty} \frac{2}{3n+5}$

Let  $f(x) = \frac{2}{3x+5}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{2}{3x+5} dx = \left[ \frac{2}{3} \ln(3x+5) \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

3.  $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Let  $f(x) = \frac{1}{2^x}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{2^x} dx = \left[ \frac{-1}{(\ln 2) 2^x} \right]_1^{\infty} = \frac{1}{2 \ln 2}$$

Converges by Theorem 9.10.

4.  $\sum_{n=1}^{\infty} 3^{-n}$

Let  $f(x) = \frac{1}{3^x}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{3^x} dx = \left[ \frac{-1}{(\ln 3) 3^x} \right]_1^{\infty} = \frac{1}{3 \ln 3}$$

Converges by Theorem 9.10.

5.  $\sum_{n=1}^{\infty} e^{-n}$

Let  $f(x) = e^{-x}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} e^{-x} dx = [-e^{-x}]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 9.10

6.  $\sum_{n=1}^{\infty} ne^{-n/2}$

Let  $f(x) = xe^{-x/2}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 3$  since

$$f'(x) = \frac{2-x}{2e^{x/2}} < 0 \text{ for } x \geq 3.$$

$$\int_3^{\infty} xe^{-x/2} dx = [-2(x+2)e^{-x/2}]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 9.10

7.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

Let  $f(x) = \frac{1}{x^2 + 1}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = [\arctan x]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 9.10

8.  $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let  $f(x) = \frac{1}{2x+1}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{2x+1} dx = [\ln \sqrt{2x+1}]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

9.  $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let  $f(x) = \frac{\ln(x+1)}{x+1}$ .

$f$  is positive, continuous, and decreasing for

$$x \geq 2 \text{ because } f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1} dx = \left[ \frac{[\ln(x+1)]^2}{2} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

10.  $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Let  $f(x) = \frac{\ln x}{\sqrt{x}}$ ,  $f'(x) = \frac{2 - \ln x}{2x^{3/2}}$ .

$f$  is positive, continuous, and decreasing for  $x > e^2 \approx 7.4$ .

$$\int_2^{\infty} \frac{\ln x}{\sqrt{x}} dx = [2\sqrt{x}(\ln x - 2)]_2^{\infty} = \infty, \text{ diverges}$$

So, the series diverges by Theorem 9.10.

11.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

Let  $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$ ,

$$f'(x) = -\frac{1+2\sqrt{x}}{2x^{3/2}(\sqrt{x}+1)^2} < 0.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx = [2 \ln(\sqrt{x}+1)]_1^{\infty} = \infty, \text{ diverges}$$

So, the series diverges by Theorem 9.10.

12.  $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

Let  $f(x) = \frac{x}{x^2+3}$ .

$f(x)$  is positive, continuous, and decreasing for  $x \geq 2$  since

$$f'(x) = \frac{3-x^2}{(x^2+3)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2+3} dx = [\ln \sqrt{x^2+3}]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

13.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

Let  $f(x) = \frac{1}{\sqrt{x+2}}$ ,  $f'(x) = \frac{-1}{2(x+2)^{3/2}} < 0$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{(x+2)^{1/2}} dx = [2\sqrt{x+2}]_1^{\infty} = \infty, \text{ diverges.}$$

So, the series diverges by Theorem 9.10.

14.  $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let  $f(x) = \frac{\ln x}{x^3}$ ,  $f'(x) = \frac{1 - 3 \ln x}{x^4}$ .

$f$  is positive, continuous, and decreasing for  $x > 2$ .

$$\int_2^{\infty} \frac{\ln x}{x^3} dx = \left[ -\frac{(2 \ln x + 1)}{4x^2} \right]_2^{\infty} = \frac{2 \ln 2 + 1}{16}, \text{ converges}$$

So, the series converges by Theorem 9.10.

15.  $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Let  $f(x) = \frac{\ln x}{x^2}$ ,  $f'(x) = \frac{1 - 2 \ln x}{x^3}$ .

$f$  is positive, continuous, and decreasing for  $x > e^{1/2} \approx 1.6$ .

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \left[ -\frac{(\ln x + 1)}{x} \right]_1^{\infty} = 1, \text{ converges}$$

So, the series converges by Theorem 9.10.

16.  $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let  $f(x) = \frac{1}{x\sqrt{\ln x}}$ ,  $f'(x) = -\frac{2 \ln x + 1}{2x^2(\ln x)^{3/2}}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \left[ 2\sqrt{\ln x} \right]_2^{\infty} = \infty, \text{ diverges}$$

So, the series diverges by Theorem 9.10.

17.  $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + 1}$

Let  $f(x) = \frac{\arctan x}{x^2 + 1}$ ,

$$f'(x) = \frac{1 - 2x \arctan x}{(x^2 + 1)^2} < 0 \text{ for } x \geq 1.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{\arctan x}{x^2 + 1} dx = \left[ \frac{(\arctan x)^2}{2} \right]_1^{\infty} = \frac{3\pi^2}{32}, \text{ converges}$$

So, the series converges by Theorem 9.10.

18.  $\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

Let  $f(x) = \frac{1}{x \ln x \ln(\ln x)}$ ,

$$f'(x) = \frac{-[(\ln x + 1) \ln(\ln x) + 1]}{x^2 (\ln x)^2 (\ln(\ln x))^2}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 3$ .

$$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)} dx = \left[ \ln(\ln(\ln x)) \right]_3^{\infty} = \infty, \text{ diverges}$$

So, the series diverges by Theorem 9.10.

19.  $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$

Let  $f(x) = (2x+3)^{-3}$ ,  $f'(x) = \frac{-6}{(2x+3)^4} < 0$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} (2x+3)^{-3} dx = \left[ \frac{-1}{4(2x+3)^2} \right]_1^{\infty} = \frac{1}{100}, \text{ converges.}$$

So,  $\sum_{n=1}^{\infty} \frac{1}{(2n+3)^3}$  converges by Theorem 9.10.

20.  $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

Let  $f(x) = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$ ,  $f'(x) = \frac{-1}{(x+1)^2} < 0$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{x+2}{x+1} dx = \left[ x + \ln(x+1) \right]_1^{\infty} \text{ diverges.}$$

So,  $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$  diverges by Theorem 9.10.

[Note:  $\lim_{n \rightarrow \infty} \frac{n+2}{n+1} = 1 \neq 0$ , so the series diverges.]

21.  $\sum_{n=1}^{\infty} \frac{4n}{2n^2 + 1}$

Let  $f(x) = \frac{4x}{2x^2 + 1}$ ,  $f'(x) = \frac{-4(2x^2 - 1)}{(2x^2 + 1)^2} < 0$

for  $x \geq 1$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{4x}{2x^2 + 1} dx = \left[ \ln(2x^2 + 1) \right]_1^{\infty} = \infty, \text{ diverges.}$$

So,  $\sum_{n=1}^{\infty} \frac{4n}{2n^2 + 1}$  diverges by Theorem 9.10.

$$22. \sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

$$\text{Let } f(x) = \frac{x}{x^4 + 1}, f'(x) = \frac{1 - 3x^4}{(x^4 + 1)^2} < 0 \text{ for } x > 1.$$

$f$  is positive, continuous, and decreasing for  $x > 1$ .

$$\int_1^{\infty} \frac{x}{x^4 + 1} dx = \left[ \frac{1}{2} \arctan(x^2) \right]_1^{\infty} = \frac{\pi}{8}, \text{ converges}$$

So, the series converges by Theorem 9.10.

$$23. \sum_{n=1}^{\infty} \frac{n}{(4n + 5)^{3/2}}$$

$$\text{Let } f(x) = \frac{x}{(4x + 5)^{3/2}}, f'(x) = \frac{-(2x + 5)}{(4x + 5)^{5/2}} < 0 \text{ for}$$

$$x \geq 3$$

$f$  is positive, continuous, and decreasing for  $x \geq 3$ .

$$\int_1^{\infty} \frac{x}{(4x + 5)^{3/2}} dx = \left[ \frac{2x + 5}{4\sqrt{4x + 5}} \right]_1^{\infty} = \infty$$

(Integration by parts:  $u = x$ ,  $dv = (4x + 5)^{-3/2} dx$ )

So,  $\sum_{n=1}^{\infty} \frac{n}{(4n + 5)^{3/2}}$  diverges by Theorem 9.10.

$$24. \sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1} = \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

$$\text{Let } f(x) = \frac{x}{(x^2 + 1)^2}, f'(x) = \frac{-(3x^2 - 1)}{(x^2 + 1)^3} < 0 \text{ for}$$

$$x \geq 1$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$

$$\int_1^{\infty} \frac{x}{(x^2 + 1)^2} dx = \left[ \frac{-1}{2(x^2 + 1)} \right]_1^{\infty} = \frac{1}{4}, \text{ converges}$$

So,  $\sum_{n=1}^{\infty} \frac{n}{n^4 + 2n^2 + 1}$  converges by Theorem 9.10.

$$25. \sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$$

$$\text{Let } f(x) = \frac{x^{k-1}}{x^k + c}.$$

$f$  is positive, continuous, and decreasing for

$$x > \sqrt[k]{c(k-1)} \text{ because}$$

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k + c)^2} < 0$$

$$\text{for } x > \sqrt[k]{c(k-1)}.$$

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c} dx = \left[ \frac{1}{k} \ln(x^k + c) \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

$$26. \sum_{n=1}^{\infty} n^k e^{-n}$$

$$\text{Let } f(x) = \frac{x^k}{e^x}.$$

$f$  is positive, continuous, and decreasing for  $x > k$  because

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0 \text{ for } x > k.$$

Use integration by parts.

$$\begin{aligned} \int_1^{\infty} x^k e^{-x} dx &= [-x^k e^{-x}]_1^{\infty} + k \int_1^{\infty} x^{k-1} e^{-x} dx \\ &= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e} \end{aligned}$$

Converges by Theorem 9.10

$$27. \text{ Let } f(x) = \frac{(-1)^x}{x}, f(n) = a_n.$$

The function  $f$  is not positive for  $x \geq 1$ .

$$28. \text{ Let } f(x) = e^{-x} \cos x, f(n) = a_n.$$

The function  $f$  is not positive for  $x \geq 1$ .

$$29. \text{ Let } f(x) = \frac{2 + \sin x}{x}, f(n) = a_n.$$

The function  $f$  is not decreasing for  $x \geq 1$ .

$$30. \text{ Let } f(x) = \left( \frac{\sin x}{x} \right)^2, f(n) = a_n.$$

The function  $f$  is not decreasing for  $x \geq 1$ .

$$31. \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\text{Let } f(x) = \frac{1}{x^3}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^3} dx = \left[ -\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 9.10

$$32. \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

$$\text{Let } f(x) = \frac{1}{x^{1/3}}.$$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{x^{1/3}} dx = \left[ \frac{3}{2} x^{2/3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 9.10

33. Let  $f(x) = \frac{1}{x^{1/4}}$ ,  $f'(x) = \frac{-1}{4x^{5/4}} < 0$  for  $x \geq 1$

$f$  is positive, continuous, and decreasing for  $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^{1/4}} dx = \left[ \frac{4x^{3/4}}{3} \right]_1^{\infty} = \infty, \text{ diverges}$$

So, the series diverges by Theorem 9.10.

34. Let  $f(x) = \frac{1}{x^4}$ ,  $f'(x) = \frac{-4}{x^5} < 0$  for  $x \geq 1$

$f$  is positive, continuous, and decreasing for  $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^4} dx = \left[ \frac{-1}{3x^3} \right]_1^{\infty} = \frac{1}{3}, \text{ converges}$$

So,  $\sum_{n=1}^{\infty} \frac{1}{n^4}$  converges by Theorem 9.10.

35.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent  $p$ -series with  $p = \frac{1}{5} < 1$

36.  $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

Convergent  $p$ -series with  $p = \frac{5}{3} > 1$

37.  $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent  $p$ -series with  $p = \frac{1}{2} < 1$

38.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Convergent  $p$ -series with  $p = 2 > 1$

39.  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent  $p$ -series with  $p = \frac{3}{2} > 1$

40.  $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Divergent  $p$ -series with  $p = \frac{2}{3} < 1$

41.  $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent  $p$ -series with  $p = 1.04 > 1$

42.  $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

Convergent  $p$ -series with  $p = \pi > 1$

43.  $\sum_{n=1}^{\infty} \frac{2}{n^{3/4}}$

$S_1 = 2$

$S_2 \approx 3.1892$

$S_3 \approx 4.0666$

$S_4 \approx 4.7740$

Matches (c), diverges

44.  $\sum_{n=1}^{\infty} \frac{2}{n}$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.6667$

$S_4 \approx 4.1667$

Matches (f), diverges

45.  $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^{\pi}}} = \sum_{n=1}^{\infty} \frac{2}{n^{\pi/2}}$

$S_1 = 2$

$S_2 \approx 2.6732$

$S_3 \approx 3.0293$

$S_4 \approx 3.2560$

Matches (b), converges

**Note:** The partial sums for 45 and 47 are very similar because  $\pi/2 \approx 3/2$ .

46.  $\sum_{n=1}^{\infty} \frac{2}{n^{2/5}}$

$S_1 = 2$

$S_2 \approx 3.5157$

$S_3 \approx 4.8045$

Matches (a), diverges

47.  $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$

$S_1 = 2$

$S_2 \approx 2.7071$

$S_3 \approx 3.0920$

$S_4 \approx 3.3420$

Matches (d), converges

**Note:** The partial sums for 45 and 47 are very similar because  $\pi/2 \approx 3/2$ .

48.  $\sum_{n=1}^{\infty} \frac{2}{n^2}$

$$S_1 = 2$$

$$S_2 = 2.5$$

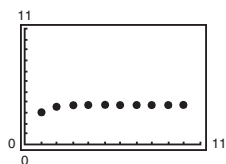
$$S_3 \approx 2.7222$$

Matches (e), converges

49. (a)

$n$	5	10	20	50	100
$S_n$	3.7488	3.75	3.75	3.75	3.75

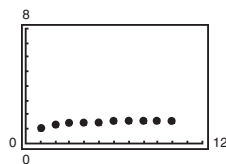
The partial sums approach the sum 3.75 very rapidly.



(b)

$n$	5	10	20	50	100
$S_n$	1.4636	1.5498	1.5962	1.6251	1.635

The partial sums approach the sum  $\pi^2/6 \approx 1.6449$  slower than the series in part (a).



50.  $\sum_{n=1}^N \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$

(a)

$M$	2	4	6	8
$N$	4	31	227	1674

(b) No. Because the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

51. Let  $f$  be positive, continuous, and decreasing for  $x \geq 1$  and  $a_n = f(n)$ . Then,

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) dx$$

either both converge or both diverge (Theorem 9.10). See Example 1, page 620.

52. A series of the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is a  $p$ -series,  $p > 0$ .

The  $p$ -series converges if  $p > 1$  and diverges if  $0 < p \leq 1$ .

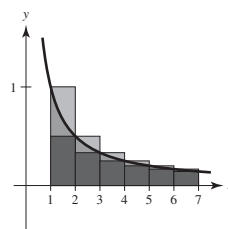
53. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000<sup>th</sup> term, and therefore diverges.

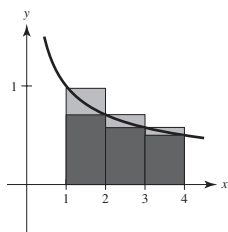
54. No. Theorem 9.9 says that if the series converges, then the terms  $a_n$  tend to zero. Some of the series in Exercises 43–48 converge because the terms tend to 0 very rapidly.

55.  $\sum_{n=1}^6 a_n \geq \int_1^7 f(x) dx \geq \sum_{n=2}^7 a_n$





56. (a)

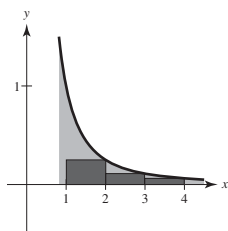


$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

The area under the rectangle is greater than the area under the curve.

Because  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^{\infty} = \infty$ , diverges,  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges.

(b)



$$\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2} dx$$

The area under the rectangles is less than the area under the curve.

Because  $\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$ , converges,

$\sum_{n=2}^{\infty} \frac{1}{n^2}$  converges (and so does  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ).

57.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If  $p = 1$ , then the series diverges by the Integral Test. If  $p \neq 1$ ,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x} dx = \left[ \frac{(\ln x)^{-p+1}}{-p+1} \right]_2^{\infty}.$$

Converges for  $-p + 1 < 0$  or  $p > 1$

58.  $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

If  $p = 1$ , then the series diverges by the Integral Test. If  $p \neq 1$ ,

$$\int_2^{\infty} \frac{\ln x}{x^p} dx = \int_2^{\infty} x^{-p} \ln x dx = \left[ \frac{x^{-p+1}}{(-p+1)^2} [-1 + (-p+1) \ln x] \right]_2^{\infty}. \text{ (Use integration by parts.)}$$

Converges for  $-p + 1 < 0$  or  $p > 1$

59.  $\sum_{n=1}^{\infty} \frac{n}{(1+n^2)^p}$

If  $p = 1$ ,  $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$  diverges (see Example 1). Let

$$f(x) = \frac{x}{(1+x^2)^p}, \quad p \neq 1$$

$$f'(x) = \frac{1 - (2p-1)x^2}{(1+x^2)^{p+1}}.$$

For a fixed  $p > 0$ ,  $p \neq 1$ ,  $f'(x)$  is eventually negative.  $f$  is positive, continuous, and eventually decreasing.

$$\int_1^{\infty} \frac{x}{(1+x^2)^p} dx = \left[ \frac{1}{(x^2+1)^{p-1}(2-2p)} \right]_1^{\infty}$$

For  $p > 1$ , this integral converges. For  $0 < p < 1$ , it diverges.

60.  $\sum_{n=1}^{\infty} n(1+n^2)^p$

Because  $p > 0$ , the series diverges for all values of  $p$ .

61.  $\sum_{n=1}^{\infty} \frac{1}{p^n} = \sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n$ , Geometric series.

Converges for  $\left|\frac{1}{p}\right| < 1 \Rightarrow p > 1$

62.  $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$

If  $p = 1$ , then

$$\int_3^{\infty} \frac{1}{x \ln x [\ln(\ln x)]} dx = [\ln(\ln(\ln x))]_3^{\infty} = \infty, \text{ so the}$$

series diverges by the Integral Test

If  $p \neq 1$ ,

$$\int_3^{\infty} \frac{1}{x \ln x [\ln(\ln x)]^p} dx = \left[ \frac{[\ln(\ln x)]^{-p+1}}{-p+1} \right]_3^{\infty}.$$

This converges for  $-p+1 < 0 \Rightarrow p > 1$ .

So, the series converges for  $p > 1$ , and diverges for  $0 < p \leq 1$ .

63.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  converges for  $p > 1$ .

So,  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ , diverges.

64.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  converges for  $p > 1$ .

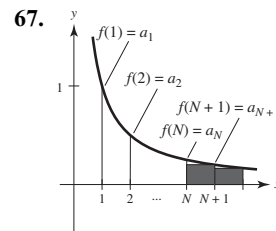
So,  $\sum_{n=2}^{\infty} \frac{1}{n^3 \sqrt{(\ln n)^2}} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$ , diverges.

65.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  converges for  $p > 1$ .

So,  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ , converges.

66.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  converges for  $p > 1$ .

So,  $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)} = \sum_{n=2}^{\infty} \frac{1}{2n \ln n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ , diverges



$$S_N = \sum_{n=1}^N a_n = a_1 + a_2 + \cdots + a_N$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n > 0$$

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \cdots \leq \int_N^{\infty} f(x) dx$$

So,  $0 \leq R_N \leq \int_N^{\infty} f(x) dx$

68. From Exercise 67, you have:

$$0 \leq S - S_N \leq \int_N^{\infty} f(x) dx$$

$$S_N \leq S \leq S_N + \int_N^{\infty} f(x) dx$$

$$\sum_{n=1}^N a_n \leq S \leq \sum_{n=1}^N a_n + \int_N^{\infty} f(x) dx$$

69.  $S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4} dx = \left[ -\frac{1}{3x^3} \right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

70.  $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \leq \int_4^\infty \frac{1}{x^5} dx = \left[ -\frac{1}{4x^4} \right]_4^\infty \approx 0.0010$$

$$1.0363 \leq \sum_{n=1}^\infty \frac{1}{n^5} \leq 1.0363 + 0.0010 = 1.0373$$

71.  $S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^2 + 1} dx = [\arctan x]_{10}^\infty = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^\infty \frac{1}{n^2 + 1} \leq 0.9818 + 0.0997 = 1.0815$$

72.  $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \leq \int_{10}^\infty \frac{1}{(x+1)[\ln(x+1)]^3} dx = \left[ -\frac{1}{2[\ln(x+1)]^2} \right]_{10}^\infty = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \leq \sum_{n=1}^\infty \frac{1}{(n+1)[\ln(n+1)]^3} \leq 1.9821 + 0.0870 = 2.0691$$

73.  $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \leq \int_4^\infty x e^{-x^2} dx = \left[ -\frac{1}{2} e^{-x^2} \right]_4^\infty = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^\infty n e^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

74.  $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \leq \int_4^\infty e^{-x} dx = [-e^{-x}]_4^\infty \approx 0.0183$$

$$0.5713 \leq \sum_{n=0}^\infty e^{-n} \leq 0.5713 + 0.0183 = 0.5896$$

75.  $0 \leq R_N \leq \int_N^\infty \frac{1}{x^4} dx = \left[ -\frac{1}{3x^3} \right]_N^\infty = \frac{1}{3N^3} < 0.001$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

76.  $0 \leq R_N \leq \int_N^\infty \frac{1}{x^{3/2}} dx = \left[ -\frac{2}{x^{1/2}} \right]_N^\infty = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \geq 4,000,000$$

77.  $R_N \leq \int_N^\infty e^{-5x} dx = \left[ -\frac{1}{5} e^{-5x} \right]_N^\infty = \frac{e^{-5N}}{5} < 0.001$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

78.  $R_N \leq \int_N^\infty e^{-x/2} dx = [-2e^{-x/2}]_N^\infty = \frac{2}{e^{N/2}} < 0.001$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2 \ln 2000 \approx 15.2$$

$$N \geq 16$$

$$79. R_N \leq \int_N^\infty \frac{1}{x^2 + 1} dx = [\arctan x]_N^\infty \\ = \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < 0.001 - \frac{\pi}{2}$$

$$\arctan N > \frac{\pi}{2} - 0.001$$

$$N > \tan\left(\frac{\pi}{2} - 0.001\right)$$

$$N \geq 1000$$

$$80. R_n \leq \int_N^\infty \frac{2}{x^2 + 5} dx = 2 \left[ \frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) \right]_N^\infty \\ = \frac{2}{\sqrt{5}} \left( \frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) \right) < 0.001$$

$$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.0005\sqrt{5}$$

$$-\arctan\left(\frac{N}{\sqrt{5}}\right) < 0.0005\sqrt{5} - \frac{\pi}{2}$$

$$\arctan\left(\frac{N}{\sqrt{5}}\right) > \frac{\pi}{2} - 0.0005\sqrt{5}$$

$$N > \sqrt{5} \tan\left(\frac{\pi}{2} - 0.0005\sqrt{5}\right)$$

$$N \geq 1000$$

81. (a)  $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ . This is a convergent  $p$ -series with  $p = 1.1 > 1$ .  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  is a divergent series. Use the Integral Test.

$f(x) = \frac{1}{x \ln x}$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^\infty \frac{1}{x \ln x} dx = [\ln|\ln x|]_2^\infty = \infty$$

$$(b) \sum_{n=2}^6 \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$$

$$\sum_{n=2}^6 \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

For  $n \geq 4$ , the terms of the convergent series **seem** to be larger than those of the divergent series.

$$(c) \frac{1}{n^{1.1}} < \frac{1}{n \ln n}$$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when  $n \geq 3.5 \times 10^{15}$ . Or,  $n > e^{40}$ . Then  $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$ .

$$82. (a) \int_{10}^\infty \frac{1}{x^p} dx = \left[ \frac{x^{-p+1}}{-p+1} \right]_{10}^\infty = \frac{1}{(p-1)10^{p-1}}, p > 1$$

$$(b) f(x) = \frac{1}{x^p}$$

$$R_{10}(p) = \sum_{n=11}^{\infty} \frac{1}{n^p}$$

$$\leq \text{Area under the graph of } f \text{ over the interval } [10, \infty)$$

(c) The horizontal asymptote is  $y = 0$ . As  $n$  increases, the error decreases.

83. (a) Let  $f(x) = 1/x$ .  $f$  is positive, continuous, and decreasing on  $[1, \infty)$ .

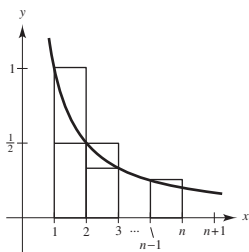
$$S_n - 1 \leq \int_1^n \frac{1}{x} dx$$

$$S_n - 1 \leq \ln n$$

So,  $S_n \leq 1 + \ln n$ . Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

So,  $\ln(n+1) \leq S_n \leq 1 + \ln n$ .



- (b) Because  $\ln(n+1) \leq S_n \leq 1 + \ln n$ , you have  $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$ . Also, because  $\ln x$  is an increasing function,  $\ln(n+1) - \ln n > 0$  for  $n \geq 1$ . So,  $0 \leq S_n - \ln n \leq 1$  and the sequence  $\{a_n\}$  is bounded.

$$(c) a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0$$

So,  $a_n \geq a_{n+1}$  and the sequence is decreasing.

- (d) Because the sequence is bounded and monotonic, it converges to a limit,  $\gamma$ .

$$(e) a_{100} = S_{100} - \ln 100 \approx 0.5822 \text{ (Actually } \gamma \approx 0.577216\text{.)}$$

$$\begin{aligned} 84. \sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) &= \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^{\infty} [\ln(n+1) + \ln(n-1) - 2\ln n] \\ &= (\ln 3 + \ln 1 - 2\ln 2) + (\ln 4 + \ln 2 - 2\ln 3) + (\ln 5 + \ln 3 - 2\ln 4) + (\ln 6 + \ln 4 - 2\ln 5) \\ &\quad + (\ln 7 + \ln 5 - 2\ln 6) + (\ln 8 + \ln 6 - 2\ln 7) + (\ln 9 + \ln 7 - 2\ln 8) + \cdots = -\ln 2 \end{aligned}$$

$$85. \sum_{n=2}^{\infty} x^{\ln n}$$

$$(a) x = 1: \sum_{n=2}^{\infty} 1^{\ln n} = \sum_{n=2}^{\infty} 1, \text{ diverges}$$

$$(b) x = \frac{1}{e}: \sum_{n=2}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum_{n=2}^{\infty} e^{-\ln n} = \sum_{n=2}^{\infty} \frac{1}{n}, \text{ diverges}$$

$$(c) \text{ Let } x \text{ be given, } x > 0. \text{ Put } x = e^{-p} \Leftrightarrow \ln x = -p.$$

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} e^{-p \ln n} = \sum_{n=2}^{\infty} n^{-p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$$

This series converges for  $p > 1 \Rightarrow x < \frac{1}{e}$ .

$$86. \xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

Converges for  $x > 1$  by Theorem 9.11

87. Let  $f(x) = \frac{1}{3x-2}$ ,  $f'(x) = \frac{-3}{(3x-2)^2} < 0$  for  $x \geq 1$

$f$  is positive, continuous, and decreasing for  $x \geq 1$ .

$$\int_1^{\infty} \frac{1}{3x-2} dx = \left[ \frac{1}{3} \ln|3x-2| \right]_1^{\infty} = \infty$$

So, the series  $\sum_{n=1}^{\infty} \frac{1}{3n-2}$

diverges by Theorem 9.10.

88.  $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Let  $f(x) = \frac{1}{x\sqrt{x^2-1}}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx = [\operatorname{arccsc} x]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 9.10

89.  $\sum_{n=1}^{\infty} \frac{1}{n^4\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

$p$ -series with  $p = \frac{5}{4}$

Converges by Theorem 9.11

90.  $3 \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$

$p$ -series with  $p = 0.95$

Diverges by Theorem 9.11

91.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with  $r = \frac{2}{3}$

Converges by Theorem 9.6

92.  $\sum_{n=0}^{\infty} (1.042)^n$  is geometric with  $r = 1.042 > 1$ . Diverges by Theorem 9.6.

93.  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 9.9

94.  $\sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$

Because these are both convergent  $p$ -series, the difference is convergent.

95.  $\sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^n$

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e \neq 0$$

Fails  $n$ th-Term Test

Diverges by Theorem 9.9

96.  $\sum_{n=2}^{\infty} \ln(n)$

$$\lim_{n \rightarrow \infty} \ln(n) = \infty$$

Diverges by Theorem 9.9

97.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let  $f(x) = \frac{1}{x(\ln x)^3}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 2$ .

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^3} dx &= \int_2^{\infty} (\ln x)^{-3} \frac{1}{x} dx \\ &= \left[ \frac{(\ln x)^{-2}}{-2} \right]_2^{\infty} \\ &= \left[ -\frac{1}{2(\ln x)^2} \right]_2^{\infty} = \frac{1}{2(\ln 2)^2} \end{aligned}$$

Converges by Theorem 9.10. See Exercise 57.

98.  $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

Let  $f(x) = \frac{\ln x}{x^3}$ .

$f$  is positive, continuous, and decreasing for  $x \geq 2$  since

$$f'(x) = \frac{1-3 \ln x}{x^4} < 0 \text{ for } x \geq 2.$$

$$\begin{aligned} \int_2^{\infty} \frac{\ln x}{x^3} dx &= \left[ -\frac{\ln x}{2x^2} \right]_2^{\infty} + \frac{1}{2} \int_2^{\infty} \frac{1}{x^3} dx \\ &= \frac{\ln 2}{8} + \left[ -\frac{1}{4x^2} \right]_2^{\infty} \\ &= \frac{\ln 2}{8} + \frac{1}{16} \text{ (Use integration by parts.)} \end{aligned}$$

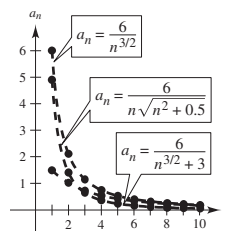
Converges by Theorem 9.10. See Exercise 36.

## Section 9.4 Comparisons of Series

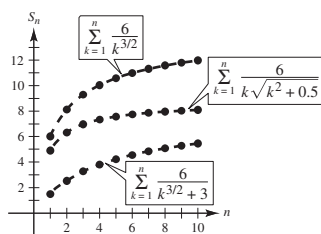
1. (a)  $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots; S_1 = 6$

$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots; S_1 = \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots; S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$$



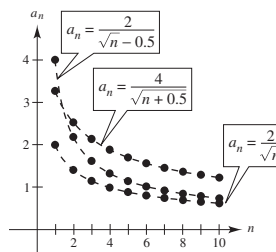
- (b) The first series is a  $p$ -series. It converges  $\left(p = \frac{3}{2} > 1\right)$ .
- (c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent  $p$ -series. So, the other two series converge.
- (d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



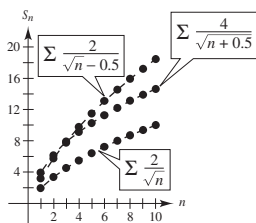
2. (a)  $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots; S_1 = 2$

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots; S_1 = 4$$

$$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots; S_1 \approx 3.3$$



- (b) The first series is a  $p$ -series. It diverges  $\left(p = \frac{1}{2} < 1\right)$ .
- (c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent  $p$ -series. So, the other two series diverge.
- (d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



3.  $0 < \frac{1}{n^2 + 1} < \frac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

4.  $\frac{1}{3n^2 + 2} < \frac{1}{3n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent  $p$ -series

$$\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5.  $\frac{1}{2n-1} > \frac{1}{2n} > 0$  for  $n \geq 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

diverges by comparison with the divergent  $p$ -series  $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ .

6.  $\frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}}$  for  $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

diverges by comparison with the divergent  $p$ -series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

7.  $\frac{1}{4^n + 1} < \frac{1}{4^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{4^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \frac{1}{4^n}.$$

8.  $\frac{4^n}{5^n + 3} < \left(\frac{4}{5}\right)^n$

Therefore,

$$\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n.$$

9. For  $n \geq 3$ ,  $\frac{\ln n}{n+1} > \frac{1}{n+1} > 0$ .

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

**Note:**  $\sum_{n=1}^{\infty} \frac{1}{n+1}$  diverges by the Integral Test.

10.  $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

11. For  $n > 3$ ,  $\frac{1}{n^2} > \frac{1}{n!} > 0$ .

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

12.  $\frac{1}{4\sqrt[3]{n-1}} > \frac{1}{4\sqrt[4]{n}}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n-1}}$$

diverges by comparison with the divergent  $p$ -series

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

13.  $0 < \frac{1}{e^{n^2}} \leq \frac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

14.  $\frac{3^n}{2^n - 1} > \left(\frac{3}{2}\right)^n$  for  $n \geq 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n.$$



$$15. \lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$16. \lim_{n \rightarrow \infty} \frac{5/(4^n + 1)}{1/4^n} = \lim_{n \rightarrow \infty} \frac{5 \cdot 4^n}{4^n + 1} = 5$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{4^n + 1}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n.$$

$$17. \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$18. \lim_{n \rightarrow \infty} \frac{(2^n + 1)/(5^n + 1)}{(2/5)^n} = \lim_{n \rightarrow \infty} \frac{2^n + 1}{5^n + 1} \cdot \frac{5^n}{2^n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2^n + 1}{5^n + 1}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n.$$

$$19. \lim_{n \rightarrow \infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} = \frac{2}{3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$20. \lim_{n \rightarrow \infty} \frac{(n + 5)/(n^3 - 2n + 3)}{1/n^2} = \lim_{n \rightarrow \infty} \left( \frac{n + 5}{n^3 - 2n + 3} \right) \left( \frac{n^2}{1} \right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n + 5}{n^3 - 2n + 3}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$21. \lim_{n \rightarrow \infty} \frac{(n + 3)/n(n^2 + 4)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{(n + 3)n^2}{n(n^2 + 4)} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n + 3}{n(n^2 + 4)}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$22. \lim_{n \rightarrow \infty} \frac{1/n^2(n + 3)}{1/n^3} = \lim_{n \rightarrow \infty} \frac{n^3}{n^2(n + 3)} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n + 3)}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$23. \lim_{n \rightarrow \infty} \frac{1/(n\sqrt{n^2 + 1})}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$$

converges by a limit comparison with the convergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$24. \lim_{n \rightarrow \infty} \frac{n / \left[ (n+1)2^{n-1} \right]}{1/(2^{n-1})} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n-1}.$$

$$25. \lim_{n \rightarrow \infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^k}{n^k + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$26. \lim_{n \rightarrow \infty} \frac{5/(n + \sqrt{n^2 + 4})}{1/n} = \lim_{n \rightarrow \infty} \frac{5n}{n + \sqrt{n^2 + 4}} = \frac{5}{2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{n + \sqrt{n^2 + 4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$27. \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2} = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$34. \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots = \frac{1}{2}$$

Converges; telescoping series

$$35. \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

Converges; Integral Test

$$28. \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2)\sec^2(1/n)}{-1/n^2} \\ = \lim_{n \rightarrow \infty} \sec^2\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$29. \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

Diverges;

$p$ -series with  $p = \frac{2}{3}$

$$30. \sum_{n=0}^{\infty} 7\left(-\frac{1}{7}\right)^n$$

Converges;

Geometric series with  $r = -\frac{1}{7}$

$$31. \sum_{n=1}^{\infty} \frac{1}{5^n + 1}$$

Converges;

Direct comparison with convergent geometric series

$$\sum_{n=1}^{\infty} \left( \frac{1}{5} \right)^n$$

$$32. \sum_{n=3}^{\infty} \frac{1}{n^3 - 8}$$

Converges; limit comparison with  $\sum_{n=3}^{\infty} \frac{1}{n^3}$

$$33. \sum_{n=1}^{\infty} \frac{2n}{3n - 2}$$

Diverges;  $n^{\text{th}}$ -Term Test

$$\lim_{n \rightarrow \infty} \frac{2n}{3n - 2} = \frac{2}{3} \neq 0$$

36.  $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+3} \right)$$

37.  $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} na_n$ . By given conditions  $\lim_{n \rightarrow \infty} na_n$  is

finite and nonzero. Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

38. If  $j < k - 1$ , then  $k - j > 1$ . The  $p$ -series with  $p = k - j$  converges and because

$$\lim_{n \rightarrow \infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0, \text{ the series } \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges by the Limit Comparison Test. Similarly, if  $j \geq k - 1$ , then  $k - j \leq 1$  which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the Limit Comparison Test.

39.  $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ ,

which diverges because the degree of the numerator is only one less than the degree of the denominator.

40.  $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ ,

which converges because the degree of the numerator is two less than the degree of the denominator.

41.  $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges because the degree of the numerator is three less than the degree of the denominator.

42.  $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

diverges because the degree of the numerator is only one less than the degree of the denominator.

43.  $\lim_{n \rightarrow \infty} n \left( \frac{n^3}{5n^4 + 3} \right) = \lim_{n \rightarrow \infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore,  $\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$  diverges.

44.  $\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty \neq 0$

Therefore,  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$  diverges.

45.  $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$

diverges, (harmonic)

46.  $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n}$

diverges

47.  $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$

converges

48.  $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3}$

converges

49. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

50. See Theorem 9.12, page 626. One example is

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \text{ converges because } \frac{1}{n^2 + 1} < \frac{1}{n^2} \text{ and}$$

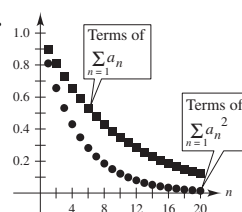
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series).}$$

51. See Theorem 9.13, page 628. One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n \rightarrow \infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1 \text{ and}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (} p\text{-series).}$$

52.



For  $0 < a_n < 1$ ,  $0 < a_n^2 < a_n < 1$ . So, the lower terms are those of  $\sum a_n^2$ .

53. (a)  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges because the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

$n$	5	10	20	50	100
$S_n$	1.1839	1.2087	1.2212	1.2287	1.2312

(c)  $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d)  $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

54. This is not correct. The beginning terms do not affect the convergence or divergence of a series. In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

$$\text{and } 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (p-series).}$$

55. False. Let  $a_n = \frac{1}{n^3}$  and  $b_n = \frac{1}{n^2}$ .  $0 < a_n \leq b_n$  and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converge.}$$

56. True

57. True

58. False. Let  $a_n = 1/n$ ,  $b_n = 1/n$ ,  $c_n = 1/n^2$ . Then,

$$a_n \leq b_n + c_n, \text{ but } \sum_{n=1}^{\infty} c_n \text{ converges.}$$

59. True

60. False.  $\sum_{n=1}^{\infty} a_n$  could converge or diverge.

For example, let  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ , which diverges.

$$0 < \frac{1}{n} < \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, but}$$

$$0 < \frac{1}{n^2} < \frac{1}{\sqrt{n}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

61. Because  $\sum_{n=1}^{\infty} b_n$  converges,  $\lim_{n \rightarrow \infty} b_n = 0$ . There exists  $N$

such that  $b_n < 1$  for  $n > N$ . So,  $a_n b_n < a_n$  for

$n > N$  and  $\sum_{n=1}^{\infty} a_n b_n$  converges by comparison to the

convergent series  $\sum_{i=1}^{\infty} a_n$ .

62. Because  $\sum_{n=1}^{\infty} a_n$  converges, then

$$\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2 \text{ converges by Exercise 61.}$$

63.  $\sum \frac{1}{n^2}$  and  $\sum \frac{1}{n^3}$  both converge, and therefore, so does

$$\sum \left( \frac{1}{n^2} \right) \left( \frac{1}{n^3} \right) = \sum \frac{1}{n^5}.$$

64.  $\sum \frac{1}{n^2}$  converge, and therefore, so does

$$\sum \left( \frac{1}{n^2} \right)^2 = \sum \frac{1}{n^4}.$$

65. Suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$  and  $\sum b_n$  converges.

From the definition of limit of a sequence, there exists  $M > 0$  such that

$$\left| \frac{a_n}{b_n} - 0 \right| < 1$$

whenever  $n > M$ . So,  $a_n < b_n$  for  $n > M$ . From the Comparison Test,  $\sum a_n$  converges.

66. Suppose  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$  and  $\sum b_n$  diverges. From the

definition of limit of a sequence, there exists  $M > 0$  such that

$$\frac{a_n}{b_n} > 1$$

for  $n > M$ . So,  $a_n > b_n$  for  $n > M$ . By the Comparison Test,  $\sum a_n$  diverges.

67. (a) Let  $\sum a_n = \sum \frac{1}{(n+1)^3}$ , and  $\sum b_n = \sum \frac{1}{n^2}$ ,

converges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/\left[\frac{1}{(n+1)^3}\right]}{1/(n^2)} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^3} = 0$$

By Exercise 65,  $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$  converges.

(b) Let  $\sum a_n = \sum \frac{1}{\sqrt{n}\pi^n}$ , and  $\sum b_n = \sum \frac{1}{\pi^n}$ ,  
converges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/(\sqrt{n}\pi^n)}{1/(\pi^n)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

By Exercise 65,  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\pi^n}$  converges.

68. (a) Let  $\sum a_n = \sum \frac{\ln n}{n}$ , and  $\sum b_n = \sum \frac{1}{n}$ , diverges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(\ln n)/n}{1/n} = \lim_{n \rightarrow \infty} \ln n = \infty$$

By Exercise 66,  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  diverges.

(b) Let  $\sum a_n = \sum \frac{1}{\ln n}$ , and  $\sum b_n = \sum \frac{1}{n}$ , diverges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \infty$$

By Exercise 66,  $\sum \frac{1}{\ln n}$  diverges.

69. Because  $\lim_{n \rightarrow \infty} a_n = 0$ , the terms of  $\sum \sin(a_n)$  are positive for sufficiently large  $n$ . Because

$$\lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1 \text{ and } \sum a_n$$

converges, so does  $\sum \sin(a_n)$ .

$$\begin{aligned} 70. \sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+n} &= \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]/2} \\ &= \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \end{aligned}$$

Because  $\sum 1/n^2$  converges, and

$$\lim_{n \rightarrow \infty} \frac{2/[n(n+1)]}{1/(n^2)} = \lim_{n \rightarrow \infty} \frac{2n^2}{n(n+1)} = 2,$$

$\sum \frac{1}{1+2+\cdots+n}$  converges.

71. First note that  $f(x) = \ln x - x^{1/4} = 0$  when  $x \approx 5503.66$ . That is,

$$\ln n < n^{1/4} \text{ for } n > 5504$$

which implies that

$$\frac{\ln n}{n^{3/2}} < \frac{1}{n^{5/4}} \text{ for } n > 5504.$$

Because  $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$  is a convergent  $p$ -series,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$$

converges by direct comparison.

72. The series diverges. For  $n > 1$ ,

$$n < 2^n$$

$$n^{1/n} < 2$$

$$\frac{1}{n^{1/n}} > \frac{1}{2}$$

$$\frac{1}{n^{(n+1)/n}} > \frac{1}{2n}$$

Because  $\sum \frac{1}{2n}$  diverges, so does  $\sum \frac{1}{n^{(n+1)/n}}$ .

73. Consider two cases:

If  $a_n \geq \frac{1}{2^{n+1}}$ , then  $a_n^{1/(n+1)} \geq \left(\frac{1}{2^{n+1}}\right)^{1/(n+1)} = \frac{1}{2}$ , and

$$a_n^{n/(n+1)} = \frac{a_n}{a_n^{1/(n+1)}} \leq 2a_n.$$

If  $a_n \leq \frac{1}{2^{n+1}}$ , then  $a_n^{n/(n+1)} \leq \left(\frac{1}{2^{n+1}}\right)^{n/(n+1)} = \frac{1}{2^n}$ , and

$$\text{combining, } a_n^{n/(n+1)} \leq 2a_n + \frac{1}{2^n}.$$

Because  $\sum_{n=1}^{\infty} \left(2a_n + \frac{1}{2^n}\right)$  converges, so does  $\sum_{n=1}^{\infty} a_n^{n/(n+1)}$  by the Comparison Test.

## Section 9.5 Alternating Series

1. 
$$\sum_{n=1}^{\infty} \frac{6}{n^2}$$

$S_1 = 6$

$S_2 = 7.5$

$S_3 \approx 8.1667$

Matches (d).

2. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 6}{n^2}$$

$S_1 = 6$

$S_2 = 4.4$

$S_3 \approx 5.1667$

Matches (f).

3. 
$$\sum_{n=1}^{\infty} \frac{3}{n!}$$

$S_1 = 3$

$S_2 = 4.5$

$S_3 = 5.0$

Matches (a).

4. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3}{n!}$$

$S_1 = 3$

$S_2 = 1.5$

$S_3 = 2.0$

Matches (b).

5. 
$$\sum_{n=1}^{\infty} \frac{10}{n2^n}$$

$S_1 = 5$

$S_2 = 6.25$

Matches (e).

6. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n 10}{n2^n}$$

$S_1 = 5$

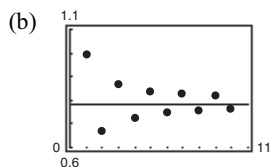
$S_2 = 3.75$

Matches (c).

7. 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$$

(a)

$n$	1	2	3	4	5	6	7	8	9	10
$S_n$	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605



(c) The points alternate sides of the horizontal line  $y = \frac{\pi}{4}$  that represents the sum of the series.

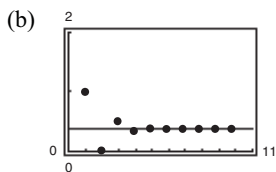
The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

8.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$

(a)

$n$	1	2	3	4	5	6	7	8	9	10
$S_n$	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679



(c) The points alternate sides of the horizontal line  $y = \frac{1}{e}$  that represents the sum of the series.

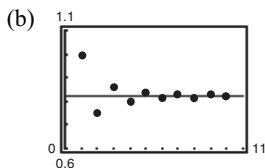
The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

9.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$

(a)

$n$	1	2	3	4	5	6	7	8	9	10
$S_n$	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180



(c) The points alternate sides of the horizontal line  $y = \frac{\pi^2}{12}$  that represents the sum of the series.

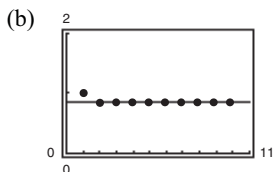
The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

10.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$

(a)

$n$	1	2	3	4	5	6	7	8	9	10
$S_n$	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415



(c) The points alternate sides of the horizontal line  $y = \sin(1)$  that represents the sum of the series.

The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$11. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$$a_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

Converges by Theorem 9.14

$$12. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$$

$$\lim_{n \rightarrow \infty} \frac{n}{3n+2} = \frac{1}{3}$$

Diverges by  $n$ th-Term test

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$$

$$a_{n+1} = \frac{1}{3^{n+1}} < \frac{1}{3^n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{3^n} = 0$$

Converges by Theorem 9.14

(Note:  $\sum_{n=1}^{\infty} \left(\frac{-1}{3}\right)^n$  is a convergent geometric series)

$$14. \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$$

$$a_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

Converges by Theorem 9.14

(Note:  $\sum_{n=1}^{\infty} \left(\frac{-1}{e}\right)^n$  is a convergent geometric series)

$$15. \sum_{n=1}^{\infty} \frac{(-1)^n (5n-1)}{4n+1}$$

$$\lim_{n \rightarrow \infty} \frac{5n-1}{4n+1} = \frac{5}{4}$$

Diverges by  $n$ th-Term test

$$16. \sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\ln(n+1)} = \infty$$

Diverges by  $n$ th-Term test

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+2}$$

$$a_{n+1} = \frac{1}{3(n+1)+2} < \frac{1}{3n+2} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{3n+2} = 0$$

Converges by Theorem 9.14

$$18. \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

Converges by Theorem 9.14

$$19. \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+5} = 1$$

Diverges by  $n$ th-Term test

$$20. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$$

$$\text{Let } f(x) = \frac{x}{x^2+5}, f'(x) = \frac{-(x^2-5)}{(x^2+5)^2} < 0 \text{ for } x \geq 3$$

So,  $a_{n+1} < a_n$  for  $n \geq 3$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2+5} = 0$$

Converges by Theorem 9.14

$$21. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 9.14

$$22. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+4} = 1$$

Diverges by  $n$ th-Term test



$$23. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{1/(n+1)} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

Diverges by the  $n$ th-Term Test

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n+1} = \lim_{n \rightarrow \infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 9.14

$$25. \sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Diverges by the  $n$ th-Term Test

$$26. \sum_{n=1}^{\infty} \frac{1}{n} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 9.14

$$27. \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$$

Diverges by the  $n$ th-Term Test

$$28. \sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 9.14

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$a_{n+1} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n+1}{2n+1} = a_n \left( \frac{n+1}{2n+1} \right) < a_n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \lim_{n \rightarrow \infty} 2 \left[ \frac{3}{3} \cdot \frac{4}{5} \cdot \frac{5}{7} \cdots \frac{n}{2n-3} \right] \cdot \frac{1}{2n-1} = 0$$

Converges by Theorem 9.14

$$29. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Converges by Theorem 9.14

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 9.14

$$31. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 9.14

$$32. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{\sqrt[3]{n}}$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n \rightarrow \infty} n^{1/6} = \infty$$

Diverges by the  $n$ th-Term Test

$$34. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$$

$$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 4 \cdot 7 \cdots (3n-2)(3n+1)} = a_n \left( \frac{2n+1}{3n+1} \right) < a_n$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 3 \left[ \frac{5}{4} \cdot \frac{7}{7} \cdot \frac{9}{10} \cdots \frac{2n-1}{3n-2} \right] \frac{1}{3n-2} = 0$$

Converges by Theorem 9.14

$$35. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2e^n)}{e^{2n} - 1}$$

Let  $f(x) = \frac{2e^x}{e^{2x} - 1}$ . Then

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

So,  $f(x)$  is decreasing. Therefore,  $a_{n+1} < a_n$ , and

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 9.14.

$$36. \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2e^n)}{e^{2n} + 1}$$

Let  $f(x) = \frac{2e^x}{e^{2x} + 1}$ . Then

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

So,  $f(x)$  is decreasing for  $x > 0$  which implies

$$a_{n+1} < a_n.$$

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 9.14.

$$37. S_6 = \sum_{n=0}^5 \frac{2(-1)^n}{n!} \approx 0.7333$$

$$0.7333 - 0.002778 \leq S \leq 0.7333 + 0.002778$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778$$

$$0.7305 \leq S \leq 0.7361$$

$$38. S_6 = \sum_{n=1}^6 \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236$$

$$0.7831 \leq S \leq 4.6303$$

$$39. S_6 = \sum_{n=1}^6 \frac{3(-1)^{n+1}}{n^2} = 2.4325$$

$$2.4325 - 0.0612 \leq S \leq 2.4325 + 0.0612$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612$$

$$2.3713 \leq S \leq 2.4937$$

$$40. S_6 = \sum_{n=1}^6 \frac{(-1)^{n+1} n}{2^n} = 0.1875$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469$$

$$0.1328 \leq S \leq 0.2422$$

$$41. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when  $N = 6$ .

(b) Approximate the series by

$$\sum_{n=0}^6 \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

$$\approx 0.368.$$

(7 terms. Note that the sum begins with  $n = 0$ .)

$$42. \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when  $N = 4$ .

(b) Approximate the series by

$$\sum_{n=0}^4 \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with  $n = 0$ .)

$$43. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when  $N = 2$ .

(b) Approximate the series by

$$\sum_{n=0}^2 \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with  $n = 0$ .)

$$44. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when  $N = 3$ .

(b) Approximate the series by

$$\sum_{n=0}^3 \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with  $n = 0$ .)

$$45. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when  $N = 1000$ .

(b) Approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \approx 0.693.$$

(1000 terms)

$$46. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$$

(a) By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when  $N = 3$ .

(b) Approximate the series by

$$\sum_{n=1}^3 \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

$$47. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)^3} < 0.001$$

$$\Rightarrow (N+1)^3 > 1000 \Rightarrow N+1 > 10.$$

Use 10 terms.

$$48. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)^2} < 0.001$$

$$\Rightarrow (N+1)^2 > 1000 \Rightarrow N = 31.$$

Use 31 terms.

$$49. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when  $N = 7$ . Use 7 terms.

$$50. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$$

By Theorem 9.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)^5} < 0.01$$

This inequality is valid when  $N = 2$ .

Use 2 terms.

$$51. \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is a convergent geometric series.}$$

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$  converges absolutely.

$$52. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a convergent } p\text{-series.}$$

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  converges absolutely.

$$53. \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

$$\frac{1}{n!} < \frac{1}{n^2} \text{ for } n \geq 4$$

and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is a convergent  $p$ -series.

So,  $\sum_{n=1}^{\infty} \frac{1}{n!}$  converges, and

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$  converges absolutely.

$$54. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n^3}$$

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$  is a convergent  $p$ -series. Therefore,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n^3}$  converges absolutely.

$$55. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+3)^2}$$

$\sum_{n=1}^{\infty} \frac{1}{(n+3)^2}$  converges by a limit comparison to the

$p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+3)^2}$  converges

absolutely.

$$56. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$$

The series converges by the Alternating Series Test. But, the series

$$\sum_{n=1}^{\infty} \frac{1}{n+3}$$

diverges by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$  converges conditionally.

$$57. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

The given series converges by the Alternating Series Test, but does not converge absolutely because

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent  $p$ -series. Therefore, the series converges conditionally.

$$58. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  which is a convergent  $p$ -series.

Therefore, the given series converges absolutely.

$$59. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$60. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n+3)}{n+10}$$

$$\lim_{n \rightarrow \infty} \frac{2n+3}{n+10} = 2$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$61. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

The series converges by the Alternating Series Test.

$$\text{Let } f(x) = \frac{1}{x \ln x}.$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = [\ln(\ln x)]_2^{\infty} = \infty$$

By the Integral Test,  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  diverges.

So, the series  $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$  converges conditionally.

$$62. \sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$$

$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$  converges by a comparison to the convergent

geometric series  $\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ . Therefore, the given series converges absolutely.

$$63. \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 5}$$

$\sum_{n=2}^{\infty} \frac{n}{n^3 - 5}$  converges by a limit comparison to the

$p$ -series  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ . Therefore, the given series converges absolutely.

$$64. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$  is a convergent  $p$ -series. Therefore, the given series converges absolutely.

$$65. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

because

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

$$66. \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

$$67. \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1, \text{ therefore, the series}$$

converges conditionally.

$$68. \sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$$

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$69. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a convergent } p\text{-series.}$$

Therefore, the given series converges absolutely.

$$70. \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left| \frac{\sin[(2n-1)\pi/2]}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent  $p$ -series. Therefore, the series converges conditionally.

71. An alternating series is a series whose terms alternate in sign.

72. See Theorem 9.14.

$$73. |S - S_N| = |R_N| \leq a_{N+1} \quad (\text{Theorem 9.15})$$

74.  $\sum a_n$  is absolutely convergent if  $\sum |a_n|$  converges.

$\sum a_n$  is conditionally convergent if  $\sum |a_n|$  diverges, but  $\sum a_n$  converges.

75. (b). The partial sums alternate above and below the horizontal line representing the sum.

76. (a) False. For example, let  $a_n = \frac{(-1)^n}{n}$ .

$$\text{Then } \sum a_n = \sum \frac{(-1)^n}{n} \text{ converges}$$

$$\text{and } \sum (-a_n) = \sum \frac{(-1)^{n+1}}{n} \text{ converges.}$$

$$\text{But, } \sum |a_n| = \sum \frac{1}{n} \text{ diverges.}$$

(b) True. For if  $\sum |a_n|$  converged, then so would  $\sum a_n$  by Theorem 9.16.

$$77. \text{ True. } S_{100} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100}$$

Because the next term  $-\frac{1}{101}$  is negative,  $S_{100}$  is an overestimate of the sum.

78. False. Let

$$\sum a_n = \sum b_n = \sum \frac{(-1)^n}{\sqrt{n}}.$$

Then both converge by the Alternating Series Test. But,

$$\sum a_n b_n = \sum \frac{1}{n}, \text{ which diverges.}$$

79.  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If  $p = 0$ , then  $\sum_{n=1}^{\infty} (-1)^n$  diverges.

If  $p < 0$ , then  $\sum_{n=1}^{\infty} (-1)^n n^{-p}$  diverges.

If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$  and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for  $p > 0$ .

80.  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+p}$

Assume that  $n+p \neq 0$  so that  $a_n = 1/(n+p)$  are defined for all  $n$ . For all  $p$ ,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n+p} = 0$$

$$a_{n+1} = \frac{1}{n+1+p} < \frac{1}{n+p} = a_n.$$

Therefore, the series converges for all  $p$ .

81. Because

$$\sum_{n=1}^{\infty} |a_n|$$

converges you have  $\lim_{n \rightarrow \infty} |a_n| = 0$ . So, there must exist

an  $N > 0$  such that  $|a_n| < 1$  for all  $n > N$  and it

follows that  $a_n^2 \leq |a_n|$  for all  $n > N$ . So, by the

Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let  $a_n = 1/n$  to see that the converse is false.

82.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converges, but  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

83.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, and so does  $\sum_{n=1}^{\infty} \frac{1}{n^4}$ .

84. (a)  $\sum_{n=1}^{\infty} \frac{x^n}{n}$

converges absolutely (by comparison) for  $-1 < x < 1$ , because

$$\left| \frac{x^n}{n} \right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for  $-1 < x < 1$ .

(b) When  $x = -1$ , you have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When  $x = 1$ , you have the divergent harmonic series  $1/n$ . Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ converges conditionally for } x = -1.$$

85. (a) No, the series does not satisfy  $a_{n+1} \leq a_n$  for all  $n$ .

For example,  $\frac{1}{9} < \frac{1}{8}$ .

(b) Yes, the series converges.

$$\begin{aligned} S_{2n} &= \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} \\ &= \left( \frac{1}{2} + \cdots + \frac{1}{2^n} \right) - \left( \frac{1}{3} + \cdots + \frac{1}{3^n} \right) \\ &= \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \right) - \left( 1 + \frac{1}{3} + \cdots + \frac{1}{3^n} \right) \end{aligned}$$

As  $n \rightarrow \infty$ ,

$$S_{2n} \rightarrow \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} = 2 - \frac{3}{2} = \frac{1}{2}.$$

86. (a) No, the series does not satisfy  $a_{n+1} \leq a_n$ :

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} a_n &= 1 - \frac{1}{8} + \frac{1}{\sqrt{3}} - \frac{1}{64} + \cdots \text{ and} \\ \frac{1}{8} &< \frac{1}{\sqrt{3}}. \end{aligned}$$

(b) No, the series diverges because  $\sum \frac{1}{\sqrt{n}}$  diverges.

87.  $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$

convergent  $p$ -series

88.  $\sum_{n=1}^{\infty} \frac{3}{n^2 + 5}$

converges by limit comparison to convergent  $p$ -series

$$\sum \frac{1}{n^2}.$$

89. Diverges by  $n$ th-Term Test

$$\lim_{n \rightarrow \infty} a_n = \infty$$

90. Converges by limit comparison to convergent geometric

$$\text{series } \sum \frac{1}{2^n}.$$

91. Convergent geometric series

$$(r = \frac{7}{8} < 1)$$

92. Diverges by  $n$ th-Term Test

$$\lim_{n \rightarrow \infty} a_n = \frac{3}{2}$$

93. Convergent geometric series ( $r = 1/\sqrt{e}$ ) or Integral

Test

$$99. s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$$

$$(i) s_{4n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{4n-1} - \frac{1}{4n}$$

$$\frac{1}{2}s_{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots + \frac{1}{4n-2} - \frac{1}{4n}$$

$$\text{Adding: } s_{4n} + \frac{1}{2}s_{2n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = s_{3n}$$

$$(ii) \lim_{n \rightarrow \infty} s_n = s \quad (\text{In fact, } s = \ln 2.)$$

$$s \neq 0 \text{ because } s > \frac{1}{2}.$$

$$S = \lim_{n \rightarrow \infty} S_{3n} = s_{4n} + \frac{1}{2}s_{2n} = s + \frac{1}{2}s = \frac{3}{2}s$$

So,  $S \neq s$ .

94. Converges (conditionally) by Alternating Series Test

95. Converges (absolutely) by Alternating Series Test

96. Diverges by comparison to Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \geq 3$$

97. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

98. Rearranging the terms of an alternating series can change the sum.

## Section 9.6 The Ratio and Root Tests

$$1. \frac{(n+1)!}{(n+2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = (n+1)(n)(n-1)$$

$$2. \frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$$

3. Use the Principle of Mathematical Induction. When  $k = 1$ , the formula is valid because  $1 = \frac{(2(1))!}{2^1 \cdot 1!}$ . Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) = \frac{(2n + 2)!}{2^{n+1}(n + 1)!}$$

To do this, note that:

$$\begin{aligned} 1 \cdot 3 \cdot 5 \cdots (2n - 1)(2n + 1) &= [1 \cdot 3 \cdot 5 \cdots (2n - 1)](2n + 1) \\ &= \frac{(2n)!}{2^n n!} \cdot (2n + 1) \quad (\text{Induction hypothesis}) \\ &= \frac{(2n)(2n + 1)}{2^n n!} \cdot \frac{(2n + 2)}{2(n + 1)} \\ &= \frac{(2n)(2n + 1)(2n + 2)}{2^{n+1} n!(n + 1)} \\ &= \frac{(2n + 2)!}{2^{n+1}(n + 1)!} \end{aligned}$$

The formula is valid for all  $n \geq 1$ .

4. Use the Principle of Mathematical Induction. When  $k = 3$ , the formula is valid because  $\frac{1}{1} = \frac{2^3 3!(3)(5)}{6!} = 1$ . Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} = \frac{2^n n!(2n - 3)(2n - 1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} = \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!}$$

To do this, note that:

$$\begin{aligned} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)(2n - 3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 5)} \cdot \frac{1}{(2n - 3)} \\ &= \frac{2^n n!(2n - 3)(2n - 1)}{(2n)!} \cdot \frac{1}{(2n - 3)} \\ &= \frac{2^n n!(2n - 1)}{(2n)!} \cdot \frac{(2n + 1)(2n + 2)}{(2n + 1)(2n + 2)} \\ &= \frac{2^n (2)(n + 1)n!(2n - 1)(2n + 1)}{(2n)!(2n + 1)(2n + 2)} \\ &= \frac{2^{n+1}(n + 1)!(2n - 1)(2n + 1)}{(2n + 2)!} \end{aligned}$$

The formula is valid for all  $n \geq 3$ .

$$5. \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots$$

$$S_1 = \frac{3}{4}, S_2 \approx 1.875$$

Matches (d).

$$6. \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16}\left(\frac{1}{2}\right) + \cdots$$

$$S_1 = \frac{3}{4}, S_2 \approx 1.03$$

Matches (c).



$$7. \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \dots$$

$$S_1 = 9$$

Matches (f).

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \dots$$

$$S_1 = 2$$

Matches (b).

$$9. \sum_{n=1}^{\infty} \left( \frac{4n}{5n-3} \right)^n = \frac{4}{2} + \left( \frac{8}{7} \right)^2 + \dots$$

$$S_1 = 2, S_2 = 3.31$$

Matches (a).

$$10. \sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \dots$$

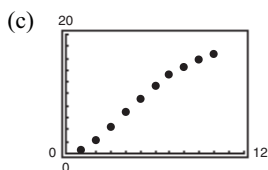
$$S_1 = 4$$

Matches (e).

$$11. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2 (5/8)^{n+1}}{n^2 (5/8)^n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^2 \frac{5}{8} = \frac{5}{8} < 1. \text{ Converges}$$

(b)

$n$	5	10	15	20	25
$S_n$	9.2104	16.7598	18.8016	19.1878	19.2491



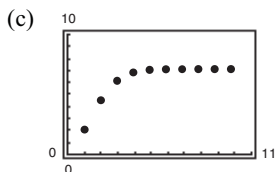
(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$12. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2 + 1}{(n+1)!}}{\frac{n^2 + 1}{n!}} = \lim_{n \rightarrow \infty} \left( \frac{n^2 + 2n + 2}{n^2 + 1} \right) \left( \frac{1}{n+1} \right) = 0 < 1. \text{ Converges}$$

(b)

$n$	5	10	15	20	25
$S_n$	7.0917	7.1548	7.1548	7.1548	7.1548



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$13. \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1/2^{n+1}}{1/2^n} \right| = \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1$$

Therefore, the series converges by the Ratio Test.

$$14. \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{1/(n+1)!}{1/n!} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \end{aligned}$$

Therefore, the series converges by the Ratio Test.

$$15. \sum_{n=0}^{\infty} \frac{n!}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

$$16. \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

$$17. \sum_{n=1}^{\infty} n \left( \frac{6}{5} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)(6/5)^{n+1}}{n(6/5)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \left( \frac{6}{5} \right) = \frac{6}{5} > 1 \end{aligned}$$

Therefore, the series diverges by the Ratio Test.

$$18. \sum_{n=1}^{\infty} n \left( \frac{10}{9} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)(10/9)^{n+1}}{n(10/9)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \left( \frac{10}{9} \right) = \frac{10}{9} > 1 \end{aligned}$$

Therefore, the series diverges by the Ratio Test.

$$19. \sum_{n=1}^{\infty} \frac{n}{4^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)/4^{n+1}}{n/4^n} = \lim_{n \rightarrow \infty} \frac{n+1}{4n} = 1/4 < 1$$

Therefore, the series converges by the Ratio Test.

$$20. \sum_{n=1}^{\infty} \frac{n^3}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^3/4^{n+1}}{n^3/4^n} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{4n^3} = \frac{1}{4} < 1 \end{aligned}$$

Therefore, the series converges by the Ratio Test.

$$21. \sum_{n=1}^{\infty} \frac{4^n}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{4^{n+1}/(n+1)^2}{4^n/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} (4) = 4 > 1 \end{aligned}$$

Therefore, the series diverges by the Ratio Test.

$$22. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \leq \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 9.14, the series converges.

**Note:** The Ratio Test is inconclusive because

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

The series converges conditionally.

$$23. \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3/2)^{n+1}}{n^2 + 2n + 1} \cdot \frac{n^2}{(3/2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n^2}{2(n^2 + 2n + 1)} = \frac{3}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

25.  $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

26.  $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

27.  $\sum_{n=0}^{\infty} \frac{e^n}{n!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{e^{n+1}/(n+1)!}{e^n/n!} \\ &= \lim_{n \rightarrow \infty} e \left( \frac{n!}{(n+1)!} \right) = \lim_{n \rightarrow \infty} \frac{e}{n+1} = 0 \end{aligned}$$

Therefore, the series converges by the Ratio Test.

28.  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e} \end{aligned}$$

Therefore, the series converges by the Ratio Test.

29.  $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{6^{n+1}/(n+2)^{n+1}}{6^n/(n+1)^n} = \lim_{n \rightarrow \infty} \frac{6}{n+2} \left( \frac{n+1}{n+2} \right)^n = 0 \left( \frac{1}{e} \right) = 0.$$

To find  $\lim_{n \rightarrow \infty} \left( \frac{n+1}{n+2} \right)^n$ : Let  $y = \left( \frac{n+1}{n+2} \right)^n$

$$\ln y = n \ln \left( \frac{n+1}{n+2} \right) = \frac{\ln(n+1) - \ln(n+2)}{1/n}$$

$$\lim_{n \rightarrow \infty} [\ln y] = \lim_{n \rightarrow \infty} \left[ \frac{1/(n+1) - 1/(n+2)}{-1/n^2} \right] = \lim_{n \rightarrow \infty} \left[ \frac{-n^2[(n+2) - (n+1)]}{(n+1)(n+2)} \right] = -1$$

by L'Hôpital's Rule. So,  $y \rightarrow \frac{1}{e}$ .

Therefore, the series converges by the Ratio Test.

30.  $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0$$

Therefore, by the Ratio Test, the series converges.

31.  $\sum_{n=0}^{\infty} \frac{5^n}{2^n + 1}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{5^{n+1}/(2^{n+1} + 1)}{5^n/(2^n + 1)} = \lim_{n \rightarrow \infty} \frac{5(2^n + 1)}{(2^{n+1} + 1)} = \lim_{n \rightarrow \infty} \frac{5(1 + 1/2^n)}{2 + 1/2^n} = \frac{5}{2} > 1$$

Therefore, the series diverges by the Ratio Test.

$$32. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}} \right| = \lim_{n \rightarrow \infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

$$33. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are  $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$ .

$$34. \sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n} \right| = \lim_{n \rightarrow \infty} \frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

**Note:** The first few terms of this series are  $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

$$35. \sum_{n=1}^{\infty} \frac{1}{5^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left[ \frac{1}{5^n} \right]^{1/n} = \frac{1}{5} < 1$$

Therefore, by the Root Test, the series converges.

$$36. \sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left[ \frac{1}{n^n} \right]^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, by the Root Test, the series converges.

$$37. \sum_{n=1}^{\infty} \left( \frac{n}{2n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n}{2n+1} \right)^n} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

$$38. \sum_{n=1}^{\infty} \left( \frac{2n}{n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{2n}{n+1} \right)^n} = \lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2$$

Therefore, by the Root Test, the series diverges.

$$39. \sum_{n=2}^{\infty} \left( \frac{2n+1}{n-1} \right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{2n+1}{n-1} \right)^n} = \lim_{n \rightarrow \infty} \left( \frac{2n+1}{n-1} \right) = 2$$

Therefore, by the Root Test, the series diverges.

$$40. \sum_{n=1}^{\infty} \left( \frac{4n+3}{2n-1} \right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{4n+3}{2n-1} \right)^n} = \lim_{n \rightarrow \infty} \frac{4n+3}{2n-1} = 2$$

Therefore, by the Root Test, the series diverges.

$$41. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(-1)^n}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{|\ln n|} = 0$$

Therefore, by the Root Test, the series converges.

42.  $\sum_{n=1}^{\infty} \left( \frac{-3n}{2n+1} \right)^{3n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{-3n}{2n+1} \right)^{3n} \right|} \\ &= \lim_{n \rightarrow \infty} \left( \frac{3n}{2n+1} \right)^3 = \left( \frac{3}{2} \right)^3 = \frac{27}{8} \end{aligned}$$

Therefore, by the Root Test, the series diverges.

43.  $\sum_{n=1}^{\infty} (2^{\sqrt{n}} + 1)^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{(2^{\sqrt{n}} + 1)^n} = \lim_{n \rightarrow \infty} (2^{\sqrt{n}} + 1)$$

To find  $\lim_{n \rightarrow \infty} 2^{\sqrt{n}}$ , let  $y = 2^{\sqrt{n}}$ . Then

$$\begin{aligned} \ln y &= \lim_{n \rightarrow \infty} (\ln 2^{\sqrt{n}}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0. \end{aligned}$$

So,  $\ln y = 0$ , so  $y = e^0 = 1$  and

$$\lim_{n \rightarrow \infty} (2^{\sqrt{n}} + 1) = 2(1) + 1 = 3.$$

Therefore, by the Root Test, the series diverges.

44.  $\sum_{n=0}^{\infty} e^{-3n} = \sum_{n=0}^{\infty} \frac{1}{e^{3n}}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{e^{3n}}} = \lim_{n \rightarrow \infty} \left( \frac{1}{e^3} \right)^{1/n} = \frac{1}{e}$$

Therefore, the series converges by the Root Test.

45.  $\sum_{n=1}^{\infty} \frac{n}{3^n}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left( \frac{n}{3^n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{3} = \frac{1}{3}$$

Therefore, the series converges by the Root Test.

**Note:** You can use L'Hôpital's Rule to show

$$\lim_{n \rightarrow \infty} n^{1/n} = 1:$$

$$\text{Let } y = n^{1/n}, \ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \Rightarrow y \rightarrow 1$$

46.  $\sum_{n=1}^{\infty} \left( \frac{n}{500} \right)^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n}{500} \right)^n} = \lim_{n \rightarrow \infty} \left( \frac{n}{500} \right) = \infty$$

Therefore, by the Root Test, the series diverges.

47.  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right)^n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{1}{n} - \frac{1}{n^2} \right)^n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{1}{n} - \frac{1}{n^2} \right) = 0 - 0 = 0 < 1 \end{aligned}$$

Therefore, by the Root Test, the series converges.

48.  $\sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{\ln n}{n} \right)^n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 < 1$$

Therefore, by the Root Test, the series converges.

49.  $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{n^{1/n}}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

50.  $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty} \frac{(n!)^n}{(n^2)^n}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{(n^2)^n}} = \lim_{n \rightarrow \infty} \frac{n!}{n^2} = \infty$$

Therefore, by the Root Test, the series diverges.

51.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

52.  $\sum_{n=1}^{\infty} \frac{100}{n} = 100 \sum_{n=1}^{\infty} \frac{1}{n}$

This is the divergent harmonic series.

53.  $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is a convergent  $p$ -series.

$$54. \sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$$

Because  $|r| = \frac{2\pi}{3} > 1$ , this is a divergent Geometric Series.

$$55. \sum_{n=1}^{\infty} \frac{5n}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{5n}{2n-1} = \frac{5}{2}$$

Therefore, the series diverges by the  $n$ th-Term Test

$$56. \sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n/(2n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2}\right)^n$$

Because  $|r| = \frac{3}{2} > 1$ , this is a divergent geometric series.

$$58. \sum_{n=1}^{\infty} \frac{10}{3\sqrt[3]{n^3}}$$

$$\lim_{n \rightarrow \infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a Limit Comparison Test with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$59. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n \rightarrow \infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a Limit Comparison Test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$60. \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{4n^2 - 1} = \lim_{n \rightarrow \infty} \frac{(\ln 2)2^n}{8n} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the  $n$ th-Term Test.

$$61. \left| \frac{\cos n}{3^n} \right| \leq \frac{1}{3^n}$$

Therefore the series  $\sum_{n=1}^{\infty} \left| \frac{\cos n}{3^n} \right|$  converges

by Direct comparison with the convergent geometric

series  $\sum_{n=1}^{\infty} \frac{1}{3^n}$ . So,  $\sum \frac{\cos n}{3^n}$  converges.

$$62. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

$$a_{n+1} = \frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n \ln(n)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

$$63. \sum_{n=1}^{\infty} \frac{n7^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \right| = \lim_{n \rightarrow \infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

$$64. \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

$$\frac{\ln(n)}{n^2} \leq \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$65. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

(Absolutely)

66.  $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

67.  $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

68.  $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1} (2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{1}{18}$$

Therefore, by the Ratio Test, the series converge.

69. (a) and (c) are the same.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n 5^n}{n!} &= \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!} \\ &= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots \end{aligned}$$

70. (b) and (c) are the same.

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) \left(\frac{3}{4}\right)^n &= \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{n-1} \\ &= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots \end{aligned}$$

71. (a) and (b) are the same.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} \\ &= 1 - \frac{1}{3!} + \frac{1}{5!} - \cdots \end{aligned}$$

72. (a) and (b) are the same.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} \\ &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \end{aligned}$$

73. Replace  $n$  with  $n+1$ .

$$\sum_{n=1}^{\infty} \frac{n}{7^n} = \sum_{n=0}^{\infty} \frac{n+1}{7^{n+1}}$$

74. Replace  $n$  with  $n+2$ .

$$\sum_{n=2}^{\infty} \frac{9^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{9^{n+2}}{n!}$$

75. (a) Because

$$\frac{3^{10}}{2^{10}10!} \approx 1.59 \times 10^{-5},$$

use 9 terms.

$$(b) \sum_{k=1}^9 \frac{(-3)^k}{2^k k!} \approx -0.7769$$

76. (a) Use 10 terms,  $k = 9$ , see Exercise 3.

$$\begin{aligned} (b) \sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} &= \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \approx 0.40967 \end{aligned}$$

$$77. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4n-1)/(3n+2)a_n}{a_n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{4n-1}{3n+2} = \frac{4}{3} > 1$$

The series diverges by the Ratio Test.

$$78. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+1)/(5n-4)a_n}{a_n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{5n-4} = \frac{2}{5} < 1$$

The series converges by the Ratio Test.

$$79. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(\sin n + 1)/(\sqrt{n})a_n}{a_n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{\sin n + 1}{\sqrt{n}} = 0 < 1$$

The series converges by the Ratio Test.

$$80. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(\cos n + 1)/(n)a_n}{a_n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{\cos n + 1}{n} = 0 < 1$$

The series converges by the Ratio Test.

$$81. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 + (1/n))a_n}{a_n} \right| = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) = 1$$

The Ratio Test is inconclusive.

But,  $\lim_{n \rightarrow \infty} a_n \neq 0$ , so the series diverges.

82. The series diverges because  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

$$a_1 = \frac{1}{4}$$

$$a_2 = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}$$

$$a_3 = \left(\frac{1}{2}\right)^{1/3} \approx 0.7937$$

In general,  $a_{n+1} > a_n > 0$ .

$$83. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1 \cdot 2 \cdots n(n+1)}{1 \cdot 3 \cdots (2n-1)(2n+1)}}{\frac{1 \cdot 2 \cdots n}{1 \cdot 3 \cdots (2n-1)}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

The series converges by the Ratio Test.

$$84. \sum_{n=0}^{\infty} \frac{n+1}{3^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3}$$

Let  $y = \lim_{n \rightarrow \infty} \sqrt[n]{n+1}$

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{n+1})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \ln(n+1)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n} = \frac{1}{n+1} = 0.$$

Because  $\ln y = 0$ ,  $y = e^0 = 1$ , so

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

$$85. \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

$$86. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)(2n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots (2n-1)}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{(2n)(2n+1)} = 0 < 1$$

The series converges by the Ratio Test.

$$87. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x/3)^{n+1}}{2(x/3)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right|$$

For the series to converge:  $\left| \frac{x}{3} \right| < 1 \Rightarrow -3 < x < 3$ .

For  $x = 3$ , the series diverges.

For  $x = -3$ , the series diverges.

$$88. \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{x+1}{4}} = \lim_{n \rightarrow \infty} \left| \frac{x+1}{4} \right| = \left| \frac{x+1}{4} \right|$$

For the series to converge,

$$\left| \frac{x+1}{4} \right| < 1 \Rightarrow -4 < x+1 < 4$$

$$\Rightarrow -5 < x < 3.$$

For  $x = 3$ ,  $\sum_{n=0}^{\infty} (1)^n$ , diverges.

For  $x = -5$ ,  $\sum_{n=0}^{\infty} (-1)^n$ , diverges.

$$89. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}/(n+1)}{x^n/n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} (x+1) \right| = |x+1|$$

For the series to converge,

$$|x+1| < 1 \Rightarrow -1 < x+1 < 1$$

$$\Rightarrow -2 < x < 0.$$

For  $x = 0$ ,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ , converges.

For  $x = -2$ ,  $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ , diverges.



$$90. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2|x-1|^{n+1}}{2|x-1|^n} \right| = \lim_{n \rightarrow \infty} |x-1| = |x-1|$$

For the series to converge,

$$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \\ \Rightarrow 0 < x < 2.$$

For  $x = 2$ ,  $\sum_{n=0}^{\infty} 2(1)^n$ , diverges.

For  $x = 0$ ,  $\sum_{n=0}^{\infty} 2(-1)^n$ , diverges.

$$91. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)! \left| \frac{x}{2} \right|^{n+1}}{n! \left| \frac{x}{2} \right|^n} \\ = \lim_{n \rightarrow \infty} (n+1) \left| \frac{x}{2} \right| = \infty$$

The series converges only at  $x = 0$ .

$$92. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x+1|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x+1|^n} = \lim_{n \rightarrow \infty} \frac{|x+1|}{n+1} = 0$$

The series converges for all  $x$ .

93. See Theorem 9.17, page 641.

94. See Theorem 9.18, page 644.

95. No. Let  $a_n = \frac{1}{n + 10,000}$ .

The series  $\sum_{n=1}^{\infty} \frac{1}{n + 10,000}$  diverges.

96. One example is  $\sum_{n=1}^{\infty} \left( -100 + \frac{1}{n} \right)$ .

97. The series converges absolutely. See Theorem 9.17.

98. (a) Converges (Ratio Test)  
 (b) Inconclusive (See Ratio Test)  
 (c) Diverges (Ratio Test)  
 (d) Diverges (Root Test)  
 (e) Inconclusive (See Root Test)  
 (f) Diverges (Root Test,  $e > 1$ )

99. Assume that

$$\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L > 1 \text{ or that } \lim_{n \rightarrow \infty} |a_{n+1}/a_n| = \infty.$$

Then there exists  $N > 0$  such that  $|a_{n+1}/a_n| > 1$  for all  $n > N$ . Therefore,

$$|a_{n+1}| > |a_n|, n > N \Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges.}$$

100. First, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1$$

and choose  $R$  such that  $0 \leq r < R < 1$ . There must exist some  $N > 0$  such that  $\sqrt[n]{|a_n|} < R$  for all  $n > N$ . So, for  $n > N$ ,  $|a_n| < R^n$  and because the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, you can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that  $\sum_{n=1}^{\infty} a_n$  converges.

Second, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some  $M > 0$  such that

$\sqrt[n]{|a_n|} > R$  for infinitely many  $n > M$ . So, for

infinitely many  $n > M$ , you have  $|a_n| > R^n > 1$  which implies that  $\lim_{n \rightarrow \infty} a_n \neq 0$  which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

$$101. \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{3/2} = 1$$

$$102. \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1} \right| \\ = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{1/2} = 1$$

$$103. \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^4} \cdot \frac{n^4}{1} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^4 = 1$$

$$104. \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^p = 1$$

105.  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ,  $p$ -series

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n \rightarrow \infty} \frac{1}{n^{p/n}} = 1$$

So, the Root Test is inconclusive.

**Note:**  $\lim_{n \rightarrow \infty} n^{p/n} = 1$  because if  $y = n^{p/n}$ , then

$$\ln y = \frac{p}{n} \ln n \text{ and } \frac{p}{n} \ln n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So  $y \rightarrow 1$  as  $n \rightarrow \infty$ .

106. Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p} = 1, \text{ inconclusive.}$$

Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n(\ln n)^p}} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}(\ln n)^{p/n}}$$

$$\lim_{n \rightarrow \infty} n^{1/n} = 1. \text{ Furthermore, let } y = (\ln n)^{p/n} \Rightarrow$$

$$\ln y = \frac{p}{n} \ln(\ln n).$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{p \ln(\ln n)}{n} = \lim_{n \rightarrow \infty} \frac{p}{\ln(n)(1/n)} = 0 \Rightarrow \lim_{n \rightarrow \infty} (\ln n)^{p/n} = 1.$$

$$\text{So, } \lim_{n \rightarrow \infty} \frac{1}{n^{1/n}(\ln n)^{p/n}} = 1, \text{ inconclusive.}$$

107.  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$ ,  $x$  positive integer

(a)  $x = 1$ :  $\sum \frac{(n!)^2}{n!} = \sum n!$ , diverges

(b)  $x = 2$ :  $\sum \frac{(n!)^2}{(2n)!}$  converges by the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(2n+2)!} \bigg/ \frac{(n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

(c)  $x = 3$ :  $\sum \frac{(n!)^2}{(3n)!}$  converges by the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{(3n+3)!} \bigg/ \frac{(n!)^2}{(3n)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

(d) Use the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{[(n+1)!]^2}{[x(n+1)!]} \bigg/ \frac{(n!)^2}{(xn)!} = \lim_{n \rightarrow \infty} (n+1)^2 \frac{(xn)!}{(xn+x)!}$$

The cases  $x = 1, 2, 3$  were solved above. For  $x > 3$ , the limit is 0. So, the series converges for all integers  $x \geq 2$ .

108. For  $n = 1, 2, 3, \dots$ ,  $-|a_n| \leq a_n \leq |a_n| \Rightarrow -\sum_{n=1}^k |a_n| \leq \sum_{n=1}^k a_n \leq \sum_{n=1}^k |a_n|$ .

Taking limits as  $k \rightarrow \infty$ ,  $-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \Rightarrow \left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$ .

109. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and  $f(x)$  is a real function such that  $f(1/n) = U_n$  for all positive integers  $n$  and  $d^2 f/dx^2$  exists at  $x = 0$ , then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if  $f(0) = f'(0) = 0$  and diverges otherwise. Below are some examples.

Convergent Series

$$\sum \frac{1}{n^3}, f(x) = x^3$$

$$\sum \left(1 - \cos \frac{1}{n}\right), f(x) = 1 - \cos x$$

Divergent Series

$$\sum \frac{1}{n}, f(x) = x$$

$$\sum \sin \frac{1}{n}, f(x) = \sin x$$

110. Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{n!}{(n+1)^n} \left( \frac{19}{7} \right)^n \right] / \left[ \frac{(n-1)!}{n^{n-1}} \left( \frac{19}{7} \right)^{n-1} \right] = \lim_{n \rightarrow \infty} \left[ \frac{n \cdot n^{n-1}}{(n+1)^n} \left( \frac{19}{7} \right) \right] = \lim_{n \rightarrow \infty} \left[ \frac{1}{(1 + (1/n))^n} \left( \frac{19}{7} \right) \right] = \frac{19}{7} \cdot \frac{1}{e} < 1$$

So, the series converges.

111. First prove Abel's Summation Theorem:

If the partial sums of  $\sum a_n$  are bounded and if  $\{b_n\}$  decreases to zero, then  $\sum a_n b_n$  converges.

Let  $S_k = \sum_{i=1}^k a_i$ . Let  $M$  be a bound for  $\{S_k\}$ .

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\ &= S_1 (b_1 - b_2) + S_2 (b_2 - b_3) + \cdots + S_{n-1} (b_{n-1} - b_n) + S_n b_n \\ &= \sum_{i=1}^{n-1} S_i (b_i - b_{i+1}) + S_n b_n \end{aligned}$$

The series  $\sum_{i=1}^{\infty} S_i (b_i - b_{i+1})$  is absolutely convergent because  $|S_i (b_i - b_{i+1})| \leq M (b_i - b_{i+1})$  and  $\sum_{i=1}^{\infty} (b_i - b_{i+1})$  converges to  $b_1$ .

Also,  $\lim_{n \rightarrow \infty} S_n b_n = 0$  because  $\{S_n\}$  bounded and  $b_n \rightarrow 0$ . Thus,  $\sum_{n=1}^{\infty} a_n b_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i b_i$  converges.

Now let  $b_n = \frac{1}{n}$  to finish the problem.

## Section 9.7 Taylor Polynomials and Approximations

1.  $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

2.  $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

 $y$ -axis symmetry

Three relative extrema

Matches (c)

3.  $y = e^{-1/2}[(x+1) + 1]$

Linear

Matches (a)

4.  $y = e^{-1/2}\left[\frac{1}{3}(x-1)^3 - (x-1) + 1\right]$

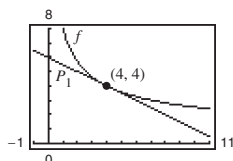
Cubic

Matches (b)

5.  $f(x) = \frac{8}{\sqrt{x}} = 8x^{-1/2} \quad f(4) = 4$

$$f'(x) = -4x^{-3/2} \quad f'(4) = -\frac{1}{2}$$

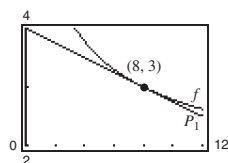
$$\begin{aligned} P_1(x) &= f(4) + f'(4)(x-4) \\ &= 4 - \frac{1}{2}(x-4) = -\frac{1}{2}x + 6 \end{aligned}$$

 $P_1$  is the first degree Taylor polynomial for  $f$  at 4.

6.  $f(x) = \frac{6}{\sqrt[3]{x}} = 6x^{-1/3} \quad f(8) = 3$

$$f'(x) = -2x^{-4/3} \quad f'(8) = -\frac{1}{8}$$

$$\begin{aligned} P_1(x) &= f(8) + f'(8)(x-8) \\ &= 3 - \frac{1}{8}(x-8) = -\frac{1}{8}x + 4 \end{aligned}$$

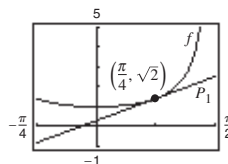
 $P_1$  is the first degree Taylor polynomial for  $f$  at 8.

7.  $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

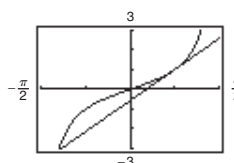
 $P_1$  is called the first degree Taylor polynomial for  $f$  at  $\frac{\pi}{4}$ .

8.  $f(x) = \tan x \quad f\left(\frac{\pi}{4}\right) = 1$

$$f'(x) = \sec^2 x \quad f'\left(\frac{\pi}{4}\right) = 2$$

$$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right)$$

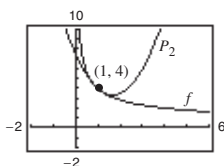
$$P_1(x) = 2x + 1 - \frac{\pi}{2}$$

 $P_1$  is called the first degree Taylor polynomial for  $f$  at  $\frac{\pi}{4}$ .

9.  $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$   
 $f'(x) = -2x^{-3/2} \quad f'(1) = -2$   
 $f''(x) = 3x^{-5/2} \quad f''(1) = 3$

$$P_2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= 4 - 2(x-1) + \frac{3}{2}(x-1)^2$$

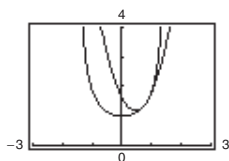


$x$	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

10.  $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$   
 $f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$   
 $f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$



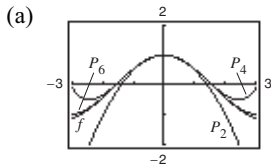
$x$	-2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	-1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	-4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

11.  $f(x) = \cos x$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$



(b)  $f'(x) = -\sin x$   $P_2'(x) = -x$

$$f''(x) = -\cos x$$
  $P_2''(x) = -1$

$$f'''(0) = P_2'''(0) = -1$$

$$f'''(x) = \sin x$$
  $P_4'''(x) = x$

$$f^{(4)}(x) = \cos x$$
  $P_4^{(4)}(x) = 1$

$$f^{(4)}(0) = 1 = P_4^{(4)}(0)$$

$$f^{(5)}(x) = -\sin x$$
  $P_6^{(5)}(x) = -x$

$$f^{(6)}(x) = -\cos x$$
  $P_6^{(6)}(x) = -1$

$$f^{(6)}(0) = -1 = P_6^{(6)}(0)$$

(c) In general,  $f^{(n)}(0) = P_n^{(n)}(0)$  for all  $n$ .

13.  $f(x) = e^{3x}$   $f(0) = 1$

$$f'(x) = 3e^{3x}$$
  $f'(0) = 3$

$$f''(x) = 9e^{3x}$$
  $f''(0) = 9$

$$f'''(x) = 27e^{3x}$$
  $f'''(0) = 27$

$$f^{(4)}(x) = 81e^{3x}$$
  $f^{(4)}(0) = 81$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

14.  $f(x) = e^{-x}$   $f(0) = 1$

$$f'(x) = -e^{-x}$$
  $f'(0) = -1$

$$f''(x) = e^{-x}$$
  $f''(0) = 1$

$$f'''(x) = -e^{-x}$$
  $f'''(0) = -1$

$$f^{(4)}(x) = e^{-x}$$
  $f^{(4)}(0) = 1$

$$f^{(5)}(x) = -e^{-x}$$
  $f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$+ \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

12.  $f(x) = x^2e^x$ ,  $f(0) = 0$

(a)  $f'(x) = (x^2 + 2x)e^x$   $f'(0) = 0$

$$f''(x) = (x^2 + 4x + 2)e^x$$
  $f''(0) = 2$

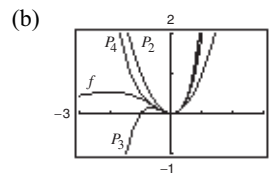
$$f'''(x) = (x^2 + 6x + 6)e^x$$
  $f'''(0) = 6$

$$f^{(4)}(x) = (x^2 + 8x + 12)e^x$$
  $f^{(4)}(0) = 12$

$$P_2(x) = \frac{2x^2}{2!} = x^2$$

$$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$$

$$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$$



(c)  $f'''(0) = 2 = P_2'''(0)$

$$f'''(0) = 6 = P_3'''(0)$$

$$f^{(4)}(0) = 12 = P_4^{(4)}(0)$$

(d)  $f^{(n)}(0) = P_n^{(n)}(0)$

15.  $f(x) = e^{-x/2} \quad f(0) = 1$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad f'''(0) = -\frac{1}{8}$$

$$f^{(4)}(x) = \frac{1}{16}e^{-x/2} \quad f^{(4)}(0) = \frac{1}{16}$$

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 \end{aligned}$$

16.  $f(x) = e^{x/3} \quad f(0) = 1$

$$f'(x) = \frac{1}{3}e^{x/3} \quad f'(0) = \frac{1}{3}$$

$$f''(x) = \frac{1}{9}e^{x/3} \quad f''(0) = \frac{1}{9}$$

$$f'''(x) = \frac{1}{27}e^{x/3} \quad f'''(0) = \frac{1}{27}$$

$$f^{(4)}(x) = \frac{1}{81}e^{x/3} \quad f^{(4)}(0) = \frac{1}{81}$$

$$\begin{aligned} P_4(x) &= 1 + \frac{1}{3}x + \frac{1/9}{2!}x^2 + \frac{1/27}{3!}x^3 + \frac{1/81}{4!}x^4 \\ &= 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4 \end{aligned}$$

17.  $f(x) = \sin x \quad f(0) = 0$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$$\begin{aligned} P_5(x) &= 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

18.  $f(x) = \sin \pi x \quad f(0) = 0$

$$f'(x) = \pi \cos \pi x \quad f'(0) = \pi$$

$$f''(x) = -\pi^2 \sin \pi x \quad f''(0) = 0$$

$$f'''(x) = -\pi^3 \cos \pi x \quad f'''(0) = -\pi^3$$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

19.  $f(x) = xe^x \quad f(0) = 0$

$$f'(x) = xe^x + e^x \quad f'(0) = 1$$

$$f''(x) = xe^x + 2e^x \quad f''(0) = 2$$

$$f'''(x) = xe^x + 3e^x \quad f'''(0) = 3$$

$$f^{(4)}(x) = xe^x + 4e^x \quad f^{(4)}(0) = 4$$

$$\begin{aligned} P_4(x) &= 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 \\ &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 \end{aligned}$$

20.  $f(x) = x^2e^{-x} \quad f(0) = 0$

$$f'(x) = 2xe^{-x} - x^2e^{-x} \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \quad f^{(4)}(0) = 12$$

$$\begin{aligned} P_4(x) &= 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 \\ &= x^2 - x^3 + \frac{1}{2}x^4 \end{aligned}$$

21.  $f(x) = \frac{1}{x+1} = (x+1)^{-1} \quad f(0) = 1$

$$f'(x) = -(x+1)^{-2} \quad f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3} \quad f''(0) = 2$$

$$f'''(x) = -6(x+1)^{-4} \quad f'''(0) = -6$$

$$f^{(4)}(x) = 24(x+1)^{-5} \quad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = -120(x+1)^{-6} \quad f^{(5)}(0) = -120$$

$$\begin{aligned} P_5(x) &= 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \frac{120x^5}{5!} \\ &= 1 - x + x^2 - x^3 + x^4 - x^5 \end{aligned}$$

$$\begin{aligned}
 22. \quad f(x) &= \frac{x}{x+1} = \frac{x+1-1}{x+1} & f(0) &= 0 \\
 &= 1 - (x+1)^{-1} \\
 f'(x) &= (x+1)^{-2} & f'(0) &= 1 \\
 f''(x) &= -2(x+1)^{-3} & f''(0) &= -2 \\
 f'''(x) &= 6(x+1)^{-4} & f'''(0) &= 6 \\
 f^{(4)}(x) &= -24(x+1)^{-5} & f^{(4)}(0) &= -24 \\
 P_4(x) &= 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 \\
 &= x - x^2 + x^3 - x^4
 \end{aligned}$$

$$\begin{aligned}
 23. \quad f(x) &= \sec x & f(0) &= 1 \\
 f'(x) &= \sec x \tan x & f'(0) &= 0 \\
 f''(x) &= \sec^3 x + \sec x \tan^2 x & f''(0) &= 1 \\
 P_2(x) &= 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2
 \end{aligned}$$

$$\begin{aligned}
 26. \quad f(x) &= \frac{1}{x^2} = x^{-2} & f(2) &= 1/4 \\
 f'(x) &= -2x^{-3} & f'(2) &= -1/4 \\
 f''(x) &= 6x^{-4} & f''(2) &= 3/8 \\
 f'''(x) &= -24x^{-5} & f'''(2) &= -3/4 \\
 f^{(4)}(x) &= 120x^{-6} & f^{(4)}(2) &= 15/8 \\
 P_4(x) &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3/8}{2!}(x-2)^2 - \frac{3/4}{3!}(x-2)^3 + \frac{15/8}{4!}(x-2)^4 \\
 &= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4
 \end{aligned}$$

$$\begin{aligned}
 27. \quad f(x) &= \sqrt{x} = x^{1/2} & f(4) &= 2 \\
 f'(x) &= \frac{1}{2}x^{-1/2} & f'(4) &= \frac{1}{4} \\
 f''(x) &= -\frac{1}{4}x^{-3/2} & f''(4) &= -\frac{1}{32} \\
 f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(4) &= \frac{3}{256} \\
 P_3(x) &= 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 \\
 &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3
 \end{aligned}$$

$$\begin{aligned}
 28. \quad f(x) &= x^{1/3} & f(8) &= 2 \\
 f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{12} \\
 f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{1}{144} \\
 f'''(x) &= \frac{10}{27}x^{-8/3} & f'''(8) &= \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}
 \end{aligned}$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

$$\begin{aligned}
 24. \quad f(x) &= \tan x & f(0) &= 0 \\
 f'(x) &= \sec^2 x & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2 x \tan x & f''(0) &= 0 \\
 f'''(x) &= 4 \sec^2 x \tan^2 x + 2 \sec^4 x & f'''(0) &= 2 \\
 P_3(x) &= 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3
 \end{aligned}$$

$$\begin{aligned}
 25. \quad f(x) &= \frac{2}{x} = 2x^{-1} & f(1) &= 2 \\
 f'(x) &= -2x^{-2} & f'(1) &= -2 \\
 f''(x) &= 4x^{-3} & f''(1) &= 4 \\
 f'''(x) &= -12x^{-4} & f'''(1) &= -12 \\
 P_3(x) &= 2 - 2(x-1) + \frac{4}{2!}(x-1)^2 - \frac{12}{3!}(x-1)^3 \\
 &= 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3
 \end{aligned}$$



29.  $f(x) = \ln x$   $f(2) = \ln 2$

$$f'(x) = \frac{1}{x} = x^{-1} \quad f'(2) = 1/2$$

$$f''(x) = -x^{-2} \quad f''(2) = -1/4$$

$$f'''(x) = 2x^{-3} \quad f'''(2) = 1/4$$

$$f^{(4)}(x) = -6x^{-4} \quad f^{(4)}(2) = -3/8$$

$$\begin{aligned} P_4(x) &= \ln 2 + \frac{1}{2}(x-2) - \frac{1/4}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3 - \frac{3/8}{4!}(x-2)^4 \\ &= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 \end{aligned}$$

30.  $f(x) = x^2 \cos x$   $f(\pi) = -\pi^2$

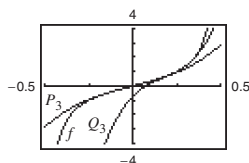
$$f'(x) = \cos x - x^2 \sin x \quad f'(\pi) = -2\pi$$

$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \quad f''(\pi) = -2 + \pi^2$$

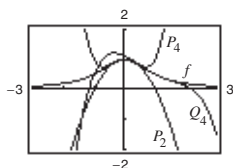
$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2-2)}{2}(x-\pi)^2$$

31. (a)  $P_3(x) = \pi x + \frac{\pi^3}{3}x^3$

(b)  $Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8}{3}\pi^3\left(x - \frac{1}{4}\right)^3$



32. (a)  $P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$



(b)  $Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$

33.  $f(x) = \sin x$

$$P_1(x) = x$$

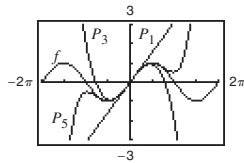
$$P_3(x) = x - \frac{1}{6}x^3$$

$$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

(a)

$x$	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the accuracy decreases.

$$34. (a) \quad f(x) = e^x \quad f(1) = e$$

$$f'(x) = e^x \quad f'(1) = e$$

$$f''(x) = f'''(x) = f^{(4)}(x) = e^x \text{ and } f''(1) = f'''(1) = f^{(4)}(1) = e$$

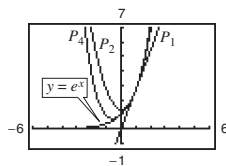
$$P_1(x) = e + e(x - 1)$$

$$P_2(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2$$

$$P_4(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2 + \frac{e}{6}(x - 1)^3 + \frac{e}{24}(x - 1)^4$$

$x$	1.00	1.25	1.50	1.75	2.00
$e^x$	$e$	3.4903	4.4817	5.7546	7.3891
$P_1(x)$	$e$	3.3979	4.0774	4.7570	5.4366
$P_2(x)$	$e$	3.4828	4.4172	5.5215	6.7957
$P_4(x)$	$e$	3.4903	4.4809	5.7485	7.3620

(b)

(c) As the degree increases, the accuracy increases. As the distance from  $x$  to 1 increases, the accuracy decreases.

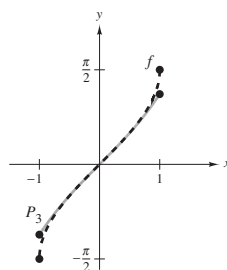
$$35. f(x) = \arcsin x$$

$$(a) \quad P_3(x) = x + \frac{x^3}{6}$$

(b)

$x$	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.848	-0.524	-0.253	0	0.253	0.524	0.848
$P_3(x)$	-0.820	-0.521	-0.253	0	0.253	0.521	0.820

(c)

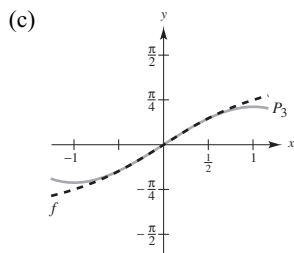


36. (a)  $f(x) = \arctan x$

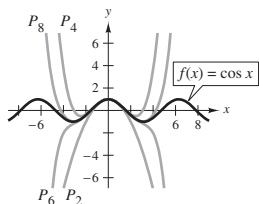
$$P_3(x) = x - \frac{x^3}{3}$$

(b)

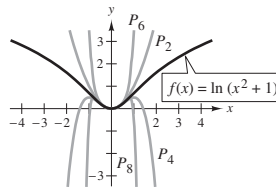
$x$	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.6435	-0.4636	-0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	-0.6094	-0.4583	-0.2448	0	0.2448	0.4583	0.6094



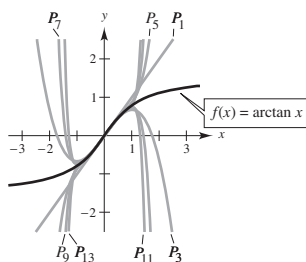
37.  $f(x) = \cos x$



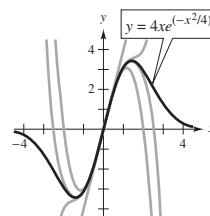
39.  $f(x) = \ln(x^2 + 1)$



38.  $f(x) = \arctan x$



40.  $f(x) = 4xe^{-x^2/4}$



41.  $f(x) = e^{3x} \approx 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

$$f\left(\frac{1}{2}\right) \approx 4.3984$$

42.  $f(x) = x^2e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$

$$f\left(\frac{1}{5}\right) \approx 0.0328$$

43.  $f(x) = \ln x \approx \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4$

$$f(2.1) \approx 0.7419$$

44.  $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\frac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$$f\left(\frac{7\pi}{8}\right) \approx -6.7954$$

45.  $f(x) = \cos x$ ;  $f^{(5)}(x) = -\sin x \Rightarrow$  Max on  $[0, 0.3]$  is 1.

$$R_4(x) \leq \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$$

Note: you could use  $R_5(x)$ :  $f^{(6)}(x) = -\cos x$ , max on  $[0, 0.3]$  is 1.

$$R_5(x) \leq \frac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$$

Exact error:  $0.000001 = 1.0 \times 10^{-6}$

46.  $f(x) = e^x$ ;  $f^{(6)}(x) = e^x \Rightarrow$  Max on  $[0, 1]$  is  $e^1$ .

$$R_5(x) \leq \frac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$$

47.  $f(x) = \arcsin x$ ;  $f^{(4)}(x) = \frac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow$  Max on  $[0, 0.4]$  is  $f^{(4)}(0.4) \approx 7.3340$ .

$$R_3(x) \leq \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}. \text{ The exact error is } 8.5 \times 10^{-4}. [\text{Note: You could use } R_4.]$$

48.  $f(x) = \arctan x$ ;  $f^{(4)}(x) = \frac{24x(x^2 + 1)}{(1 - x^2)^4}$   
 $\Rightarrow$  Max on  $[0, 0.4]$  is  $f^{(4)}(0.4) \approx 22.3672$ .

$$R_3(x) \leq \frac{22.3672}{4!}(0.4)^4 \approx 0.0239$$

49.  $g(x) = \sin x$

$$|g^{(n+1)}(x)| \leq 1 \text{ for all } x.$$

$$R_n(x) \leq \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error,  $n = 3$ .

50.  $f(x) = \cos x$

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x \text{ and all } n.$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!} \right| \leq \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

By trial and error,  $n = 2$ .

51.  $f(x) = e^x$

$$f^{(n+1)}(x) = e^x$$

Max on  $[0, 0.6]$  is  $e^{0.6} \approx 1.8221$ .

$$R_n \leq \frac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$$

By trial and error,  $n = 5$ .

52.  $f(x) = \ln x$ ,  $f'(x) = x^{-1}$ ,  $f''(x) = -x^{-2}$ , ...

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$$

The maximum value of  $|f^{(n+1)}(x)|$  on  $[1, 1.25]$  is  $n!$

$$|R_n| \leq \frac{n!}{(n+1)!}(0.25)^{n+1} < 0.001$$

$$\frac{(0.25)^{n+1}}{n+1} < 0.001$$

By trial and error,  $n = 3$

53.  $f(x) = \ln(x+1)$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \leq \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error,  $n = 9$ . (See Example 9.) Using 9 terms,  $\ln(1.5) \approx 0.4055$ .

54.  $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \dots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \dots$$

$$f(0.6) = 1 - \frac{\pi^2 (0.6)^4}{2!} + \frac{\pi^4 (0.6)^8}{4!} - \frac{\pi^6 (0.6)^{12}}{6!} + \dots$$

Because this is an alternating series,

$$R_n \leq a_{n+1} = \frac{\pi^{2n}}{(2n)!} (0.6)^{4n} < 0.0001.$$

By trial and error,  $n = 4$ . Using 4 terms

$$f(0.6) \approx 0.4257.$$

55.  $f(x) = e^{-\pi x}$ ,  $f(1.3)$

$$f'(x) = (-\pi)e^{-\pi x}$$

$$f^{(n+1)}(x) = (-\pi)^{n+1} e^{-\pi x} \leq |(-\pi)^{n+1}| \text{ on } [0, 1.3]$$

$$|R_n| \leq \frac{(\pi)^{n+1}}{(n+1)!} (1.3)^{n+1} < 0.0001$$

By trial and error,  $n = 16$ . Using 16 terms,

$$e^{-\pi(1.3)} \approx 0.01684.$$

56.  $f(x) = e^{-x}$

$$f'(x) = -e^{-x}$$

$$f^{(n+1)}(x) = (-1)^{n+1} e^{-x} \leq 1 \text{ on } [0, 1]$$

$$|R_n| \leq \frac{1}{(n+1)!} 1^{n+1} \leq 0.0001$$

By trial and error,  $n = 7$ .

57.  $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ ,  $x < 0$

$$R_3(x) = \frac{e^z}{4!} x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$|xe^{z/4}| < 0.3936$$

$$|x| < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$$-0.3936 < x < 0$$

58.  $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$|R_3(x)| = \left| \frac{\sin z}{4!} x^4 \right| \leq \frac{|x^4|}{4!} < 0.001$$

$$x^4 < 0.024$$

$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

59.  $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ , fifth degree polynomial

$$|f^{(n+1)}(x)| \leq 1 \text{ for all } x \text{ and all } n.$$

$$|R_5(x)| \leq \frac{1}{6!} |x|^6 < 0.001$$

$$|x|^6 < 0.72$$

$$|x| < 0.9467$$

$$-0.9467 < x < 0.9467$$

**Note:** Use a graphing utility to graph

$y = \cos x - (1 - x^2/2 + x^4/24)$  in the viewing

window  $[-0.9467, 0.9467] \times [-0.001, 0.001]$  to verify the answer.

60.  $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

$$f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x},$$

$$f'''(x) = -8e^{-2x}, f^{(4)}(x) = 16e^{-2x}$$

$$R_3(x) = \frac{f^{(4)}(z)}{4!} (x-0)^4 = \frac{16e^{-2z}}{24} x^4 = \frac{2}{3} e^{-2z} x^4 < 0.001$$

$$e^{-2z} x^4 < 0.0015$$

$$x < \left( \frac{0.0015}{e^{-2z}} \right)^{1/4} \approx 0.1970 e^{2z} < 0.1970, \text{ for } z < 0.$$

So,  $0 < x < 0.1970$ .

In fact, by graphing  $f(x) = e^{-2x}$  and

$y = 1 - 2x + 2x^2 - \frac{4}{3}x^3$ , you can verify that

$$|f(x) - y| < 0.001 \text{ on } (-0.19294, 0.20068).$$

61. The graph of the approximating polynomial  $P$  and the elementary function  $f$  both pass through the point  $(c, f(c))$  and the slopes of  $P$  and  $f$  agree at  $(c, f(c))$ . Depending on the degree of  $P$ , the  $n$ th derivatives of  $P$  and  $f$  agree at  $(c, f(c))$ .

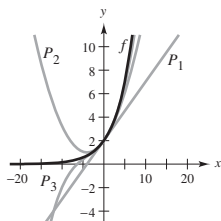
62.  $f(c) = P_2(c)$ ,  $f'(c) = P_2'(c)$ , and  $f''(c) = P_2''(c)$

63. See definition on page 652.

64. See Theorem 9.19, page 656.

65. As the degree of the polynomial increases, the graph of the Taylor polynomial becomes a better and better approximation of the function within the interval of convergence. Therefore, the accuracy is increased.

66.



67. (a)  $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = x P_4(x)$$

(b)  $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = x P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c)  $g(x) = \frac{\sin x}{x} = \frac{1}{x} P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$

68. (a)  $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$  for  $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for  $g(x) = \cos x$ .

(b)  $Q_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$  for  $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c)  $R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

69. (a)  $Q_2(x) = -1 + \frac{\pi^2(x+2)^2}{32}$

(b)  $R_2(x) = -1 + \frac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear. Horizontal translations of the result in part (a) are possible only at  $x = -2 + 8n$  (where  $n$  is an integer) because the period of  $f$  is 8.

70. Let  $f$  be an odd function and  $P_n$  be the  $n$ th Maclaurin polynomial for  $f$ . Because  $f$  is odd,  $f'$  is even:

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x). \end{aligned}$$

Similarly,  $f''$  is odd,  $f'''$  is even, etc. Therefore,  $f, f'', f^{(4)}, \dots$  are all odd functions, which implies that  $f(0) = f''(0) = \dots = 0$ . So, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$

all the coefficients of the even power of  $x$  are zero.

71. Let  $f$  be an even function and  $P_n$  be the  $n$ th Maclaurin polynomial for  $f$ . Because  $f$  is even,  $f'$  is odd,  $f''$  is even,  $f'''$  is odd, etc. All of the odd derivatives of  $f$  are odd and so, all of the odd powers of  $x$  will have coefficients of zero.  $P_n$  will only have terms with even powers of  $x$ .

72. Let

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n$$

$$\text{where } a_i = \frac{f^{(i)}(c)}{i!}.$$

$$P_n(c) = a_0 = f(c)$$

For

$$1 \leq k \leq n, \quad P_n^{(k)}(c) = a_n k! = \left( \frac{f^{(k)}(c)}{k!} \right) k! = f^{(k)}(c).$$

73. As you move away from  $x = c$ , the Taylor Polynomial becomes less and less accurate.

## Section 9.8 Power Series

1. Centered at 0

2. Centered at 0

3. Centered at 2

4. Centered at  $\pi$

5.  $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| |x| = |x|$$

$$|x| < 1 \Rightarrow R = 1$$

6.  $\sum_{n=0}^{\infty} (4x)^n$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1}}{(4x)^n} \right| = \lim_{n \rightarrow \infty} |4x| = 4|x|$$

$$4|x| < 1 \Rightarrow R = \frac{1}{4}$$

7.  $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1}/(n+1)^2}{(4x)^n/n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} (4x) \right| = 4|x|$$

$$4|x| < 1 \Rightarrow R = \frac{1}{4}$$

8.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}/5^{n+1}}{(-1)^n x^n/5^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{5} \right| = \frac{|x|}{5}$$

$$\frac{|x|}{5} < 1 \Rightarrow R = 5$$

9.  $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{(2n+2)}/(2n+2)!}{x^{2n}/(2n)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0$$

So, the series converges for all  $x \Rightarrow R = \infty$ .

10.  $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2}/(n+1)!}{(2n)! x^{2n}/n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x^2}{(n+1)} \right| = \infty$$

The series only converges at  $x = 0$ .  $R = 0$ .

11.  $\sum_{n=0}^{\infty} \left( \frac{x}{4} \right)^n$

Because the series is geometric, it converges only if

$$\left| \frac{x}{4} \right| < 1, \text{ or } -4 < x < 4.$$

12.  $\sum_{n=0}^{\infty} \left( \frac{x}{7} \right)^n$

Because the series is geometric, it converges only if

$$\left| \frac{x}{7} \right| < 1, \text{ or } -7 < x < 7.$$

13.  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{nx}{n+1} \right| = |x|$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges.

When  $x = -1$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

Therefore, the interval of convergence is  $(-1, 1]$ .

$$14. \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)x^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (n+2)x^{n+1}}{(-1)^n (n+1)x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x}{n+1} \right| = |x| \end{aligned}$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the series  $\sum_{n=0}^{\infty} (-1)^{n+1} (n+1)$  diverges.

When  $x = -1$ , the series  $\sum_{n=0}^{\infty} -(n+1)$  diverges.

Therefore, the interval of convergence is  $(-1, 1)$ .

$$15. \sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{5(n+1)} / (n+1)!}{x^{5n} / n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^5}{n+1} \right| = 0$$

The series converges for all  $x$ . The interval of convergence is  $(-\infty, \infty)$ .

$$18. \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{(-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{n+3} \right| = |x|$$

Interval:  $-1 < x < 1$

When  $x = 1$ , the alternating series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$  converges.

When  $x = -1$ , the series  $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$  converges by limit comparison to  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

Therefore, the interval of convergence is  $[-1, 1]$ .

$$19. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Because the series is geometric, it converges only if  $|x/4| < 1$  or  $-4 < x < 4$ .

$$20. \sum_{n=0}^{\infty} \frac{(-1)^n n! (x-5)^n}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)! (x-5)^{n+1} / 3^{n+1}}{(-1)^n n! (x-5)^n / 3^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-5)}{3} \right| = \infty$$

The series converges only for  $x = 5$ .

$$16. \sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+1)!} \cdot \frac{(2n)!}{(3x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{3x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

$$17. \sum_{n=0}^{\infty} (2n)! \left( \frac{x}{3} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! (x/3)^{n+1}}{(2n)! (x/3)^n} \right| \\ &= \left| \frac{(2n+2)(2n+1)x}{3} \right| = \infty \end{aligned}$$

The series converges only for  $x = 0$ .



$$21. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-4)^{n+1} / ((n+1)9^{n+1})}{(-1)^n(x-4)^n / (n9^n)} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \cdot \frac{(x-4)}{9} \right| = \frac{1}{9} |x-4|$$

$$R = 9$$

$$\text{Interval: } -5 < x < 13$$

$$\text{When } x = 13, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}9^n}{n9^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

$$\text{When } x = -5, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-9)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{-1}{n} \text{ diverges.}$$

Therefore, the interval of convergence is  $(-5, 13]$ .

$$22. \sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+2} / [(n+2)4^{n+2}]}{(x-3)^{n+1} / [(n+1)4^{n+1}]} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(x-3)(n+1)}{4(n+2)} \right| = \left| \frac{x-3}{4} \right|$$

$$R = 4$$

$$\text{Interval: } -1 < x < 7$$

$$\text{When } x = 7, \sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ diverges.}$$

$$\text{When } x = -1, \sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)} \text{ converges.}$$

Therefore, the interval of convergence is  $[-1, 7)$ .

$$23. \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-1)}{n+2} \right| = |x-1|$$

$$R = 1$$

$$\text{Center: } x = 1$$

$$\text{Interval: } -1 < x-1 < 1 \text{ or } 0 < x < 2$$

$$\text{When } x = 0, \text{ the series } \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ diverges by the integral test.}$$

$$\text{When } x = 2, \text{ the alternating series } \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \text{ converges.}$$

Therefore, the interval of convergence is  $(0, 2]$ .

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n}{(-1)^{n+1}(x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x-2}{2} \cdot \frac{n}{n+1} \right| = \left| \frac{x-2}{2} \right|$$

$$\left| \frac{x-2}{2} \right| < 1 \Rightarrow -2 < x-2 < 2 \Rightarrow 0 < x < 4$$

$$\text{when } x = 0,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)(2^n)}{n2^n} = \sum_{n=1}^{\infty} \frac{-1}{n} \text{ diverges.}$$

$$\text{when } x = 4,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

Therefore the interval of convergence is  $(0, 4]$ .

25.  $\sum_{n=1}^{\infty} \left(\frac{x-3}{3}\right)^{n-1}$  is geometric. It converges if

$$\left|\frac{x-3}{3}\right| < 1 \Rightarrow |x-3| < 3 \Rightarrow 0 < x < 6.$$

Therefore, the interval of convergence is  $(0, 6)$ .

26.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2n+1)}{(2n+3)} x^2 \right| = |x^2| \end{aligned}$$

$$R = 1$$

$$\text{Interval: } -1 < x < 1$$

When  $x = 1$ ,  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$  converges.

When  $x = -1$ ,  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$  converges.

Therefore, the interval of convergence is  $[-1, 1]$ .

27.  $\sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-2x)(n+1)^2}{n(n+2)} \right| = 2|x| \end{aligned}$$

$$R = \frac{1}{2}$$

$$\text{Interval: } -\frac{1}{2} < x < \frac{1}{2}$$

When  $x = -\frac{1}{2}$ , the series  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  diverges by the  $n$ th

Term Test.

When  $x = \frac{1}{2}$ , the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1} \text{ diverges.}$$

Therefore, the interval of convergence is  $\left(-\frac{1}{2}, \frac{1}{2}\right)$ .

28.  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

29.  $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{3n+4}/(3n+4)!}{x^{3n+1}/(3n+1)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^3}{(3n+4)(3n+3)(3n+2)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

30.  $\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n! x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

31.  $\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1) x^n}{n!} = \sum_{n=1}^{\infty} (n+1) x^n$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} x \right| = |x|$$

$$\text{Converges if } |x| < 1 \Rightarrow -1 < x < 1.$$

At  $x = \pm 1$ , diverges.

Therefore the interval of convergence is  $(-1, 1)$ .

$$32. \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)x^2}{(2n+3)} \right| = |x^2|$$

$$R = 1$$

When  $x = \pm 1$ , the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges.

Therefore, the interval of convergence is  $(-1, 1)$ .

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(4n+3)(x-3)}{4} \right| = \infty \end{aligned}$$

$$R = 0$$

Center:  $x = 3$

Therefore, the series converges only for  $x = 3$ .

$$34. \sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x+1)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n)!(x+1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+1)}{2n+1} \right| = \frac{1}{2}|x+1|$$

Converges if  $\frac{1}{2}|x+1| < 1 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1$ .

At  $x = 1$ ,  $a_n = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1$ , diverges.

At  $x = -3$ ,  $a_n = \frac{n!(-2)^n}{1 \cdot 3 \cdots (2n-1)} = (-1)^n \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)}$ , diverges.

Therefore, the interval of convergence is  $(-3, 1)$ .

$$35. \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

Center:  $x = c$

Interval:  $-c < x - c < c$  or  $0 < x < 2c$

When  $x = 0$ , the series  $\sum_{n=1}^{\infty} (-1)^{n-1}$  diverges.

When  $x = 2c$ , the series  $\sum_{n=1}^{\infty} 1$  diverges.

Therefore, the interval of convergence is  $(0, 2c)$ .

36.  $\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$ ,  $k$  is a positive integer.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^k x^{n+1}}{[k(n+1)!]} \cdot \frac{(n!)^k x^n}{(kn)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^k x}{(k+nk)(k-1+nk) \cdots (1+nk)} \right| = \frac{|x|}{k^k}$$

Converges if  $\frac{|x|}{k^k} < 1 \Rightarrow R = k^k$ .

37.  $\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n$

Because the series is geometric, it converges only if  $|x/k| < 1$  or  $-k < x < k$ .

Therefore, the interval of convergence is  $(-k, k)$ .

38.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-c)^n}{nc^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1} (x-c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-c)}{c(n+1)} \right| = \frac{1}{c} |x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When  $x = 0$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{-1}{n}$  diverges.

When  $x = 2c$ , the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges.

Therefore, the interval of convergence is  $(0, 2c)$ .

39.  $\sum_{n=1}^{\infty} \frac{k(k+1) \cdots (k+n-1)x^n}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k(k+1) \cdots (k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1) \cdots (k+n-1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(k+n)x}{n+1} \right| = |x|$$

$$R = 1$$

When  $x = \pm 1$ , the series diverges and the interval of convergence is  $(-1, 1)$ .

$$\left[ \frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \geq 1 \right]$$

40.  $\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-c)}{2n+1} \right| = \frac{1}{2} |x-c|$$

$$R = 2$$

$$\text{Interval: } -2 < x - c < 2 \text{ or } c - 2 < x < c + 2$$

The series diverges at the endpoints. Therefore, the interval of convergence is  $(c-2, c+2)$ .

$$\left[ \frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1 \right]$$

$$41. \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$42. \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)x^n = \sum_{n=1}^{\infty} (-1)^n (n)x^{n-1}$$

$$43. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

Replace  $n$  with  $n-1$ .

$$44. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$$

Replace  $n$  with  $n-1$ .

$$45. (a) f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n, (-3, 3) \quad (\text{Geometric})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{n}{3} \left(\frac{x}{3}\right)^{n-1}, (-3, 3)$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{9} \left(\frac{x}{3}\right)^{n-2}, (-3, 3)$$

$$(d) \int f(x) dx = \sum_{n=0}^{\infty} \frac{3}{n+1} \left(\frac{x}{3}\right)^{n+1}, [-3, 3]$$

$$\left[ \sum_{n=1}^{\infty} \frac{3}{n} \left(\frac{-3}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3}{n}, \text{converges} \right]$$

$$46. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-5)^n}{n5^n}, (0, 10]$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-5)^{n-1}}{5^n}, (0, 10)$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1} (n-1)(x-5)^{n-2}}{5^n}, (0, 10)$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-5)^{n+1}}{n(n+1)5^n}, [0, 10]$$

$$47. (a) f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-1)^{n+1}}{n+1}, (0, 2]$$

$$(b) f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n, (0, 2)$$

$$(c) f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n(x-1)^{n-1}, (0, 2)$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^{n+2}}{(n+1)(n+2)}, [0, 2]$$

$$48. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-2)^n}{n}, (1, 3]$$

$$(b) f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} (x-2)^{n-1}, (1, 3)$$

$$(c) f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1} (n-1)(x-2)^{n-2}, (1, 3)$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-2)^{n+1}}{n(n+1)}, [1, 3]$$

$$49. g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

$S_1 = 1, S_2 = 1.33$ . Matches (c).

$$50. g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$S_1 = 1, S_2 = 1.67$ . Matches (a).

$$51. g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n \text{ diverges. Matches (b).}$$

$$52. g(-2) = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n \text{ alternating. Matches (d).}$$

$$53. g\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \frac{1}{16} + \cdots, \text{converges}$$

$S_1 = 1, S_2 = 1.25, S_3 = 1.3125$  Matches (b).

$$54. g\left(-\frac{1}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(-\frac{1}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = 1 - \frac{1}{4} + \frac{1}{16} - \cdots, \text{converges}$$

$S_1 = 1, S_2 = 0.75, S_3 = 0.8125$  Matches (c).

$$55. g\left(\frac{9}{16}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{9}{16}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{9}{8}\right)^n, \text{diverges}$$

$S_1 = 1, S_2 = \frac{17}{8}$  Matches (d).

$$56. g\left(\frac{3}{8}\right) = \sum_{n=0}^{\infty} \left[2\left(\frac{3}{8}\right)\right]^n = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n, \text{converges}$$

$S_1 = 1, S_2 = 1.75$  Matches (a).

57. A series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at  $c$ , where  $c$  is constant.

58. The set of all values of  $x$  for which the power series converges is the interval of convergence. If the power series converges for all  $x$ , then the radius of convergence is  $R = \infty$ . If the power series converges at only  $c$ , then  $R = 0$ . Otherwise, according to Theorem 8.20, there exists a real number  $R > 0$  (radius of convergence) such that the series converges absolutely for  $|x - c| < R$  and diverges for  $|x - c| > R$ .

59. A single point, an interval, or the entire real line.

60. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

61. Answers will vary.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$  converges for  $-1 \leq x < 1$ . At  $x = -1$ , the

convergence is conditional because  $\sum \frac{1}{n}$  diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$  converges for  $-1 \leq x \leq 1$ . At  $x = \pm 1$ , the

convergence is absolute.

62. Many answers possible.

(a)  $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$  Geometric:  $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$  converges for  $-1 < x \leq 1$

(c)  $\sum_{n=1}^{\infty} (2x + 1)^n$  Geometric:  
 $|2x + 1| < 1 \Rightarrow -1 < x < 0$

(d)  $\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n4^n}$  converges for  $-2 \leq x < 6$

63. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n+2} = 0 \end{aligned}$$

Therefore, the interval of convergence is  $(-\infty, \infty)$ .

(b)  $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c)  $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$   
 $= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d)  $f(x) = \sin x$  and

$g(x) = \cos x$

64. (a)  $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0 \end{aligned}$$

The series converges for all  $x$ . Therefore, the interval of convergence is  $(-\infty, \infty)$ .

(b)  $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(c)  $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$   
 $f(0) = 1$

(d)  $f(x) = e^x$

$$\begin{aligned}
 65. \quad y &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \\
 y' &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \\
 y'' + y &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} = 0
 \end{aligned}$$

$$\begin{aligned}
 66. \quad y &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!} \\
 y' &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1) x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!} \\
 y'' + y &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!} = 0
 \end{aligned}$$

$$\begin{aligned}
 69. \quad y &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \quad y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!} \quad y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} \\
 y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\
 &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\
 &= \sum_{n=0}^{\infty} \left[ \frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right] \\
 &= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}[(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0
 \end{aligned}$$

$$\begin{aligned}
 70. \quad y &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
 y' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} \\
 y'' + x^2 y &= -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2 \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0
 \end{aligned}$$

$$\begin{aligned}
 67. \quad y &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \\
 y' &= \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = y \\
 y'' - y &= 0
 \end{aligned}$$

$$\begin{aligned}
 68. \quad y &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!} \\
 y' &= \sum_{n=1}^{\infty} \frac{(2n-2)x^{2n-1}}{(2n-2)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \\
 y'' &= \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!} = y \\
 y'' - y &= 0
 \end{aligned}$$

$$71. J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} [(k+1)!]^2} \cdot \frac{2^{2k} (k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2 (k+1)^2} \right| = 0$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .

$$(b) \quad J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1} [(k+1)!]^2}$$

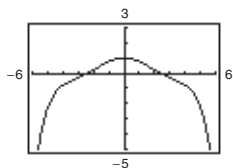
$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1} [(k+1)!]^2}$$

$$x^2 J_0'' + x J_0' + x^2 J_0 = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1} (k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1} (k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ (-1) \frac{2(2k+1)}{4(k+1)} + (-1) \frac{2}{4(k+1)} + 1 \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[ \frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0$$

$$(c) \quad P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$



$$(d) \int_0^1 J_0 dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx = \left[ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} = 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$$

(integral is approximately 0.9197304101)

$$72. J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3} (k+1)!(k+2)!} \cdot \frac{2^{2k+1} k!(k+1)!}{(-1)^k x^{2k+1}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2 (k+2)(k+1)} \right| = 0$$

Therefore, the interval of convergence is  $-\infty < x < \infty$ .



$$(b) \quad J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{2^{2k+1} k!(k+1)!}$$

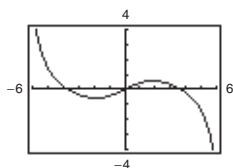
$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k-1}}{2^{2k+1} k!(k+1)!}$$

$$\begin{aligned} x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \\ &= \left[ \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \right] \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} [(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1} (k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+3} [(-1) + 1]}{2^{2k+1} k!(k+1)!} = 0 \end{aligned}$$

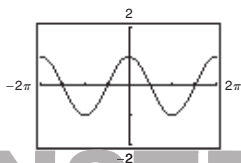
$$(c) \quad P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18,432}x^7$$

$$(d) \quad J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1) x^{2k+1}}{2^{2k+2} (k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

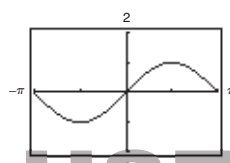
$$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!} \quad \text{Note: } J_0'(x) = -J_1(x)$$



$$73. \quad f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

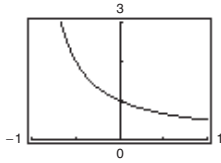


$$74. \quad f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

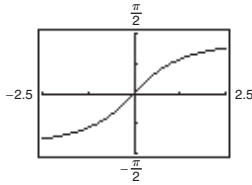


75.  $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$  Geometric

$$= \frac{1}{1 - (-x)} = \frac{1}{1 + x} \text{ for } -1 < x < 1$$

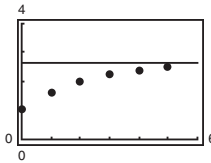


76.  $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, -1 \leq x \leq 1$

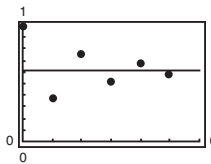


77.  $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n, (-4, 4)$

(a)  $\sum_{n=0}^{\infty} \left(\frac{(5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n = \frac{1}{1 - 5/8} = \frac{8}{3}$



(b)  $\sum_{n=0}^{\infty} \left(\frac{(-5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{5}{8}\right)^n = \frac{1}{1 - 5/8} = \frac{8}{13}$



- (c) The alternating series converges more rapidly. The partial sums of the series of positive terms approaches the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

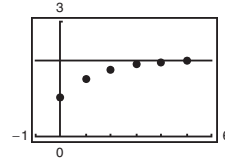
M	10	100	1000	10,000
N	5	14	24	35

83.  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1+p)!}{(n+1)!(n+1+q)!} x^{n+1} \cdot \frac{n!(n+q)!}{(n+p)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1+p)x}{(n+1)(n+1+q)} \right| = 0$

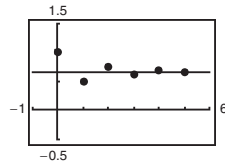
So, the series converges for all  $x$ :  $R = \infty$ .

78.  $\sum_{n=0}^{\infty} (3x)^n$  converges on  $\left(-\frac{1}{3}, \frac{1}{3}\right)$ .

(a)  $x = \frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = 2$



(b)  $x = -\frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(-\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = \frac{2}{3}$



- (c) The alternating series converges more rapidly. The partial sums in (a) approach the sum 2 from below. The partial sums in (b) alternate sides of the horizontal line  $y = \frac{2}{3}$ .

(d)  $\sum_{n=0}^N \left(3 \cdot \frac{2}{3}\right)^n = \sum_{n=0}^N 2^n > M$

M	10	100	1000	10,000
N	3	6	9	13

79. False;

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

converges for  $x = 2$  but diverges for  $x = -2$ .

80. False; it is not possible. See Theorem 9.20.

81. True; the radius of convergence is  $R = 1$  for both series.

82. True

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left( \sum_{n=0}^{\infty} a_n x^n \right) dx \\ &= \left[ \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \end{aligned}$$

84. (a)  $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$

$$S_{2n} = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots + x^{2n} + 2x^{2n+1} = (1 + x^2 + x^4 + \cdots + x^{2n}) + 2x(1 + x^2 + x^4 + \cdots + x^{2n})$$

$$\lim_{n \rightarrow \infty} S_{2n} = \sum_{n=0}^{\infty} x^{2n} + 2x \sum_{n=0}^{\infty} x^{2n}$$

Each series is geometric,  $R = 1$ , and the interval of convergence is  $(-1, 1)$ .

(b) For  $|x| < 1$ ,  $g(x) = \frac{1}{1-x^2} + 2x \frac{1}{1-x^2} = \frac{1+2x}{1-x^2}$ .

85. (a)  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ ,  $c_{n+3} = c_n$

$$= c_0 + c_1 x + c_2 x^2 + c_0 x^3 + c_1 x^4 + c_2 x^5 + c_0 x^6 + \cdots$$

$$S_{3n} = c_0(1 + x^3 + \cdots + x^{3n}) + c_1 x(1 + x^3 + \cdots + x^{3n}) + c_2 x^2(1 + x^3 + \cdots + x^{3n})$$

$$\lim_{n \rightarrow \infty} S_{3n} = c_0 \sum_{n=0}^{\infty} x^{3n} + c_1 x \sum_{n=0}^{\infty} x^{3n} + c_2 x^2 \sum_{n=0}^{\infty} x^{3n}$$

Each series is geometric,  $R = 1$ , and the interval of convergence is  $(-1, 1)$ .

(b) For  $|x| < 1$ ,  $f(x) = c_0 \frac{1}{1-x^3} + c_1 x \frac{1}{1-x^3} + c_2 x^2 \frac{1}{1-x^3} = \frac{c_0 + c_1 x + c_2 x^2}{1-x^3}$ .

86. For the series  $\sum c_n x^n$ ,

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} x^{n+1}}{c_n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} x \right| < 1 \Rightarrow |x| < \left| \frac{c_n}{c_{n+1}} \right| = R$$

For the series  $\sum c_n x^{2n}$ ,

$$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} x^{2n+2}}{c_n x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} x^2 \right| < 1 \Rightarrow |x^2| < \left| \frac{c_n}{c_{n+1}} \right| = R \Rightarrow |x| < \sqrt{R}.$$

87. At  $x = x_0 + R$ ,  $\sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n R^n$ , diverges.

At  $x = x_0 - R$ ,  $\sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (-R)^n$ , converges.

Furthermore, at  $x = x_0 - R$ ,

$$\sum_{n=0}^{\infty} |c_n (x - x_0)^n| = \sum_{n=0}^{\infty} C_n R^n, \text{ diverges.}$$

So, the series converges conditionally at  $x_0 - R$ .

## Section 9.9 Representation of Functions by Power Series

$$1. (a) \frac{1}{4-x} = \frac{1/4}{1-(x/4)} \\ = \frac{a}{1-r} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right) \left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$$

This series converges on  $(-4, 4)$ .

$$(b) \begin{array}{r} \frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \cdots \\ 4-x \overline{) 1} \\ 1 - \frac{x}{4} \\ \hline \frac{x}{4} \\ \frac{x}{4} - \frac{x^2}{16} \\ \hline \frac{x^2}{16} \\ \frac{16}{x^2} - \frac{x^3}{64} \\ \hline \frac{16}{x^2} - \frac{x^3}{64} \\ \vdots \end{array}$$

$$2. (a) \frac{1}{2+x} = \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} \\ = \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on  $(-2, 2)$ .

$$(b) \begin{array}{r} \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\ 2+x \overline{) 1} \\ 1 + \frac{x}{2} \\ \hline -\frac{x}{2} \\ -\frac{x}{2} - \frac{x^2}{4} \\ \hline \frac{x^2}{4} \\ \frac{4}{x^2} + \frac{x^3}{8} \\ \hline \frac{x^3}{8} \\ \frac{8}{x^3} - \frac{x^4}{16} \\ \hline \frac{8}{x^3} - \frac{x^4}{16} \\ \vdots \end{array}$$

$$3. (a) \frac{3}{4+x} = \frac{3/4}{1-(-x/4)} = \frac{a}{1-r} \\ = \sum_{n=0}^{\infty} \frac{3}{4} \left(-\frac{x}{4}\right)^n = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{4^{n+1}}$$

This series converges on  $(-4, 4)$ .

$$(b) \begin{array}{r} \frac{3}{4} - \frac{3x}{16} + \frac{3x^2}{64} - \cdots \\ 4+x \overline{) 3} \\ 3 + \frac{3x}{4} \\ \hline -\frac{3x}{4} \\ -\frac{3x}{4} - \frac{3x^2}{16} \\ \hline \frac{3x^2}{16} \\ \frac{16}{3x^2} + \frac{3x^3}{64} \\ \hline \frac{16}{3x^2} + \frac{3x^3}{64} \\ \vdots \end{array}$$

$$4. (a) \frac{2}{5-x} = \frac{2/5}{1-(x/5)} = \frac{a}{1-r} \\ = \sum_{n=0}^{\infty} \frac{2}{5} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{5^{n+1}}$$

This series converges on  $(-5, 5)$ .

$$(b) \begin{array}{r} \frac{2}{5} + \frac{2x}{25} + \frac{2x^2}{125} + \cdots \\ 5-x \overline{) 2} \\ 2 - \frac{2x}{5} \\ \hline \frac{2x}{5} \\ \frac{2x}{5} - \frac{2x^2}{25} \\ \hline \frac{2x^2}{25} \\ \frac{25}{2x^2} \\ \vdots \end{array}$$

$$5. \frac{1}{3-x} = \frac{1}{2-(x-1)} = \frac{1/2}{1-\left(\frac{x-1}{2}\right)} = \frac{a}{1-r} \\ = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{x-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$$

Interval of convergence:  $\left|\frac{x-1}{2}\right| < 1 \Rightarrow (-1, 3)$

$$6. \frac{4}{5-x} = \frac{4}{8-(x+3)} = \frac{1/2}{1-\left(\frac{x+3}{8}\right)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{x+3}{8} \right)^n = \sum_{n=0}^{\infty} \frac{(x+3)^n}{2^{3n+1}}$$

Interval of convergence:  $\left| \frac{x+3}{8} \right| < 1 \Rightarrow (-11, 5)$

$$7. \frac{1}{1-3x} = \frac{a}{1-r} = \sum_{n=0}^{\infty} (3x)^n$$

Interval of convergence:  $|3x| < 1 \Rightarrow \left( \frac{1}{3}, \frac{1}{3} \right)$

$$8. \frac{1}{1-5x} = \frac{a}{1-r} = \sum_{n=0}^{\infty} (5x)^n$$

Interval of convergence:  $|5x| < 1 \Rightarrow \left( \frac{1}{5}, \frac{1}{5} \right)$

$$9. \frac{5}{2x-3} = \frac{5}{-9+2(x+3)} = \frac{-5/9}{1-\frac{2}{9}(x+3)} = \frac{a}{1-r}$$

$$= -\frac{5}{9} \sum_{n=0}^{\infty} \left( \frac{2}{9}(x+3) \right)^n, \left| \frac{2}{9}(x+3) \right| < 1$$

$$= -5 \sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}} (x+3)^n$$

Interval of convergence:  $\left| \frac{2}{9}(x+3) \right| < 1 \Rightarrow \left( -\frac{15}{2}, \frac{3}{2} \right)$

$$10. \frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \left[ -\frac{2}{3}(x-2) \right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n (x-2)^n}{3^n}$$

Interval of convergence:  $|x-2| < \frac{3}{2} \Rightarrow \left( \frac{1}{2}, \frac{7}{2} \right)$

$$11. \frac{2}{2x+3} = \frac{2/3}{1+\frac{2}{3}x} = \frac{2/3}{1-\left(-\frac{2x}{3}\right)} = \frac{a}{1-r}$$

$$= \frac{2}{3} \sum_{n=0}^{\infty} \left( -\frac{2x}{3} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{3^{n+1}} x^n$$

Interval of convergence:  $\left| -\frac{2x}{3} \right| < 1 \Rightarrow \left( -\frac{3}{2}, \frac{3}{2} \right)$

$$12. \frac{4}{3x+2} = \frac{4}{11+3(x-3)} = \frac{4/11}{1-(-3/11)(x-3)} = \frac{a}{1-r}$$

$$= \frac{4}{11} \sum_{n=0}^{\infty} \left[ \frac{-3(x-3)}{11} \right]^n$$

$$= 4 \sum_{n=0}^{\infty} \frac{(-3)^n (x-3)^n}{11^{n+1}}$$

Interval of convergence:  $\left| -\frac{3}{11}(x-3) \right| < 1 \Rightarrow \left( 3, \frac{20}{3} \right)$

$$13. \frac{4x}{x^2+2x-3} = \frac{3}{x+3} + \frac{1}{x-1}$$

$$= \frac{1}{1-(-x/3)} + \frac{-1}{1-x}$$

$$= \sum_{n=0}^{\infty} \left( -\frac{x}{3} \right)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left[ \frac{1}{(-3)^n} - 1 \right] x^n$$

Interval of convergence:  $\left| \frac{x}{3} \right| < 1$  and  $|x| < 1 \Rightarrow (-1, 1)$

$$14. \frac{3x-8}{3x^2+5x-2} = \frac{2}{x+2} - \frac{3}{3x-1}$$

$$= \frac{1}{1-(-x/2)} + \frac{3}{1-3x}$$

$$= \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n + 3 \sum_{n=0}^{\infty} (3x)^n$$

$$= \sum_{n=0}^{\infty} \left[ \left( -\frac{1}{2} \right)^n + 3^{n+1} \right] x^n$$

Interval of convergence:  $\left| \frac{x}{2} \right| < 1$  and

$|3x| < 1 \Rightarrow \left( -\frac{1}{3}, \frac{1}{3} \right)$

$$15. \frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

$$= \sum_{n=0}^{\infty} (1+(-1)^n) x^n = 2 \sum_{n=0}^{\infty} x^{2n}$$

Interval of convergence:  $|x^2| < 1$  or  $(-1, 1)$  because

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right| = |x^2|$$

$$16. \frac{5}{5+x^2} = \frac{1}{1-\left(\frac{-x^2}{5}\right)} = \frac{a}{1-r} = \sum_{n=0}^{\infty} \left(\frac{-x^2}{5}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n x^{2n}$$

$$\text{Interval of convergence: } \left| \frac{x^2}{5} \right| < 1 \Rightarrow -5 < x^2 < 5 \Rightarrow (-\sqrt{5}, \sqrt{5})$$

$$17. \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^{2n} x^n = \sum_{n=0}^{\infty} x^n$$

$$\begin{aligned} h(x) &= \frac{-2}{x^2-1} = \frac{1}{1+x} + \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [(-1)^n + 1] x^n \\ &= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = 2 \sum_{n=0}^{\infty} x^{2n}, (-1, 1) \text{ (See Exercise 15.)} \end{aligned}$$

$$\begin{aligned} 18. h(x) &= \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n \text{ (See Exercise 17.)} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} [(-1)^n - 1] x^n = \frac{1}{2} [0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1}, (-1, 1) \end{aligned}$$

$$19. \text{ By taking the first derivative, you have } \frac{d}{dx} \left[ \frac{1}{x+1} \right] = \frac{-1}{(x+1)^2}. \text{ Therefore,}$$

$$\frac{-1}{(x+1)^2} = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) x^n, (-1, 1).$$

$$20. \text{ By taking the second derivative, you have } \frac{d^2}{dx^2} \left[ \frac{1}{x+1} \right] = \frac{2}{(x+1)^3}. \text{ Therefore,}$$

$$\frac{2}{(x+1)^3} = \frac{d^2}{dx^2} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] = \frac{d}{dx} \left[ \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \right] = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n, (-1, 1).$$

$$21. \text{ By integrating, you have } \int \frac{1}{x+1} dx = \ln(x+1). \text{ Therefore,}$$

$$\ln(x+1) = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, -1 < x \leq 1.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, (-1, 1].$$

22. By integrating, you have

$$\int \frac{1}{1+x} dx = \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x} dx = -\ln(1-x) + C_2.$$

$$f(x) = \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]. \text{ Therefore,}$$

$$\begin{aligned} \ln(1-x^2) &= \int \frac{1}{1+x} dx - \int \frac{1}{1-x} dx \\ &= \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx - \int \left[ \sum_{n=0}^{\infty} x^n \right] dx = \left[ C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right] - \left[ C_2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right] \\ &= C + \sum_{n=0}^{\infty} \frac{[(-1)^n - 1] x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty} \frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty} \frac{(-1)x^{2n+2}}{n+1} \end{aligned}$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(1-x^2) = -\sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1}, (-1, 1).$$

$$23. \frac{1}{x^2+1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, (-1, 1)$$

$$\begin{aligned} 24. \frac{2x}{x^2+1} &= 2x \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ (See Exercise 23.)} \\ &= \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \end{aligned}$$

Because  $\frac{d}{dx}(\ln(x^2+1)) = \frac{2x}{x^2+1}$ , you have

$$\ln(x^2+1) = \int \left[ \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, -1 \leq x \leq 1.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, [-1, 1].$$

$$25. \text{ Because, } \frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ you have } \frac{1}{4x^2+1} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}, \left(-\frac{1}{2}, \frac{1}{2}\right).$$

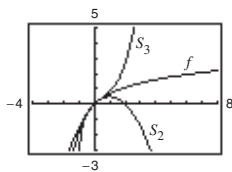
26. Because  $\int \frac{1}{4x^2+1} dx = \frac{1}{2} \arctan(2x)$ , you can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2 \int \frac{1}{4x^2+1} dx = 2 \int \left[ \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \right] dx = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, -\frac{1}{2} < x < \frac{1}{2}.$$

To solve for  $C$ , let  $x = 0$  and conclude that  $C = 0$ . Therefore,

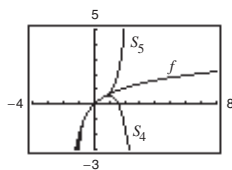
$$\arctan(2x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \left(-\frac{1}{2}, \frac{1}{2}\right).$$

$$27. x - \frac{x^2}{2} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$



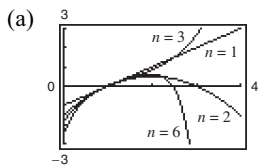
$x$	0.0	0.2	0.4	0.6	0.8	1.0
$S_2 = x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.182	0.336	0.470	0.588	0.693
$S_3 = x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

$$28. x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$



$x$	0.0	0.2	0.4	0.6	0.8	1.0
$S_4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x+1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$S_5 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

$$29. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$



(b) From Example 4,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} = \ln x, \quad 0 < x \leq 2, \quad R = 1.$$

(c)  $x = 0.5$ :

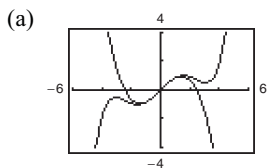
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{-(1/2)^n}{n} \approx -0.693147$$

(d) This is an approximation of  $\ln\left(\frac{1}{2}\right)$ . The error is approximately 0. [The error is less than the first omitted term,

$$1/(51 \cdot 2^{51}) \approx 8.7 \times 10^{-18}.]$$



$$30. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$



$$(b) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x, R = \infty$$

$$(c) \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+1}}{(2n+1)!} \approx 0.4794255386$$

(d) This is an approximation of  $\sin\left(\frac{1}{2}\right)$ . The error is approximately 0.

$$31. g(x) = x \text{ line}$$

Matches (c)

$$32. g(x) = x - \frac{x^3}{3}, \text{ cubic with 3 zeros.}$$

Matches (d)

$$33. g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Matches (a)

$$34. g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7},$$

Matches (b)

In Exercises 35–38,  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ .

$$35. \arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \dots$$

Because  $\frac{1}{5120} < 0.001$ , you can approximate the series by its first two terms:  $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$ .

$$36. \arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\begin{aligned} \int_0^{3/4} \arctan x^2 dx &= \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}} \\ &= \frac{27}{192} - \frac{2187}{344,064} + \frac{177,147}{230,686,720} \end{aligned}$$

Because  $177,147/230,686,720 < 0.001$ , you can approximate the series by its first two terms: 0.134.

$$37. \frac{\arctan x^2}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)} + C \text{ (Note: } C = 0\text{)}$$

$$\int_0^{1/2} \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \dots$$

Because  $\frac{1}{1152} < 0.001$ , you can approximate the series by its first term:  $\int_0^{1/2} \frac{\arctan x^2}{x} dx \approx 0.125$ .

$$38. \quad x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Because  $\frac{1}{1152} < 0.001$ , you can approximate the series by its first term:  $\int_0^{1/2} x^2 \arctan x \, dx \approx 0.016$ .

In Exercises 39–42, use  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$ .

$$39. \quad \frac{1}{(1-x)^2} = \frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} x^n \right] = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

$$40. \quad \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$$

$$\begin{aligned} 41. \quad \frac{1+x}{(1-x)^2} &= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} \\ &= \sum_{n=1}^{\infty} n(x^{n-1} + x^n), \quad |x| < 1 \\ &= \sum_{n=0}^{\infty} (2n+1)x^n, \quad |x| < 1 \end{aligned}$$

$$42. \quad \frac{x(1+x)}{(1-x)^2} = x \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$$

(See Exercise 41.)

$$\begin{aligned} 43. \quad P(n) &= \left(\frac{1}{2}\right)^n \\ E(n) &= \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} \\ &= \frac{1}{2} \frac{1}{[1-(1/2)]^2} = 2 \end{aligned}$$

Because the probability of obtaining a head on a single toss is  $\frac{1}{2}$ , it is expected that, on average, a head will be obtained in two tosses.

$$44. \quad (a) \quad \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n = \frac{2}{9} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^{n-1} = \frac{2}{9} \frac{1}{[1-(2/3)]^2} = 2$$

$$\begin{aligned} (b) \quad \frac{1}{10} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^n &= \frac{9}{100} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^{n-1} \\ &= \frac{9}{100} \cdot \frac{1}{[1-(9/10)]^2} = 9 \end{aligned}$$

45. Because  $\frac{1}{1+x} = \frac{1}{1-(-x)}$ , substitute  $(-x)$  into the geometric series.

46. Because  $\frac{1}{1-x^2} = \frac{1}{1-(x^2)}$ , substitute  $(x^2)$  into the geometric series.

47. Because  $\frac{1}{1+x} = 5 \left( \frac{1}{1-(-x)} \right)$ , substitute  $(-x)$  into the geometric series and then multiply the series by 5.

48. Because  $\ln(1-x) = -\int \frac{1}{1-x} \, dx$ , integrate the series and then multiply by  $(-1)$ .

49. Let  $\arctan x + \arctan y = \theta$ . Then,

$$\tan(\arctan x + \arctan y) = \tan \theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \tan \theta$$

$$\frac{x+y}{1-xy} = \tan \theta$$

$$\arctan \left( \frac{x+y}{1-xy} \right) = \theta.$$

Therefore,

$$\arctan x + \arctan y = \arctan \left( \frac{x+y}{1-xy} \right) \text{ for } xy \neq 1.$$

50. (a) From Exercise 49, you have

$$\begin{aligned} \arctan \frac{120}{119} - \arctan \frac{1}{239} &= \arctan \frac{120}{119} + \arctan \left( -\frac{1}{239} \right) = \arctan \left[ \frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)} \right] \\ &= \arctan \left( \frac{28,561}{28,561} \right) = \arctan 1 = \frac{\pi}{4} \end{aligned}$$

$$(b) \quad 2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan \left[ \frac{2(1/5)}{1 - (1/5)^2} \right] = \arctan \frac{10}{24} = \arctan \frac{5}{12}$$

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan \left[ \frac{2(5/12)}{1 - (5/12)^2} \right] = \arctan \frac{120}{119}$$

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

$$51. (a) \quad 2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan \left[ \frac{\frac{1}{2} + \frac{1}{2}}{1 - (1/2)^2} \right] = \arctan \frac{4}{3}$$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan \left( -\frac{1}{7} \right) = \arctan \left[ \frac{(4/3) - (1/7)}{1 + (4/3)(1/7)} \right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

$$(b) \quad \pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8 \left[ \frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7} \right] - 4 \left[ \frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7} \right] \approx 3.14$$

$$52. (a) \quad \arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan \left[ \frac{(1/2) + (1/3)}{1 - (1/2)(1/3)} \right] = \arctan \left( \frac{5/6}{5/6} \right) = \frac{\pi}{4}$$

$$(b) \quad \pi = 4 \left[ \arctan \frac{1}{2} + \arctan \frac{1}{3} \right] \\ = 4 \left[ \frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} \right] + 4 \left[ \frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} \right] \approx 4(0.4635) + 4(0.3217) \approx 3.14$$

53. From Exercise 21, you have

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \\ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

$$\text{So, } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/2)^n}{n} \\ = \ln \left( \frac{1}{2} + 1 \right) = \ln \frac{3}{2} \approx 0.4055.$$

54. From Exercise 53, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/3)^n}{n} \\ = \ln \left( \frac{1}{3} + 1 \right) = \ln \frac{4}{3} \approx 0.2877.$$

55. From Exercise 53, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2/5)^n}{n} \\ = \ln \left( \frac{2}{5} + 1 \right) = \ln \frac{7}{5} \approx 0.3365.$$

$$56. \text{ From Example 5, you have } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} \\ = \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

57. From Exercise 56, you have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} \\ = \arctan \frac{1}{2} \approx 0.4636.$$

58. From Exercise 56, you have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} \\ = \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1} \\ = \arctan \frac{1}{3} \approx 0.3218.$$

59.  $f(x) = \arctan x$  is an odd function (symmetric to the origin).

60. The approximations of degree 3, 7, 11, ...,  $(4n - 1, n = 1, 2, \dots)$  have relative extrema.

61. The series in Exercise 56 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

62. Because  $\frac{d}{dx} \left[ \sum_{n=0}^{\infty} a_n x^n \right] = \sum_{n=1}^{\infty} n a_n x^{n-1}$ , the radius of convergence is the same, 3.

63. Because the first series is the derivative of the second series, the second series converges for  $|x + 1| < 4$  (and perhaps at the endpoints,  $x = 3$  and  $x = -5$ .)

64. You can verify that the statement is incorrect by calculating the constant terms of each side:

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left( \frac{x}{5} \right)^n = (1 + 1) + \left( x + \frac{x}{5} \right) + \dots$$

$$\sum_{n=0}^{\infty} \left( 1 + \frac{1}{5} \right) x^n = \left( 1 + \frac{1}{5} \right) + \left( 1 + \frac{1}{5} \right) x + \dots$$

The formula should be

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left( \frac{x}{5} \right)^n = \sum_{n=0}^{\infty} \left[ 1 + \left( \frac{1}{5} \right)^n \right] x^n.$$

$$65. \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n (2n + 1)}$$

From Example 5 you have  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}$ .

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n (2n + 1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n} (2n + 1)} \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n + 1} \\ &= \sqrt{3} \arctan \left( \frac{1}{\sqrt{3}} \right) \\ &= \sqrt{3} \left( \frac{\pi}{6} \right) \approx 0.9068997 \end{aligned}$$

$$\begin{aligned} 66. \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1} (2n + 1)!} &= \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n+1}}{(2n + 1)!} \\ &= \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \approx 0.866025 \end{aligned}$$

67. Using a graphing utility, you obtain the following partial sums for the left hand side. Note that  $1/\pi \approx 0.3183098862$ .

$$n = 0: S_0 \approx 0.3183098784$$

$$n = 1: S_1 \approx 0.3183098862$$

## Section 9.10 Taylor and Maclaurin Series

1. For  $c = 0$ , you have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

2. For  $c = 0$ , you have:

$$f(x) = e^{3x}$$

$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(0) = 3^n$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}.$$

3. For  $c = \pi/4$ , you have:

$$f(x) = \cos(x) \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \quad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \quad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, you have:

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[ 1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \dots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^n}{n!}. \end{aligned}$$

[Note:  $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \dots$ ]

4. For  $c = \pi/4$ , you have:

$$f(x) = \sin x \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore you have:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[ 1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \dots \right] \\ &= \frac{\sqrt{2}}{2} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2} [x - (\pi/4)]^{n+1}}{(n+1)!} + 1 \right\}. \end{aligned}$$

5. For  $c = 1$ , you have

$$\begin{aligned} f(x) &= \frac{1}{x} = x^{-1} & f(1) &= 1 \\ f'(x) &= -x^{-2} & f'(1) &= -1 \\ f''(x) &= 2x^{-3} & f''(1) &= 2 \\ f'''(x) &= -6x^{-4} & f'''(1) &= -6 \end{aligned}$$

and so on. Therefore, you have

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \end{aligned}$$

7. For  $c = 1$ , you have,

$$\begin{aligned} f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= 1 \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -1 \\ f'''(x) &= \frac{2}{x^3} & f'''(1) &= 2 \\ f^{(4)}(x) &= -\frac{6}{x^4} & f^{(4)}(1) &= -6 \\ f^{(5)}(x) &= \frac{24}{x^5} & f^{(5)}(1) &= 24 \end{aligned}$$

and so on. Therefore, you have:

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \cdots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}. \end{aligned}$$

8. For  $c = 1$ , you have:

$$\begin{aligned} f(x) &= e^x \\ f^{(n)}(x) &= e^x \Rightarrow f^{(n)}(1) = e \\ e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e \left[ 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots \right] = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}. \end{aligned}$$

6. For  $c = 2$ , you have

$$\begin{aligned} f(x) &= \frac{1}{1-x} = (1-x)^{-1} & f(2) &= -1 \\ f'(x) &= (1-x)^{-2} & f'(2) &= 1 \\ f''(x) &= 2(1-x)^{-3} & f''(2) &= -2 \\ f'''(x) &= 6(1-x)^{-4} & f'''(2) &= 6 \end{aligned}$$

and so on. Therefore you have

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)(x-2)^n}{n!} \\ &= -1 + (x-2) - (x-2)^2 + (x-2)^3 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n \end{aligned}$$

9. For  $c = 0$ , you have

$$\begin{aligned} f(x) &= \sin 3x & f(0) &= 0 \\ f'(x) &= 3 \cos 3x & f'(0) &= 3 \\ f''(x) &= -9 \sin 3x & f''(0) &= 0 \\ f'''(x) &= -27 \cos 3x & f'''(0) &= -27 \\ f^{(4)}(x) &= 81 \sin 3x & f^{(4)}(0) &= 0 \end{aligned}$$

and so on. Therefore you have

$$\sin 3x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 3x + 0 - \frac{27x^3}{3!} + 0 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

10. For  $c = 0$ , you have.

$$\begin{aligned} f(x) &= \ln(x^2 + 1) & f(0) &= 0 \\ f'(x) &= \frac{2x}{x^2 + 1} & f'(0) &= 0 \\ f''(x) &= \frac{2 - 2x^2}{(x^2 + 1)^2} & f''(0) &= 2 \\ f'''(x) &= \frac{4x(x^2 - 3)}{(x^2 + 1)^3} & f'''(0) &= 0 \\ f^{(4)}(x) &= \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} & f^{(4)}(0) &= -12 \\ f^{(5)}(x) &= \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} & f^{(6)}(0) &= 240 \end{aligned}$$

and so on. Therefore, you have:

$$\begin{aligned} \ln(x^2 + 1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}. \end{aligned}$$

11. For  $c = 0$ , you have:

$$\begin{aligned} f(x) &= \sec(x) & f(0) &= 1 \\ f'(x) &= \sec(x) \tan(x) & f'(0) &= 0 \\ f''(x) &= \sec^3(x) + \sec(x) \tan^2(x) & f''(0) &= 1 \\ f'''(x) &= 5 \sec^3(x) \tan(x) + \sec(x) \tan^3(x) & f'''(0) &= 0 \\ f^{(4)}(x) &= 5 \sec^5(x) + 18 \sec^3(x) \tan^2(x) + \sec(x) \tan^4(x) & f^{(4)}(0) &= 5 \\ \sec(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots \end{aligned}$$

12. For
- $c = 0$
- , you have;

$$\begin{aligned}
 f(x) &= \tan(x) & f(0) &= 0 \\
 f'(x) &= \sec^2(x) & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2(x) \tan(x) & f''(0) &= 0 \\
 f'''(x) &= 2[\sec^4(x) + 2 \sec^2(x) \tan^2(x)] & f'''(0) &= 2 \\
 f^{(4)}(x) &= 8[\sec^4(x) \tan(x) + \sec^2(x) \tan^3(x)] & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 8[2 \sec^6(x) + 11 \sec^4(x) \tan^2(x) + 2 \sec^2(x) \tan^4(x)] & f^{(5)}(0) &= 16 \\
 \tan(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots.
 \end{aligned}$$

13. The Maclaurin series for
- $f(x) = \cos x$
- is
- $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
- .

Because  $f^{(n+1)}(x) = \pm \sin x$  or  $\pm \cos x$ , you have  $|f^{(n+1)}(z)| \leq 1$  for all  $z$ . So by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Because  $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ , it follows that  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ . So, the Maclaurin series for  $\cos x$  converges to  $\cos x$  for all  $x$ .

14. The Maclaurin series for
- $f(x) = e^{-2x}$
- is
- $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$
- .

$f^{(n+1)}(x) = (-2)^{n+1} e^{-2x}$ . So, by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{(-2)^{n+1} e^{-2z}}{(n+1)!} x^{n+1} \right|.$$

Because  $\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} x^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \right| = 0$ , it follows that  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ .

So, the Maclaurin Series for  $e^{-2x}$  converges to  $e^{-2x}$  for all  $x$ .

15. The Maclaurin series for
- $f(x) = \sinh x$
- is
- $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
- .

$f^{(n+1)}(x) = \sinh x$  (or  $\cosh x$ ). For fixed  $x$ ,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{\sinh(z)}{(n+1)!} x^{n+1} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(The argument is the same if  $f^{(n+1)}(x) = \cosh x$ ). So, the Maclaurin series for  $\sinh x$  converges to  $\sinh x$  for all  $x$ .

16. The Maclaurin series for
- $f(x) = \cosh x$
- is
- $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
- .

$f^{(n+1)}(x) = \sinh x$  (or  $\cosh x$ ). For fixed  $x$ ,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{\sinh(z)}{(n+1)!} x^{n+1} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(The argument is the same if  $f^{(n+1)}(x) = \cosh x$ ). So, the Maclaurin series for  $\cosh x$  converges to  $\cosh x$  for all  $x$ .



17. Because  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$ , you have

$$\begin{aligned}(1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{4!} - \dots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n.\end{aligned}$$

18. Because  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$

you have

$$(1+x)^{-4} = 1 - 4x + \frac{4(5)x^2}{2!} - \frac{4(5)(6)x^3}{3!} + \frac{4(5)(6)(7)x^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(n+3)!}{3!n!} x^n$$

19. Because  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$ , you have

$$\begin{aligned}[1 + (-x)]^{-1/2} &= 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \dots \\ &= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}.\end{aligned}$$

20. Because  $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$  you have

$$\begin{aligned}[1 + (-x^2)]^{-1/2} &= 1 - \frac{1}{2}x^2 + \frac{(1/2)(3/2)x^4}{2!} - \frac{(1/2)(3/2)(5/2)x^6}{3!} + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{(1)(3)x^4}{2^2 2!} - \frac{(1)(3)(5)x^6}{2^3 3!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^{2n}\end{aligned}$$

21.  $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$  and because  $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$ , you have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!}\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

22.  $\frac{1}{(2+x)^3} = \frac{1}{8} \left(1 + \frac{x}{2}\right)^{-3}, \quad k = -3$

$$\frac{1}{(2+x)^3} = \frac{1}{8} \left\{1 - 3\left(\frac{x}{2}\right) + \frac{3(4)}{2!}\left(\frac{x}{2}\right)^2 - \frac{3(4)(5)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right\} = \frac{1}{8} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+2)!}{2^{n+1} n!} x^n\right]$$

23.  $\sqrt{1+x} = (1+x)^{1/2}, \quad k = 1/2$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1/2(-1/2)}{2!}x^2 + \frac{1/2(-1/2)(-3/2)}{3!}x^3 + \dots = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n$$

$$\begin{aligned}
 24. \quad (1+x)^{1/4} &= 1 + \frac{1}{4}x + \frac{(1/4)(-3/4)}{2!}x^2 + \frac{(1/4)(-3/4)(-7/4)}{3!}x^3 + \cdots \\
 &= 1 + \frac{1}{4}x - \frac{3}{4^2 2!}x^2 + \frac{3 \cdot 7}{4^3 3!}x^3 - \frac{3 \cdot 7 \cdot 11}{4^4 4!}x^4 + \cdots \\
 &= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-5)}{4^n n!} x^n
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \text{Because } (1+x)^{1/2} &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n \\
 \text{you have } (1+x^2)^{1/2} &= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{2n}.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{Because } (1+x)^{1/2} &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^n \\
 \text{you have } (1+x^3)^{1/2} &= 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{3n}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
 e^{x^2/2} &= \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots
 \end{aligned}$$

$$\begin{aligned}
 28. \quad e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
 e^{-3x} &= \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots
 \end{aligned}$$

$$29. \quad \ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$30. \quad \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\ln(x^2+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}, \quad -1 < x \leq 1$$

$$31. \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

$$32. \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!}$$

$$33. \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\begin{aligned}
 \cos 4x &= \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} \\
 &= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \cdots
 \end{aligned}$$

$$34. \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

$$35. \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\begin{aligned} \cos x^{3/2} &= \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} \\ &= 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \dots \end{aligned}$$

$$\begin{aligned} 36. \quad \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ 2 \sin x^3 &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} \\ &= 2 \left( x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots \right) \\ &= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \dots \end{aligned}$$

$$\begin{aligned} 39. \quad \cos^2(x) &= \frac{1}{2} [1 + \cos(2x)] \\ &= \frac{1}{2} \left[ 1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} - \dots \right] = \frac{1}{2} \left[ 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right] \end{aligned}$$

40. The formula for the binomial series gives  $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$ , which implies that

$$\begin{aligned} (1+x^2)^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!} \\ \ln(x + \sqrt{x^2 + 1}) &= \int \frac{1}{\sqrt{x^2 + 1}} dx \\ &= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2^n (2n+1)n!} \\ &= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots \end{aligned}$$

$$41. \quad x \sin x = x \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

$$42. \quad x \cos x = x \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

$$\begin{aligned} 43. \quad \frac{\sin x}{x} &= \frac{x - (x^3/3!) + (x^5/5!) - \dots}{x} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, \quad x \neq 0 \\ &= 1, \quad x = 0 \end{aligned}$$

$$37. \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \dots$$

$$\begin{aligned} \sinh(x) &= \frac{1}{2}(e^x - e^{-x}) \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

$$38. \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$

$$2 \cosh(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2 \frac{x^{2n}}{(2n)!}$$

$$44. \frac{\arcsin x}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{(2^n n!)^2 (2n+1)} \cdot \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{(2^n n!)^2 (2n+1)}, x \neq 0$$

$$= 1, x = 0$$

$$45. e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

$$46. e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots \quad (\text{See Exercise 45.})$$

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

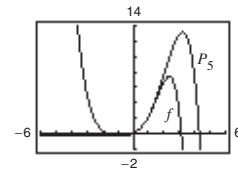
$$47. f(x) = e^x \sin x$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

$$P_5(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$$



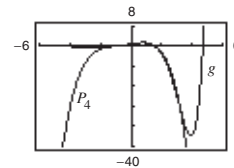
$$48. g(x) = e^x \cos x$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots\right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$

$$= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{x^3}{2}\right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

$$P_4(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$



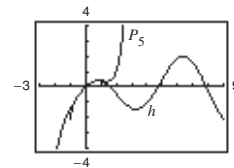
$$49. h(x) = \cos x \ln(1+x)$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots$$

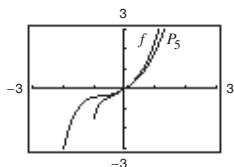
$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$$

$$P_5(x) = x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40}$$



50.  $f(x) = e^x \ln(1+x)$

$$\begin{aligned}
 &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\
 &= x + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6}\right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\
 &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \\
 P_5(x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40}
 \end{aligned}$$

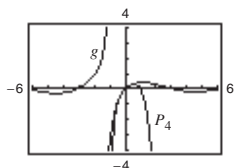


51.  $g(x) = \frac{\sin x}{1+x}$ . Divide the series for  $\sin x$  by  $(1+x)$ .

$$\begin{array}{r}
 x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots \\
 1+x \overline{) \begin{array}{l} x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots \\ \underline{x + x^2} \\ -x^2 - \frac{x^3}{6} \\ \underline{-x^2 - x^3} \\ \frac{5x^3}{6} + 0x^4 \\ \frac{5x^3}{6} + \frac{5x^4}{6} \\ \underline{-\frac{5x^4}{6} + \frac{x^5}{120}} \\ -\frac{5x^4}{6} - \frac{5x^5}{6} \\ \underline{\phantom{-\frac{5x^4}{6} - \frac{5x^5}{6}} \vdots} \end{array} \\
 \end{array}$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$

$$P_4(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6}$$

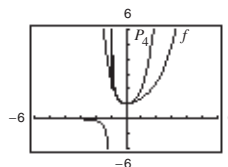


52.  $f(x) = \frac{e^x}{1+x}$ . Divide the series for  $e^x$  by  $(1+x)$ .

$$\begin{array}{r}
 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots \\
 1+x \overline{) \begin{array}{l} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \\ \underline{1 + x} \\ 0 + \frac{x^2}{2} + \frac{x^3}{6} \\ \underline{\frac{x^2}{2} + \frac{x^3}{2}} \\ -\frac{x^3}{2} + \frac{x^4}{6} \\ \underline{-\frac{x^3}{2} + \frac{x^4}{3}} \\ \frac{3x^4}{8} + \frac{x^5}{120} \\ \underline{\frac{3x^4}{8} + \frac{3x^5}{8}} \\ \vdots \end{array} \\
 \end{array}$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$

$$P_4(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8}$$



53.  $y = x^2 - \frac{x^4}{3!} = x\left(x - \frac{x^3}{3!}\right)$

$$f(x) = x \sin x$$

Matches (c)

$$54. \quad y = x - \frac{x^3}{2!} + \frac{x^5}{4!} = x \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \right)$$

$$f(x) = x \cos x$$

Matches (d)

$$55. \quad y = x + x^2 + \frac{x^3}{2!} = x \left( 1 + x + \frac{x^2}{2!} \right)$$

$$f(x) = xe^x$$

Matches (a)

$$56. \quad y = x^2 - x^3 + x^4 = x^2(1 - x + x^2)$$

$$f(x) = x^2 \frac{1}{1+x}$$

Matches (b)

$$\begin{aligned} 58. \quad \int_0^x \sqrt{1+t^3} \, dt &= \int_0^x \left[ 1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n}}{2^n n!} \right] dt \\ &= \left[ t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n+1}}{(3n+1)2^n n!} \right]_0^x \\ &= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n+1}}{(3n+1)2^n n!} \end{aligned}$$

$$59. \text{ Because } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad (0 < x \leq 2)$$

$$\text{you have } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931. \quad (10,001 \text{ terms})$$

$$60. \text{ Because } \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \text{ you have}$$

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415. \quad (4 \text{ terms})$$

$$61. \text{ Because } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\text{you have } e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891. \quad (12 \text{ terms})$$

$$62. \text{ Because } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \text{ you have } e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$\text{and } \frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321. \quad (6 \text{ terms})$$

$$\begin{aligned} 57. \quad \int_0^x (e^{-t^2} - 1) \, dt &= \int_0^x \left[ \left( \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) - 1 \right] dt \\ &= \int_0^x \left[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!} \right] dt \\ &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!} \end{aligned}$$

63. Because

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\ 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\ \frac{1 - \cos x}{x} &= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} \\ \text{you have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} = 0.\end{aligned}$$

64. Because

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \\ \text{you have } \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.\end{aligned}$$

65. Because  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\begin{aligned}e^x - 1 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \\ \text{and } \frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \\ \text{you have } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = 1.\end{aligned}$$

66. Because  $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(See Exercise 29.)

$$\begin{aligned}\frac{\ln(x+1)}{x} &= 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} \\ \text{you have } \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} &= \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = 1.\end{aligned}$$

68.  $\int_0^{1/4} x \ln(x+1) dx = \int_0^{1/4} \left( x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots \right]_0^{1/4}$

Because  $\frac{(1/4)^5}{15} < 0.0001$ ,  $\int_0^{1/4} x \ln(x+1) dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472$ .

69.  $\int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \right] dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$

Because  $1/(7 \cdot 7!) < 0.0001$ , you need three terms:

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \cdots \approx 0.9461. \quad (\text{using three nonzero terms})$$

**Note:** You are using  $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$ .

67.  $\int_0^1 e^{-x^3} dx = \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} \right] dx$

$$\begin{aligned}&= \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \right] dx \\&= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)n!} \right]_0^1 \\&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)n!} \\&= 1 - \frac{1}{4} + \frac{1}{14} - \cdots + (-1)^n \frac{1}{(3n+1)n!} + \cdots\end{aligned}$$

Because  $\frac{1}{[3(6)+1]6!} < 0.0001$ , you need 6 terms.

$$\int_0^1 e^{-x^3} dx \approx \sum_{n=0}^5 \frac{(-1)^n}{(3n+1)n!} \approx 0.8075$$

$$\begin{aligned}
 70. \int_0^1 \cos x^2 \, dx &= \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \right] dx \\
 &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} \right]_0^1 \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}
 \end{aligned}$$

$$\int_0^1 \cos x^2 \, dx \approx \sum_{n=0}^3 \frac{(-1)^n}{(4n+1)(2n)!} \approx 0.904523$$

Because  $\frac{1}{[4(4)+1][2(4)]!} < 0.0001$ , you need 4 terms.

$$\begin{aligned}
 71. \int_0^{1/2} \frac{\arctan x}{x} \, dx &= \int_0^{1/2} \left( 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots \right) dx \\
 &= \left[ x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots \right]_0^{1/2}
 \end{aligned}$$

Because  $1/(9^2 2^9) < 0.0001$ , you have

$$\begin{aligned}
 \int_0^{1/2} \frac{\arctan x}{x} \, dx &\approx \left( \frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9} \right) \\
 &\approx 0.4872.
 \end{aligned}$$

**Note:** You are using  $\lim_{x \rightarrow 0^+} \frac{\arctan x}{x} = 1$ .

$$73. \int_{0.1}^{0.3} \sqrt{1+x^3} \, dx = \int_{0.1}^{0.3} \left( 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \cdots \right) dx = \left[ x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \cdots \right]_{0.1}^{0.3}$$

Because  $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$ , you need two terms.

$$\int_{0.1}^{0.3} \sqrt{1+x^3} \, dx = \left[ (0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) \right] \approx 0.201.$$

$$74. \sqrt{1+x^2} = (1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)x^4}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^6}{3!} + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots$$

$$\int_0^{0.2} \sqrt{1+x^2} \, dx = \int_0^{0.2} \left[ 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \cdots \right] dx = \left[ x + \frac{x^3}{6} - \frac{x^5}{40} + \frac{x^7}{112} - \cdots \right]_0^{0.2}$$

Because  $\frac{(0.2)^5}{40} < 0.0001$ , you need 2 terms.

$$\int_0^{0.2} \sqrt{1+x^2} \, dx \approx 0.2 + \frac{(0.2)^3}{6} \approx 0.201333$$

$$\begin{aligned}
 72. \int_0^{1/2} \arctan(x^2) \, dx &= \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} \right] dx \\
 &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)} \right]_0^{1/2} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)2^{4n+3}}
 \end{aligned}$$

Because  $\frac{1}{(4n+3)(2n+1)2^{4n+3}} < 0.0001$

when  $n = 2$ , you need 2 terms.

$$\int_0^{1/2} \arctan(x^2) \, dx \approx \frac{1}{3(1) \cdot 2^3} - \frac{1}{7(3)2^7} \approx 0.041295$$



$$75. \int_0^{\pi/2} \sqrt{x} \cos x \, dx = \int_0^{\pi/2} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!} \right] dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!} \right]_0^{\pi/2} = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!} \right]_0^{\pi/2}$$

Because  $2(\pi/2)^{23/2}/(23 \cdot 10!) < 0.0001$ , you need five terms.

$$\int_0^1 \sqrt{x} \cos x \, dx = 2 \left[ \frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10,800} + \frac{(\pi/2)^{19/2}}{766,080} \right] \approx 0.7040.$$

$$76. \int_{0.5}^1 \cos \sqrt{x} \, dx = \int_{0.5}^1 \left( 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \dots \right) dx = \left[ x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \dots \right]_{0.5}^1$$

Because  $\frac{1}{201,600}(1 - 0.5^5) < 0.0001$ , you have

$$\int_{0.5}^1 \cos \sqrt{x} \, dx \approx \left[ (1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5^5) \right] \approx 0.3243.$$

77. From Exercise 27, you have

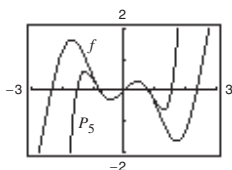
$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2} \, dx &= \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \, dx = \frac{1}{\sqrt{2\pi}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7} \right) \approx 0.3412. \end{aligned}$$

78. From Exercise 27, you have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} \, dx &= \frac{1}{\sqrt{2\pi}} \int_1^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \, dx = \frac{1}{\sqrt{2\pi}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_1^2 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2^{n+1} - 1)}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{7}{2 \cdot 1 \cdot 3} + \frac{31}{2^2 \cdot 2! \cdot 5} - \frac{127}{2^3 \cdot 3! \cdot 7} + \frac{511}{2^4 \cdot 4! \cdot 9} - \frac{2047}{2^5 \cdot 5! \cdot 11} \right) \\ &\quad + \frac{8191}{2^6 \cdot 6! \cdot 13} - \frac{32,767}{2^7 \cdot 7! \cdot 15} + \frac{131,071}{2^8 \cdot 8! \cdot 17} - \frac{524,287}{2^9 \cdot 9! \cdot 19} \approx 0.1359. \end{aligned}$$

$$79. f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$$

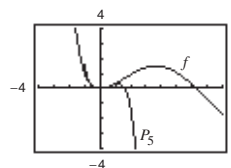
$$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$$



The polynomial is a reasonable approximation on the interval  $\left[-\frac{3}{4}, \frac{3}{4}\right]$ .

$$80. f(x) = \sin \frac{x}{2} \ln(1+x)$$

$$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$$

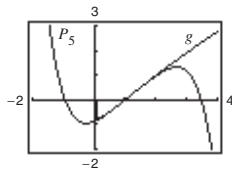


The polynomial is a reasonable approximation on the interval  $(-0.60, 0.73)$ .

81.  $f(x) = \sqrt{x} \ln x, c = 1$

$$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$$

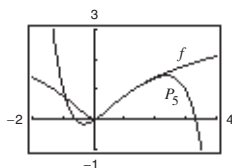
The polynomial is a reasonable approximation on the interval  $\left[\frac{1}{4}, 2\right]$ .



82.  $f(x) = \sqrt[3]{x} \cdot \arctan x, c = 1$

$$P_5(x) \approx 0.7854 + 0.7618(x-1) - 0.3412 \left[ \frac{(x-1)^2}{2!} \right] - 0.0424 \left[ \frac{(x-1)^3}{3!} \right] + 1.3025 \left[ \frac{(x-1)^4}{4!} \right] - 5.5913 \left[ \frac{(x-1)^5}{5!} \right]$$

The polynomial is a reasonable approximation on the interval  $(0.48, 1.75)$ .



83. See Guidelines, page 682.

84.  $a_{2n+1} = 0$  (odd coefficients are zero)

85. The binomial series is  $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$ . The radius of convergence is  $R = 1$ .

86. (a) Replace  $x$  with  $(-x)$ .

(b) Replace  $x$  with  $3x$ .

(c) Multiply series by  $x$ .

(d) Replace  $x$  with  $2x$ , then replace  $x$  with  $-2x$ , and add the two together.

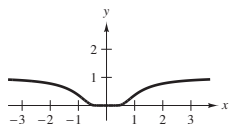
$$\begin{aligned} 87. y &= \left( \tan \theta - \frac{g}{kv_0 \cos \theta} \right) x - \frac{g}{k^2} \ln \left( 1 - \frac{kx}{v_0 \cos \theta} \right) \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} - \frac{g}{k^2} \left[ -\frac{kx}{v_0 \cos \theta} - \frac{1}{2} \left( \frac{kx}{v_0 \cos \theta} \right)^2 - \frac{1}{3} \left( \frac{kx}{v_0 \cos \theta} \right)^3 - \frac{1}{4} \left( \frac{kx}{v_0 \cos \theta} \right)^4 - \dots \right] \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} + \frac{gx}{kv_0 \cos \theta} + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{gk^2x^4}{4v_0^4 \cos^4 \theta} + \dots \\ &= (\tan \theta)x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2gx^4}{4v_0^4 \cos^4 \theta} + \dots \end{aligned}$$

88.  $\theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$

$$\begin{aligned} y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \dots \\ &= \sqrt{3}x - 32 \left[ \frac{2^2 x^2}{2(64)^2} + \frac{2^3 x^3}{3(64)^3 16} + \frac{2^4 x^4}{4(64)^4 (16)^2} + \dots \right] \\ &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n (16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n (16)^{n-2}} \end{aligned}$$

89.  $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b)  $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$

Let  $y = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$ . Then

$$\ln y = \lim_{x \rightarrow 0} \ln \left( \frac{e^{-1/x^2}}{x} \right) = \lim_{x \rightarrow 0^+} \left[ -\frac{1}{x^2} - \ln x \right] = \lim_{x \rightarrow 0^+} \left[ \frac{-1 - x^2 \ln x}{x^2} \right] = -\infty.$$

So,  $y = e^{-\infty} = 0$  and you have  $f'(0) = 0$ .

(c)  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots = 0 \neq f(x)$  This series converges to  $f$  at  $x = 0$  only.

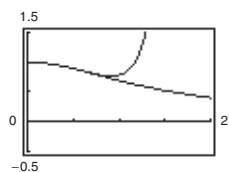
90. (a)  $f(x) = \frac{\ln(x^2 + 1)}{x^2}$ .

From Exercise 10, you obtain:

$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}.$$

(b)



$$(c) F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$$

$$G(x) = \int_0^x P_8(t) dt$$

$x$	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6924	0.8865	1.6878	9.6063

(d) The curves are nearly identical for  $0 < x < 1$ . Hence, the integrals nearly agree on that interval.

91. By the Ratio Test:  $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$  which shows that  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for all  $x$ .

$$92. \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots = 2x \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}, R = 1$$

$$\ln 3 = \ln\left(\frac{1+1/2}{1-1/2}\right) \approx 2\left(\frac{1}{2}\right) \left[1 + \frac{(1/2)^2}{3} + \frac{(1/2)^4}{5} + \frac{(1/2)^6}{7}\right] = 1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448} \approx 1.098065$$

$$(\ln 3 \approx 1.098612)$$

$$93. \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$$

$$96. \binom{-1/3}{5} = \frac{(-1/3)(-4/3)(-7/3)(-10/3)(-13/3)}{5!}$$

$$= \frac{-91}{729} \approx -0.12483$$

$$94. \binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$$

$$97. (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$95. \binom{0.5}{4} = \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!} = -0.0390625 = -\frac{5}{128}$$

$$\text{Example: } (1+x)^2 = \sum_{n=0}^{\infty} \binom{2}{n} x^n = 1 + 2x + x^2$$

98. Assume  $e = p/q$  is rational. Let  $N > q$  and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!}\right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

$$\text{Set } a = N! \left[ e - \left(1 + 1 + \cdots + \frac{1}{N!}\right) \right], \text{ a positive integer. But,}$$

$$a = N! \left[ \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots$$

$$= \frac{1}{N+1} \left[ 1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \right] = \frac{1}{N+1} \left[ \frac{1}{1 - \left(\frac{1}{N+1}\right)} \right] = \frac{1}{N}, \text{ a contradiction.}$$

99.  $g(x) = \frac{x}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \cdots$

$$x = (1-x-x^2)(a_0 + a_1x + a_2x^2 + \cdots)$$

$$x = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + \cdots$$

Equating coefficients,

$$a_0 = 0$$

$$a_1 - a_0 = 1 \Rightarrow a_1 = 1$$

$$a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = 1$$

$$a_3 - a_2 - a_1 = 0 \Rightarrow a_3 = 2$$

$$a_4 = a_3 + a_2 = 3, \text{ etc.}$$

In general,  $a_n = a_{n-1} + a_{n-2}$ . The coefficients are the Fibonacci numbers.

100. Assume the interval is  $[-1, 1]$ . Let  $x \in [-1, 1]$ ,

$$f(1) = f(x) + (1-x)f'(x) + \frac{1}{2}(1-x)^2 f''(c), c \in (x, 1)$$

$$f(-1) = f(x) + (-1-x)f'(x) + \frac{1}{2}(-1-x)^2 f''(d), d \in (-1, x).$$

$$\text{So, } f(1) - f(-1) = 2f'(x) + \frac{1}{2}(1-x)^2 f''(c) - \frac{1}{2}(1+x)^2 f''(d)$$

$$2f'(x) = f(1) - f(-1) - \frac{1}{2}(1-x)^2 f''(c) + \frac{1}{2}(1+x)^2 f''(d).$$

Because  $|f(x)| \leq 1$  and  $|f''(x)| \leq 1$ ,

$$2|f'(x)| \leq |f(1)| + |f(-1)| + \frac{1}{2}(1-x)^2 |f''(c)| + \frac{1}{2}(1+x)^2 |f''(d)| \leq 1 + 1 + \frac{1}{2}(1-x^2) + \frac{1}{2}(1+x)^2 = 3 + x^2 \leq 4.$$

So,  $|f'(x)| \leq 2$ .

**Note:** Let  $f(x) = \frac{1}{2}(x+1)^2 - 1$ . Then  $|f'(x)| \leq 1$ ,  $|f''(x)| = 1$  and  $f'(1) = 2$ .

## Review Exercises for Chapter 9

1.  $a_n = \frac{1}{n! + 1}$

2.  $a_n = \frac{n}{n^2 + 1}$

3.  $a_n = 4 + \frac{2}{n}$ : 6, 5, 4.67, ...

Matches (a)

4.  $a_n = 4 - \frac{n}{2}$ : 3.5, 3, ...

Matches (c)

5.  $a_n = 10(0.3)^{n-1}$ : 10, 3, ...

Matches (d)

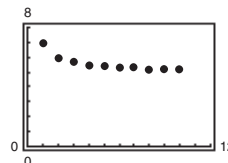
6.  $a_n = 6\left(-\frac{2}{3}\right)^{n-1}$ : 6, -4, ...

Matches (b)

7.  $a_n = \frac{5n+2}{n}$

The sequence seems to converge to 5.

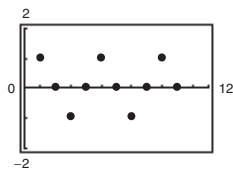
$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5n+2}{n} \\ &= \lim_{n \rightarrow \infty} \left(5 + \frac{2}{n}\right) = 5 \end{aligned}$$



8.  $a_n = \sin \frac{n\pi}{2}$

The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$



9.  $\lim_{n \rightarrow \infty} \left[ \left( \frac{7}{8} \right)^n + 3 \right] = 3$

Converges

10.  $\lim_{n \rightarrow \infty} \left[ 1 + \frac{5}{n+1} \right] = 1$

Converges

11.  $\lim_{n \rightarrow \infty} \frac{n^3 + 1}{n^2} = \infty$

Diverges

12.  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

Converges

13.  $\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = 0$

Converges

14.  $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$

Diverges

$$\begin{aligned} 15. \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) &= \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \end{aligned}$$

Converges

16.  $\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2n} \right)^n = \lim_{k \rightarrow \infty} \left[ \left( 1 + \frac{1}{k} \right)^k \right]^{1/2} = e^{1/2}$

Converges;  $k = 2n$

17.  $\lim_{n \rightarrow \infty} \frac{\sin \sqrt{n}}{\sqrt{n}} = 0$

Converges

18. Let  $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c).$$

Assume  $b \geq c$  and note that the terms

$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as  $n \rightarrow \infty$ . Hence  $a_n$  converges.

19. (a)  $A_n = 8000 \left( 1 + \frac{0.05}{4} \right)^n, \quad n = 1, 2, 3, \dots$

$$A_1 = 8000 \left( 1 + \frac{0.05}{4} \right)^1 = \$8100.00$$

$$A_2 = \$8201.25$$

$$A_3 = \$8303.77$$

$$A_4 = \$8407.56$$

$$A_5 = \$8512.66$$

$$A_6 = \$8619.07$$

$$A_7 = \$8726.80$$

$$A_8 = \$8835.89$$

(b)  $A_{40} = \$13,148.96$

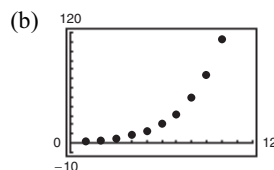
20. (a)  $V_n = 175,000(0.70)^n, \quad n = 1, 2, \dots$

(b)  $V_5 = 175,000(0.70)^5 \approx \$29,412.25$

21. (a)

$n$	5	10	15	20	25
$S_n$	13.2	113.3	873.8	6648.5	50,500.3

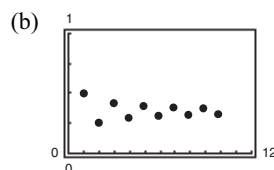
The series diverges (geometric  $r = \frac{3}{2} > 1$ ).



22. (a)

$n$	5	10	15	20	25
$S_n$	0.3917	0.3228	0.3627	0.3344	0.3564

The series converges by the Alternating Series Test.

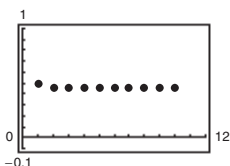


23. (a)

$n$	5	10	15	20	25
$S_n$	0.4597	0.4597	0.4597	0.4597	0.4597

The series converges by the Alternating Series Test.

(b)

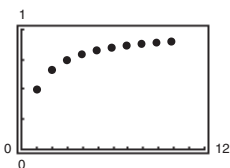


24. (a)

$n$	5	10	15	20	25
$S_n$	0.8333	0.9091	0.9375	0.9524	0.9615

The series converges, by the Limit Comparison Test with  $\sum \frac{1}{n^2}$ .

(b)



25.  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$  Geometric series with  $a = 1$  and  $r = \frac{2}{3}$ .

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

26.  $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 25.

27.  $\sum_{n=1}^{\infty} [(0.6)^n + (0.8)^n] = \sum_{n=0}^{\infty} 0.6(0.6)^n + \sum_{n=0}^{\infty} 0.8(0.8)^n = (0.6) \frac{1}{1-0.6} + (0.8) \frac{1}{1-0.8} = \frac{6}{10} \cdot \frac{10}{4} + \frac{8}{10} \cdot \frac{10}{2} = \frac{11}{2} = 5.5$

28.  $\sum_{n=0}^{\infty} \left[ \left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$   
 $= \frac{1}{1-(2/3)} - \left[ \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \right] = 3 - 1 = 2$

29. (a)  $0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty} (0.09)(0.01)^n$

(b)  $0.\overline{09} = \frac{0.09}{1-0.01} = \frac{1}{11}$

30. (a)  $0.\overline{64} = 0.64 + 0.0064 + 0.000064 + \cdots = 0.64(1 + 0.01 + 0.0001 + \cdots) = 0.64 \sum_{n=0}^{\infty} (0.01)^n$

(b)  $0.\overline{64} = \frac{0.64}{1-0.01} = \frac{64}{99}$

31. Diverges. Geometric series with  $a = 1$  and  $|r| = 1.67 > 1$ .

32. Converges. Geometric series with  $a = 1$  and  $|r| = 0.67 < 1$ .

33. Diverges.  $n$ th-Term Test.  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

34. Diverges.  $n$ th-Term Test.  $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$ .

35.  $D_1 = 8$

$$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$$

$\vdots$

$$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1 - 0.7} = 45\frac{1}{3} \text{ meters}$$

36.  $S = \sum_{n=0}^{39} 42,000(1.055)^n$   
 $= \frac{42,000(1 - 1.055^{40})}{1 - 1.055} \approx \$5,737,435.79$

37. (See Exercise 116 in Section 9.2)

$$A = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

$$= \frac{300(e^{0.06(2)} - 1)}{e^{0.06/12} - 1} \approx \$7630.70$$

38. (See Exercise 116 in Section 9.2)

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 125\left(\frac{12}{0.035}\right)\left[\left(1 + \frac{0.035}{12}\right)^{12(10)} - 1\right] \approx \$17,929.06$$

39.  $\int_1^{\infty} x^{-4} \ln(x) dx = \lim_{b \rightarrow \infty} \left[ -\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]_1^b = 0 + \frac{1}{9} = \frac{1}{9}$

By the Integral Test, the series converges.

40.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$

Divergent  $p$ -series,  $p = \frac{3}{4} < 1$

41.  $\sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n}$

Because the second series is a divergent  $p$ -series while the first series is a convergent  $p$ -series, the difference diverges.

42.  $\sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$

Because the first series is a convergent  $p$ -series while the second series is a convergent geometric series, the difference converges.

43.  $\sum_{n=1}^{\infty} \frac{6}{5n-1}$   
 $\lim_{n \rightarrow \infty} \left[ \frac{6/(5n-1)}{1/n} \right] = \frac{6}{5}$

By a limit comparison test with the divergent harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ , the series diverges.

44.  $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 3n}}$   
 $\lim_{n \rightarrow \infty} \frac{n/\sqrt{n^3 + 3n}}{1/\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 3n}} = 1$

By a limit comparison test with the divergent  $p$ -series

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ , the series diverges.

45.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$   
 $\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^3 + 2n}}{1/(n^{3/2})} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$

By a limit comparison test with the convergent  $p$ -series

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ , the series converges.

46.  $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$   
 $\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$

By a limit comparison test with  $\sum_{n=1}^{\infty} \frac{1}{n}$ , the series diverges.

47.  $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$   
 $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \left( \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right) \frac{1}{2n} > \frac{1}{2n}$

Because  $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  diverges (harmonic series), so does the original series.



48. Because  $\sum_{n=1}^{\infty} \frac{1}{3^n}$  converges,  $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$  converges by the Limit Comparison Test.

49.  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$  converges by the Alternating Series Test.  
 $\lim_{n \rightarrow \infty} \frac{1}{n^5} = 0$  and  $a_{n+1} = \frac{1}{(n+1)^5} < \frac{1}{n^5} = a_n$ .

50.  $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2 + 1}$  converges by the Alternating Series

Test.  $\lim_{n \rightarrow \infty} \frac{n+1}{n^2 + 1} = 0$  and if

$$f(x) = \frac{x+1}{x^2+1}, f'(x) = \frac{-(x^2+2x-1)}{(x^2+1)^2} < 0 \Rightarrow \text{terms}$$

are decreasing. So,  $a_{n+1} < a_n$ .

51.  $\sum_{n=2}^{\infty} \frac{(-1)^n - n}{n^2 - 3}$  converges by the Alternating Series Test.

$\lim_{n \rightarrow \infty} \frac{n}{n^2 - 3} = 0$  and if

$$f(x) = \frac{n}{n^2 - 3}, f'(x) = \frac{-(n^2 + 3)}{(n^2 - 3)^2} < 0 \Rightarrow \text{terms are}$$

decreasing. So,  $a_{n+1} < a_n$ .

52.  $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

53. Diverges by the  $n$ th-Term Test.

$$\lim_{n \rightarrow \infty} \frac{n}{n-3} = 1 \neq 0$$

60.  $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

55.  $\lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{3n-1}{2n+5} \right)^n} = \lim_{n \rightarrow \infty} \left( \frac{3n-1}{2n+5} \right) = \frac{3}{2} > 1$

Diverges by Root Test.

56.  $\lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{4n}{7n-1} \right)^n} = \lim_{n \rightarrow \infty} \left( \frac{4n}{7n-1} \right) = \frac{4}{7} < 1$

Converges by Root Test.

57.  $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n} \right| \\ &= \lim_{n \rightarrow \infty} \left( \frac{1}{e^{2n+1}} \right) \left( \frac{n+1}{n} \right) \\ &= (0)(1) = 0 < 1 \end{aligned}$$

By the Ratio Test, the series converges.

58.  $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty \end{aligned}$$

By the Ratio Test, the series diverges.

59.  $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

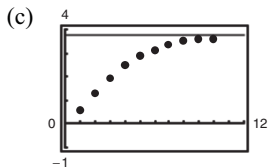
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right| = \lim_{n \rightarrow \infty} \frac{2n^3}{(n+1)^3} = 2$$

By the Ratio Test, the series diverges.

61. (a) Ratio Test:  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right) \left( \frac{3}{5} \right) = \frac{3}{5} < 1$ , Converges

(b)

$n$	5	10	15	20	25
$S_n$	2.8752	3.6366	3.7377	3.7488	3.7499

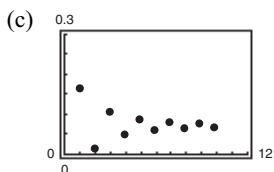


(d) The sum is approximately 3.75.

62. (a) The series converges by the Alternating Series Test.

(b)

$n$	5	10	15	20	25
$S_n$	0.0871	0.0669	0.0734	0.0702	0.0721



(d) The sum is approximately 0.0714.

63. (a)  $\int_N^\infty \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_N^\infty = \frac{1}{N}$

$N$	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^\infty \frac{1}{x^2} dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b)  $\int_N^\infty \frac{1}{x^5} dx = \left[ -\frac{1}{4x^4} \right]_N^\infty = \frac{1}{4N^4}$

$N$	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^\infty \frac{1}{x^5} dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

64. No. Let  $a_n = \frac{3937.5}{n^2}$ , then  $a_{75} = 0.7$ . The series  $\sum_{n=1}^\infty \frac{3937.5}{n^2}$  is a convergent  $p$ -series.

65.  $f(x) = e^{-3x} \quad f(0) = 1$   
 $f'(x) = -3e^{-3x} \quad f'(0) = -3$   
 $f''(x) = 9e^{-3x} \quad f''(0) = 9$   
 $f'''(x) = -27e^{-3x} \quad f'''(0) = -27$

$$P_3(x) = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!}$$

$$= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$$

66.  $f(x) = \tan x \quad f\left(-\frac{\pi}{4}\right) = -1$

$$f'(x) = \sec^2 x \quad f'\left(-\frac{\pi}{4}\right) = 2$$

$$f''(x) = 2 \sec^2 x \tan x \quad f''\left(-\frac{\pi}{4}\right) = -4$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \quad f'''\left(-\frac{\pi}{4}\right) = 16$$

$$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$$

67. Because  $\frac{(95\pi)^9}{180^9 \cdot 9!} < 0.001$ , use four terms.

$$\sin 95^\circ = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} \approx 0.99594$$

68. Because  $\frac{(0.75)^6}{6!} < 0.001$ , use three terms.

$$\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} \approx 0.7319$$

69. Because  $\frac{(0.75)^{15}}{15} < 0.001$ , use 14 terms.

$$\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \cdots - \frac{(0.75)^{14}}{14} \approx 0.559062$$

70. Because  $\frac{(0.25)^5}{5!} < 0.001$ , use five terms.

$$e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$$

71.  $f(x) = \cos x, \quad c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$$|f^{(n+1)}(z)| \leq 1 \Rightarrow R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

(a)  $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for  $n = 4$ .

(b)  $R_n(x) \leq \frac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for  $n = 6$ .

(c)  $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for  $n = 5$ .

(d)  $R_n(x) \leq \frac{2^{n+1}}{(n+1)!} < 0.0001$

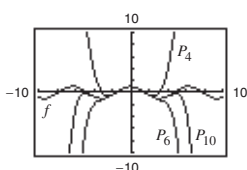
This inequality is true for  $n = 10$ .

72.  $f(x) = \cos x$

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$



73.  $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

Geometric series which converges only if  $|x/10| < 1$  or  $-10 < x < 10$ .

74.  $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if  $|2x| < 1$  or

$$-\frac{1}{2} < x < \frac{1}{2}$$

75.  $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n (x-2)^n} \right|$$

$$= |x-2|$$

$$R = 1$$

Center: 2

Because the series converges when  $x = 1$  and when  $x = 3$ , the interval of convergence is  $[1, 3]$ .

76.  $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} (x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n (x-2)^n} \right|$$

$$= 3|x-2|$$

$$R = \frac{1}{3}$$

Center: 2

Because the series converges at  $\frac{5}{3}$  and diverges at  $\frac{7}{3}$ , the interval of convergence is  $\left[\frac{5}{3}, \frac{7}{3}\right)$ .

77.  $\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n} \right| = \infty$$

which implies that the series converges only at the center  $x = 2$ .

78.  $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

79.

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2} \\
 y' &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+1}}{4^{n+1} [(n+1)!]^2} \\
 y'' &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n}}{4^{n+1} [(n+1)!]^2} \\
 x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n (n!)^2} \\
 &= \sum_{n=0}^{\infty} \left[ (-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^{n+1} (2n+2)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^n}{4^n (n!)^2} \right] x^{2n+2} \\
 &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} (2n+2)(2n+1+1)}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\
 &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 4(n+1)^2}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\
 &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^{n+1} 1}{4^n (n!)^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} = 0
 \end{aligned}$$

80.

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \\
 y' &= \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!} \\
 y'' &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} \\
 y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2)(2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0
 \end{aligned}$$

81.  $\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{x}{3} \right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

$$82. \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

$$83. g(x) = \frac{2}{3-x}. \text{ Power series } \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{x}{3}\right)^n$$

$$\text{Derivative: } \sum_{n=1}^{\infty} \frac{2}{3} n \left(\frac{x}{3}\right)^{n-1} \left(\frac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{2}{9} n \left(\frac{x}{3}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{9} (n+1) \left(\frac{x}{3}\right)^n$$

$$84. \text{ Integral: } \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$$

$$85. 1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n = \frac{1}{1-(2x/3)} = \frac{3}{3-2x}, \quad \left(-\frac{3}{2}, \frac{3}{2}\right)$$

$$86. 8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots = \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4}\right]^n = \frac{8}{1-[-(x-3)/4]}$$

$$= \frac{32}{4+(x-3)} = \frac{32}{1+x}, (-1, 7)$$

$$87. f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x, \dots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x) [x - (3\pi/4)]^n}{n!}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (3\pi/4)]^n}{n!}$$

$$88. f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4) [x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots$$

$$= \frac{\sqrt{2}}{2} \left[ 1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]$$

$$89. 3^x = (e^{\ln(3)})^x = e^{x \ln(3)} \text{ and because } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ you have}$$

$$3^x = \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} = 1 + x \ln 3 + \frac{x^2 [\ln 3]^2}{2!} + \frac{x^3 [\ln 3]^3}{3!} + \frac{x^4 [\ln 3]^4}{4!} + \cdots$$

90.  $f(x) = \csc(x)$   
 $f'(x) = -\csc(x) \cot(x)$   
 $f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$   
 $f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$   
 $f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$   

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2) \left[ x - (\pi/2) \right]^n}{n!} = 1 + \frac{1}{2!} \left( x - \frac{\pi}{2} \right)^2 + \frac{5}{4!} \left( x - \frac{\pi}{2} \right)^4 + \dots$$

91.  $f(x) = \frac{1}{x}$   
 $f'(x) = -\frac{1}{x^2}$   
 $f''(x) = \frac{2}{x^3}$   
 $f'''(x) = -\frac{6}{x^4}, \dots$   

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n, -2 < x < 0$$

92.  $f(x) = x^{1/2}$   
 $f'(x) = \frac{1}{2}x^{-1/2}$   
 $f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$   
 $f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$   
 $f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$   

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} = 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots$$
  

$$= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}$$

93.  $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \dots$   
 $(1+x)^{1/5} = 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \dots$   

$$= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \dots = 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!} = 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \dots$$

94.  $h(x) = (1+x)^{-3}$   
 $h'(x) = -3(1+x)^{-4}$   
 $h''(x) = 12(1+x)^{-5}$   
 $h'''(x) = -60(1+x)^{-6}$   
 $h^{(4)}(x) = 360(1+x)^{-7}$   
 $h^{(5)}(x) = -2520(1+x)^{-8}$   

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}$$

$$95. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{5}{4} - 1\right)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231$$

$$96. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\ln\left(\frac{6}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{6}{5} - 1\right)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823$$

$$97. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

$$102. \frac{1}{\sqrt{1+x^3}} = (1+x^3)^{-1/2}, k = -\frac{1}{2}$$

$$= 1 - \frac{1}{2}(x^3) + \frac{(-1/2)(-3/2)}{2!}x^6 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^9 + \cdots = 1 - \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{2^n n!} x^{3n}$$

$$103. (a) \quad f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$P(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$(b) \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$(c) \quad e^x \cdot e^x = \left(1 + x + \frac{x^2}{2!} + \cdots\right) \left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$P(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

$$104. (a) \quad f(x) = \sin 2x \quad f(0) = 0$$

$$f'(x) = 2 \cos 2x \quad f'(0) = 2$$

$$f''(x) = -4 \sin 2x \quad f''(0) = 0$$

$$f'''(x) = -8 \cos 2x \quad f'''(0) = -8$$

$$f^{(4)}(x) = 16 \sin 2x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 32 \cos 2x \quad f^{(5)}(0) = 32$$

$$f^{(6)}(x) = -64 \sin 2x \quad f^{(6)}(0) = 0$$

$$f^{(7)}(x) = -128 \cos 2x \quad f^{(7)}(0) = -128$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

$$98. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

$$99. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

$$100. \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

101. The series in Exercise 45 converges very slowly because the terms approach 0 at a slow rate.



$$\begin{aligned} \text{(b)} \quad \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\ &= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \sin 2x &= 2 \sin x \cos x \\ &= 2 \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\ &= 2 \left[ x + \left( -\frac{x^3}{2} - \frac{x^3}{6} \right) + \left( \frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left( -\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\ &= 2 \left( x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots \end{aligned}$$

$$\begin{aligned} 105. \quad \sin t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\ \frac{\sin t}{t} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \\ \int_0^x \frac{\sin t}{t} dt &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \end{aligned}$$

$$\begin{aligned} 106. \quad \cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\ \cos \frac{\sqrt{t}}{2} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!} \\ \int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)} \right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)} \end{aligned}$$

$$\begin{aligned} 107. \quad \frac{1}{1+t} &= \sum_{n=0}^{\infty} (-1)^n t^n \\ \ln(1+t) &= \int \frac{1}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1} \\ \frac{\ln(1+t)}{t} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1} \\ \int_0^x \frac{\ln(1+t)}{t} dt &= \left[ \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2} \end{aligned}$$

$$\begin{aligned} 108. \quad e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ e^t - 1 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\ \frac{e^t - 1}{t} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\ \int_0^x \frac{e^t - 1}{t} dt &= \left[ \sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \end{aligned}$$

$$\begin{aligned} 109. \quad \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \\ \frac{\arctan x}{\sqrt{x}} &= \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots \end{aligned}$$

$$\lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\left( \frac{1}{1+x^2} \right)}{\left( \frac{1}{2\sqrt{x}} \right)} = \lim_{x \rightarrow 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$$

$$\begin{aligned} 110. \quad \arcsin x &= x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots \\ \frac{\arcsin x}{x} &= 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

By L'Hôpital's Rule,

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left( \frac{1}{\sqrt{1-x^2}} \right)}{1} = 1.$$

## Problem Solving for Chapter 9

1. (a)  $1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$

(b)  $0, \frac{1}{3}, \frac{2}{3}, 1$ , etc.

(c)  $\lim_{n \rightarrow \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - 1 = 0$

2. (a) Let  $\varepsilon > 0$  be given.  $\lim_{n \rightarrow \infty} a_{2n} = L$  means there exists  $M_1$  such that  $|a_{2n} - L| < \varepsilon$  for  $n > M_1$ .  $\lim_{n \rightarrow \infty} a_{2n+1} = L$  means there exists  $M_2$  such that  $|a_{2n+1} - L| < \varepsilon$  for  $n > M_2$ . Let  $M = \max\{2M_1, 2M_2 + 1\}$ . Then for  $n > M$ , and  $n = 2m$  even, you have  $2m > M > 2M_1 \Rightarrow m > M_1 \Rightarrow |a_{2m} - L| < \varepsilon$ . And for  $n > M$ ,  $n = 2m+1$  odd, you have  $2m+1 > M > 2M_2 + 1 \Rightarrow m > M_2 \Rightarrow |a_{2m+1} - L| < \varepsilon$ . So,  $\lim_{n \rightarrow \infty} a_n = L$ .

(b)  $a_1 = 1, a_{n+1} = 1 + \frac{1}{1 + a_n}$ .

$$a_2 = 1 + \frac{1}{1 + a_1} = 1 + \frac{1}{1 + 1} = \frac{3}{2} = 1.5$$

$$a_3 = 1 + \frac{1}{1 + a_2} = 1 + \frac{1}{1 + (3/2)} = \frac{7}{5} = 1.4$$

$$a_4 = \frac{17}{12} = 1.41\bar{6}$$

$$a_5 = \frac{41}{29} \approx 1.4140$$

$$a_6 = \frac{99}{70} \approx 1.41429$$

$$a_7 = \frac{239}{169} \approx 1.414201$$

$$a_8 = \frac{577}{408} \approx 1.414216$$

Using mathematical induction, you can show that the odd terms are increasing and the even terms are decreasing. Both sequences are bounded in  $[1, 2]$ . So, both sequences converge.

Let  $\lim_{n \rightarrow \infty} a_{2n} = L$ . Then  $\lim_{n \rightarrow \infty} a_{2n+2} = L$ , and

$$a_{n+2} = 1 + \frac{1}{1 + a_{n+1}} = 1 + \frac{1}{1 + \left[1 + \frac{1}{1 + a_n}\right]} = 1 + \frac{1}{1 + \left[\frac{2 + a_n}{1 + a_n}\right]} = 1 + \frac{1}{\left(\frac{3 + 2a_n}{1 + a_n}\right)} = 1 + \frac{1 + a_n}{3 + 2a_n} = \frac{4 + 3a_n}{3 + 2a_n}$$

$$\Rightarrow a_{2n+2} = \frac{4 + 3a_{2n}}{3 + 2a_{2n}}$$

So,  $L = \frac{4 + 3L}{3 + 2L} \Rightarrow 2L^2 = 4 \Rightarrow L = \sqrt{2}$ . Similarly,  $\lim_{n \rightarrow \infty} a_{2n+1} = \sqrt{2}$ . So by part (a),  $\lim_{n \rightarrow \infty} a_n = L = \sqrt{2}$

3. Let  $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ .

Then

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

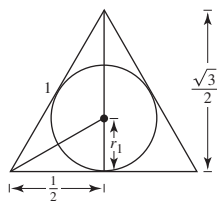
$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$

$$= S + \frac{1}{2^2} \left[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right] = S + \frac{1}{2^2} \left( \frac{\pi^2}{6} \right).$$

So,  $S = \frac{\pi^2}{6} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{6} \left( \frac{3}{4} \right) = \frac{\pi^2}{8}$ .

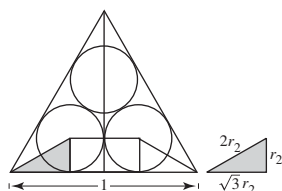
4. If there are  $n$  rows, then  $a_n = \frac{n(n+1)}{2}$ .

For one circle,  $a_1 = 1$  and  $r_1 = \frac{1}{3} \left( \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$ .



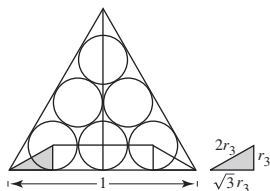
For three circles,  $a_2 = 3$  and  $1 = 2\sqrt{3}r_2 + 2r_2$

$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$



For six circles,  $a_3 = 6$  and  $1 = 2\sqrt{3}r_3 + 4r_3$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$



Continuing this pattern,  $r_n = \frac{1}{2\sqrt{3} + 2(n-1)}$ .

$$\text{Total Area} = (\pi r_n^2) a_n = \pi \left( \frac{1}{2\sqrt{3} + 2(n-1)} \right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi n(n+1)}{2 [2\sqrt{3} + 2(n-1)]^2}$$

$$\lim_{n \rightarrow \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

6. (a)  $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \dots$

$$= (1 + x^3 + x^6 + \dots) + 2(x + x^4 + x^7 + \dots) + 3(x^2 + x^5 + x^8 + \dots)$$

$$= (1 + x^3 + x^6 + \dots)(1 + 2x + 3x^2) = (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$R = 1$  because each series in the second line has  $R = 1$ .

(b)  $\sum a_n x^n = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \dots) + \dots$

$$= a_0(1 + x^p + \dots) + a_1 x(1 + x^p + \dots) + \dots + a_{p-1} x^{p-1}(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1})(1 + x^p + \dots) = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) \frac{1}{1 - x^p}$$

$R = 1$

(Assume all  $a_n > 0$ .)

5. (a) Position the three blocks as indicated in the figure. The bottom block extends  $1/6$  over the edge of the table, the middle block extends  $1/4$  over the edge of the bottom block, and the top block extends  $1/2$  over the edge of the middle block.

The centers of gravity are located at

bottom block:  $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block:  $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block:  $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}$ .

The center of gravity of the top 2 blocks is

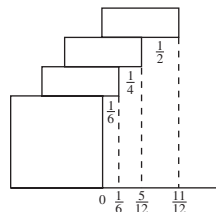
$$\left( -\frac{1}{12} + \frac{5}{12} \right) / 2 = \frac{1}{6}, \text{ which lies over the bottom}$$

block. The center of gravity of the 3 blocks is

$$\left( -\frac{1}{3} - \frac{1}{12} + \frac{5}{12} \right) / 3 = 0 \text{ which lies over the table.}$$

So, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12} \text{ beyond the edge of the table.}$$



- (b) Yes. If there are  $n$  blocks, then the edge of the top block

lies  $\sum_{i=1}^n \frac{1}{2i}$  from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^4 \frac{1}{2i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}$$

which shows that the top block extends beyond the table.

- (c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty} \frac{1}{2i} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

$$7. a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$$

$$\text{If } a = b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges conditionally.}$$

$$\text{If } a \neq b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n} \text{ diverges.}$$

No values of  $a$  and  $b$  give absolute convergence.  $a = b$  implies conditional convergence.

$$8. (a) \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting  $x = 0$ , you have  $C = 1$ . Letting  $x = 1$ ,

$$e - e + 1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

$$\text{So, } \sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$$

$$(b) \text{ Differentiating, } xe^x + e^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

$$\text{Letting } x = 1, 2e = \sum_{n=0}^{\infty} \frac{n+1}{n!} \approx 5.4366.$$

$$9. \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \Rightarrow f^{(12)}(0) = \frac{12!}{6!} = 665,280$$

$$10. \text{ Let } a_1 = \int_0^{\pi} \frac{\sin x}{x} dx, a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x} dx,$$

$$a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx, \text{ etc. Then,}$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = a_1 - a_2 + a_3 - a_4 + \cdots.$$

Because  $\lim_{n \rightarrow \infty} a_n = 0$  and  $a_{n+1} < a_n$ , this series converges.

$$11. (a) \text{ If } p = 1, \int_2^{\infty} \frac{1}{x \ln x} dx = [\ln \ln x]_2^{\infty} \text{ diverges.}$$

If

$$p > 1, \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{b \rightarrow \infty} \left[ \frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right]$$

converges.

If  $p < 1$ , diverges.

$$(b) \sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2} \sum_{n=4}^{\infty} \frac{1}{n \ln n} \text{ diverges by part (a).}$$

$$12. (a) \quad a_1 = 3.0$$

$$a_2 \approx 1.73205$$

$$a_3 \approx 2.17533$$

$$a_4 \approx 2.27493$$

$$a_5 \approx 2.29672$$

$$a_6 \approx 2.30146$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{13}}{2} \text{ [See part (b) for proof.]}$$

(b) Use mathematical induction to show the sequence is increasing. Clearly,

$$a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1. \text{ Now}$$

assume  $a_n > a_{n-1}$ . Then

$$\begin{aligned} a_n + a &> a_{n-1} + a \\ \sqrt{a_n + a} &> \sqrt{a_{n-1} + a} \end{aligned}$$

$$a_{n+1} > a_n.$$

Use mathematical induction to show that the sequence is bounded above by  $a$ . Clearly,

$$a_1 = \sqrt{a} < a. \text{ Now assume } a_n < a. \text{ Then}$$

$$a > a_n \text{ and } a - 1 > 1 \text{ implies}$$

$$a(a-1) > a_n(1)$$

$$a^2 - a > a_n$$

$$a^2 > a_n + a$$

$$a > \sqrt{a_n + a} = a_{n+1}.$$

So, the sequence converges to some number  $L$ . To find  $L$ , assume  $a_{n+1} \approx a_n \approx L$ :

$$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

$$\text{So, } L = \frac{1 + \sqrt{1 + 4a}}{2}.$$

$$13. \text{ Let } b_n = a_n r^n.$$

$$(b_n)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \rightarrow Lr \text{ as } n \rightarrow \infty.$$

$$Lr < \frac{1}{r} = 1.$$

By the Root Test,  $\sum b_n$  converges  $\Rightarrow \sum a_n r^n$  converges.

14. (a)  $\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \dots$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_1 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

(b)  $\frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2(-1)^n}{2^{1+(-1)^{n+1}}}$

This sequence is  $\frac{1}{8}, 2, \frac{1}{8}, 2, \dots$  which diverges.

(c)  $\sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left( \frac{1}{2^n \cdot 2^{(-1)^n}} \right)^{1/n} = \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1$

converges because  $\left\{ 2^{(-1)^n} \right\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \dots$

and  $\sqrt[n]{1/2} \rightarrow 1$  and  $\sqrt[n]{2} \rightarrow 1$ .

15. (a)  $\frac{1}{0.99} = \frac{1}{1-0.01} = \sum_{n=0}^{\infty} (0.01)^n$

$$= 1 + 0.01 + (0.01)^2 + \dots$$

$$= 1.010101 \dots$$

(b)  $\frac{1}{0.98} = \frac{1}{1-0.02} = \sum_{n=0}^{\infty} (0.02)^n$

$$= 1 + 0.02 + (0.02)^2 + \dots$$

$$= 1 + 0.02 + 0.0004 + \dots$$

$$= 1.0204081632 \dots$$

16.  $S_6 = 130 + 70 + 40 = 240$

$$S_7 = 240 + 130 + 70 = 440$$

$$S_8 = 440 + 240 + 130 = 810$$

$$S_9 = 810 + 440 + 240 = 1490$$

$$S_{10} = 1490 + 810 + 440 = 2740$$

17. (a) Height  $= 2 \left[ 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \right]$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \left( p\text{-series, } p = \frac{1}{2} < 1 \right)$$

(b)  $S = 4\pi \left[ 1 + \frac{1}{2} + \frac{1}{3} + \dots \right] = 4\pi \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

(c)  $W = \frac{4}{3}\pi \left[ 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots \right]$

$$= \frac{4}{3}\pi \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$

18. (a)  $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  diverges (harmonic)

(b) Let  $f(x) = \sin x$ . By the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(c)| |x - y|$$

$$= |\cos(c)| |x - y| \leq |x - y|,$$

where  $c$  is between  $x$  and  $y$ . So,

$$0 \leq \left| \sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right) \right|$$

$$\leq \left| \frac{1}{2n} - \frac{1}{2n+1} \right| = \frac{1}{2n(2n+1)}$$

Because  $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}$  converges, the Comparison

Theorem tells us that

$$\sum_{n=1}^{\infty} \left[ \sin\left(\frac{1}{2n}\right) - \sin\left(\frac{1}{2n+1}\right) \right] \text{ converges.}$$

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**C H A P T E R 10**

**Conics, Parametric Equations,  
and Polar Coordinates**

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**INSTRUCTOR USE ONLY**

## CHAPTER 10

### Conics, Parametric Equations, and Polar Coordinates

#### Section 10.1 Conics and Calculus

1.  $y^2 = 4x$  Parabola

Vertex:  $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches (h)

2.  $(x + 4)^2 = 2(y + 2)$  Parabola

Vertex:  $(-4, -2)$

Opens upward

Matches (a)

3.  $(x + 4)^2 = -2(y - 2)$  Parabola

Vertex:  $(-4, 2)$

Opens downward

Matches (e)

4.  $\frac{(x - 2)^2}{16} + \frac{(y + 1)^2}{4} = 1$  Ellipse

Center:  $(2, -1)$

Matches (b)

5.  $\frac{x^2}{4} + \frac{y^2}{9} = 1$  Ellipse

Center:  $(0, 0)$

Vertices:  $(0, \pm 3)$

Matches (f)

6.  $\frac{x^2}{16} + \frac{y^2}{16} = 1$  Circle

Matches (g)

7.  $\frac{y^2}{16} - \frac{x^2}{1} = 1$  Hyperbola

Vertices:  $(0, \pm 4)$

Matches (c)

8.  $\frac{(x - 2)^2}{9} - \frac{y^2}{4} = 1$  Hyperbola

Vertices:  $(5, 0), (-1, 0)$

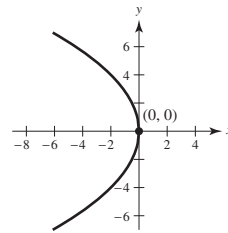
Matches (d)

9.  $y^2 = -8x = 4(-2)x$

Vertex:  $(0, 0)$

Focus:  $(-2, 0)$

Directrix:  $x = 2$



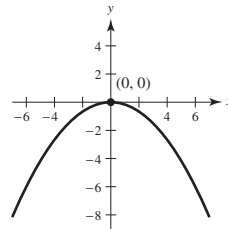
10.  $x^2 + 6y = 0$

$$x^2 = -6y = 4\left(-\frac{3}{2}\right)y$$

Vertex:  $(0, 0)$

Focus:  $\left(0, -\frac{3}{2}\right)$

Directrix:  $y = \frac{3}{2}$



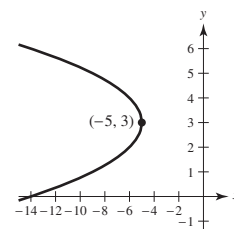
11.  $(x + 5) + (y - 3)^2 = 0$

$$(y - 3)^2 = -(x + 5) = 4\left(-\frac{1}{4}\right)(x + 5)$$

Vertex:  $(-5, 3)$

Focus:  $\left(-\frac{21}{4}, 3\right)$

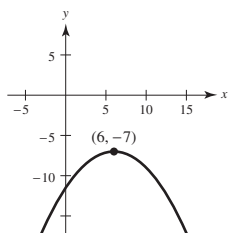
Directrix:  $x = -\frac{19}{4}$



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12.  $(x - 6)^2 + 8(y + 7) = 0$

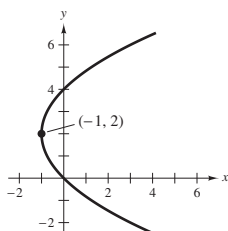
$$(x - 6)^2 = -8(y + 7) = 4(-2)(y + 7)$$

Vertex:  $(6, -7)$ Focus:  $(6, -9)$ Directrix:  $y = -5$ 

13.  $y^2 - 4y - 4x = 0$

$$y^2 - 4y + 4 = 4x + 4$$

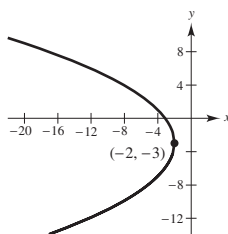
$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex:  $(-1, 2)$ Focus:  $(0, 2)$ Directrix:  $x = -2$ 

14.  $y^2 + 6y + 8x + 25 = 0$

$$y^2 + 6y + 9 = -8x - 25 + 9$$

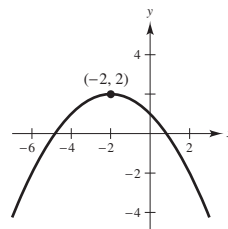
$$(y + 3)^2 = 4(-2)(x + 2)$$

Vertex:  $(-2, -3)$ Focus:  $(-4, -3)$ Directrix:  $x = 0$ 

15.  $x^2 + 4x + 4y - 4 = 0$

$$x^2 + 4x + 4 = -4y + 4 + 4$$

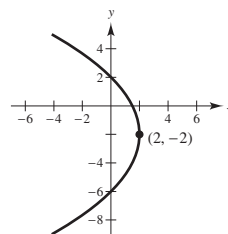
$$(x + 2)^2 = 4(-1)(y - 2)$$

Vertex:  $(-2, 2)$ Focus:  $(-2, 1)$ Directrix:  $y = 3$ 

16.  $y^2 + 4y + 8x - 12 = 0$

$$y^2 + 4y + 4 = -8x + 12 + 4$$

$$(y + 2)^2 = 4(-2)(x - 2)$$

Vertex:  $(2, -2)$ Focus:  $(0, -2)$ Directrix:  $x = 4$ 

17.  $y^2 + x + y = 0$

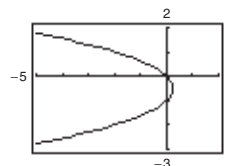
$$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$$

Vertex:  $\left(\frac{1}{4}, -\frac{1}{2}\right)$ Focus:  $\left(0, -\frac{1}{2}\right)$ Directrix:  $x = \frac{1}{2}$ 

$$y_1 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x}$$

$$y_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - x}$$



INSTRUCTOR USE ONLY



18.  $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$$-6y = (x - 4)^2 - 10$$

$$-6y + 10 = (x - 4)^2$$

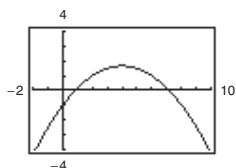
$$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$$

$$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$$

Vertex:  $\left(4, \frac{5}{3}\right)$

Focus:  $\left(4, \frac{1}{6}\right)$

Directrix:  $y = \frac{19}{6}$



19.  $y^2 - 4x - 4 = 0$

$$y^2 = 4x + 4$$

$$= 4(1)(x + 1)$$

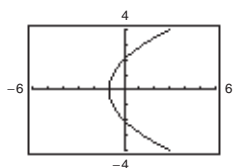
Vertex:  $(-1, 0)$

Focus:  $(0, 0)$

Directrix:  $x = -2$

$$y_1 = 2\sqrt{x + 1}$$

$$y_2 = -2\sqrt{x + 1}$$



20.  $x^2 - 2x + 8y + 9 = 0$

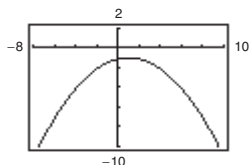
$$x^2 - 2x + 1 = -8y - 9 + 1$$

$$(x - 1)^2 = 4(-2)(y + 1)$$

Vertex:  $(1, -1)$

Focus:  $(1, -3)$

Directrix:  $y = 1$



21.  $(y - 4)^2 = 4(-2)(x - 5)$

$$y^2 - 8y + 16 = -8x + 40$$

$$y^2 - 8y + 8x - 24 = 0$$

22.  $(x + 2)^2 = 4(-2)(y - 1)$

$$x^2 + 4x + 8y - 4 = 0$$

23.  $(x - 0)^2 = 4(8)(y - 5)$

$$x^2 = 4(8)(y - 5)$$

$$x^2 - 32y + 160 = 0$$

24. Vertex:  $(0, 2)$

$$(y - 2)^2 = 4(2)(x - 0)$$

$$y^2 - 8x - 4y + 4 = 0$$

25.  $y = 4 - x^2$

$$x^2 + y - 4 = 0$$

26.  $y = 4 - (x - 2)^2 = 4x - x^2$

$$x^2 - 4x + y = 0$$

27. Because the axis of the parabola is vertical, the form of the equation is  $y = ax^2 + bx + c$ . Now, substituting the values of the given coordinates into this equation, you obtain

$$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$$

Solving this system, you have  $a = \frac{5}{3}, b = -\frac{14}{3}, c = 3$ .

So,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

28. From Example 2:  $4p = 8$  or  $p = 2$

Vertex:  $(4, 0)$

$$(x - 4)^2 = 8(y - 0)$$

$$x^2 - 8x - 8y + 16 = 0$$

29.  $16x^2 + y^2 = 16$

$$x^2 + \frac{y^2}{16} = 1$$

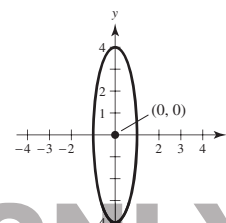
$$a^2 = 16, b^2 = 1, c^2 = 16 - 1 = 15$$

Center:  $(0, 0)$

Foci:  $(0, \pm\sqrt{15})$

Vertices:  $(0, \pm 4)$

$$e = \frac{c}{a} = \frac{\sqrt{15}}{4}$$



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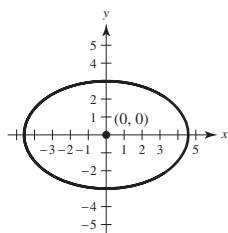
30.  $3x^2 + 7y^2 = 63$

$$\frac{x^2}{21} + \frac{y^2}{9} = 1$$

$$a^2 = 21, b^2 = 9, c^2 = 21 - 9 = 12$$

Center:  $(0, 0)$ Foci:  $(\pm 2\sqrt{3}, 0)$ Vertices:  $(\pm\sqrt{21}, 0)$ 

$$e = \frac{c}{a} = \frac{2\sqrt{3}}{\sqrt{21}} = \frac{2\sqrt{7}}{7}$$

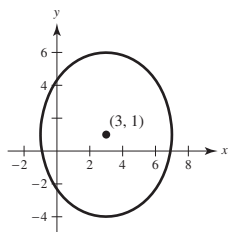


31.  $\frac{(x-3)^2}{16} + \frac{(y-1)^2}{25} = 1$

$$a^2 = 25, b^2 = 16, c^2 = 25 - 16 = 9$$

Center:  $(3, 1)$ Foci:  $(3, 1 + 3) = (3, 4), (3, 1 - 3) = (3, -2)$ Vertices:  $(3, 6), (3, -4)$ 

$$e = \frac{c}{a} = \frac{3}{5}$$

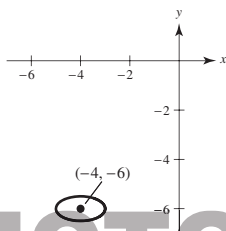


32.  $(x+4)^2 + \frac{(y+6)^2}{1/4} = 1$

$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Center:  $(-4, -6)$ Foci:  $\left(-4 \pm \frac{\sqrt{3}}{2}, -6\right)$ Vertices:  $(-5, -6), (-3, -6)$ 

$$e = \frac{c}{a} = \frac{\sqrt{3}}{2}$$



33.  $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

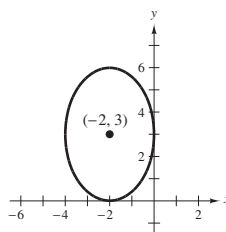
$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36 = 36$$

$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

$$a^2 = 9, b^2 = 4, c^2 = 5$$

Center:  $(-2, 3)$ Foci:  $(-2, 3 \pm \sqrt{5})$ Vertices:  $(-2, 6), (-2, 0)$ 

$$e = \frac{\sqrt{5}}{3}$$



34.  $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

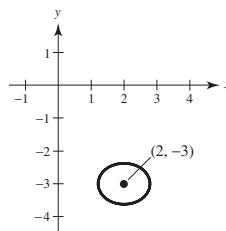
$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 9) = -279 + 64 + 225 = 10$$

$$\frac{(x-2)^2}{(5/8)} + \frac{(y+3)^2}{(2/5)} = 1$$

$$a^2 = \frac{5}{8}, b^2 = \frac{2}{5}, c^2 = a^2 - b^2 = \frac{9}{40}$$

Center:  $(2, -3)$ Foci:  $\left(2 \pm \frac{3\sqrt{10}}{20}, -3\right)$ Vertices:  $\left(2 \pm \frac{\sqrt{10}}{4}, -3\right)$ 

$$e = \frac{c}{a} = \frac{3}{5}$$



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35.  $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20 = 60$$

$$\frac{\left[x - (1/2)\right]^2}{5} + \frac{(y+1)^2}{3} = 1$$

$$a^2 = 5, b^2 = 3, c^2 = 2$$

$$\text{Center: } \left(\frac{1}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{1}{2} \pm \sqrt{2}, -1\right)$$

$$\text{Vertices: } \left(\frac{1}{2} \pm \sqrt{5}, -1\right)$$

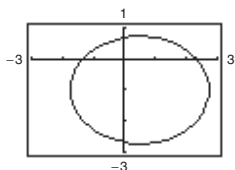
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y+1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



36.  $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36 = 9$$

$$\frac{\left[x + (2/3)\right]^2}{1/4} + \frac{(y-2)^2}{1} = 1$$

$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$$

$$\text{Center: } \left(-\frac{2}{3}, 2\right)$$

$$\text{Foci: } \left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$$

$$\text{Vertices: } \left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$$

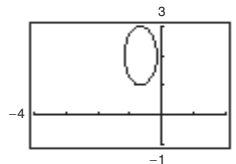
Solve for y:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y-2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)}$$

(Graph each of these separately.)



37.  $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{\left[x - (3/2)\right]^2}{4} + \frac{(y+1)^2}{2} = 1$$

$$a^2 = 4, b^2 = 2, c^2 = 2$$

$$\text{Center: } \left(\frac{3}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{3}{2} \pm \sqrt{2}, -1\right)$$

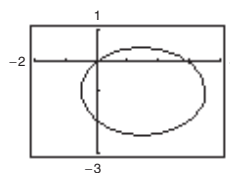
$$\text{Vertices: } \left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$$

$$\text{Solve for y: } 2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$$

$$(y+1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



$$\begin{aligned}
 38. \quad & 2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0 \\
 & 50x^2 + 25y^2 + 120x - 160y + 78 = 0 \\
 & 50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250 \\
 & \frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1
 \end{aligned}$$

$$a^2 = 10, b^2 = 5, c^2 = 5$$

$$\text{Center: } \left(-\frac{6}{5}, \frac{16}{5}\right)$$

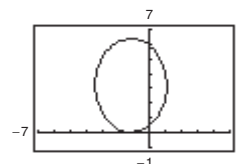
$$\text{Foci: } \left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$$

$$\text{Vertices: } \left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$$

$$\text{Solve for } y: (y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2} \quad (\text{Graph each of these separately.})$$



$$39. \text{ Center: } (0, 0)$$

$$\text{Focus: } (5, 0)$$

$$\text{Vertex: } (6, 0)$$

Horizontal major axis

$$a = 6, c = 5 \Rightarrow b = \sqrt{a^2 - c^2} = \sqrt{11}$$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

$$40. \text{ Vertices: } (0, 3), (8, 3)$$

$$\text{Eccentricity: } \frac{3}{4}$$

Horizontal major axis

$$\text{Center: } (4, 3)$$

$$a = 4, e = \frac{c}{a} \Rightarrow c = 4\left(\frac{3}{4}\right) = 3$$

$$\Rightarrow b = \sqrt{16 - 9} = \sqrt{7}$$

$$\frac{(x - 4)^2}{16} + \frac{(y - 3)^2}{7} = 1$$

$$41. \text{ Vertices: } (3, 1), (3, 9)$$

Minor axis length: 6

Vertical major axis

$$\text{Center: } (3, 5)$$

$$a = 4, b = 3$$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

$$42. \text{ Foci: } (0, \pm 9)$$

Major axis length: 22

Vertical major axis

$$\text{Center: } (0, 0)$$

$$c = 9, a = 11 \Rightarrow b = \sqrt{40}$$

$$\frac{x^2}{40} + \frac{y^2}{121} = 1$$

$$43. \text{ Center: } (0, 0)$$

Horizontal major axis

Points on ellipse:  $(3, 1), (4, 0)$

Because the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, you have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

$$\text{The solution to this system is } a^2 = 16, b^2 = \frac{16}{7}.$$

So,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

44. Center: (1, 2)

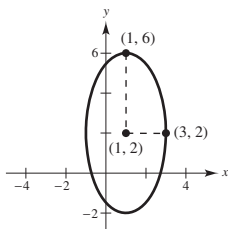
Vertical major axis

Points on ellipse: (1, 6), (3, 2)

From the sketch, you can see that

$$h = 1, k = 2, a = 4, b = 2$$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



45.  $y^2 - \frac{x^2}{9} = 1$

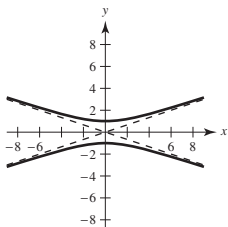
$$a = 1, b = 3, c = \sqrt{10}$$

Center: (0, 0)

Vertices: (0, ±1)

Foci: (0, ±√10)

$$\text{Asymptotes: } y = \pm \frac{a}{b}x = \pm \frac{1}{3}x$$



46.  $\frac{x^2}{25} - \frac{y^2}{16} = 1$

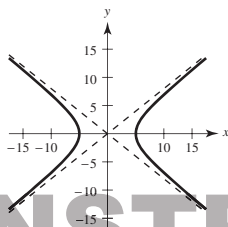
$$a = 5, b = 4, c = \sqrt{25 + 16} = \sqrt{41}$$

Center: (0, 0)

Vertices: (±5, 0)

Foci: (±√41, 0)

$$\text{Asymptotes: } y = \pm \frac{b}{a}x = \pm \frac{4}{5}x$$



47.  $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

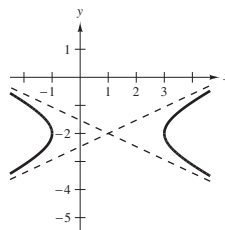
$$a = 2, b = 1, c = \sqrt{5}$$

Center: (1, -2)

Vertices: (-1, -2), (3, -2)

Foci: (1 ± √5, -2)

$$\text{Asymptotes: } y = -2 \pm \frac{1}{2}(x-1)$$



48.  $\frac{(y+3)^2}{225} - \frac{(x-5)^2}{64} = 1$

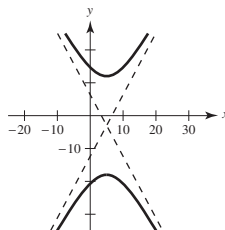
$$a = 15, b = 8, c = \sqrt{225 + 64} = 17$$

Center: (5, -3)

Vertices: (5, 12), (5, -18)

Foci: (5, 14), (5, -20)

$$\text{Asymptotes: } y = k \pm \frac{a}{b}(x-h) = -3 \pm \frac{15}{8}(x-5)$$



49.  $9x^2 - y^2 - 36x + 18 = 0$

$$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

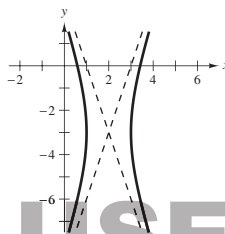
$$a = 1, b = 3, c = \sqrt{10}$$

Center: (2, -3)

Vertices: (1, -3), (3, -3)

Foci: (2 ± √10, -3)

$$\text{Asymptotes: } y = -3 \pm 3(x-2)$$



50.  $y^2 - 16x^2 + 64x - 208 = 0$

$$y^2 - 16(x^2 - 4x + 4) = 208 - 64 = 144$$

$$\frac{y^2}{144} - \frac{(x-2)^2}{9} = 1$$

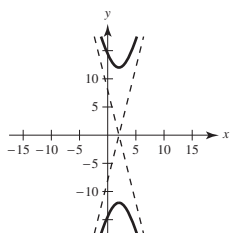
$$a = 12, b = 3, c = \sqrt{144 + 9} = \sqrt{153}$$

Center: (2, 0)

Vertices: (2, 12), (2, -12)

Foci:  $(2, \pm\sqrt{153})$ 

$$\text{Asymptotes: } y = \pm \frac{12}{3}(x-2) = \pm 4(x-2)$$



51.  $x^2 - 9y^2 + 2x - 54y - 80 = 0$

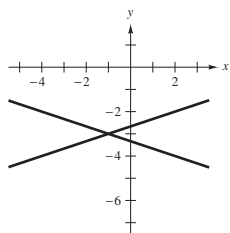
$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x+1)^2 - 9(y+3)^2 = 0$$

$$y+3 = \pm \frac{1}{3}(x+1)$$

$$y = 3 \pm \frac{1}{3}(x+1)$$

Degenerate hyperbola is two lines intersecting at  $(-1, -3)$ .



52.  $9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

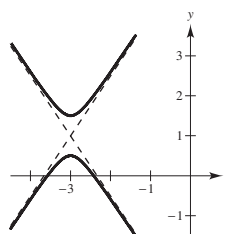
$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

$$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{\sqrt{13}}{6}$$

Center: (-3, 1)

Vertices:  $(-3, \frac{1}{2}), (-3, \frac{3}{2})$ Foci:  $(-3, 1 \pm \frac{1}{6}\sqrt{13})$ 

$$\text{Asymptotes: } y = 1 \pm \frac{3}{2}(x+3)$$



53.  $9y^2 - x^2 + 2x + 54y + 62 = 0$

$$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$$

$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$$

Center: (1, -3)

Vertices:  $(1, -3 \pm \sqrt{2})$ Foci:  $(1, -3 \pm 2\sqrt{5})$ 

$$\text{Asymptotes: } y = \frac{1}{3}x - \frac{1}{3} - 3$$

$$y = -\frac{1}{3}x + \frac{1}{3} - 3$$

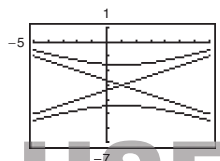
Solve for y:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)



**54.**  $9x^2 - y^2 + 54x + 10y + 55 = 0$   
 $9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$   

$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$$

Center:  $(-3, 5)$

Vertices:  $\left(-3 \pm \frac{1}{3}, 5\right)$

Foci:  $\left(-3 \pm \frac{\sqrt{10}}{3}, 5\right)$

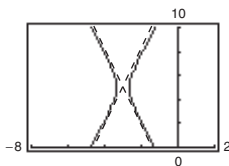
Solve for  $y$ :

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)



**55.**  $3x^2 - 2y^2 - 6x - 12y - 27 = 0$   
 $3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$   

$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$$a = 2, b = \sqrt{6}, c = \sqrt{10}$$

Center:  $(1, -3)$

Vertices:  $(-1, -3), (3, -3)$

Foci:  $(1 \pm \sqrt{10}, -3)$

Asymptotes:  $y = \frac{\sqrt{6}x}{2} - \frac{\sqrt{6}}{2} - 3$

$$y = -\frac{\sqrt{6}x}{2} + \frac{\sqrt{6}}{2} - 3$$

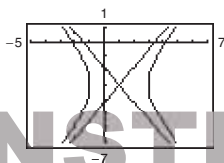
Solve for  $y$ :

$$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$$

$$(y+3)^2 = \frac{3x^2 - 6x - 9}{2}$$

$$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$$

(Graph each curve separately.)



**56.**  $3y^2 - x^2 + 6x - 12y = 0$   
 $3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$   

$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$$a = 1, b = \sqrt{3}, c = 2$$

Center:  $(3, 2)$

Vertices:  $(3, 1), (3, 3)$

Foci:  $(3, 0), (3, 4)$

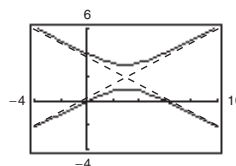
Solve for  $y$ :

$$3(y^2 - 4y + 4) = x^2 - 6x + 12$$

$$(y-2)^2 = \frac{x^2 - 6x + 12}{3}$$

$$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$$

(Graph each curve separately.)



**57.** Vertices:  $(\pm 1, 0)$

Asymptotes:  $y = \pm 5x$

Horizontal transverse axis

Center:  $(0, 0)$

$$a = 1, \frac{b}{a} = 5 \Rightarrow b = 5$$

$$\frac{x^2}{1} - \frac{y^2}{25} = 1$$

**58.** Vertices:  $(0, \pm 4)$

Asymptotes:  $y = \pm 2x$

Vertical transverse axis

$$a = 4, \frac{a}{b} = 2 \Rightarrow b = 2$$

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

59. Vertices:
- $(2, \pm 3)$

Point on graph:  $(0, 5)$ 

Vertical transverse axis

Center:  $(2, 0)$ 

$$a = 3$$

So, the equation is of the form

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1.$$

Substituting the coordinates of the point  $(0, 5)$ , you have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

$$\text{So, the equation is } \frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1.$$

60. Vertices:
- $(2, \pm 3)$

Foci:  $(2, \pm 5)$ 

Vertical transverse axis

Center:  $(2, 0)$ 

$$a = 3, c = 5, b^2 = c^2 - a^2 = 16$$

$$\text{So, } \frac{y^2}{9} - \frac{(x-2)^2}{16} = 1.$$

61. Center:
- $(0, 0)$

Vertex:  $(0, 2)$ Focus:  $(0, 4)$ 

Vertical transverse axis

$$a = 2, c = 4, b^2 = c^2 - a^2 = 12$$

$$\text{So, } \frac{y^2}{4} - \frac{x^2}{12} = 1.$$

62. Center:
- $(0, 0)$

Vertex:  $(6, 0)$ Focus:  $(10, 0)$ 

Horizontal transverse axis

$$a = 6, c = 10, b^2 = c^2 - a^2 = 100 - 36 = 64$$

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

63. Vertices:
- $(0, 2), (6, 2)$

$$\text{Asymptotes: } y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$$

Horizontal transverse axis

Center:  $(3, 2)$ 

$$a = 3$$

$$\text{Slopes of asymptotes: } \pm \frac{b}{a} = \pm \frac{2}{3}$$

So,  $b = 2$ . Therefore,

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$

64. Focus:
- $(20, 0)$

$$\text{Asymptotes: } y = \pm \frac{3}{4}x$$

Horizontal transverse axis

Center:  $(0, 0)$ 

$$c = 20$$

$$\frac{b}{a} = \frac{3}{4} \Rightarrow b = \frac{3}{4}a$$

$$c^2 = 400 = a^2 + b^2 = a^2 + \frac{9}{16}a^2 = \frac{25}{16}a^2$$

$$\Rightarrow a^2 = 256 \quad \text{and} \quad b^2 = 144$$

$$\frac{x^2}{256} - \frac{y^2}{144} = 1$$

$$65. (a) \quad \frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$$

$$\text{At } x = 6: y = \pm\sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$$

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x - 3\sqrt{3}y - 3 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x + 3\sqrt{3}y - 3 = 0$$

(b) From part (a) you know that the slopes of the normal lines must be  $\mp 9/(2\sqrt{3})$ .

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x + 2\sqrt{3}y - 60 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x - 2\sqrt{3}y - 60 = 0$$



66. (a)  $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$   
 $y' = \frac{4x}{2y} = \frac{2x}{y}$

At  $x = 4$ :  $y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm \frac{4}{3}$

At  $(4, 6)$ :  $y - 6 = -\frac{4}{3}(x - 4)$  or  $4x + 3y - 34 = 0$

At  $(4, -6)$ :  $y + 6 = -\frac{4}{3}(x - 4)$  or  $4x + 3y + 2 = 0$

(b) From part (a) you know that the slopes of the normal lines must be  $\mp 3/4$ .

At  $(4, 6)$ :  $y - 6 = -\frac{3}{4}(x - 4)$  or  $3x + 4y - 36 = 0$

At  $(4, -6)$ :  $y + 6 = \frac{3}{4}(x - 4)$  or  $3x - 4y - 36 = 0$

67.  $x^2 + 4y^2 - 6x + 16y + 21 = 0$   
 $(x^2 - 6x + 9) + 4(y^2 + 4y + 4) = -21 + 9 + 16$   
 $(x - 3)^2 + 4(y + 2)^2 = 4$

Ellipse

68.  $4x^2 - y^2 - 4x - 3 = 0$   
 $4(x^2 - x + \frac{1}{4}) - y^2 = 3 + 1$   
 $4(x - \frac{1}{2})^2 - y^2 = 4$

Hyperbola

69.  $y^2 - 8y - 8x = 0$   
 $y^2 - 8y + 16 = 8x + 16$   
 $(y - 4)^2 = 4(2)(x + 2)$

Parabola

70.  $25x^2 - 10x - 200y - 119 = 0$   
 $25(x^2 - \frac{2}{5}x + \frac{1}{25}) = 200y + 119 + 1$   
 $25(x - \frac{1}{5})^2 = 200(y + 1)$

Parabola

71.  $4x^2 + 4y^2 - 16y + 15 = 0$   
 $4x^2 + 4(y^2 - 4y + 4) = -15 + 16$   
 $4x^2 + 4(y - 2)^2 = 1$

Circle (Ellipse)

72.  $y^2 - 4y = x + 5$   
 $y^2 - 4y + 4 = x + 5 + 4$   
 $(y - 2)^2 = x + 9$

Parabola

73.  $9x^2 + 9y^2 - 36x + 6y + 34 = 0$   
 $9(x^2 - 4x + 4) + 9(y^2 + \frac{2}{3}y + \frac{1}{9}) = -34 + 36 + 1$   
 $9(x - 2)^2 + 9(y + \frac{1}{3})^2 = 3$

Circle (Ellipse)

74.  $2x(x - y) = y(3 - y - 2x)$   
 $2x^2 - 2xy = 3y - y^2 - 2xy$   
 $2x^2 + y^2 - 3y = 0$   
 $2x^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$

Ellipse

75.  $3(x - 1)^2 = 6 + 2(y + 1)^2$   
 $3(x - 1)^2 - 2(y + 1)^2 = 6$   
 $\frac{(x - 1)^2}{2} - \frac{(y + 1)^2}{3} = 1$

Hyperbola

76.  $9(x + 3)^2 = 36 - 4(y - 2)^2$   
 $9(x + 3)^2 + 4(y - 2)^2 = 36$   
 $\frac{(x + 3)^2}{4} + \frac{(y - 2)^2}{9} = 1$

Ellipse

77. (a) A parabola is the set of all points  $(x, y)$  that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.  
 (b) For directrix  $y = k - p$ :  $(x - h)^2 = 4p(y - k)$   
 For directrix  $x = h - p$ :  $(y - k)^2 = 4p(x - h)$   
 (c) If  $P$  is a point on a parabola, then the tangent line to the parabola at  $P$  makes equal angles with the line passing through  $P$  and the focus, and with the line passing through  $P$  parallel to the axis of the parabola.

78. (a) An ellipse is the set of all points  $(x, y)$ , the sum of whose distance from two distinct fixed points (foci) is constant.

(b)  $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$  or  $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

79. (a) A hyperbola is the set of all points  $(x, y)$  for which the absolute value of the difference between the distances from two distinct fixed points (foci) is constant.

- (b) Transverse axis is horizontal:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Transverse axis is vertical:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

- (c) Transverse axis is horizontal:

$$y = k + (b/a)(x-h) \text{ and } y = k - (b/a)(x-h)$$

Transverse axis is vertical:

$$y = k + (a/b)(x-h) \text{ and } y = k - (a/b)(x-h)$$

80.  $e = \frac{c}{a}$ ,  $c = \sqrt{a^2 - b^2}$ ,  $0 < e < 1$

For  $e \approx 0$ , the ellipse is nearly circular.

For  $e \approx 1$ , the ellipse is elongated.

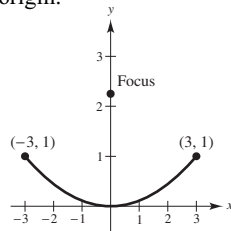
81. Assume that the vertex is at the origin.

$$x^2 = 4py$$

$$(3)^2 = 4p(1)$$

$$\frac{9}{4} = p$$

The pipe is located  $\frac{9}{4}$  meters from the vertex.



82. Assume that the vertex is at the origin.

(a)  $x^2 = 4py$

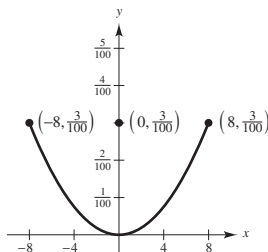
$$8^2 = 4p\left(\frac{3}{100}\right)$$

$$\frac{1600}{3} = p$$

$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

- (b) The deflection is 1 cm when

$$y = \frac{2}{100} \Rightarrow x = \pm \sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



83.  $y = ax^2$

$$y' = 2ax$$

The equation of the tangent line is

$$y - ax_0^2 = 2ax_0(x - x_0)$$

$$\text{or } y = 2ax_0x - ax_0^2.$$

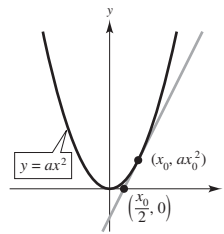
Let  $y = 0$ . Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$

$$ax_0^2 = 2ax_0x$$

$$x = \frac{x_0}{2}$$

So,  $\left(\frac{x_0}{2}, 0\right)$  is the  $x$ -intercept.



84. (a) Without loss of generality, place the coordinate

system so that the equation of the parabola is

$$x^2 = 4py \text{ and, so,}$$

$$y' = \left(\frac{1}{2p}\right)x.$$

So, for distinct tangent lines, the slopes are unequal and the lines intersect.

(b)  $x^2 - 4x - 4y = 0$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At  $(0, 0)$ , the slope is  $-1$ :  $y = -x$ . At  $(6, 3)$ , the slope is  $2$ :  $y = 2x - 9$ . Solving for  $x$ ,

$$-x = 2x - 9$$

$$-3x = -9$$

$$x = 3$$

$$y = -3.$$

Point of intersection:  $(3, -3)$

85. (a) Consider the parabola  $x^2 = 4py$ . Let  $m_0$  be the slope of the one tangent line at  $(x_1, y_1)$  and so,  $-\frac{1}{m_0}$  is the slope of the

second at  $(x_2, y_2)$ . Differentiating,  $2x = 4py'$  or  $y' = \frac{x}{2p}$ , and you have:

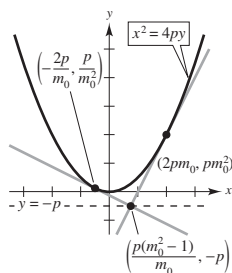
$$m_0 = \frac{1}{2p}x_1 \quad \text{or} \quad x_1 = 2pm_0$$

$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \quad \text{or} \quad x_2 = \frac{-2p}{m_0}.$$

Substituting these values of  $x$  into the equation  $x^2 = 4py$ , we have the coordinates of the points of tangency  $(2pm_0, pm_0^2)$  and  $(-2p/m_0, p/m_0^2)$  and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and} \quad \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is  $\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right)$  and is on the directrix,  $y = -p$ .



(b)  $x^2 - 4x - 4y + 8 = 0$

$$(x - 2)^2 = 4(y - 1)$$

Vertex:  $(2, 1)$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At  $(-2, 5)$ ,  $\frac{dy}{dx} = -2$ . At  $\left(3, \frac{5}{4}\right)$ ,  $\frac{dy}{dx} = \frac{1}{2}$ .

Tangent line at  $(-2, 5)$ :  $y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0$ .

Tangent line at  $\left(3, \frac{5}{4}\right)$ :  $y - \frac{5}{4} = \frac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0$ .

Because  $m_1 m_2 = (-2)\left(\frac{1}{2}\right) = -1$ , the lines are perpendicular.

Point of intersection:  $-2x + 1 = \frac{1}{2}x - \frac{1}{4}$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix:  $y = 0$  and the point of intersection  $\left(\frac{1}{2}, 0\right)$  lies on this line.

86. The focus of  $x^2 = 8y = 4(2)y$  is  $(0, 2)$ . The distance from a point on the parabola,  $(x, x^2/8)$ , and the focus,  $(0, 2)$ , is

$$d = \sqrt{(x - 0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

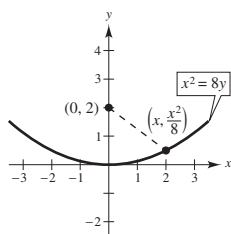
Because  $d$  is minimized when  $d^2$  is minimized, it is sufficient to minimize the function

$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

$$f'(x) = 0 \text{ implies that } \frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \Rightarrow x = 0.$$

This is a minimum by the First Derivative Test. So, the closest point to the focus is the vertex,  $(0, 0)$ .



87.  $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At the point of tangency  $(x_1, y_1)$  on the mountain,  $m = 1 - 2x_1$ . Also,  $m = \frac{y_1 - 1}{x_1 + 1}$ .

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for  $x_1$ , we have  $x_1 = -1 + \sqrt{3}$ .

$$m = 1 - 2(-1 + \sqrt{3}) = 3 - 2\sqrt{3}$$

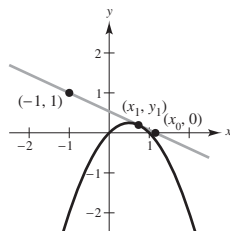
$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$

$$\text{So, } -\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

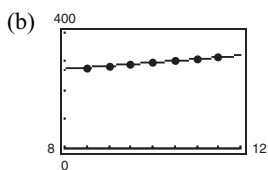
$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

$$\frac{2\sqrt{3}}{3} = x_0.$$

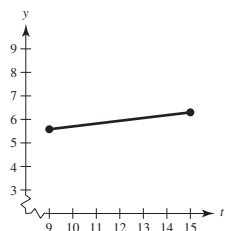


The closest the receiver can be to the hill is  $(2\sqrt{3}/3) - 1 \approx 0.155$ .

88. (a)  $A = 0.06t^2 + 4.5t + 234$  ( $t = 9 \leftrightarrow 1999$ )



(c)  $\frac{dA}{dt} = 0.12t + 4.5$



The average change per year is linear.

$\frac{dA}{dt}$  is increasing.

## 89. Parabola

Vertex:  $(0, 4)$

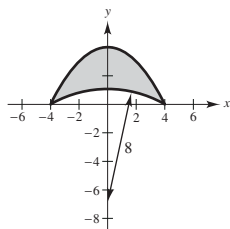
$$x^2 = 4p(y - 4)$$

$$4^2 = 4p(0 - 4)$$

$$p = -1$$

$$x^2 = -4(y - 4)$$

$$y = 4 - \frac{x^2}{4}$$



Circle

Center:  $(0, k)$

Radius: 8

$$x^2 + (y - k)^2 = 64$$

$$4^2 + (0 - k)^2 = 64$$

$$k^2 = 48$$

$$k = -4\sqrt{3} \quad (\text{Center is on the negative } y\text{-axis.})$$

$$x^2 + (y + 4\sqrt{3})^2 = 64$$

$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Because the  $y$ -value is positive when  $x = 0$ , we have  $y = -4\sqrt{3} + \sqrt{64 - x^2}$ .

$$A = 2 \int_0^4 \left[ \left( 4 - \frac{x^2}{4} \right) - \left( -4\sqrt{3} + \sqrt{64 - x^2} \right) \right] dx$$

$$= 2 \left[ 4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2} \left( x\sqrt{64 - x^2} + 64 \arcsin \frac{x}{8} \right) \right]_0^4$$

$$= 2 \left( 16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin \frac{1}{2} \right) = \frac{16(4 + 3\sqrt{3} - 2\pi)}{3} \approx 15.536 \text{ square feet}$$

# INSTRUCTOR USE ONLY

90.  $x = \frac{1}{4}y^2$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

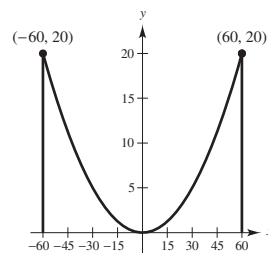
$$\begin{aligned} s &= \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)} dy = \frac{1}{2} \int_0^4 \sqrt{4 + y^2} dy \\ &= \frac{1}{4} \left[ y\sqrt{4 + y^2} + 4 \ln|y + \sqrt{4 + y^2}| \right]_0^4 \\ &= \frac{1}{4} (4\sqrt{20} + 4 \ln|4 + \sqrt{20}| - 4 \ln 2) = 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916 \end{aligned}$$

91. (a) Assume that  $y = ax^2$ .

$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

(b)  $f(x) = \frac{1}{180}x^2$ ,  $f'(x) = \frac{1}{90}x$

$$\begin{aligned} S &= 2 \int_0^{60} \sqrt{1 + \left(\frac{1}{90}x\right)^2} dx = \frac{2}{90} \int_0^{60} \sqrt{90^2 + x^2} dx \\ &= \frac{2}{90} \frac{1}{2} \left[ x\sqrt{90^2 + x^2} + 90^2 \ln|x + \sqrt{90^2 + x^2}| \right]_0^{60} \quad (\text{Formula 26}) \\ &= \frac{1}{90} \left[ 60\sqrt{11,700} + 90^2 \ln(60 + \sqrt{11,700}) - 90^2 \ln 90 \right] \\ &= \frac{1}{90} \left[ 1800\sqrt{13} + 90^2 \ln(60 + 30\sqrt{13}) - 90^2 \ln 90 \right] \\ &= 20\sqrt{13} + 90 \ln\left(\frac{60 + 30\sqrt{13}}{90}\right) \\ &= 10 \left[ 2\sqrt{13} + 9 \ln\left(\frac{2 + \sqrt{13}}{3}\right) \right] \approx 128.4 \text{ m} \end{aligned}$$



92.  $x^2 = 20y$

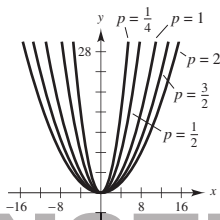
$$y = \frac{x^2}{20}$$

$$y' = \frac{x}{10}$$

$$S = 2\pi \int_0^r x \sqrt{1 + \left(\frac{x}{10}\right)^2} dx = 2\pi \int_0^r \frac{x\sqrt{100 + x^2}}{10} dx = \left[ \frac{\pi}{10} \cdot \frac{2}{3} (100 + x^2)^{3/2} \right]_0^r = \frac{\pi}{15} \left[ (100 + r^2)^{3/2} - 1000 \right]$$

93.  $x^2 = 4py$ ,  $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As  $p$  increases, the graph becomes wider.

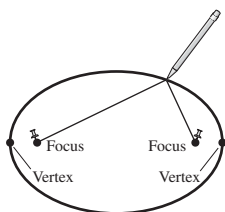


$$94. A = 2 \int_0^h \sqrt{4py} \, dy$$

$$= 4\sqrt{p} \int_0^h y^{1/2} \, dy = \left[ 4\sqrt{p} \left( \frac{2}{3} \right) y^{3/2} \right]_0^h = \frac{8}{3} \sqrt{p} h^{3/2}$$

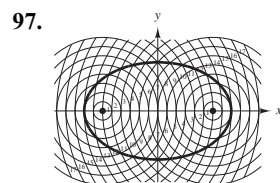
95. (a) At the vertices you notice that the string is horizontal and has a length of  $2a$ .

(b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



$$96. a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$$

The tacks should be placed 1.5 feet from the center. The string should be  $2a = 5$  feet long.



$$98. e = \frac{c}{a}$$

$$0.0167 = \frac{c}{149,598,000}$$

$$c \approx 2,498,286.6$$

Least distance:  $a - c = 147,099,713.4$  km

Greatest distance:  $a + c = 152,096,286.6$  km

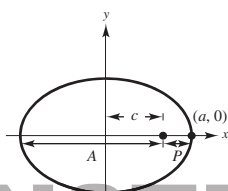
$$99. e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$



$$100. e = \frac{A - P}{A + P}$$

$$= \frac{(123,000 + 4000) - (119 + 4000)}{(123,000 + 4000) + (119 + 4000)}$$

$$= \frac{122,881}{131,119} \approx 0.9372$$

$$101. e = \frac{A - P}{A + P} \quad (\text{Exercise 99})$$

$$= \frac{(1865 + 4000) - (96 + 4000)}{(1865 + 4000) + (96 + 4000)} = \frac{1769}{9961} \approx 0.1776$$

$$102. 9x^2 + 4y^2 - 36x - 24y - 36 = 0$$

$$(a) 9(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 36 + 36 + 36$$

$$9(x - 2)^2 + 4(y - 3)^2 = 108$$

$$\frac{(x - 2)^2}{12} + \frac{(y - 3)^2}{27} = 1$$

Ellipse

$$(b) 9x^2 - 4y^2 - 36x - 24y - 36 = 0$$

$$9(x^2 - 4x + 4) - 4(y^2 + 6y + 9) = 36 + 36 - 36$$

$$\frac{(x - 2)^2}{4} - \frac{(y + 3)^2}{9} = 1$$

Hyperbola

$$(c) 4x^2 + 4y^2 - 36x - 24y - 36 = 0$$

$$4\left(x^2 - 9x + \frac{81}{4}\right) + 4(y^2 - 6y + 9) = 36 + 81 + 36$$

$$\left(x - \frac{9}{2}\right)^2 + (y - 3)^2 = \frac{153}{4}$$

Circle

(d) Sample answer: Eliminate the  $y^2$ -term

$$103. e = \frac{A - P}{A + P} = \frac{35.29 - 0.59}{35.29 + 0.59} = 0.9671$$

$$104. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As  $e \rightarrow 0$ ,  $1 - e^2 \rightarrow 1$  and you have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ or the circle } x^2 + y^2 = a^2.$$

105.  $\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2 x}{10^2 y} = \frac{-x}{4y}$$

At  $(-8, 3)$ :  $y' = \frac{8}{12} = \frac{2}{3}$

The equation of the tangent line is  $y - 3 = \frac{2}{3}(x + 8)$ . It will cross the  $y$ -axis when  $x = 0$  and  $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$ .

106.  $\frac{x^2}{(4.5)^2} + \frac{y^2}{(2.5)^2} = 1$

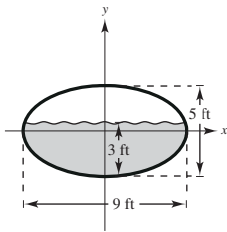
$$x^2 = (4.5)^2 \left[ 1 - \frac{y^2}{(2.5)^2} \right]$$

$$x = \pm \frac{9}{5} \sqrt{(2.5)^2 - y^2}$$

$$V = (\text{Area of bottom})(\text{Length}) + (\text{Area of top})(\text{Length})$$

$$V = \left[ \frac{\pi(4.5)(2.5)}{2} \right] (16) + 16 \int_0^{0.5} 2 \frac{9}{5} \sqrt{(2.5)^2 - y^2} dy \quad (\text{Recall: Area of ellipse is } \pi ab.)$$

$$= 90\pi + \frac{144}{5} \cdot \left[ y \sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin \frac{y}{2.5} \right]_0^{0.5} = 90\pi + \frac{144}{5} \left[ 0.5\sqrt{6} + (2.5)^2 \arcsin \frac{1}{5} \right] \approx 354.3 \text{ ft}^3$$



107.  $16x^2 + 9y^2 + 96x + 36y + 36 = 0$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$  when  $x = -3$ .  $y'$  is undefined when  $y = -2$ .

At  $x = -3$ ,  $y = 2$  or  $-6$ .

Endpoints of major axis:  $(-3, 2)$ ,  $(-3, -6)$

At  $y = -2$ ,  $x = 0$  or  $-6$ .

Endpoints of minor axis:  $(0, -2)$ ,  $(-6, -2)$

**Note:** Equation of ellipse is  $\frac{(x+3)^2}{9} + \frac{(y+2)^2}{16} = 1$

108.  $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$  when  $x = -2$ .  $y'$  undefined when  $y = 3$ .

At  $x = -2$ ,  $y = 0$  or  $6$ .

Endpoints of major axis:  $(-2, 0)$ ,  $(-2, 6)$

At  $y = 3$ ,  $x = 0$  or  $-4$ .

Endpoints of minor axis:  $(0, 3)$ ,  $(-4, 3)$

**Note:** Equation of ellipse is  $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$



$$109. (a) \quad A = 4 \int_0^2 \frac{1}{2} \sqrt{4 - x^2} \, dx = \left[ x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]_0^2 = 2\pi \quad [\text{or, } A = \pi ab = \pi(2)(1) = 2\pi]$$

$$(b) \text{ Disk:} \quad V = 2\pi \int_0^2 \frac{1}{4}(4 - x^2) \, dx = \frac{1}{2}\pi \left[ 4x - \frac{1}{3}x^3 \right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi) \int_0^2 y \left( \frac{\sqrt{16 - 3x^2}}{4y} \right) dx = \pi \int_0^2 \sqrt{16 - 3x^2} \, dx$$

$$= \frac{\pi}{2\sqrt{3}} \left[ \sqrt{3}x\sqrt{16 - 3x^2} + 16 \arcsin\left(\frac{\sqrt{3}x}{4}\right) \right]_0^2 = \frac{2\pi}{9}(9 + 4\sqrt{3}\pi) \approx 21.48$$

$$(c) \text{ Shell:} \quad V = 2\pi \int_0^2 x\sqrt{4 - x^2} \, dx = -\pi \int_0^2 -2x(4 - x^2)^{1/2} \, dx = -\frac{2\pi}{3} \left[ (4 - x^2)^{3/2} \right]_0^2 = \frac{16\pi}{3}$$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi) \int_0^1 2\sqrt{1 - y^2} \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}} dy = 8\pi \int_0^1 \sqrt{1 + 3y^2} \, dy$$

$$= \frac{8\pi}{2\sqrt{3}} \left[ \sqrt{3}y\sqrt{1 + 3y^2} + \ln|\sqrt{3}y + \sqrt{1 + 3y^2}| \right]_0^1$$

$$= \frac{4\pi}{3} \left[ 6 + \sqrt{3} \ln(2 + \sqrt{3}) \right] \approx 34.69$$

$$110. (a) \quad A = 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \, dx = \frac{3}{2} \left[ x\sqrt{16 - x^2} + 16 \arcsin \frac{x}{4} \right]_0^4 = 12\pi$$

$$(b) \text{ Disk:} \quad V = 2\pi \int_0^4 \frac{9}{16}(16 - x^2) \, dx = \frac{9\pi}{8} \left[ 16x - \frac{1}{3}x^3 \right]_0^4 = 48\pi$$

$$y = \frac{3}{4}\sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi) \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}} \, dx = 4\pi \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}} \, dx = \frac{3\pi}{4} \int_0^4 \sqrt{256 - 7x^2} \, dx$$

$$= \frac{3\pi}{8\sqrt{7}} \left[ \sqrt{7}x\sqrt{256 - 7x^2} + 256 \arcsin \frac{\sqrt{7}x}{16} \right]_0^4 = \frac{3\pi}{8\sqrt{7}} \left( 48\sqrt{7} + 256 \arcsin \frac{\sqrt{7}}{4} \right) \approx 138.93$$

(c) **Shell:** 
$$V = 4\pi \int_0^4 x \left( \frac{3}{4} \sqrt{16 - x^2} \right) dx = 3\pi \left[ \left( -\frac{1}{2} \right) \left( \frac{2}{3} \right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi$$

$$x = \frac{4}{3} \sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$\begin{aligned} S &= 2(2\pi) \int_0^3 \frac{4}{3} \sqrt{9 - y^2} \sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}} dy \\ &= 4\pi \int_0^3 \frac{4}{9} \sqrt{81 + 7y^2} dy \\ &= \frac{16}{9} \left( \frac{\pi}{2\sqrt{7}} \right) \left[ \sqrt{7}y \sqrt{81 + 7y^2} + 81 \ln \left| \sqrt{7}y + \sqrt{81 + 7y^2} \right| \right]_0^3 \\ &= \frac{8\pi}{9\sqrt{7}} 3\sqrt{7}(12) + 81 \ln(3\sqrt{7} + 12) - 81 \ln 9 \approx 168.53 \end{aligned}$$

111. From Example 5,

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

For  $\frac{x^2}{25} + \frac{y^2}{49} = 1$ , you have

$$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$$

$$C = 4(7) \int_0^{\pi/2} \sqrt{1 - \frac{24}{49} \sin^2 \theta} d\theta \approx 28(1.3558) \approx 37.96$$

112. (1)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

$$\text{At } P, y' = -\frac{b^2}{a^2} \cdot \frac{x_0}{y_0} = m.$$

(2) Slope of line through  $(-c, 0)$  and  $(x_0, y_0)$ :  $m_1 = \frac{y_0}{x_0 + c}$

Slope of line through  $(c, 0)$  and  $(x_0, y_0)$ :  $m_2 = \frac{y_0}{x_0 - c}$

$$\begin{aligned}
 (3) \quad \tan \alpha &= \frac{m_2 - m}{1 + m_2 m} = \frac{\frac{y_0}{x_0 - c} - \left(-\frac{b^2 x_0}{a^2 y_0}\right)}{1 + \left(\frac{y_0}{x_0 - c}\right)\left(-\frac{b^2 x_0}{a^2 y_0}\right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 - c)}{a^2 y_0(x_0 - c) - b^2 x_0 y_0} \\
 &= \frac{a^2 y_0^2 + b^2 x_0^2 - b^2 x_0 c}{x_0 y_0(a^2 - b^2) - a^2 y_0 c} = \frac{a^2 b^2 - b^2 x_0 c}{x_0 y_0 c^2 - a^2 y_0 c} = \frac{b^2(a^2 - x_0 c)}{y_0 c(x_0 c - a^2)} = -\frac{b^2}{y_0 c} \\
 \alpha &= \arctan\left(-\frac{b^2}{y_0 c}\right) = -\arctan\left(\frac{b^2}{y_0 c}\right) \\
 \tan \beta &= \frac{m_1 - m}{1 + m_1 m} = \frac{\frac{y_0}{x_0 + c} - \left(-\frac{b^2 x_0}{a^2 y_0}\right)}{1 + \left(\frac{y_0}{x_0 + c}\right)\left(-\frac{b^2 x_0}{a^2 y_0}\right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 + c)}{a^2 y_0(x_0 + c) - b^2 x_0 y_0} \\
 &= \frac{a^2 y_0^2 + b^2 x_0^2 + b^2 x_0 c}{a^2 x_0 y_0 + a^2 c y_0 - b^2 x_0 y_0} = \frac{a^2 b^2 + b^2 x_0 c}{x_0 y_0(a^2 - b^2) + a^2 c y_0} = \frac{b^2(a^2 + x_0 c)}{y_0 c(x_0 c + a^2)} = \frac{b^2}{y_0 c} \\
 \beta &= \arctan\left(\frac{b^2}{y_0 c}\right)
 \end{aligned}$$

Because  $|\alpha| = |\beta|$ , the tangent line to an ellipse at a point  $P$  makes equal angles with the line through  $P$  and the foci.

113. Area circle =  $\pi r^2 = 100\pi$

Area ellipse =  $\pi ab = \pi a(10)$

$2(100\pi) = 10\pi a \Rightarrow a = 20$

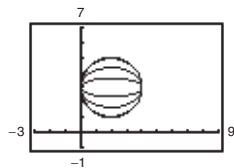
So, the length of the major axis is  $2a = 40$ .

114. (a)  $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$ . So,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

(b)  $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{4(1-e^2)} = 1$



(c) As  $e$  approaches 0, the ellipse approaches a circle.

115. The transverse axis is horizontal since  $(2, 2)$  and  $(10, 2)$  are the foci (see definition of hyperbola).

Center:  $(6, 2)$

$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$

So, the equation is  $\frac{(x-6)^2}{9} - \frac{(y-2)^2}{7} = 1$ .

116. The transverse axis is vertical because  $(-3, 0)$  and  $(-3, 3)$  are the foci.

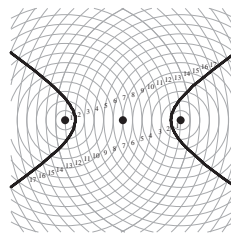
Center:  $\left(-3, \frac{3}{2}\right)$

$c = \frac{3}{2}, 2a = 2, b^2 = c^2 - a^2 = \frac{5}{4}$

So, the equation is  $\frac{[y - (3/2)]^2}{1} - \frac{(x+3)^2}{5/4} = 1$ .

117.  $2a = 10 \Rightarrow a = 5$

$c = 6 \Rightarrow b = \sqrt{11}$



118. Center:  $(0, 0)$ 

Horizontal transverse axis

Foci:  $(\pm c, 0)$ Vertices:  $(\pm a, 0)$ 

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is  $(a + c) - (c - a) = 2a$ .

Now, for any point  $(x, y)$  on the hyperbola, the difference of the distances between  $(x, y)$  and the two foci must also be  $2a$ .

$$\sqrt{(x - c)^2 + (y - 0)^2} - \sqrt{(x + c)^2 + (y - 0)^2} = 2a$$

$$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

$$-4xc - 4a^2 = 4a\sqrt{(x + c)^2 + y^2}$$

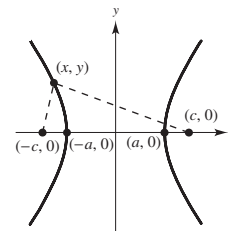
$$-(xc + a^2) = a\sqrt{(x + c)^2 + y^2}$$

$$x^2c^2 + 2a^2cx + a^4 = a^2[x^2 + 2cx + c^2 + y^2]$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Because  $a^2 + b^2 = c^2$ , we have  $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1$ .



119. Time for sound of bullet hitting target to reach  $(x, y)$ :  $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$

Time for sound of rifle to reach  $(x, y)$ :  $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

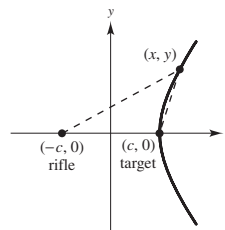
Because the times are the same, you have:  $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s} \sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{v_s^2} = \frac{(x + c)^2 + y^2}{v_s^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$

$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$

$$\frac{x^2}{c^2 v_s^2 / v_m^2} - \frac{y^2}{c^2 (v_m^2 - v_s^2) / v_m^2} = 1$$



120.  $c = 150$ ,  $2a = 0.001(186,000)$ ,  $a = 93$ ,

$$b = \sqrt{150^2 - 93^2} = \sqrt{13,851}$$

$$\frac{x^2}{93^2} - \frac{y^2}{13,851} = 1$$

When  $y = 75$ , you have

$$x^2 = 93^2 \left(1 + \frac{75^2}{13,851}\right)$$

$$x \approx 110.3 \text{ mi.}$$

- 121.** The point  $(x, y)$  lies on the line between  $(0, 10)$  and  $(10, 0)$ . So,  $y = 10 - x$ . The point also lies on the hyperbola  $(x^2/36) - (y^2/64) = 1$ . Using substitution, you have:

$$\begin{aligned}\frac{x^2}{36} - \frac{(10-x)^2}{64} &= 1 \\ 16x^2 - 9(10-x)^2 &= 576 \\ 7x^2 + 180x - 1476 &= 0\end{aligned}$$

$$\begin{aligned}x &= \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} \\ &= \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}\end{aligned}$$

Choosing the positive value for  $x$  we have:

$$\begin{aligned}x &= \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and} \\ y &= \frac{160 - 96\sqrt{2}}{7} \approx 3.462\end{aligned}$$

- 123.**  $\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}$ . Let  $c^2 = a^2 - b^2$ .

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \Rightarrow 2 = x^2 \left( \frac{1}{a^2} + \frac{1}{a^2 - b^2} \right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \Rightarrow x = \pm \frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2} \left( \frac{2a^2c^2}{2a^2 - b^2} \right) \Rightarrow \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \Rightarrow y = \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection:  $\left( \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right), \left( -\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \Rightarrow y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \Rightarrow y'_h = \frac{b^2x}{2c^2y}$$

At  $\left( \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$ , the slopes of the tangent lines are:

$$y'_e = \frac{-b^2 \left( \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2a^2 \left( \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2 \left( \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2c^2 \left( \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = \frac{a}{c}.$$

Because the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

**122.**  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

124.  $Ax^2 + Cy^2 + Dx + Ey + F = 0$  (Assume  $A \neq 0$  and  $C \neq 0$ ; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

- (a) If  $A = C$ , you have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

- (b) If  $C = 0$ , you have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If  $A = 0$ , you have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

- (c) If  $AC < 0$ , you have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

- (d) If  $AC > 0$ , you have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

125. False. See the definition of a parabola.

126. True

127. True

128. False.  $y^2 - x^2 + 2x + 2y = 0$  yields two intersecting lines:  $y + 1 = \pm(x - 1)$

129. True

130. True

131. Let  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  be the equation of the ellipse with  $a > b > 0$ . Let  $(\pm c, 0)$  be the foci,  $c^2 = a^2 - b^2$ . Let  $(u, v)$  be a point on the tangent line at  $P(x, y)$ , as indicated in the figure.

$$x^2b^2 + y^2a^2 = a^2b^2$$

$$2xb^2 + 2yy'a^2 = 0$$

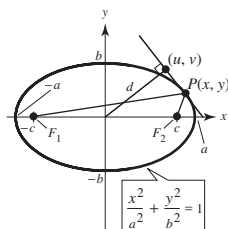
$$y' = -\frac{b^2x}{a^2y} \quad \text{Slope at } P(x, y)$$

Now,  $\frac{y-v}{x-u} = -\frac{b^2x}{a^2y}$

$$y^2a^2 - a^2vy = -b^2x^2 + b^2xu$$

$$y^2a^2 + x^2b^2 = a^2vy + b^2ux$$

$$a^2b^2 = a^2vy + b^2ux$$



Because there is a right angle at  $(u, v)$ ,

$$\frac{v}{u} = \frac{a^2y}{b^2x}$$

$$vb^2x = a^2uy.$$

You have two equations:

$$a^2vy + b^2ux = a^2b^2$$

$$a^2uy - b^2vx = 0.$$

Multiplying the first by  $v$  and the second by  $u$ , and adding,

$$a^2v^2y + a^2u^2y = a^2b^2v$$

$$y[u^2 + v^2] = b^2v$$

$$yd^2 = b^2v$$

$$v = \frac{yd^2}{b^2}.$$

Similarly,  $u = \frac{xd^2}{a^2}.$

From the figure,  $u = d \cos \theta$  and  $v = d \sin \theta$ . So,  $\cos \theta = \frac{xd}{a^2}$  and  $\sin \theta = \frac{yd}{b^2}.$

$$\cos^2 \theta + \sin^2 \theta = \frac{x^2d^2}{a^4} + \frac{y^2d^2}{b^4} = 1$$

$$x^2b^4d^2 + y^2a^4d^2 = a^4b^4$$

$$d^2 = \frac{a^4b^4}{x^2b^4 + y^2a^4}$$

Let  $r_1 = PF_1$  and  $r_2 = PF_2$ ,  $r_1 + r_2 = 2a$ .

$$r_1r_2 = \frac{1}{2}[(r_1 + r_2)^2 - r_1^2 - r_2^2] = \frac{1}{2}[4a^2 - (x+c)^2 - y^2 - (x-c)^2 - y^2] = 2a^2 - x^2 - y^2 - c^2 = a^2 + b^2 - x^2 - y^2$$

Finally,  $d^2r_1r_2 = \frac{a^4b^4}{x^2b^4 + y^2a^4} \cdot [a^2 + b^2 - x^2 - y^2]$

$$= \frac{a^4b^4}{b^2(b^2x^2) + a^2(a^2y^2)} \cdot [a^2 + b^2 - x^2 - y^2]$$

$$= \frac{a^4b^4}{b^2(a^2b^2 - a^2y^2) + a^2(a^2b^2 - b^2x^2)} \cdot [a^2 + b^2 - x^2 - y^2]$$

$$= \frac{a^4b^4}{a^2b^2[a^2 + b^2 - x^2 - y^2]} \cdot [a^2 + b^2 - x^2 - y^2] = a^2b^2, \text{ a constant!}$$

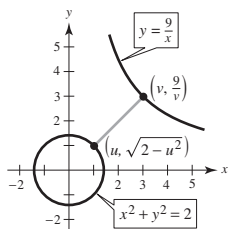
132. Consider circle  $x^2 + y^2 = 2$  and hyperbola  $y = \frac{9}{x}$ .

Let  $(u, \sqrt{2 - u^2})$  and  $(v, \frac{9}{v})$  be points on the circle and hyperbola, respectively. We need to minimize the distance between these 2 points:

$$(\text{Distance})^2 = f(u, v) = (u - v)^2 + \left( \sqrt{2 - u^2} - \frac{9}{v} \right)^2.$$

The tangent lines at  $(1, 1)$  and  $(3, 3)$  are both perpendicular to  $y = x$ , and so they are parallel.

The minimum value is  $(3 - 1)^2 + (3 - 1)^2 = 8$ .



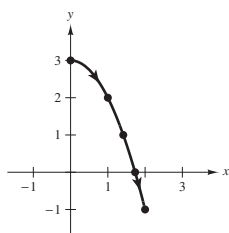
## Section 10.2 Plane Curves and Parametric Equations

1.  $x = \sqrt{t}, y = 3 - t$

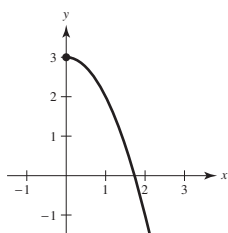
(a)

$t$	0	1	2	3	4
$x$	0	1	$\sqrt{2}$	$\sqrt{3}$	2
$y$	3	2	1	0	-1

(b), (c)



(d)  $x^2 = t, y = 3 - x^2, x \geq 0$

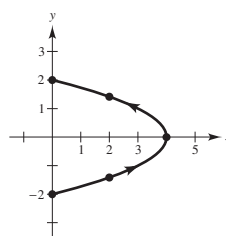


2.  $x = 4 \cos^2 \theta, y = 2 \sin \theta$   
 $0 \leq x \leq 4, -2 \leq y \leq 2$

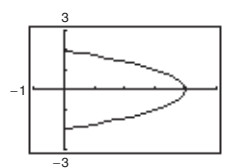
(a)

$\theta$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x$	0	2	4	2	0
$y$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2

(b)



(c)



(d)  $\frac{x}{4} = \cos^2 \theta$

$$\frac{y^2}{4} = \sin^2 \theta$$

$$\frac{x}{4} + \frac{y^2}{4} = 1$$

$$x = 4 - y^2, -2 \leq y \leq 2$$

(e) The graph would be oriented in the opposite direction.

# INSTRUCTOR USE ONLY



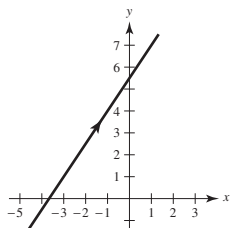
3.  $x = 2t - 3$

$y = 3t + 1$

$t = \frac{x+3}{2}$

$y = 3\left(\frac{x+3}{2}\right) + 1 = \frac{3}{2}x + \frac{11}{2}$

$3x - 2y + 11 = 0$

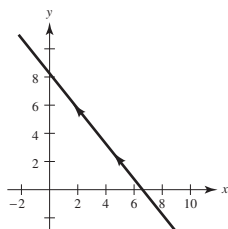


4.  $x = 5 - 4t$

$y = 2 + 5t$

$t = \frac{5-x}{4}$

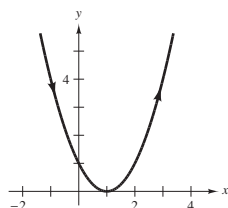
$y = 2 + 5\left(\frac{5-x}{4}\right) = -\frac{5}{4}x + \frac{33}{4}$



5.  $x = t + 1$

$y = t^2$

$y = (x-1)^2$



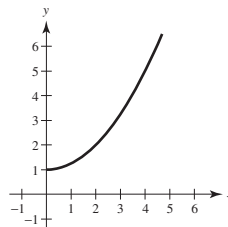
6.  $x = 2t^2$

$y = t^4 + 1$

$y = \left(\frac{x}{2}\right)^2 + 1 = \frac{x^2}{4} + 1, x \geq 0$

For  $t < 0$ , the orientation is right to left.

For  $t > 0$ , the orientation is left to right.

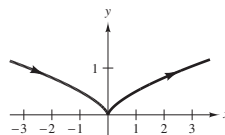


7.  $x = t^3$

$y = \frac{1}{2}t^2$

$y = t^3$  implies  $t = x^{1/3}$

$y = \frac{1}{2}x^{2/3}$



8.  $x = t^2 + t, y = t^2 - t$

Subtracting the second equation from the first, you have

$x - y = 2t$  or  $t = \frac{x-y}{2}$ .

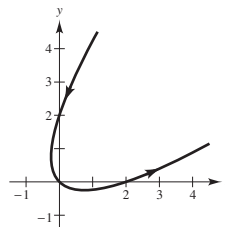
$y = \frac{(x-y)^2}{4} - \frac{x-y}{2}$

$t$	-2	-1	0	1	2
$x$	2	0	0	2	6
$y$	6	2	0	0	2

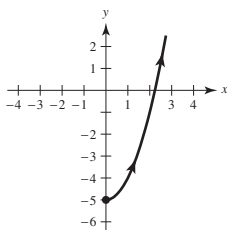
Because the discriminant is

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$ ,

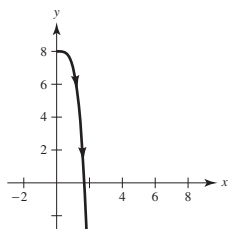
the graph is a rotated parabola.



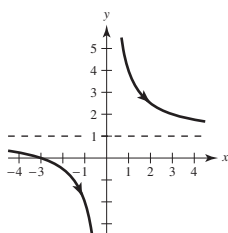
9.  $x = \sqrt{t}$   
 $y = t - 5$   
 $x^2 = t$   
 $y = x^2 - 5, x \geq 0$



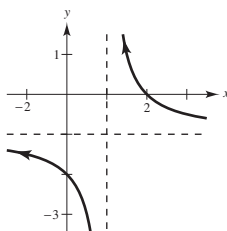
10.  $x = \sqrt[4]{t}$   
 $y = 8 - t$   
 $x^4 = t$   
 $y = 8 - x^4, x \geq 0$



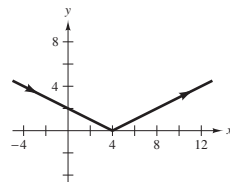
11.  $x = t - 3$   
 $y = \frac{t}{t - 3}$   
 $t = x + 3$   
 $y = \frac{x + 3}{(x + 3) - 3} = 1 + \frac{3}{x} = \frac{x + 3}{x}$



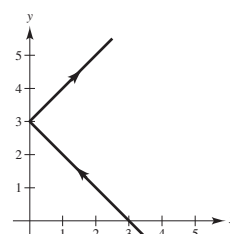
12.  $x = 1 + \frac{1}{t}$   
 $y = t - 1$   
 $x = 1 + \frac{1}{t}$  implies  $t = \frac{1}{x - 1}$   
 $y = \frac{1}{x - 1} - 1$



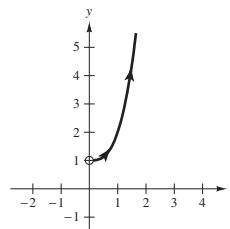
13.  $x = 2t$   
 $y = |t - 2|$   
 $y = \left| \frac{x}{2} - 2 \right| = \frac{|x - 4|}{2}$



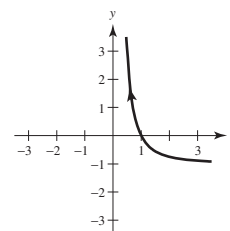
14.  $x = |t - 1|$   
 $y = t + 2$   
 $x = |(y - 2) - 1| = |y - 3|$



15.  $x = e^t, x > 0$   
 $y = e^{3t} + 1$   
 $y = x^3 + 1, x > 0$



16.  $x = e^{-t}, x > 0$   
 $y = e^{2t} - 1$   
 $y = x^{-2} - 1 = \frac{1}{x^2} - 1, x > 0$



17.  $x = \sec \theta$

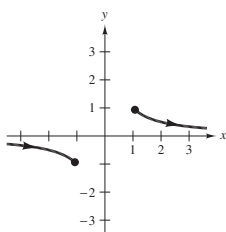
$y = \cos \theta$

$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$

$xy = 1$

$y = \frac{1}{x}$

$|x| \geq 1, |y| \leq 1$



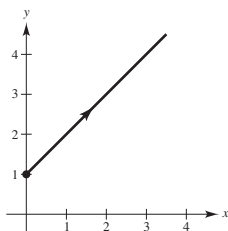
18.  $x = \tan^2 \theta$

$y = \sec^2 \theta$

$\sec^2 \theta = \tan^2 \theta + 1$

$y = x + 1$

$x \geq 0$

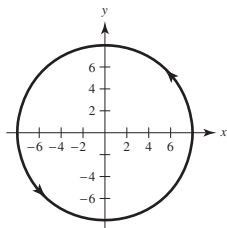


19.  $x = 8 \cos \theta$

$y = 8 \sin \theta$

$x^2 + y^2 = 64 \cos^2 \theta + 64 \sin^2 \theta = 64(1) = 64$

Circle



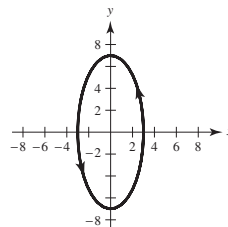
20.  $x = 3 \cos \theta$

$y = 7 \sin \theta$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{7}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$

$\frac{x^2}{9} + \frac{y^2}{49} = 1$

Ellipse



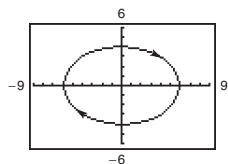
21.  $x = 6 \sin 2\theta$

$y = 4 \cos 2\theta$

$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$

$\frac{x^2}{36} + \frac{y^2}{16} = 1$

Ellipse



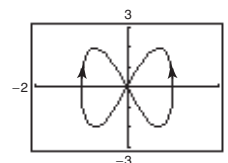
22.  $x = \cos \theta$

$y = 2 \sin 2\theta$

$y = 4 \sin \theta \cos \theta$

$1 - x^2 = \sin^2 \theta$

$y = \pm 4x\sqrt{1 - x^2}$



23.

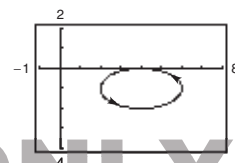
$x = 4 + 2 \cos \theta$

$y = -1 + \sin \theta$

$\frac{(x-4)^2}{4} = \cos^2 \theta$

$\frac{(y+1)^2}{1} = \sin^2 \theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$



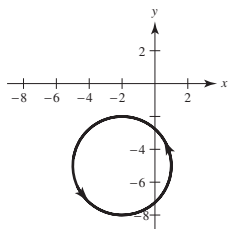
24.  $x = -2 + 3 \cos \theta$

$y = -5 + 3 \sin \theta$

$(x + 2)^2 + (y + 5)^2 = 9 \cos^2 \theta + 9 \sin^2 \theta = 9$

$(x + 2)^2 + (y + 5)^2 = 9$

Circle



25.  $x = -3 + 4 \cos \theta$

$y = 2 + 5 \sin \theta$

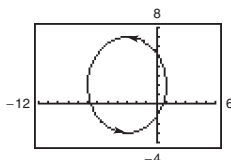
$x + 3 = 4 \cos \theta$

$y - 2 = 5 \sin \theta$

$\left(\frac{x + 3}{4}\right)^2 + \left(\frac{y - 2}{5}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$

$\frac{(x + 3)^2}{16} + \frac{(y - 2)^2}{25} = 1$

Ellipse



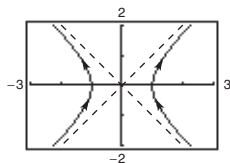
26.  $x = \sec \theta$

$y = \tan \theta$

$x^2 = \sec^2 \theta$

$y^2 = \tan^2 \theta$

$x^2 - y^2 = 1$



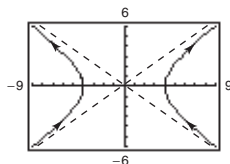
27.  $x = 4 \sec \theta$

$y = 3 \tan \theta$

$\frac{x^2}{16} = \sec^2 \theta$

$\frac{y^2}{9} = \tan^2 \theta$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$



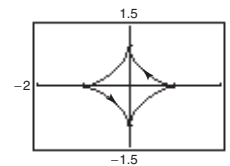
28.  $x = \cos^3 \theta$

$y = \sin^3 \theta$

$x^{2/3} = \cos^2 \theta$

$y^{2/3} = \sin^2 \theta$

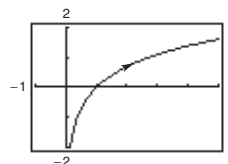
$x^{2/3} + y^{2/3} = 1$



29.  $x = t^3$

$y = 3 \ln t$

$y = 3 \ln \sqrt[3]{x} = \ln x$

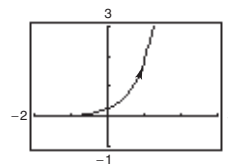


30.  $x = \ln 2t$

$y = t^2$

$t = \frac{e^x}{2}$

$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$



31.  $x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

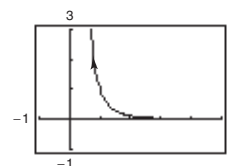
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

$x > 0$

$y > 0$



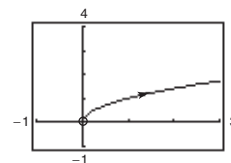
32.  $x = e^{2t}$

$y = e^t$

$y^2 = x$

$y > 0$

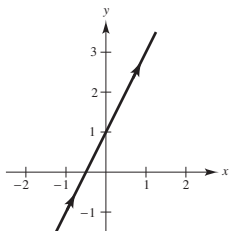
$y = \sqrt{x}, x > 0$



INSTRUCTOR USE ONLY

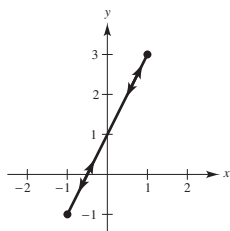
33. By eliminating the parameters in (a) – (d), you get  $y = 2x + 1$ . They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

(a)  $x = t, y = 2t + 1$

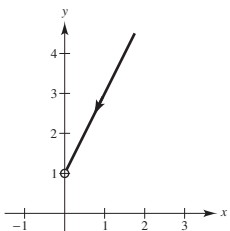


(b)  $x = \cos \theta, y = 2 \cos \theta + 1$   
 $-1 \leq x \leq 1, -1 \leq y \leq 3$

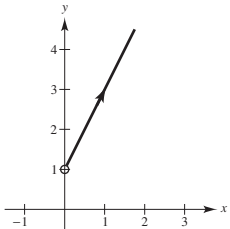
$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$  when  $\theta = 0, \pm\pi, \pm2\pi, \dots$



(c)  $x = e^{-t}, y = 2e^{-t} + 1$   
 $x > 0, y > 1$

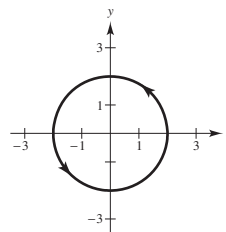


(d)  $x = e^t, y = 2e^t + 1$   
 $x > 0, y > 1$

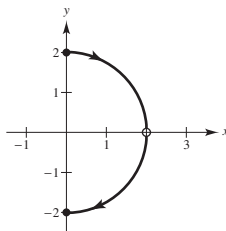


34. By eliminating the parameters in (a) – (d), you get  $x^2 + y^2 = 4$ . They differ from each other in orientation and in restricted domains. These curves are all smooth.

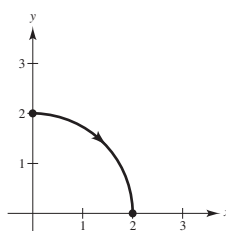
(a)  $x = 2 \cos \theta, y = 2 \sin \theta$



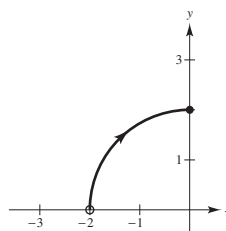
(b)  $x = \frac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \frac{1}{t^2}}, y = \frac{1}{t}$   
 $x \geq 0, x \neq 2, y \neq 0$



(c)  $x = \sqrt{t}, y = \sqrt{4 - t}$   
 $x \geq 0, y \geq 0$



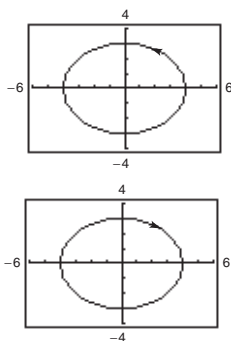
(d)  $x = -\sqrt{4 - e^{2t}}, y = e^t$   
 $-2 < x \leq 0, y > 0$



35. The curves are identical on  $0 < \theta < \pi$ . They are both smooth. They represent  $y = 2(1 - x^2)$  for  $-1 \leq x \leq 1$ . The orientation is from right to left in part (a) and in part (b).

36. The orientations are reversed. The graphs are the same. They are both smooth.

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Answers will vary. For example,

$$x = 2 \sec t \quad x = 2 \sec(-t)$$

$$y = 5 \sin t \quad y = 5 \sin(-t)$$

have the same graphs, but their orientations are reversed.

38. The set of points  $(x, y)$  corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

39.  $x = x_1 + t(x_2 - x_1)$

$$y = y_1 + t(y_2 - y_1)$$

$$\frac{x - x_1}{x_2 - x_1} = t$$

$$y = y_1 + \left( \frac{x - x_1}{x_2 - x_1} \right) (y_2 - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - y_1 = m(x - x_1)$$

40.

$$x = h + r \cos \theta$$

$$y = k + r \sin \theta$$

$$\cos \theta = \frac{x - h}{r}$$

$$\sin \theta = \frac{y - k}{r}$$

$$\cos^2 \theta + \sin^2 \theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$

$$(x - h)^2 + (y - k)^2 = r^2$$

41.

$$x = h + a \cos \theta$$

$$y = k + b \sin \theta$$

$$\frac{x - h}{a} = \cos \theta$$

$$\frac{y - k}{b} = \sin \theta$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

42.

$$x = h + a \sec \theta$$

$$y = k + b \tan \theta$$

$$\frac{x - h}{a} = \sec \theta$$

$$\frac{y - k}{b} = \tan \theta$$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

43. From Exercise 39 you have

$$x = 4t$$

$$y = -7t$$

Solution not unique

44. From Exercise 39 you have

$$x = 1 + 4t$$

$$y = 4 - 6t.$$

Solution not unique

45. From Exercise 40 you have

$$x = 3 + 2 \cos \theta$$

$$y = 1 + 2 \sin \theta$$

Solution not unique

46. From Exercise 40 you have

$$x = -6 + 4 \cos \theta$$

$$y = 2 + 4 \sin \theta$$

47. From Exercise 41 you have

$$a = 10, c = 8 \Rightarrow b = 6$$

$$x = 10 \cos \theta$$

$$y = 6 \sin \theta$$

Center:  $(0, 0)$

Solution not unique

48. From Exercise 41 you have

$$a = 5, c = 3 \Rightarrow b = 4$$

$$x = 4 + 5 \cos$$

$$y = 2 + 4 \sin \theta.$$

Center:  $(4, 2)$

Solution not unique

49. From Exercise 42 you have

$$a = 4, c = 5 \Rightarrow b = 3$$

$$x = 4 \sec \theta$$

$$y = 3 \tan \theta.$$

Center:  $(0, 0)$

Solution not unique

50. From Exercise 42 you have

$$a = 1, c = 2 \Rightarrow b = \sqrt{3}$$

$$x = \sqrt{3} \tan \theta$$

$$y = \sec \theta.$$

Center:  $(0, 0)$

Solution not unique

The transverse axis is vertical, so,  $x$  and  $y$  are interchanged.

51.  $y = 6x - 5$

Examples:

$$x = t, y = 6t - 5$$

$$x = t + 1, y = 6t + 1$$

52.  $y = \frac{4}{x-1}$

Examples:

$$x = t, y = \frac{4}{t-1}$$

$$x = t + 1, y = \frac{4}{t}$$

53.  $y = x^3$

Example

$$x = t, y = t^3$$

$$x = \sqrt[3]{t}, y = t$$

$$x = \tan t, y = \tan^3 t$$

54.  $y = x^2$

Example

$$x = t, y = t^2$$

$$x = t^3, y = t^6$$

55.  $y = 2x - 5$

$$\text{At } (3, 1), t = 0: \quad x = 3 - t$$

$$y = 2(3 - t) - 5 = -2t + 1$$

$$\text{or, } x = t + 3$$

$$y = 2t + 1$$

56.  $y = 4x + 1$

$$\text{At } (-2, -7), t = -1: \quad x = -1 + t$$

$$y = 4(-1 + t) + 1 = 4t - 3$$

57.  $y = x^2$

$$t = 4 \text{ at } (4, 16): \quad x = t$$

$$y = t^2$$

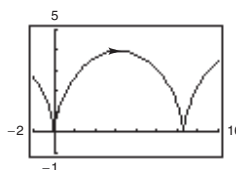
58.  $y = 4 - x^2$

$$t = 1 \text{ at } (1, 3): \quad x = t$$

$$y = 4 - t^2$$

59.  $x = 2(\theta - \sin \theta)$

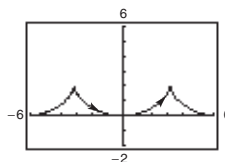
$$y = 2(1 - \cos \theta)$$



Not smooth at  $\theta = 2n\pi$

60.  $x = \theta + \sin \theta$

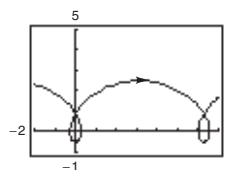
$$y = 1 - \cos \theta$$



Not smooth at  $x = (2n - 1)\pi$

61.  $x = \theta - \frac{3}{2} \sin \theta$

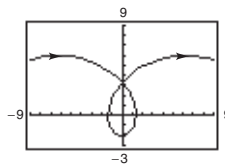
$$y = 1 - \frac{3}{2} \cos \theta$$



Smooth Everywhere

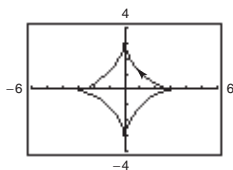
62.  $x = 2\theta - 4 \sin \theta$

$$y = 2 - 4 \cos \theta$$



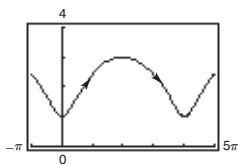
Smooth everywhere

63.  $x = 3 \cos^3 \theta$   
 $y = 3 \sin^3 \theta$



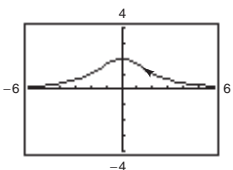
Not smooth at  $(x, y) = (\pm 3, 0)$  and  $(0, \pm 3)$ , or  
 $\theta = \frac{1}{2}n\pi$ .

64.  $x = 2\theta - \sin \theta$   
 $y = 2 - \cos \theta$



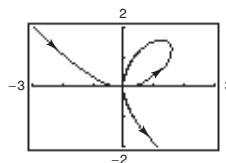
Smooth everywhere

65.  $x = 2 \cot \theta$   
 $y = 2 \sin^2 \theta$



Smooth everywhere

66.  $x = \frac{3t}{1+t^3}$   
 $y = \frac{3t^2}{1+t^3}$



Smooth everywhere

67. Each point  $(x, y)$  in the plane is determined by the plane curve  $x = f(t)$ ,  $y = g(t)$ . For each  $t$ , plot  $(x, y)$ . As  $t$  increases, the curve is traced out in a specific direction called the orientation of the curve.

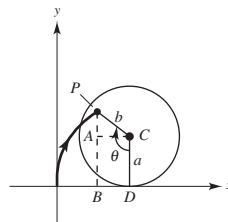
68. (a) Matches (iv) because  $(0, 2)$  is on the graph.  
 (b) Matches (v) because  $(1, 0)$  is on the graph.  
 (c) Matches (ii) because  $-1 \leq x \leq 0$  and  $1 \leq y \leq 3$ .  
 (d) Matches (iii) because  $(4, 0)$  is on the graph.  
 (e) Matches (vi) because undefined at  $\theta = 0$ .  
 (f) Matches (i) because  $x = (y - 2)^2 - 1$  for all  $y$ .

69. When the circle has rolled  $\theta$  radians, you know that the center is at  $(a\theta, a)$ .

$$\sin \theta = \sin(180^\circ - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \text{ or } |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180^\circ - \theta) = \frac{|AP|}{-b} \text{ or } |AP| = -b \cos \theta$$

So,  $x = a\theta - b \sin \theta$  and  $y = a - b \cos \theta$ .



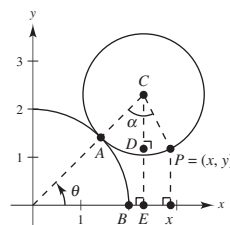
70. Let the circle of radius 1 be centered at  $C$ .  $A$  is the point of tangency on the line  $OC$ .  $OA = 2$ ,  $AC = 1$ ,  $OC = 3$ .  $P = (x, y)$  is the point on the curve being traced out as the angle  $\theta$  changes.  $\widehat{AB} = \widehat{AP}$ ,  $\widehat{AB} = 2\theta$  and  $\widehat{AP} = \alpha \Rightarrow \alpha = 2\theta$ . Form the right triangle  $\triangle CDP$ . The angle  $OCE = (\pi/2) - \theta$  and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3 \sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3 \cos \theta - \cos 3\theta$$

$$y = EC - CD = 3 \sin \theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3 \sin \theta - \sin 3\theta$$

So,  $x = 3 \cos \theta - \cos 3\theta$ ,  $y = 3 \sin \theta - \sin 3\theta$ .





71. False

$$x = t^2 \Rightarrow x \geq 0$$

$$y = t^2 \Rightarrow y \geq 0$$

The graph of the parametric equations is only a portion of the line  $y = x$  when  $x \geq 0$ .

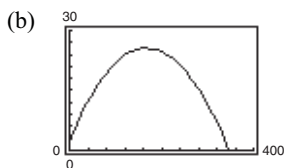
72. False. Let  $x = t^2$  and  $y = t$ . Then  $x = y^2$  and  $y$  is not a function of  $x$ .

73. True.  $y = \cos x$

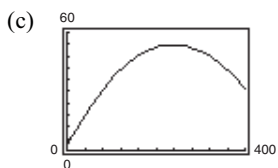
75. (a)  $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

$$x = (v_0 \cos \theta)t = \left(\frac{440}{3} \cos \theta\right)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2 = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$$



It is not a home run—when  $x = 400$ ,  $y < 10$ .



Yes, it's a home run when  $x = 400$ ,  $y > 10$ .

(d) You need to find the angle  $\theta$  (and time  $t$ ) such that

$$x = \left(\frac{440}{3} \cos \theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 = 10.$$

From the first equation  $t = 1200/440 \cos \theta$ . Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{1200}{440 \cos \theta}\right) - 16\left(\frac{1200}{440 \cos \theta}\right)^2$$

$$7 = 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 \sec^2 \theta = 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 (\tan^2 \theta + 1).$$

You now solve the quadratic for  $\tan \theta$ :

$$16\left(\frac{120}{44}\right)^2 \tan^2 \theta - 400 \tan \theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0.$$

$$\tan \theta \approx 0.35185 \Rightarrow \theta \approx 19.4^\circ$$

74.  $x = 8 \cos t$ ,  $y = 8 \sin t$

(a)  $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 = \cos^2 t + \sin^2 t = 1$

$$x^2 + y^2 = 64 \text{ Circle radius 8,}$$

Center:  $(0, 0)$  Oriented counterclockwise

(b) Circle of radius 8, but Center:  $(3, 6)$

(c) The orientation is reversed.

76. (a)  $x = (v_0 \cos \theta)t$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$t = \frac{x}{v_0 \cos \theta} \Rightarrow y = h + (v_0 \sin \theta) \frac{x}{v_0 \cos \theta} - 16 \left( \frac{x}{v_0 \cos \theta} \right)^2$$

$$y = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$$

(b)  $y = 5 + x - 0.005x^2 = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$

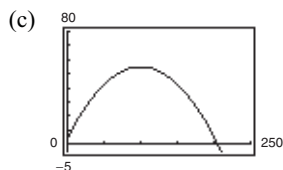
$$h = 5, \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \text{ and}$$

$$0.005 = \frac{16 \sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

$$\text{So, } x = (80 \cos(45^\circ))t$$

$$y = 5 + (80 \sin(45^\circ))t - 16t^2.$$

(d) Maximum height:  $y = 55$  (at  $x = 100$ )

Range: 204.88

### Section 10.3 Parametric Equations and Calculus

1.  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-6}{2t} = -\frac{3}{t}$

2.  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

3.  $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2 \cos \theta \sin \theta}{2 \sin \theta \cos \theta} = -1$

$$\left[ \text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \frac{dy}{dx} = -1 \right]$$

4.  $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(-1/2)e^{-\theta/2}}{2e^\theta} = -\frac{1}{4}e^{-3\theta/2} = \frac{-1}{4e^{3\theta/2}}$

5.  $x = 4t, y = 3t - 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{4}$$

$$\frac{d^2y}{dx^2} = 0$$

At  $t = 3$ , slope is  $\frac{3}{4}$ . (Line)

Neither concave upward nor downward.

6.  $x = \sqrt{t}, y = 3t - 1$

$$\frac{dy}{dx} = \frac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6 \text{ when } t = 1.$$

$$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6 \text{ concave upward}$$

7.  $x = t + 1, y = t^2 + 3t$

$$\frac{dy}{dx} = \frac{2t + 3}{1} = 1 \text{ when } t = -1.$$

$$\frac{d^2y}{dx^2} = 2 \text{ concave upward}$$

8.  $x = t^2 + 5t + 4, y = 4t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{2t+5}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{4}{2t+5}\right]}{dx/dt} = \frac{\frac{-8}{(2t+5)^2}}{2t+5} = \frac{-8}{(2t+5)^3}$$

At  $t = 0, \frac{dy}{dx} = \frac{4}{5}$ .

At  $t = 0, \frac{d^2y}{dx^2} = -\frac{8}{125}$

concave downward

9.  $x = 4 \cos \theta, y = 4 \sin \theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4 \cos \theta}{-4 \sin \theta} = \frac{-\cos \theta}{\sin \theta} = -\cot \theta$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-\cot \theta]}{dx/d\theta} = \frac{\csc^2 \theta}{-4 \sin \theta} = \frac{-1}{4 \sin^3 \theta} = -\frac{1}{4} \csc^3 \theta$$

At  $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -1$ .

$$\frac{d^2y}{dx^2} = \frac{-1}{4(\sqrt{2}/2)^3} = \frac{-\sqrt{2}}{2}$$

concave downward

10.  $x = \cos \theta, y = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta}{-\sin \theta} = -3 \cot \theta \cdot \frac{dy}{dx} \text{ is undefined when } \theta = 0.$$

$$\frac{d^2y}{dx^2} = \frac{3 \csc^2 \theta}{-\sin \theta} = \frac{-3}{\sin^3 \theta} \cdot \frac{d^2y}{dx^2} \text{ is undefined when } \theta = 0.$$

11.  $x = 2 + \sec \theta, y = 1 + 2 \tan \theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2 \sec^2 \theta}{\sec \theta \tan \theta} \\ &= \frac{2 \sec \theta}{\tan \theta} = 2 \csc \theta = 4 \text{ when } \theta = \frac{\pi}{6}. \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} = \frac{-2 \csc \theta \cot \theta}{\sec \theta \tan \theta} \\ &= -2 \cot^3 \theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}. \end{aligned}$$

concave downward

12.  $x = \sqrt{t}, y = \sqrt{t-1}$

$$\frac{dy}{dx} = \frac{1/(2\sqrt{t-1})}{1/(2\sqrt{t})} = \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\left[\sqrt{t-1}/(2\sqrt{t}) - \sqrt{t}/(2\sqrt{t-1})\right]/(t-1)}{1/(2\sqrt{t})} \\ &= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2. \end{aligned}$$

concave downward

13.  $x = \cos^3 \theta, y = \sin^3 \theta$

$$\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta} = -\tan \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-\sec^2 \theta}{-3 \cos^2 \theta \sin \theta} = \frac{1}{3 \cos^4 \theta \sin \theta} \\ &= \frac{\sec^4 \theta \csc \theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}. \end{aligned}$$

concave upward

14.  $x = \theta - \sin \theta, y = 1 - \cos \theta$

$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = 0 \text{ when } \theta = \pi.$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{[(1 - \cos \theta) \cos \theta - \sin^2 \theta]}{(1 - \cos \theta)^2}}{(1 - \cos \theta)} \\ &= \frac{-1}{(1 - \cos \theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi. \end{aligned}$$

concave downward

15.  $x = 2 \cot \theta, y = 2 \sin^2 \theta$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos \theta}{-2 \csc^2 \theta} = -2 \sin^3 \theta \cos \theta$$

At  $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), \theta = \frac{2\pi}{3}, \text{ and } \frac{dy}{dx} = \frac{3\sqrt{3}}{8}.$

Tangent line:  $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$$3\sqrt{3}x - 8y + 18 = 0$$

At  $(0, 2), \theta = \frac{\pi}{2}, \text{ and } \frac{dy}{dx} = 0.$

Tangent line:  $y - 2 = 0$

At  $\left(2\sqrt{3}, \frac{1}{2}\right), \theta = \frac{\pi}{6}, \text{ and } \frac{dy}{dx} = -\frac{\sqrt{3}}{8}.$

Tangent line:  $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}(x - 2\sqrt{3})$

$$\sqrt{3}x + 8y - 10 = 0$$

16.  $x = 2 - 3 \cos \theta$ ,  $y = 3 + 2 \sin \theta$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{3 \sin \theta} = \frac{2}{3} \cot \theta$$

At  $(-1, 3)$ ,  $\theta = 0$ , and  $\frac{dy}{dx}$  is undefined.Tangent line:  $x = -1$ At  $(2, 5)$ ,  $\theta = \frac{\pi}{2}$ , and  $\frac{dy}{dx} = 0$ .Tangent line:  $y = 5$ At  $\left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$ ,  $\theta = \frac{7\pi}{6}$ , and  $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$ .

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3} \left( x - \frac{4 + 3\sqrt{3}}{2} \right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

17.  $x = t^2 - 4$

$$y = t^2 - 2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 2}{2t}$$

At  $(0, 0)$ ,  $t = 2$ ,  $\frac{dy}{dx} = \frac{1}{2}$ .Tangent line:  $y = \frac{1}{2}x$   
 $2y - x = 0$ At  $(-3, -1)$ ,  $t = 1$ ,  $\frac{dy}{dx} = 0$ .Tangent line:  $y = -1$   
 $y + 1 = 0$ At  $(-3, 3)$ ,  $t = -1$ ,  $\frac{dy}{dx} = 2$ .Tangent line:  $y - 3 = 2(x + 3)$   
 $2x - y + 9 = 0$ 

18.  $x = t^4 + 2$

$$y = t^3 + t$$

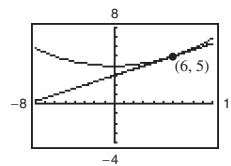
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{4t^3}$$

At  $(2, 0)$ ,  $t = 0$ ,  $\frac{dy}{dx}$  undefined.Tangent line:  $x = 2$  (vertical tangent)At  $(3, -2)$ ,  $t = -1$ ,  $\frac{dy}{dx} = -1$ .Tangent line:  $y + 2 = -(x - 3)$   
 $y = -x + 1$ At  $(18, 10)$ ,  $t = 2$ ,  $\frac{dy}{dx} = \frac{13}{32}$ .Tangent line:  $y - 10 = \frac{13}{32}(x - 18)$ 

$$y = \frac{13}{32}x + \frac{43}{16}$$

19.  $x = 6t$ ,  $y = t^2 + 4$ ,  $t = 1$

(a), (d)

(b) At  $t = 1$ ,  $(x, y) = (6, 5)$ 

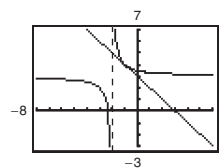
$$\frac{dx}{dt} = 6, \frac{dy}{dt} = 2, \frac{dy}{dx} = \frac{1}{3}$$

(c)  $y - 5 = \frac{1}{3}(x - 6)$ 

$$y = \frac{1}{3}x + 3$$

20.  $x = t - 2$ ,  $y = \frac{1}{t} + 3$ ,  $t = 1$

(a), (d)

(b) At  $t = 1$ ,  $(x, y) = (-1, 4)$ 

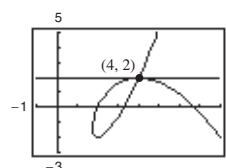
$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -1, \frac{dy}{dx} = -1$$

(c)  $y - 4 = -(x + 1)$ 

$$y = -x + 3$$

21.  $x = t^2 - t + 2$ ,  $y = t^3 - 3t$ ,  $t = -1$

(a), (d)

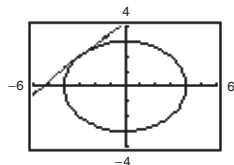
(b) At  $t = -1$ ,  $(x, y) = (4, 2)$ , and

$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 0, \frac{dy}{dx} = 0$$

(c)  $\frac{dy}{dx} = 0$ . At  $(4, 2)$ ,  $y - 2 = 0(x - 4)$   
 $y = 2$ .

22.  $x = 4 \cos \theta, y = 3 \sin \theta, \theta = \frac{3\pi}{4}$

(a), (d)



(b) At  $\theta = \frac{3\pi}{4}, (x, y) = \left(-\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ , and

$$\frac{dx}{dt} = -2\sqrt{2}, \frac{dy}{dt} = -\frac{3\sqrt{2}}{2}, \frac{dy}{dx} = \frac{3}{4}.$$

(c)  $\frac{dy}{dx} = \frac{3}{4}$ . At  $\left(-\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ ,

$$y - \frac{3}{\sqrt{2}} = \frac{3}{4}\left(x + \frac{4}{\sqrt{2}}\right)$$

$$y = \frac{3}{4}x + 3\sqrt{2}.$$

23.  $x = 2 \sin 2t, y = 3 \sin t$  crosses itself at the origin,  $(x, y) = (0, 0)$ .

At this point,  $t = 0$  or  $t = \pi$ .

$$\frac{dy}{dx} = \frac{3 \cos t}{4 \cos 2t}$$

At  $t = 0: \frac{dy}{dx} = \frac{3}{4}$  and  $y = \frac{3}{4}x$ . Tangent Line

At  $t = \pi, \frac{dy}{dx} = -\frac{3}{4}$  and  $y = -\frac{3}{4}x$ . Tangent Line

24.  $x = 2 - \pi \cos t, y = 2t - \pi \sin t$  crosses itself at a point on the  $x$ -axis:  $(2, 0)$ . The corresponding  $t$ -values are  $t = \pm\pi/2$ .

$$\frac{dy}{dt} = 2 - \pi \cos t, \frac{dx}{dt} = \pi \sin t, \frac{dy}{dx} = \frac{2 - \pi \cos t}{\pi \sin t}$$

At  $t = \frac{\pi}{2}: \frac{dy}{dx} = \frac{2}{\pi}$ .

Tangent line:  $y - 0 = \frac{2}{\pi}(x - 2)$

$$y = \frac{2}{\pi}x - \frac{4}{\pi}$$

At  $t = -\frac{\pi}{2}: \frac{dy}{dx} = -\frac{2}{\pi}$ .

Tangent line:  $y - 0 = -\frac{2}{\pi}(x - 2)$

$$y = -\frac{2}{\pi}x + \frac{4}{\pi}$$

25.  $x = t^2 - t, y = t^3 - 3t - 1$  crosses itself at the point  $(x, y) = (2, 1)$ .

At this point,  $t = -1$  or  $t = 2$ .

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At  $t = -1, \frac{dy}{dx} = 0$  and  $y = 1$ . Tangent Line

At  $t = 2, \frac{dy}{dx} = \frac{9}{3} = 3$  and  $y - 1 = 3(x - 2)$  or  $y = 3x - 5$ .

Tangent Line

26.  $x = t^3 - 6t, y = t^2$  crosses itself at  $(0, 6)$ . The corresponding  $t$ -values are  $t = \pm\sqrt{6}$ .

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$$

At  $t = \sqrt{6}, \frac{dy}{dx} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$ .

Tangent line:  $y - 6 = \frac{\sqrt{6}}{6}(x - 0)$

$$y = \frac{\sqrt{6}}{6}x + 6$$

At  $t = -\sqrt{6}, \frac{dy}{dx} = -\frac{2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6}$ .

Tangent line:  $y - 6 = -\frac{\sqrt{6}}{6}x + 6$

27.  $x = \cos \theta + \theta \sin \theta, y = \sin \theta - \theta \cos \theta$

Horizontal tangents:  $\frac{dy}{d\theta} = \theta \sin \theta = 0$  when

$$\theta = \pm\pi, \pm2\pi, \pm3\pi, \dots$$

Points:  $(-1, [2n - 1]\pi), (1, 2n\pi)$  where  $n$  is an integer.

Points shown:  $(1, 0), (-1, \pi), (1, -2\pi)$

Vertical tangents:  $\frac{dx}{d\theta} = \theta \cos \theta = 0$  when

$$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$$

**Note:**  $\theta = 0$  corresponds to the cusp at  $(x, y) = (1, 0)$ .

$$\frac{dy}{dx} = \frac{\theta \sin \theta}{\theta \cos \theta} = \tan \theta = 0 \text{ at } \theta = 0$$

Points:  $\left(\frac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1}\right)$

Points shown:  $\left(\frac{\pi}{2}, 1\right), \left(-\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right)$

28.  $x = 2\theta, y = 2(1 - \cos \theta)$

Horizontal tangents:  $\frac{dy}{d\theta} = 2 \sin \theta = 0$  when

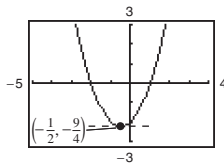
$\theta = 0, \pm\pi, \pm2\pi, \dots$

Points:  $(4n\pi, 0), (2[2n-1]\pi, 4)$  where  $n$  is an integerPoints shown:  $(0, 0), (2\pi, 4), (4\pi, 0)$ Vertical tangents:  $\frac{dx}{d\theta} = 2 \neq 0$ ; none

29.  $x = 4 - t, y = t^2$

Horizontal tangents:  $\frac{dy}{dt} = 2t = 0$  when  $t = 0$ .Point:  $(4, 0)$ Vertical tangents:  $\frac{dx}{dt} = -1 \neq 0$  None

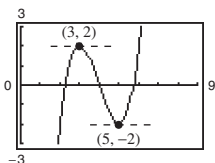
30.  $x = t + 1, y = t^2 + 3t$

Horizontal tangents:  $\frac{dy}{dt} = 2t + 3 = 0$  when  $t = -\frac{3}{2}$ Point:  $(-\frac{1}{2}, -\frac{9}{4})$ Vertical tangents:  $\frac{dx}{dt} = 1 \neq 0$ ; none

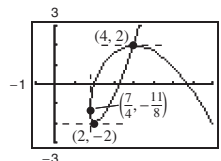
31.  $x = t + 4, y = t^3 - 3t$

Horizontal tangents:

$\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0 \Rightarrow t = \pm 1$

Points:  $(5, -2), (3, 2)$ Vertical tangents:  $\frac{dx}{dt} = 1 \neq 0$  None

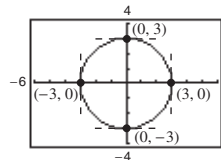
32.  $x = t^2 - t + 2, y = t^3 - 3t$

Horizontal tangents:  $\frac{dy}{dt} = 3t^2 - 3 = 0$  when  $t = \pm 1$ .Points:  $(2, -2), (4, 2)$ Vertical tangents:  $\frac{dx}{dt} = 2t - 1 = 0$  when  $t = \frac{1}{2}$ .Point:  $(\frac{7}{4}, -\frac{11}{8})$ 

33.  $x = 3 \cos \theta, y = 3 \sin \theta$

Horizontal tangents:  $\frac{dy}{d\theta} = 3 \cos \theta = 0$  when

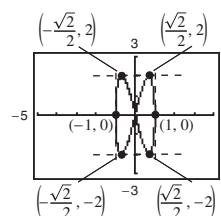
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Points:  $(0, 3), (0, -3)$ Vertical tangents:  $\frac{dx}{d\theta} = -3 \sin \theta = 0$  when  $\theta = 0, \pi$ .Points:  $(3, 0), (-3, 0)$ 

34.  $x = \cos \theta, y = 2 \sin 2\theta$

Horizontal tangents:  $\frac{dy}{d\theta} = 4 \cos 2\theta = 0$  when

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Points:  $(\frac{\sqrt{2}}{2}, 2), (-\frac{\sqrt{2}}{2}, -2), (-\frac{\sqrt{2}}{2}, 2), (\frac{\sqrt{2}}{2}, -2)$ Vertical tangents:  $\frac{dx}{d\theta} = -\sin \theta = 0$  when  $\theta = 0, \pi$ .Points:  $(1, 0), (-1, 0)$ 

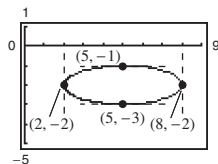
35.  $x = 5 + 3 \cos \theta$ ,  $y = -2 + \sin \theta$

Horizontal tangents:  $\frac{dy}{dt} = \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Points:  $(5, -1)$ ,  $(5, -3)$

Vertical tangents:  $\frac{dx}{dt} = -3 \sin \theta = 0 \Rightarrow \theta = 0, \pi$

Points:  $(8, -2)$ ,  $(2, -2)$



36.  $x = 4 \cos^2 \theta$ ,  $y = 2 \sin \theta$

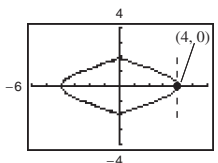
Horizontal tangents:  $\frac{dy}{d\theta} = 2 \cos \theta = 0$  when

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ .

Because  $dx/d\theta = 0$  at  $\pi/2$  and  $3\pi/2$ , exclude them.

Vertical tangents:  $\frac{dx}{d\theta} = -8 \cos \theta \sin \theta = 0$  when  
 $\theta = 0, \pi$ .

Point:  $(4, 0)$



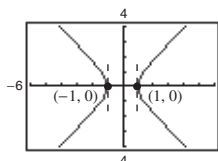
37.  $x = \sec \theta$ ,  $y = \tan \theta$

Horizontal tangents:  $\frac{dy}{d\theta} = \sec^2 \theta \neq 0$ ; None

Vertical tangents:  $\frac{dx}{d\theta} = \sec \theta \tan \theta = 0$  when

$x = 0, \pi$ .

Points:  $(1, 0)$ ,  $(-1, 0)$



38.  $x = \cos^2 \theta$ ,  $y = \cos \theta$

Horizontal tangents:  $\frac{dy}{d\theta} = -\sin \theta = 0$  when  $x = 0, \pi$ .

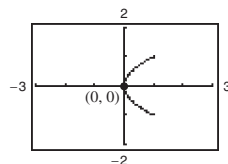
Since  $dx/d\theta = 0$  at these values, exclude them.

Vertical tangents:  $\frac{dx}{d\theta} = -2 \cos \theta \sin \theta = 0$  when

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ .

(Exclude  $0, \pi$ .)

Point:  $(0, 0)$



39.  $x = 3t^2$ ,  $y = t^3 - t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{6t} = \frac{t}{2} - \frac{1}{6t}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[ \frac{t}{2} - \frac{1}{6t} \right]}{dx/dt} = \frac{\frac{1}{2} + \frac{1}{6t^2}}{6t} = \frac{6t^2 + 2}{36t^3}$

Concave upward for  $t > 0$

Concave downward for  $t < 0$

40.  $x = 2 + t^2$ ,  $y = t^2 + t^3$

$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$

$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$

Concave upward for  $t > 0$

Concave downward for  $t < 0$

41.  $x = 2t + \ln t$ ,  $y = 2t - \ln t$ ,  $t > 0$

$\frac{dy}{dx} = \frac{2 - (1/t)}{2 + (1/t)} = \frac{2t - 1}{2t + 1}$

$\frac{d^2y}{dx^2} = \frac{\left[ \frac{(2t+1)2 - (2t-1)2}{(2t+1)^2} \right]}{\left( 2 + \frac{1}{t} \right)^2}$   
 $= \frac{4}{(2t+1)^2} \cdot \frac{t}{2t+1} = \frac{4t}{(2t+1)^3}$

Because  $t > 0$ ,  $\frac{d^2y}{dx^2} > 0$

Concave upward for  $t > 0$

42.  $x = t^2, y = \ln t, t > 0$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1/t^3}{2t} = -\frac{1}{2t^4}$$

Because  $t > 0$ ,  $\frac{d^2y}{dx^2} < 0$

Concave downward for  $t > 0$

43.  $x = \sin t, y = \cos t, 0 < t < \pi$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t}$$

Concave upward on  $\pi/2 < t < \pi$

Concave downward on  $0 < t < \pi/2$

44.  $x = 4 \cos t, y = 2 \sin t, 0 < t < 2\pi$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 \cos t}{-4 \sin t} = -\frac{1}{2} \cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-\frac{1}{2} \cot t\right]}{dx/dt} = \frac{\frac{1}{2} \csc^2 t}{-4 \sin t} = \frac{-1}{8 \sin^3 t}$$

Concave upward on  $\pi < t < 2\pi$

Concave downward on  $0 < t < \pi$

45.  $x = 3t - t^2, y = 2t^{3/2}, 1 \leq t \leq 3$

$$\frac{dx}{dt} = 3 - 2t, \frac{dy}{dt} = 3t^{1/2}$$

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_1^3 \sqrt{(3-2t)^2 + 9t} dt = \int_1^3 \sqrt{4t^2 - 3t + 9} dt \end{aligned}$$

46.  $x = \ln t, y = 4t - 3, 1 \leq t \leq 5$

$$\frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 4$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^5 \sqrt{\frac{1}{t^2} + 16} dt$$

47.  $x = e^t + 2, y = 2t + 1, -2 \leq t \leq 2$

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2$$

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{-2}^2 \sqrt{e^{2t} + 4} dt \end{aligned}$$

48.  $x = t + \sin t, y = t - \cos t, 0 \leq t \leq \pi$

$$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = 1 + \sin t$$

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^\pi \sqrt{(1 + \cos t)^2 + (1 + \sin t)^2} dt \\ &= \int_0^\pi \sqrt{3 + 2 \cos t + 2 \sin t} dt \end{aligned}$$

49.  $x = 3t + 5, y = 7 - 2t, -1 \leq t \leq 3$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -2$$

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{-1}^3 \sqrt{9 + 4} dt \end{aligned}$$

$$\left[\sqrt{13} t\right]_{-1}^3 = 4\sqrt{13} \approx 14.422$$

50.  $x = t^2, y = 2t, 0 \leq t \leq 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2,$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$\begin{aligned} s &= 2 \int_0^2 \sqrt{t^2 + 1} dt \\ &= \left[t\sqrt{t^2 + 1} + \ln|t + \sqrt{t^2 + 1}|\right]_0^2 \\ &= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916 \end{aligned}$$

51.  $x = 6t^2, y = 2t^3, 1 \leq t \leq 4$

$$\frac{dx}{dt} = 12t, \frac{dy}{dt} = 6t^2$$

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_1^4 \sqrt{144t^2 + 36t^4} dt \\ &= \int_1^4 6t\sqrt{4 + t^2} dt \\ &= \left[2(4 + t^2)^{3/2}\right]_1^4 \\ &= 2(20^{3/2} - 5^{3/2}) \\ &= 70\sqrt{5} \approx 156.525 \end{aligned}$$



52.  $x = t^2 + 1, y = 4t^3 + 3, -1 \leq t \leq 0$

$$\begin{aligned}\frac{dx}{dt} &= 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4 \\ s &= \int_{-1}^0 \sqrt{4t^2 + 144t^4} dt = \int_{-1}^0 -2t\sqrt{1 + 36t^2} dt \\ &= \left[ \frac{-(1 + 36t^2)^{3/2}}{54} \right]_{-1}^0 \\ &= \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149\end{aligned}$$

53.  $x = e^{-t} \cos t, y = e^{-t} \sin t, 0 \leq t \leq \frac{\pi}{2}$

$$\begin{aligned}\frac{dx}{dt} &= -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t) \\ s &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{2e^{-2t}} dt = -\sqrt{2} \int_0^{\pi/2} e^{-t}(-1) dt \\ &= \left[ -\sqrt{2}e^{-t} \right]_0^{\pi/2} \\ &= \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12\end{aligned}$$

54.  $x = \arcsin t, y = \ln\sqrt{1-t^2}, 0 \leq t \leq \frac{1}{2}$

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\sqrt{1-t^2}}, \frac{dy}{dt} = \frac{1}{2} \left( \frac{-2t}{1-t^2} \right) = \frac{t}{1-t^2} \\ s &= \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{1/2} \sqrt{\frac{1}{(1-t^2)^2}} dt = \int_0^{1/2} \frac{1}{1-t^2} dt \\ &= \left[ -\frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \right]_0^{1/2} \\ &= -\frac{1}{2} \ln \left( \frac{1}{3} \right) = \frac{1}{2} \ln(3) \approx 0.549\end{aligned}$$

55.  $x = \sqrt{t}, y = 3t - 1, \frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 3$

$$\begin{aligned}s &= \int_0^1 \sqrt{\frac{1}{4t} + 9} dt = \frac{1}{2} \int_0^1 \frac{\sqrt{1+36t}}{\sqrt{t}} dt \\ &= \frac{1}{6} \int_0^6 \sqrt{1+u^2} du \\ &= \frac{1}{12} \left[ \ln(\sqrt{1+u^2} + u) + u\sqrt{1+u^2} \right]_0^6 \\ &= \frac{1}{12} \left[ \ln(\sqrt{37} + 6) + 6\sqrt{37} \right] \approx 3.249\end{aligned}$$

$$u = 6\sqrt{t}, du = \frac{3}{\sqrt{t}} dt$$

56.  $x = t, y = \frac{t^5}{10} + \frac{1}{6t^3}, \frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$

$$\begin{aligned}s &= \int_1^2 \sqrt{1 + \left( \frac{t^4}{2} - \frac{1}{2t^4} \right)^2} dt \\ &= \int_1^2 \sqrt{\left( \frac{t^4}{2} + \frac{1}{2t^4} \right)^2} dt \\ &= \int_1^2 \left( \frac{t^4}{2} + \frac{1}{2t^4} \right) dt = \left[ \frac{t^5}{10} - \frac{1}{6t^3} \right]_1^2 = \frac{779}{240}\end{aligned}$$

57.  $x = a \cos^3 \theta, y = a \sin^3 \theta, \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta,$

$$\begin{aligned}\frac{dy}{d\theta} &= 3a \sin^2 \theta \cos \theta \\ s &= 4 \int_0^{\pi/2} \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta \\ &= 12a \int_0^{\pi/2} \sin \theta \cos \theta \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta \\ &= 6a \int_0^{\pi/2} \sin 2\theta d\theta = [-3a \cos 2\theta]_0^{\pi/2} = 6a\end{aligned}$$

58.  $x = a \cos \theta, y = a \sin \theta, \frac{dx}{d\theta} = -a \sin \theta,$

$$\begin{aligned}\frac{dy}{d\theta} &= a \cos \theta \\ s &= 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} d\theta \\ &= 4a \int_0^{\pi/2} d\theta = [4a\theta]_0^{\pi/2} = 2\pi a\end{aligned}$$

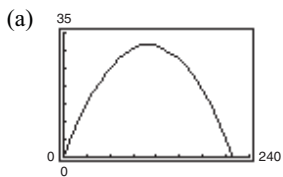
59.  $x = a(\theta - \sin \theta), y = a(1 - \cos \theta),$

$$\begin{aligned}\frac{dx}{d\theta} &= a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta \\ s &= 2 \int_0^{\pi} \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= 2\sqrt{2}a \int_0^{\pi} \sqrt{1 - \cos \theta} d\theta \\ &= 2\sqrt{2}a \int_0^{\pi} \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta \\ &= [-4\sqrt{2}a\sqrt{1 + \cos \theta}]_0^{\pi} = 8a\end{aligned}$$

60.  $x = \cos \theta + \theta \sin \theta, y = \sin \theta - \theta \cos \theta,$

$$\begin{aligned}\frac{dx}{d\theta} &= \theta \cos \theta \\ \frac{dy}{d\theta} &= \theta \sin \theta \\ s &= \int_0^{2\pi} \sqrt{\theta^2 \cos^2 \theta + \theta^2 \sin^2 \theta} d\theta \\ &= \int_0^{2\pi} \theta d\theta = \left[ \frac{\theta^2}{2} \right]_0^{2\pi} = 2\pi^2\end{aligned}$$

61.  $x = (90 \cos 30^\circ)t$ ,  $y = (90 \sin 30^\circ)t - 16t^2$



(b) Range: 219.2 ft,  $\left(t = \frac{45}{16}\right)$

(c)  $\frac{dx}{dt} = 90 \cos 30^\circ$ ,  $\frac{dy}{dt} = 90 \sin 30^\circ - 32t$

$$y = 0 \text{ for } t = \frac{45}{16}.$$

$$s = \int_0^{45/16} \sqrt{(90 \cos 30^\circ)^2 + (90 \sin 30^\circ - 32t)^2} dt \approx 230.8 \text{ ft}$$

62.  $y = 0 \Rightarrow (90 \sin \theta)t = 16t^2 \Rightarrow t = 0, \frac{90}{16} \sin \theta$

$$x = (90 \cos \theta)t = (90 \cos \theta) \frac{90}{16} \sin \theta$$

$$= \frac{90^2}{16} \sin \theta \cos \theta = \frac{90^2}{32} \sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32} 2 \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test,  $\theta = \frac{\pi}{4} (45^\circ)$  maximizes the

range ( $x = 253.125$  feet).

To maximize the arc length, you have

$$\frac{dx}{dt} = 90 \cos \theta, \frac{dy}{dt} = 90 \sin \theta - 32t.$$

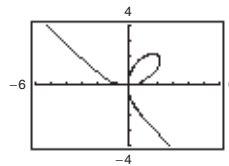
$$s = \int_0^{(90/16)\sin \theta} \sqrt{(90 \cos \theta)^2 + (90 \sin \theta - 32t)^2} dt$$

$$= \frac{2025}{8} \sin \theta + \frac{2025}{16} \cos^2 \theta \ln \left[ \frac{1 + \sin \theta}{1 - \sin \theta} \right]$$

Using a graphing utility, we see that  $s$  is a maximum of approximately 303.67 feet at  $\theta \approx 0.9855 (56.5^\circ)$ .

63.  $x = \frac{4t}{1+t^3}$ ,  $y = \frac{4t^2}{1+t^3}$

(a)  $x^3 + y^3 = 4xy$



(b)  $\frac{dy}{dt} = \frac{(1+t^3)(8t) - 4t^2(3t^2)}{(1+t^3)^2}$

$$= \frac{4t(2-t^3)}{(1+t^3)^2} = 0 \text{ when } t = 0 \text{ or } t = \sqrt[3]{2}.$$

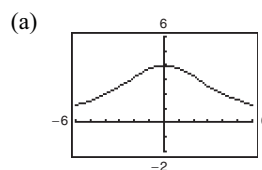
Points:  $(0, 0)$ ,  $\left(\frac{4\sqrt[3]{2}}{3}, \frac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

(c)  $s = 2 \int_0^1 \sqrt{\left[\frac{4(1-2t^3)}{(1+t^3)^2}\right]^2 + \left[\frac{4t(2-t^3)}{(1+t^3)^2}\right]^2} dt$

$$= 2 \int_0^1 \sqrt{\frac{16}{(1+t^3)^4} [t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1]} dt$$

$$= 8 \int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2} dt \approx 6.557$$

64.  $x = 4 \cot \theta = \frac{4}{\tan \theta}$ ,  $y = 4 \sin^2 \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



(b)  $\frac{dy}{d\theta} = 8 \sin \theta \cdot \cos \theta$

$$\frac{dx}{d\theta} = -4 \csc^2 \theta$$

$$\frac{dy}{d\theta} = 0 \text{ for } \theta = 0, \pm \frac{\pi}{2}$$

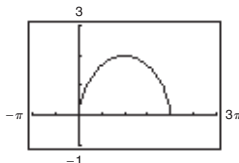
Horizontal tangent at  $(x, y) = (0, 4) \left( \theta = \pm \frac{\pi}{2} \right)$

(Function is not defined at  $\theta = 0$ )

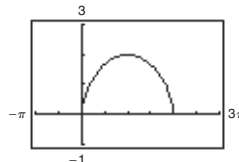
(c) Arc length over  $\frac{\pi}{4} \leq t \leq \frac{\pi}{2}$ : 4.5183

# INSTRUCTOR USE ONLY

65. (a)  $x = t - \sin t$   
 $y = 1 - \cos t$   
 $0 \leq t \leq 2\pi$



$x = 2t - \sin(2t)$   
 $y = 1 - \cos(2t)$   
 $0 \leq t \leq \pi$

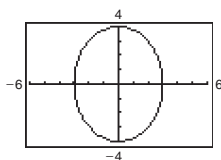


(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

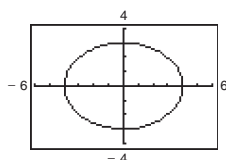
(c)  $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$   
 $y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is  $t = 4\pi$ .

66. (a) First particle:  $x = 3 \cos t$ ,  $y = 4 \sin t$ ,  $0 \leq t \leq 2\pi$



Second particle:  $x = 4 \sin t$ ,  $y = 3 \cos t$ ,  
 $0 \leq t \leq 2\pi$



(b) There are 4 points of intersection.

(c) Suppose at time  $t$  that

$3 \cos t = 4 \sin t$  and  $4 \sin t = 3 \cos t$   
 $\tan t = \frac{3}{4}$  and  $\tan t = \frac{3}{4}$ .

Yes, the particles are at the same place at the same time for  $\tan t = \frac{3}{4}$ ,  $t \approx 0.6435, 3.7851$ . The

intersection points are  $(2.4, 2.4)$  and  $(-2.4, -2.4)$

(d) The curves intersect twice, but not at the same time.

67.  $x = 3t$ ,  $\frac{dx}{dt} = 3$

$y = t + 2$ ,  $\frac{dy}{dt} = 1$

$S = 2\pi \int_0^4 (t + 2) \sqrt{3^2 + 1^2} dt$

$= 2\pi \sqrt{10} \left[ \frac{t^2}{2} + 2t \right]_0^4$

$= 2\pi \sqrt{10} [8 + 8] = 32\sqrt{10}\pi \approx 317.9068$

68.  $x = \frac{1}{4}t^2$ ,  $\frac{dx}{dt} = \frac{t}{2}$

$y = t + 3$ ,  $\frac{dy}{dt} = 1$

$S = 2\pi \int_0^3 (t + 3) \sqrt{\left(\frac{t}{2}\right)^2 + 1} dt$

$= 2\pi \int_0^3 (t + 3) \sqrt{\frac{t^2}{4} + 1} dt$

$\approx 114.1999$

69.  $x = \cos^2 \theta$ ,  $\frac{dx}{d\theta} = -2 \cos \theta \sin \theta$

$y = \cos \theta$ ,  $\frac{dy}{d\theta} = -\sin \theta$

$S = 2\pi \int_0^{\pi/2} \cos \theta \sqrt{4 \cos^2 \theta \sin^2 \theta + \sin^2 \theta} d\theta$

$= 2\pi \int_0^{\pi/2} \cos \theta \sin \theta \sqrt{4 \cos^2 \theta + 1} d\theta$

$= \frac{(5\sqrt{5} - 1)\pi}{6}$

$\approx 5.3304$

70.  $x = \theta + \sin \theta$ ,  $\frac{dx}{d\theta} = 1 + \cos \theta$

$y = \theta + \cos \theta$ ,  $\frac{dy}{d\theta} = 1 - \sin \theta$

$S = 2\pi \int_0^{\pi/2} (\theta + \cos \theta) \sqrt{(1 + \cos \theta)^2 + (1 - \sin \theta)^2} d\theta$

$= 2\pi \int_0^{\pi/2} (\theta + \cos \theta) \sqrt{3 + 2 \cos \theta - 2 \sin \theta} d\theta$

$\approx 23.2433$

71.  $x = 2t$ ,  $\frac{dx}{dt} = 2$

$y = 3t$ ,  $\frac{dy}{dt} = 3$

(a)  $S = 2\pi \int_0^3 3t \sqrt{4 + 9} dt$

$= 6\sqrt{13}\pi \left[ \frac{t^2}{2} \right]_0^3 = 6\sqrt{13}\pi \left( \frac{9}{2} \right) = 27\sqrt{13}\pi$

(b)  $S = 2\pi \int_0^3 2t \sqrt{4 + 9} dt$

$= 4\sqrt{13}\pi \left[ \frac{t^2}{2} \right]_0^3 = 4\sqrt{13}\pi \left( \frac{9}{2} \right) = 18\sqrt{13}\pi$

$$72. x = t, y = 4 - 2t, \frac{dx}{dt} = 1, \frac{dy}{dt} = -2$$

$$(a) S = 2\pi \int_0^2 (4 - 2t)\sqrt{1 + 4} dt$$

$$= \left[ 2\sqrt{5}\pi(4t - t^2) \right]_0^2 = 8\pi\sqrt{5}$$

$$(b) S = 2\pi \int_0^2 t\sqrt{1 + 4} dt = \left[ \sqrt{5}\pi t^2 \right]_0^2 = 4\pi\sqrt{5}$$

$$73. x = 5 \cos \theta, \frac{dx}{d\theta} = -5 \sin \theta$$

$$y = 5 \sin \theta, \frac{dy}{d\theta} = 5 \cos \theta$$

$$S = 2\pi \int_0^{\pi/2} 5 \cos \theta \sqrt{25 \sin^2 \theta + 25 \cos^2 \theta} d\theta$$

$$= 10\pi \int_0^{\pi/2} 5 \cos \theta d\theta$$

$$= 50\pi [\sin \theta]_0^{\pi/2} = 50\pi$$

[Note: This is the surface area of a hemisphere of radius 5]

$$74. x = \frac{1}{3}t^3, y = t + 1, 1 \leq t \leq 2, y\text{-axis}$$

$$\frac{dx}{dt} = t^2, \frac{dy}{dt} = 1$$

$$S = 2\pi \int_1^2 \frac{1}{3}t^3 \sqrt{t^4 + 1} dt = \frac{\pi}{9} \left[ (x^4 + 1)^{3/2} \right]_1^2$$

$$= \frac{\pi}{9} (17^{3/2} - 2^{3/2}) \approx 23.48$$

$$75. x = a \cos^3 \theta, y = a \sin^3 \theta, \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta, \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$S = 4\pi \int_0^{\pi/2} a \sin^3 \theta \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta$$

$$= 12a^2 \pi \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta = \frac{12\pi a^2}{5} [\sin^5 \theta]_0^{\pi/2} = \frac{12}{5} \pi a^2$$

$$76. x = a \cos \theta, y = b \sin \theta, \frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = b \cos \theta$$

$$(a) S = 4\pi \int_0^{\pi/2} b \sin \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$$

$$= 4\pi \int_0^{\pi/2} ab \sin \theta \sqrt{1 - \left( \frac{a^2 - b^2}{a^2} \right) \cos^2 \theta} d\theta = \frac{-4ab\pi}{e} \int_0^{\pi/2} (-e \sin \theta) \sqrt{1 - e^2 \cos^2 \theta} d\theta$$

$$= \frac{-2ab\pi}{e} \left[ e \cos \theta \sqrt{1 - e^2 \cos^2 \theta} + \arcsin(e \cos \theta) \right]_0^{\pi/2} = \frac{2ab\pi}{e} \left[ e\sqrt{1 - e^2} + \arcsin(e) \right]$$

$$= 2\pi b^2 + \left( \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \right) \arcsin \left( \frac{\sqrt{a^2 - b^2}}{a} \right) = 2\pi b^2 + 2\pi \left( \frac{ab}{e} \right) \arcsin(e)$$

$$\left( e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a} : \text{eccentricity} \right)$$

$$(b) S = 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$$

$$= 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta = \frac{4a\pi}{c} \int_0^{\pi/2} c \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta$$

$$= \frac{2a\pi}{c} \left[ c \sin \theta \sqrt{b^2 + c^2 \sin^2 \theta} + b^2 \ln \left| c \sin \theta + \sqrt{b^2 + c^2 \sin^2 \theta} \right| \right]_0^{\pi/2}$$

$$= \frac{2a\pi}{c} \left[ c\sqrt{b^2 + c^2} + b^2 \ln \left| c + \sqrt{b^2 + c^2} \right| - b^2 \ln b \right]$$

$$= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}} \ln \left| \frac{a + \sqrt{a^2 - b^2}}{b} \right| = 2\pi a^2 + \left( \frac{\pi b^2}{e} \right) \ln \left| \frac{1 + e}{1 - e} \right|$$

77.  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

See Theorem 10.7.

78.  $x = t, y = 3 \Rightarrow \frac{dy}{dx} = 0$

79.  $x = t, y = 6t - 5 \Rightarrow \frac{dy}{dx} = \frac{6}{1} = 6$

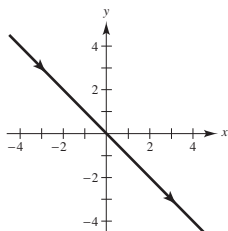
80.  $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

See Theorem 10.8.

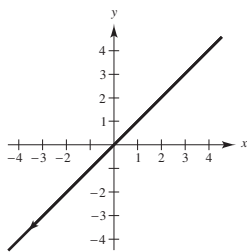
81. (a)  $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

(b)  $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

82. One possible answer is the graph given by  
 $x = t, y = -t$ .



One possible answer is the graph given by  
 $x = -t, y = -t$ .

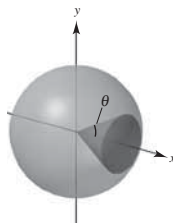


83. Let  $y$  be a continuous function of  $x$  on  $a \leq x \leq b$ . Suppose that  $x = f(t)$ ,  $y = g(t)$ , and  $f(t_1) = a$ ,  $f(t_2) = b$ . Then using integration by substitution,  $dx = f'(t) dt$  and

$$\int_a^b y dx = \int_{t_1}^{t_2} g(t) f'(t) dt.$$

84.  $x = r \cos \phi, y = r \sin \phi$

$$\begin{aligned} S &= 2\pi \int_0^\theta r \sin \phi \sqrt{r^2 \sin^2 \phi + r^2 \cos^2 \phi} d\phi \\ &= 2\pi r^2 \int_0^\theta \sin \phi d\phi \\ &= \left[ -2\pi r^2 \cos \phi \right]_0^\theta \\ &= 2\pi r^2 (1 - \cos \theta) \end{aligned}$$



85.  $x = 2 \sin^2 \theta$

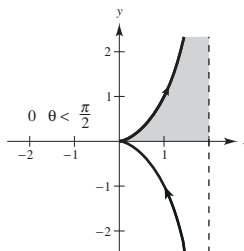
$$y = 2 \sin^2 \theta \tan \theta$$

$$\frac{dx}{d\theta} = 4 \sin \theta \cos \theta$$

$$A = \int_0^{\pi/2} 2 \sin^2 \theta \tan \theta (4 \sin \theta \cos \theta) d\theta$$

$$= 8 \int_0^{\pi/2} \sin^4 \theta d\theta$$

$$= 8 \left[ \frac{-\sin^3 \theta \cos \theta}{4} - \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta \right]_0^{\pi/2} = \frac{3\pi}{2}$$



86.  $x = 2 \cot \theta, y = 2 \sin^2 \theta, \frac{dx}{d\theta} = -2 \csc^2 \theta$

$$A = 2 \int_{\pi/2}^0 (2 \sin^2 \theta) (-2 \csc^2 \theta) d\theta$$

$$= -8 \int_{\pi/2}^0 d\theta = [-8\theta]_{\pi/2}^0 = 4\pi$$

87.  $\pi ab$  is area of ellipse (d).

88.  $\frac{3}{8}\pi a^2$  is area of asteroïd (b).

89.  $6\pi a^2$  is area of cardioid (f).

90.  $2\pi a^2$  is area of deltoid (c).

91.  $\frac{8}{3}ab$  is area of hourglass (a).

92.  $2\pi ab$  is area of teardrop (e).

93.  $x = \sqrt{t}, y = 4 - t, 0 < t < 4$

$$A = \int_0^2 y \, dx = \int_0^4 (4 - t) \frac{1}{2\sqrt{t}} \, dt = \frac{1}{2} \int_0^4 (4t^{-1/2} - t^{1/2}) \, dt = \left[ \frac{1}{2} \left( 8\sqrt{t} - \frac{2}{3} t\sqrt{t} \right) \right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A} \int_0^2 yx \, dx = \frac{3}{16} \int_0^4 (4 - t) \sqrt{t} \left( \frac{1}{2\sqrt{t}} \right) \, dt = \frac{3}{32} \int_0^4 (4 - t) \, dt = \left[ \frac{3}{32} \left( 4t - \frac{t^2}{2} \right) \right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{1}{A} \int_0^2 \frac{y^2}{2} \, dx = \frac{3}{32} \int_0^4 (4 - t)^2 \frac{1}{2\sqrt{t}} \, dt = \frac{3}{64} \int_0^4 (16t^{-1/2} - 8t^{1/2} + t^{3/2}) \, dt = \frac{3}{64} \left[ 32\sqrt{t} - \frac{16}{3} t\sqrt{t} + \frac{2}{5} t^2\sqrt{t} \right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left( \frac{3}{4}, \frac{8}{5} \right)$$

94.  $x = \sqrt{4 - t}, y = \sqrt{t}, \frac{dx}{dt} = -\frac{1}{2\sqrt{4 - t}}, 0 \leq t \leq 4$

$$A = \int_4^0 \sqrt{t} \left( -\frac{1}{2\sqrt{4 - t}} \right) \, dt = \int_0^2 \sqrt{4 - u^2} \, du = \frac{1}{2} \left[ u\sqrt{4 - u^2} + 4 \arcsin \frac{u}{2} \right]_0^2 = \pi$$

Let  $u = \sqrt{4 - t}$ , then  $du = -1/(2\sqrt{4 - t}) \, dt$  and  $\sqrt{t} = \sqrt{4 - u^2}$ .

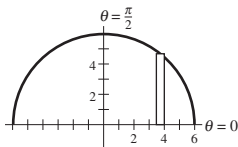
$$\bar{x} = \frac{1}{\pi} \int_4^0 \sqrt{4 - t} \sqrt{t} \left( -\frac{1}{2\sqrt{4 - t}} \right) \, dt = -\frac{1}{2\pi} \int_4^0 \sqrt{t} \, dt = \left[ -\frac{1}{2\pi} \frac{2}{3} t^{3/2} \right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi} \int_4^0 (\sqrt{t})^2 \left( -\frac{1}{2\sqrt{4 - t}} \right) \, dt = -\frac{1}{4\pi} \int_4^0 \frac{t}{\sqrt{4 - t}} \, dt = -\frac{1}{4\pi} \left[ \frac{-2(8 + t)}{3} \sqrt{4 - t} \right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left( \frac{8}{3\pi}, \frac{8}{3\pi} \right)$$

95.  $x = 6 \cos \theta, y = 6 \sin \theta, \frac{dx}{d\theta} = -6 \sin \theta \, d\theta$

$$\begin{aligned} V &= 2\pi \int_{\pi/2}^0 (6 \sin \theta)^2 (-6 \sin \theta) \, d\theta \\ &= -432\pi \int_{\pi/2}^0 \sin^3 \theta \, d\theta \\ &= -432\pi \int_{\pi/2}^0 (1 - \cos^2 \theta) \sin \theta \, d\theta \\ &= -432\pi \left[ -\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 \\ &= -432\pi \left( -1 + \frac{1}{3} \right) = 288\pi \end{aligned}$$



Note: Volume of sphere is  $\frac{4}{3}\pi(6^3) = 288\pi$ .

96.  $x = \cos \theta, y = 3 \sin \theta, \frac{dx}{d\theta} = -\sin \theta$

$$\begin{aligned} V &= 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-\sin \theta) \, d\theta \\ &= -18\pi \int_{\pi/2}^0 \sin^3 \theta \, d\theta \\ &= -18\pi \left[ -\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 12\pi \end{aligned}$$

97.  $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$

(a)  $\frac{dy}{d\theta} = a \sin \theta, \frac{dx}{d\theta} = a(1 - \cos \theta)$

$$\frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}$$

$$\frac{d^2y}{dx^2} = \left[ \frac{(1 - \cos \theta) \cos \theta - \sin \theta (\sin \theta)}{(1 - \cos \theta)^2} \right] \bigg/ \left[ a(1 - \cos \theta) \right] = \frac{\cos \theta - 1}{a(1 - \cos \theta)^3} = \frac{-1}{a(\cos \theta - 1)^2}$$

(b) At  $\theta = \frac{\pi}{6}, x = a\left(\frac{\pi}{6} - \frac{1}{2}\right), y = a\left(1 - \frac{\sqrt{3}}{2}\right), \frac{dy}{dx} = \frac{1/2}{1 - \sqrt{3}/2} = 2 + \sqrt{3}.$

Tangent line:  $y - a\left(1 - \frac{\sqrt{3}}{2}\right) = (2 + \sqrt{3})\left(x - a\left(\frac{\pi}{6} - \frac{1}{2}\right)\right)$

(c)  $\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = 0 \Rightarrow \sin \theta = 0, 1 - \cos \theta \neq 0$

Points of horizontal tangency:  $(x, y) = (a(2n + 1)\pi, 2a)$

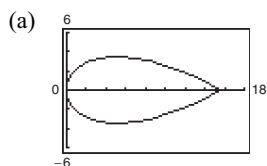
(d) Concave downward on all open  $\theta$ -intervals:

$\dots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \dots$

(e)  $s = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + a^2(1 - \cos \theta)^2} d\theta$

$$= a \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} d\theta = a \int_0^{2\pi} \sqrt{4 \sin^2 \frac{\theta}{2}} d\theta = 2a \int_0^{2\pi} \sin \frac{\theta}{2} d\theta = \left[ -4a \cos \left( \frac{\theta}{2} \right) \right]_0^{2\pi} = 8a$$

98.  $x = t^2\sqrt{3}, y = 3t - \frac{1}{3}t^3$



(b)  $\frac{dx}{dt} = 2\sqrt{3}t, \frac{dy}{dt} = 3 - t^2, \frac{dy}{dx} = \frac{3 - t^2}{2\sqrt{3}t}$

$$\frac{d^2y}{dx^2} = \left[ \frac{2\sqrt{3}(t)(-2t) - (3 - t^2)2\sqrt{3}}{12t^2} \right] \bigg/ \left[ 2\sqrt{3}t \right] = \frac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(12t^2)(2\sqrt{3}t)} = -\frac{t^2 + 3}{12t^3}$$

(c)  $(x, y) = \left( \sqrt{3}, \frac{8}{3} \right)$  at  $t = 1. \frac{dy}{dx} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$

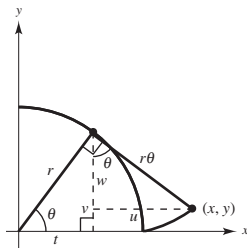
$$y - \frac{8}{3} = \frac{\sqrt{3}}{3}(x - \sqrt{3})$$

$$y = \frac{\sqrt{3}}{3}x + \frac{5}{3}$$

(d)  $s = \int_{-3}^3 \sqrt{12t^2 + (3 - t)^2} dt = \int_{-3}^3 \sqrt{t^4 - 6t^2 + 9 + 12t^2} dt = \int_{-3}^3 \sqrt{(t^2 + 3)^2} dt = \int_{-3}^3 (t^2 + 3) dt = 36$

(e)  $S = 2\pi \int_0^3 \left( 3t - \frac{1}{3}t^3 \right) (t^2 + 3) dt = 81\pi$

$$\begin{aligned} 99. \quad x &= t + u = r \cos \theta + r\theta \sin \theta \\ &= r(\cos \theta + \theta \sin \theta) \\ y &= v - w = r \sin \theta - r\theta \cos \theta \\ &= r(\sin \theta - \theta \cos \theta) \end{aligned}$$



100. Focus on the region above the  $x$ -axis. From Exercise 99, the equation of the involute from  $(1, 0)$  to  $(-1, \pi)$  is

$$\begin{aligned} x &= \cos \theta + \theta \sin \theta \\ y &= \sin \theta - \theta \cos \theta \\ 0 &\leq \theta \leq \pi. \end{aligned}$$

At  $(-1, \pi)$ , the string is fully extended and has length  $\pi$ .

$$\text{So, the area of region A is } \frac{1}{4}\pi(\pi^2) = \frac{1}{4}\pi^3.$$

You now need to find the area of region B.

$$\frac{dx}{d\theta} = -\sin \theta + \sin \theta + \theta \cos \theta = \theta \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}. (\theta = 0 \text{ is cusp.})$$

So, the far right point on the involute is  $(\pi/2, 1)$ .

The area of the region B + C + D is given by

$$\int_{\theta=\pi}^{\theta=\pi/2} y \, dx - \int_{\theta=0}^{\theta=\pi/2} y \, dx = \int_{\theta=\pi}^{\theta=0} y \, dx$$

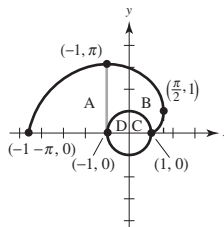
where  $y = \sin \theta - \theta \cos \theta$  and  $dx = \theta \cos \theta \, d\theta$ .

So, you can calculate

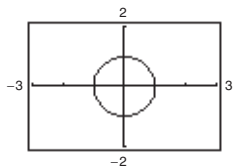
$$\int_{\pi}^0 [\sin \theta - \theta \cos \theta] \theta \cos \theta \, d\theta = \frac{\pi}{6}(\pi^2 + 3).$$

Because the area of C + D is  $\pi/2$ , you have

$$\text{Total area covered} = 2 \left[ \frac{1}{4}\pi^3 + \frac{\pi}{6}(\pi^2 + 3) - \frac{\pi}{2} \right] = \frac{5}{6}\pi^3.$$



101. (a)



$$(b) \quad x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}, \quad -20 \leq t \leq 20$$

The graph (for  $-\infty < t < \infty$ ) is the circle  $x^2 + y^2 = 1$ , except the point  $(-1, 0)$ .

Verify:

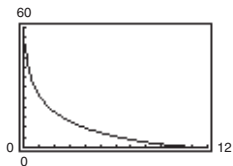
$$\begin{aligned} x^2 + y^2 &= \left( \frac{1-t^2}{1+t^2} \right)^2 + \left( \frac{2t}{1+t^2} \right)^2 \\ &= \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1 \end{aligned}$$

- (c) As  $t$  increases from  $-20$  to  $0$ , the speed increases, and as  $t$  increases from  $0$  to  $20$ , the speed decreases.

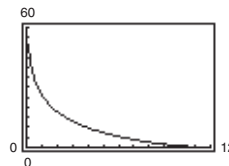


102. (a)  $y = -12 \ln \left( \frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$

$0 < x \leq 12$



(b)  $x = 12 \operatorname{sech} \frac{t}{12}, y = t - 12 \tanh \frac{t}{12}, 0 \leq t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time  $t$ .

(c)  $\frac{dy}{dx} = \frac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12) \tanh(t/12)} = -\sinh \frac{t}{12}$

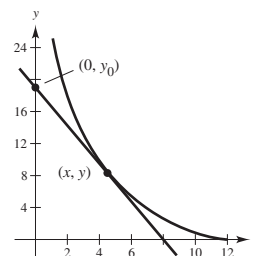
Tangent line:  $y - \left( t_0 - 12 \tanh \frac{t_0}{12} \right) = -\sinh \frac{t_0}{12} \left( x - 12 \operatorname{sech} \frac{t_0}{12} \right)$

$$y = t_0 - \left( \sinh \frac{t_0}{12} \right) x$$

$y$ -intercept:  $(0, t_0)$

Distance between  $(0, t_0)$  and  $(x, y)$ :  $d = \sqrt{\left( 12 \operatorname{sech} \frac{t_0}{12} \right)^2 + \left( -12 \tanh \frac{t_0}{12} \right)^2} = 12$

$d = 12$  for any  $t \geq 0$ .



103. False.  $\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left[ \frac{g'(t)}{f'(t)} \right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$

105.  $A = \pi(2)^2 - \pi\left(\frac{1}{2}\right)^2 = \left(4 - \frac{1}{4}\right)\pi = \frac{15}{4}\pi$

$L = \frac{\frac{15}{4}\pi}{0.001} \approx 11780.97 \text{ in.} \approx 981.7 \text{ ft}$

104. False. Both  $dx/dt$  and  $dy/dt$  are zero when  $t = 0$ . By eliminating the parameter, you have  $y = x^{2/3}$  which does not have a horizontal tangent at the origin.

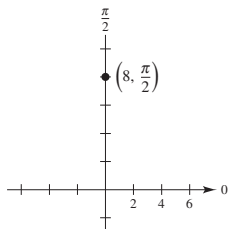
## Section 10.4 Polar Coordinates and Polar Graphs

1.  $\left( 8, \frac{\pi}{2} \right)$

$x = 8 \cos \frac{\pi}{2} = 0$

$y = 8 \sin \frac{\pi}{2} = 8$

$(x, y) = (0, 8)$

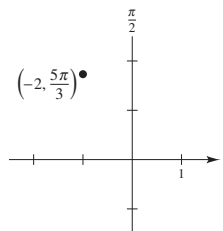


2.  $\left( -2, \frac{5\pi}{3} \right)$

$x = -2 \cos \frac{5\pi}{3} = -2\left(\frac{1}{2}\right) = -1$

$y = -2 \sin \frac{5\pi}{3} = -2\left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3}$

$(x, y) = (-1, \sqrt{3})$

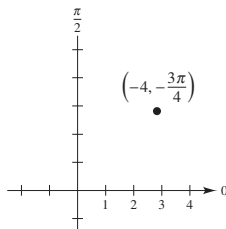


3.  $\left(-4, -\frac{3\pi}{4}\right)$

$$x = -4 \cos\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$y = -4 \sin\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$(x, y) = (2\sqrt{2}, 2\sqrt{2})$$

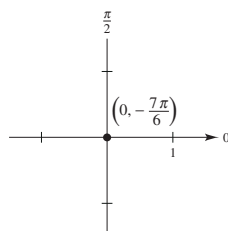


4.  $\left(0, -\frac{7\pi}{6}\right)$

$$x = 0 \cos\left(-\frac{7\pi}{6}\right) = 0$$

$$y = 0 \sin\left(-\frac{7\pi}{6}\right) = 0$$

$$(x, y) = (0, 0)$$

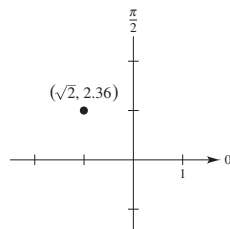


5.  $(\sqrt{2}, 2.36)$

$$x = \sqrt{2} \cos(2.36) \approx -1.004$$

$$y = \sqrt{2} \sin(2.36) \approx 0.996$$

$$(x, y) = (-1.004, 0.996)$$

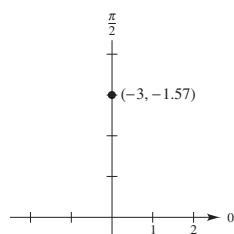


6.  $(-3, -1.57)$

$$x = -3 \cos(-1.57) \approx -0.0024$$

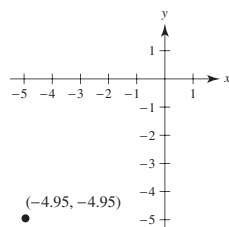
$$y = -3 \sin(-1.57) \approx 3$$

$$(x, y) = (-0.0024, 3)$$



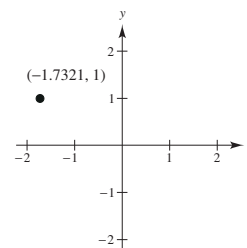
7.  $(r, \theta) = \left(7, \frac{5\pi}{4}\right)$

$$(x, y) = (-4.9497, -4.9497)$$



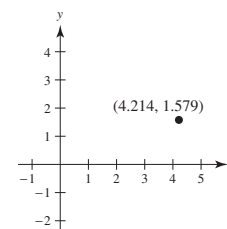
8.  $(r, \theta) = \left(-2, \frac{11\pi}{6}\right)$

$$(x, y) = (-1.7321, 1)$$



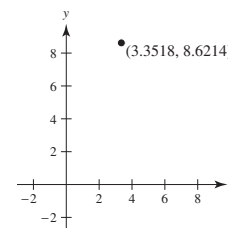
9.  $(r, \theta) = (-4.5, 3.5)$

$$(x, y) = (4.2141, 1.5785)$$



10.  $(r, \theta) = (9.25, 1.2)$

$$(x, y) = (3.3518, 8.6214)$$



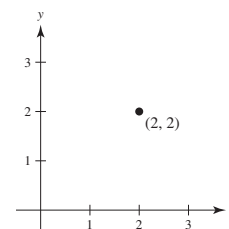
11.  $(x, y) = (2, 2)$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\tan \theta = \frac{2}{2} = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\left(2\sqrt{2}, \frac{\pi}{4}\right), \left(-2\sqrt{2}, \frac{5\pi}{4}\right)$$



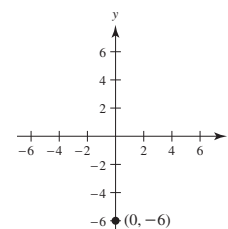
12.  $(x, y) = (0, -6)$

$$r = \pm 6$$

$$\tan \theta \text{ undefined}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\left(6, \frac{3\pi}{2}\right), \left(-6, \frac{\pi}{2}\right)$$



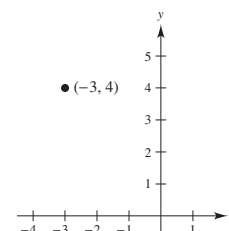
13.  $(x, y) = (-3, 4)$

$$r = \pm\sqrt{9 + 16} = \pm 5$$

$$\tan \theta = -\frac{4}{3}$$

$$\theta \approx 2.214, 5.356, (5, 2.214),$$

$$(-5, 5.356)$$



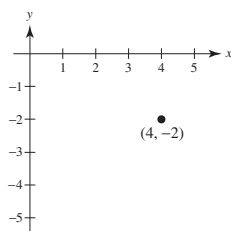
14.  $(x, y) = (4, -2)$

$$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$$

$$\tan \theta = -\frac{2}{4} = -\frac{1}{2}$$

$$\theta \approx -0.464$$

$$(2\sqrt{5}, -0.464), (-2\sqrt{5}, 2.678)$$



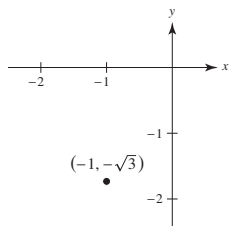
15.  $(x, y) = (-1, -\sqrt{3})$

$$r = \sqrt{4} = 2$$

$$\tan \theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$$

$$\left(2, \frac{4\pi}{3}\right), \left(-2, \frac{\pi}{3}\right)$$

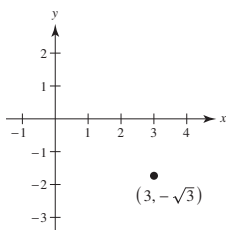


16.  $(x, y) = (3, -\sqrt{3})$

$$r = \sqrt{9 + 3} = 2\sqrt{3}$$

$$\tan \theta = -\frac{\sqrt{3}}{3}$$

$$(r, \theta) = \left(2\sqrt{3}, \frac{11\pi}{6}\right) = \left(-2\sqrt{3}, \frac{5\pi}{6}\right)$$



17.  $(x, y) = (3, -2)$

$$(r, \theta) = (3.606, -0.588)$$

18.  $(x, y) = (3\sqrt{2}, 3\sqrt{2})$

$$(r, \theta) = (6, 0.785)$$

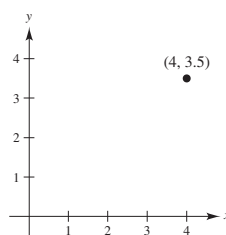
19.  $(x, y) = \left(\frac{7}{4}, \frac{5}{2}\right)$

$$(r, \theta) = (3.0516, 0.9601)$$

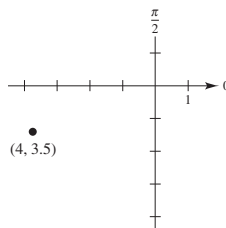
20.  $(x, y) = (0, -5)$

$$(r, \theta) = (5, -1.571)$$

21. (a)  $(x, y) = (4, 3.5)$



(b)  $(r, \theta) = (4, 3.5)$



22. (a) Moving horizontally, the  $x$ -coordinate changes.  
Moving vertically, the  $y$ -coordinate changes.

(b) Both  $r$  and  $\theta$  values change.

(c) In polar mode, horizontal (or vertical) changes result in changes in both  $r$  and  $\theta$ .

23.  $r = 2 \sin \theta$  circle

Matches (c)

24.  $r = 4 \cos 2\theta$

Rose curve

Matches (b)

25.  $r = 3(1 + \cos \theta)$

Cardioid

Matches (a)

26.  $r = 2 \sec \theta$

Line

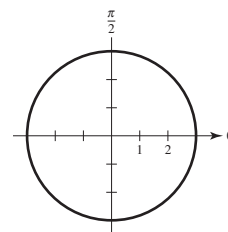
Matches (d)

27.  $x^2 + y^2 = 9$

$$r^2 = 9$$

$$r = 3$$

Circle



NOT FOR SALE

28.  $x^2 - y^2 = 9$

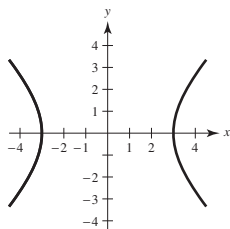
$$(r \cos \theta)^2 - (r \sin \theta)^2 = 9$$

$$r^2(\cos^2 \theta - \sin^2 \theta) = 9$$

$$r^2 \cos 2\theta = 9$$

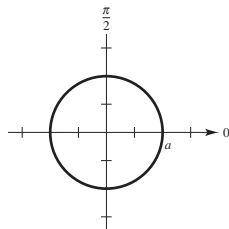
$$r = \frac{3}{\sqrt{\cos 2\theta}}$$

Hyperbola



29.  $x^2 + y^2 = a^2$

$$r = a$$

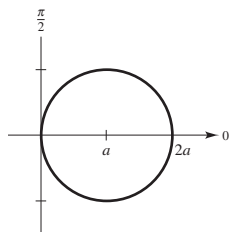


30.  $x^2 + y^2 - 2ax = 0$

$$r^2 - 2ar \cos \theta = 0$$

$$r(r - 2a \cos \theta) = 0$$

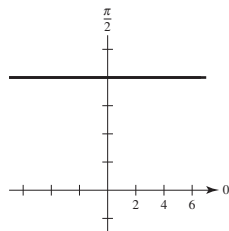
$$r = 2a \cos \theta$$



31.  $y = 8$

$$r \sin \theta = 8$$

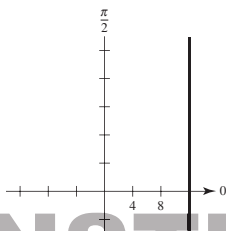
$$r = 8 \csc \theta$$



32.  $x = 12$

$$r \cos \theta = 12$$

$$r = 12 \sec \theta$$

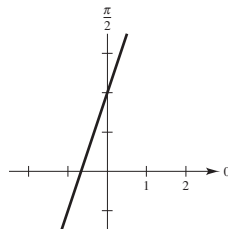


33.  $3x - y + 2 = 0$

$$3r \cos \theta - r \sin \theta + 2 = 0$$

$$r(3 \cos \theta - \sin \theta) = -2$$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

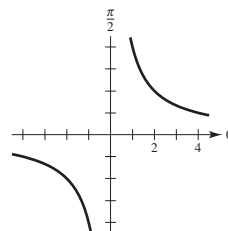


34.  $xy = 4$

$$(r \cos \theta)(r \sin \theta) = 4$$

$$r^2 = 4 \sec \theta \csc \theta$$

$$= 8 \csc 2\theta$$

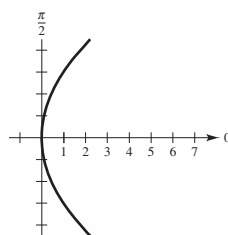


35.  $y^2 = 9x$

$$r^2 \sin^2 \theta = 9r \cos \theta$$

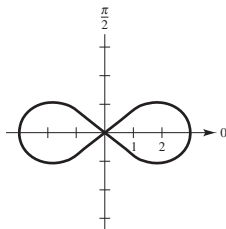
$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

$$r = 9 \csc^2 \theta \cos \theta$$

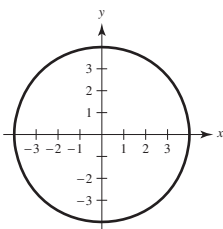


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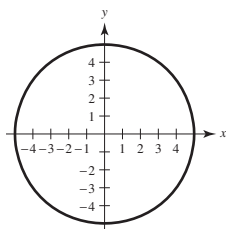
36.  $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$   
 $(r^2)^2 - 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta) = 0$   
 $r^2[r^2 - 9(\cos 2\theta)] = 0$   
 $r^2 = 9 \cos 2\theta$



37.  $r = 4$   
 $r^2 = 16$   
 $x^2 + y^2 = 16$   
 Circle

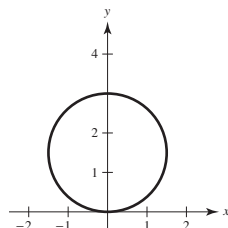


38.  $r = -5$   
 $r^2 = 25$   
 $x^2 + y^2 = 25$   
 Circle

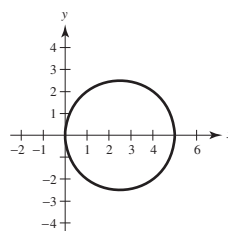


39.  $r = 3 \sin \theta$   
 $r^2 = 3r \sin \theta$   
 $x^2 + y^2 = 3y$   
 $x^2 + (y^2 - 3y + \frac{9}{4}) = \frac{9}{4}$   
 $x^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$

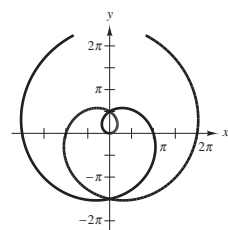
Circle



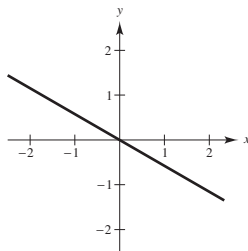
40.  $r = 5 \cos \theta$   
 $r^2 = 5r \cos \theta$   
 $x^2 + y^2 = 5x$   
 $x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$   
 $(x - \frac{5}{2})^2 + y^2 = (\frac{5}{2})^2$



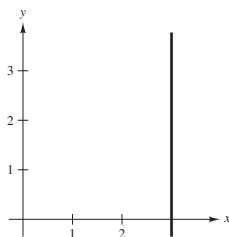
41.  $r = \theta$   
 $\tan r = \tan \theta$   
 $\tan \sqrt{x^2 + y^2} = \frac{y}{x}$   
 $\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$



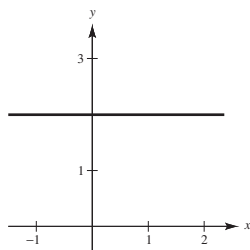
42.  $\theta = \frac{5\pi}{6}$   
 $\tan \theta = \tan \frac{5\pi}{6}$   
 $\frac{y}{x} = -\frac{\sqrt{3}}{3}$   
 $y = -\frac{\sqrt{3}}{3}x$



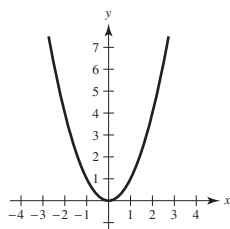
43.  $r = 3 \sec \theta$   
 $r \cos \theta = 3$   
 $x = 3$   
 $x - 3 = 0$



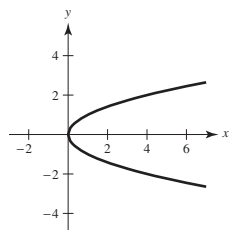
44.  $r = 2 \csc \theta$   
 $r \sin \theta = 2$   
 $y = 2$   
 $y - 2 = 0$



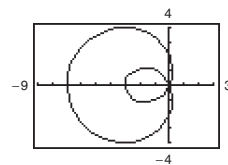
45.  $r = \sec \theta \tan \theta$   
 $r \cos \theta = \tan \theta$   
 $x = \frac{y}{x}$   
 $y = x^2$   
 Parabola



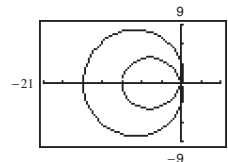
46.  $r = \cot \theta \csc \theta$   
 $r \sin \theta = \cot \theta$   
 $y = \frac{x}{y}$   
 $x = y^2$   
 Parabola



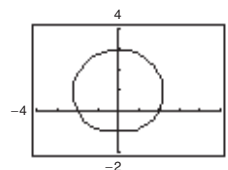
47.  $r = 2 - 5 \cos \theta$   
 $0 \leq \theta < 2\pi$



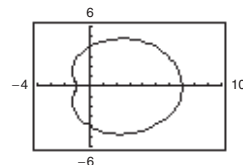
48.  $r = 3(1 - 4 \cos \theta)$   
 $0 \leq \theta < 2\pi$



49.  $r = 2 + \sin \theta$   
 $0 \leq \theta < 2\pi$

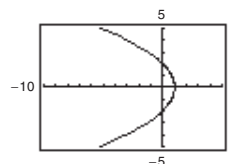


50.  $r = 4 + 3 \cos \theta$   
 $0 \leq \theta < 2\pi$



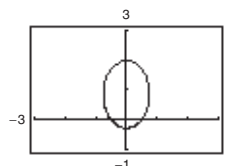
51.  $r = \frac{2}{1 + \cos \theta}$

Traced out once on  $-\pi < \theta < \pi$



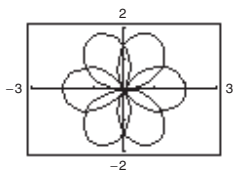
52.  $r = \frac{2}{4 - 3 \sin \theta}$

Traced out once on  $0 \leq \theta \leq 2\pi$



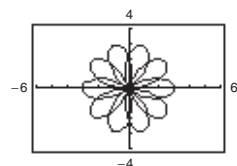
53.  $r = 2 \cos\left(\frac{3\theta}{2}\right)$

$0 \leq \theta < 4\pi$



54.  $r = 3 \sin\left(\frac{5\theta}{2}\right)$

$0 \leq \theta < 4\pi$

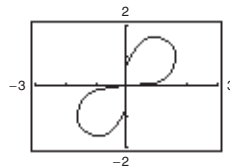


55.  $r^2 = 4 \sin 2\theta$

$r_1 = 2\sqrt{\sin 2\theta}$

$r_2 = -2\sqrt{\sin 2\theta}$

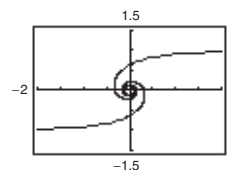
$0 \leq \theta < \frac{\pi}{2}$



56.  $r^2 = \frac{1}{\theta}$

Graph as  $r_1 = \frac{1}{\sqrt{\theta}}, r_2 = -\frac{1}{\sqrt{\theta}}$ .

It is traced out once on  $[0, \infty)$ .



57.

$$r = 2(h \cos \theta + k \sin \theta)$$

$$r^2 = 2r(h \cos \theta + k \sin \theta)$$

$$r^2 = 2[h(r \cos \theta) + k(r \sin \theta)]$$

$$x^2 + y^2 = 2(hx + ky)$$

$$x^2 + y^2 - 2hx - 2ky = 0$$

$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = 0 + h^2 + k^2$$

$$(x - h)^2 + (y - k)^2 = h^2 + k^2$$

Radius:  $\sqrt{h^2 + k^2}$

Center:  $(h, k)$

58. (a) The rectangular coordinates of  $(r_1, \theta_1)$  are  $(r_1 \cos \theta_1, r_1 \sin \theta_1)$ . The rectangular coordinates of  $(r_2, \theta_2)$  are  $(r_2 \cos \theta_2, r_2 \sin \theta_2)$ .

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2$$

$$= r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1$$

$$= r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2)$$

$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

(b) If  $\theta_1 = \theta_2$ , the points lie on the same line passing through the origin. In this case,

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(0)}$$

$$= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|.$$

(c) If  $\theta_1 - \theta_2 = 90^\circ$ , then  $\cos(\theta_1 - \theta_2) = 0$  and  $d = \sqrt{r_1^2 + r_2^2}$ , the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points  $(r_1, \theta_1) = (1, 0)$  and  $(r_2, \theta_2) = \left(2, \frac{\pi}{2}\right)$ .

$$d = \sqrt{1^2 + 2^2 - 2(1)(2)\cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

$$\text{Using } (r_1, \theta_1) = (-1, \pi) \text{ and } (r_2, \theta_2) = \left[2, \left(\frac{5\pi}{2}\right)\right], d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2)\cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}.$$

You always obtain the same distance.

59.  $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

$$d = \sqrt{1^2 + 4^2 - 2(1)(4)\cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)}$$

$$= \sqrt{17 - 8\cos\frac{\pi}{2}} = \sqrt{17}$$

60.  $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

$$d = \sqrt{8^2 + 5^2 - 2(8)(5)\cos\left(\frac{7\pi}{4} - \pi\right)}$$

$$= \sqrt{89 - 80\cos\frac{3\pi}{4}}$$

$$= \sqrt{89 - 80\left(-\frac{\sqrt{2}}{2}\right)}$$

$$= \sqrt{89 + 40\sqrt{2}} \approx 12.0652$$

61.  $(2, 0.5), (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7)\cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28\cos(-0.7)} \approx 5.6$$

62.  $(4, 2.5), (12, 1)$

$$d = \sqrt{4^2 + 12^2 - 2(4)(12)\cos(2.5 - 1)}$$

$$= \sqrt{160 - 96\cos 1.5} \approx 12.3$$

63.  $r = 2 + 3\sin\theta$

$$\frac{dy}{dx} = \frac{3\cos\theta\sin\theta + \cos\theta(2 + 3\sin\theta)}{3\cos\theta\cos\theta - \sin\theta(2 + 3\sin\theta)}$$

$$= \frac{2\cos\theta(3\sin\theta + 1)}{3\cos 2\theta - 2\sin\theta} = \frac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$$

At  $\left(5, \frac{\pi}{2}\right), \frac{dy}{dx} = 0.$

At  $(2, \pi), \frac{dy}{dx} = -\frac{2}{3}.$

At  $\left(-1, \frac{3\pi}{2}\right), \frac{dy}{dx} = 0.$

64.  $r = 2(1 - \sin\theta)$

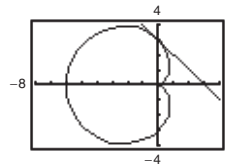
$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta(1 - \sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1 - \sin\theta)}$$

At  $(2, 0), \frac{dy}{dx} = -1.$

At  $\left(3, \frac{7\pi}{6}\right), \frac{dy}{dx}$  is undefined.

At  $\left(4, \frac{3\pi}{2}\right), \frac{dy}{dx} = 0.$

65. (a), (b)  $r = 3(1 - \cos\theta)$

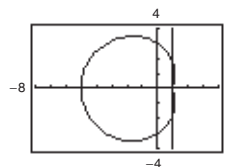


$$(r, \theta) = \left(3, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$$

Tangent line:  $y - 3 = -1(x - 0)$   
 $y = -x + 3$

(c) At  $\theta = \frac{\pi}{2}, \frac{dy}{dx} = -1.0.$

66. (a), (b)  $r = 3 - 2\cos\theta$



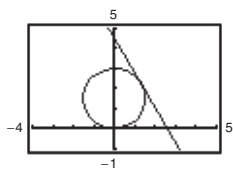
$$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$$

Tangent line:  $x = 1$

(c) At  $\theta = 0, \frac{dy}{dx}$  does not exist (vertical tangent).



67. (a), (b)  $r = 3 \sin \theta$



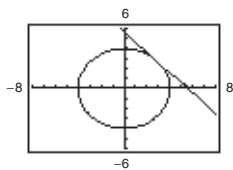
$$(r, \theta) = \left( \frac{3\sqrt{3}}{2}, \frac{\pi}{3} \right) \Rightarrow (x, y) = \left( \frac{3\sqrt{3}}{4}, \frac{9}{4} \right)$$

$$\text{Tangent line: } y - \frac{9}{4} = -\sqrt{3} \left( x - \frac{3\sqrt{3}}{4} \right)$$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At  $\theta = \frac{\pi}{3}$ ,  $\frac{dy}{dx} = -\sqrt{3} \approx -1.732$ .

68. (a), (b)  $r = 4$



$$(r, \theta) = \left( 4, \frac{\pi}{4} \right) \Rightarrow (x, y) = (2\sqrt{2}, 2\sqrt{2})$$

$$\text{Tangent line: } y - 2\sqrt{2} = -1(x - 2\sqrt{2})$$

$$y = -x + 4\sqrt{2}$$

(c) At  $\theta = \frac{\pi}{4}$ ,  $\frac{dy}{dx} = -1$ .

69.  $r = 1 - \sin \theta$

$$\begin{aligned} \frac{dy}{d\theta} &= (1 - \sin \theta) \cos \theta - \cos \theta \sin \theta \\ &= \cos \theta (1 - 2 \sin \theta) = 0 \end{aligned}$$

$$\cos \theta = 0 \text{ or } \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Horizontal tangents: } \left( 2, \frac{3\pi}{2} \right), \left( \frac{1}{2}, \frac{\pi}{6} \right), \left( \frac{1}{2}, \frac{5\pi}{6} \right)$$

$$\begin{aligned} \frac{dx}{d\theta} &= (-1 + \sin \theta) \sin \theta - \cos \theta \cos \theta \\ &= -\sin \theta + \sin^2 \theta + \sin^2 \theta - 1 \\ &= 2 \sin^2 \theta - \sin \theta - 1 \\ &= (2 \sin \theta + 1)(\sin \theta - 1) = 0 \end{aligned}$$

$$\sin \theta = 1 \text{ or } \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\text{Vertical tangents: } \left( \frac{3}{2}, \frac{7\pi}{6} \right), \left( \frac{3}{2}, \frac{11\pi}{6} \right)$$

70.  $r = a \sin \theta$

$$\begin{aligned} \frac{dy}{d\theta} &= a \sin \theta \cos \theta + a \cos \theta \sin \theta \\ &= 2a \sin \theta \cos \theta = 0 \end{aligned}$$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a \sin^2 \theta + a \cos^2 \theta = a(1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = \pm \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{Horizontal: } (0, 0), \left( a, \frac{\pi}{2} \right)$$

$$\text{Vertical: } \left( \frac{a\sqrt{2}}{2}, \frac{\pi}{4} \right), \left( \frac{a\sqrt{2}}{2}, \frac{3\pi}{4} \right)$$

71.  $r = 2 \csc \theta + 3$

$$\begin{aligned} \frac{dy}{d\theta} &= (2 \csc \theta + 3) \cos \theta + (-2 \csc \theta \cot \theta) \sin \theta \\ &= 3 \cos \theta = 0 \end{aligned}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{Horizontal: } \left( 5, \frac{\pi}{2} \right), \left( 1, \frac{3\pi}{2} \right)$$

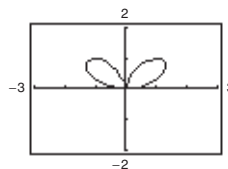
72.  $r = a \sin \theta \cos^2 \theta$

$$\begin{aligned} \frac{dy}{d\theta} &= a \sin \theta \cos^3 \theta + [-2a \sin^2 \theta \cos \theta + a \cos^3 \theta] \sin \theta \\ &= 2a [\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta] \\ &= 2a \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) = 0 \end{aligned}$$

$$\theta = 0, \tan^2 \theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\text{Horizontal: } \left( \frac{\sqrt{2}a}{4}, \frac{\pi}{4} \right), \left( \frac{\sqrt{2}a}{4}, \frac{3\pi}{4} \right), (0, 0)$$

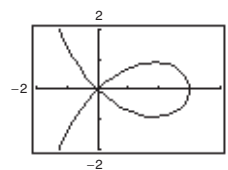
73.  $r = 4 \sin \theta \cos^2 \theta$



Horizontal tangents:

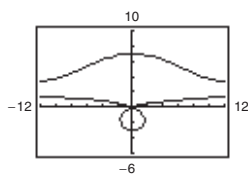
$$(r, \theta) = (0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$$

74.  $r = 3 \cos 2\theta \sec \theta$



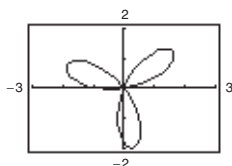
$$\text{Horizontal tangents: } (r, \theta) = (2.061, \pm 0.452)$$

75.  $r = 2 \csc \theta + 5$



Horizontal tangents:  $(r, \theta) = \left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$

76.  $r = 2 \cos(3\theta - 2)$



Horizontal tangents:

$(r, \theta) = (1.894, 0.776), (1.755, 2.594),$   
 $(1.998, -1.442), (-0.423, 0.072)$

77.  $r = 5 \sin \theta$

$$r^2 = 5r \sin \theta$$

$$x^2 + y^2 = 5y$$

$$x^2 + \left(y^2 - 5y + \frac{25}{4}\right) = \frac{25}{4}$$

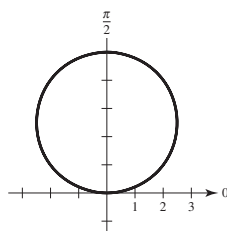
$$x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$$

Circle: center:  $\left(0, \frac{5}{2}\right)$ , radius:  $\frac{5}{2}$

Tangent at pole:  $\theta = 0$

Note:  $f(\theta) = r = 5 \sin \theta$

$$f(0) = 0, f'(0) \neq 0$$



78.  $r = 5 \cos \theta$

$$r^2 = 5r \cos \theta$$

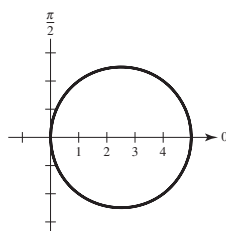
$$x^2 + y^2 = 5x$$

$$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$$

Circle: center:  $\left(\frac{5}{2}, 0\right)$ , radius:  $\frac{5}{2}$

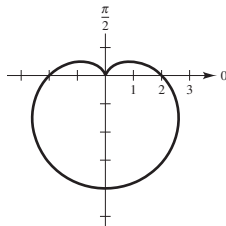
Tangent at pole:  $\theta = \frac{\pi}{2}$



79.  $r = 2(1 - \sin \theta)$

Cardioid

Symmetric to y-axis,  $\theta = \frac{\pi}{2}$

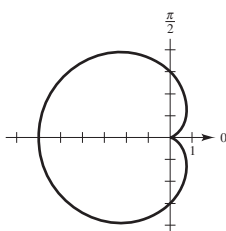


80.  $r = 3(1 - \cos \theta)$

Cardioid

Symmetric to polar axis since  $r$  is a function of  $\cos \theta$ .

$\theta$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$
$r$	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6

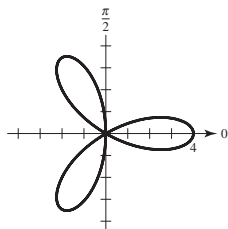


81.  $r = 4 \cos 3\theta$

Rose curve with three petals.

Tangents at pole: ( $r = 0$ ,  $r' \neq 0$ ):

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



82.  $r = -\sin(5\theta)$

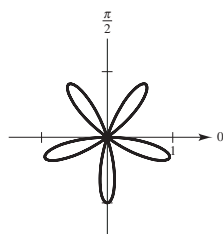
Rose curve with five petals

Symmetric to  $\theta = \frac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5 \cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole:  $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$



83.  $r = 3 \sin 2\theta$

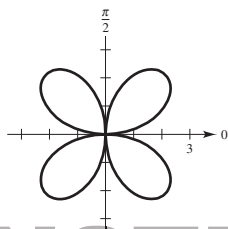
Rose curve with four petals

Symmetric to the polar axis,  $\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $\left(\pm 3, \frac{\pi}{4}\right), \left(\pm 3, \frac{5\pi}{4}\right)$

Tangents at the pole:  $\theta = 0, \frac{\pi}{2}$

$\left(\theta = \pi, \frac{3\pi}{2} \text{ give the same tangents.}\right)$



84.  $r = 3 \cos 2\theta$

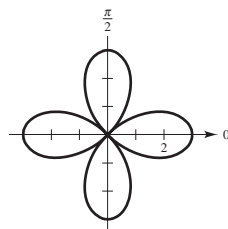
Rose curve with four petals

Symmetric to the polar axis,  $\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $(3, 0), \left(-3, \frac{\pi}{2}\right), (3, \pi), \left(-3, \frac{3\pi}{2}\right)$

Tangents at the pole:  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

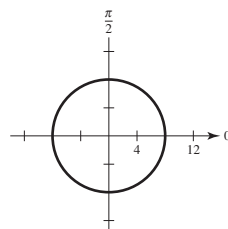
$\theta = \frac{5\pi}{4}$  and  $\frac{7\pi}{4}$  given the same tangents.



85.  $r = 8$

Circle radius 8

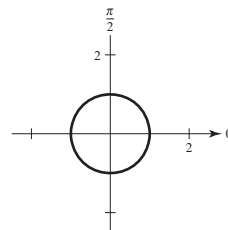
$$x^2 + y^2 = 64$$



86.  $r = 1$

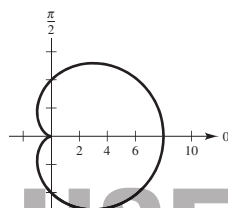
Circle radius 1

$$x^2 + y^2 = 1$$



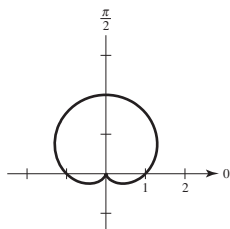
87.  $r = 4(1 + \cos \theta)$

Cardioid



88.  $r = 1 + \sin \theta$

Cardioid

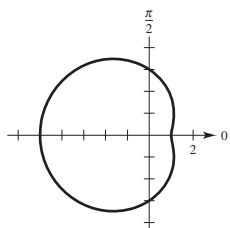


89.  $r = 3 - 2 \cos \theta$

Limaçon

Symmetric to polar axis

$\theta$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$
$r$	1	2	3	4	5

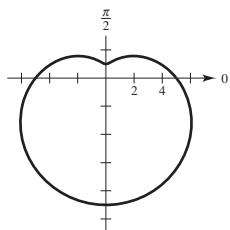


90.  $r = 5 - 4 \sin \theta$

Limaçon

Symmetric to  $\theta = \frac{\pi}{2}$ 

$\theta$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
$r$	9	7	5	3	1

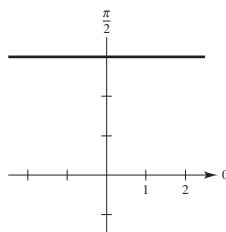


91.  $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line

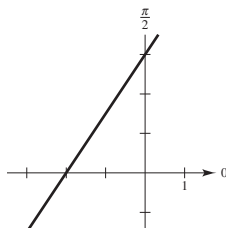


92.  $r = \frac{6}{2 \sin \theta - 3 \cos \theta}$

$2r \sin \theta - 3r \cos \theta = 6$

$2y - 3x = 6$

Line

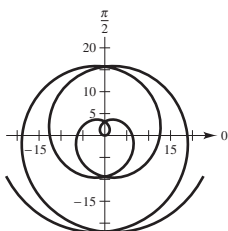


93.  $r = 2\theta$

Spiral of Archimedes

Symmetric to  $\theta = \frac{\pi}{2}$ 

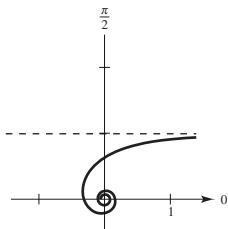
$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$r$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$	$\frac{5\pi}{2}$	$3\pi$

Tangent at the pole:  $\theta = 0$ 

94.  $r = \frac{1}{\theta}$

Hyperbolic spiral

$\theta$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$r$	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$



95.  $r^2 = 4 \cos(2\theta)$

$r = 2\sqrt{\cos 2\theta}$ ,  $0 \leq \theta \leq 2\pi$

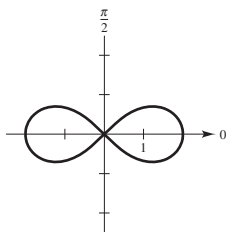
Lemniscate

Symmetric to the polar axis,  $\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $(\pm 2, 0)$

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$r$	$\pm 2$	$\pm\sqrt{2}$	0

Tangents at the pole:  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$



96.  $r^2 = 4 \sin \theta$

Lemniscate

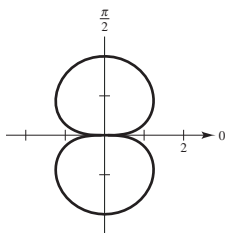
Symmetric to the polar axis,

$\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $(\pm 2, \frac{\pi}{2})$

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\pi$
$r$	0	$\pm\sqrt{2}$	$\pm 2$	$\pm\sqrt{2}$	0

Tangent at the pole:  $\theta = 0$



97. Because

$$r = 2 - \sec \theta = 2 - \frac{1}{\cos \theta},$$

the graph has polar axis symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^-$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow -\frac{\pi}{2}^+.$$

Also,

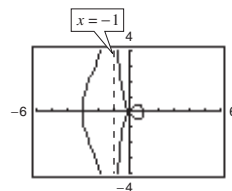
$$r = 2 - \frac{1}{\cos \theta}$$

$$= 2 - \frac{r}{r \cos \theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

So,  $r \Rightarrow \pm\infty$  as  $x \Rightarrow -1$ .



98. Because

$$r = 2 + \csc \theta = 2 + \frac{1}{\sin \theta},$$

the graphs has symmetry with respect to  $\theta = \pi/2$ . Furthermore,

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

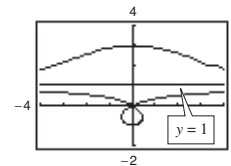
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

$$\text{Also, } r = 2 + \frac{1}{\sin \theta} = 2 + \frac{r}{\sin \theta} = 2 + \frac{r}{y}$$

$$ry = 2y + r$$

$$r = \frac{2y}{y-1}.$$

So,  $r \Rightarrow \pm\infty$  as  $y \Rightarrow 1$ .



99.  $r = \frac{2}{\theta}$

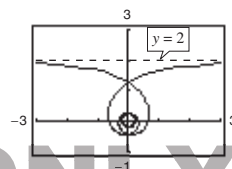
Hyperbolic spiral

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0$$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2 \sin \theta}{r \sin \theta} = \frac{2 \sin \theta}{y}$$

$$y = \frac{2 \sin \theta}{\theta}$$

$$\lim_{\theta \rightarrow 0} \frac{2 \sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{2 \cos \theta}{1} = 2$$



100.  $r = 2 \cos 2\theta \sec \theta$

Strophoid

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

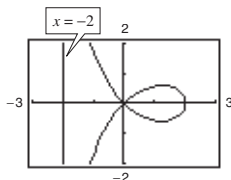
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$

$$r = 2 \cos 2\theta \sec \theta = 2(2 \cos^2 \theta - 1) \sec \theta$$

$$r \cos \theta = 4 \cos^2 \theta - 2$$

$$x = 4 \cos^2 \theta - 2$$

$$\lim_{\theta \rightarrow \pm\pi/2} (4 \cos^2 \theta - 2) = -2$$



101. The rectangular coordinate system consists of all points of the form  $(x, y)$  where  $x$  is the directed distance from the  $y$ -axis to the point, and  $y$  is the directed distance from the  $x$ -axis to the point.

Every point has a unique representation.

The polar coordinate system uses  $(r, \theta)$  to designate the location of a point. $r$  is the directed distance to the origin and  $\theta$  is the angle the point makes with the positive  $x$ -axis, measured counterclockwise.

Points do not have a unique polar representation.

102.  $x = r \cos \theta, y = r \sin \theta$

$$x^2 + y^2 = r^2, \tan \theta = \frac{y}{x}$$

103. Slope of tangent line to graph of
- $r = f(\theta)$
- at
- $(r, \theta)$
- is

$$\frac{dy}{dx} = \frac{f(\theta)\cos \theta + f'(\theta)\sin \theta}{-f(\theta)\sin \theta + f'(\theta)\cos \theta}.$$

If  $f(\alpha) = 0$  and  $f'(\alpha) \neq 0$ , then  $\theta = \alpha$  is tangent at the pole.

104. (a)
- $r = 7$
- : Circle radius 7 centered at origin

- (b)
- $r^2 = 7$
- : Circle radius
- $\sqrt{7}$
- centered at origin

- (c)
- $r = \frac{7}{\cos \theta} \Rightarrow r \cos \theta = x = 7$
- : Vertical line through the point
- $(7, 0)$

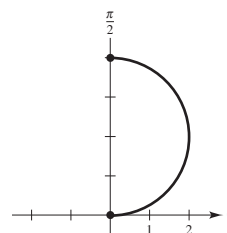
- (d)
- $r = \frac{7}{\sin \theta} \Rightarrow r \sin \theta = y = 7$
- : Horizontal line through the point
- $(0, 7)$

- (e)
- $r = 7 \cos \theta$
- : Circle radius
- $\frac{7}{2}$
- centered at
- $(\frac{7}{2}, 0)$

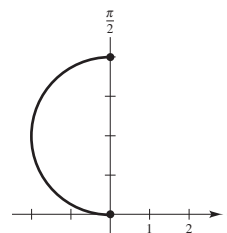
- (f)
- $r = 7 \sin \theta$
- : Circle radius
- $\frac{7}{2}$
- centered at
- $(0, \frac{7}{2})$

105.  $r = 4 \sin \theta$

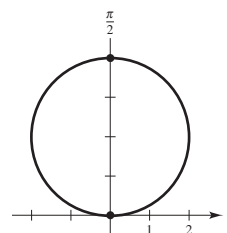
- (a)
- $0 \leq \theta \leq \frac{\pi}{2}$



- (b)
- $\frac{\pi}{2} \leq \theta \leq \pi$

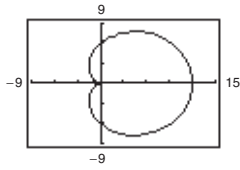


- (c)
- $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

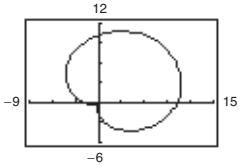


106.  $r = 6[1 + \cos(\theta - \phi)]$

(a)  $\phi = 0, r = 6[1 + \cos \theta]$



(b)  $\theta = \frac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$

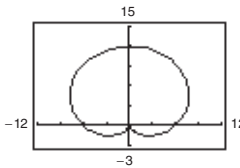


The graph of  $r = 6[1 + \cos \theta]$  is rotated through the angle  $\pi/4$ .

(c)  $\theta = \frac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= 6\left[1 + \cos \theta \cos \frac{\pi}{2} + \sin \theta \sin \frac{\pi}{2}\right] = 6[1 + \sin \theta]$$



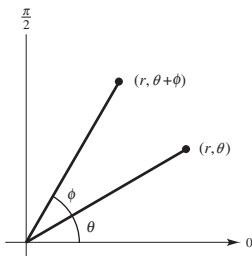
The graph of  $r = 6[1 + \cos \theta]$  is rotated through the angle  $\pi/2$ .

107. Let the curve  $r = f(\theta)$  be rotated by  $\phi$  to form the curve  $r = g(\theta)$ . If  $(r_1, \theta_1)$  is a point on  $r = f(\theta)$ , then  $(r_1, \theta_1 + \phi)$  is on  $r = g(\theta)$ . That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

Letting  $\theta = \theta_1 + \phi$ , or  $\theta_1 = \theta - \phi$ , you see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$



108. (a)  $\sin\left(\theta - \frac{\pi}{2}\right) = \sin \theta \cos\left(\frac{\pi}{2}\right) - \cos \theta \sin\left(\frac{\pi}{2}\right)$

$$= -\cos \theta$$

$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= f(-\cos \theta)$$

(b)  $\sin(\theta - \pi) = \sin \theta \cos \pi - \cos \theta \sin \pi$

$$= -\sin \theta$$

$$r = f[\sin(\theta - \pi)]$$

$$= f(-\sin \theta)$$

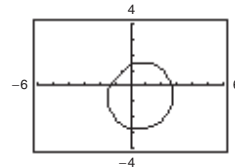
(c)  $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin \theta \cos\left(\frac{3\pi}{2}\right) - \cos \theta \sin\left(\frac{3\pi}{2}\right)$

$$= \cos \theta$$

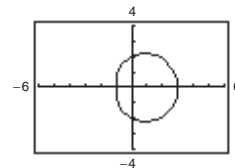
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos \theta)$$

109.  $r = 2 - \sin \theta$

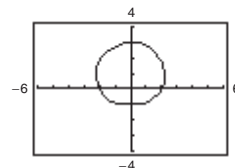
(a)  $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin \theta - \cos \theta)$



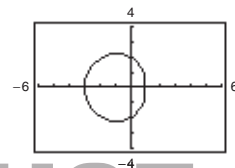
(b)  $r = 2 - \sin\left(\theta - \frac{\pi}{2}\right) = 2 - (-\cos \theta) = 2 + \cos \theta$



(c)  $r = 2 - \sin(\theta - \pi) = 2 - (-\sin \theta) = 2 + \sin \theta$

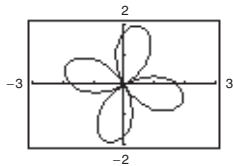


(d)  $r = 2 - \sin\left(\theta - \frac{3\pi}{2}\right) = 2 - \cos \theta$

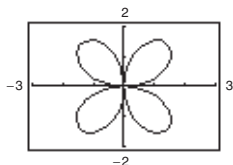


110.  $r = 2 \sin 2\theta = 4 \sin \theta \cos \theta$

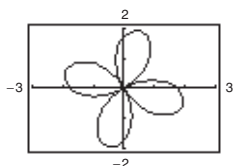
(a)  $r = 4 \sin\left(\theta - \frac{\pi}{6}\right) \cos\left(\theta - \frac{\pi}{6}\right)$



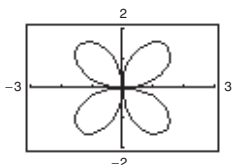
(b)  $r = 4 \sin\left(\theta - \frac{\pi}{2}\right) \cos\left(\theta - \frac{\pi}{2}\right) = -4 \sin \theta \cos \theta$



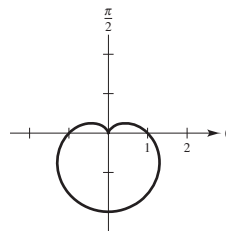
(c)  $r = 4 \sin\left(\theta - \frac{2\pi}{3}\right) \cos\left(\theta - \frac{2\pi}{3}\right)$



(d)  $r = 4 \sin(\theta - \pi) \cos(\theta - \pi) = 4 \sin \theta \cos \theta$

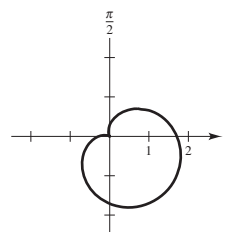


111. (a)  $r = 1 - \sin \theta$



(b)  $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

Rotate the graph of  
 $r = 1 - \sin \theta$   
through the angle  $\pi/4$ .



112. By Theorem 9.11, the slope of the tangent line through  $A$  and  $P$  is

$$\frac{f \cos \theta + f' \sin \theta}{-f \sin \theta + f' \cos \theta}.$$

This is equal to

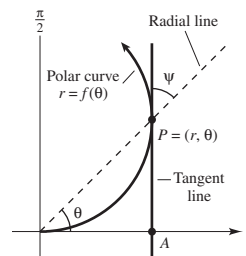
$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\sin \theta + \cos \theta \tan \psi}{\cos \theta - \sin \theta \tan \psi}.$$

Equating the expressions and cross-multiplying, you obtain

$$\begin{aligned} (f \cos \theta + f' \sin \theta)(\cos \theta - \sin \theta \tan \psi) &= (\sin \theta + \cos \theta \tan \psi)(-f \sin \theta + f' \cos \theta) \\ f \cos^2 \theta - f \cos \theta \sin \theta \tan \psi + f' \sin \theta \cos \theta - f' \sin^2 \theta \tan \psi &= -f \sin^2 \theta - f \sin \theta \cos \theta \tan \psi + f' \sin \theta \cos \theta \\ &\quad + f' \cos^2 \theta \tan \psi \end{aligned}$$

$$f(\cos^2 \theta + \sin^2 \theta) = f' \tan \psi (\cos^2 \theta + \sin^2 \theta)$$

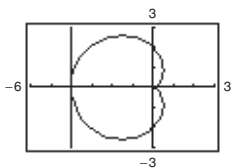
$$\tan \psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$





$$113. \tan \psi = \frac{r}{dr/d\theta} = \frac{2(1 - \cos \theta)}{2 \sin \theta}$$

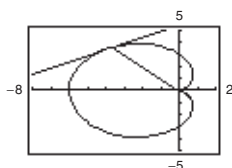
$$\text{At } \theta = \pi, \tan \psi \text{ is undefined} \Rightarrow \psi = \frac{\pi}{2}.$$



$$114. \tan \psi = \frac{r}{dr/d\theta} = \frac{3(1 - \cos \theta)}{3 \sin \theta}$$

$$\text{At } \theta = \frac{3\pi}{4}, \tan \psi = \frac{1 + (\sqrt{2}/2)}{\sqrt{2}} = \frac{2 + \sqrt{2}}{\sqrt{2}}.$$

$$\psi = \arctan\left(\frac{2 + \sqrt{2}}{\sqrt{2}}\right) \approx 1.178 (\approx 67.5^\circ)$$

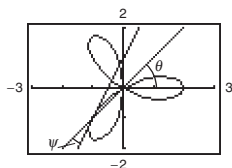


$$115. r = 2 \cos 3\theta$$

$$\tan \psi = \frac{r}{dr/d\theta} = \frac{2 \cos 3\theta}{-6 \sin 3\theta} = -\frac{1}{3} \cot 3\theta$$

$$\text{At } \theta = \frac{\pi}{4}, \tan \psi = -\frac{1}{3} \cot\left(\frac{3\pi}{4}\right) = \frac{1}{3}.$$

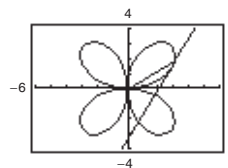
$$\psi = \arctan\left(\frac{1}{3}\right) \approx 18.4^\circ$$



$$116. \tan \psi = \frac{r}{dr/d\theta} = \frac{4 \sin 2\theta}{8 \cos 2\theta}$$

$$\text{At } \theta = \frac{\pi}{6}, \tan \psi = \frac{\sin(\pi/3)}{2 \cos(\pi/3)} = \frac{\sqrt{3}}{2}.$$

$$\psi = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89^\circ)$$

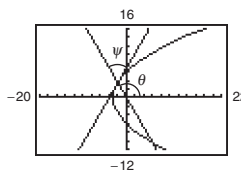


$$117. r = \frac{6}{1 - \cos \theta} = 6(1 - \cos \theta)^{-1} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 - \cos \theta)^2}$$

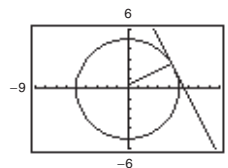
$$\tan \psi = \frac{r}{dr/d\theta} = \frac{\frac{6}{(1 - \cos \theta)}}{\frac{6 \sin \theta}{(1 - \cos \theta)^2}} = \frac{1 - \cos \theta}{-\sin \theta}$$

$$\text{At } \theta = \frac{2\pi}{3}, \tan \psi = \frac{1 - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} = -\sqrt{3}.$$

$$\psi = \frac{\pi}{3}, (60^\circ)$$



$$118. \tan \psi = \frac{r}{dr/d\theta} = \frac{5}{0} \text{ undefined} \Rightarrow \psi = \frac{\pi}{2}$$



119. True

120. True

121. True

122. True

## Section 10.5 Area and Arc Length in Polar Coordinates

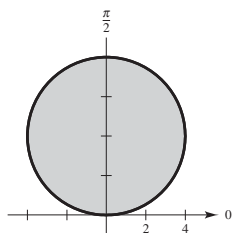
$$1. A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} [4 \sin \theta]^2 d\theta = 8 \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$2. A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{3\pi/4}^{5\pi/4} (\cos 2\theta)^2 d\theta$$

$$3. A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\pi/2}^{3\pi/2} [3 - 2 \sin \theta]^2 d\theta$$

$$4. (a) r = 8 \sin \theta$$



$$A = \pi(4)^2 = 16\pi$$

$$(b) A = 2 \left( \frac{1}{2} \right) \int_0^{\pi/2} [8 \sin \theta]^2 d\theta$$

$$= 64 \int_0^{\pi/2} \sin^2 \theta d\theta$$

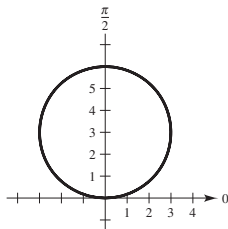
$$= 32 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta$$

$$= 32 \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 16\pi$$

$$5. A = \frac{1}{2} \int_0^{\pi} [6 \sin \theta]^2 d\theta$$

$$= 18 \int_0^{\pi} \frac{1 - \cos 2\theta}{2} d\theta = 9 \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} = 9\pi$$

Note:  $r = 6 \sin \theta$  is circle of radius 3,  $0 \leq \theta \leq \pi$ .

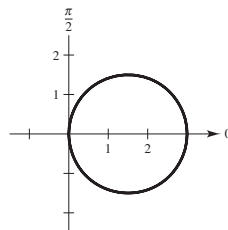


$$6. A = \frac{1}{2} \int_0^{\pi} [3 \cos \theta]^2 d\theta$$

$$= \frac{9}{2} \int_0^{\pi} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= \frac{9}{4} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi} = \frac{9}{4}\pi$$

Note:  $r = 3 \cos \theta$  is circle of radius  $\frac{3}{2}$ ,  $0 \leq \theta \leq \pi$ .



$$7. A = 2 \left[ \frac{1}{2} \int_0^{\pi/6} (2 \cos 3\theta)^2 d\theta \right] = 2 \left[ \theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{3}$$

$$8. A = \frac{1}{2} \int_0^{\pi/3} [4 \sin 3\theta]^2 d\theta$$

$$= 8 \int_0^{\pi/3} \sin^2 3\theta d\theta$$

$$= 8 \int_0^{\pi/3} \frac{1 - \cos 6\theta}{2} d\theta$$

$$= 4 \left[ \theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3}$$

$$= 4 \left[ \frac{\pi}{3} \right] = \frac{4\pi}{3}$$

$$9. A = \frac{1}{2} \int_0^{\pi/2} [\sin 2\theta]^2 d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta$$

$$= \frac{1}{4} \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$= \frac{1}{4} \left[ \frac{\pi}{2} \right] = \frac{\pi}{8}$$

$$10. A = 2 \left[ \frac{1}{2} \int_0^{\pi/10} (\cos 5\theta)^2 d\theta \right]$$

$$= \frac{1}{2} \left[ \theta + \frac{1}{10} \sin(10\theta) \right]_0^{\pi/10} = \frac{\pi}{20}$$

$$11. A = 2 \left[ \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \sin \theta)^2 d\theta \right]$$

$$= \left[ \frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

$$12. A = 2 \left[ \frac{1}{2} \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta \right]$$

$$= \left[ \frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi - 8}{4}$$

$$13. A = \frac{1}{2} \int_0^{2\pi} [5 + 2 \sin \theta]^2 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} [25 + 20 \sin \theta + 4 \sin^2 \theta] d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} [25 + 20 \sin \theta + 2(1 - \cos 2\theta)] d\theta$$

$$= \frac{1}{2} [27\theta - 20 \cos \theta - \sin 2\theta]_0^{2\pi}$$

$$= \frac{1}{2} [27(2\pi)] = 27\pi$$

$$14. A = \frac{1}{2} \int_0^{2\pi} [4 - 4 \cos \theta]^2 d\theta$$

$$= 8 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$$

$$= 8 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta$$

$$= 8 \int_0^{2\pi} \left( 1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta$$

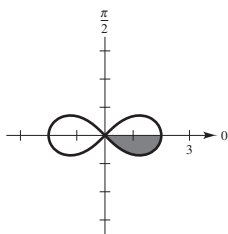
$$= 8 \left[ \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

$$= 8 \left[ \frac{3}{2} (2\pi) \right] = 24\pi$$

15. On the interval  $-\frac{\pi}{4} \leq \theta \leq 0$ ,  $r = 2\sqrt{\cos 2\theta}$  traces out one-half of one leaf of the lemniscate. So,

$$A = 4 \cdot \frac{1}{2} \int_{-\pi/4}^0 4 \cos 2\theta d\theta$$

$$= 8 \left[ \frac{\sin 2\theta}{2} \right]_{-\pi/4}^0 = 8 \left[ \frac{1}{2} \right] = 4.$$



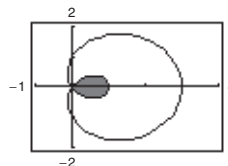
16. On the interval  $0 \leq \theta \leq \pi/2$ ,  $r = \sqrt{6 \sin 2\theta}$  traces out half of the lemniscate. So

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} 6 \sin 2\theta d\theta$$

$$= 6 \left[ \frac{-\cos 2\theta}{2} \right]_0^{\pi/2} = 6 \left[ \frac{1}{2} + \frac{1}{2} \right] = 6.$$

$$17. A = \left[ 2 \frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right]$$

$$= [3\theta + 4 \sin \theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$



18. Half of the inner loop of  $r = 2 - 4 \cos \theta$  is traced out on the interval  $0 \leq \theta \leq \frac{\pi}{3}$ . So

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (2 - 4 \cos \theta)^2 d\theta$$

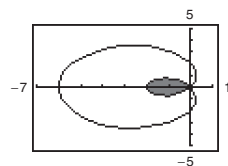
$$= \int_0^{\pi/3} [4 - 16 \cos \theta + 16 \cos^2 \theta] d\theta$$

$$= \int_0^{\pi/3} [4 - 16 \cos \theta + 8(1 + \cos 2\theta)] d\theta$$

$$= [12\theta - 16 \sin \theta + 4 \sin 2\theta]_0^{\pi/3}$$

$$= 12(\pi/3) - 16(\sqrt{3}/2) + 4(\sqrt{3}/2)$$

$$= 4\pi - 6\sqrt{3}.$$



19. The inner loop of  $r = 1 + 2 \sin \theta$  is traced out on the interval  $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$ . So,

$$A = \frac{1}{2} \int_{7\pi/6}^{11\pi/6} [1 + 2 \sin \theta]^2 d\theta$$

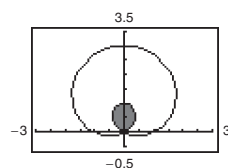
$$= \frac{1}{2} \int_{7\pi/6}^{11\pi/6} [1 + 4 \sin \theta + 4 \sin^2 \theta] d\theta$$

$$= \frac{1}{2} \int_{7\pi/6}^{11\pi/6} [1 + 4 \sin \theta + 2(1 - \cos 2\theta)] d\theta$$

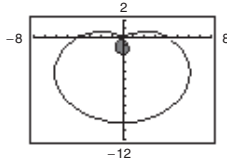
$$= \frac{1}{2} [3\theta - 4 \cos \theta - \sin 2\theta]_{7\pi/6}^{11\pi/6}$$

$$= \frac{1}{2} \left[ \left( \frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2} \right) - \left( \frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right) \right]$$

$$= \frac{1}{2} [2\pi - 3\sqrt{3}].$$



$$\begin{aligned}
 20. \quad A &= 2 \left[ \frac{1}{2} \int_{\arcsin(2/3)}^{\pi/2} (4 - 6 \sin \theta)^2 d\theta \right] \\
 &= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48 \sin \theta + 36 \sin^2 \theta] d\theta \\
 &= \int_{\arcsin(2/3)}^{\pi/2} \left[ 16 - 48 \sin \theta + 36 \left( \frac{1 - \cos 2\theta}{2} \right) \right] d\theta \\
 &= [34\theta + 48 \cos \theta - 9 \sin 2\theta]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635
 \end{aligned}$$

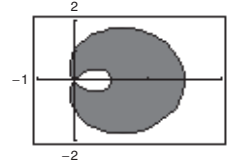


21. The area inside the outer loop is

$$\begin{aligned}
 2 \left[ \frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \right] &= [3\theta + 4 \sin \theta + \sin 2\theta]_0^{2\pi/3} \\
 &= \frac{4\pi + 3\sqrt{3}}{2}.
 \end{aligned}$$

From the result of Exercise 17, the area between the loops is

$$A = \left( \frac{4\pi + 3\sqrt{3}}{2} \right) - \left( \frac{2\pi - 3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}.$$



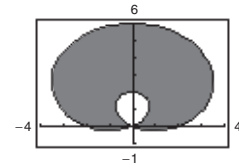
22. Four times the area in Exercise 21,  $A = 4(\pi + 3\sqrt{3})$ . More specifically, you see that the area inside the outer loop is

$$2 \left[ \frac{1}{2} \int_{-\pi/6}^{\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = \int_{-\pi/6}^{\pi/2} (4 + 16 \sin \theta + 16 \sin^2 \theta) d\theta = 8\pi + 6\sqrt{3}.$$

The area inside the inner loop is

$$2 \left[ \frac{1}{2} \int_{7\pi/6}^{3\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = 4\pi - 6\sqrt{3}.$$

So, the area between the loops is  $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$ .



23. The area inside the outer loop is

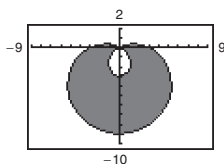
$$\begin{aligned}
 A &= 2 \cdot \frac{1}{2} \int_{5\pi/6}^{3\pi/2} [3 - 6 \sin \theta]^2 d\theta \\
 &= \int_{5\pi/6}^{3\pi/2} [9 - 36 \sin \theta + 36 \sin^2 \theta] d\theta \\
 &= \int_{5\pi/6}^{3\pi/2} [9 - 36 \sin \theta + 18(1 - \cos 2\theta)] d\theta \\
 &= [27\theta + 36 \cos \theta - 9 \sin 2\theta]_{5\pi/6}^{3\pi/2} = \left[ \frac{81\pi}{2} - \left( \frac{45\pi}{2} - 18\sqrt{3} + \frac{9\sqrt{3}}{2} \right) \right] = 18\pi + \frac{27\sqrt{3}}{2}.
 \end{aligned}$$

The area inside the inner loop is

$$\begin{aligned}
 A &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} [3 - 6 \sin \theta]^2 d\theta \\
 &= [27\theta + 36 \cos \theta - 9 \sin 2\theta]_{\pi/6}^{\pi/2} = \left[ \frac{27\pi}{2} - \left( \frac{9\pi}{2} + 18\sqrt{3} - \frac{9\sqrt{3}}{2} \right) \right] = 9\pi - \frac{27\sqrt{3}}{2}.
 \end{aligned}$$

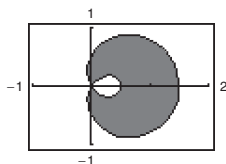
Finally, the area between the loops is

$$\left[ 18\pi + \frac{27\sqrt{3}}{2} \right] - \left[ 9\pi - \frac{27\sqrt{3}}{2} \right] = 9\pi + 27\sqrt{3}.$$



24. The area inside the outer loop is

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_0^{2\pi/3} \left[ \frac{1}{2} + \cos \theta \right]^2 d\theta \\ &= \int_0^{2\pi/3} \left[ \frac{1}{4} + \cos \theta + \frac{1 + \cos 2\theta}{2} \right] d\theta \\ &= \left[ \frac{3}{4}\theta + \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{2\pi/3} \\ &= \frac{3}{4} \left( \frac{2\pi}{3} \right) + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} \\ &= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}. \end{aligned}$$



The area inside the inner loop is

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_{2\pi/3}^{\pi} \left[ \frac{1}{2} + \cos \theta \right]^2 d\theta \\ &= \left[ \frac{3}{4}\theta + \sin \theta + \frac{\sin 2\theta}{4} \right]_{2\pi/3}^{\pi} \\ &= \frac{3}{4}\pi - \left( \frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} \right) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8} \end{aligned}$$

Finally, the area between the loops is

$$\left[ \frac{\pi}{2} + \frac{3\sqrt{3}}{8} \right] - \left[ \frac{\pi}{4} - \frac{3\sqrt{3}}{8} \right] = \frac{\pi}{4} + \frac{3\sqrt{3}}{4}.$$

25.  $r = 1 + \cos \theta$   
 $r = 1 - \cos \theta$

Solving simultaneously,

$$\begin{aligned} 1 + \cos \theta &= 1 - \cos \theta \\ 2 \cos \theta &= 0 \end{aligned}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing  $r$  by  $-r$  and  $\theta$  by  $\theta + \pi$  in the first equation and solving,  $-1 + \cos \theta = 1 - \cos \theta$ ,  $\cos \theta = 1$ ,

$\theta = 0$ . Both curves pass through the pole,  $(0, \pi)$ , and  $(0, 0)$ , respectively.

Points of intersection:  $\left(1, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right), (0, 0)$

26.  $r = 3(1 + \sin \theta)$   
 $r = 3(1 - \sin \theta)$

Solving simultaneously,

$$3(1 + \sin \theta) = 3(1 - \sin \theta)$$

$$2 \sin \theta = 0$$

$$\theta = 0, \pi.$$

Replacing  $r$  by  $-r$  and  $\theta$  by  $\theta + \pi$  in the first equation and solving,  $-3(1 + \sin \theta) = 3(1 - \sin \theta)$ ,  $\sin \theta = 1$ ,

$\theta = \pi/2$ . Both curves pass through the pole,  $(0, 3\pi/2)$ , and  $(0, \pi/2)$ , respectively.

Points of intersection:  $(3, 0), (3, \pi), (0, 0)$

27.  $r = 1 + \cos \theta$   
 $r = 1 - \sin \theta$

Solving simultaneously,

$$1 + \cos \theta = 1 - \sin \theta$$

$$\cos \theta = -\sin \theta$$

$$\tan \theta = -1$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing  $r$  by  $-r$  and  $\theta$  by  $\theta + \pi$  in the first equation and solving,  $-1 + \cos \theta = 1 - \sin \theta$ ,

$\sin \theta + \cos \theta = 2$ , which has no solution. Both curves pass through the pole,  $(0, \pi)$ , and  $(0, \pi/2)$ , respectively.

Points of intersection:

$$\left( \frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4} \right), \left( \frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4} \right), (0, 0)$$

28.  $r = 2 - 3 \cos \theta$   
 $r = \cos \theta$

Solving simultaneously,

$$2 - 3 \cos \theta = \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole,  $(0, \arccos 2/3)$ , and  $(0, \pi/2)$ , respectively.

Points of intersection:  $\left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right), (0, 0)$

29.  $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$

Solving simultaneously,

$4 - 5 \sin \theta = 3 \sin \theta$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Both curves pass through the pole,  $(0, \arcsin 4/5)$ , and  $(0, 0)$ , respectively.

Points of intersection:  $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

30.  $r = 1 + \cos \theta$

$r = 3 \cos \theta$

Solving simultaneously,

$1 + \cos \theta = 3 \cos \theta$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Both curves pass through the pole,  $(0, \pi)$ , and  $(0, \pi/2)$ , respectively.

Points of intersection:  $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

31.  $r = \frac{\theta}{2}$

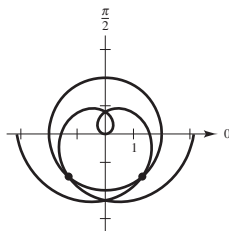
$r = 2$

Solving simultaneously, you have

$\theta/2 = 2, \theta = 4$

Points of intersection:

$(2, 4), (-2, -4)$

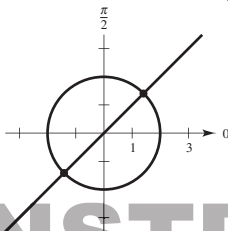


32.  $\theta = \frac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection:  $\left(2, \frac{\pi}{4}\right), \left(-2, \frac{5\pi}{4}\right)$



33.  $r = 2 \sin 2\theta$

$r = 1$

$r = 2 \sin 2\theta$  is a 4-leaved rose curve. The circle  $r = 1$  intersects at 8 points. For the petal in the first quadrant,

$2 \sin 2\theta = 1$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

The points of intersection for one petal are  $\left(1, \frac{\pi}{12}\right),$

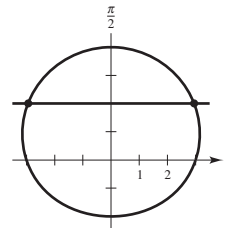
$\left(1, \frac{5\pi}{12}\right)$ . By symmetry, the other points are

$$\left(1, \frac{7\pi}{12}\right), \left(1, \frac{11\pi}{12}\right), \left(1, \frac{13\pi}{12}\right), \left(1, \frac{17\pi}{12}\right),$$

$$\left(1, \frac{19\pi}{12}\right), \left(1, \frac{23\pi}{12}\right).$$

34.  $r = 3 + \sin \theta$

$r = 2 \csc \theta$



The graph of  $r = 3 + \sin \theta$  is a limaçon symmetric to  $\theta = \pi/2$ , and the graph of  $r = 2 \csc \theta$  is the horizontal line  $y = 2$ . So, there are two points of intersection.

Solving simultaneously,

$3 + \sin \theta = 2 \csc \theta$

$\sin^2 \theta + 3 \sin \theta - 2 = 0$

$$\sin \theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

$$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$$(3.56, 0.596), (3.56, 2.545)$$

35.  $r = 2 + 3 \cos \theta$

$$r = \frac{\sec \theta}{2}$$

The graph of  $r = 2 + 3 \cos \theta$  is a limaçon with an inner loop ( $b > a$ ) and is symmetric to the polar axis. The graph of  $r = (\sec \theta)/2$  is the vertical line  $x = 1/2$ . So, there are four points of intersection. Solving simultaneously,

$$2 + 3 \cos \theta = \frac{\sec \theta}{2}$$

$$6 \cos^2 \theta + 4 \cos \theta - 1 = 0$$

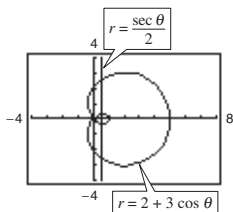
$$\cos \theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection:

$$(-0.581, \pm 2.607), (2.581, \pm 1.376)$$



36.  $r = 3(1 - \cos \theta)$

$$r = \frac{6}{1 - \cos \theta}$$

The graph of  $r = 3(1 - \cos \theta)$  is a cardioid with polar axis symmetry. The graph of

$$r = 6/(1 - \cos \theta)$$

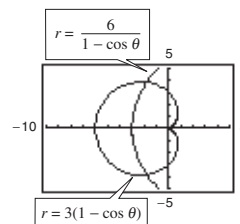
is a parabola with focus at the pole, vertex  $(3, \pi)$ , and polar axis symmetry. So, there are two points of intersection. Solving simultaneously,

$$3(1 - \cos \theta) = \frac{6}{1 - \cos \theta}$$

$$(1 - \cos \theta)^2 = 2$$

$$\cos \theta = 1 \pm \sqrt{2}$$

$$\theta = \arccos(1 - \sqrt{2}).$$



Points of intersection:

$$(3\sqrt{2}, \arccos(1 - \sqrt{2})) \approx (4.243, 1.998),$$

$$(3\sqrt{2}, 2\pi - \arccos(1 - \sqrt{2})) \approx (4.243, 4.285)$$

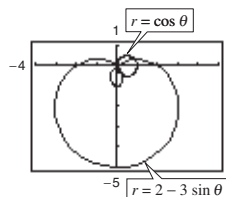
37.  $r = \cos \theta$

$$r = 2 - 3 \sin \theta$$

Points of intersection:

$$(0, 0), (0.935, 0.363), (0.535, -1.006)$$

The graphs reach the pole at different times ( $\theta$  values).

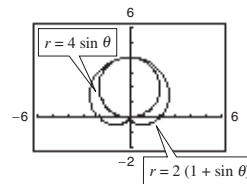


38.  $r = 4 \sin \theta$

$$r = 2(1 + \sin \theta)$$

Points of intersection:  $(0, 0), \left(4, \frac{\pi}{2}\right)$

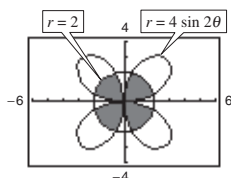
The graphs reach the pole at different times ( $\theta$  values).



39. From Exercise 25, the points of intersection for one petal are  $(2, \pi/12)$  and  $(2, 5\pi/12)$ . The area within one petal is

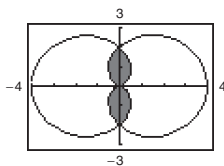
$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/12} (4 \sin 2\theta)^2 d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4 \sin 2\theta)^2 d\theta \\ &= 16 \int_0^{\pi/12} \sin^2(2\theta) d\theta + 2 \int_{\pi/12}^{5\pi/12} d\theta \quad (\text{by symmetry of the petal}) \\ &= 8 \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/12} + [2\theta]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}. \end{aligned}$$

$$\text{Total area} = 4 \left( \frac{4\pi}{3} - \sqrt{3} \right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3} (4\pi - 3\sqrt{3})$$

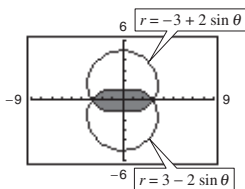


40. The common interior is 4 times the area in the first quadrant.

$$\begin{aligned} A &= 4 \int_0^{\pi/2} [2(1 - \cos \theta)]^2 d\theta \\ &= 8 \int_0^{\pi/2} \left( 1 - 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \right) d\theta \\ &= 8 \left[ \frac{3\theta}{2} - 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} \\ &= 8 \left[ \frac{3}{2} \left( \frac{\pi}{2} \right) - 2 \right] = 6\pi - 16 \end{aligned}$$

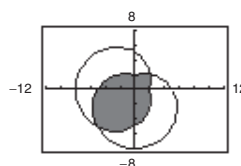


41.  $A = 4 \left[ \frac{1}{2} \int_0^{\pi/2} (3 - 2 \sin \theta)^2 d\theta \right]$   
 $= 2 [11\theta + 12 \cos \theta - \sin(2\theta)]_0^{\pi/2} = 11\pi - 24$

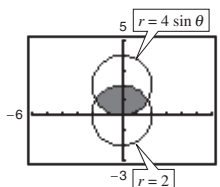


42.  $r = 5 - 3 \sin \theta$  and  $r = 5 - 3 \cos \theta$  intersect at  $\theta = \pi/4$  and  $\theta = 5\pi/4$ .

$$\begin{aligned} A &= 2 \left[ \frac{1}{2} \int_{\pi/4}^{5\pi/4} (5 - 3 \sin \theta)^2 d\theta \right] \\ &= \left[ \frac{59}{2} \theta + 30 \cos \theta - \frac{9}{4} \sin 2\theta \right]_{\pi/4}^{5\pi/4} \\ &= \left( \frac{59}{2} \left( \frac{5\pi}{4} \right) - 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) - \left( \frac{59}{2} \left( \frac{\pi}{4} \right) + 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) \\ &= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251 \end{aligned}$$



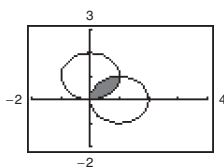
43.  $A = 2 \left[ \frac{1}{2} \int_0^{\pi/6} (4 \sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (2)^2 d\theta \right]$   
 $= 16 \left[ \frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/6} + [4\theta]_{\pi/6}^{\pi/2}$   
 $= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3} (4\pi - 3\sqrt{3})$





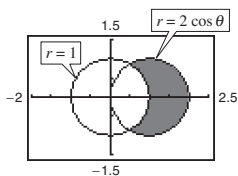
44. The common interior is given by

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_{\pi/4}^{\pi/2} [2 \cos \theta]^2 d\theta \\ &= 4 \int_{\pi/4}^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta \\ &= 2 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2} \\ &= 2 \left[ \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} \right] \\ &= \frac{\pi}{2} - 1 \end{aligned}$$



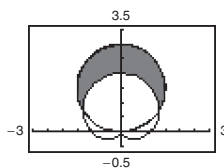
45.  $r = 2 \cos \theta = 1 \Rightarrow \theta = \pi/3$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_0^{\pi/3} ([2 \cos \theta]^2 - 1) d\theta \\ &= \int_0^{\pi/3} [2(1 + \cos 2\theta) - 1] d\theta \\ &= [\theta + \sin 2\theta]_0^{\pi/3} \\ &= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

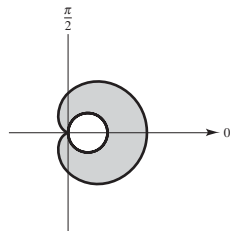


46.  $3 \sin \theta = 1 + \sin \theta \Rightarrow \sin \theta = 1/2 \Rightarrow \theta = \pi/6$

$$\begin{aligned} A &= 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} ([3 \sin \theta]^2 - [1 + \sin \theta]^2) d\theta \\ &= \int_{\pi/6}^{\pi/2} [9 \sin^2 \theta - 1 - 2 \sin \theta - \sin^2 \theta] d\theta \\ &= \int_{\pi/6}^{\pi/2} [4(1 - \cos 2\theta) - 1 - 2 \sin \theta] d\theta \\ &= [3\theta - 2 \sin 2\theta + 2 \cos \theta]_{\pi/6}^{\pi/2} \\ &= 3 \cdot \frac{\pi}{2} - 3 \cdot \frac{\pi}{6} + 2 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} \\ &= \pi \end{aligned}$$



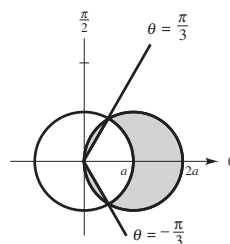
$$\begin{aligned} 47. A &= 2 \left[ \frac{1}{2} \int_0^{\pi} [a(1 + \cos \theta)]^2 d\theta \right] - \frac{a^2 \pi}{4} \\ &= a^2 \left[ \frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi} - \frac{a^2 \pi}{4} \\ &= \frac{3a^2 \pi}{2} - \frac{a^2 \pi}{4} = \frac{5a^2 \pi}{4} \end{aligned}$$



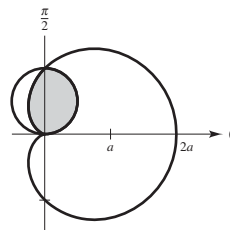
48. Area = Area of  $r = 2a \cos \theta$  - Area of sector - twice area between  $r = 2a \cos \theta$  and the lines

$$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}.$$

$$\begin{aligned} A &= \pi a^2 - \left( \frac{\pi}{3} \right) a^2 - 2 \left[ \frac{1}{2} \int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 d\theta \right] \\ &= \frac{2\pi a^2}{3} - 2a^2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/3}^{\pi/2} \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[ \frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6} \end{aligned}$$



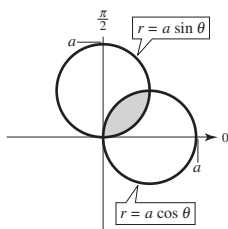
$$\begin{aligned} 49. A &= \frac{\pi a^2}{8} + \frac{1}{2} \int_{\pi/2}^{\pi} [a(1 + \cos \theta)]^2 d\theta \\ &= \frac{\pi a^2}{8} + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left( \frac{3}{2} + 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta \\ &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[ \frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi} \\ &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[ \frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right] = \frac{a^2}{2} [\pi - 2] \end{aligned}$$



50.  $r = a \cos \theta, r = a \sin \theta$

$\tan \theta = 1, \theta = \pi/4$

$$\begin{aligned}
 A &= 2 \left[ \frac{1}{2} \int_0^{\pi/4} (a \sin \theta)^2 d\theta \right] \\
 &= a^2 \int_0^{\pi/4} \frac{1 - \cos 2\theta}{2} d\theta \\
 &= \frac{1}{2} a^2 \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4} \\
 &= \frac{1}{2} a^2 \left[ \frac{\pi}{4} - \frac{1}{2} \right] \\
 &= \frac{1}{8} a^2 \pi - \frac{1}{4} a^2
 \end{aligned}$$



52. By symmetry,  $A_1 = A_2$  and  $A_3 = A_4$ .

$$\begin{aligned}
 A_1 &= A_2 = \frac{1}{2} \int_{-\pi/3}^{\pi/6} [(2a \cos \theta)^2 - (a)^2] d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} [(2a \cos \theta)^2 - (2a \sin \theta)^2] d\theta \\
 &= \frac{a^2}{2} \int_{-\pi/3}^{\pi/6} (4 \cos^2 \theta - 1) d\theta + 2a^2 \int_{\pi/6}^{\pi/4} \cos 2\theta d\theta \\
 &= \frac{a^2}{2} [\theta + \sin 2\theta]_{-\pi/3}^{\pi/6} + a^2 [\sin 2\theta]_{\pi/6}^{\pi/4} = \frac{a^2}{2} \left( \frac{\pi}{2} + \sqrt{3} \right) + a^2 \left( 1 - \frac{\sqrt{3}}{2} \right) = a^2 \left( \frac{\pi}{4} + 1 \right)
 \end{aligned}$$

$$A_3 = A_4 = \frac{1}{2} \left( \frac{\pi}{2} \right) a^2 = \frac{\pi a^2}{4}$$

$$\begin{aligned}
 A_5 &= \frac{1}{2} \left( \frac{5\pi}{6} \right) a^2 - 2 \left( \frac{1}{2} \right) \int_{5\pi/6}^{\pi} (2a \sin \theta)^2 d\theta \\
 &= \frac{5\pi a^2}{12} - 2a^2 \int_{5\pi/6}^{\pi} (1 - \cos 2\theta) d\theta \\
 &= \frac{5\pi a^2}{12} - a^2 [2\theta - \sin 2\theta]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2 \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = a^2 \left( \frac{\pi}{12} + \frac{\sqrt{3}}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 A_6 &= 2 \left( \frac{1}{2} \right) \int_0^{\pi/6} (2a \sin \theta)^2 d\theta + 2 \left( \frac{1}{2} \right) \int_{\pi/6}^{\pi/4} a^2 d\theta \\
 &= 2a^2 \int_0^{\pi/6} (1 - \cos 2\theta) d\theta + [a^2 \theta]_{\pi/6}^{\pi/4} \\
 &= a^2 [2\theta - \sin 2\theta]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2 \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) + \frac{\pi a^2}{12} = a^2 \left( \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \right)
 \end{aligned}$$

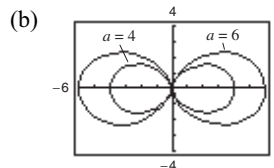
$$\begin{aligned}
 A_7 &= 2 \left( \frac{1}{2} \right) \int_{\pi/6}^{\pi/4} [(2a \sin \theta)^2 - (a)^2] d\theta \\
 &= a^2 \int_{\pi/6}^{\pi/4} (4 \sin^2 \theta - 1) d\theta = a^2 [\theta - \sin 2\theta]_{\pi/6}^{\pi/4} = a^2 \left( \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} \right)
 \end{aligned}$$

[Note:  $A_1 + A_6 + A_7 + A_4 = \pi a^2 = \text{area of circle of radius } a$ ]

51. (a)  $r = a \cos^2 \theta$

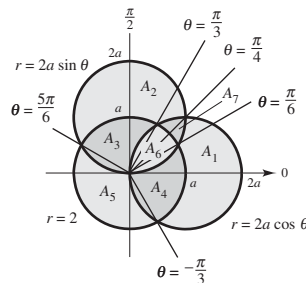
$r^3 = ar^2 \cos^2 \theta$

$(x^2 + y^2)^{3/2} = ax^2$



(c)

$$\begin{aligned}
 A &= 4 \left( \frac{1}{2} \right) \int_0^{\pi/2} [(6 \cos^2 \theta)^2 - (4 \cos^2 \theta)^2] d\theta \\
 &= 40 \int_0^{\pi/2} \cos^4 \theta d\theta \\
 &= 10 \int_0^{\pi/2} (1 + \cos 2\theta)^2 d\theta \\
 &= 10 \int_0^{\pi/2} \left( 1 + 2 \cos 2\theta + \frac{1 - \cos 4\theta}{2} \right) d\theta \\
 &= 10 \left[ \frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = \frac{15\pi}{2}
 \end{aligned}$$

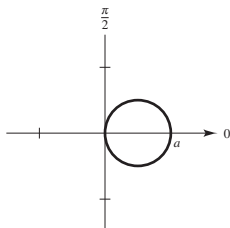


53.  $r = a \cos(n\theta)$

For  $n = 1$ :

$$r = a \cos \theta$$

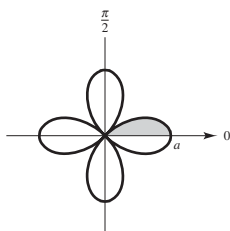
$$A = \pi \left( \frac{a}{2} \right)^2 = \frac{\pi a^2}{4}$$



For  $n = 2$ :

$$r = a \cos 2\theta$$

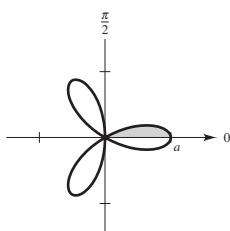
$$A = 8 \left( \frac{1}{2} \right) \int_0^{\pi/4} (a \cos 2\theta)^2 d\theta = \frac{\pi a^2}{2}$$



For  $n = 3$ :

$$r = a \cos 3\theta$$

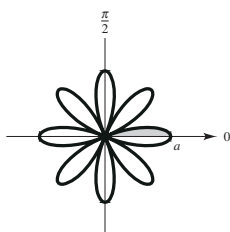
$$A = 6 \left( \frac{1}{2} \right) \int_0^{\pi/6} (a \cos 3\theta)^2 d\theta = \frac{\pi a^2}{4}$$



For  $n = 4$ :

$$r = a \cos 4\theta$$

$$A = 16 \left( \frac{1}{2} \right) \int_0^{\pi/8} (a \cos 4\theta)^2 d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by  $r = a \cos(n\theta)$  for  $n = 1, 2, 3, \dots$  is  $(\pi a^2)/4$  if  $n$  is odd and is  $(\pi a^2)/2$  if  $n$  is even.

54.  $r = \sec \theta - 2 \cos \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$r \cos \theta = 1 - 2 \cos^2 \theta$$

$$x = 1 - 2 \frac{r^2 \cos^2 \theta}{r^2} = 1 - 2 \left( \frac{x^2}{x^2 + y^2} \right)$$

$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

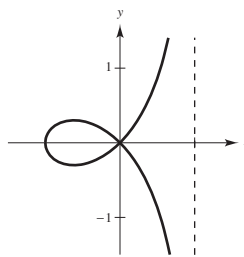
$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2 \left( \frac{1}{2} \right) \int_0^{\pi/4} (\sec \theta - 2 \cos \theta)^2 d\theta$$

$$= \int_0^{\pi/4} (\sec^2 \theta - 4 + 4 \cos^2 \theta) d\theta$$

$$= \int_0^{\pi/4} (\sec^2 \theta - 4 + 2(1 + \cos 2\theta)) d\theta$$

$$= [\tan \theta - 2\theta + \sin 2\theta]_0^{\pi/4} = 2 - \frac{\pi}{2}$$



55.  $r = 8, r^1 = 0$

$$s = \int_0^{2\pi} \sqrt{8^2 + 0^2} d\theta = 8\theta \Big|_0^{2\pi} = 16\pi$$

(circumference of circle of radius 8)

56.  $r = a$

$$r' = 0$$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2} d\theta = [a\theta]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius  $a$ )

57.  $r = 4 \sin \theta$

$$r' = 4 \cos \theta$$

$$s = \int_0^{\pi} \sqrt{(4 \sin \theta)^2 + (4 \cos \theta)^2} d\theta$$

$$= \int_0^{\pi} 4 d\theta = [4\theta]_0^{\pi} = 4\pi$$

(circumference of circle of radius 2)

58.  $r = 2a \cos \theta$

$$r' = -2a \sin \theta$$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a d\theta = [2a\theta]_{-\pi/2}^{\pi/2} = 2\pi a$$

59.  $r = 1 + \sin \theta$

$r' = \cos \theta$

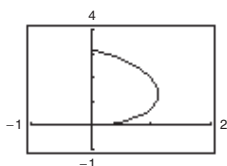
$$\begin{aligned}
 s &= 2 \int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2} d\theta \\
 &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin \theta} d\theta \\
 &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}} d\theta \\
 &= \left[ 4\sqrt{2} \sqrt{1 - \sin \theta} \right]_{\pi/2}^{3\pi/2} \\
 &= 4\sqrt{2}(\sqrt{2} - 0) = 8
 \end{aligned}$$

60.  $r = 8(1 + \cos \theta), 0 \leq \theta \leq 2\pi$

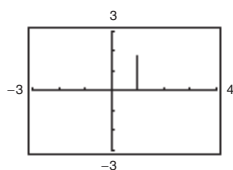
$r' = -8 \sin \theta$

$$\begin{aligned}
 s &= 2 \int_0^\pi \sqrt{[8(1 + \cos \theta)]^2 + (-8 \sin \theta)^2} d\theta \\
 &= 16 \int_0^\pi \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta \\
 &= 16\sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} d\theta \\
 &= 16\sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} \cdot \left( \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}} \right) d\theta \\
 &= 16\sqrt{2} \int_0^\pi \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta \\
 &= \left[ 32\sqrt{2} \sqrt{1 - \cos \theta} \right]_0^\pi \\
 &= 64
 \end{aligned}$$

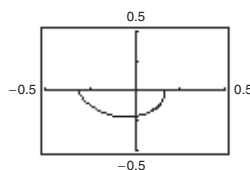
61.  $r = 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$

Length  $\approx 4.16$ 

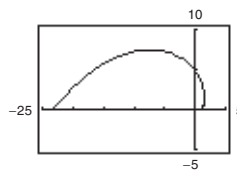
62.  $r = \sec \theta, 0 \leq \theta \leq \frac{\pi}{3}$

Length  $\approx 1.73$  (exact  $\sqrt{3}$ )

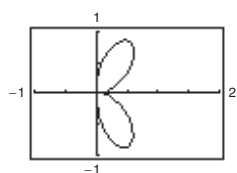
63.  $r = \frac{1}{\theta}, \pi \leq \theta \leq 2\pi$

Length  $\approx 0.71$ 

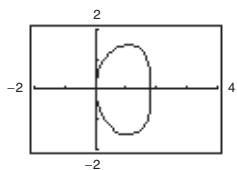
64.  $r = e^\theta, 0 \leq \theta \leq \pi$

Length  $\approx 31.31$ 

65.  $r = \sin(3 \cos \theta), 0 \leq \theta \leq \pi$

Length  $\approx 4.39$ 

66.  $r = 2 \sin(2 \cos \theta), 0 \leq \theta \leq \pi$

Length  $\approx 7.78$ 

67.  $r = 6 \cos \theta$

$r' = -6 \sin \theta$

$$\begin{aligned}
 S &= 2\pi \int_0^{\pi/2} 6 \cos \theta \sin \theta \sqrt{36 \cos^2 \theta + 36 \sin^2 \theta} d\theta \\
 &= 72\pi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \\
 &= \left[ 36\pi \sin^2 \theta \right]_0^{\pi/2} \\
 &= 36\pi
 \end{aligned}$$

68.  $r = a \cos \theta$

$r' = -a \sin \theta$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} a \cos \theta (\cos \theta) \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta \\ &= 2\pi a^2 \int_0^{\pi/2} \cos^2 \theta d\theta = \pi a^2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \left[ \pi a^2 \left( \theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2} = \frac{\pi^2 a^2}{2} \end{aligned}$$

70.  $r = a(1 + \cos \theta)$

$r' = -a \sin \theta$

$$\begin{aligned} S &= 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta = 2\pi a^2 \int_0^\pi \sin \theta (1 + \cos \theta) \sqrt{2 + 2 \cos \theta} d\theta \\ &= -2\sqrt{2}\pi a^2 \int_0^\pi (1 + \cos \theta)^{3/2} (-\sin \theta) d\theta = -\frac{4\sqrt{2}\pi a^2}{5} \left[ (1 + \cos \theta)^{5/2} \right]_0^\pi = \frac{32\pi a^2}{5} \end{aligned}$$

71.  $r = 4 \cos 2\theta$

$r' = -8 \sin 2\theta$

$$S = 2\pi \int_0^{\pi/4} 4 \cos 2\theta \sin \theta \sqrt{16 \cos^2 2\theta + 64 \sin^2 2\theta} d\theta = 32\pi \int_0^{\pi/4} \cos 2\theta \sin \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta \approx 21.87$$

72.  $r = \theta$

$r' = 1$

$$S = 2\pi \int_0^\pi \theta \sin \theta \sqrt{\theta^2 + 1} d\theta \approx 42.32$$

73. You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

74. (a) is correct:  $s \approx 33.124$ .

75. (a)  $S = 2\pi \int_\alpha^\beta f(\theta) \sin \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

(b)  $S = 2\pi \int_\alpha^\beta f(\theta) \cos \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

69.  $r = e^{a\theta}$

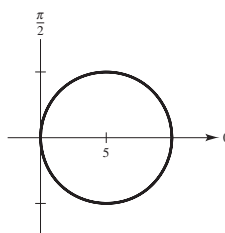
$r' = ae^{a\theta}$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} e^{a\theta} \cos \theta \sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2} d\theta \\ &= 2\pi \sqrt{1 + a^2} \int_0^{\pi/2} e^{2a\theta} \cos \theta d\theta \\ &= 2\pi \sqrt{1 + a^2} \left[ \frac{e^{2a\theta}}{4a^2 + 1} (2a \cos \theta + \sin \theta) \right]_0^{\pi/2} \\ &= \frac{2\pi \sqrt{1 + a^2}}{4a^2 + 1} (e^{\pi a} - 2a) \end{aligned}$$

76. (a)  $r = 10 \cos \theta, 0 \leq \theta < \pi$

Circle of radius 5

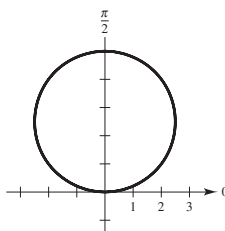
Area =  $25\pi$



(b)  $r = 5 \sin \theta, 0 \leq \theta < \pi$

Circle radius  $5/2$

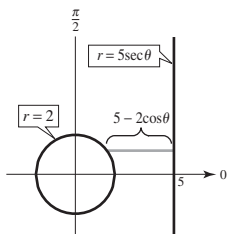
Area =  $\frac{25}{4}\pi$



77. Revolve
- $r = 2$
- about the line
- $r = 5 \sec \theta$
- .

$$f(\theta) = 2, f'(\theta) = 0$$

$$\begin{aligned} S &= 2\pi \int_0^{2\pi} (5 - 2 \cos \theta) \sqrt{2^2 + 0^2} d\theta \\ &= 4\pi \int_0^{2\pi} (5 - 2 \cos \theta) d\theta \\ &= 4\pi [5\theta - 2 \sin \theta]_0^{2\pi} \\ &= 40\pi^2 \end{aligned}$$

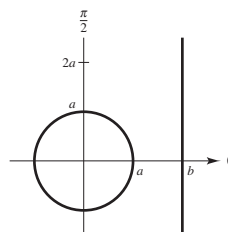


78. Revolve
- $r = a$
- about the line
- $r = b \sec \theta$
- where
- $b > a > 0$
- .

$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$\begin{aligned} S &= 2\pi \int_0^{2\pi} [b - a \cos \theta] \sqrt{a^2 + 0^2} d\theta \\ &= 2\pi a [b\theta - a \sin \theta]_0^{2\pi} \\ &= 2\pi a (2\pi b) = 4\pi^2 ab \end{aligned}$$



- 79.
- $r = 8 \cos \theta, 0 \leq \theta \leq \pi$

$$(a) A = \frac{1}{2} \int_0^\pi r^2 d\theta = \frac{1}{2} \int_0^\pi 64 \cos^2 \theta d\theta = 32 \int_0^\pi \frac{1 + \cos 2\theta}{2} d\theta = 16 \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^\pi = 16\pi$$

$$(\text{Area circle} = \pi r^2 = \pi 4^2 = 16\pi)$$

(b)

$\theta$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$A$	6.32	12.14	17.06	20.80	23.27	24.60	25.08

- (c), (d) For  $\frac{1}{4}$  of area ( $4\pi \approx 12.57$ ): 0.42

$$\text{For } \frac{1}{2} \text{ of area } (8\pi \approx 25.13): 1.57 \left( \frac{\pi}{2} \right)$$

$$\text{For } \frac{3}{4} \text{ of area } (12\pi \approx 37.70): 2.73$$

- (e) No, it does not depend on the radius.

- 80.
- $r = 3 \sin \theta, 0 \leq \theta \leq \pi$

$$(a) A = \frac{1}{2} \int_0^\pi r^2 d\theta = \frac{9}{2} \int_0^\pi \sin^2 \theta d\theta = \frac{9}{4} \int_0^\pi (1 - \cos 2\theta) d\theta = \frac{9}{4} \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^\pi = \frac{9}{4} \pi$$

$$\left[ \text{Note: radius of circle is } \frac{3}{2} \Rightarrow A = \pi \left( \frac{3}{2} \right)^2 = \frac{9}{4} \pi \right]$$

(b)

$\theta$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$A$	0.0119	0.0930	0.3015	0.6755	1.2270	1.9401	2.7731

- (c), (d) For  $\frac{1}{8}$  of area  $\left( \frac{1}{8} \frac{9}{4} \pi \approx 0.8836 \right): \theta \approx 0.88$

$$\text{For } \frac{1}{4} \text{ of area } \left( \frac{1}{4} \frac{9}{4} \pi \approx 1.7671 \right): \theta \approx 1.15$$

$$\text{For } \frac{1}{2} \text{ of area } \left( \frac{1}{2} \frac{9}{4} \pi \approx 3.5343 \right): \theta = \frac{\pi}{2} \approx 1.57$$

81.  $r = a \sin \theta + b \cos \theta$   
 $r^2 = ar \sin \theta + br \cos \theta$   
 $x^2 + y^2 = ay + bx$   
 $x^2 + y^2 - bx - ay = 0$  represents a circle.

82.  $r = \sin \theta + \cos \theta$ , Circle

$$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi} (\sin \theta + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi} (1 + 2 \sin \theta \cos \theta) d\theta = \frac{1}{2} [\theta + \sin^2 \theta]_0^{\pi} = \frac{\pi}{2} \end{aligned}$$

Converting to rectangular form:

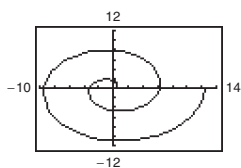
$$\begin{aligned} r^2 &= r \sin \theta + r \cos \theta \\ x^2 + y^2 &= y + x \\ \left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) &= \frac{1}{2} \\ \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 &= \frac{1}{2} \end{aligned}$$

Circle of radius  $\frac{1}{\sqrt{2}}$  and center  $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Area} = \pi \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

83. (a)  $r = \theta, \theta \geq 0$

As  $a$  increases, the spiral opens more rapidly. If  $\theta < 0$ , the spiral is reflected about the  $y$ -axis.



(b)  $r = a\theta, \theta \geq 0$ , crosses the polar axis for  $\theta = n\pi, n$  and integer. To see this

$$r = a\theta \Rightarrow r \sin \theta = y = a\theta \sin \theta = 0$$

for  $\theta = n\pi$ . The points are

$$(r, \theta) = (an\pi, n\pi), n = 1, 2, 3, \dots$$

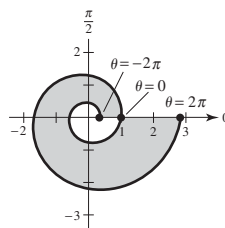
(c)  $f(\theta) = \theta, f'(\theta) = 1$

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta \\ &= \frac{1}{2} \left[ \ln(\sqrt{x^2 + 1} + x) + x\sqrt{x^2 + 1} \right]_0^{2\pi} \\ &= \frac{1}{2} \ln(\sqrt{4\pi^2 + 1} + 2\pi) + \pi\sqrt{4\pi^2 + 1} \approx 21.2563 \end{aligned}$$

(d)  $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 dr = \frac{1}{2} \int_0^{2\pi} \theta^2 d\theta = \left[ \frac{\theta^3}{6} \right]_0^{2\pi} = \frac{4}{3}\pi^3$

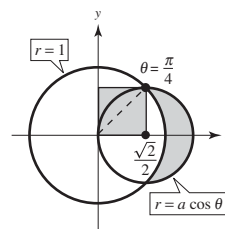
84.  $r = e^{\theta/6}$

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (e^{\theta/6})^2 d\theta - \frac{1}{2} \int_{-2\pi}^0 (e^{\theta/6})^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} e^{\theta/3} d\theta - \frac{1}{2} \int_{-2\pi}^0 e^{\theta/3} d\theta \\ &= \left[ \frac{3}{2} e^{\theta/3} \right]_0^{2\pi} - \left[ \frac{3}{2} e^{\theta/3} \right]_{-2\pi}^0 \\ &= \frac{3}{2} e^{2\pi/3} - \frac{3}{2} - \frac{3}{2} + \frac{3}{2} e^{-2\pi/3} = \frac{3}{2} [e^{2\pi/3} + e^{-2\pi/3} - 2] \\ &\approx 9.3655 \end{aligned}$$



85. The smaller circle has equation  $r = a \cos \theta$ . The area of the shaded lune is:

$$\begin{aligned} A &= 2 \left( \frac{1}{2} \right) \int_0^{\pi/4} [(a \cos \theta)^2 - 1] d\theta \\ &= \int_0^{\pi/4} \left[ \frac{a^2}{2} (1 + \cos 2\theta) - 1 \right] d\theta \\ &= \left[ \frac{a^2}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) - \theta \right]_0^{\pi/4} \\ &= \frac{a^2}{2} \left( \frac{\pi}{4} + \frac{1}{2} \right) - \frac{\pi}{4} \end{aligned}$$



This equals the area of the square,  $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$ .

$$\frac{a^2}{2} \left( \frac{\pi}{4} + \frac{1}{2} \right) - \frac{\pi}{4} = \frac{1}{2}$$

$$\pi a^2 + 2a^2 - 2\pi - 4 = 0$$

$$a^2 = \frac{4 + 2\pi}{2 + \pi} = 2$$

$$a = \sqrt{2}$$

Smaller circle:  $r = \sqrt{2} \cos \theta$

86.  $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$

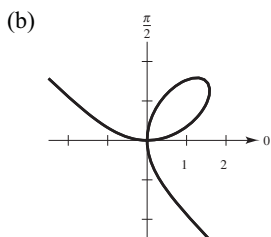
(a)  $x^3 + y^3 = \frac{27(t^3 + t^6)}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2}$

$$3xy = \frac{27t^3}{(1+t^3)^2}$$

So,  $x^3 + y^3 = 3xy$ .

$$(r \cos \theta)^3 + (r \sin \theta)^3 = 3(r \cos \theta)(r \sin \theta)$$

$$r = \frac{3 \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}$$



(c)  $A = \frac{1}{2} \int_0^{\pi/2} r^2 d\theta = \frac{3}{2}$

87. False.  $f(\theta) = 1$  and  $g(\theta) = -1$  have the same graphs.

88. False.  $f(\theta) = 0$  and  $g(\theta) = \sin 2\theta$  have only one point of intersection.

89. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Using  $\theta$  instead of  $t$ , you have

$$x = r \cos \theta = f(\theta) \cos \theta \text{ and}$$

$$y = r \sin \theta = f(\theta) \sin \theta. \text{ So,}$$

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and}$$

$$\frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

$$\text{So, } s = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta.$$

## Section 10.6 Polar Equations of Conics and Kepler's Laws

1.  $r = \frac{2e}{1 + e \cos \theta}$

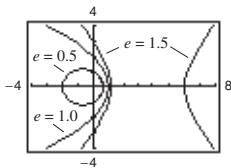
(a)  $e = 1, r = \frac{2}{1 + \cos \theta}$ , parabola

(b)  $e = 0.5$ ,

$$r = \frac{1}{1 + 0.5 \cos \theta} = \frac{2}{2 + \cos \theta}, \text{ ellipse}$$

(c)  $e = 1.5$ ,

$$r = \frac{3}{1 + 1.5 \cos \theta} = \frac{6}{2 + 3 \cos \theta}, \text{ hyperbola}$$



2.  $r = \frac{2e}{1 - e \cos \theta}$

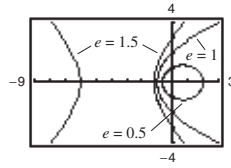
(a)  $e = 1, r = \frac{2}{1 - \cos \theta}$ , parabola

(b)  $e = 0.5$ ,

$$r = \frac{1}{1 - 0.5 \cos \theta} = \frac{2}{2 - \cos \theta}, \text{ ellipse}$$

(c)  $e = 1.5$ ,

$$r = \frac{3}{1 - 1.5 \cos \theta} = \frac{6}{2 - 3 \cos \theta}, \text{ hyperbola}$$





3.  $r = \frac{2e}{1 - e \sin \theta}$

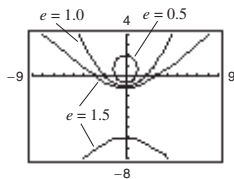
(a)  $e = 1, r = \frac{2}{1 - \sin \theta}$ , parabola

(b)  $e = 0.5$ ,

$$r = \frac{1}{1 - 0.5 \sin \theta} = \frac{2}{2 - \sin \theta}, \text{ ellipse}$$

(c)  $e = 1.5$ ,

$$r = \frac{3}{1 - 1.5 \sin \theta} = \frac{6}{2 - 3 \sin \theta}, \text{ hyperbola}$$



4.  $r = \frac{2e}{1 + e \sin \theta}$

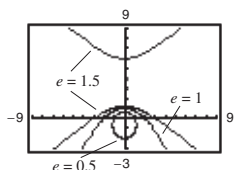
(a)  $e = 1, r = \frac{2}{1 + \sin \theta}$ , parabola

(b)  $e = 0.5$ ,

$$r = \frac{1}{1 + 0.5 \sin \theta} = \frac{2}{2 + \sin \theta}, \text{ ellipse}$$

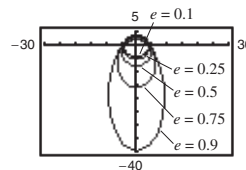
(c)  $e = 1.5$ ,

$$r = \frac{3}{1 + 1.5 \sin \theta} = \frac{6}{2 + 3 \sin \theta}, \text{ hyperbola}$$

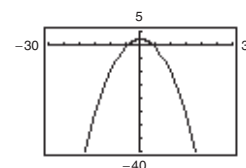


5.  $r = \frac{4}{1 + e \sin \theta}$

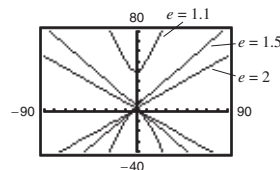
(a) The conic is an ellipse. As  $e \rightarrow 1^-$ , the ellipse becomes more elliptical, and as  $e \rightarrow 0^+$ , it becomes more circular.



(b) The conic is a parabola.



(c) The conic is a hyperbola. As  $e \rightarrow 1^+$ , the hyperbola opens more slowly, and as  $e \rightarrow \infty$ , it opens more rapidly.



6.  $r = \frac{4}{1 - 0.4 \cos \theta}$

(a) Because  $e = 0.4 < 1$ , the conic is an ellipse with vertical directrix to the left of the pole.

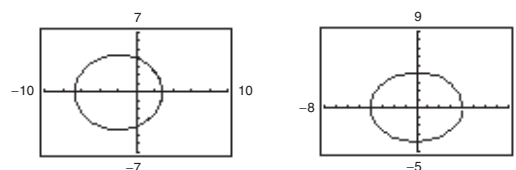
(b)  $r = \frac{4}{1 + 0.4 \cos \theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole.

$$r = \frac{4}{1 - 0.4 \sin \theta}.$$

The ellipse has a horizontal directrix below the pole.

(c)



7. Parabola; Matches (c)

8. Ellipse; Matches (f)

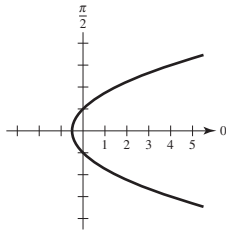
9. Hyperbola; Matches (a)

10. Parabola; Matches (e)

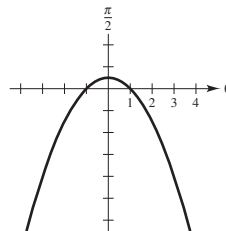
11. Ellipse; Matches (b)

12. Hyperbola; Matches (d)

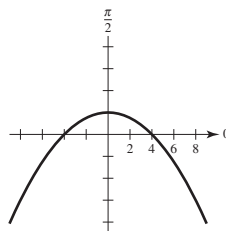
$$13. r = \frac{1}{1 - \cos \theta}$$

Parabola because  $e = 1, d = 1$ .Distance from pole to directrix:  $|d| = 1$ Directrix:  $x = -d = -1$ 

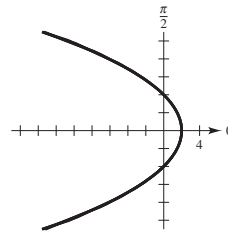
$$14. r = \frac{1}{1 + \sin \theta}$$

Parabola because  $e = 1, d = 1$ Distance from pole to directrix:  $|d| = 1$ Directrix:  $y = 1$ 

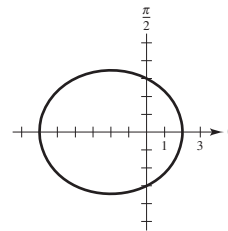
$$15. r = \frac{-4}{1 - \sin \theta} = \frac{1(-4)}{1 - \sin \theta}$$

 $e = 1, d = -4$  ParabolaDistance from pole to directrix:  $|d| = 4$ Directrix:  $y = 4$ 

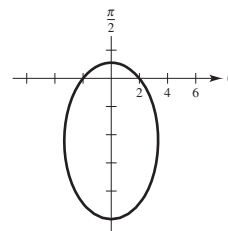
$$16. r = \frac{4}{1 + \cos \theta}$$

 $e = 1, d = 4$  ParabolaDistance from pole to directrix:  $|d| = 4$ Directrix:  $x = 4$ 

$$17. r = \frac{6}{2 + \cos \theta} = \frac{3}{1 + (1/2) \cos \theta}$$

Ellipse because  $e = \frac{1}{2}; d = 6$ Directrix:  $x = 6$ Distance from pole to directrix:  $|d| = 6$ Vertices:  $(r, \theta) = (2, 0), (6, \pi)$ 

$$18. r = \frac{10}{5 + 4 \sin \theta} = \frac{2}{1 + (\frac{4}{5}) \sin \theta} = \frac{(\frac{4}{5})(\frac{5}{2})}{1 + (\frac{4}{5}) \sin \theta}$$

 $e = \frac{4}{5} < 1, d = \frac{5}{2}$  EllipseDistance from pole to directrix:  $|d| = \frac{5}{2}$ 

19.  $r(2 + \sin \theta) = 4$

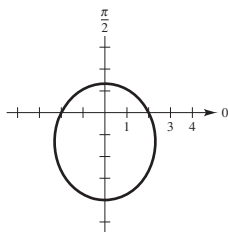
$$r = \frac{4}{2 + \sin \theta} = \frac{2}{1 + (1/2) \sin \theta}$$

Ellipse because  $e = 1/2$ ;  $d = 4$

Directrix:  $y = 4$

Distance from pole to directrix:  $|d| = 4$

Vertices:  $(r, \theta) = (4/3, \pi/2), (4, 3\pi/2)$



20.  $r(3 - 2 \cos \theta) = 6$

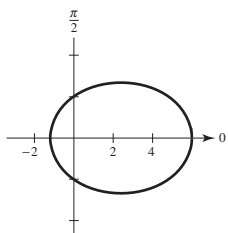
$$r = \frac{6}{3 - 2 \cos \theta} = \frac{2}{1 - (2/3) \cos \theta}$$

Ellipse because  $e = 2/3 < 1$ ;  $d = 3$

Directrix:  $x = -3$

Distance from pole to directrix:  $|d| = 3$

Vertices:  $(r, \theta) = (6, 0), (6/5, \pi)$



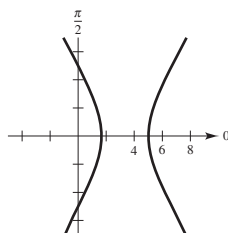
21.  $r = \frac{5}{-1 + 2 \cos \theta} = \frac{-5}{1 - 2 \cos \theta}$

Hyperbola because  $e = 2 > 1$ ;  $d = -5/2$

Directrix:  $x = 5/2$

Distance from pole to directrix:  $|d| = 5/2$

Vertices:  $(r, \theta) = (5, 0), (-5/3, \pi)$



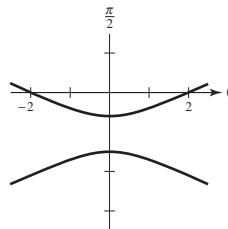
22.  $r = \frac{-6}{3 + 7 \sin \theta} = \frac{-2}{1 + (7/3) \sin \theta}$

Hyperbola because  $e = 7/3 > 1$ ;  $d = -6/7$

Directrix:  $y = -6/7$

Distance from pole to directrix:  $|d| = 6/7$

Vertices:  $(r, \theta) = (-3/5, \pi/2), (3/2, 3\pi/2)$



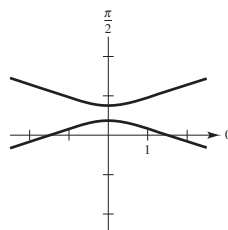
23.  $r = \frac{3}{2 + 6 \sin \theta} = \frac{3/2}{1 + 3 \sin \theta}$

Hyperbola because  $e = 3 > 1$ ;  $d = 1/2$

Directrix:  $y = 1/2$

Distance from pole to directrix:  $|d| = 1/2$

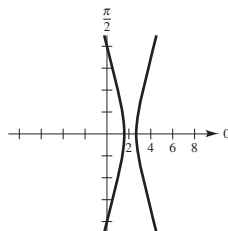
Vertices:  $(r, \theta) = (3/8, \pi/2), (-3/4, 3\pi/2)$



24.  $r = \frac{8}{1 + 4 \cos \theta} = \frac{4(2)}{1 + 4 \cos \theta}$

$e = 4 > 1$ ,  $d = 2$  Hyperbola

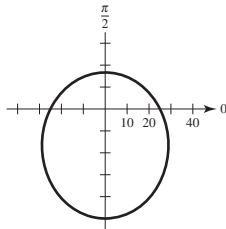
Distance from pole to directrix:  $|d| = 2$



$$25. r = \frac{300}{-12 + 6 \sin \theta} = \frac{-25}{1 - \frac{1}{2} \sin \theta} = \frac{\frac{1}{2}(-50)}{1 - \frac{1}{2} \sin \theta}$$

$$e = \frac{1}{2}, d = -50, \text{ Ellipse}$$

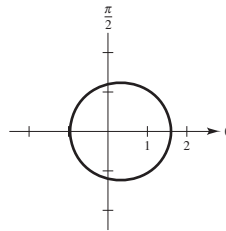
Distance from pole to directrix:  $|d| = 50$



$$26. r = \frac{180}{15 - 3.75 \cos \theta} = \frac{12}{1 - \frac{1}{4} \cos \theta} = \frac{\frac{1}{4}(48)}{1 - \frac{1}{4} \cos \theta}$$

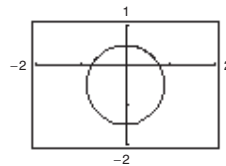
$$e = \frac{1}{4}, d = 48, \text{ Ellipse}$$

Distance from pole to directrix:  $|d| = 48$



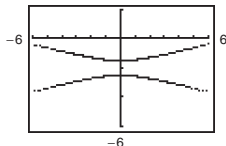
$$27. r = \frac{3}{-4 + 2 \sin \theta} = \frac{-\frac{3}{4}}{1 - \frac{1}{2} \sin \theta}$$

$$e = \frac{1}{2}, \text{ Ellipse}$$



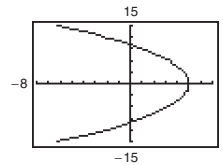
$$28. r = \frac{-15}{2 + 8 \sin \theta} = \frac{-\frac{15}{2}}{1 + 4 \sin \theta}$$

$$e = 4, \text{ Hyperbola}$$



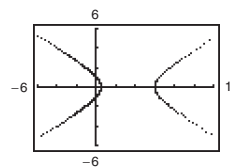
$$29. r = \frac{-10}{1 - \cos \theta}$$

$$e = 1, \text{ Parabola}$$

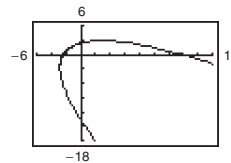


$$30. r = \frac{6}{6 + 7 \cos \theta} = \frac{1}{1 + \left(\frac{7}{6}\right) \cos \theta}$$

$$e = \frac{7}{6}, \text{ Hyperbola}$$



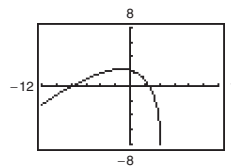
$$31. r = \frac{-4}{1 - \sin\left(\theta - \frac{\pi}{4}\right)}$$



Rotate the graph of  $r = \frac{-4}{1 - \sin \theta}$

$\frac{\pi}{4}$  radian counterclockwise.

$$32. r = \frac{4}{1 + \cos\left(\theta - \frac{\pi}{3}\right)}$$



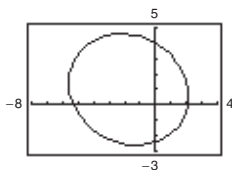
Rotate the graph of  $r = \frac{4}{1 + \cos \theta}$

$\frac{\pi}{3}$  radian counterclockwise.

33.  $r = \frac{6}{2 + \cos\left(\theta + \frac{\pi}{6}\right)}$

Rotate the graph of  $r = \frac{6}{2 + \cos \theta}$

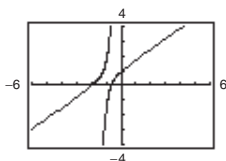
$\frac{\pi}{6}$  radian clockwise.



34.  $r = \frac{-6}{3 + 7 \sin(\theta + (2\pi/3))}$

Rotate graph of  $r = \frac{-6}{3 + 7 \sin \theta}$

$\frac{2\pi}{3}$  radians clockwise.



35. Change  $\theta$  to  $\theta + \frac{\pi}{6}$

$$r = \frac{8}{8 + 5 \cos\left(\theta + \frac{\pi}{6}\right)}$$

36. Change  $\theta$  to  $\theta - \frac{\pi}{4}$

$$r = \frac{9}{1 + \sin\left(\theta - \frac{\pi}{4}\right)}$$

37. Parabola

$$e = 1$$

$$x = -3 \Rightarrow d = 3$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{3}{1 - \cos \theta}$$

38. Parabola

$$e = 1, y = 4 \Rightarrow d = 4$$

$$r = \frac{ed}{1 + e \sin \theta} = \frac{4}{1 + \sin \theta}$$

39. Ellipse

$$e = \frac{1}{2}, y = 1, d = 1$$

$$r = \frac{ed}{1 + e \sin \theta} = \frac{1/2}{1 + (1/2) \sin \theta} = \frac{1}{2 + \sin \theta}$$

40. Ellipse

$$e = \frac{3}{4}, y = -2, d = 2$$

$$r = \frac{ed}{1 - e \sin \theta} = \frac{2(3/4)}{1 - (3/4) \sin \theta} = \frac{6}{4 - 3 \sin \theta}$$

41. Hyperbola

$$e = 2, x = 1, d = 1$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{2}{1 + 2 \cos \theta}$$

42. Hyperbola

$$e = \frac{3}{2}, x = -1, d = 1$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{3/2}{1 - (3/2) \cos \theta} = \frac{3}{2 - 3 \cos \theta}$$

43. Parabola

$$\text{Vertex: } \left(1, -\frac{\pi}{2}\right)$$

$$e = 1, d = 2, r = \frac{2}{1 - \sin \theta}$$

44. Parabola

$$\text{Vertex: } (5, \pi)$$

$$e = 1, d = 10$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{10}{1 - \cos \theta}$$

45. Ellipse

$$\text{Vertices: } (2, 0), (8, \pi)$$

$$e = \frac{3}{5}, d = \frac{16}{3}$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{16/5}{1 + (3/5) \cos \theta} = \frac{16}{5 + 3 \cos \theta}$$

46. Ellipse

$$\text{Vertices: } \left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$$

$$e = \frac{1}{3}, d = 8$$

$$r = \frac{ed}{1 + e \sin \theta} = \frac{8/3}{1 + (1/3) \sin \theta} = \frac{8}{3 + \sin \theta}$$

## 47. Hyperbola

$$\text{Vertices: } \left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$$

$$e = \frac{5}{4}, d = \frac{9}{5}$$

$$r = \frac{ed}{1 - e \sin \theta} = \frac{9/4}{1 - (5/4) \sin \theta} = \frac{9}{4 - 5 \sin \theta}$$

## 48. Hyperbola

$$\text{Vertices: } (2, 0), (10, 0)$$

$$e = \frac{3}{2}, d = \frac{10}{3}$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{5}{1 + (3/2) \cos \theta} = \frac{10}{2 + 3 \cos \theta}$$

49. Ellipse,  $e = \frac{1}{2}$ ,

$$\text{Directrix: } r = 4 \sec \theta \Rightarrow x = r \cos \theta = 4$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{\left(\frac{1}{2}\right)4}{1 + \frac{1}{2} \cos \theta} = \frac{4}{2 + \cos \theta}$$

50. Hyperbola,  $e = 2$ 

$$\text{Directrix: } r = -8 \csc \theta \Rightarrow y = r \sin \theta = -8$$

$$r = \frac{ed}{1 - e \sin \theta} = \frac{2(-8)}{1 - 2 \sin \theta} = \frac{-16}{1 - 2 \sin \theta}$$

51. Ellipse if  $0 < e < 1$ , parabola if  $e = 1$ , hyperbola if  $e > 1$ .52. (a) Hyperbola ( $e = 2 > 1$ )

$$(b) \text{ Ellipse } (e = \frac{1}{10} < 1)$$

$$(c) \text{ Parabola } (e = 1)$$

$$(d) \text{ Rotated hyperbola } (e = 3)$$

53. If the foci are fixed and  $e \rightarrow 0$ , then  $d \rightarrow \infty$ . To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2) \cos \theta}, e = 1/2, d = 1$$

$$r = \frac{5/16}{1 + (1/4) \cos \theta}, e = 1/4, d = 5/4.$$

54.  $r = \frac{4}{1 + \sin \theta}$  is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left of pole.

(b) Parabola with horizontal directrix below pole.

(c) Parabola with vertical directrix to right of pole.

(d) Parabola (b) rotated counterclockwise  $\pi/4$ .

$$55. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 b^2 + y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta + a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta + a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [a^2 + \cos^2 \theta (b^2 - a^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{a^2 + (b^2 - a^2) \cos^2 \theta} = \frac{a^2 b^2}{a^2 - c^2 \cos^2 \theta}$$

$$= \frac{b^2}{1 - (c/a)^2 \cos^2 \theta} = \frac{b^2}{1 - e^2 \cos^2 \theta}$$

$$56. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x^2 b^2 - y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta - a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta - a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [-a^2 + \cos^2 \theta (a^2 + b^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{-a^2 + c^2 \cos^2 \theta} = \frac{b^2}{-1 + (c^2/a^2) \cos^2 \theta}$$

$$= \frac{-b^2}{1 - e^2 \cos^2 \theta}$$

$$57. a = 5, c = 4, e = \frac{4}{5}, b = 3$$

$$r^2 = \frac{9}{1 - (16/25) \cos^2 \theta}$$

$$58. a = 4, c = 5, b = 3, e = \frac{5}{4}$$

$$r^2 = \frac{-9}{1 - (25/16) \cos^2 \theta}$$

$$59. a = 3, b = 4, c = 5, e = \frac{5}{3}$$

$$r^2 = \frac{-16}{1 - (25/9) \cos^2 \theta}$$

$$60. a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$$

$$r^2 = \frac{1}{1 - (3/4) \cos^2 \theta}$$

$$61. A = 2 \left[ \frac{1}{2} \int_0^\pi \left( \frac{3}{2 - \cos \theta} \right)^2 d\theta \right]$$

$$= 9 \int_0^\pi \frac{1}{(2 - \cos \theta)^2} d\theta \approx 10.88$$

$$62. A = \frac{1}{2} \int_0^{2\pi} \left( \frac{9}{4 + \cos \theta} \right)^2 d\theta \approx 17.52$$

$$63. A = 2 \left[ \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left( \frac{2}{3 - 2 \sin \theta} \right)^2 d\theta \right]$$

$$= 4 \int_{-\pi/2}^{\pi/2} \frac{1}{(3 - 2 \sin \theta)^2} d\theta \approx 3.37$$

$$64. A = \frac{1}{2} \int_0^{2\pi} \left[ \frac{3}{6 + 5 \sin \theta} \right]^2 d\theta \approx 4.65$$

65. Vertices:  $(123,000 + 4000, 0) = (127,000, 0)$

$$(119 + 4000, \pi) = (4119, \pi)$$

$$a = \frac{127,000 + 4119}{2} = 65,559.5$$

$$c = 65,559.5 - 4119 = 61,440.5$$

$$e = \frac{c}{a} = \frac{122,881}{131,119} \approx 0.93717$$

$$r = \frac{ed}{1 - e \cos \theta}$$

$$\theta = 0: r = \frac{ed}{1 - e}, \theta = \pi: r = \frac{ed}{1 + e}$$

$$2a = 2(65,559.5) = \frac{ed}{1 - e} + \frac{ed}{1 + e}$$

$$131,119 = d \left( \frac{e}{1 - e} + \frac{e}{1 + e} \right) = d \left( \frac{2e}{1 - e^2} \right)$$

$$d = \frac{131,119(1 - e^2)}{2e} \approx 8514.1397$$

$$r = \frac{7979.21}{1 - 0.93717 \cos \theta} = \frac{1,046,226,000}{131,119 - 122,881 \cos \theta}$$

When  $\theta = 60^\circ = \frac{\pi}{3}$ ,  $r \approx 15,015$ .

Distance between earth and the satellite is  
 $r - 4000 \approx 11,015$  miles.

$$66. (a) r = \frac{ed}{1 - e \cos \theta}$$

When  $\theta = 0$ ,  $r = c + a = ea + a = a(1 + e)$ .

So,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

$$\text{So, } r = \frac{(1 - e^2)a}{1 - e \cos \theta}.$$

(b) The perihelion distance is

$$a - c = a - ea = a(1 - e).$$

When  $\theta = \pi$ ,  $r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$ .

The aphelion distance is

$$a + c = a + ea = a(1 + e).$$

When  $\theta = 0$ ,  $r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$ .

67.  $a = 1.496 \times 10^8$ ,  $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{149,558,278.1}{1 - 0.0167 \cos \theta}$$

Perihelion distance:  $a(1 - e) \approx 147,101,680$  km

Aphelion distance:  $a(1 + e) \approx 152,098,320$  km

68.  $a = 1.427 \times 10^9$ ,  $e = 0.0542$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{1,422,807,988}{1 - 0.0542 \cos \theta}$$

Perihelion distance:  $a(1 - e) \approx 1,349,656,600$  km

Aphelion distance:  $a(1 + e) \approx 1,504,343,400$  km

69.  $a = 4.498 \times 10^9$ ,  $e = 0.0086$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{4,497,667,328}{1 - 0.0086 \cos \theta}$$

Perihelion distance:  $a(1 - e) \approx 4,459,317,200$  km

Aphelion distance:  $a(1 + e) \approx 4,536,682,800$  km

70.  $a = 5.791 \times 10^7$ ,  $e = 0.2056$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{55,462,065.54}{1 - 0.2056 \cos \theta}$$

Perihelion distance  $\approx a(1 - e) \approx 46,003,704$  km

Aphelion distance  $\approx a(1 + e) \approx 69,816,296$  km

$$71. r = \frac{4.498 \times 10^9}{1 - 0.0086 \cos \theta}$$

$$(a) A = \frac{1}{2} \int_0^{\pi/9} r^2 d\theta \approx 3.591 \times 10^{18} \text{ km}^2$$

$$165 \left[ \frac{\frac{1}{2} \int_0^{\pi/2} r^2 d\theta}{\frac{1}{2} \int_0^{2\pi} r^2 d\theta} \right] \approx 9.322 \text{ yrs}$$

$$(b) \frac{1}{2} \int_{\pi}^{\alpha} r^2 d\theta = 3.591 \times 10^{18}$$

By trial and error,  $\alpha \approx \pi + 0.361$

$0.361 > \pi/9 \approx 0.349$  because the rays in part (a) are longer than those in part (b)

(c) For part (a),

$$s = \int_0^{\pi/9} \sqrt{r^2 + (dr/d\theta)^2} d\theta \approx 1.583 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.583 \times 10^9}{9.322} \approx 1.698 \times 10^8 \text{ km/yr}$$

For part (b),

$$s = \int_{\pi}^{\pi+0.361} \sqrt{r^2 + (dr/d\theta)^2} d\theta \approx 1.610 \times 10^9 \text{ km}$$

$$\text{Average per year} = \frac{1.610 \times 10^9}{9.322} \approx 1.727 \times 10^8 \text{ km/yr}$$

$$72. a = \frac{1}{2}(500) = 250 \text{ au}, e \approx 0.995$$

$$(a) e = \frac{c}{a} \Rightarrow c \approx 248.75$$

$$b^2 = a^2 - c^2 \Rightarrow b \approx 24.969 \Rightarrow \text{minor axis} = 2b \approx 49.9 \text{ au}$$

$$(b) r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{2.49375}{1 - 0.995 \cos \theta}$$

$$(c) \text{Perihelion distance: } a(1 - e) \approx 1.25 \text{ au}$$

$$\text{Aphelion distance: } a(1 + e) \approx 498.75 \text{ au}$$

$$73. r_1 = a + c, r_0 = a - c, r_1 - r_0 = 2c, r_1 + r_0 = 2a$$

$$e = \frac{c}{a} = \frac{r_1 - r_0}{r_1 + r_0}$$

$$\frac{1+e}{1-e} = \frac{1+\frac{c}{a}}{1-\frac{c}{a}} = \frac{a+c}{a-c} = \frac{r_1}{r_0}$$

74. For a hyperbola,

$$r_0 = c - a \text{ and } r_1 = c + a.$$

$$\text{So } r_1 + r_0 = 2c \text{ and } r_1 - r_0 = 2a.$$

$$e = \frac{c}{a} = \frac{r_1 + r_0}{r_1 - r_0}$$

$$\frac{e+1}{e-1} = \frac{\frac{c}{a}+1}{\frac{c}{a}-1} = \frac{c+a}{c-a} = \frac{r_1}{r_0}$$



75.  $r_1 = \frac{ed}{1 + \sin \theta}$  and  $r_2 = \frac{ed}{1 - \sin \theta}$

Points of intersection:  $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At  $(ed, 0)$ ,  $\frac{dy}{dx} = -1$ . At  $(ed, \pi)$ ,  $\frac{dy}{dx} = 1$ .

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At  $(ed, 0)$ ,  $\frac{dy}{dx} = 1$ . At  $(ed, \pi)$ ,  $\frac{dy}{dx} = -1$ .

So, at  $(ed, 0)$  you have  $m_1 m_2 = (-1)(1) = -1$ , and  
at  $(ed, \pi)$  you have  $m_1 m_2 = 1(-1) = -1$ . The curves  
intersect at right angles.

76.  $r_1 = \frac{c}{1 + \cos \theta}, r_2 = \frac{d}{1 - \cos \theta}$  (Parabolas)

To find the intersection points:

$$\begin{aligned} \frac{c}{1 + \cos \theta} &= \frac{d}{1 - \cos \theta} \\ c - c \cdot \cos \theta &= d + d \cdot \cos \theta \\ \cos \theta &= \frac{c - d}{c + d} \end{aligned}$$

$$r_1 = \frac{c}{1 + \left(\frac{c - d}{c + d}\right)} = \frac{c(c + d)}{2c} = \frac{c + d}{2} = r_2$$

$$\frac{dr_1}{d\theta} = \frac{c \cdot \sin \theta}{(1 + \cos \theta)^2}, \frac{dr_2}{d\theta} = \frac{-d \cdot \sin \theta}{(1 - \cos \theta)^2}$$

For the first parabola,

$$\begin{aligned} \frac{dy}{dx} &= \frac{r_1 \cos \theta + r_1' \sin \theta}{-r_1 \sin \theta + r_1' \cos \theta} \\ &= \frac{c \cdot \cos \theta (1 + \cos \theta) + c \cdot \sin^2 \theta}{-c \cdot \sin \theta (1 + \cos \theta) + c \cdot \sin \theta \cos \theta} \\ &= \frac{1 + \cos \theta}{-\sin \theta}. \end{aligned}$$

Similarly for the second parabola,  $\frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta}$ .

Because the product of the slopes is  $-1$ , they intersect at right angles.

## Review Exercises for Chapter 10

1.  $4x^2 + y^2 = 4$

Ellipse

Vertex:  $(1, 0)$ .

Matches (c)

2.  $4x^2 - y^2 = 4$

Hyperbola

Vertex:  $(1, 0)$

Matches (c)

3.  $y^2 = -4x$

Parabola opening to left.

Matches (b)

4.  $y^2 - 4x^2 = 4$

Hyperbola

Vertex:  $(0, 2)$

Matches (d)

5.  $x^2 + 4y^2 = 4$

Ellipse

Vertex:  $(0, 1)$

Matches (a)

6.  $x^2 = 4y$

Parabola opening upward.

Matches (f)

7.  $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

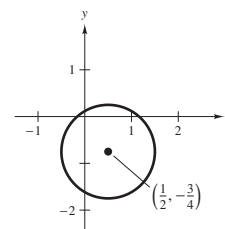
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center:  $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



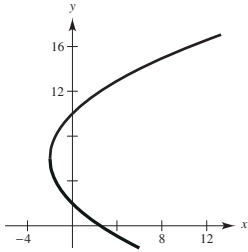
8.  $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex:  $(-2, 6)$



9.  $3x^2 - 2y^2 + 24x + 12y + 24 = 0$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

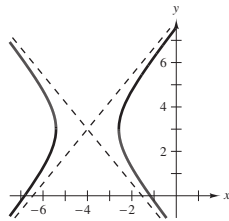
Hyperbola

Center:  $(-4, 3)$

Vertices:  $(-4 \pm \sqrt{2}, 3)$

Asymptotes:

$$y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$$



10.  $5x^2 + y^2 - 20x + 19 = 0$

$$5(x^2 - 4x + 4) + y^2 = -19 + 20$$

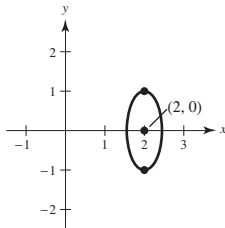
$$5(x - 2)^2 + y^2 = 1$$

$$\frac{(x - 2)^2}{(1/5)} + y^2 = 1$$

Ellipse

Center:  $(2, 0)$

Vertices:  $(2, \pm 1)$



11.  $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

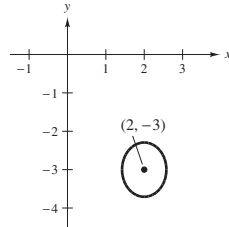
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center:  $(2, -3)$

Vertices:  $(2, -3 \pm \frac{\sqrt{2}}{2})$



12.  $12x^2 - 12y^2 - 12x + 24y - 45 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) - 12(y^2 - 2y + 1) = 45 + 3 - 12$$

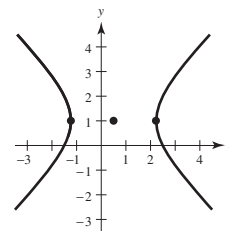
$$12\left(x - \frac{1}{2}\right)^2 - 12(y - 1)^2 = 36$$

$$\frac{(x - 1/2)^2}{3} - \frac{(y - 1)^2}{3} = 1$$

Hyperbola

Center:  $(\frac{1}{2}, 1)$

Vertices:  $(\frac{1}{2} \pm \sqrt{3}, 1)$



13. Vertex:  $(0, 2)$

Directrix:  $x = -3$

Parabola opens to the right.

$$p = 3$$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

14. Vertex: (2, 6)

Focus: (2, 4)

Parabola opens downward,  $p = -2$

$$(x - 2)^2 = 4(-2)(y - 6)$$

$$x^2 - 4x + 4 = -8y + 48$$

$$x^2 - 4x + 8y - 44 = 0$$

15. Vertices: (-5, 0), (7, 0)

Foci: (-3, 0), (5, 0)

Horizontal major axis

$$a = 6, c = 4, b = \sqrt{36 - 16} = 2\sqrt{5}$$

Center: (1, 0)

$$\frac{(x - 1)^2}{36} + \frac{y^2}{20} = 1$$

16. Center: (0, 0)

Solution points: (1, 2), (2, 0)

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

you obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \frac{4}{b^2} = 1.$$

Solving the system, you have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

17. Vertices: ( $\pm 7$ , 0)

Foci: ( $\pm 9$ , 0)

Horizontal transverse axis

Center (0, 0)

$$a = 7, c = 9, b = \sqrt{81 - 49} = \sqrt{32}$$

$$\frac{x^2}{49} - \frac{y^2}{32} = 1$$

18. Foci: (0,  $\pm 8$ )

Asymptotes:  $y = \pm 4x$

Center: (0, 0)

Vertical transverse axis

$$c = 8$$

$$y = \frac{a}{b}x = 4x \text{ asymptote} \rightarrow a = 4b$$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$

$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

19.  $\frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$

By Example 5 of Section 10.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

20.  $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 10.1,

$$C = 20 \int_0^{\pi/2} \sqrt{1 - \frac{21}{25} \sin^2 \theta} d\theta \approx 23.01.$$

21.  $y = x - 2$  has a slope of 1. The perpendicular slope is -1.

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

$$4x + 4y - 7 = 0$$

22.  $2x + y = 5$  has slope -2. The perpendicular slope is  $\frac{1}{2}$ .

$$y = -3x^2 + x - 6 \quad \text{Parabola}$$

$$y' = -6x + 1 = \frac{1}{2}$$

$$6x = \frac{1}{2}$$

$$x = \frac{1}{12}, y = -\frac{95}{16}$$

$$\text{Perpendicular line: } y + \frac{95}{16} = \frac{1}{2}\left(x - \frac{1}{12}\right)$$

$$y = \frac{1}{2}x - \frac{287}{48}$$

23.  $y = \frac{1}{200}x^2$

(a)  $x^2 = 200y$

$x^2 = 4(50)y$

Focus:  $(0, 50)$

(b)  $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10,000}}$

$S = 2\pi \int_0^{100} x \sqrt{1 + \frac{x^2}{10,000}} dx \approx 38,294.49$

24. (a)  $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

(b)  $F = 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[ 3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right]$   
 $= \frac{8}{3}(62.4) \left[ \frac{3}{2} \left( y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3$   
 $= \frac{8}{3}(62.4) \left[ \frac{3}{2} \left( \frac{9\pi}{2} \right) - \frac{3}{2} \left( -\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left( \frac{27\pi}{2} \right) \approx 7057.274$

(c) You want  $\frac{3}{4}$  of the total area of  $12\pi$  covered. Find  $h$  so that

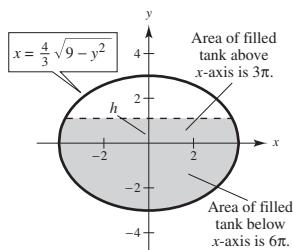
$2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy = 3\pi$

$\int_0^h \sqrt{9 - y^2} dy = \frac{9\pi}{8}$

$\frac{1}{2} \left[ y \sqrt{9 - y^2} + 9 \arcsin \left( \frac{y}{3} \right) \right]_0^h = \frac{9\pi}{8}$

$h \sqrt{9 - h^2} + 9 \arcsin \left( \frac{h}{3} \right) = \frac{9\pi}{4}$

By Newton's Method,  $h \approx 1.212$ . So, the total height of the water is  $1.212 + 3 = 4.212 \text{ ft}$ .



(d) Area of ends  $= 2(12\pi) = 24\pi$

Area of sides  $= (\text{Perimeter})(\text{Length})$

$= 16 \int_0^{\pi/2} \left( \sqrt{1 - \left( \frac{7}{16} \right) \sin^2 \theta} \right) d\theta (16) \text{ [from Example 5 of Section 9.1]} \approx 353.656$

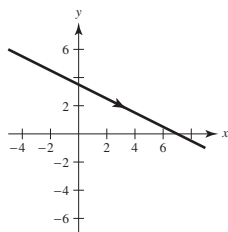
Total area  $= 24\pi + 353.656 \approx 429.054$

INSTRUCTOR USE ONLY

25.  $x = 1 + 8t, y = 3 - 4t$

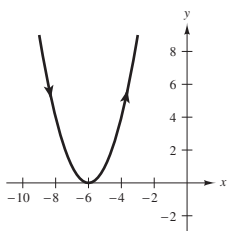
$$t = \frac{x-1}{8} \Rightarrow y = 3 - 4\left(\frac{x-1}{8}\right) = \frac{7}{2} - \frac{x}{2}$$

$$x + 2y - 7 = 0, \text{ Line}$$



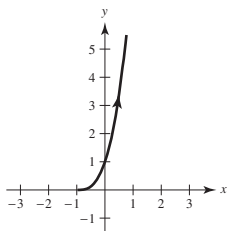
26.  $x = t - 6, y = t^2$

$$t = x + 6 \Rightarrow y = (x + 6)^2, \text{ Parabola}$$



27.  $x = e^t - 1, y = e^{3t}$

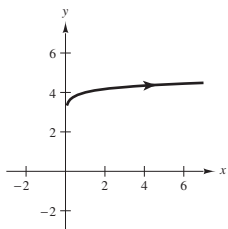
$$e^t = x + 1 \Rightarrow y = (x + 1)^3, x > -1$$



28.  $x = e^{4t}, y = t + 4$

$$t = y - 4 \Rightarrow x = e^{4y-16}$$

$$\text{or, } 4t = \ln x \Rightarrow y = \frac{\ln x}{4} + 4, x > 0$$

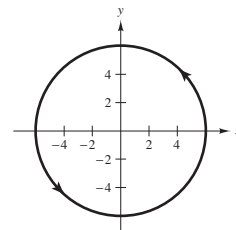


29.  $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

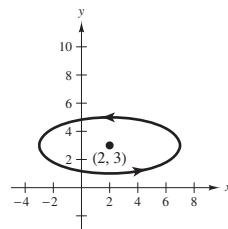
Circle



30.  $x = 2 + 5 \cos t, y = 3 + 2 \sin t$

$$\left(\frac{x-2}{5}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\frac{(x-2)^2}{25} + \frac{(y-3)^2}{4} = 1 \quad \text{Ellipse}$$

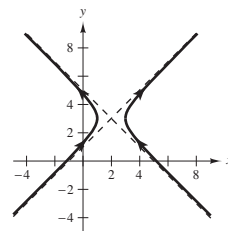


31.  $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

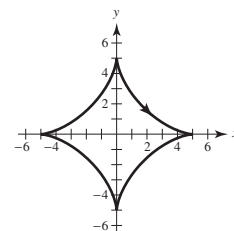
Hyperbola



32.  $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

$$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$$

$$x^{2/3} + y^{2/3} = 5^{2/3}$$



33.  $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

34.  $(x-h)^2 + (y-k)^2 = r^2$

$$(x+4)^2 + (y+5)^2 = 9$$

$$x = -4 + 3 \cos t$$

$$y = -5 + 3 \sin t$$

35.  $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

Let  $\frac{(x+3)^2}{16} = \cos^2 \theta$  and  $\frac{(y-4)^2}{9} = \sin^2 \theta$ .

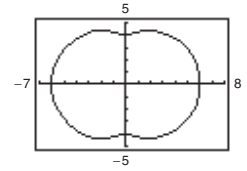
Then  $x = -3 + 4 \cos \theta$  and  $y = 4 + 3 \sin \theta$ .

36.  $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \frac{y^2}{16} - \frac{x^2}{9} = 1$

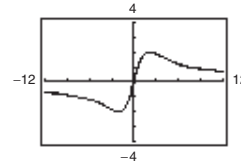
Let  $\frac{y^2}{16} = \sec^2 \theta$  and  $\frac{x^2}{9} = \tan^2 \theta$ .

Then  $x = 3 \tan \theta$  and  $y = 4 \sec \theta$ .

37.  $x = \cos 3\theta + 5 \cos \theta$   
 $y = \sin 3\theta + 5 \sin \theta$



38. (a)  $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



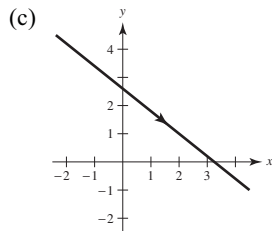
(b)  $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$   
 $= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$   
 $= 16 \frac{\cos \theta}{\sin \theta} = 8(2 \cot \theta) = 8x$

39.  $x = 2 + 5t, y = 1 - 4t$

(a)  $\frac{dy}{dx} = -\frac{4}{5}$

No horizontal tangent

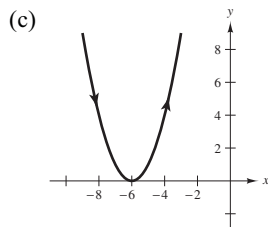
(b)  $t = \frac{x-2}{5} \Rightarrow y = 1 - 4\left(\frac{x-2}{5}\right) = 1 - \frac{4}{5}x + \frac{8}{5} = -\frac{4}{5}x + \frac{13}{5}$   
 $4x + 5y - 13 = 0$



40.  $x = t - 6, y = t^2$

(a)  $\frac{dy}{dx} = \frac{2t}{1} = 2t$

(b)  $t = x + 6 \Rightarrow y = (x + 6)^2$ , parabola

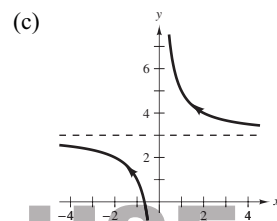


41.  $x = \frac{1}{t}$   
 $y = 2t + 3$

(a)  $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents, ( $t \neq 0$ )

(b)  $t = \frac{1}{x}$   
 $y = \frac{2}{x} + 3$

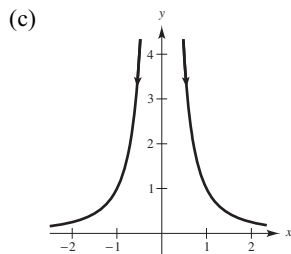


42.  $x = \frac{1}{t}$   
 $y = t^2$

(a)  $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents, ( $t \neq 0$ )

(b)  $t = \frac{1}{x}$   
 $y = \frac{1}{x^2}$



43.  $x = \frac{1}{2t+1}$   
 $y = \frac{1}{t^2 - 2t}$

(a)  $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$

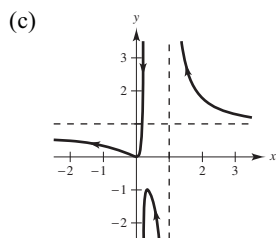
when  $t = 1$ .

Point of horizontal tangency:  $\left(\frac{1}{3}, -1\right)$

(b)  $2t+1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}, (x \neq 0)$$



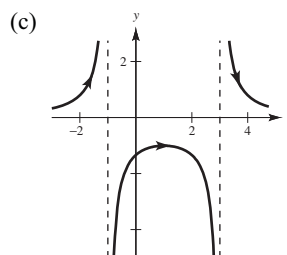
44.  $x = 2t - 1$   
 $y = \frac{1}{t^2 - 2t}$

(a)  $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$   
 $= \frac{1-t}{t^2(t-2)^2} = 0$  when  $t = 1$ .

Point of horizontal tangency:  $(1, -1)$

(b)  $t = \frac{x+1}{2}$

$$y = \frac{1}{\left[\frac{(x+1)}{2}\right]^2 - 2\left[\frac{(x+1)}{2}\right]} = \frac{4}{(x-3)(x+1)}$$



45.  $x = 5 + \cos \theta$ ,  $y = 3 + 4 \sin \theta$

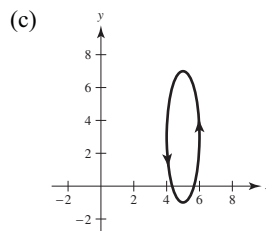
(a)  $\frac{dy}{dx} = \frac{4 \cos \theta}{-\sin \theta} = -4 \cot \theta$

Horizontal tangents:

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow (5, 7), (5, -1)$$

(b)  $(x-5)^2 + \left(\frac{(y-3)}{4}\right)^2 = 1$

$$(x-5)^2 + \frac{(y-3)^2}{16} = 1, \text{ Ellipse}$$



46.  $x = 10 \cos \theta, y = 10 \sin \theta$

(a)  $\frac{dy}{dx} = \frac{10 \cos \theta}{-10 \sin \theta} = -\cot \theta$

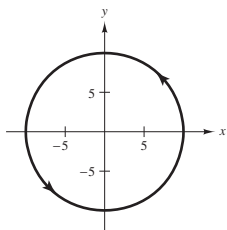
Horizontal tangents:

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow (0, 10), (0, -10)$$

(b)  $\left(\frac{x}{10}\right)^2 + \left(\frac{y}{10}\right)^2 = 1$

$$x^2 + y^2 = 100, \text{ Circle}$$

(c)



47.  $x = \cos^3 \theta$

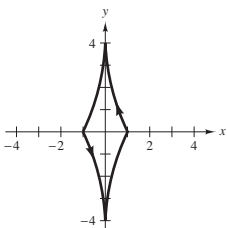
$$y = 4 \sin^3 \theta$$

(a)  $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$

when  $\theta = 0, \pi$ .But,  $\frac{dy}{dt} = \frac{dx}{dt} = 0$  at  $\theta = 0, \pi$ . So no points of horizontal tangency.

(b)  $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$

(c)



48.  $x = e^t$

$$y = e^{-t}$$

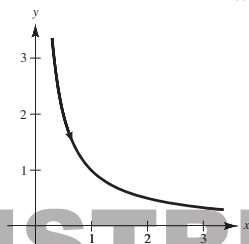
(a)  $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b)  $t = \ln x$

$$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$$

(c)



49.  $x = 5 - t, y = 2t^2$

$$\frac{dx}{dt} = -1, \frac{dy}{dt} = 4t$$

Horizontal tangent at  $t = 0$ :  $(5, 0)$ 

No vertical tangents

50.  $x = t + 2, y = t^3 - 2t$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 3t^2 - 2$$

$$\frac{dy}{dt} = 0 \text{ for } t = \pm \sqrt{\frac{2}{3}} = \frac{\pm \sqrt{6}}{3}$$

Horizontal tangents:

$$t = \frac{\sqrt{2}}{3}: (x, y) = \left(\frac{\sqrt{6}}{3} + 2, \frac{2\sqrt{6}}{9} - \frac{2}{3}\sqrt{6}\right) \approx (2.8165, -1.0887)$$

$$t = -\frac{\sqrt{6}}{3}: (x, y) = \left(-\frac{\sqrt{6}}{3} + 2, \frac{2}{3}\sqrt{6} - \frac{2\sqrt{6}}{9}\right) \approx (1.1835, 1.0887)$$

No vertical tangents

51.  $x = 2 + 2 \sin \theta, y = 1 + \cos \theta$

$$\frac{dx}{d\theta} = 2 \cos \theta, \frac{dy}{d\theta} = -\sin \theta$$

$$\frac{dy}{d\theta} = 0 \text{ for } \theta = 0, \pi, 2\pi, \dots$$

Horizontal tangents:  $(x, y) = (2, 2), (2, 0)$ 

$$\frac{dx}{d\theta} = 0 \text{ for } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

Vertical tangents:  $(x, y) = (4, 1), (0, 1)$ 

52.  $x = 2 - 2 \cos \theta, y = 2 \sin 2\theta$

$$\frac{dx}{d\theta} = 2 \sin \theta, \frac{dy}{d\theta} = 4 \cos 2\theta$$

$$\frac{dy}{d\theta} = 0 \text{ for } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots$$

Horizontal tangents:  $(x, y) = (2 \pm \sqrt{2}, 2), (2 \pm \sqrt{2}, -2)$ 

$$\frac{dx}{d\theta} = 0 \text{ for } \theta = 0, \pi, 2\pi, \dots$$

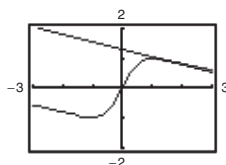
Vertical tangents:  $(x, y) = (0, 0), (4, 0)$



53.  $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)

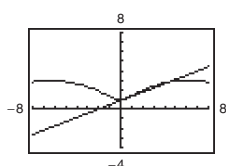


(b) At  $\theta = \frac{\pi}{6}$ ,  $\frac{dx}{d\theta} = -4$ ,  $\frac{dy}{d\theta} = 1$ , and  $\frac{dy}{dx} = -\frac{1}{4}$ .

54.  $x = 2\theta - \sin \theta$

$$y = 2 - \cos \theta$$

(a), (c)



(b) At  $\theta = \frac{\pi}{6}$ ,  $\frac{dx}{d\theta} \approx 1.134$ ,  $\left(2 - \frac{\sqrt{3}}{2}\right)$ ,

$$\frac{dy}{d\theta} = 0.5, \text{ and } \frac{dy}{dx} \approx 0.441.$$

57.  $x = t, y = 3t, 0 \leq t \leq 2$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 3, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+9} = \sqrt{10}$$

(a)  $S = 2\pi \int_0^2 3t\sqrt{10} dt = 6\sqrt{10} \pi \left[\frac{t^2}{2}\right]_0^2 = 12\sqrt{10} \pi \approx 119.215$

(b)  $S = 2\pi \int_0^2 \sqrt{10} dt = 2\pi [\sqrt{10}t]_0^2 = 4\pi\sqrt{10} \approx 39.738$

58.  $x = 2 \cos \theta, y = 2 \sin \theta, 0 \leq \theta \leq \frac{\pi}{2}$

$$\frac{dx}{d\theta} = -2 \sin \theta, \frac{dy}{d\theta} = 2 \cos \theta, \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2$$

(a)  $S = 2\pi \int_0^{\pi/2} 2 \sin \theta (2) d\theta = 8\pi [-\cos \theta]_0^{\pi/2} = 8\pi$

(b)  $S = 2\pi \int_0^{\pi/2} 2 \cos \theta (2) d\theta = 8\pi [\sin \theta]_0^{\pi/2} = 8\pi$

[Note: The surface is a hemisphere:  $\frac{1}{2}(4\pi(2^2)) = 8\pi$ ]

55.  $x = r(\cos \theta + \theta \sin \theta)$

$$y = r(\sin \theta - \theta \cos \theta)$$

$$\frac{dx}{d\theta} = r\theta \cos \theta$$

$$\frac{dy}{d\theta} = r\theta \sin \theta$$

$$s = r \int_0^\pi \sqrt{\theta^2 \cos^2 \theta + \theta^2 \sin^2 \theta} d\theta$$

$$= r \int_0^\pi \theta d\theta = \frac{r}{2} [\theta^2]_0^\pi = \frac{1}{2}\pi^2 r$$

56.  $x = 6 \cos \theta$

$$y = 6 \sin \theta$$

$$\frac{dx}{d\theta} = -6 \sin \theta$$

$$\frac{dy}{d\theta} = 6 \cos \theta$$

$$s = \int_0^\pi \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta} d\theta = [6\theta]_0^\pi = 6\pi$$

(one-half circumference of circle)

59.  $x = 3 \sin \theta, y = 2 \cos \theta$

$$A = \int_a^b y dx = \int_{-\pi/2}^{\pi/2} 2 \cos \theta (3 \cos \theta) d\theta$$

$$= 6 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= 3 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{-\pi/2}^{\pi/2}$$

$$= 3 \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = 3\pi$$

60.  $A = \int_a^b y dx = \int_\pi^0 \sin \theta (-2 \sin \theta) d\theta$

$$= - \int_\pi^0 \frac{1 - \cos 2\theta}{2} d\theta$$

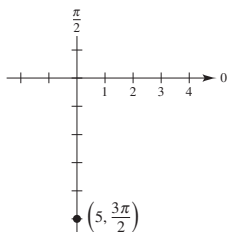
$$= - \left[ \theta - \frac{\sin 2\theta}{2} \right]_\pi^0 = \pi$$

61.  $(r, \theta) = \left(5, \frac{3\pi}{2}\right)$

$$x = r \cos \theta = 5 \cos \frac{3\pi}{2} = 0$$

$$y = r \sin \theta = 5 \sin \frac{3\pi}{2} = -5$$

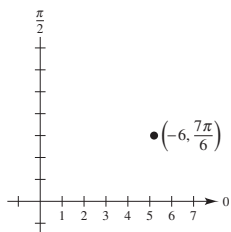
$$(x, y) = (0, -5)$$



62.  $(r, \theta) = \left(-6, \frac{7\pi}{6}\right)$

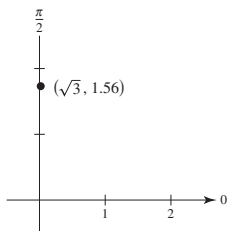
$$x = r \cos \theta = -6 \cos \frac{7\pi}{6} = (-6) \left(-\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

$$y = r \sin \theta = -6 \sin \frac{7\pi}{6} = 3$$



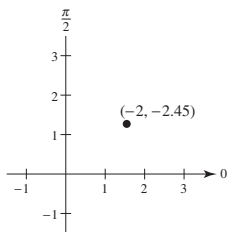
63.  $(r, \theta) = (\sqrt{3}, 1.56)$

$$(x, y) = (\sqrt{3} \cos(1.56), \sqrt{3} \sin(1.56))$$
$$\approx (0.0187, 1.7319)$$



64.  $(r, \theta) = (-2, -2.45)$

$$(x, y) = (-2 \cos(-2.45), -\sin(-2.45))$$
$$\approx (1.5405, 1.2755)$$

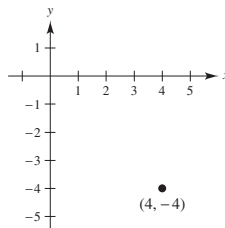


65.  $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = \frac{7\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$

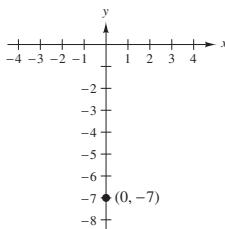


66.  $(x, y) = (0, -7)$

$$r = \sqrt{0^2 + (-7)^2} = 7$$

$$\tan \theta \text{ undefined} \Rightarrow \theta = \frac{3\pi}{2}$$

$$(r, \theta) = \left(7, \frac{3\pi}{2}\right), \left(-7, \frac{\pi}{2}\right)$$

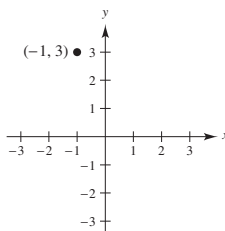


67.  $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 (108.43^\circ)$$

$$(r, \theta) = (\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$$

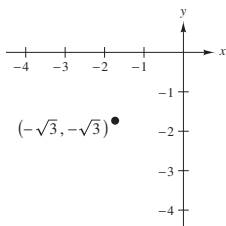


68.  $(x, y) = (-\sqrt{3}, -\sqrt{3})$

$$r = \sqrt{3+3} = \sqrt{6}$$

$$\tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$(r, \theta) = \left(\sqrt{6}, \frac{5\pi}{4}\right), \left(-\sqrt{6}, \frac{\pi}{4}\right)$$



69.  $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

70.  $r = 10$

$$r^2 = 100$$

$$x^2 + y^2 = 100$$

71.  $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm\sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

72.  $r = \frac{1}{2 - \cos \theta}$

$$2r - r \cos \theta = 1$$

$$2(\pm\sqrt{x^2 + y^2}) - x = 1$$

$$4(x^2 + y^2) = (x + 1)^2$$

$$3x^2 + 4y^2 - 2x - 1 = 0$$

73.  $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

74.  $r = 4 \sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos\left[\theta - \left(\frac{\pi}{3}\right)\right]}$

$$= \frac{4}{(1/2)\cos \theta + (\sqrt{3}/2)\sin \theta}$$

$$r(\cos \theta + \sqrt{3} \sin \theta) = 8$$

$$x + \sqrt{3}y = 8$$

75.  $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1)\left(\frac{1}{\cos \theta}\right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8\left(\frac{x^2}{x^2 + y^2}\right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2\left(\frac{4-x}{4+x}\right)$$

76.  $\theta = \frac{3\pi}{4}$

$$\tan \theta = -1$$

$$\frac{y}{x} = -1$$

$$y = -x$$

77.  $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

78.  $x^2 + y^2 - 4x = 0$

$$r^2 - 4r \cos \theta = 0$$

$$r = 4 \cos \theta$$

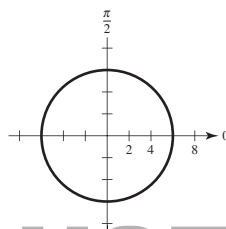
79.  $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x}\right)^2$

$$r^2 = a^2 \theta^2$$

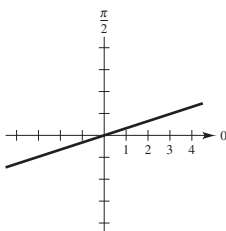
80.  $(x^2 + y^2) \left(\arctan \frac{y}{x}\right)^2 = a^2$

$$r^2 \theta^2 = a^2$$

81.  $r = 6$ , Circle radius 6



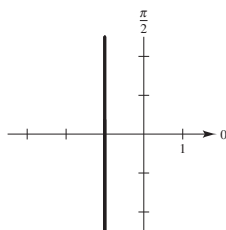
82.  $\theta = \frac{\pi}{10}$ , Line



83.  $r = -\sec \theta = \frac{-1}{\cos \theta}$

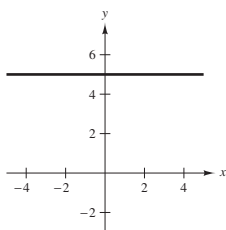
$r \cos \theta = -1, x = -1$

Vertical line



84.  $r = 5 \csc \theta \Rightarrow r \sin \theta = y = 5$

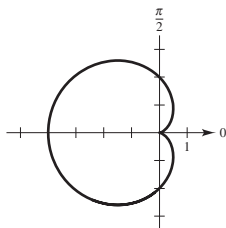
Horizontal line



85.  $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

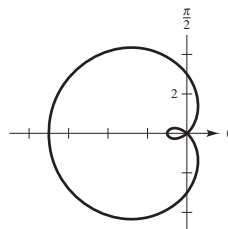


$\theta$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$
$r$	-4	-3	-2	-1	0

86.  $r = 3 - 4 \cos \theta$

Limaçon

Symmetric to polar axis

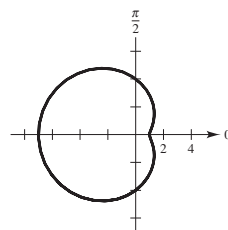


$\theta$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$
$r$	-1	1	3	5	7

87.  $r = 4 - 3 \cos \theta$

Limaçon

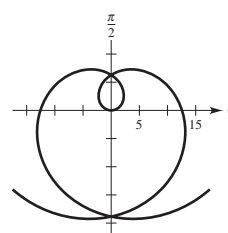
Symmetric to polar axis



$\theta$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\pi$
$r$	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

88.  $r = 4\theta$

Spiral

Symmetric to  $\theta = \frac{\pi}{2}$ 

$\theta$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$r$	0	$\pi$	$2\pi$	$3\pi$	$4\pi$	$6\pi$	$8\pi$

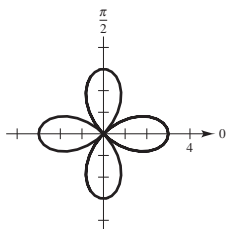
89.  $r = -3 \cos 2\theta$

Rose curve with four petals

Symmetric to polar axis,  $\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $(-3, 0)$ ,  $(3, \frac{\pi}{2})$ ,  $(-3, \pi)$ ,  $(3, \frac{3\pi}{2})$

Tangents at the pole:  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$



90.  $r = \cos 5\theta$

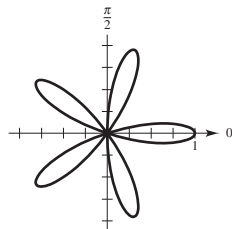
Rose curve with five petals

Symmetric to polar axis

Relative extrema:

$(1, 0)$ ,  $(-1, \frac{\pi}{5})$ ,  $(1, \frac{2\pi}{5})$ ,  $(-1, \frac{3\pi}{5})$ ,  $(1, \frac{4\pi}{5})$

Tangents at the pole:  $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$



91.  $r^2 = 4 \sin^2 2\theta$

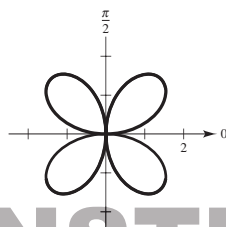
$r = \pm 2 \sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis,  $\theta = \frac{\pi}{2}$ , and pole

Relative extrema:  $(\pm 2, \frac{\pi}{4})$ ,  $(\pm 2, \frac{3\pi}{4})$

Tangents at the pole:  $\theta = 0, \frac{\pi}{2}$



92.  $r^2 = \cos 2\theta$

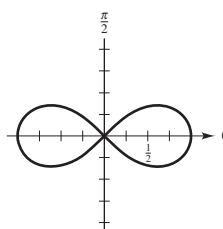
Lemniscate

Symmetric to the polar axis

Relative extrema:  $(\pm 1, 0)$

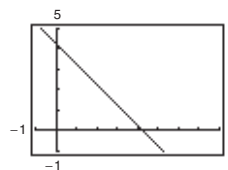
Tangents at the pole:  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$r$	$\pm 1$	$\pm \frac{\sqrt{2}}{2}$	0



93.  $r = \frac{3}{\cos \theta - (\pi/4)}$

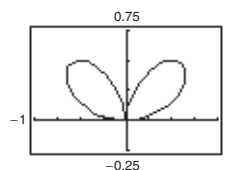
Graph of  $r = 3 \sec \theta$  rotated through an angle of  $\pi/4$



94.  $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to  $\theta = \pi/2$



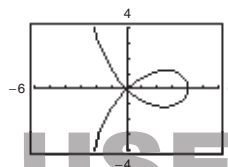
95.  $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow \infty$  as  $\theta \Rightarrow \frac{\pi}{2}$

$r \Rightarrow \infty$  as  $\theta \Rightarrow \frac{-\pi}{2}$



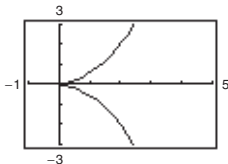
96.  $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



97.  $r = 1 - 2 \cos \theta$

- (a) The graph has polar symmetry and the tangents at the pole are  $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$ .

(b)  $\frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$

Horizontal tangents:  $-4 \cos^2 \theta + \cos \theta + 2 = 0$ ,

$$\cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

When

$$\cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left( \frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4}$$

$$\left[ \frac{3 - \sqrt{33}}{4}, \arccos \left( \frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[ \frac{3 - \sqrt{33}}{4}, -\arccos \left( \frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[ \frac{3 + \sqrt{33}}{4}, \arccos \left( \frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

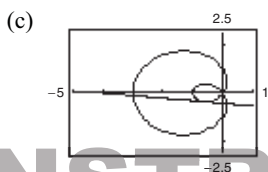
$$\left[ \frac{3 + \sqrt{33}}{4}, -\arccos \left( \frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206)$$

Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4}$$

$$\theta = 0, \pi, \theta = \pm \arccos \left( \frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left( \frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$



98.  $r^2 = 4 \sin(2\theta)$

(a)  $2r \left( \frac{dr}{d\theta} \right) = 8 \cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4 \cos(2\theta)}{r}$$

Tangents at the pole:  $\theta = 0, \frac{\pi}{2}$

(b)  $\frac{dy}{dx} = \frac{r \cos \theta + [(4 \cos 2\theta \sin \theta)/r]}{-r \sin \theta + [(4 \cos 2\theta \cos \theta)/r]}$   

$$= \frac{\cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta}{\cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta}$$

Horizontal tangents:

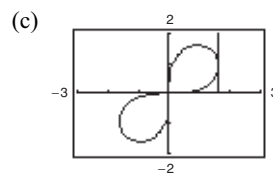
$$\frac{dy}{dx} = 0 \text{ when } \cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta = 0,$$

$$\tan \theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left( \pm \sqrt{2\sqrt{3}}, \frac{\pi}{3} \right)$$

Vertical tangents when

$$\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0:$$

$$\tan 2\theta \tan \theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left( \pm \sqrt{2\sqrt{3}}, \frac{\pi}{6} \right)$$



99.  $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points  $(1, \pi/2)$  and  $(1, 3\pi/2)$  are the two points of intersection (other than the pole). The slope of the graph of  $r = 1 + \cos \theta$  is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta (1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta (1 + \cos \theta)}$$

At  $(1, \pi/2)$ ,  $m_1 = -1/-1 = 1$  and at  $(1, 3\pi/2)$ ,

$m_1 = -1/1 = -1$ . The slope of the graph of  $r = 1 - \cos \theta$  is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta (1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta (1 - \cos \theta)}$$

At  $(1, \pi/2)$ ,  $m_2 = 1/-1 = -1$  and at  $(1, 3\pi/2)$ ,

$m_2 = 1/1 = 1$ . In both cases,  $m_1 = -1/m_2$  and you conclude that the graphs are orthogonal at  $(1, \pi/2)$  and  $(1, 3\pi/2)$ .

100.  $r = a \sin \theta, r = a \cos \theta$

The points of intersection are  $(a/\sqrt{2}, \pi/4)$  and  $(0, 0)$ . For  $r = a \sin \theta$ ,

$$m_1 = \frac{dy}{dx} = \frac{a \cos \theta \sin \theta + a \sin \theta \cos \theta}{a \cos^2 \theta - a \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta}.$$

At  $(a/\sqrt{2}, \pi/4)$ ,  $m_1$  is undefined and at  $(0, 0)$ ,  $m_1 = 0$ .

For  $r = a \cos \theta$ ,

$$m_2 = \frac{dy}{dx} = \frac{-a \sin^2 \theta + a \cos^2 \theta}{-a \sin \theta \cos \theta - a \cos \theta \sin \theta} = \frac{\cos 2\theta}{-2 \sin \theta \cos \theta}.$$

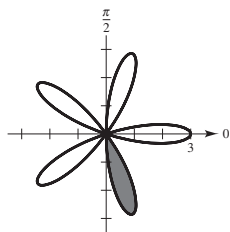
At  $(a/\sqrt{2}, \pi/4)$ ,  $m_2 = 0$  and at  $(0, 0)$ ,  $m_2$  is undefined.

So, the graphs are orthogonal at  $(a/\sqrt{2}, \pi/4)$  and  $(0, 0)$ .

101.  $A = 2 \cdot \frac{1}{2} \int_0^{\pi/10} [3 \cos 5\theta]^2 d\theta$

$$= \int_0^{\pi/10} 9 \left( \frac{1 + \cos(10\theta)}{2} \right) d\theta$$

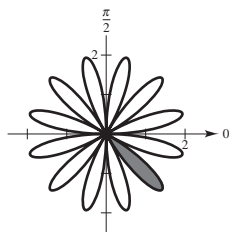
$$= \frac{9}{2} \left[ \theta + \frac{\sin(10\theta)}{10} \right]_0^{\pi/10} = \frac{9}{2} \left[ \frac{\pi}{10} \right] = \frac{9\pi}{20}$$



102.  $A = 2 \cdot \frac{1}{2} \int_0^{\pi/12} [2 \sin 6\theta]^2 d\theta$

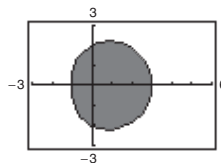
$$= \int_0^{\pi/12} 4 \left( \frac{1 - \cos 12\theta}{2} \right) d\theta$$

$$= 2 \left[ \theta - \frac{\sin 12\theta}{12} \right]_0^{\pi/12} = 2 \left[ \frac{\pi}{12} \right] = \frac{\pi}{6}$$



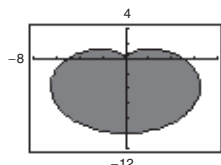
103.  $r = 2 + \cos \theta$

$$A = 2 \left[ \frac{1}{2} \int_0^{\pi} (2 + \cos \theta)^2 d\theta \right] \approx 14.14, \left( \frac{9\pi}{2} \right)$$



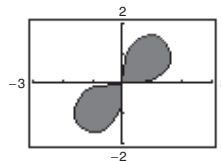
104.  $r = 5(1 - \sin \theta)$

$$A = 2 \left[ \frac{1}{2} \int_{\pi/2}^{3\pi/2} [5(1 - \sin \theta)]^2 d\theta \right] \approx 117.81, \left( \frac{75\pi}{2} \right)$$



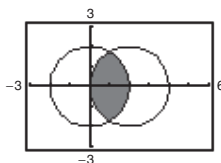
105.  $r^2 = 4 \sin 2\theta$

$$A = 2 \left[ \frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



106.  $r = 4 \cos \theta, r = 2$

$$A = 2 \left[ \frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$



107.  $r = 1 - \cos \theta$

$$r = 1 + \sin \theta$$

The cardioids intersect at 3 points:

$$1 - \cos \theta = 1 + \sin \theta$$

$$\tan \theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

$$\left( 1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4} \right), \left( 1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4} \right), (0, 0)$$

108. The circle  $r = 3 \sin \theta$  and cardioid

$$r = 1 + \sin \theta$$
 intersect at 3 points:

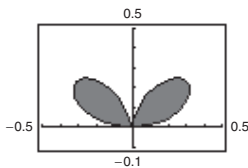
$$3 \sin \theta = 1 + \sin \theta$$

$$\sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

$$\left( \frac{3}{2}, \frac{\pi}{6} \right), \left( \frac{3}{2}, \frac{5\pi}{6} \right), (0, 0)$$

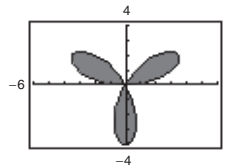
109.  $r = \sin \theta \cos^2 \theta$

$$A = 2 \left[ \frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10, \left( \frac{\pi}{32} \right)$$



110.  $r = 4 \sin 3\theta$

$$A = 3 \left[ \frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta \right] \approx 12.57 (4\pi)$$



111.  $r = 3, r^2 = 18 \sin 2\theta$

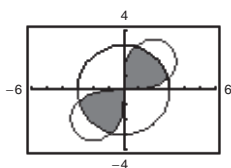
$$9 = r^2 = 18 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{12}$$

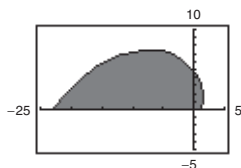
$$A = 2 \left[ \frac{1}{2} \int_0^{\pi/12} 18 \sin 2\theta d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} 9 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} 18 \sin 2\theta d\theta \right]$$

$$\approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$



112.  $r = e^\theta, 0 \leq \theta \leq \pi$

$$A = \frac{1}{2} \int_0^\pi (e^\theta)^2 d\theta \approx 133.62$$



114.  $r = a \cos 2\theta, -\pi/4 \leq \theta \leq \pi/2$

$$\frac{dr}{d\theta} = -2a \sin 2\theta$$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 \cos^2 2\theta + 4a^2 \sin^2 2\theta} d\theta$$

$$= a \int_{-\pi/2}^{\pi/2} \sqrt{1 + 3 \sin^2 2\theta} d\theta$$

Using a graphing utility,  $s \approx 4.8442a$ .

113.  $r = a(1 - \cos \theta), 0 \leq \theta \leq \pi$

$$\frac{dr}{d\theta} = a \sin \theta$$

$$s = \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$$

$$= \sqrt{2a} \int_0^\pi \sqrt{1 - \cos \theta} d\theta$$

$$= \sqrt{2a} \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta$$

$$= -2\sqrt{2a} \left[ (1 + \cos \theta)^{1/2} \right]_0^\pi = 4a$$

115.  $f(\theta) = 1 + 4 \cos \theta$

$$f'(\theta) = -4 \sin \theta$$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{(1 + 4 \cos \theta)^2 + (-4 \sin \theta)^2}$$

$$= \sqrt{17 + 8 \cos \theta}$$

$$S = 2\pi \int_0^{\pi/2} (1 + 4 \cos \theta) \sin \theta \sqrt{17 + 8 \cos \theta} d\theta$$

$$= \frac{34\pi\sqrt{17}}{5} \approx 88.08$$

116.  $f(\theta) = 2 \sin \theta$

$$f'(\theta) = 2 \cos \theta$$

$$\sqrt{f(\theta)^2 + f'(\theta)^2} = \sqrt{4 \sin^2 \theta + 4 \cos^2 \theta} = 2$$

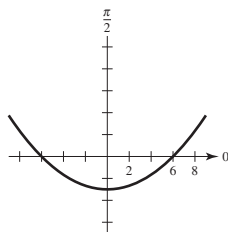
$$S = 2\pi \int_0^{\pi/2} 2 \sin \theta \cos \theta (2) d\theta = 4\pi$$



$$117. r = \frac{6}{1 - \sin \theta}$$

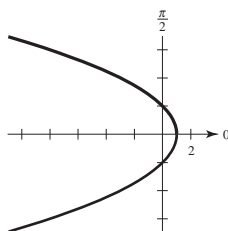
$$e = 1,$$

Parabola



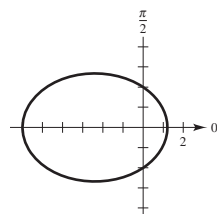
$$118. r = \frac{2}{1 + \cos \theta}, e = 1$$

Parabola



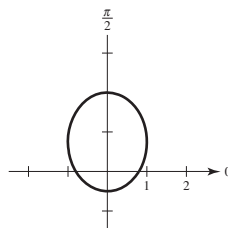
$$119. r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$$

Ellipse



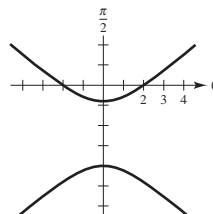
$$120. r = \frac{4}{5 - 3 \sin \theta} = \frac{4/5}{1 - (3/5) \sin \theta}, e = \frac{3}{5}$$

Ellipse



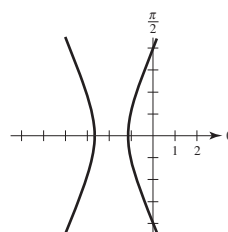
$$121. r = \frac{4}{2 - 3 \sin \theta} = \frac{2}{1 - (3/2) \sin \theta}, e = \frac{3}{2}$$

Hyperbola



$$122. r = \frac{8}{2 - 5 \cos \theta} = \frac{4}{1 - (5/2) \cos \theta}, e = \frac{5}{2}$$

Hyperbola



123. Circle

Center:  $\left(5, \frac{\pi}{2}\right) = (0, 5)$  in rectangular coordinates

Solution point:  $(0, 0)$

$$x^2 + (y - 5)^2 = 25$$

$$x^2 + y^2 - 10y = 0$$

$$r^2 - 10r \sin \theta = 0$$

$$r = 10 \sin \theta$$

124. Line

Slope:  $\sqrt{3}$

Solution point:  $(0, 0)$

$$y = \sqrt{3}x, r \sin \theta = \sqrt{3} r \cos \theta,$$

$$\tan \theta = \sqrt{3}, \theta = \frac{\pi}{3}$$

125. Parabola

Vertex:  $(2, \pi)$

Focus:  $(0, 0)$

$$e = 1, d = 4$$

$$r = \frac{4}{1 - \cos \theta}$$

## 126. Parabola

$$\text{Vertex: } \left(2, \frac{\pi}{2}\right)$$

$$\text{Focus: } (0, 0)$$

$$e = 1, d = 4$$

$$r = \frac{4}{1 + \sin \theta}$$

## 127. Ellipse

$$\text{Vertices: } (5, 0), (1, \pi)$$

$$\text{Focus: } (0, 0)$$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right)\cos \theta} = \frac{5}{3 - 2\cos \theta}$$

## 128. Hyperbola

$$\text{Vertices: } (1, 0), (7, 0)$$

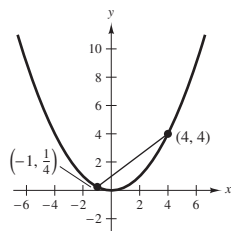
$$\text{Focus: } (0, 0)$$

$$a = 3, c = 4, e = \frac{4}{3}, d = \frac{7}{4}$$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right)\cos \theta} = \frac{7}{3 + 4\cos \theta}$$

## Problem Solving for Chapter 10

1. (a)



$$(b) \quad x^2 = 4y$$

$$2x = 4y'$$

$$y' = \frac{1}{2}x$$

$$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4 \quad \text{Tangent line at } (4, 4)$$

$$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4} \quad \text{Tangent line at } \left(-1, \frac{1}{4}\right)$$

Tangent lines have slopes of 2 and  $-\frac{1}{2} \Rightarrow$  perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$

$$8x - 16 = -2x - 1$$

$$10x = 15$$

$$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$$

Point of intersection,  $\left(\frac{3}{2}, -1\right)$ , is on directrix  $y = -1$ .

2. Assume  $p > 0$ .

Let  $y = mx + p$  be the equation of the focal chord.

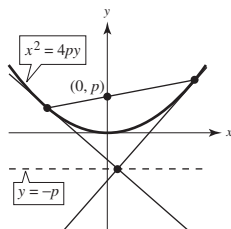
First find  $x$ -coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \Rightarrow y' = \frac{x}{2p}.$$



(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p} \left[ 2pm + 2p\sqrt{m^2 + 1} \right] \frac{1}{2p} \left[ 2pm - 2p\sqrt{m^2 + 1} \right] = \frac{1}{4p^2} \left[ 4p^2m^2 - 4p^2(m^2 + 1) \right] = \frac{1}{4p^2} [-4p^2] = -1$$

(b) Finally, you show that the tangent lines intersect at a point on the directrix  $y = -p$ .

$$\text{Let } b = 2pm + 2p\sqrt{m^2 + 1} \text{ and } c = 2pm - 2p\sqrt{m^2 + 1}.$$

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

$$\text{Tangent line at } x = b: y - \frac{b^2}{4p} = \frac{b}{2p}(x - b) \Rightarrow y = \frac{bx}{2p} - \frac{b^2}{4p}$$

$$\text{Tangent line at } x = c: y - \frac{c^2}{4p} = \frac{c}{2p}(x - c) \Rightarrow y = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$\text{Intersection of tangent lines: } \frac{bx}{2p} - \frac{b^2}{4p} = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$2bx - b^2 = 2cx - c^2$$

$$2x(b - c) = b^2 - c^2$$

$$2x(4p\sqrt{m^2 + 1}) = 16p^2m\sqrt{m^2 + 1}$$

$$x = 2pm$$

Finally, the corresponding  $y$ -value is  $y = -p$ , which shows that the intersection point lies on the directrix.

3. Consider  $x^2 = 4py$  with focus  $F = (0, p)$ .

Let  $P(a, b)$  be point on parabola.

$$2x = 4py' \Rightarrow y' = \frac{x}{2p}$$

$$y - b = \frac{a}{2p}(x - a) \quad \text{Tangent line at } P$$

$$\text{For } x = 0, y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b.$$

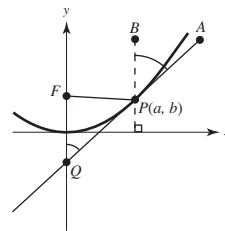
So,  $Q = (0, -b)$ .

$\triangle FQP$  is isosceles because

$$|FQ| = p + b$$

$$|FP| = \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} = \sqrt{4pb + b^2 - 2bp + p^2} = \sqrt{(b+p)^2} = b + p.$$

So,  $\angle FQP = \angle BPA = \angle FPQ$ .



4.  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a^2 + b^2 = c^2, MF_2 - MF_1 = 2a$

$$y' = \frac{b^2x}{a^2y}$$

$$\text{Tangent line at } M(x_0, y_0): y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0x - x_0^2}{a^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

$$\text{At } x = 0, y = -\frac{b^2}{y_0} \Rightarrow Q = \left(0, -\frac{b^2}{y_0}\right).$$

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^2}} = d$$

$$MQ = \sqrt{x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$

By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ) \cos \alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2) \cos \alpha$$

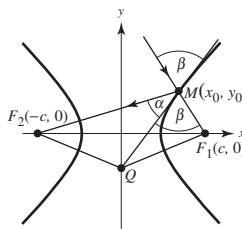
$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1) \cos \beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1) \cos \beta.$$

$$\cos \alpha = \frac{(MF_2)^2 f^2 - d^2}{2f(MF_2)}, \cos \beta = \frac{(MF_1)^2 + f^2 - d^2}{2f(MF_1)}$$

$$MF_2 = MF_1 + 2a. \text{ Let } z = MF_1.$$

$$\text{Slopes: } MF_1: \frac{y_0}{x_0 - c}; QF_1: \frac{-b^2}{y_0 c}; QF_2: \frac{b^2}{y_0 c}$$



To show  $\alpha = \beta$ , consider

$$\begin{aligned} & [(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)] \\ \Leftrightarrow & [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a] \\ \Leftrightarrow & z^2 + 2az = f^2 - d^2 \\ \Leftrightarrow & (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right) \\ \Leftrightarrow & az - x_0c + a^2 = 0 \\ \Leftrightarrow & a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2 \\ \Leftrightarrow & x_0^2b^2 - a^2y_0^2 = a^2b^2 \\ \Leftrightarrow & \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1. \end{aligned}$$

So,  $\alpha = \beta$  and the reflective property is verified.

5. (a) In  $\triangle OCB$ ,  $\cos \theta = \frac{2a}{OB} \Rightarrow OB = 2a \cdot \sec \theta$ .

In  $\triangle OAC$ ,  $\cos \theta = \frac{OA}{2a} \Rightarrow OA = 2a \cdot \cos \theta$ .

$$r = OP = AB = OB - OA = 2a(\sec \theta - \cos \theta)$$

$$= 2a\left(\frac{1}{\cos \theta} - \cos \theta\right)$$

$$= 2a \cdot \frac{\sin^2 \theta}{\cos \theta}$$

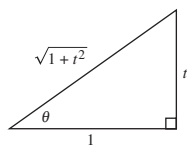
$$= 2a \cdot \tan \theta \sin \theta$$

(b)  $x = r \cos \theta = (2a \tan \theta \sin \theta) \cos \theta = 2a \sin^2 \theta$

$$y = r \sin \theta = (2a \tan \theta \sin \theta) \sin \theta = 2a \tan \theta \cdot \sin^2 \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Let  $t = \tan \theta$ ,  $-\infty < t < \infty$ .

Then  $\sin^2 \theta = \frac{t^2}{1 + t^2}$  and  $x = 2a \frac{t^2}{1 + t^2}$ ,  $y = 2a \frac{t^3}{1 + t^2}$ .



(c)  $r = 2a \tan \theta \sin \theta$

$$r \cos \theta = 2a \sin^2 \theta$$

$$r^3 \cos \theta = 2a r^2 \sin^2 \theta$$

$$(x^2 + y^2)x = 2ay^2$$

$$y^2 = \frac{x^3}{(2a - x)}$$

$$6. (a) \quad A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx = \frac{4b}{a} \left( \frac{1}{2} \right) \left[ x \sqrt{a^2 - x^2} + a^2 \arcsin \left( \frac{x}{a} \right) \right]_0^a = \pi ab$$

$$(b) \quad \text{Disk: } V = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) \, dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) \, dy = \frac{2\pi a^2}{b^2} \left[ b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{4}{3} \pi a^2 b$$

$$\begin{aligned} S &= 4\pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \left( \frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b \sqrt{b^2 - y^2}} \right) dy \\ &= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + c^2 y^2} \, dy = \frac{2\pi a}{b^2 c} \left[ cy \sqrt{b^4 + c^2 y^2} + b^4 \ln \left| cy + \sqrt{b^4 + c^2 y^2} \right| \right]_0^b \\ &= \frac{2\pi a}{b^2 c} \left[ b^2 c \sqrt{b^2 + c^2} + b^4 \ln \left| cb + b \sqrt{b^2 + c^2} \right| - b^4 \ln(b^2) \right] \\ &= 2\pi a^2 + \frac{\pi ab^2}{c} \ln \left( \frac{c + a}{e} \right)^2 = 2\pi a^2 + \left( \frac{\pi b^2}{e} \right) \ln \left( \frac{1 + e}{1 - e} \right) \end{aligned}$$

$$(c) \quad \text{Disk: } V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) \, dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) \, dx = \frac{2\pi b^2}{a^2} \left[ a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi ab^2$$

$$\begin{aligned} S &= 2(2\pi) \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \left( \frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} \right) dx \\ &= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - c^2 x^2} \, dx = \frac{2\pi b}{a^2 c} \left[ cx \sqrt{a^4 - c^2 x^2} + a^4 \arcsin \left( \frac{cx}{a^2} \right) \right]_0^a \\ &= \frac{a\pi b}{a^2 c} \left[ a^2 c \sqrt{a^2 - c^2} + a^4 \arcsin \left( \frac{c}{a} \right) \right] = 2\pi b^2 + 2\pi \left( \frac{ab}{e} \right) \arcsin(e) \end{aligned}$$

$$7. (a) \quad y^2 = \frac{t^2(1-t^2)^2}{(1+t^2)^2}, \quad x^2 = \frac{(1-t^2)^2}{(1+t^2)^2}$$

$$\frac{1-x}{1+x} = \frac{1 - \left( \frac{1-t^2}{1+t^2} \right)}{1 + \left( \frac{1-t^2}{1+t^2} \right)} = \frac{2t^2}{2} = t^2$$

$$\text{So, } y^2 = x^2 \left( \frac{1-x}{1+x} \right).$$

$$(b) \quad r^2 \sin^2 \theta = r^2 \cos^2 \theta \left( \frac{1 - r \cos \theta}{1 + r \cos \theta} \right)$$

$$\sin^2 \theta (1 + r \cos \theta) = \cos^2 \theta (1 - r \cos \theta)$$

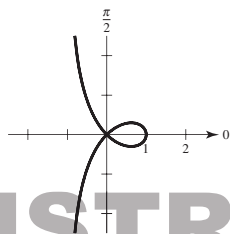
$$r \cos \theta \sin^2 \theta + \sin^2 \theta = \cos^2 \theta - r \cos^3 \theta$$

$$r \cos \theta (\sin^2 \theta + \cos^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$r \cos \theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec \theta$$

(c)



(d)  $r(\theta) = 0$  for  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ .

So,  $y = x$  and  $y = -x$  are tangent lines to curve at the origin.

(e)  $y'(t) = \frac{(1+t^2)(1-3t^2) - (t-t^3)(2t)}{(1+t^2)^2} = \frac{1-4t^2-t^4}{(1+t^2)^2} = 0$

$$t^4 + 4t^2 - 1 = 0 \Rightarrow t^2 = -2 \pm \sqrt{5} \Rightarrow x = \frac{1 - (-2 \pm \sqrt{5})}{1 + (-2 \pm \sqrt{5})} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}} = \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}$$

$$\left( \frac{\sqrt{5} - 1}{2}, \pm \frac{\sqrt{5} - 1}{2} \sqrt{-2 + \sqrt{5}} \right)$$

8.  $y = a(1 - \cos \theta) \Rightarrow \cos \theta = \frac{a - y}{a}$

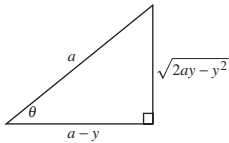
$$\theta = \arccos\left(\frac{a - y}{a}\right)$$

$$x = a(\theta - \sin \theta)$$

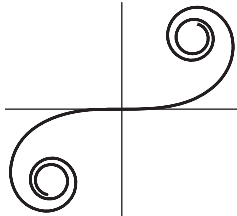
$$= a\left(\arccos\left(\frac{a - y}{a}\right) - \sin\left(\arccos\left(\frac{a - y}{a}\right)\right)\right)$$

$$= a\left(\arccos\left(\frac{a - y}{a}\right) - \frac{\sqrt{2ay - y^2}}{a}\right)$$

$$x = a \cdot \arccos\left(\frac{a - y}{a}\right) - \sqrt{2ay - y^2}, 0 \leq y \leq 2a$$



9. (a)



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(b)  $(-x, -y) = \left(-\int_0^t \cos \frac{\pi u^2}{2} du, -\int_0^t \sin \frac{\pi u^2}{2} du\right)$  is

on the curve whenever  $(x, y)$  is on the curve.

(c)  $x'(t) = \cos \frac{\pi t^2}{2}, y'(t) = \sin \frac{\pi t^2}{2},$

$$x'(t)^2 + y'(t)^2 = 1$$

$$\text{So, } s = \int_0^a dt = a.$$

$$\text{On } [-\pi, \pi], s = 2\pi.$$

10. For  $t = \frac{\pi}{2}, \frac{3}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$   
 $y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \dots$

So, the curve has length greater than

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \dots \\ &= \frac{2}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right) \\ &> \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots\right) \\ &= \infty. \text{ (Harmonic series)} \end{aligned}$$

11.  $r = \frac{ab}{a \sin \theta + b \cos \theta}, 0 \leq \theta \leq \frac{\pi}{2}$

$$r(a \sin \theta + b \cos \theta) = ab$$

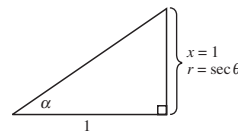
$$ay + bx = ab$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

$$\text{Area} = \frac{1}{2}ab$$

12. (a)  $\text{Area} = \int_0^\alpha \frac{1}{2} r^2 d\theta$   
 $= \frac{1}{2} \int_0^\alpha \sec^2 \theta d\theta$



(b)  $\tan \alpha = \frac{h}{1} \Rightarrow \text{Area} = \frac{1}{2}(1) \tan \alpha$

$$\Rightarrow \tan \alpha = \int_0^\alpha \sec^2 \theta d\theta$$

(c) Differentiating,  $\frac{d}{d\alpha}(\tan \alpha) = \sec^2 \alpha.$

13. Let
- $(r, \theta)$
- be on the graph.

$$\sqrt{r^2 + 1 + 2r \cos \theta} \sqrt{r^2 + 1 - 2r \cos \theta} = 1$$

$$(r^2 + 1)^2 - 4r^2 \cos^2 \theta = 1$$

$$r^4 + 2r^2 + 1 - 4r^2 \cos^2 \theta = 1$$

$$r^2(r^2 - 4 \cos^2 \theta + 2) = 0$$

$$r^2 = 4 \cos^2 \theta - 2$$

$$r^2 = 2(2 \cos^2 \theta - 1)$$

$$r^2 = 2 \cos 2\theta$$

14. If a dog is located at
- $(r, \theta)$
- in the first quadrant, then its

neighbor is at  $\left(r, \theta + \frac{\pi}{2}\right)$ :

$$(x_1, y_1) = (r \cos \theta, r \sin \theta) \text{ and}$$

$$(x_2, y_2) = (-r \sin \theta, r \cos \theta).$$

The slope joining these points is

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

= slope of tangent line at  $(r, \theta)$ .

$$\frac{dy}{dx} = \frac{\frac{dy}{dr}}{\frac{dx}{dr}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r$$

$$\frac{dr}{r} = -d\theta$$

$$\ln r = -\theta + C_1$$

$$r = e^{-\theta + C_1}$$

$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}} e^{\pi/4}$$

$$\text{Finally, } r = \frac{d}{\sqrt{2}} e^{((\pi/4) - \theta)}, \theta \geq \frac{\pi}{4}.$$

15. (a) The first plane makes an angle of
- $70^\circ$
- with the positive
- $x$
- axis, and is 150 miles from P:

$$x_1 = \cos 70^\circ(150 - 375t)$$

$$y_1 = \sin 70^\circ(150 - 375t)$$

Similarly for the second plane,

$$x_2 = \cos 135^\circ(190 - 450t)$$

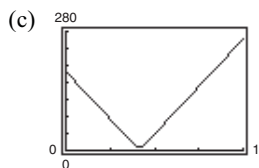
$$= \cos 45^\circ(-190 + 450t)$$

$$y_2 = \sin 135^\circ(190 - 450t)$$

$$= \sin 45^\circ(190 - 450t).$$

$$(b) d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \left[ [\cos 45^\circ(-190 + 450t) - \cos 70^\circ(150 - 375t)]^2 + [\sin 45^\circ(190 - 450t) - \sin 70^\circ(150 - 375t)]^2 \right]^{1/2}$$

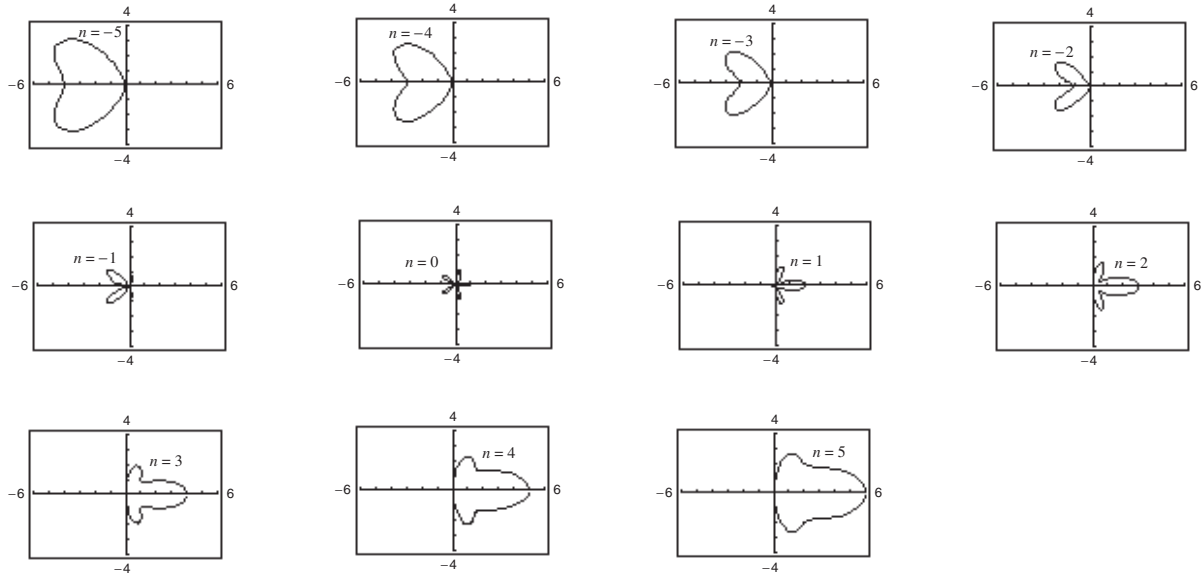


The minimum distance is 7.59 miles when  $t = 0.4145$ ; Yes.

16. The curve is produced over the interval
- $0 \leq \theta \leq 10\pi$
- .



17.



$n = 1, 2, 3, 4, 5$  produce "bells";  $n = -1, -2, -3, -4, -5$  produce "hearts".

**NOT FOR SALE**

**C H A P T E R 11**

**Vectors and the Geometry of Space**

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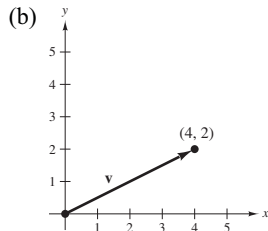
**INSTRUCTOR USE ONLY**

## CHAPTER 11

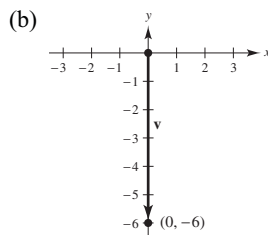
### Vectors and Geometry of Space

#### Section 11.1 Vectors in the Plane

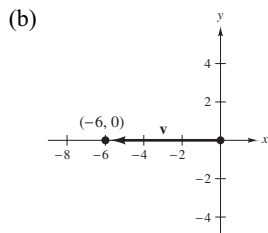
1. (a)  $\mathbf{v} = \langle 5 - 1, 4 - 2 \rangle = \langle 4, 2 \rangle$



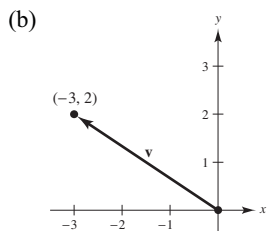
2. (a)  $\mathbf{v} = \langle 3 - 3, -2 - 4 \rangle = \langle 0, -6 \rangle$



3. (a)  $\mathbf{v} = \langle -4 - 2, -3 - (-3) \rangle = \langle -6, 0 \rangle$



4. (a)  $\mathbf{v} = \langle -1 - 2, 3 - 1 \rangle = \langle -3, 2 \rangle$



5.  $\mathbf{u} = \langle 5 - 3, 6 - 2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 3 - 1, 8 - 4 \rangle = \langle 2, 4 \rangle$

$\mathbf{u} = \mathbf{v}$

6.  $\mathbf{u} = \langle 1 - (-4), 8 - 0 \rangle = \langle 5, 8 \rangle$

$\mathbf{v} = \langle 7 - 2, 7 - (-1) \rangle = \langle 5, 8 \rangle$

$\mathbf{u} = \mathbf{v}$

7.  $\mathbf{u} = \langle 6 - 0, -2 - 3 \rangle = \langle 6, -5 \rangle$

$\mathbf{v} = \langle 9 - 3, 5 - 10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

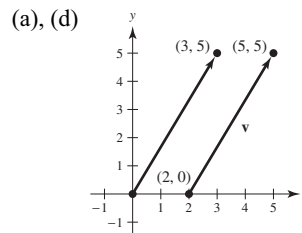
8.  $\mathbf{u} = \langle 11 - (-4), -4 - (-1) \rangle = \langle 15, -3 \rangle$

$\mathbf{v} = \langle 25 - 0, 10 - 13 \rangle = \langle 15, -3 \rangle$

$\mathbf{u} = \mathbf{v}$

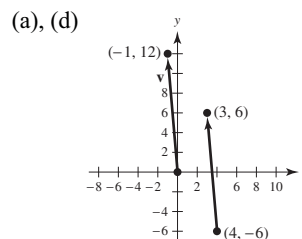
9. (b)  $\mathbf{v} = \langle 5 - 2, 5 - 0 \rangle = \langle 3, 5 \rangle$

(c)  $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$



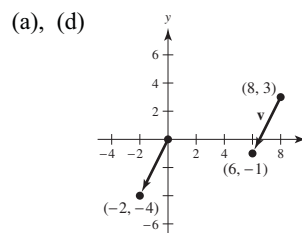
10. (b)  $\mathbf{v} = \langle 3 - 4, 6 - (-6) \rangle = \langle -1, 12 \rangle$

(c)  $\mathbf{v} = -\mathbf{i} + 12\mathbf{j}$



11. (b)  $\mathbf{v} = \langle 6 - 8, -1 - 3 \rangle = \langle -2, -4 \rangle$

(c)  $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

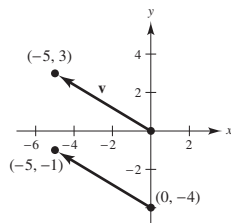


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12. (b)  $\mathbf{v} = \langle -5 - 0, -1 - (-4) \rangle = \langle -5, 3 \rangle$

(c)  $\mathbf{v} = -5\mathbf{i} + 3\mathbf{j}$

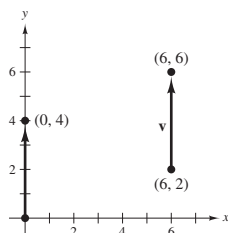
(a) and (d).



13. (b)  $\mathbf{v} = \langle 6 - 6, 6 - 2 \rangle = \langle 0, 4 \rangle$

(c)  $\mathbf{v} = 4\mathbf{j}$

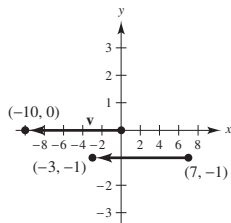
(a) and (d).



14. (b)  $\mathbf{v} = \langle -3 - 7, -1 - (-1) \rangle = \langle -10, 0 \rangle$

(c)  $\mathbf{v} = -10\mathbf{i}$

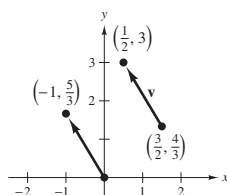
(a) and (d).



15. (b)  $\mathbf{v} = \langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \rangle = \langle -1, \frac{5}{3} \rangle$

(c)  $\mathbf{v} = -\mathbf{i} + \frac{5}{3}\mathbf{j}$

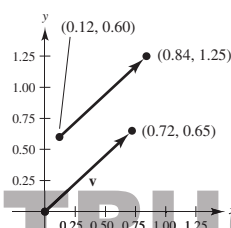
(a) and (d).



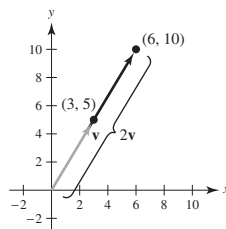
16. (b)  $\mathbf{v} = \langle 0.84 - 0.12, 1.25 - 0.60 \rangle = \langle 0.72, 0.65 \rangle$

(c)  $\mathbf{v} = 0.72\mathbf{i} + 0.65\mathbf{j}$

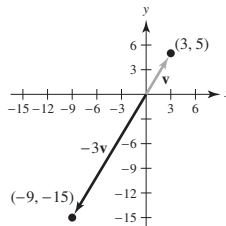
(a) and (d).



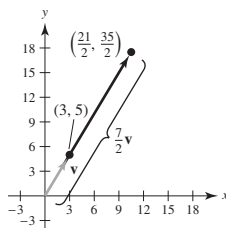
17. (a)  $2\mathbf{v} = 2\langle 3, 5 \rangle = \langle 6, 10 \rangle$



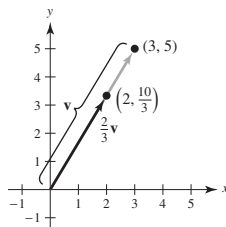
(b)  $-3\mathbf{v} = \langle -9, -15 \rangle$



(c)  $\frac{7}{2}\mathbf{v} = \langle \frac{21}{2}, \frac{35}{2} \rangle$

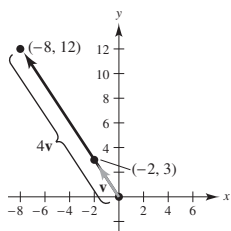


(d)  $\frac{2}{3}\mathbf{v} = \langle 2, \frac{10}{3} \rangle$

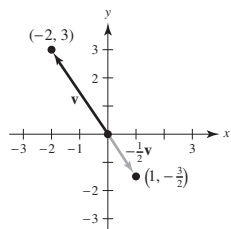


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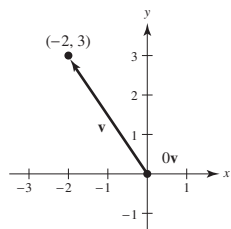
18. (a)  $4\mathbf{v} = 4\langle -2, 3 \rangle = \langle -8, 12 \rangle$



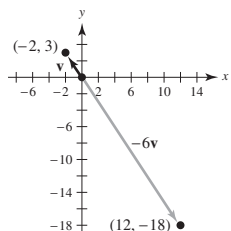
(b)  $-\frac{1}{2}\mathbf{v} = \langle 1, -\frac{3}{2} \rangle$



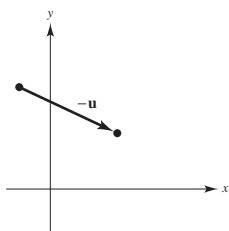
(c)  $0\mathbf{v} = \langle 0, 0 \rangle$



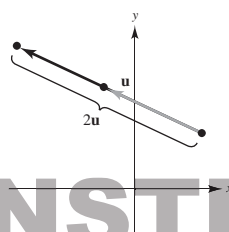
(d)  $-6\mathbf{u} = \langle 12, -18 \rangle$



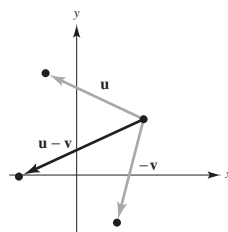
19.



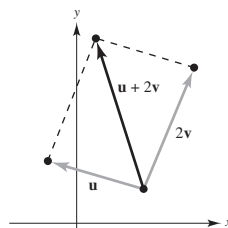
20. Twice as long as given vector  $\mathbf{u}$ .



21.



22.



23. (a)  $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9 \rangle = \langle \frac{8}{3}, 6 \rangle$

(b)  $\mathbf{v} - \mathbf{u} = \langle 2, -5 \rangle - \langle 4, 9 \rangle = \langle -2, -14 \rangle$

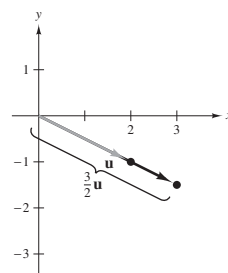
(c)  $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9 \rangle + 5\langle 2, -5 \rangle = \langle 18, -7 \rangle$

24. (a)  $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle -3, -8 \rangle = \langle -2, -\frac{16}{3} \rangle$

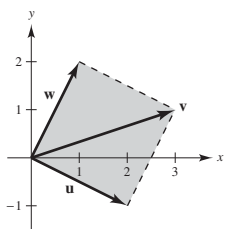
(b)  $\mathbf{v} - \mathbf{u} = \langle 8, 25 \rangle - \langle -3, -8 \rangle = \langle 11, 33 \rangle$

(c)  $2\mathbf{u} + 5\mathbf{v} = 2\langle -3, -8 \rangle + 5\langle 8, 25 \rangle = \langle 34, 109 \rangle$

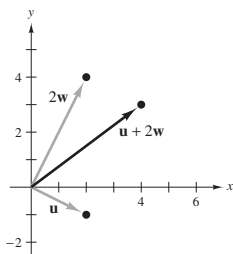
25.  $\mathbf{v} = \frac{3}{2}(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \frac{3}{2}\mathbf{j} = \langle 3, -\frac{3}{2} \rangle$



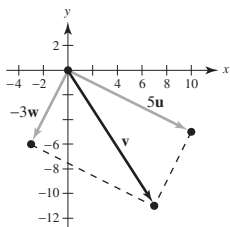
26.  $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + (\mathbf{i} + 2\mathbf{j}) = 3\mathbf{i} + \mathbf{j} = \langle 3, 1 \rangle$



$$27. \mathbf{v} = (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3 \rangle$$



$$28. \mathbf{v} = 5\mathbf{u} - 3\mathbf{w} = 5\langle 2, -1 \rangle - 3\langle 1, 2 \rangle = \langle 7, -11 \rangle$$



$$\begin{array}{ll} 29. u_1 - 4 = -1 & u_1 = 3 \\ u_2 - 2 = 3 & u_2 = 5 \\ & Q = (3, 5) \end{array}$$

$$\begin{array}{ll} 30. u_1 - 5 = 4 & u_1 = 9 \\ u_2 - 3 = -9 & u_2 = -6 \\ Q = (9, -6) & \text{Terminal point} \end{array}$$

$$31. \|\mathbf{v}\| = \sqrt{0^2 + 7^2} = 7$$

$$32. \|\mathbf{v}\| = \sqrt{(-3)^2 + 0^2} = 3$$

$$33. \|\mathbf{v}\| = \sqrt{4^2 + 3^2} = 5$$

$$34. \|\mathbf{v}\| = \sqrt{12^2 + (-5)^2} = 13$$

$$35. \|\mathbf{v}\| = \sqrt{6^2 + (-5)^2} = \sqrt{61}$$

$$36. \|\mathbf{v}\| = \sqrt{(-10)^2 + 3^2} = \sqrt{109}$$

$$37. \mathbf{v} = \langle 3, 12 \rangle$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{3^2 + 12^2} = \sqrt{153} \\ \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, 12 \rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle \\ &= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector} \end{aligned}$$

$$38. \mathbf{v} = \langle -5, 15 \rangle$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10} \\ \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -5, 15 \rangle}{5\sqrt{10}} = \left\langle -\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10} \right\rangle \text{ unit vector} \end{aligned}$$

$$39. \mathbf{v} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2} \\ \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\left\langle \frac{3}{2}, \frac{5}{2} \right\rangle}{\frac{\sqrt{34}}{2}} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle \\ &= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector} \end{aligned}$$

$$40. \mathbf{v} = \langle -6.2, 3.4 \rangle$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(-6.2)^2 + (3.4)^2} = \sqrt{50} = 5\sqrt{2} \\ \mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -6.2, 3.4 \rangle}{5\sqrt{2}} = \left\langle -\frac{31\sqrt{2}}{50}, \frac{17\sqrt{2}}{50} \right\rangle \text{ unit vector} \end{aligned}$$

$$41. \mathbf{u} = \langle 1, -1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$$

$$(a) \|\mathbf{u}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$(b) \|\mathbf{v}\| = \sqrt{1^2 + 4^2} = \sqrt{5}$$

$$(c) \mathbf{u} + \mathbf{v} = \langle 0, 1 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0^2 + 1^2} = 1$$

$$(d) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

$$(e) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

$$(f) \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \langle 0, 1 \rangle$$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

42.  $\mathbf{u} = \langle 0, 1 \rangle, \mathbf{v} = \langle 3, -3 \rangle$

(a)  $\|\mathbf{u}\| = \sqrt{0+1} = 1$

(b)  $\|\mathbf{v}\| = \sqrt{9+9} = 3\sqrt{2}$

(c)  $\mathbf{u} + \mathbf{v} = \langle 3, -2 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9+4} = \sqrt{13}$

(d)  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \langle 0, 1 \rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3\sqrt{2}} \langle 3, -3 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f)  $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 3, -2 \rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

43.  $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$

(a)  $\|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

(b)  $\|\mathbf{v}\| = \sqrt{4+9} = \sqrt{13}$

(c)  $\mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2} \right\rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}} = \frac{\sqrt{85}}{2}$

(d)  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}} \left\langle 1, \frac{1}{2} \right\rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f)  $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{2}{\sqrt{85}} \left\langle 3, \frac{7}{2} \right\rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

44.  $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

(a)  $\|\mathbf{u}\| = \sqrt{4+16} = 2\sqrt{5}$

(b)  $\|\mathbf{v}\| = \sqrt{25+25} = 5\sqrt{2}$

(c)  $\mathbf{u} + \mathbf{v} = \langle 7, 1 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{49+1} = 5\sqrt{2}$

(d)  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle$

$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$

(e)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 5, 5 \rangle$

$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$

(f)  $\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{1}{5\sqrt{2}} \langle 7, 1 \rangle$

$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$

45.  $\mathbf{u} = \langle 2, 1 \rangle$

$\|\mathbf{u}\| = \sqrt{5} \approx 2.236$

$\mathbf{v} = \langle 5, 4 \rangle$

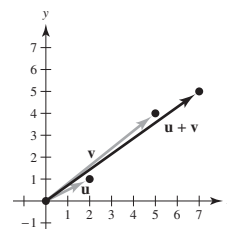
$\|\mathbf{v}\| = \sqrt{41} \approx 6.403$

$\mathbf{u} + \mathbf{v} = \langle 7, 5 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = \sqrt{74} \approx 8.602$

$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

$\sqrt{74} \leq \sqrt{5} + \sqrt{41}$



46.  $\mathbf{u} = \langle -3, 2 \rangle$

$\|\mathbf{u}\| = \sqrt{13} \approx 3.606$

$\mathbf{v} = \langle 1, -2 \rangle$

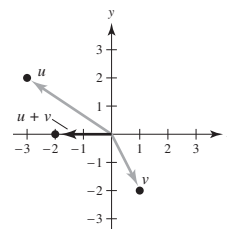
$\|\mathbf{v}\| = \sqrt{5} \approx 2.236$

$\mathbf{u} + \mathbf{v} = \langle -2, 0 \rangle$

$\|\mathbf{u} + \mathbf{v}\| = 2$

$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

$2 \leq \sqrt{13} + \sqrt{5}$



47.  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3} \langle 0, 3 \rangle = \langle 0, 1 \rangle$

$6 \left( \frac{\mathbf{u}}{\|\mathbf{u}\|} \right) = 6 \langle 0, 1 \rangle = \langle 0, 6 \rangle$

$\mathbf{v} = \langle 0, 6 \rangle$

$$48. \quad \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$$

$$4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 2\sqrt{2}\langle 1, 1 \rangle$$

$$\mathbf{v} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$$

$$49. \quad \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{5}}\langle -1, 2 \rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$5\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 5\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \langle -\sqrt{5}, 2\sqrt{5} \rangle$$

$$\mathbf{v} = \langle -\sqrt{5}, 2\sqrt{5} \rangle$$

$$50. \quad \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{3}}\langle \sqrt{3}, 3 \rangle$$

$$2\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{1}{\sqrt{3}}\langle \sqrt{3}, 3 \rangle$$

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$

$$51. \quad \mathbf{v} = 3[(\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j}] = 3\mathbf{i} = \langle 3, 0 \rangle$$

$$52. \quad \mathbf{v} = 5[(\cos 120^\circ)\mathbf{i} + (\sin 120^\circ)\mathbf{j}]$$

$$= -\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$$

$$53. \quad \mathbf{v} = 2[(\cos 150^\circ)\mathbf{i} + (\sin 150^\circ)\mathbf{j}]$$

$$= -\sqrt{3}\mathbf{i} + \mathbf{j} = \langle -\sqrt{3}, 1 \rangle$$

$$54. \quad \mathbf{v} = 4[(\cos 3.5^\circ)\mathbf{i} + (\sin 3.5^\circ)\mathbf{j}]$$

$$\approx 3.9925\mathbf{i} + 0.2442\mathbf{j}$$

$$= \langle 3.9925, 0.2442 \rangle$$

$$55. \quad \mathbf{u} = (\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j} = \mathbf{i}$$

$$\mathbf{v} = 3(\cos 45^\circ)\mathbf{i} + 3(\sin 45^\circ)\mathbf{j} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = \left(\frac{2 + 3\sqrt{2}}{2}\right)\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} = \left\langle \frac{2 + 3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right\rangle$$

$$56. \quad \mathbf{u} = 4(\cos 0^\circ)\mathbf{i} + 4(\sin 0^\circ)\mathbf{j} = 4\mathbf{i}$$

$$\mathbf{v} = 2(\cos 30^\circ)\mathbf{i} + 2(\sin 30^\circ)\mathbf{j} = \mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 5\mathbf{i} + \sqrt{3}\mathbf{j} = \langle 5, \sqrt{3} \rangle$$

$$57. \quad \mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$$

$$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = (2\cos 4 + \cos 2)\mathbf{i} + (2\sin 4 + \sin 2)\mathbf{j}$$

$$= \langle 2\cos 4 + \cos 2, 2\sin 4 + \sin 2 \rangle$$

$$58. \quad \mathbf{u} = 5[\cos(-0.5)]\mathbf{i} + 5[\sin(-0.5)]\mathbf{j}$$

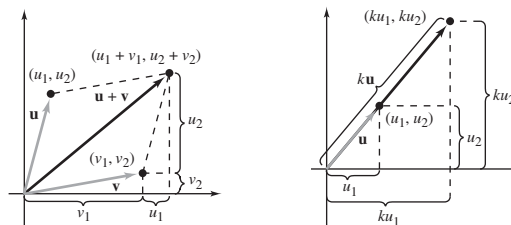
$$= 5(\cos 0.5)\mathbf{i} - 5(\sin 0.5)\mathbf{j}$$

$$\mathbf{v} = 5(\cos 0.5)\mathbf{i} + 5(\sin 0.5)\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = 10(\cos 0.5)\mathbf{i} = \langle 10\cos 0.5, 0 \rangle$$

59. Answers will vary. *Sample answer:* A scalar is a real number such as 2. A vector is represented by a directed line segment. A vector has both magnitude and direction. For example  $\langle \sqrt{3}, 1 \rangle$  has direction  $\frac{\pi}{6}$  and a magnitude of 2.

60. See page 766:



61. (a) Vector. The velocity has both magnitude and direction.

(b) Scalar. The price is a number.

62. (a) Scalar. The temperature is a number.

(b) Vector. The weight has magnitude and direction.

For Exercises 63–68,

$$\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}.$$

63.  $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ . So,  $a + b = 2$ ,  $2a - b = 1$ . Solving simultaneously, you have  $a = 1$ ,  $b = 1$ .

64.  $\mathbf{v} = 3\mathbf{j}$ . So,  $a + b = 0$ ,  $2a - b = 3$ . Solving simultaneously, you have  $a = 1$ ,  $b = -1$ .

65.  $\mathbf{v} = 3\mathbf{i}$ . So,  $a + b = 3$ ,  $2a - b = 0$ . Solving simultaneously, you have  $a = 1$ ,  $b = 2$ .

66.  $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j}$ . So,  $a + b = 3$ ,  $2a - b = 3$ . Solving simultaneously, you have  $a = 2$ ,  $b = 1$ .

67.  $\mathbf{v} = \mathbf{i} + \mathbf{j}$ . So,  $a + b = 1$ ,  $2a - b = 1$ . Solving simultaneously, you have  $a = \frac{2}{3}$ ,  $b = \frac{1}{3}$ .

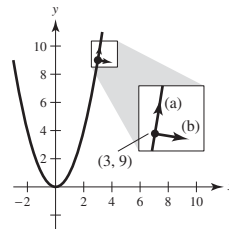
68.  $\mathbf{v} = -\mathbf{i} + 7\mathbf{j}$ . So,  $a + b = -1$ ,  $2a - b = 7$ . Solving simultaneously, you have  $a = 2$ ,  $b = -3$ .



69.  $f(x) = x^2, f'(x) = 2x, f'(3) = 6$

(a)  $m = 6$ . Let  $\mathbf{w} = \langle 1, 6 \rangle, \|\mathbf{w}\| = \sqrt{37}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{37}} \langle 1, 6 \rangle$ .

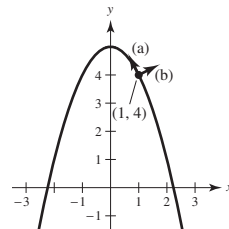
(b)  $m = -\frac{1}{6}$ . Let  $\mathbf{w} = \langle -6, 1 \rangle, \|\mathbf{w}\| = \sqrt{37}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{37}} \langle -6, 1 \rangle$ .



70.  $f(x) = -x^2 + 5, f'(x) = -2x, f'(1) = -2$

(a)  $m = -2$ . Let  $\mathbf{w} = \langle 1, -2 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$ .

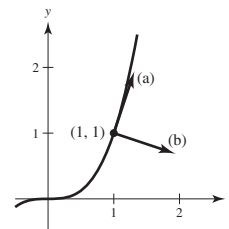
(b)  $m = \frac{1}{2}$ . Let  $\mathbf{w} = \langle 2, 1 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 2, 1 \rangle$ .



71.  $f(x) = x^3, f'(x) = 3x^2 = 3$  at  $x = 1$ .

(a)  $m = 3$ . Let  $\mathbf{w} = \langle 1, 3 \rangle, \|\mathbf{w}\| = \sqrt{10}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$ .

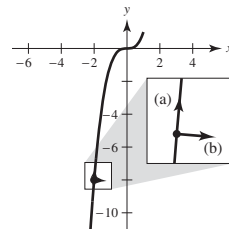
(b)  $m = -\frac{1}{3}$ . Let  $\mathbf{w} = \langle 3, -1 \rangle, \|\mathbf{w}\| = \sqrt{10}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 3, -1 \rangle$ .



72.  $f(x) = x^3, f'(x) = 3x^2 = 12$  at  $x = -2$ .

(a)  $m = 12$ . Let  $\mathbf{w} = \langle 1, 12 \rangle, \|\mathbf{w}\| = \sqrt{145}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 1, 12 \rangle$ .

(b)  $m = -\frac{1}{12}$ . Let  $\mathbf{w} = \langle 12, -1 \rangle, \|\mathbf{w}\| = \sqrt{145}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{145}} \langle 12, -1 \rangle$ .

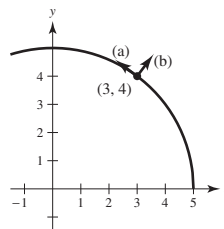


73.  $f(x) = \sqrt{25 - x^2}$

$f'(x) = \frac{-x}{\sqrt{25 - x^2}} = -\frac{3}{4}$  at  $x = 3$ .

(a)  $m = -\frac{3}{4}$ . Let  $\mathbf{w} = \langle -4, 3 \rangle, \|\mathbf{w}\| = 5$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle -4, 3 \rangle$ .

(b)  $m = \frac{4}{3}$ . Let  $\mathbf{w} = \langle 3, 4 \rangle, \|\mathbf{w}\| = 5$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle 3, 4 \rangle$ .

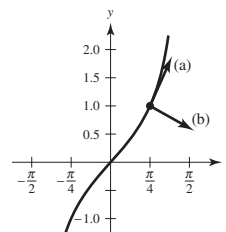


74.  $f(x) = \tan x$

$f'(x) = \sec^2 x = 2$  at  $x = \frac{\pi}{4}$

(a)  $m = 2$ . Let  $\mathbf{w} = \langle 1, 2 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$ .

(b)  $m = -\frac{1}{2}$ . Let  $\mathbf{w} = \langle -2, 1 \rangle, \|\mathbf{w}\| = \sqrt{5}$ , then  $\pm \frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$ .



$$75. \quad \mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$$

$$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$76. \quad \mathbf{u} = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$$

$$\begin{aligned} \mathbf{v} &= (\mathbf{u} + \mathbf{v}) - \mathbf{u} = (-3 - 2\sqrt{3})\mathbf{i} + (3\sqrt{3} - 2)\mathbf{j} \\ &= \langle -3 - 2\sqrt{3}, 3\sqrt{3} - 2 \rangle \end{aligned}$$

77. (a)–(c) Programs will vary.

(d) Magnitude  $\approx 63.5$

Direction  $\approx -8.26^\circ$

$$80. \quad \|\mathbf{F}_1\| = 2, \theta_{\mathbf{F}_1} = -10^\circ$$

$$\|\mathbf{F}_2\| = 4, \theta_{\mathbf{F}_2} = 140^\circ$$

$$\|\mathbf{F}_3\| = 3, \theta_{\mathbf{F}_3} = 200^\circ$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 4.09$$

$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 163.0^\circ$$

$$81. \quad \mathbf{F}_1 + \mathbf{F}_2 = (500 \cos 30^\circ \mathbf{i} + 500 \sin 30^\circ \mathbf{j}) + (200 \cos(-45^\circ) \mathbf{i} + 200 \sin(-45^\circ) \mathbf{j}) = (250\sqrt{3} + 100\sqrt{2})\mathbf{i} + (250 - 100\sqrt{2})\mathbf{j}$$

$$\|\mathbf{F}_1 + \mathbf{F}_2\| = \sqrt{(250\sqrt{3} + 100\sqrt{2})^2 + (250 - 100\sqrt{2})^2} \approx 584.6 \text{ lb}$$

$$\tan \theta = \frac{250 - 100\sqrt{2}}{250\sqrt{3} + 100\sqrt{2}} \Rightarrow \theta \approx 10.7^\circ$$

$$82. \text{ (a) } 180(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) + 275\mathbf{i} \approx 430.88\mathbf{i} + 90\mathbf{j}$$

$$\text{Direction: } \alpha \approx \arctan\left(\frac{90}{430.88}\right) \approx 0.206 (\approx 11.8^\circ)$$

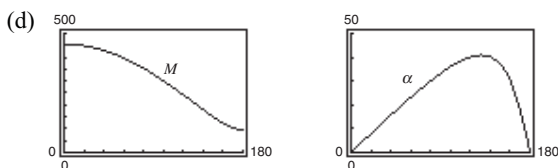
$$\text{Magnitude: } \sqrt{430.88^2 + 90^2} \approx 440.18 \text{ newtons}$$

$$\text{(b) } M = \sqrt{(275 + 180 \cos \theta)^2 + (180 \sin \theta)^2}$$

$$\alpha = \arctan\left[\frac{180 \sin \theta}{275 + 180 \cos \theta}\right]$$

(c)

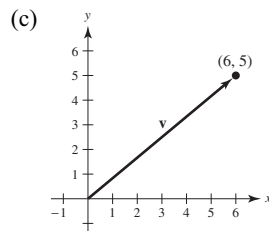
$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$M$	455	440.2	396.9	328.7	241.9	149.3	95
$\alpha$	$0^\circ$	$11.8^\circ$	$23.1^\circ$	$33.2^\circ$	$40.1^\circ$	$37.1^\circ$	$0$



(e)  $M$  decreases because the forces change from acting in the same direction to acting in the opposite direction as  $\theta$  increases from  $0^\circ$  to  $180^\circ$ .

$$78. \text{ (a) } \mathbf{v} = \langle 9 - 3, 1 - (-4) \rangle = \langle 6, 5 \rangle$$

$$\text{(b) } \mathbf{v} = 6\mathbf{i} + 5\mathbf{j}$$



$$\text{(d) } \|\mathbf{v}\| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

$$79. \quad \|\mathbf{F}_1\| = 2, \theta_{\mathbf{F}_1} = 33^\circ$$

$$\|\mathbf{F}_2\| = 3, \theta_{\mathbf{F}_2} = -125^\circ$$

$$\|\mathbf{F}_3\| = 2.5, \theta_{\mathbf{F}_3} = 110^\circ$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 1.33$$

$$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 132.5^\circ$$

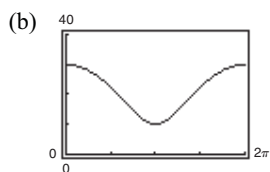
$$\begin{aligned}
 83. \quad \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &= (75 \cos 30^\circ \mathbf{i} + 75 \sin 30^\circ \mathbf{j}) + (100 \cos 45^\circ \mathbf{i} + 100 \sin 45^\circ \mathbf{j}) + (125 \cos 120^\circ \mathbf{i} + 125 \sin 120^\circ \mathbf{j}) \\
 &= \left(\frac{75}{2}\sqrt{3} + 50\sqrt{2} - \frac{125}{2}\right)\mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2}\sqrt{3}\right)\mathbf{j} \\
 \|\mathbf{R}\| &= \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb} \\
 \theta_{\mathbf{R}} &= \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3^\circ
 \end{aligned}$$

$$\begin{aligned}
 84. \quad \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &= [400(\cos(-30^\circ)\mathbf{i} + \sin(-30^\circ)\mathbf{j})] + [280(\cos(45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j})] + [350(\cos(135^\circ)\mathbf{i} + \sin(135^\circ)\mathbf{j})] \\
 &= [200\sqrt{3} + 140\sqrt{2} - 175\sqrt{2}]\mathbf{i} + [-200 + 140\sqrt{2} + 175\sqrt{2}]\mathbf{j} \\
 \|\mathbf{R}\| &= \sqrt{(200\sqrt{3} - 35\sqrt{2})^2 + (-200 + 315\sqrt{2})^2} \approx 385.2483 \text{ newtons} \\
 \theta_{\mathbf{R}} &= \arctan\left(\frac{-200 + 315\sqrt{2}}{200\sqrt{3} - 35\sqrt{2}}\right) \approx 0.6908 \approx 39.6^\circ
 \end{aligned}$$

85. (a) The forces act along the same direction.  $\theta = 0^\circ$ .  
 (b) The forces cancel out each other.  $\theta = 180^\circ$ .  
 (c) No, the magnitude of the resultant can not be greater than the sum.

$$86. \quad \mathbf{F}_1 = \langle 20, 0 \rangle, \mathbf{F}_2 = 10\langle \cos \theta, \sin \theta \rangle$$

$$\begin{aligned}
 (a) \quad \|\mathbf{F}_1 + \mathbf{F}_2\| &= \|\langle 20 + 10 \cos \theta, 10 \sin \theta \rangle\| \\
 &= \sqrt{400 + 400 \cos \theta + 100 \cos^2 \theta + 100 \sin^2 \theta} \\
 &= \sqrt{500 + 400 \cos \theta}
 \end{aligned}$$



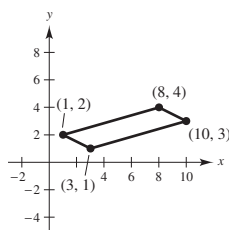
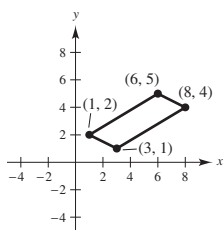
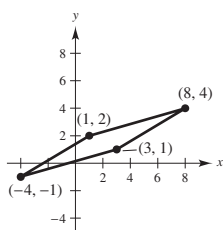
- (c) The range is  $10 \leq \|\mathbf{F}_1 + \mathbf{F}_2\| \leq 30$ .

The maximum is 30, which occur at  $\theta = 0$  and  $\theta = 2\pi$ .

The minimum is 10 at  $\theta = \pi$ .

- (d) The minimum of the resultant is 10.

$$87. \quad (-4, -1), (6, 5), (10, 3)$$



$$88. \quad \mathbf{u} = \langle 7 - 1, 5 - 2 \rangle = \langle 6, 3 \rangle$$

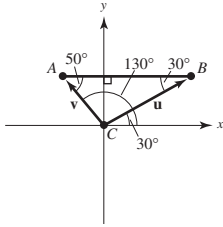
$$\frac{1}{3}\mathbf{u} = \langle 2, 1 \rangle$$

$$P_1 = (1, 2) + (2, 1) = (3, 3)$$

$$P_2 = (1, 2) + 2(2, 1) = (5, 4)$$

$$89. \mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$$

$$\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130^\circ \mathbf{i} + \sin 130^\circ \mathbf{j})$$



$$\text{Vertical components: } \|\mathbf{u}\| \sin 30^\circ + \|\mathbf{v}\| \sin 130^\circ = 3000$$

$$\text{Horizontal components: } \|\mathbf{u}\| \cos 30^\circ + \|\mathbf{v}\| \cos 130^\circ = 0$$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 1958.1 \text{ pounds}$$

$$\|\mathbf{v}\| \approx 2638.2 \text{ pounds}$$

$$90. \theta_1 = \arctan\left(\frac{24}{20}\right) \approx 0.8761 \text{ or } 50.2^\circ$$

$$\theta_2 = \arctan\left(\frac{24}{-10}\right) + \pi \approx 1.9656 \text{ or } 112.6^\circ$$

$$\mathbf{u} = \|\mathbf{u}\|(\cos \theta_1 \mathbf{i} + \sin \theta_1 \mathbf{j})$$

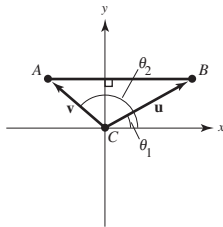
$$\mathbf{v} = \|\mathbf{v}\|(\cos \theta_2 \mathbf{i} + \sin \theta_2 \mathbf{j})$$

$$\text{Vertical components: } \|\mathbf{u}\| \sin \theta_1 + \|\mathbf{v}\| \sin \theta_2 = 5000$$

$$\text{Horizontal components: } \|\mathbf{u}\| \cos \theta_1 + \|\mathbf{v}\| \cos \theta_2 = 0$$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 2169.4 \text{ and } \|\mathbf{v}\| \approx 3611.2.$$



$$93. \mathbf{u} = 900(\cos 148^\circ \mathbf{i} + \sin 148^\circ \mathbf{j})$$

$$\mathbf{v} = 100(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})$$

$$\mathbf{u} + \mathbf{v} = (900 \cos 148^\circ + 100 \cos 45^\circ) \mathbf{i} + (900 \sin 148^\circ + 100 \sin 45^\circ) \mathbf{j}$$

$$\approx -692.53 \mathbf{i} + 547.64 \mathbf{j}$$

$$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34^\circ; 38.34^\circ \text{ North of West}$$

$$\|\mathbf{u} + \mathbf{v}\| \approx \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/h}$$

$$91. \text{ Horizontal component} = \|\mathbf{v}\| \cos \theta$$

$$= 1200 \cos 6^\circ \approx 1193.43 \text{ ft/sec}$$

$$\text{Vertical component} = \|\mathbf{v}\| \sin \theta$$

$$= 1200 \sin 6^\circ \approx 125.43 \text{ ft/sec}$$

92. To lift the weight vertically, the sum of the vertical components of  $\mathbf{u}$  and  $\mathbf{v}$  must be 100 and the sum of the horizontal components must be 0.

$$\mathbf{u} = \|\mathbf{u}\|(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$$

$$\mathbf{v} = \|\mathbf{v}\|(\cos 110^\circ \mathbf{i} + \sin 110^\circ \mathbf{j})$$

$$\text{So, } \|\mathbf{u}\| \sin 60^\circ + \|\mathbf{v}\| \sin 110^\circ = 100, \text{ or}$$

$$\|\mathbf{u}\| \left(\frac{\sqrt{3}}{2}\right) + \|\mathbf{v}\| \sin 110^\circ = 100.$$

$$\text{And } \|\mathbf{u}\| \cos 60^\circ + \|\mathbf{v}\| \cos 110^\circ = 0 \text{ or}$$

$$\|\mathbf{u}\| \left(\frac{1}{2}\right) + \|\mathbf{v}\| \cos 110^\circ = 0.$$

Multiplying the last equation by  $(\sqrt{3})$  and adding to the first equation gives

$$\|\mathbf{u}\|(\sin 110^\circ - \sqrt{3} \cos 110^\circ) = 100 \Rightarrow \|\mathbf{v}\| \approx 65.27 \text{ lb.}$$

$$\text{Then, } \|\mathbf{u}\| \left(\frac{1}{2}\right) + 65.27 \cos 110^\circ = 0 \text{ gives}$$

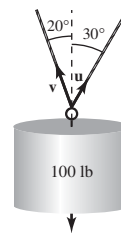
$$\|\mathbf{u}\| \approx 44.65 \text{ lb.}$$

$$(a) \text{ The tension in each rope: } \|\mathbf{u}\| = 44.65 \text{ lb,}$$

$$\|\mathbf{v}\| = 65.27 \text{ lb}$$

$$(b) \text{ Vertical components: } \|\mathbf{u}\| \sin 60^\circ \approx 38.67 \text{ lb,}$$

$$\|\mathbf{v}\| \sin 110^\circ \approx 61.33 \text{ lb}$$



94.  $\mathbf{u} = 400\mathbf{i}$  (plane)

$\mathbf{v} = 50(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}) = -25\sqrt{2}\mathbf{i} + 25\sqrt{2}\mathbf{j}$  (wind)

$\mathbf{u} + \mathbf{v} = (400 - 25\sqrt{2})\mathbf{i} + 25\sqrt{2}\mathbf{j} \approx 364.64\mathbf{i} + 35.36\mathbf{j}$

$\tan \theta = \frac{35.36}{364.64} \Rightarrow \theta \approx 5.54^\circ$

Direction North of East:  $\approx$  N  $84.46^\circ$  E

Speed:  $\approx$  336.35 mi/h

95. True

96. True

97. True

98. False

$a = b = 0$

99. False

$\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

100. True

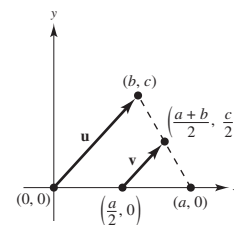
101.  $\|\mathbf{u}\| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$ ,

$\|\mathbf{v}\| = \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$

102. Let the triangle have vertices at  $(0, 0)$ ,  $(a, 0)$ , and

$(b, c)$ . Let  $\mathbf{u}$  be the vector joining  $(0, 0)$  and  $(b, c)$ , as indicated in the figure. Then  $\mathbf{v}$ , the vector joining the midpoints, is

$$\begin{aligned}\mathbf{v} &= \left( \frac{a+b}{2} - \frac{a}{2} \right) \mathbf{i} + \frac{c}{2} \mathbf{j} \\ &= \frac{b}{2} \mathbf{i} + \frac{c}{2} \mathbf{j} \\ &= \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u}.\end{aligned}$$



103. Let  $\mathbf{u}$  and  $\mathbf{v}$  be the vectors that determine the parallelogram, as indicated in the figure. The two diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{v} - \mathbf{u}$ . So,

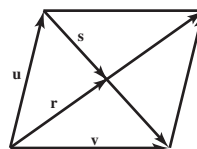
$\mathbf{r} = x(\mathbf{u} + \mathbf{v})$ ,  $\mathbf{s} = 4(\mathbf{v} - \mathbf{u})$ . But,

$\mathbf{u} = \mathbf{r} - \mathbf{s}$

$= x(\mathbf{u} + \mathbf{v}) - y(\mathbf{v} - \mathbf{u}) = (x + y)\mathbf{u} + (x - y)\mathbf{v}.$

So,  $x + y = 1$  and  $x - y = 0$ . Solving you have

$x = y = \frac{1}{2}.$



104.  $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$= \|\mathbf{u}\|[\|\mathbf{v}\|\cos \theta_v \mathbf{i} + \|\mathbf{v}\|\sin \theta_v \mathbf{j}] + \|\mathbf{v}\|[\|\mathbf{u}\|\cos \theta_u \mathbf{i} + \|\mathbf{u}\|\sin \theta_u \mathbf{j}]$

$= \|\mathbf{u}\|\|\mathbf{v}\|[(\cos \theta_u + \cos \theta_v)\mathbf{i} + (\sin \theta_u + \sin \theta_v)\mathbf{j}]$

$= 2\|\mathbf{u}\|\|\mathbf{v}\|\left[\cos\left(\frac{\theta_u + \theta_v}{2}\right)\cos\left(\frac{\theta_u - \theta_v}{2}\right)\mathbf{i} + \sin\left(\frac{\theta_u + \theta_v}{2}\right)\cos\left(\frac{\theta_u - \theta_v}{2}\right)\mathbf{j}\right]$

$\tan \theta_w = \frac{\sin\left(\frac{\theta_u + \theta_v}{2}\right)\cos\left(\frac{\theta_u - \theta_v}{2}\right)}{\cos\left(\frac{\theta_u + \theta_v}{2}\right)\cos\left(\frac{\theta_u - \theta_v}{2}\right)} = \tan\left(\frac{\theta_u + \theta_v}{2}\right)$

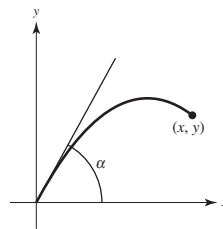
So,  $\theta_w = (\theta_u + \theta_v)/2$  and  $\mathbf{w}$  bisects the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

105. The set is a circle of radius 5, centered at the origin.

$\|\mathbf{u}\| = \|\langle x, y \rangle\| = \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$

106. Let  $x = v_0 t \cos \alpha$  and  $y = v_0 t \sin \alpha - \frac{1}{2} g t^2$ .

$$\begin{aligned} t &= \frac{x}{v_0 \cos \alpha} \Rightarrow y = v_0 \sin \alpha \left( \frac{x}{v_0 \cos \alpha} \right) - \frac{1}{2} g \left( \frac{x}{v_0 \cos \alpha} \right)^2 \\ &= x \tan \alpha - \frac{g}{2v_0^2} x^2 \sec^2 \alpha \\ &= x \tan \alpha - \frac{gx^2}{2v_0^2} (1 + \tan^2 \alpha) \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2} \tan^2 \alpha + x \tan \alpha - \frac{v_0^2}{2g} \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2} \left[ \tan^2 \alpha - 2 \tan \alpha \left( \frac{v_0^2}{gx} \right) + \frac{v_0^4}{g^2 x^2} \right] \\ &= \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2} \left( \tan \alpha - \frac{v_0^2}{gx} \right)^2 \end{aligned}$$



If  $y \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$ , then  $\alpha$  can be chosen to hit the point  $(x, y)$ . To hit  $(0, y)$ : Let  $\alpha = 90^\circ$ . Then

$$y = v_0 t - \frac{1}{2} g t^2 = \frac{v_0^2}{2g} - \frac{v_0^2}{2g} \left( \frac{g}{v_0} t - 1 \right)^2, \text{ and you need } y \leq \frac{v_0^2}{2g}.$$

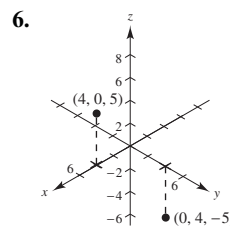
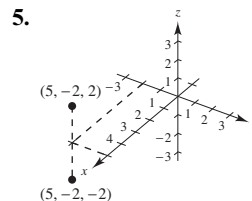
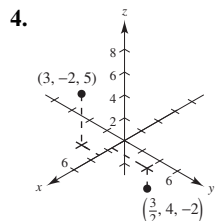
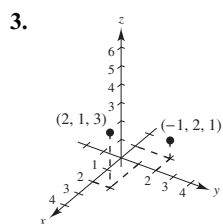
The set H is given by  $0 \leq x$ ,  $0 < y$  and  $y \leq \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$

**Note:** The parabola  $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$  is called the “parabola of safety.”

## Section 11.2 Space Coordinates and Vectors in Space

1.  $A(2, 3, 4)$   
 $B(-1, -2, 2)$

2.  $A(2, -3, -1)$   
 $B(-3, 1, 4)$



7.  $x = -3, y = 4, z = 5$ :  $(-3, 4, 5)$

8.  $x = 7, y = -2, z = -1$ :  
 $(7, -2, -1)$

9.  $y = z = 0, x = 12$ :  $(12, 0, 0)$

10.  $x = 0, y = 3, z = 2$ :  $(0, 3, 2)$

11. The  $z$ -coordinate is 0.
12. The  $x$ -coordinate is 0.
13. The point is 6 units above the  $xy$ -plane.
14. The point is 2 units in front of the  $xz$ -plane.
15. The point is on the plane parallel to the  $yz$ -plane that passes through  $x = -3$ .
16. The point is on the plane parallel to the  $xy$ -plane that passes through  $z = -5/2$ .
17. The point is to the left of the  $xz$ -plane.
18. The point is in front of the  $yz$ -plane.
19. The point is on or between the planes  $y = 3$  and  $y = -3$ .
20. The point is in front of the plane  $x = 4$ .
21. The point  $(x, y, z)$  is 3 units below the  $xy$ -plane, and below either quadrant I or III.
22. The point  $(x, y, z)$  is 4 units above the  $xy$ -plane, and above either quadrant II or IV.
23. The point could be above the  $xy$ -plane and so above quadrants II or IV, or below the  $xy$ -plane, and so below quadrants I or III.
24. The point could be above the  $xy$ -plane, and so above quadrants I and III, or below the  $xy$ -plane, and so below quadrants II or IV.

$$\begin{aligned} 25. \quad d &= \sqrt{(-4 - 0)^2 + (2 - 0)^2 + (7 - 0)^2} \\ &= \sqrt{16 + 4 + 49} = \sqrt{69} \end{aligned}$$

$$\begin{aligned} 26. \quad d &= \sqrt{(2 - (-2))^2 + (-5 - 3)^2 + (-2 - 2)^2} \\ &= \sqrt{16 + 64 + 16} = \sqrt{96} = 4\sqrt{6} \end{aligned}$$

$$\begin{aligned} 27. \quad d &= \sqrt{(6 - 1)^2 + (-2 - (-2))^2 + (-2 - 4)^2} \\ &= \sqrt{25 + 0 + 36} = \sqrt{61} \end{aligned}$$

$$\begin{aligned} 28. \quad d &= \sqrt{(4 - 2)^2 + (-5 - 2)^2 + (6 - 3)^2} \\ &= \sqrt{4 + 49 + 9} = \sqrt{62} \end{aligned}$$

$$29. \quad A(0, 0, 4), B(2, 6, 7), C(6, 4, -8)$$

$$|AB| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$$

$$|AC| = \sqrt{6^2 + 4^2 + (-12)^2} = \sqrt{196} = 14$$

$$|BC| = \sqrt{4^2 + (-2)^2 + (-15)^2} = \sqrt{245} = 7\sqrt{5}$$

$$|BC|^2 = 245 = 49 + 196 = |AB|^2 + |AC|^2$$

Right triangle

$$30. \quad A(3, 4, 1), B(0, 6, 2), C(3, 5, 6)$$

$$|AB| = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$|AC| = \sqrt{0 + 1 + 25} = \sqrt{26}$$

$$|BC| = \sqrt{9 + 1 + 16} = \sqrt{26}$$

Because  $|AC| = |BC|$ , the triangle is isosceles.

$$31. \quad A(-1, 0, -2), B(-1, 5, 2), C(-3, -1, 1)$$

$$|AB| = \sqrt{0 + 25 + 16} = \sqrt{41}$$

$$|AC| = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|BC| = \sqrt{4 + 36 + 1} = \sqrt{41}$$

Because  $|AB| = |BC|$ , the triangle is isosceles.

$$32. \quad A(4, -1, -1), B(2, 0, -4), C(3, 5, -1)$$

$$|AB| = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|AC| = \sqrt{1 + 36 + 0} = \sqrt{37}$$

$$|BC| = \sqrt{1 + 25 + 9} = \sqrt{35}$$

Neither

$$33. \quad \text{The } z\text{-coordinate is changed by 5 units:}$$

$$(0, 0, 9), (2, 6, 12), (6, 4, -3)$$

$$34. \quad \text{The } y\text{-coordinate is changed by 3 units:}$$

$$(3, 7, 1), (0, 9, 2), (3, 8, 6)$$

$$35. \quad \left( \frac{5 + (-2)}{2}, \frac{-9 + 3}{2}, \frac{7 + 3}{2} \right) = \left( \frac{3}{2}, -3, 5 \right)$$

$$36. \quad \left( \frac{4 + 8}{2}, \frac{0 + 8}{2}, \frac{-6 + 20}{2} \right) = (6, 4, 7)$$

$$37. \quad \text{Center: } (0, 2, 5)$$

Radius: 2

$$(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$$

$$38. \quad \text{Center: } (4, -1, 1)$$

Radius: 5

$$(x - 4)^2 + (y + 1)^2 + (z - 1)^2 = 25$$

$$39. \quad \text{Center: } \frac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$$

Radius:  $\sqrt{10}$

$$(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$$

40. Center:
- $(-3, 2, 4)$

$$r = 3$$

(tangent to  $yz$ -plane)

$$(x + 3)^2 + (y - 2)^2 + (z - 4)^2 = 9$$

- 41.
- $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) = -1 + 1 + 9 + 16$$

$$(x - 1)^2 + (y + 3)^2 + (z + 4)^2 = 25$$

Center:  $(1, -3, -4)$ 

Radius: 5

- 42.
- $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$

$$\left(x^2 + 9x + \frac{81}{4}\right) + (y^2 - 2y + 1) + (z^2 + 10z + 25) = -19 + \frac{81}{4} + 1 + 25$$

$$\left(x + \frac{9}{2}\right)^2 + (y - 1)^2 + (z + 5)^2 = \frac{109}{4}$$

Center:  $\left(-\frac{9}{2}, 1, -5\right)$ Radius:  $\frac{\sqrt{109}}{2}$ 

- 43.
- $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

$$x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$$

$$\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

Center:  $\left(\frac{1}{3}, -1, 0\right)$ 

Radius: 1

- 44.
- $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0$

$$(x^2 - 6x + 9) + (y^2 - y + \frac{1}{4}) + (z^2 + 2z + 1) = \frac{23}{4} + 9 + \frac{1}{4} + 1$$

$$(x - 3)^2 + \left(y - \frac{1}{2}\right)^2 + (z + 1)^2 = 16$$

Center:  $\left(3, \frac{1}{2}, -1\right)$ 

Radius: 4

- 45.
- $x^2 + y^2 + z^2 \leq 36$

Solid sphere of radius 6 centered at origin.

- 46.
- $x^2 + y^2 + z^2 > 4$

Set of all points in space outside the ball of radius 2 centered at the origin.



47.  $x^2 + y^2 + z^2 < 4x - 6y + 8z - 13$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 8z + 16) < 4 + 9 + 16 - 13$$

$$(x - 2)^2 + (y + 3)^2 + (z - 4)^2 < 16$$

Interior of sphere of radius 4 centered at  $(2, -3, 4)$ .

48.  $x^2 + y^2 + z^2 > -4x + 6y - 8z - 13$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 8z + 16) > -13 + 4 + 9 + 16$$

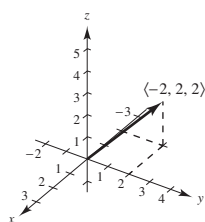
$$(x + 2)^2 + (y - 3)^2 + (z + 4)^2 > 16$$

Set of all points in space outside the ball of radius 4 centered at  $(-2, 3, -4)$ .

49. (a)  $\mathbf{v} = \langle 2 - 4, 4 - 2, 3 - 1 \rangle = \langle -2, 2, 2 \rangle$

(b)  $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

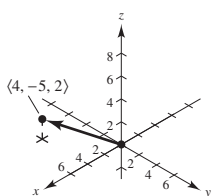
(c)



50. (a)  $\mathbf{v} = \langle 4 - 0, 0 - 5, 3 - 1 \rangle = \langle 4, -5, 2 \rangle$

(b)  $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$

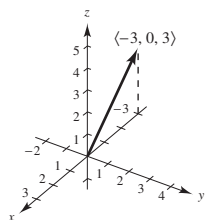
(c)



51. (a)  $\mathbf{v} = \langle 0 - 3, 3 - 3, 3 - 0 \rangle = \langle -3, 0, 3 \rangle$

(b)  $\mathbf{v} = -3\mathbf{i} + 3\mathbf{k}$

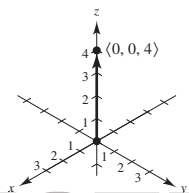
(c)



52. (a)  $\mathbf{v} = \langle 2 - 2, 3 - 3, 4 - 0 \rangle = \langle 0, 0, 4 \rangle$

(b)  $\mathbf{v} = 4\mathbf{k}$

(c)



53.  $\langle 4 - 3, 1 - 2, 6 - 0 \rangle = \langle 1, -1, 6 \rangle$

$$\|\langle 1, -1, 6 \rangle\| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

$$\text{Unit vector: } \frac{\langle 1, -1, 6 \rangle}{\sqrt{38}} = \left\langle \frac{1}{\sqrt{38}}, \frac{-1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right\rangle$$

54.  $\langle -1 - 4, 7 - (-5), -3 - 2 \rangle = \langle -5, 12, -5 \rangle$

$$\|\langle -5, 12, -5 \rangle\| = \sqrt{25 + 144 + 25} = \sqrt{194}$$

$$\text{Unit vector: } \frac{\langle -5, 12, -5 \rangle}{\sqrt{194}} = \left\langle \frac{-5}{\sqrt{194}}, \frac{12}{\sqrt{194}}, \frac{-5}{\sqrt{194}} \right\rangle$$

55.  $\langle -5 - (-4), 3 - 3, 0 - 1 \rangle = \langle -1, 0, -1 \rangle$

$$\|\langle -1, 0, -1 \rangle\| = \sqrt{1 + 1} = \sqrt{2}$$

$$\text{Unit vector: } \frac{\langle -1, 0, -1 \rangle}{\sqrt{2}} = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$$

56.  $\langle 2 - 1, 4 - (-2), -2 - 4 \rangle = \langle 1, 6, -6 \rangle$

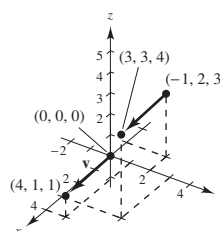
$$\|\langle 1, 6, -6 \rangle\| = \sqrt{1 + 36 + 36} = \sqrt{73}$$

$$\text{Unit vector: } \frac{\langle 1, 6, -6 \rangle}{\sqrt{73}} = \left\langle \frac{1}{\sqrt{73}}, \frac{6}{\sqrt{73}}, \frac{-6}{\sqrt{73}} \right\rangle$$

57. (b)  $\mathbf{v} = \langle 3 - (-1), 3 - 2, 4 - 3 \rangle = \langle 4, 1, 1 \rangle$

(c)  $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$

(a), (d)

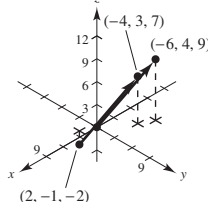


NOT FOR SALE

58. (b)  $\mathbf{v} = \langle -4 - 2, 3 - (-1), 7 - (-2) \rangle = \langle -6, 4, 9 \rangle$

(c)  $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$

(a), (d)



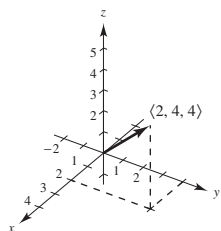
59.  $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$Q = (3, 1, 8)$

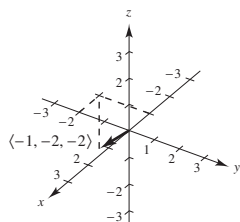
60.  $(q_1, q_2, q_3) - (0, 2, \frac{5}{2}) = (1, -\frac{2}{3}, \frac{1}{2})$

$Q = (1, -\frac{4}{3}, 3)$

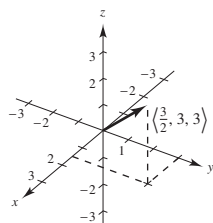
61. (a)  $2\mathbf{v} = \langle 2, 4, 4 \rangle$



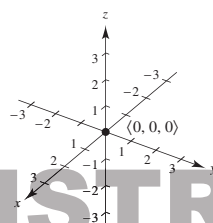
(b)  $-\mathbf{v} = \langle -1, -2, -2 \rangle$



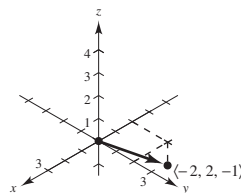
(c)  $\frac{3}{2}\mathbf{v} = \langle \frac{3}{2}, 3, 3 \rangle$



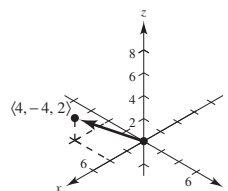
(d)  $0\mathbf{v} = \langle 0, 0, 0 \rangle$



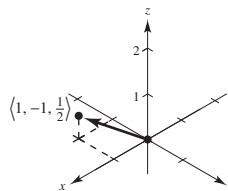
62. (a)  $-\mathbf{v} = \langle -2, 2, -1 \rangle$



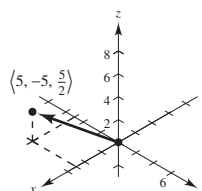
(b)  $2\mathbf{v} = \langle 4, -4, 2 \rangle$



(c)  $\frac{1}{2}\mathbf{v} = \langle 1, -1, \frac{1}{2} \rangle$



(d)  $\frac{5}{2}\mathbf{v} = \langle 5, -5, \frac{5}{2} \rangle$



63.  $\mathbf{z} = \mathbf{u} - \mathbf{v} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle = \langle -1, 0, 4 \rangle$

64.  $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

$$= \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle + \langle 8, 0, -8 \rangle = \langle 7, 0, -4 \rangle$$

65.  $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$

$$= \langle 2, 4, 6 \rangle + \langle 8, 8, -4 \rangle - \langle 4, 0, -4 \rangle = \langle 6, 12, 6 \rangle$$

66.  $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

$$= \langle 5, 10, 15 \rangle - \langle 6, 6, -3 \rangle - \langle 2, 0, -2 \rangle$$

$$= \langle -3, 4, 20 \rangle$$

67.  $2\mathbf{z} - 3\mathbf{u} = 2\langle z_1, z_2, z_3 \rangle - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$$2z_1 - 3 = 4 \Rightarrow z_1 = \frac{7}{2}$$

$$2z_2 - 6 = 0 \Rightarrow z_2 = 3$$

$$2z_3 - 9 = -4 \Rightarrow z_3 = \frac{5}{2}$$

$$\mathbf{z} = \langle \frac{7}{2}, 3, \frac{5}{2} \rangle$$

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68.  $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = 2\langle 1, 2, 3 \rangle + \langle 2, 2, -1 \rangle - \langle 4, 0, -4 \rangle + 3\langle z_1, z_2, z_3 \rangle = \langle 0, 0, 0 \rangle$   
 $\langle 0, 6, 9 \rangle + \langle 3z_1, 3z_2, 3z_3 \rangle = \langle 0, 0, 0 \rangle$   
 $0 + 3z_1 = 0 \Rightarrow z_1 = 0$   
 $6 + 3z_2 = 0 \Rightarrow z_2 = -2$   
 $9 + 3z_3 = 0 \Rightarrow z_3 = -3$   
 $\mathbf{z} = \langle 0, -2, -3 \rangle$

69. (a) and (b) are parallel because  
 $\langle -6, -4, 10 \rangle = -2\langle 3, 2, -5 \rangle$  and  
 $\langle 2, \frac{4}{3}, -\frac{10}{3} \rangle = \frac{2}{3}\langle 3, 2, -5 \rangle$ .

70. (b) and (d) are parallel because  
 $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k} = -2(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k})$  and  
 $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k} = \frac{3}{2}(\frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k})$ .

71.  $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$   
 (a) is parallel because  $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$ .

72.  $\mathbf{z} = \langle -7, -8, 3 \rangle$   
 (b) is parallel because  $(-z)\mathbf{z} = \langle 14, 16, -6 \rangle$ .

73.  $P(0, -2, -5), Q(3, 4, 4), R(2, 2, 1)$   
 $\overrightarrow{PQ} = \langle 3, 6, 9 \rangle$   
 $\overrightarrow{PR} = \langle 2, 4, 6 \rangle$   
 $\langle 3, 6, 9 \rangle = \frac{3}{2}\langle 2, 4, 6 \rangle$   
 So,  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are parallel, the points are collinear.

74.  $P(4, -2, 7), Q(-2, 0, 3), R(7, -3, 9)$   
 $\overrightarrow{PQ} = \langle -6, 2, -4 \rangle$   
 $\overrightarrow{PR} = \langle 3, -1, 2 \rangle$   
 $\langle 3, -1, 2 \rangle = -\frac{1}{2}\langle -6, 2, -4 \rangle$   
 So,  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are parallel. The points are collinear.

75.  $P(1, 2, 4), Q(2, 5, 0), R(0, 1, 5)$   
 $\overrightarrow{PQ} = \langle 1, 3, -4 \rangle$   
 $\overrightarrow{PR} = \langle -1, -1, 1 \rangle$   
 Because  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are not parallel, the points are not collinear.

76.  $P(0, 0, 0), Q(1, 3, -2), R(2, -6, 4)$   
 $\overrightarrow{PQ} = \langle 1, 3, -2 \rangle$   
 $\overrightarrow{PR} = \langle 2, -6, 4 \rangle$   
 Because  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  are not parallel, the points are not collinear.

77.  $A(2, 9, 1), B(3, 11, 4), C(0, 10, 2), D(1, 12, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{CD} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{AC} = \langle -2, 1, 1 \rangle$$

$$\overrightarrow{BD} = \langle -2, 1, 1 \rangle$$

Because  $\overrightarrow{AB} = \overrightarrow{CD}$  and  $\overrightarrow{AC} = \overrightarrow{BD}$ , the given points form the vertices of a parallelogram.

78.  $A(1, 1, 3), B(9, -1, -2), C(11, 2, -9), D(3, 4, -4)$

$$\overrightarrow{AB} = \langle 8, -2, -5 \rangle$$

$$\overrightarrow{DC} = \langle 8, -2, -5 \rangle$$

$$\overrightarrow{AD} = \langle 2, 3, -7 \rangle$$

$$\overrightarrow{BC} = \langle 2, 3, -7 \rangle$$

Because  $\overrightarrow{AB} = \overrightarrow{DC}$  and  $\overrightarrow{AD} = \overrightarrow{BC}$ , the given points form the vertices of a parallelogram.

79.  $\mathbf{v} = \langle 0, 0, 0 \rangle$

$$\|\mathbf{v}\| = 0$$

80.  $\mathbf{v} = \langle 1, 0, 3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$$

81.  $\mathbf{v} = 3\mathbf{j} - 5\mathbf{k} = \langle 0, 3, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$$

82.  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k} = \langle 2, 5, -1 \rangle$

$$\|\mathbf{v}\| = \sqrt{4 + 25 + 1} = \sqrt{30}$$

83.  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} = \langle 1, -2, -3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

84.  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} = \langle -4, 3, 7 \rangle$

$$\|\mathbf{v}\| = \sqrt{16 + 9 + 49} = \sqrt{74}$$

85.  $\mathbf{v} = \langle 2, -1, 2 \rangle$

$$\|\mathbf{v}\| = \sqrt{4 + 1 + 4} = 3$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}\langle 2, -1, 2 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{3}\langle 2, -1, 2 \rangle$

86.  $\mathbf{v} = \langle 6, 0, 8 \rangle$

$$\|\mathbf{v}\| = \sqrt{36 + 0 + 64} = 10$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{10}\langle 6, 0, 8 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{10}\langle 6, 0, 8 \rangle$

87.  $\mathbf{v} = \langle 3, 2, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{9 + 4 + 25} = \sqrt{38}$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

88.  $\mathbf{v} = \langle 8, 0, 0 \rangle$

$$\|\mathbf{v}\| = 8$$

(a)  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{8}\langle 8, 0, 0 \rangle$

(b)  $-\frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{8}\langle 8, 0, 0 \rangle$

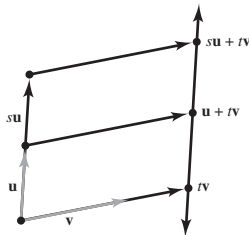
89. (a)–(d) Programs will vary.

(e)  $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2 \rangle$

$$\|\mathbf{u} + \mathbf{v}\| \approx 8.732$$

$$\|\mathbf{u}\| \approx 5.099$$

$$\|\mathbf{v}\| \approx 9.019$$

90. The terminal points of the vectors  $t\mathbf{u}$ ,  $\mathbf{u} + t\mathbf{v}$  and  $s\mathbf{u} + t\mathbf{v}$  are collinear.

91.  $\|c\mathbf{v}\| = \|c(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\| = \sqrt{4c^2 + 4c^2 + c^2} = 7$

$$\sqrt{9c^2} = 7$$

$$9c^2 = 49$$

$$c = \pm \frac{7}{3}$$

92.  $\|c\mathbf{u}\| = \|c(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\| = \sqrt{c^2 + 4c^2 + 9c^2} = 4$

$$\sqrt{14c^2} = 4$$

$$14c^2 = 16$$

$$c = \pm \frac{8}{7}$$

93.  $\mathbf{v} = 10 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 10 \frac{\langle 0, 3, 3 \rangle}{3\sqrt{2}}$

$$= 10 \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle$$

94.  $\mathbf{v} = 3 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 3 \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$

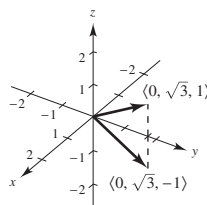
$$= 3 \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \left\langle \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{3}{\sqrt{3}} \right\rangle$$

95.  $\mathbf{v} = \frac{3}{2} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{3}{2} \frac{\langle 2, -2, 1 \rangle}{3} = \frac{1}{2} \left\langle 2, -2, 1 \right\rangle = \left\langle 1, -1, \frac{1}{2} \right\rangle$

96.  $\mathbf{v} = 7 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 7 \frac{\langle -4, 6, 2 \rangle}{2\sqrt{14}} = \left\langle \frac{-14}{\sqrt{14}}, \frac{21}{\sqrt{14}}, \frac{7}{\sqrt{14}} \right\rangle$

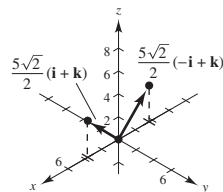
97.  $\mathbf{v} = 2[\cos(\pm 30^\circ)\mathbf{j} + \sin(\pm 30^\circ)\mathbf{k}]$

$$= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \langle 0, \sqrt{3}, \pm 1 \rangle$$



98.  $\mathbf{v} = 5(\cos 45^\circ\mathbf{i} + \sin 45^\circ\mathbf{k}) = \frac{5\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$  or

$$\mathbf{v} = 5(\cos 135^\circ\mathbf{i} + \sin 135^\circ\mathbf{k}) = \frac{5\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$$



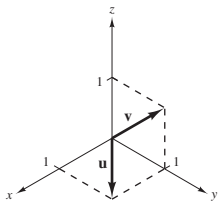
99.  $\mathbf{v} = \langle -3, -6, 3 \rangle$

$$\frac{2}{3}\mathbf{v} = \langle -2, -4, 2 \rangle$$

$$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$$

100.  $\mathbf{v} = \langle 5, 6, -3 \rangle$   
 $\frac{2}{3}\mathbf{v} = \langle \frac{10}{3}, 4, -2 \rangle$   
 $(1, 2, 5) + (\frac{10}{3}, 4, -2) = (\frac{13}{3}, 6, 3)$

101. (a)



(b)  $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{0}$

$a = 0, a + b = 0, b = 0$

So,  $a$  and  $b$  are both zero.

(c)  $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$a = 1, a + b = 2, b = 1$

$\mathbf{w} = \mathbf{u} + \mathbf{v}$

(d)  $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$a = 1, a + b = 2, b = 3$

Not possible

102. A sphere of radius 4 centered at  $(x_1, y_1, z_1)$ .

$$\|\mathbf{v}\| = \|\langle x - x_1, y - y_1, z - z_1 \rangle\|$$

$$= \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = 4$$

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = 16$$

103.  $x_0$  is directed distance to  $yz$ -plane.

$y_0$  is directed distance to  $xz$ -plane.

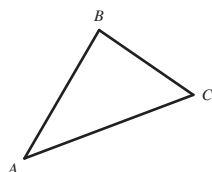
$z_0$  is directed distance to  $xy$ -plane.

104.  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

105.  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

106. Two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are parallel if  $\mathbf{u} = c\mathbf{v}$  for some scalar  $c$ .

107.



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

So,  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA} = \mathbf{0}$

108.  $\|\mathbf{r} - \mathbf{r}_0\| = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2} = 2$   
 $(x-1)^2 + (y-1)^2 + (z-1)^2 = 4$

This is a sphere of radius 2 and center  $(1, 1, 1)$ .

109. (a) The height of the right triangle is  $h = \sqrt{L^2 - 18^2}$ .

The vector  $\overrightarrow{PQ}$  is given by

$\overrightarrow{PQ} = \langle 0, -18, h \rangle$ .

The tension vector  $\mathbf{T}$  in each wire is

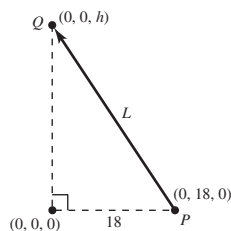
$\mathbf{T} = c\langle 0, -18, h \rangle$  where  $ch = \frac{24}{3} = 8$ .

So,  $\mathbf{T} = \frac{8}{h}\langle 0, -18, h \rangle$  and

$$T = \|\mathbf{T}\| = \frac{8}{h}\sqrt{18^2 + h^2}$$

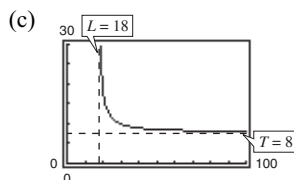
$$= \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + (L^2 - 18^2)}$$

$$= \frac{8L}{\sqrt{L^2 - 18^2}}, L > 18.$$



(b)

$L$	20	25	30	35	40	45	50
$T$	18.4	11.5	10	9.3	9.0	8.7	8.6



$x = 18$  is a vertical asymptote and  $y = 8$  is a horizontal asymptote.

(d)  $\lim_{L \rightarrow 18^+} \frac{8L}{\sqrt{L^2 - 18^2}} = \infty$

$$\lim_{L \rightarrow \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \rightarrow \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$$

(e) From the table,  $T = 10$  implies  $L = 30$  inches.

110. As in Exercise 109(c),  $x = a$  will be a vertical asymptote. So,  $\lim_{r_0 \rightarrow a^-} T = \infty$ .

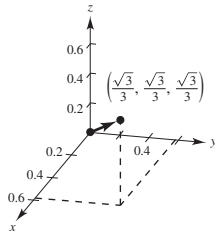
111. Let
- $\alpha$
- be the angle between
- $\mathbf{v}$
- and the coordinate axes.

$$\mathbf{v} = (\cos \alpha)\mathbf{i} + (\cos \alpha)\mathbf{j} + (\cos \alpha)\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{3} \cos \alpha = 1$$

$$\cos \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}\langle 1, 1, 1 \rangle$$



- 112.
- $550 = \|c(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})\|$

$$302,500 = 18,125c^2$$

$$c^2 = 16.689655$$

$$c \approx 4.085$$

$$\mathbf{F} \approx 4.085(75\mathbf{i} - 50\mathbf{j} - 100\mathbf{k})$$

$$\approx 306\mathbf{i} - 204\mathbf{j} - 409\mathbf{k}$$

114. Let
- $A$
- lie on the
- $y$
- axis and the wall on the
- $x$
- axis. Then
- $A = (0, 10, 0)$
- ,
- $B = (8, 0, 6)$
- ,
- $C = (-10, 0, 6)$
- and

$$\overrightarrow{AB} = \langle 8, -10, 6 \rangle, \overrightarrow{AC} = \langle -10, -10, 6 \rangle$$

$$\|AB\| = 10\sqrt{2}, \|AC\| = 2\sqrt{59}$$

$$\text{Thus, } \mathbf{F}_1 = 420 \frac{\overrightarrow{AB}}{\|AB\|}, \mathbf{F}_2 = 650 \frac{\overrightarrow{AC}}{\|AC\|}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \approx \langle 237.6, -297.0, 178.2 \rangle + \langle -423.1, -423.1, 253.9 \rangle \approx \langle -185.5, -720.1, 432.1 \rangle$$

$$\|\mathbf{F}\| \approx 860.0 \text{ lb}$$

- 115.
- $d(AP) = 2d(BP)$

$$\sqrt{x^2 + (y+1)^2 + (z-1)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + (y^2 - 6y + 9) + \left(z^2 + \frac{2}{3}z + \frac{1}{9}\right)$$

$$\frac{44}{9} = \left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2$$

$$\text{Sphere; center: } \left(\frac{4}{3}, 3, -\frac{1}{3}\right), \text{radius: } \frac{2\sqrt{11}}{3}$$

- 113.
- $\overrightarrow{AB} = \langle 0, 70, 115 \rangle, \mathbf{F}_1 = C_1 \langle 0, 70, 115 \rangle$

$$\overrightarrow{AC} = \langle -60, 0, 115 \rangle, \mathbf{F}_2 = C_2 \langle -60, 0, 115 \rangle$$

$$\overrightarrow{AD} = \langle 45, -65, 115 \rangle, \mathbf{F}_3 = C_3 \langle 45, -65, 115 \rangle$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$$

$$\text{So: } -60C_2 + 45C_3 = 0$$

$$70C_1 - 65C_3 = 0$$

$$115(C_1 + C_2 + C_3) = 500$$

Solving this system yields  $C_1 = \frac{104}{69}$ ,  $C_2 = \frac{28}{23}$ , and

$$C_3 = \frac{112}{69}. \text{ So:}$$

$$\|\mathbf{F}_1\| \approx 202.919 \text{ N}$$

$$\|\mathbf{F}_2\| \approx 157.909 \text{ N}$$

$$\|\mathbf{F}_3\| \approx 226.521 \text{ N}$$

## Section 11.3 The Dot Product of Two Vectors

1.  $\mathbf{u} = \langle 3, 4 \rangle, \mathbf{v} = \langle -1, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 3(-1) + 4(5) = 17$

(b)  $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c)  $\|\mathbf{u}\|^2 = 3^2 + 4^2 = 25$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 17\langle -1, 5 \rangle = \langle -17, 85 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(17) = 34$

2.  $\mathbf{u} = \langle 4, 10 \rangle, \mathbf{v} = \langle -2, 3 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 4(-2) + 10(3) = 22$

(b)  $\mathbf{u} \cdot \mathbf{u} = 4(4) + 10(10) = 116$

(c)  $\|\mathbf{u}\|^2 = 4^2 + 10^2 = 116$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 22\langle -2, 3 \rangle = \langle -44, 66 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(22) = 44$

3.  $\mathbf{u} = \langle 6, -4 \rangle, \mathbf{v} = \langle -3, 2 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 6(-3) + (-4)(2) = -26$

(b)  $\mathbf{u} \cdot \mathbf{u} = 6(6) + (-4)(-4) = 52$

(c)  $\|\mathbf{u}\|^2 = 6^2 + (-4)^2 = 52$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -26\langle -3, 2 \rangle = \langle 78, -52 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-26) = -52$

4.  $\mathbf{u} = \langle -4, 8 \rangle, \mathbf{v} = \langle 7, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = -4(7) + 8(5) = 12$

(b)  $\mathbf{u} \cdot \mathbf{u} = (-4)(-4) + 8(8) = 80$

(c)  $\|\mathbf{u}\|^2 = (-4)^2 + 8^2 = 80$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 12\langle 7, 5 \rangle = \langle 84, 60 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(12) = 24$

5.  $\mathbf{u} = \langle 2, -3, 4 \rangle, \mathbf{v} = \langle 0, 6, 5 \rangle$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + 4(5) = 2$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c)  $\|\mathbf{u}\|^2 = 2^2 + (-3)^2 + 4^2 = 29$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5 \rangle = \langle 0, 12, 10 \rangle$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

6.  $\mathbf{u} = \mathbf{i}, \mathbf{v} = \mathbf{i}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 1$

(b)  $\mathbf{u} \cdot \mathbf{u} = 1$

(c)  $\|\mathbf{u}\|^2 = 1$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{i}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

7.  $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + 1(1) = 6$

(c)  $\|\mathbf{u}\|^2 = 2^2 + (-1)^2 + 1^2 = 6$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

8.  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a)  $\mathbf{u} \cdot \mathbf{v} = 2(1) + 1(-3) + (-2)(2) = -5$

(b)  $\mathbf{u} \cdot \mathbf{u} = 2(2) + 1(1) + (-2)(-2) = 9$

(c)  $\|\mathbf{u}\|^2 = 2^2 + 1^2 + (-2)^2 = 9$

(d)  $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -5(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) = -5\mathbf{i} + 15\mathbf{j} - 10\mathbf{k}$

(e)  $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-5) = -10$

9.  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos \theta$

$\mathbf{u} \cdot \mathbf{v} = (8)(5) \cos \frac{\pi}{3} = 20$

10.  $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos \theta$

$\mathbf{u} \cdot \mathbf{v} = (40)(25) \cos \frac{5\pi}{6} = -500\sqrt{3}$

11.  $\mathbf{u} = \langle 1, 1 \rangle, \mathbf{v} = \langle 2, -2 \rangle$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$

$\theta = \frac{\pi}{2}$

12.  $\mathbf{u} = \langle 3, 1 \rangle, \mathbf{v} = \langle 2, -1 \rangle$

$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

13.  $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ ,  $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-2}{\sqrt{10}\sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

$$\theta = \arccos\left(-\frac{1}{5\sqrt{2}}\right) \approx 98.1^\circ$$

14.  $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

$$\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\begin{aligned}\cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \\ &= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}(1 - \sqrt{3})\end{aligned}$$

$$\theta = \arccos\left[\frac{\sqrt{2}}{4}(1 - \sqrt{3})\right] = 105^\circ$$

15.  $\mathbf{u} = \langle 1, 1, 1 \rangle$ ,  $\mathbf{v} = \langle 2, 1, -1 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\theta = \arccos \frac{\sqrt{2}}{3} \approx 61.9^\circ$$

16.  $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{3(2) + 2(-3) + 0}{\|\mathbf{u}\| \|\mathbf{v}\|} = 0$$

$$\theta = \frac{\pi}{2}$$

17.  $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ ,  $\mathbf{v} = 2\mathbf{j} + 3\mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

$$\theta = \arccos\left(-\frac{8\sqrt{13}}{65}\right) \approx 116.3^\circ$$

18.  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{9}{\sqrt{14}\sqrt{6}} = \frac{9}{2\sqrt{21}} = \frac{3\sqrt{21}}{14}$$

$$\theta = \arccos\left(\frac{3\sqrt{21}}{14}\right) \approx 10.9^\circ$$

19.  $\mathbf{u} = \langle 4, 0 \rangle$ ,  $\mathbf{v} = \langle 1, 1 \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0 \Rightarrow$  not orthogonal

Neither

20.  $\mathbf{u} = \langle 2, 18 \rangle$ ,  $\mathbf{v} = \langle \frac{3}{2}, -\frac{1}{6} \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

21.  $\mathbf{u} = \langle 4, 3 \rangle$ ,  $\mathbf{v} = \langle \frac{1}{2}, -\frac{2}{3} \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

22.  $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$ ,  $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} = -\frac{1}{6}\mathbf{v} \Rightarrow$  parallel

23.  $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$ ,  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = -8 \neq 0 \Rightarrow$  not orthogonal

Neither

24.  $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ ,  $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

25.  $\mathbf{u} = \langle 2, -3, 1 \rangle$ ,  $\mathbf{v} = \langle -1, -1, -1 \rangle$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

26.  $\mathbf{u} = \langle \cos \theta, \sin \theta, -1 \rangle$ ,

$$\mathbf{v} = \langle \sin \theta, -\cos \theta, 0 \rangle$$

$\mathbf{u} \neq c\mathbf{v} \Rightarrow$  not parallel

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow$  orthogonal

27. The vector  $\langle 1, 2, 0 \rangle$  joining  $(1, 2, 0)$  and  $(0, 0, 0)$  is

perpendicular to the vector  $\langle -2, 1, 0 \rangle$  joining

$(-2, 1, 0)$  and  $(0, 0, 0)$ :  $\langle 1, 2, 0 \rangle \cdot \langle -2, 1, 0 \rangle = 0$

The triangle has a right angle, so it is a right triangle.

28. Consider the vector  $\langle -3, 0, 0 \rangle$  joining  $(0, 0, 0)$  and

$(-3, 0, 0)$ , and the vector  $\langle 1, 2, 3 \rangle$  joining  $(0, 0, 0)$  and

$(1, 2, 3)$ :  $\langle -3, 0, 0 \rangle \cdot \langle 1, 2, 3 \rangle = -3 < 0$

The triangle has an obtuse angle, so it is an obtuse triangle.

29.  $A(2, 0, 1)$ ,  $B(0, 1, 2)$ ,  $C(-\frac{1}{2}, \frac{3}{2}, 0)$

$$\overrightarrow{AB} = \langle -2, 1, 1 \rangle \quad \overrightarrow{BA} = \langle 2, -1, -1 \rangle$$

$$\overrightarrow{AC} = \langle -\frac{5}{2}, \frac{3}{2}, -1 \rangle \quad \overrightarrow{CA} = \langle \frac{5}{2}, -\frac{3}{2}, 1 \rangle$$

$$\overrightarrow{BC} = \langle -\frac{1}{2}, \frac{1}{2}, -2 \rangle \quad \overrightarrow{CB} = \langle \frac{1}{2}, -\frac{1}{2}, 2 \rangle$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 5 + \frac{3}{2} - 1 > 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = -1 - \frac{1}{2} + 2 > 0$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = \frac{5}{4} + \frac{3}{4} + 2 > 0$$

The triangle has three acute angles, so it is an acute triangle.



30.  $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$$\overrightarrow{AB} = \langle -3, 12, 5 \rangle \quad \overrightarrow{BA} = \langle 3, -12, -5 \rangle$$

$$\overrightarrow{AC} = \langle 2, 13, -4 \rangle \quad \overrightarrow{CA} = \langle -2, -13, 4 \rangle$$

$$\overrightarrow{BC} = \langle 5, 1, -9 \rangle \quad \overrightarrow{CB} = \langle -5, -1, 9 \rangle$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = -6 + 156 - 20 > 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = 15 - 12 + 45 > 0$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = 10 + 13 + 36 > 0$$

The triangle has three acute angles, so it is an acute triangle.

31.  $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \|\mathbf{u}\| = 3$

$$\cos \alpha = \frac{1}{3}$$

$$\cos \beta = \frac{2}{3}$$

$$\cos \gamma = \frac{2}{3}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

34.  $\mathbf{u} = \langle a, b, c \rangle, \|\mathbf{u}\| = \sqrt{a^2 + b^2 + c^2}$

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos \gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2} = 1$$

35.  $\mathbf{u} = \langle 3, 2, -2 \rangle, \|\mathbf{u}\| = \sqrt{17}$

$$\cos \alpha = \frac{3}{\sqrt{17}} \Rightarrow \alpha \approx 0.7560 \text{ or } 43.3^\circ$$

$$\cos \beta = \frac{2}{\sqrt{17}} \Rightarrow \beta \approx 1.0644 \text{ or } 61.0^\circ$$

$$\cos \gamma = \frac{-2}{\sqrt{17}} \Rightarrow \gamma \approx 2.0772 \text{ or } 119.0^\circ$$

36.  $\mathbf{u} = \langle -4, 3, 5 \rangle, \|\mathbf{u}\| = \sqrt{50} = 5\sqrt{2}$

$$\cos \alpha = \frac{-4}{5\sqrt{2}} \Rightarrow \alpha \approx 2.1721 \text{ or } 124.4^\circ$$

$$\cos \beta = \frac{3}{5\sqrt{2}} \Rightarrow \beta \approx 1.1326 \text{ or } 64.9^\circ$$

$$\cos \gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \gamma \approx \frac{\pi}{4} \text{ or } 45^\circ$$

32.  $\mathbf{u} = \langle 5, 3, -1 \rangle, \|\mathbf{u}\| = \sqrt{35}$

$$\cos \alpha = \frac{5}{\sqrt{35}}$$

$$\cos \beta = \frac{3}{\sqrt{35}}$$

$$\cos \gamma = \frac{-1}{\sqrt{35}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{25}{35} + \frac{9}{35} + \frac{1}{35} = 1$$

33.  $\mathbf{u} = \langle 0, 6, -4 \rangle, \|\mathbf{u}\| = \sqrt{52} = 2\sqrt{13}$

$$\cos \alpha = 0$$

$$\cos \beta = \frac{3}{\sqrt{13}}$$

$$\cos \gamma = -\frac{2}{\sqrt{13}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 0 + \frac{9}{13} + \frac{4}{13} = 1$$

$$37. \mathbf{u} = \langle -1, 5, 2 \rangle \quad \|\mathbf{u}\| = \sqrt{30}$$

$$\cos \alpha = \frac{-1}{\sqrt{30}} \Rightarrow \alpha \approx 1.7544 \text{ or } 100.5^\circ$$

$$\cos \beta = \frac{5}{\sqrt{30}} \Rightarrow \beta \approx 0.4205 \text{ or } 24.1^\circ$$

$$\cos \gamma = \frac{2}{\sqrt{30}} \Rightarrow \gamma \approx 1.1970 \text{ or } 68.6^\circ$$

$$38. \mathbf{u} = \langle -2, 6, 1 \rangle \quad \|\mathbf{u}\| = \sqrt{41}$$

$$\cos \alpha = \frac{-2}{\sqrt{41}} \Rightarrow \alpha \approx 1.8885 \text{ or } 108.2^\circ$$

$$\cos \beta = \frac{6}{\sqrt{41}} \Rightarrow \beta \approx 0.3567 \text{ or } 20.4^\circ$$

$$\cos \gamma = \frac{1}{\sqrt{41}} \Rightarrow \gamma \approx 1.4140 \text{ or } 81.0^\circ$$

$$39. \mathbf{F}_1: C_1 = \frac{50}{\|\mathbf{F}_1\|} \approx 4.3193$$

$$\mathbf{F}_2: C_2 = \frac{80}{\|\mathbf{F}_2\|} \approx 5.4183$$

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &\approx 4.3193\langle 10, 5, 3 \rangle + 5.4183\langle 12, 7, -5 \rangle \\ &= \langle 108.2126, 59.5246, -14.1336 \rangle \end{aligned}$$

$$\|\mathbf{F}\| \approx 124.310 \text{ lb}$$

$$\cos \alpha \approx \frac{108.2126}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 29.48^\circ$$

$$\cos \beta \approx \frac{59.5246}{\|\mathbf{F}\|} \Rightarrow \beta \approx 61.39^\circ$$

$$\cos \gamma \approx \frac{-14.1336}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 96.53^\circ$$

$$40. \mathbf{F}_1: C_1 = \frac{300}{\|\mathbf{F}_1\|} \approx 13.0931$$

$$\mathbf{F}_2: C_2 = \frac{100}{\|\mathbf{F}_2\|} \approx 6.3246$$

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &\approx 13.0931\langle -20, -10, 5 \rangle + 6.3246\langle 5, 15, 0 \rangle \\ &= \langle -230.239, 36.062, 65.4655 \rangle \end{aligned}$$

$$\|\mathbf{F}\| \approx 242.067 \text{ lb}$$

$$\cos \alpha \approx \frac{-230.239}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 162.02^\circ$$

$$\cos \beta \approx \frac{-36.062}{\|\mathbf{F}\|} \Rightarrow \beta \approx 98.57^\circ$$

$$\cos \gamma \approx \frac{65.4655}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 74.31^\circ$$

$$41. \overline{OA} = \langle 0, 10, 10 \rangle$$

$$\cos \alpha = \frac{0}{\sqrt{0^2 + 10^2 + 10^2}} = 0 \Rightarrow \alpha = 90^\circ$$

$$\begin{aligned} \cos \beta &= \cos \gamma = \frac{10}{\sqrt{0^2 + 10^2 + 10^2}} \\ &= \frac{1}{\sqrt{2}} \Rightarrow \beta = \gamma = 45^\circ \end{aligned}$$

$$42. \mathbf{F}_1 = C_1\langle 0, 10, 10 \rangle.$$

$$\|\mathbf{F}_1\| = 200 = C_1 10\sqrt{2} \Rightarrow C_1 = 10\sqrt{2} \text{ and}$$

$$\mathbf{F}_1 = \langle 0, 100\sqrt{2}, 100\sqrt{2} \rangle$$

$$\mathbf{F}_2 = C_2\langle -4, -6, 10 \rangle$$

$$\mathbf{F}_3 = C_3\langle 4, -6, 10 \rangle$$

$$\mathbf{F} = \langle 0, 0, w \rangle$$

$$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$$

$$-4C_2 + 4C_3 = 0 \Rightarrow C_2 = C_3$$

$$100\sqrt{2} - 6C_2 - 6C_3 = 0 \Rightarrow C_2 = C_3 = \frac{25\sqrt{2}}{3} \text{ N}$$

$$W = 10C_2 + 10C_3 + 100\sqrt{2} = \frac{800\sqrt{2}}{3}$$

$$43. \mathbf{u} = \langle 6, 7 \rangle, \mathbf{v} = \langle 1, 4 \rangle$$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{6(1) + 7(4)}{1^2 + 4^2} \langle 1, 4 \rangle \\ &= \frac{34}{17} \langle 1, 4 \rangle = \langle 2, 8 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$$

$$44. \mathbf{u} = \langle 9, 7 \rangle, \mathbf{v} = \langle 1, 3 \rangle$$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{9(1) + 7(3)}{1 + 3^2} \langle 1, 3 \rangle \\ &= \frac{30}{10} \langle 1, 3 \rangle = \langle 3, 9 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 9, 7 \rangle - \langle 3, 9 \rangle = \langle 6, -2 \rangle$$

45.  $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} = \langle 2, 3 \rangle$ ,  $\mathbf{v} = 5\mathbf{i} + \mathbf{j} = \langle 5, 1 \rangle$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(5) + 3(1)}{5^2 + 1} \langle 5, 1 \rangle \\ &= \frac{13}{26} \langle 5, 1 \rangle = \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 3 \rangle - \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle = \left\langle -\frac{1}{2}, \frac{5}{2} \right\rangle$$

46.  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} = \langle 2, -3 \rangle$ ,  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} = \langle 3, 2 \rangle$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(3) + (-3)(2)}{3^2 + 2^2} \langle 3, 2 \rangle \\ &= 0 \langle 3, 2 \rangle = \langle 0, 0 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, -3 \rangle$$

47.  $\mathbf{u} = \langle 0, 3, 3 \rangle$ ,  $\mathbf{v} = \langle -1, 1, 1 \rangle$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{0(-1) + 3(1) + 3(1)}{1 + 1 + 1} \langle -1, 1, 1 \rangle \\ &= \frac{6}{3} \langle -1, 1, 1 \rangle = \langle -2, 2, 2 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3 \rangle - \langle -2, 2, 2 \rangle = \langle 2, 1, 1 \rangle$$

48.  $\mathbf{u} = \langle 8, 2, 0 \rangle$ ,  $\mathbf{v} = \langle 2, 1, -1 \rangle$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{8(2) + 2(1) + 0(-1)}{2^2 + 1 + 1} \langle 2, 1, -1 \rangle \\ &= \frac{18}{6} \langle 2, 1, -1 \rangle = \langle 6, 3, -3 \rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 8, 2, 0 \rangle - \langle 6, 3, -3 \rangle = \langle 2, -1, 3 \rangle$$

49.  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \langle 2, 1, 2 \rangle$

$$\mathbf{v} = 3\mathbf{j} + 4\mathbf{k} = \langle 0, 3, 4 \rangle$$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{2(0) + 1(3) + 2(4)}{3^2 + 4^2} \langle 0, 3, 4 \rangle \\ &= \frac{11}{25} \langle 0, 3, 4 \rangle = \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 2, 1, 2 \rangle - \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle = \left\langle 2, -\frac{8}{25}, \frac{6}{25} \right\rangle$$

50.  $\mathbf{u} = \mathbf{i} + 4\mathbf{k} = \langle 1, 0, 4 \rangle$

$$\mathbf{v} = 3\mathbf{i} + 2\mathbf{k} = \langle 3, 0, 2 \rangle$$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{1(3) + 4(2)}{3^2 + 2^2} \langle 3, 0, 2 \rangle \\ &= \frac{11}{13} \langle 3, 0, 2 \rangle = \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{(b) } \mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1 = \langle 1, 0, 4 \rangle - \left\langle \frac{33}{13}, 0, \frac{22}{13} \right\rangle \\ &= \left\langle -\frac{20}{13}, 0, \frac{30}{13} \right\rangle \end{aligned}$$

51.  $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$

52. The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal if  $\mathbf{u} \cdot \mathbf{v} = 0$ . The angle  $\theta$  between  $\mathbf{u}$  and  $\mathbf{v}$  is given by  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$ .

53. (a) and (b) are defined. (c) and (d) are not defined because it is not possible to find the dot product of a scalar and a vector or to add a scalar to a vector.

54. See page 786. Direction cosines of  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  are  $\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}$ ,  $\cos \beta = \frac{v_2}{\|\mathbf{v}\|}$ ,  $\cos \gamma = \frac{v_3}{\|\mathbf{v}\|}$ .  $\alpha$ ,  $\beta$ , and  $\gamma$  are the direction angles. See Figure 11.26.

55. See figure 11.29, page 787.

56. (a)  $\left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{u} \Rightarrow \mathbf{u} = c\mathbf{v} \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are parallel.

(b)  $\left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

57. Yes, 
$$\left\| \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right\| = \left\| \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \right\|$$
  

$$|\mathbf{u} \cdot \mathbf{v}| \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v} \cdot \mathbf{u}| \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$$
  

$$\frac{1}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{u}\|}$$
  

$$\|\mathbf{u}\| = \|\mathbf{v}\|$$

58. (a) Orthogonal,  $\theta = \frac{\pi}{2}$

(b) Acute,  $0 < \theta < \frac{\pi}{2}$

(c) Obtuse,  $\frac{\pi}{2} < \theta < \pi$

59.  $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$   
 $\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$   
 $\mathbf{u} \cdot \mathbf{v} = 3240(1.35) + 1450(2.65) + 2235(1.85)$   
 $= \$12,351.25$   
 This represents the total amount that the restaurant earned on its three products.
60.  $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$   
 $\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$   
 Increase prices by 4%:  $1.04\mathbf{v}$   
 New total amount:  $1.04(\mathbf{u} \cdot \mathbf{v}) = 1.04(12,351.25)$   
 $= \$12,845.30$
61. (a)–(c) Programs will vary.
62.  $\|\mathbf{u}\| \approx 9.165$   
 $\|\mathbf{v}\| \approx 5.745$   
 $\theta = 90^\circ$
63. Programs will vary.
64.  $\left\langle -\frac{21}{26}, \frac{63}{26}, \frac{42}{13} \right\rangle$
65. Because  $\mathbf{u}$  and  $\mathbf{v}$  are parallel,  $\text{proj}_{\mathbf{u}} \mathbf{u} = \mathbf{u}$
66. Because  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular,  $\text{proj}_{\mathbf{u}} \mathbf{u} = \mathbf{0}$
67. Answers will vary. *Sample answer:*  
 $\mathbf{u} = -\frac{1}{4}\mathbf{i} + \frac{3}{2}\mathbf{j}$ . Want  $\mathbf{u} \cdot \mathbf{v} = 0$ .  
 $\mathbf{v} = 12\mathbf{i} + 2\mathbf{j}$  and  $-\mathbf{v} = -12\mathbf{i} - 2\mathbf{j}$  are orthogonal to  $\mathbf{u}$ .
68. Answers will vary. *Sample answer:*  
 $\mathbf{u} = 9\mathbf{i} - 4\mathbf{j}$ . Want  $\mathbf{u} \cdot \mathbf{v} = 0$ .  
 $\mathbf{v} = 4\mathbf{i} + 9\mathbf{j}$  and  $-\mathbf{v} = -4\mathbf{i} - 9\mathbf{j}$   
 are orthogonal to  $\mathbf{u}$ .
69. Answers will vary. *Sample answer:*  
 $\mathbf{u} = \langle 3, 1, -2 \rangle$ . Want  $\mathbf{u} \cdot \mathbf{v} = 0$ .  
 $\mathbf{v} = \langle 0, 2, 1 \rangle$  and  $-\mathbf{v} = \langle 0, -2, -1 \rangle$  are orthogonal to  $\mathbf{u}$ .
70. Answers will vary. *Sample answer:*  
 $\mathbf{u} = \langle 4, -3, 6 \rangle$ . Want  $\mathbf{u} \cdot \mathbf{v} = 0$   
 $\mathbf{v} = \langle 0, 6, 3 \rangle$  and  $-\mathbf{v} = \langle 0, -6, -3 \rangle$   
 are orthogonal to  $\mathbf{u}$ .

71. (a) Gravitational Force  $\mathbf{F} = -48,000\mathbf{j}$   
 $\mathbf{v} = \cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j}$   
 $\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = (\mathbf{F} \cdot \mathbf{v}) \mathbf{v}$   
 $= (-48,000)(\sin 10^\circ) \mathbf{v}$   
 $\approx -8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$   
 $\|\mathbf{w}_1\| \approx 8335.1 \text{ lb}$
- (b)  $\mathbf{w}_2 = \mathbf{F} - \mathbf{w}_1$   
 $= -48,000\mathbf{j} + 8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$   
 $= 8208.5\mathbf{i} - 46,552.6\mathbf{j}$   
 $\|\mathbf{w}_2\| \approx 47,270.8 \text{ lb}$
72.  $\overline{OA} = \langle 10, 5, 20 \rangle$ ,  $\mathbf{v} = \langle 0, 0, 1 \rangle$   
 $\text{proj}_{\mathbf{v}} \overline{OA} = \frac{20}{1^2} \langle 0, 0, 1 \rangle = \langle 0, 0, 20 \rangle$   
 $\|\text{proj}_{\mathbf{v}} \overline{OA}\| = 20$
73.  $\mathbf{F} = 85 \left( \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} \right)$   
 $\mathbf{v} = 10\mathbf{i}$   
 $W = \mathbf{F} \cdot \mathbf{v} = 425 \text{ ft-lb}$
74.  $\mathbf{F} = 25(\cos 20^\circ \mathbf{i} + \sin 20^\circ \mathbf{j})$   
 $\mathbf{v} = 50\mathbf{i}$   
 $W = \mathbf{F} \cdot \mathbf{v} = 1250 \cos 20^\circ \approx 1174.6 \text{ ft-lb}$
75.  $\mathbf{F} = 1600(\cos 25^\circ \mathbf{i} + \sin 25^\circ \mathbf{j})$   
 $\mathbf{v} = 2000\mathbf{i}$   
 $W = \mathbf{F} \cdot \mathbf{v} = 1600(2000)\cos 25^\circ$   
 $\approx 2,900,184.9 \text{ Newton meters (Joules)}$   
 $\approx 2900.2 \text{ km-N}$
76.  $\overline{PQ} = 40\mathbf{i}$   
 $\mathbf{F} = 100 \cos 25^\circ \mathbf{i}$   
 $W = \mathbf{F} \cdot \overline{PQ} = 4000 \cos 25^\circ \approx 3625.2 \text{ Joules}$
77. False.  
 For example, let  $\mathbf{u} = \langle 1, 1 \rangle$ ,  $\mathbf{v} = \langle 2, 3 \rangle$  and  
 $\mathbf{w} = \langle 1, 4 \rangle$ . Then  $\mathbf{u} \cdot \mathbf{v} = 2 + 3 = 5$  and  
 $\mathbf{u} \cdot \mathbf{w} = 1 + 4 = 5$ .
78. True  
 $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} = 0 + 0 = 0$  so,  $\mathbf{w}$  and  
 $\mathbf{u} + \mathbf{v}$  are orthogonal.

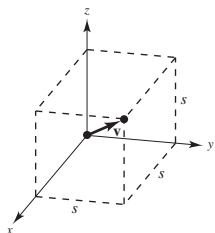
79. Let  $s$  = length of a side.

$$\mathbf{v} = \langle s, s, s \rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$



80.  $\mathbf{v}_1 = \langle s, s, s \rangle$

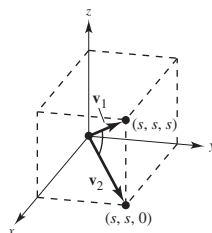
$$\|\mathbf{v}_1\| = s\sqrt{3}$$

$$\mathbf{v}_2 = \langle s, s, 0 \rangle$$

$$\|\mathbf{v}_2\| = s\sqrt{2}$$

$$\cos \theta = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\theta = \arccos\frac{\sqrt{6}}{3} \approx 35.26^\circ$$



81. (a) The graphs  $y_1 = x^2$  and  $y_2 = x^{1/3}$  intersect at  $(0, 0)$  and  $(1, 1)$ .

(b)  $y'_1 = 2x$  and  $y'_2 = \frac{1}{3x^{2/3}}$ .

At  $(0, 0)$ ,  $\pm \langle 1, 0 \rangle$  is tangent to  $y_1$  and  $\pm \langle 0, 1 \rangle$  is tangent to  $y_2$ .

At  $(1, 1)$ ,  $y'_1 = 2$  and  $y'_2 = \frac{1}{3}$ .

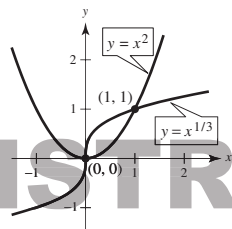
$\pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}} \langle 3, 1 \rangle$  is tangent to  $y_2$ .

- (c) At  $(0, 0)$ , the vectors are perpendicular ( $90^\circ$ ).

At  $(1, 1)$ ,

$$\cos \theta = \frac{\frac{1}{\sqrt{5}} \langle 1, 2 \rangle \cdot \frac{1}{\sqrt{10}} \langle 3, 1 \rangle}{(1)(1)} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$



82. (a) The graphs  $y_1 = x^3$  and  $y_2 = x^{1/3}$  intersect at  $(-1, -1)$ ,  $(0, 0)$  and  $(1, 1)$ .

(b)  $y'_1 = 3x^2$  and  $y'_2 = \frac{1}{3x^{2/3}}$ .

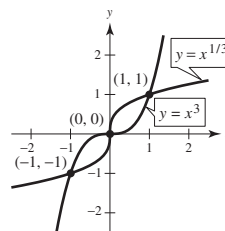
At  $(0, 0)$ ,  $\pm \langle 1, 0 \rangle$  is tangent to  $y_1$  and  $\pm \langle 0, 1 \rangle$  is tangent to  $y_2$ .

At  $(1, 1)$ ,  $y'_1 = 3$  and  $y'_2 = \frac{1}{3}$ .

$\pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}} \langle 3, 1 \rangle$  is tangent to  $y_2$ .

At  $(-1, -1)$ ,  $y'_1 = 3$  and  $y'_2 = \frac{1}{3}$ .

$\pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$  is tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{10}} \langle 3, 1 \rangle$  is tangent to  $y_2$ .



- (c) At  $(0, 0)$ , the vectors are perpendicular ( $90^\circ$ ).

At  $(1, 1)$ ,

$$\cos \theta = \frac{\frac{1}{\sqrt{10}} \langle 1, 3 \rangle \cdot \frac{1}{\sqrt{10}} \langle 3, 1 \rangle}{(1)(1)} = \frac{6}{10} = \frac{3}{5}$$

$$\theta \approx 0.9273 \text{ or } 53.13^\circ$$

By symmetry, the angle is the same at  $(-1, -1)$ .

83. (a) The graphs of  $y_1 = 1 - x^2$  and  $y_2 = x^2 - 1$  intersect at  $(1, 0)$  and  $(-1, 0)$ .

(b)  $y'_1 = -2x$  and  $y'_2 = 2x$ .

At  $(1, 0)$ ,  $y'_1 = -2$  and  $y'_2 = 2$ .  $\pm \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$  is

tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$  is tangent to  $y_2$ .

At  $(-1, 0)$ ,  $y'_1 = 2$  and  $y'_2 = -2$ .  $\pm \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$  is

tangent to  $y_1$ ,  $\pm \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$  is tangent to  $y_2$ .

(c) At  $(1, 0)$ ,  $\cos \theta = \frac{\frac{1}{\sqrt{5}} \langle 1, -2 \rangle \cdot \frac{-1}{\sqrt{5}} \langle 1, -2 \rangle}{\frac{1}{\sqrt{5}} \cdot \frac{-1}{\sqrt{5}}} = \frac{3}{5}$

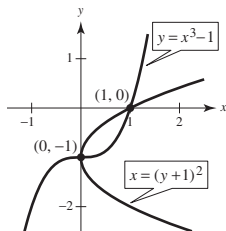
$$\theta \approx 0.9273 \text{ or } 53.13^\circ$$

By symmetry, the angle is the same at  $(-1, 0)$ .

84. (a) To find the intersection points, rewrite the second equation as  $y + 1 = x^3$ . Substituting into the first equation

$$(y + 1)^2 = x \Rightarrow x^6 = x \Rightarrow x = 0, 1.$$

There are two points of intersection,  $(0, -1)$  and  $(1, 0)$ , as indicated in the figure.



- (b) First equation:

$$(y + 1)^2 = x \Rightarrow 2(y + 1)y' = 1 \Rightarrow y' = \frac{1}{2(y + 1)}$$

$$\text{At } (1, 0), y' = \frac{1}{2}.$$

Second equation:  $y = x^3 - 1 \Rightarrow y' = 3x^2$ . At  $(1, 0)$ ,  $y' = 3$ .

$$\pm \frac{1}{\sqrt{5}} \langle 2, 1 \rangle \text{ unit tangent vectors to first curve,}$$

$$\pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle \text{ unit tangent vectors to second curve}$$

At  $(0, 1)$ , the unit tangent vectors to the first curve are  $\pm \langle 0, 1 \rangle$ , and the unit tangent vectors to the second curve are  $\pm \langle 1, 0 \rangle$ .

- (c) At  $(1, 0)$ ,

$$\cos \theta = \frac{1}{\sqrt{5}} \langle 2, 1 \rangle \cdot \frac{1}{\sqrt{10}} \langle 1, 3 \rangle = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

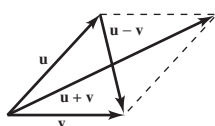
$$\theta \approx \frac{\pi}{4} \text{ or } 45^\circ$$

At  $(0, -1)$  the vectors are perpendicular,  $\theta = 90^\circ$ .

85. In a rhombus,  $\|\mathbf{u}\| = \|\mathbf{v}\|$ . The diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ .

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0 \end{aligned}$$

So, the diagonals are orthogonal.



86. If  $\mathbf{u}$  and  $\mathbf{v}$  are the sides of the parallelogram, then the diagonals are  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ , as indicated in the figure.

the parallelogram is a rectangle.

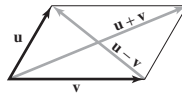
$$\Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$$

$$\Leftrightarrow 2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$$

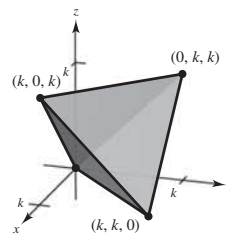
$$\Leftrightarrow (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$$

$$\Leftrightarrow \text{The diagonals are equal in length.}$$



87. (a)



- (b) Length of each edge:  $\sqrt{k^2 + k^2 + 0^2} = k\sqrt{2}$

$$(c) \cos \theta = \frac{k^2}{(k\sqrt{2})(k\sqrt{2})} = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right) = 60^\circ$$

$$(d) \vec{r}_1 = \langle k, k, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle \frac{k}{2}, \frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\vec{r}_2 = \langle 0, 0, 0 \rangle - \left\langle \frac{k}{2}, \frac{k}{2}, \frac{k}{2} \right\rangle = \left\langle -\frac{k}{2}, -\frac{k}{2}, -\frac{k}{2} \right\rangle$$

$$\cos \theta = \frac{-\frac{k^2}{4}}{\left(\frac{k}{2}\right)^2 \cdot 3} = -\frac{1}{3}$$

$$\theta = 109.5^\circ$$

88.  $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ ,  $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$

The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is  $\alpha - \beta$ . (Assuming that  $\alpha > \beta$ ). Also,

$$\begin{aligned} \cos(\alpha - \beta) &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \\ &= \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{(1)(1)} \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta. \end{aligned}$$

$$\begin{aligned}
 89. \quad \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\
 &= (\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
 &= \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\
 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}
 \end{aligned}$$

$$\begin{aligned}
 90. \quad \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\
 |\mathbf{u} \cdot \mathbf{v}| &= |\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta| \\
 &= \|\mathbf{u}\| \|\mathbf{v}\| |\cos \theta| \\
 &\leq \|\mathbf{u}\| \|\mathbf{v}\| \text{ because } |\cos \theta| \leq 1.
 \end{aligned}$$

$$\begin{aligned}
 91. \quad \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
 &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\
 &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
 &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\
 &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
 \end{aligned}$$

$$\text{So, } \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

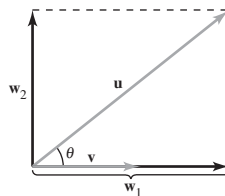
92. Let  $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}$ , as indicated in the figure. Because  $\mathbf{w}_1$  is a scalar multiple of  $\mathbf{v}$ , you can write  $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = c\mathbf{v} + \mathbf{w}_2$ .

Taking the dot product of both sides with  $\mathbf{v}$  produces

$$\begin{aligned}
 \mathbf{u} \cdot \mathbf{v} &= (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v} \\
 &= c\|\mathbf{v}\|^2, \text{ because } \mathbf{w}_2 \text{ and } \mathbf{v} \text{ are orthogonal.}
 \end{aligned}$$

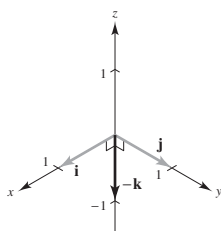
$$\text{So, } \mathbf{u} \cdot \mathbf{v} = c\|\mathbf{v}\|^2 \Rightarrow c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \text{ and}$$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

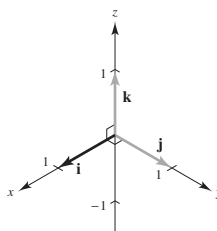


## Section 11.4 The Cross Product of Two Vectors in Space

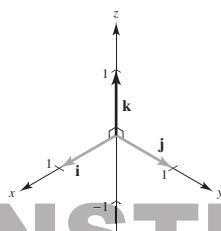
$$1. \quad \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$



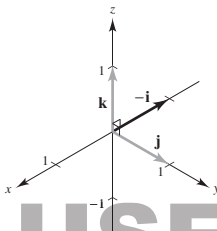
$$3. \quad \mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$$



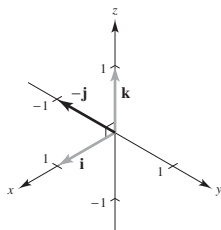
$$2. \quad \mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$



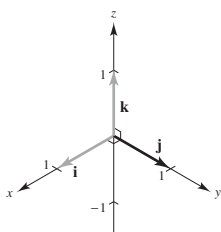
$$4. \quad \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$



$$5. \mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$$



$$6. \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$



$$7. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 0 \\ 3 & 2 & 5 \end{vmatrix} = 20\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -20\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$8. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 5 \\ 2 & 3 & -2 \end{vmatrix} = -15\mathbf{i} + 16\mathbf{j} + 9\mathbf{k}$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 15\mathbf{i} - 16\mathbf{j} - 9\mathbf{k}$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$9. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 17\mathbf{i} - 33\mathbf{j} - 10\mathbf{k}$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -17\mathbf{i} + 33\mathbf{j} + 10\mathbf{k}$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$10. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -2 \\ 1 & 5 & 1 \end{vmatrix} = 8\mathbf{i} - 5\mathbf{j} + 17\mathbf{k}$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -8\mathbf{i} + 5\mathbf{j} + 17\mathbf{k}$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$11. \mathbf{u} = \langle 12, -3, 0 \rangle, \mathbf{v} = \langle -2, 5, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 12(0) + (-3)(0) + 0(54) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(0) + 5(0) + 0(54) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$12. \mathbf{u} = \langle -1, 1, 2 \rangle, \mathbf{v} = \langle 0, 1, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{k} = \langle -2, 0, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (-1)(-2) + (1)(0) + (2)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (0)(-2) + (1)(0) + (0)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$13. \mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle 1, -2, 1 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$14. \mathbf{u} = \langle -10, 0, 6 \rangle, \mathbf{v} = \langle 5, -3, 0 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -10 & 0 & 6 \\ 5 & -3 & 0 \end{vmatrix} = 18\mathbf{i} + 30\mathbf{j} + 30\mathbf{k} = \langle 18, 30, 30 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = -10(18) + 0(30) + 6(30) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 5(18) - 3(30) + 0(30) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$15. \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-2) + 1(3) + 1(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-2) + 1(3) + (-1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

$$(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}$$



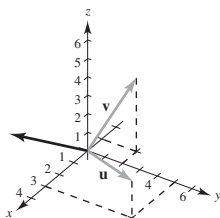
16.  $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$ ,  $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - \mathbf{j} + 13\mathbf{k}$$

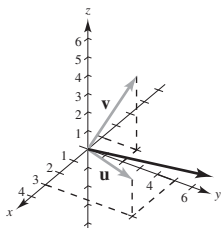
$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 1(6) + 6(-1) = 0 \Rightarrow \mathbf{u} \perp (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -2(6) + 1(-1) + 1(13) = 0 \Rightarrow \mathbf{v} \perp (\mathbf{u} \times \mathbf{v})$$

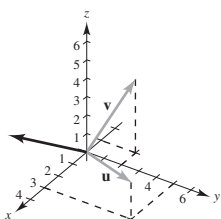
17.



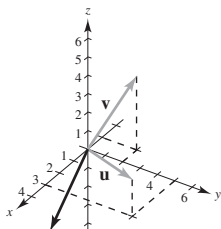
18.



19.



20.



21.  $\mathbf{u} = \langle 4, -3.5, 7 \rangle$ ,  $\mathbf{v} = \langle 2.5, 9, 3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \langle -73.5, 5.5, 44.75 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle \frac{-2.94}{\sqrt{11.8961}}, \frac{0.22}{\sqrt{11.8961}}, \frac{1.79}{\sqrt{11.8961}} \right\rangle$$

22.  $\mathbf{u} = \langle -8, -6, 4 \rangle$

$$\mathbf{v} = \langle 10, -12, -2 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \langle 60, 24, 156 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{36\sqrt{22}} \langle 60, 24, 156 \rangle$$

$$= \left\langle \frac{5}{3\sqrt{22}}, \frac{2}{3\sqrt{22}}, \frac{13}{3\sqrt{22}} \right\rangle$$

23.  $\mathbf{u} = \langle -3, 2, -5 \rangle$ ,  $\mathbf{v} = \langle 0.4, -0.8, 0.2 \rangle$

$$\mathbf{u} \times \mathbf{v} = \langle -3.6, -1.4, 1.6 \rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle -\frac{1.8}{\sqrt{4.37}}, -\frac{0.7}{\sqrt{4.37}}, \frac{0.8}{\sqrt{4.37}} \right\rangle$$

24.  $\mathbf{u} = \left\langle 0, 0, \frac{7}{10} \right\rangle$ ,  $\mathbf{v} = \left\langle \frac{3}{2}, 0, \frac{31}{5} \right\rangle$

$$\mathbf{u} \times \mathbf{v} = \left\langle 0, \frac{21}{20}, 0 \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \langle 0, 1, 0 \rangle$$

25. Programs will vary.

26.  $\mathbf{u} \times \mathbf{v} = \langle -50, 40, -34 \rangle$

$$\|\mathbf{u} \times \mathbf{v}\| \approx 72.498$$

27.  $\mathbf{u} = \mathbf{j}$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{i}\| = 1$$

28.  $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|-\mathbf{j} + \mathbf{k}\| = \sqrt{2}$$

29.  $\mathbf{u} = \langle 3, 2, -1 \rangle$

$$\mathbf{v} = \langle 1, 2, 3 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 8, -10, 4 \rangle\| = \sqrt{180} = 6\sqrt{5}$$

30.  $\mathbf{u} = \langle 2, -1, 0 \rangle$

$\mathbf{v} = \langle -1, 2, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 3 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 0, 0, 3 \rangle\| = 3$$

31.  $A(0, 3, 2), B(1, 5, 5), C(6, 9, 5), D(5, 7, 2)$

$\overrightarrow{AB} = \langle 1, 2, 3 \rangle$

$\overrightarrow{DC} = \langle 1, 2, 3 \rangle$

$\overrightarrow{BC} = \langle 5, 4, 0 \rangle$

$\overrightarrow{AD} = \langle 5, 4, 0 \rangle$

Because  $\overrightarrow{AB} = \overrightarrow{DC}$  and  $\overrightarrow{BC} = \overrightarrow{AD}$ , the figure  $ABCD$  is a parallelogram.

$\overrightarrow{AB}$  and  $\overrightarrow{AD}$  are adjacent sides

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 0 \end{vmatrix} = \langle -12, 15, -6 \rangle$$

$$A = \|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{144 + 225 + 36} = 9\sqrt{5}$$

32.  $A(2, -3, 1), B(6, 5, -1), C(7, 2, 2), D(3, -6, 4)$

$\overrightarrow{AB} = \langle 4, 8, -2 \rangle$

$\overrightarrow{DC} = \langle 4, 8, -2 \rangle$

$\overrightarrow{BC} = \langle 1, -3, 3 \rangle$

$\overrightarrow{AD} = \langle 1, -3, 3 \rangle$

Because  $\overrightarrow{AB} = \overrightarrow{DC}$  and  $\overrightarrow{BC} = \overrightarrow{AD}$ , the figure  $ABCD$  is a parallelogram.

$\overrightarrow{AB}$  and  $\overrightarrow{AD}$  are adjacent sides

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & -2 \\ 1 & -3 & 3 \end{vmatrix} = \langle 18, -14, -20 \rangle$$

$$A = \|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{324 + 196 + 400} = 2\sqrt{230}$$

33.  $A(0, 0, 0), B(1, 0, 3), C(-3, 2, 0)$

$\overrightarrow{AB} = \langle 1, 0, 3 \rangle, \overrightarrow{AC} = \langle -3, 2, 0 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ -3 & 2 & 0 \end{vmatrix} = \langle -6, -9, 2 \rangle$$

$$A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{36 + 81 + 4} = \frac{11}{2}$$

34.  $A(2, -3, 4), B(0, 1, 2), C(-1, 2, 0)$

$\overrightarrow{AB} = \langle -2, 4, -2 \rangle, \overrightarrow{AC} = \langle -3, 5, -4 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = -6\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

$$A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{44} = \sqrt{11}$$

35.  $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$\overrightarrow{AB} = \langle -3, 12, 5 \rangle, \overrightarrow{AC} = \langle 2, 13, -4 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix} = \langle -113, -2, -63 \rangle$$

$$A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{16,742}$$

36.  $A(1, 2, 0), B(-2, 1, 0), C(0, 0, 0)$

$\overrightarrow{AB} = \langle -3, -1, 0 \rangle, \overrightarrow{AC} = \langle -1, -2, 0 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 5\mathbf{k}$$

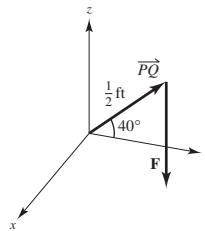
$$A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{5}{2}$$

37.  $\mathbf{F} = -20\mathbf{k}$

$\overrightarrow{PQ} = \frac{1}{2}(\cos 40^\circ \mathbf{j} + \sin 40^\circ \mathbf{k})$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40^\circ/2 & \sin 40^\circ/2 \\ 0 & 0 & -20 \end{vmatrix} = -10 \cos 40^\circ \mathbf{i}$$

$$\|\overrightarrow{PQ} \times \mathbf{F}\| = 10 \cos 40^\circ \approx 7.66 \text{ ft-lb}$$

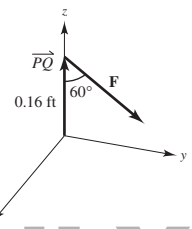


38.  $\mathbf{F} = -2000(\cos 30^\circ \mathbf{j} + \sin 30^\circ \mathbf{k}) = -1000\sqrt{3}\mathbf{j} - 1000\mathbf{k}$

$\overrightarrow{PQ} = 0.16\mathbf{k}$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.16 \\ 0 & -1000\sqrt{3} & -1000 \end{vmatrix} = 160\sqrt{3}\mathbf{i}$$

$$\|\overrightarrow{PQ} \times \mathbf{F}\| = 160\sqrt{3} \text{ ft-lb}$$



39. (a) Place the wrench in the  $xy$ -plane, as indicated in the figure.

The angle from  $\overrightarrow{AB}$  to  $\mathbf{F}$  is  $30^\circ + 180^\circ + \theta = 210^\circ + \theta$

$$\|\overrightarrow{OA}\| = 18 \text{ inches} = 1.5 \text{ feet}$$

$$\overrightarrow{OA} = 1.5[\cos(30^\circ)\mathbf{i} + \sin(30^\circ)\mathbf{j}] = \frac{3\sqrt{3}}{4}\mathbf{i} + \frac{3}{4}\mathbf{j}$$

$$\mathbf{F} = 56[\cos(210^\circ + \theta)\mathbf{i} + \sin(210^\circ + \theta)\mathbf{j}]$$

$$\begin{aligned} \overrightarrow{OA} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 56 \cos(210^\circ + \theta) & 56 \sin(210^\circ + \theta) & 0 \end{vmatrix} \\ &= [42\sqrt{3} \sin(210^\circ + \theta) - 42 \cos(210^\circ + \theta)]\mathbf{k} \\ &= [42\sqrt{3}(\sin 210^\circ \cos \theta + \cos 210^\circ \sin \theta) - 42(\cos 210^\circ \cos \theta - \sin 210^\circ \sin \theta)]\mathbf{k} \\ &= \left[ 42\sqrt{3}\left(-\frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta\right) - 42\left(-\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta\right) \right]\mathbf{k} = (-84 \sin \theta)\mathbf{k} \end{aligned}$$

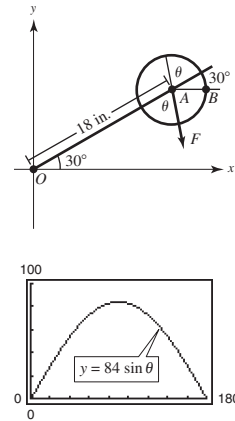
$$\|\overrightarrow{OA} \times \mathbf{F}\| = 84 \sin \theta, \quad 0 \leq \theta \leq 180^\circ$$

(b) When  $\theta = 45^\circ$ ,  $\|\overrightarrow{OA} \times \mathbf{F}\| = 84 \frac{\sqrt{2}}{2} = 42\sqrt{2} \approx 59.40$

(c) Let  $T = 84 \sin \theta$

$$\frac{dT}{d\theta} = 84 \cos \theta = 0 \text{ when } \theta = 90^\circ.$$

This is reasonable. When  $\theta = 90^\circ$ , the force is perpendicular to the wrench.



40. (a)  $AC = 15 \text{ inches} = \frac{5}{4} \text{ feet}$

$BC = 12 \text{ inches} = 1 \text{ foot}$

$$\overrightarrow{AB} = -\frac{5}{4}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} = -180(\cos \theta \mathbf{j} + \sin \theta \mathbf{k})$$

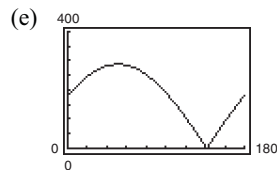
$$\begin{aligned} \text{(b)} \quad \overrightarrow{AB} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{5}{4} & 1 \\ 0 & -180 \cos \theta & -180 \sin \theta \end{vmatrix} \\ &= (225 \sin \theta + 180 \cos \theta)\mathbf{i} \end{aligned}$$

$$\|\overrightarrow{AB} \times \mathbf{F}\| = |225 \sin \theta + 180 \cos \theta|$$

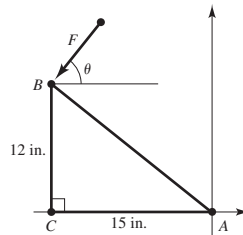
(c) When  $\theta = 30^\circ$ ,  $\|\overrightarrow{AB} \times \mathbf{F}\| = 225\left(\frac{1}{2}\right) + 180\left(\frac{\sqrt{3}}{2}\right) \approx 268.38$

(d) If  $T = |225 \sin \theta + 180 \cos \theta|$ ,  $T = 0$  for  $225 \sin \theta = -180 \cos \theta \Rightarrow \tan \theta = -\frac{4}{5} \Rightarrow \theta \approx 141.34^\circ$ .

For  $0 < \theta < 141.34$ ,  $T'(\theta) = 225 \cos \theta - 180 \sin \theta = 0 \Rightarrow \tan \theta = \frac{5}{4} \Rightarrow \theta \approx 51.34^\circ$ .  $\overrightarrow{AB}$  and  $\mathbf{F}$  are perpendicular.



From part (d), the zero is  $\theta \approx 141.34^\circ$ , when the vectors are parallel.



$$41. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$42. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$43. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$$

$$44. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$$

$$45. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$$

$$46. \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = -72$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 72$$

$$47. \mathbf{u} = \langle 3, 0, 0 \rangle$$

$$\mathbf{v} = \langle 0, 5, 1 \rangle$$

$$\mathbf{w} = \langle 2, 0, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$$

$$48. \mathbf{u} = \langle 0, 4, 0 \rangle$$

$$\mathbf{v} = \langle -3, 0, 0 \rangle$$

$$\mathbf{w} = \langle -1, 1, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 0 & 4 & 0 \\ -3 & 0 & 0 \\ -1 & 1 & 5 \end{vmatrix} = -4(-15) = 60$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 60$$

$$49. \mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are parallel.}$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

So,  $\mathbf{u}$  or  $\mathbf{v}$  (or both) is the zero vector.

$$50. (a) \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \quad (b)$$

$$= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \quad (c)$$

$$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} \quad (d)$$

$$= \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} \quad (h)$$

$$(e) \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \quad (f)$$

$$= \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) = (-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \quad (g)$$

So,  $a = b = c = d = h$  and  $e = f = g$

$$51. \mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$52. \text{ See Theorem 11.8, page 794.}$$

$$53. \text{ The magnitude of the cross product will increase by a factor of 4.}$$

$$54. \text{ From the vectors for two sides of the triangle, and compute their cross product.}$$

$$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle$$

$$55. \text{ False. If the vectors are ordered pairs, then the cross product does not exist.}$$

$$56. \text{ False. In general, } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$57. \text{ False. Let } \mathbf{u} = \langle 1, 0, 0 \rangle, \mathbf{v} = \langle 1, 0, 0 \rangle, \mathbf{w} = \langle -1, 0, 0 \rangle.$$

Then,  $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}$ , but  $\mathbf{v} \neq \mathbf{w}$ .

$$58. \text{ True}$$

$$59. \mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix}$$

$$= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + (u_2w_3 - u_3w_2)\mathbf{i} - (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k}$$

$$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$$

60.  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ ,  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ ,  $c$  is a scalar:

$$\begin{aligned} (c\mathbf{u}) \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] = c(\mathbf{u} \times \mathbf{v}) \end{aligned}$$

61.  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

62.  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1) \\ &= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \end{aligned}$$

63.  $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$   
 $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = \mathbf{0}$   
 $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = \mathbf{0}$   
 So,  $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$  and  $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$ .

64. If  $\mathbf{u}$  and  $\mathbf{v}$  are scalar multiples of each other,  $\mathbf{u} = c\mathbf{v}$  for some scalar  $c$ .

$$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , then  $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = 0$ . (Assume  $\mathbf{u} \neq \mathbf{0}$ ,  $\mathbf{v} \neq \mathbf{0}$ .) So,  $\sin\theta = 0$ ,  $\theta = 0$ , and  $\mathbf{u}$  and  $\mathbf{v}$  are parallel. So,  $\mathbf{u} = c\mathbf{v}$  for some scalar  $c$ .

65.  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

If  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal,  $\theta = \pi/2$  and  $\sin\theta = 1$ . So,  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|$ .

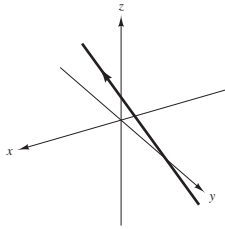
66.  $\mathbf{u} = \langle a_1, b_1, c_1 \rangle$ ,  $\mathbf{v} = \langle a_2, b_2, c_2 \rangle$ ,  $\mathbf{w} = \langle a_3, b_3, c_3 \rangle$

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\mathbf{i} - (a_2c_3 - a_3c_2)\mathbf{j} + (a_2b_3 - a_3b_2)\mathbf{k} \\ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ (b_2c_3 - b_3c_2) & (a_3c_2 - a_2c_3) & (a_2b_3 - a_3b_2) \end{vmatrix} \\ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= [b_1(a_2b_3 - a_3b_2) - c_1(a_3c_2 - a_2c_3)]\mathbf{i} - [a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)]\mathbf{j} \\ &\quad + [a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)]\mathbf{k} \\ &= [a_2(a_1a_3 + b_1b_3 + c_1c_3) - a_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{i} + [b_2(a_1b_3 + b_1b_3 + c_1c_3) - b_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{j} \\ &\quad + [c_2(a_1a_3 + b_1b_3 + c_1c_3) - c_3(a_1a_2 + b_1b_2 + c_1c_2)]\mathbf{k} \\ &= (a_1a_3 + b_1b_3 + c_1c_3)\langle a_2, b_2, c_2 \rangle - (a_1a_2 + b_1b_2 + c_1c_2)\langle a_3, b_3, c_3 \rangle = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \end{aligned}$$

## Section 11.5 Lines and Planes in Space

1.  $x = 1 + 3t, y = 2 - t, z = 2 + 5t$

(a)

(b) When  $t = 0, P = (1, 2, 2)$ . When  $t = 3, Q = (10, -1, 17)$ .

$$\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$$

The components of the vector and the coefficients of  $t$  are proportional because the line is parallel to  $\overrightarrow{PQ}$ .

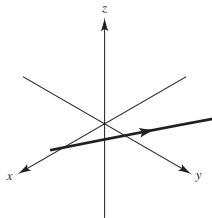
(c)  $y = 0$  when  $t = 2$ . So,  $x = 7$  and  $z = 12$ .Point:  $(7, 0, 12)$ 

$x = 0$  when  $t = -\frac{1}{3}$ . So,  $y = \frac{7}{3}$  and  $z = \frac{1}{3}$ . Point:  $(0, \frac{7}{3}, \frac{1}{3})$

$z = 0$  when  $t = -\frac{2}{5}$ . So,  $x = -\frac{1}{5}$  and  $y = \frac{12}{5}$ . Point:  $(-\frac{1}{5}, \frac{12}{5}, 0)$

2.  $x = 2 + 3t, y = 2, z = 1 - t$

(a)

(b) When  $t = 0, P = (2, 2, 1)$ . When  $t = 2,$ 

$$Q = (-4, 2, -1).$$

$$\overrightarrow{PQ} = \langle -6, 0, -2 \rangle$$

The components of the vector and the coefficients of  $t$  are proportional because the line is parallel to  $\overrightarrow{PQ}$ .

(c)  $z = 0$  when  $t = 1$ . So,  $x = -1$  and  $y = 2$ .Point:  $(-1, 2, 0)$ 

$x = 0$  when  $t = \frac{2}{3}$ . So,  $y = 2$  and  $z = \frac{1}{3}$

Point:  $(0, 2, \frac{1}{3})$ 

3.  $x = -2 + t, y = 3t, z = 4 + t$

(a)  $(0, 6, 6)$ : For  $x = 0 = -2 + t$ , you have

$$t = 2. \text{ Then } y = 3(2) = 6 \text{ and}$$

$$z = 4 + 2 = 6. \text{ Yes, } (0, 6, 6) \text{ lies on the line.}$$

(b)  $(2, 3, 5)$ : For  $x = 2 = -2 + t$ , you have

$t = 4$ . Then  $y = 3(4) = 12 \neq 3$ . No,  $(2, 3, 5)$  does not lie on the line.

4.  $\frac{x-3}{2} = \frac{y-7}{8} = z+2$

(a)  $(7, 23, 0)$ : Substituting, you have

$$\frac{7-3}{2} = \frac{23-7}{8} = 0+2$$

$$2 = 2 = 2$$

Yes,  $(7, 23, 0)$  lies on the line.(b)  $(1, -1, -3)$ : Substituting, you have

$$\frac{1-3}{2} = \frac{-1-7}{8} = -3+2$$

$$-1 = -1 = -1$$

Yes,  $(1, -1, -3)$  lies on the line.5. Point:  $(0, 0, 0)$ Direction vector:  $\langle 3, 1, 5 \rangle$ 

Direction numbers: 3, 1, 5

(a) Parametric:  $x = 3t, y = t, z = 5t$ (b) Symmetric:  $\frac{x}{3} = y = \frac{z}{5}$ 6. Point:  $(0, 0, 0)$ Direction vector:  $\mathbf{v} = \left\langle -2, \frac{5}{2}, 1 \right\rangle$ 

Direction numbers: -4, 5, 2

(a) Parametric:  $x = -4t, y = 5t, z = 2t$ (b) Symmetric:  $\frac{x}{-4} = \frac{y}{5} = \frac{z}{2}$

7. Point:  $(-2, 0, 3)$

Direction vector:  $\mathbf{v} = \langle 2, 4, -2 \rangle$

Direction numbers: 2, 4, -2

(a) Parametric:  $x = -2 + 2t, y = 4t, z = 3 - 2t$

(b) Symmetric:  $\frac{x+2}{2} = \frac{y}{4} = \frac{z-3}{-2}$

8. Point:  $(-3, 0, 2)$

Direction vector:  $\mathbf{v} = \langle 0, 6, 3 \rangle$

Direction numbers: 0, 2, 1

(a) Parametric:  $x = -3, y = 2t, z = 2 + t$

(b) Symmetric:  $\frac{y}{2} = z - 2, x = -3$

9. Point:  $(1, 0, 1)$

Direction vector:  $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Direction numbers: 3, -2, 1

(a) Parametric:  $x = 1 + 3t, y = -2t, z = 1 + t$

(b) Symmetric:  $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-1}{1}$

10. Point:  $(-3, 5, 4)$

Directions numbers: 3, -2, 1

(a) Parametric:  $x = -3 + 3t, y = 5 - 2t, z = 4 + t$

(b) Symmetric:  $\frac{x+3}{3} = \frac{y-5}{-2} = z - 4$

11. Points:  $(5, -3, -2), \left(-\frac{2}{3}, \frac{2}{3}, 1\right)$

Direction vector:  $\mathbf{v} = \frac{17}{3}\mathbf{i} - \frac{11}{3}\mathbf{j} - 3\mathbf{k}$

Direction numbers: 17, -11, -9

(a) Parametric:

$$x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$$

(b) Symmetric:  $\frac{x-5}{17} = \frac{y+3}{-11} = \frac{z+2}{-9}$

12. Points:  $(0, 4, 3), (-1, 2, 5)$

Direction vector:  $\langle 1, 2, -2 \rangle$

Direction numbers: 1, 2, -2

(a) Parametric:  $x = t, y = 4 + 2t, z = 3 - 2t$

(b) Symmetric:  $x = \frac{y-4}{2} = \frac{z-3}{-2}$

13. Points:  $(7, -2, 6), (-3, 0, 6)$

Direction vector:  $\langle -10, 2, 0 \rangle$

Direction numbers: -10, 2, 0

(a) Parametric:  $x = 7 - 10t, y = -2 + 2t, z = 6$

(b) Symmetric: Not possible because the direction number for  $z$  is 0. But, you could describe the line as  $\frac{x-7}{10} = \frac{y+2}{-2}, z = 6$ .

14. Points:  $(0, 0, 25), (10, 10, 0)$

Direction vector:  $\langle 10, 10, -25 \rangle$

Direction numbers: 2, 2, -5

(a) Parametric:  $x = 2t, y = 2t, z = 25 - 5t$

(b) Symmetric:  $\frac{x}{2} = \frac{y}{2} = \frac{z-25}{-5}$

15. Point:  $(2, 3, 4)$

Direction vector:  $\mathbf{v} = \mathbf{k}$

Direction numbers: 0, 0, 1

Parametric:  $x = 2, y = 3, z = 4 + t$

16. Point:  $(-4, 5, 2)$

Direction vector:  $\mathbf{v} = \mathbf{j}$

Direction numbers: 0, 1, 0

Parametric:  $x = -4, y = 5 + t, z = 2$

17. Point:  $(2, 3, 4)$

Direction vector:  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

Direction numbers: 3, 2, -1

Parametric:  $x = 2 + 3t, y = 3 + 2t, z = 4 - t$

18. Point  $(-4, 5, 2)$

Direction vector:  $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Direction numbers: -1, 2, 1

Parametric:  $x = -4 - t, y = 5 + 2t, z = 2 + t$

19. Point:  $(5, -3, -4)$

Direction vector:  $\mathbf{v} = \langle 2, -1, 3 \rangle$

Direction numbers: 2, -1, 3

Parametric:  $x = 5 + 2t, y = -3 - t, z = -4 + 3t$

20. Point:  $(-1, 4, -3)$

Direction vector:  $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$

Direction numbers: 5, -1, 0

Parametric:  $x = -1 + 5t, y = 4 - t, z = -3$

21. Point:  $(2, 1, 2)$   
 Direction vector:  $\langle -1, 1, 1 \rangle$   
 Direction numbers:  $-1, 1, 1$   
 Parametric:  $x = 2 - t, y = 1 + t, z = 2 + t$
22. Point:  $(-6, 0, 8)$   
 Direction vector:  $\langle -2, 2, 0 \rangle$   
 Direction numbers:  $-2, 2, 0$   
 Parametric:  $x = -6 - 2t, y = 2t, z = 8$
23. Let  $t = 0$ :  $P = (3, -1, -2)$  (other answers possible)  
 $\mathbf{v} = \langle -1, 2, 0 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)
24. Let  $t = 0$ :  $P = (0, 5, 4)$  (other answers possible)  
 $\mathbf{v} = \langle 4, -1, 3 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)
25. Let each quantity equal 0:  
 $P = (7, -6, -2)$  (other answers possible)  
 $\mathbf{v} = \langle 4, 2, 1 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)
26. Let each quantity equal 0:  
 $P = (-3, 0, 3)$  (other answers possible)  
 $\mathbf{v} = \langle 5, 8, 6 \rangle$  (any nonzero multiple of  $\mathbf{v}$  is correct)
27.  $L_1: \mathbf{v} = \langle -3, 2, 4 \rangle$   $(6, -2, 5)$  on line  
 $L_2: \mathbf{v} = \langle 6, -4, -8 \rangle$   $(6, -2, 5)$  on line  
 $L_3: \mathbf{v} = \langle -6, 4, 8 \rangle$   $(6, -2, 5)$  not on line  
 $L_4: \mathbf{v} = \langle 6, 4, -6 \rangle$  not parallel to  $L_1, L_2$ , nor  $L_3$   
 $L_1$  and  $L_2$  are identical.  $L_1 = L_2$  and is parallel to  $L_3$ .
28.  $L_1: \mathbf{v} = \langle 2, -6, -2 \rangle$   $(3, 0, 1)$  on line  
 $L_2: \mathbf{v} = \langle 2, -1, 3 \rangle$   $(1, -1, 0)$  on line  
 $L_3: \mathbf{v} = \langle 2, -10, -4 \rangle$   $(-1, 3, 1)$  on line  
 $L_4: \mathbf{v} = \langle 2, -1, 3 \rangle$   $(5, 1, 8)$  on line  
 $L_2$  and  $L_4$  are parallel, not identical, because  $(1, -1, 0)$  is not on  $L_4$ .
29.  $L_1: \mathbf{v} = \langle 4, -2, 3 \rangle$   $(8, -5, -9)$  on line  
 $L_2: \mathbf{v} = \langle 2, 1, 5 \rangle$   
 $L_3: \mathbf{v} = \langle -8, 4, -6 \rangle$   $(8, -5, -9)$  on line  
 $L_4: \mathbf{v} = \langle -2, 1, 1.5 \rangle$   
 $L_1$  and  $L_3$  are identical.
30.  $L_1: \mathbf{v} = \langle 2, 1, 2 \rangle$   $(3, 2, -2)$  on line  
 $L_2: \mathbf{v} = \langle 4, 2, 4 \rangle$   $(1, 1, -3)$  on line  
 $L_3: \mathbf{v} = \langle 1, \frac{1}{2}, 1 \rangle$   $(-2, 1, 3)$  on line  
 $L_4: \mathbf{v} = \langle 2, 4, -1 \rangle$   $(3, -1, 2)$  on line  
 $L_1, L_2$  and  $L_3$  have same direction.  
 $(3, 2, -2)$  is not on  $L_2$  nor  $L_3$   
 $(1, 1, -3)$  is not on  $L_3$   
 So, the three lines are parallel, not identical.
31. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. So,  
 (i)  $4t + 2 = 2s + 2$ , (ii)  $3 = 2s + 3$ , and  
 (iii)  $-t + 1 = s + 1$ .  
 From (ii), you find that  $s = 0$  and consequently, from (iii),  $t = 0$ . Letting  $s = t = 0$ , you see that equation (i) is satisfied and so the two lines intersect. Substituting zero for  $s$  or for  $t$ , you obtain the point  $(2, 3, 1)$ .  
 $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$  (First line)  
 $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$  (Second line)  
 $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{8 - 1}{\sqrt{17}\sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$
32. By equating like variables, you have  
 (i)  $-3t + 1 = 3s + 1$ , (ii)  $4t + 1 = 2s + 4$ , and  
 (iii)  $2t + 4 = -s + 1$ .  
 From (i) you have  $s = -t$ , and consequently from (ii),  
 $t = \frac{1}{2}$  and from (iii),  $t = -3$ . The lines do not intersect.
33. Writing the equations of the lines in parametric form you have  
 $x = 3t$   $y = 2 - t$   $z = -1 + t$   
 $x = 1 + 4s$   $y = -2 + s$   $z = -3 - 3s$ .  
 For the coordinates to be equal,  $3t = 1 + 4s$  and  
 $2 - t = -2 + s$ . Solving this system yields  $t = \frac{17}{7}$  and  
 $s = \frac{11}{7}$ . When using these values for  $s$  and  $t$ , the  $z$  coordinates are not equal. The lines do not intersect.



34. Writing the equations of the lines in parametric form you have

$$\begin{aligned} x &= 2 - 3t & y &= 2 + 6t & z &= 3 + t \\ x &= 3 + 2s & y &= -5 + s & z &= -2 + 4s. \end{aligned}$$

By equating like variables, you have

$$2 - 3t = 3 + 2s, \quad 2 + 6t = -5 + s, \quad 3 + t = -2 + 4s.$$

So,  $t = -1$ ,  $s = 1$  and the point of intersection is

$$(5, -4, 2).$$

$$\mathbf{u} = \langle -3, 6, 1 \rangle \quad (\text{First line})$$

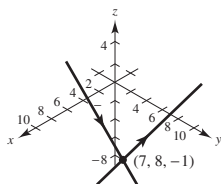
$$\mathbf{v} = \langle 2, 1, 4 \rangle \quad (\text{Second line})$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{4}{\sqrt{46} \sqrt{21}} = \frac{4}{\sqrt{966}} = \frac{2\sqrt{966}}{483}$$

35.  $x = 2t + 3 \quad x = -2s + 7$   
 $y = 5t - 2 \quad y = s + 8$   
 $z = -t + 1 \quad z = 2s - 1$

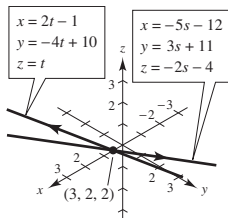
Point of intersection:  $(7, 8, -1)$

**Note:**  $t = 2$  and  $s = 0$



36.  $x = 2t - 1 \quad x = -5s - 12$   
 $y = -4t + 10 \quad y = 3s + 11$   
 $z = t \quad z = -2s - 4$

Point of intersection:  $(3, 2, 2)$



37.  $4x - 3y - 6z = 6$

- (a)  $P(0, 0, -1), Q(0, -2, 0), R(3, 4, -1)$

$$\overrightarrow{PQ} = \langle 0, -2, 1 \rangle, \quad \overrightarrow{PR} = \langle 3, 4, 0 \rangle$$

$$(b) \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \langle -4, 3, 6 \rangle$$

The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

38.  $2x + 3y + 4z = 4$

$$P(0, 0, 1), Q(2, 0, 0), R(3, 2, -2)$$

$$(a) \quad \overrightarrow{PQ} = \langle 2, 0, -1 \rangle, \quad \overrightarrow{PR} = \langle 3, 2, -3 \rangle$$

$$(b) \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} = \langle 2, 3, 4 \rangle$$

The components of the cross product are proportional (for this choice of  $P$ ,  $Q$ , and  $R$ , they are the same) to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

39.  $x + 2y - 4z - 1 = 0$

$$(a) \quad (-7, 2, -1): (-7) + 2(2) - 4(-1) - 1 = 0$$

Point is in plane

$$(b) \quad (5, 2, 2): 5 + 2(2) - 4(2) - 1 = 0$$

Point is in plane

40.  $2x + y + 3z - 6 = 0$

$$(a) \quad (3, 6, -2): 2(3) + 6 + 3(-2) - 6 = 0$$

Point is in plane

$$(b) \quad (-1, 5, -1): 2(-1) + 5 + 3(-1) - 6 = -6 \neq 0$$

Point is not in plane

41. Point:  $(1, 3, -7)$

$$\text{Normal vector: } \mathbf{n} = \mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\begin{aligned} 0(x - 1) + 1(y - 3) + 0(z - (-7)) &= 0 \\ y - 3 &= 0 \end{aligned}$$

42. Point:  $(0, -1, 4)$

$$\text{Normal vector: } \mathbf{n} = \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\begin{aligned} 0(x - 0) + 0(y + 1) + 1(z - 4) &= 0 \\ z - 4 &= 0 \end{aligned}$$

43. Point:  $(3, 2, 2)$

$$\text{Normal vector: } \mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$\begin{aligned} 2(x - 3) + 3(y - 2) - 1(z - 2) &= 0 \\ 2x + 3y - z &= 10 \end{aligned}$$

44. Point:  $(0, 0, 0)$

$$\text{Normal vector: } \mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$$

$$\begin{aligned} -3(x - 0) + 0(y - 0) + 2(z - 0) &= 0 \\ -3x + 2z &= 0 \end{aligned}$$

45. Point:
- $(-1, 4, 0)$

Normal vector:  $\mathbf{v} = \langle 2, -1, -2 \rangle$

$$2(x + 1) - 1(y - 4) - 2(z - 0) = 0$$

$$2x - y - 2z + 6 = 0$$

46. Point:
- $(3, 2, 2)$

Normal vector:  $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$$4(x - 3) + (y - 2) - 3(z - 2) = 0$$

$$4x + y - 3z = 8$$

47. Let
- $\mathbf{u}$
- be the vector from
- $(0, 0, 0)$
- to

$$(2, 0, 3): \mathbf{u} = \langle 2, 0, 3 \rangle$$

Let  $\mathbf{u}$  be the vector from  $(0, 0, 0)$  to

$$(-3, -1, 5): \mathbf{v} = \langle -3, -1, 5 \rangle$$

Normal vectors:  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ -3 & -1 & 5 \end{vmatrix} = \langle 3, -19, -2 \rangle$

$$3(x - 0) - 19(y - 0) - 2(z - 0) = 0$$

$$3x - 19y - 2z = 0$$

48. Let
- $\mathbf{u}$
- be the vector from
- $(3, -1, 2)$
- to
- $(2, 1, 5)$
- :

$$\mathbf{u} = \langle -1, 2, 3 \rangle$$

Let  $\mathbf{u}$  be the vector from  $(3, -1, 2)$  to  $(1, -2, -2)$ :

$$\mathbf{v} = \langle -2, -1, -4 \rangle$$

Normal vector:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ -2 & -1 & -4 \end{vmatrix} = \langle -5, -10, 5 \rangle = -5\langle 1, 2, -1 \rangle$$

$$1(x - 3) + 2(y + 1) - (z - 2) = 0$$

$$x + 2y - z + 1 = 0$$

49. Let
- $\mathbf{u}$
- be the vector from
- $(1, 2, 3)$
- to

$$(3, 2, 1): \mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$$

Let  $\mathbf{v}$  be the vector from  $(1, 2, 3)$  to

$$(-1, -2, 2): \mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

Normal vector:

$$\left(\frac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$4(x - 1) - 3(y - 2) + 4(z - 3) = 0$$

$$4x - 3y + 4z = 10$$

- 50.
- $(1, 2, 3)$
- , Normal vector:
- $\mathbf{v} = \mathbf{i}$
- ,
- $1(x - 1) = 0$
- ,
- $x = 1$

- 51.
- $(1, 2, 3)$
- , Normal vector:
- $\mathbf{v} = \mathbf{k}$
- ,
- $1(z - 3) = 0$
- ,
- $z = 3$

52. The plane passes through the three points

$$(0, 0, 0), (0, 1, 0), (\sqrt{3}, 0, 1).$$

The vector from  $(0, 0, 0)$  to  $(0, 1, 0)$ :  $\mathbf{u} = \mathbf{j}$ The vector from  $(0, 0, 0)$  to  $(\sqrt{3}, 0, 1)$ :  $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{k}$ 

Normal vector:  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{vmatrix} = \mathbf{i} - \sqrt{3}\mathbf{k}$

$$x - \sqrt{3}z = 0$$

53. The direction vectors for the lines are

$$\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}.$$

Normal vector:  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Point of intersection of the lines:  $(-1, 5, 1)$ 

$$(x + 1) + (y - 5) + (z - 1) = 0$$

$$x + y + z = 5$$

54. The direction of the line is
- $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
- . Choose any point on the line,
- $[(0, 4, 0)$
- , for example], and let
- $\mathbf{v}$
- be the vector from
- $(0, 4, 0)$
- to the given point
- $(2, 2, 1)$
- :

$$\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Normal vector:  $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{k}$

$$(x - 2) - 2(z - 1) = 0$$

$$x - 2z = 0$$

55. Let
- $\mathbf{v}$
- be the vector from
- $(-1, 1, -1)$
- to

$$(2, 2, 1): \mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Let  $\mathbf{n}$  be a vector normal to the plane

$$2x - 3y + z = 3: \mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

Because  $\mathbf{v}$  and  $\mathbf{n}$  both lie in the plane  $P$ , the normal vector to  $P$  is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$$

$$7(x - 2) - 1(y - 2) - 11(z - 1) = 0$$

$$7x + y - 11z = 5$$

56. Let  $\mathbf{v}$  be the vector from  $(3, 2, 1)$  to

$$(3, 1, -5): \mathbf{v} = -\mathbf{j} - 6\mathbf{k}$$

Let  $\mathbf{n}$  be the normal to the given plane:

$$\mathbf{n} = 6\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$$

Because  $\mathbf{v}$  and  $\mathbf{n}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\begin{aligned} \mathbf{v} \times \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} = 40\mathbf{i} - 36\mathbf{j} + 6\mathbf{k} \\ &= 2(20\mathbf{i} - 18\mathbf{j} + 3\mathbf{k}) \end{aligned}$$

$$20(x - 3) - 18(y - 2) + 3(z - 1) = 0$$

$$20x - 18y + 3z = 27$$

58. Let  $\mathbf{u} = \mathbf{k}$  and let  $\mathbf{v}$  be the vector from  $(4, 2, 1)$  to  $(-3, 5, 7): \mathbf{v} = -7\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Because  $\mathbf{u}$  and  $\mathbf{v}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -7 & 3 & 6 \end{vmatrix} = -3\mathbf{i} - 7\mathbf{j} = -(3\mathbf{i} + 7\mathbf{j})$$

$$3(x - 4) + 7(y - 2) = 0$$

$$3x + 7y = 26$$

59.  $xy$ -plane: Let  $z = 0$ .

$$\text{Then } 0 = 4 - t \Rightarrow t = 4 \Rightarrow x = 1 - 2(4) = -7 \text{ and}$$

$$y = -2 + 3(4) = 10. \text{ Intersection: } (-7, 10, 0)$$

$xz$ -plane: Let  $y = 0$ .

$$\text{Then } 0 = -2 + 3t \Rightarrow t = \frac{2}{3} \Rightarrow x = 1 - 2\left(\frac{2}{3}\right) = -\frac{1}{3} \text{ and}$$

$$z = -4 + \frac{2}{3} = -\frac{10}{3}. \text{ Intersection: } \left(-\frac{1}{3}, 0, -\frac{10}{3}\right)$$

$yz$ -plane: Let  $x = 0$ .

$$\text{Then } 0 = 1 - 2t \Rightarrow t = \frac{1}{2} \Rightarrow y = -2 + 3\left(\frac{1}{2}\right) = -\frac{1}{2} \text{ and}$$

$$z = -4 + \frac{1}{2} = -\frac{7}{2}. \text{ Intersection: } \left(0, -\frac{1}{2}, -\frac{7}{2}\right)$$

57. Let  $\mathbf{u} = \mathbf{i}$  and let  $\mathbf{v}$  be the vector from  $(1, -2, -1)$  to

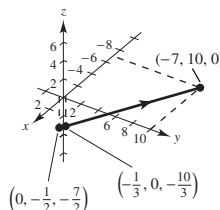
$$(2, 5, 6): \mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$

Because  $\mathbf{u}$  and  $\mathbf{v}$  both lie in the plane  $P$ , the normal vector to  $P$  is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k})$$

$$[y - (-2)] - [z - (-1)] = 0$$

$$y - z = -1$$



60. Parametric equations:  $x = 2 + 3t$ ,  $y = -1 + t$ ,  $z = 3 + 2t$

$xy$ -plane: Let  $z = 0$ .

$$\text{Then } 3 + 2t = 0 \Rightarrow t = -\frac{3}{2} \Rightarrow x = 2 + 3\left(-\frac{3}{2}\right) = -\frac{5}{2} \text{ and}$$

$$y = -1 + \left(-\frac{3}{2}\right) = -\frac{5}{2}. \text{ Intersection: } \left(-\frac{5}{2}, -\frac{5}{2}, 0\right)$$

$xz$ -plane: Let  $y = 0$ .

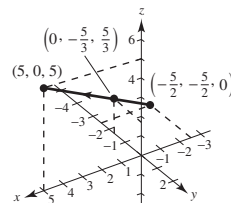
$$\text{Then } t = 1 \Rightarrow x = 2 + 3(1) = 5 \text{ and}$$

$$z = 3 + 2(1) = 5. \text{ Intersection: } (5, 0, 5)$$

$yz$ -plane: Let  $x = 0$ .

$$\text{Then } 2 + 3t = 0 \Rightarrow t = -\frac{2}{3} \Rightarrow y = -1 - \frac{2}{3} = -\frac{5}{3} \text{ and}$$

$$z = 3 + 2\left(-\frac{2}{3}\right) = \frac{5}{3}. \text{ Intersection: } \left(0, -\frac{5}{3}, \frac{5}{3}\right)$$



61. Let  $(x, y, z)$  be equidistant from  $(2, 2, 0)$  and  $(0, 2, 2)$ .

$$\begin{aligned}\sqrt{(x-2)^2 + (y-2)^2 + (z-0)^2} &= \sqrt{(x-0)^2 + (y-2)^2 + (z-2)^2} \\ x^2 - 4x + 4 + y^2 - 4y + 4 + z^2 &= x^2 + y^2 - 4y + 4 + z^2 - 4z + 4 \\ -4x + 8 &= -4z + 8 \\ x - z &= 0 \text{ Plane}\end{aligned}$$

62. Let  $(x, y, z)$  be equidistant from  $(1, 0, 2)$  and  $(2, 0, 1)$ .

$$\begin{aligned}\sqrt{(x-1)^2 + (y-0)^2 + (z-2)^2} &= \sqrt{(x-2)^2 + (y-0)^2 + (z-1)^2} \\ x^2 - 2x + 1 + y^2 + z^2 - 4z + 4 &= x^2 - 4x + 4 + y^2 + z^2 - 2z + 1 \\ -2x - 4z + 5 &= -4x - 2z + 5 \\ 2x - 2z &= 0 \\ x - z &= 0 \text{ Plane}\end{aligned}$$

63. Let  $(x, y, z)$  be equidistant from  $(-3, 1, 2)$  and  $(6, -2, 4)$ .

$$\begin{aligned}\sqrt{(x+3)^2 + (y-1)^2 + (z-2)^2} &= \sqrt{(x-6)^2 + (y+2)^2 + (z-4)^2} \\ x^2 + 6x + 9 + y^2 - 2y + 1 + z^2 - 4z + 4 &= x^2 - 12x + 36 + y^2 + 4y + 4 + z^2 - 8z + 16 \\ 6x - 2y - 4z + 14 &= -12x + 4y - 8z + 56 \\ 18x - 6y + 4z - 42 &= 0 \\ 9x - 3y + 2z - 21 &= 0 \text{ Plane}\end{aligned}$$

64. Let  $(x, y, z)$  be equidistant from  $(-5, 1, -3)$  and  $(2, -1, 6)$

$$\begin{aligned}\sqrt{(x+5)^2 + (y-1)^2 + (z+3)^2} &= \sqrt{(x-2)^2 + (y+1)^2 + (z-6)^2} \\ x^2 + 10x + 25 + y^2 - 2y + 1 + z^2 + 6z + 9 &= x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 12z + 36 \\ 10x - 2y + 6z + 35 &= -4x + 2y - 12z + 41 \\ 14x - 4y + 18z - 6 &= 0 \\ 7x - 2y + 9z - 3 &= 0 \text{ Plane}\end{aligned}$$

65. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 5, -3, 1 \rangle, \mathbf{n}_2 = \langle 1, 4, 7 \rangle, \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0.$$

So,  $\theta = \pi/2$  and the planes are orthogonal.

66. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 3, 1, -4 \rangle, \mathbf{n}_2 = \langle -9, -3, 12 \rangle.$$

Because  $\mathbf{n}_2 = -3\mathbf{n}_1$ , the planes are parallel, but not equal.

67. The normal vectors to the planes are

$$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46}\sqrt{27}} = \frac{4\sqrt{138}}{414} = \frac{2\sqrt{138}}{207}.$$

$$\text{So, } \theta = \arccos\left(\frac{2\sqrt{138}}{207}\right) \approx 83.5^\circ.$$

68. The normal vectors to the planes are

$$\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|3 - 8 - 2|}{\sqrt{14}\sqrt{21}} = \frac{7\sqrt{6}}{42} = \frac{\sqrt{6}}{6}.$$

$$\text{So, } \theta = \arccos\left(\frac{\sqrt{6}}{6}\right) \approx 65.9^\circ.$$

69. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 1, -5, -1 \rangle$  and

$$\mathbf{n}_2 = \langle 5, -25, -5 \rangle. \text{ Because } \mathbf{n}_2 = 5\mathbf{n}_1, \text{ the planes are parallel, but not equal.}$$

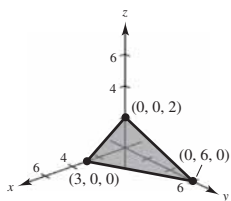
70. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 2, 0, -1 \rangle, \mathbf{n}_2 = \langle 4, 1, 8 \rangle,$$

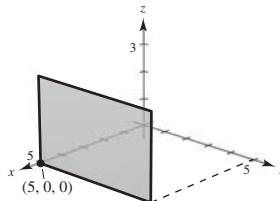
$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0$$

So,  $\theta = \frac{\pi}{2}$  and the planes are orthogonal.

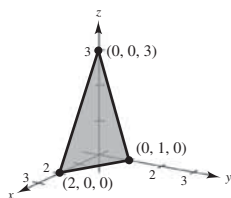
71.  $4x + 2y + 6z = 12$



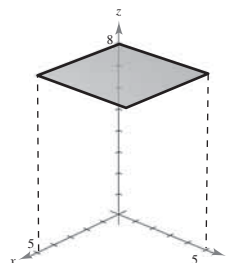
77.  $x = 5$



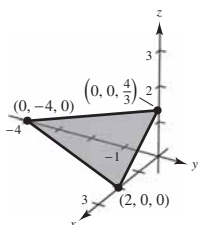
72.  $3x + 6y + 2z = 6$



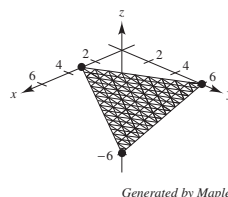
78.  $z = 8$



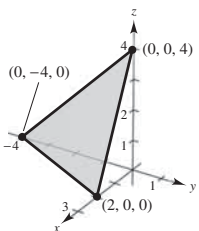
73.  $2x - y + 3z = 4$



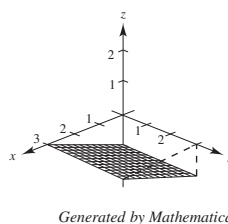
79.  $2x + y - z = 6$



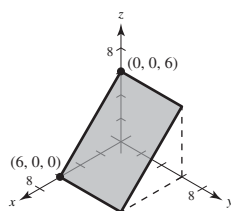
74.  $2x - y + z = 4$



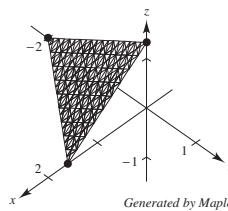
80.  $x - 3z = 3$



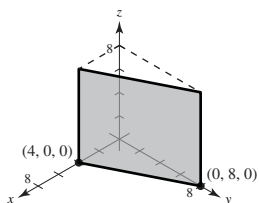
75.  $x + z = 6$



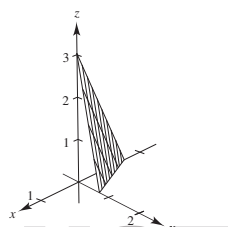
81.  $-5x + 4y - 6z + 8 = 0$



76.  $2x + y = 8$



82.  $2.1x - 4.7y - z + 3 = 0$



83.  $P_1: \mathbf{n} = \langle 15, -6, 24 \rangle$   $(0, -1, -1)$  not on plane  
 $P_2: \mathbf{n} = \langle -5, 2, -8 \rangle$   $(0, -1, -1)$  on plane  
 $P_3: \mathbf{n} = \langle 6, -4, 4 \rangle$   
 $P_4: \mathbf{n} = \langle 3, -2, -2 \rangle$

Planes  $P_1$  and  $P_2$  are parallel.

84.  $P_1: \mathbf{n} = \langle 2, -1, 3 \rangle$   $(4, 0, 0)$  on plane  
 $P_2: \mathbf{n} = \langle 3, -5, -2 \rangle$   
 $P_3: \mathbf{n} = \langle 8, -4, 12 \rangle$   $(4, 0, 0)$  not on plane  
 $P_4: \mathbf{n} = \langle -4, -2, 6 \rangle$   
 $P_1$  and  $P_3$  are parallel.

85.  $P_1: \mathbf{n} = \langle 3, -2, 5 \rangle$   $(1, -1, 1)$  on plane  
 $P_2: \mathbf{n} = \langle -6, 4, -10 \rangle$   $(1, -1, 1)$  not on plane  
 $P_3: \mathbf{n} = \langle -3, 2, 5 \rangle$   
 $P_4: \mathbf{n} = \langle 75, -50, 125 \rangle$   $(1, -1, 1)$  on plane  
 $P_1$  and  $P_4$  are identical.  
 $P_1 = P_4$  and is parallel to  $P_2$ .

86.  $P_1: \mathbf{n} = \langle -60, 90, 30 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, \frac{9}{10})$  on plane  
 $P_2: \mathbf{n} = \langle 6, -9, -3 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, -\frac{2}{3})$  on plane  
 $P_3: \mathbf{n} = \langle -20, 30, 10 \rangle$  or  $\langle -2, 3, 1 \rangle$   $(0, 0, \frac{5}{6})$  on plane  
 $P_4: \mathbf{n} = \langle 12, -18, 6 \rangle$  or  $\langle -2, 3, -1 \rangle$   
 $P_1, P_2,$  and  $P_3$  are parallel.

87. Each plane passes through the points  $(c, 0, 0)$ ,  $(0, c, 0)$ , and  $(0, 0, c)$ .

88.  $x = y = c$

Each plane is parallel to the  $z$ -axis.

89. If  $c = 0$ ,  $z = 0$  is  $xy$ -plane.

If  $c \neq 0$ ,  $cy + z = 0 \Rightarrow y = \frac{-1}{c}z$  is a plane parallel to  $x$ -axis and passing through the points  $(0, 0, 0)$  and  $(0, 1, -c)$ .

90.  $x + cz = 0$

If  $c = 0$ ,  $z = 0$  is the  $yz$ -plane.

If  $c \neq 0$ ,  $x + cz = 0$  is a plane parallel to the  $y$ -axis.

91. (a)  $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|-7|}{\sqrt{14}\sqrt{21}} = \frac{\sqrt{6}}{6}$$

$$\Rightarrow \theta \approx 1.1503 \approx 65.91^\circ$$

- (b) The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Find a point of intersection of the planes.

$$6x + 4y - 2z = 14$$

$$x - 4y + 2z = 0$$

$$7x = 14$$

$$x = 2$$

Substituting 2 for  $x$  in the second equation, you have  $-4y + 2z = -2$  or  $z = 2y - 1$ . Letting  $y = 1$ , a point of intersection is  $(2, 1, 1)$ .

$$x = 2, y = 1 + t, z = 1 + 2t$$

92. (a)  $\mathbf{n}_1 = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ ,  $\mathbf{n}_2 = -\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{-4}{\sqrt{46}\sqrt{27}} = \frac{-2\sqrt{138}}{207}$$

$$\theta \approx 1.6845 \approx 96.52^\circ$$

- (b) The direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \langle -16, -31, 3 \rangle.$$

Find a point of intersection of the planes.

$$6x - 3y + z = 5 \Rightarrow 6x - 3y + z = 5$$

$$-x + y + 5z = 5 \Rightarrow \frac{-6x + 6y + 30z = 30}{3y + 31z = 35}$$

$$\text{Let } y = -9, z = 2 \Rightarrow x = -4 \Rightarrow (-4, -9, 2).$$

$$x = -4 - 16t, y = -9 - 31t, z = 2 + 3t$$

93. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = \frac{1}{2} + t, y = \frac{-3}{2} - t, z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12, t = \frac{3}{2}$$

Substituting  $t = 3/2$  into the parametric equations for the line you have the point of intersection  $(2, -3, 2)$ .

The line does not lie in the plane.

94. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 4t, y = 2t, z = 3 + 6t$$

$$2(1 + 4t) + 3(2t) = -5, t = -\frac{1}{2}$$

Substituting  $t = -\frac{1}{2}$  into the parametric equations for the line you have the point of intersection  $(-1, -1, 0)$ .

The line does not lie in the plane.

95. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 1 + 3t, y = -1 - 2t, z = 3 + t$$

$$2(1 + 3t) + 3(-1 - 2t) = 10, -1 = 10, \text{contradiction}$$

So, the line does not intersect the plane.

96. Writing the equation of the line in parametric form and substituting into the equation of the plane you have:

$$x = 4 + 2t, y = -1 - 3t, z = -2 + 5t$$

$$5(4 + 2t) + 3(-1 - 3t) = 17, t = 0$$

Substituting  $t = 0$  into the parametric equations for the line you have the point of intersection  $(4, -1, -2)$ .

The line does not lie in the plane.

97. Point:  $Q(0, 0, 0)$

$$\text{Plane: } 2x + 3y + z - 12 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 3, 1 \rangle$$

$$\text{Point in plane: } P(6, 0, 0)$$

$$\text{Vector } \overrightarrow{PQ} = \langle -6, 0, 0 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

98. Point:  $Q(0, 0, 0)$

$$\text{Plane: } 5x + y - z - 9 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 5, 1, -1 \rangle$$

$$\text{Point in plane: } P(0, 9, 0)$$

$$\text{Vector } \overrightarrow{PQ} = \langle 0, -9, 0 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-9|}{\sqrt{27}} = \sqrt{3}$$

99. Point:  $Q(2, 8, 4)$

$$\text{Plane: } 2x + y + z = 5$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 1, 1 \rangle$$

$$\text{Point in plane: } P(0, 0, 5)$$

$$\text{Vector: } \overrightarrow{PQ} = \langle 2, 8, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

100. Point:  $Q(1, 3, -1)$

$$\text{Plane: } 3x - 4y + 5z - 6 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 3, -4, 5 \rangle$$

$$\text{Point in plane: } P(2, 0, 0)$$

$$\text{Vector } \overrightarrow{PQ} = \langle -1, 3, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-20|}{\sqrt{50}} = 2\sqrt{2}$$

101. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 1, -3, 4 \rangle$  and  $\mathbf{n}_2 = \langle 1, -3, 4 \rangle$ . Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are parallel. Choose a point in each plane.

$$P(10, 0, 0) \text{ is a point in } x - 3y + 4z = 10.$$

$$Q(6, 0, 0) \text{ is a point in } x - 3y + 4z = 6.$$

$$\overrightarrow{PQ} = \langle -4, 0, 0 \rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

102. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 4, -4, 9 \rangle$  and  $\mathbf{n}_2 = \langle 4, -4, 9 \rangle$ . Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are parallel. Choose a point in each plane.

$$P(-5, 0, 3) \text{ is a point in } 4x - 4y + 9z = 7.$$

$$Q(0, 0, 2) \text{ is a point in } 4x - 4y + 9z = 18.$$

$$\overrightarrow{PQ} = \langle 5, 0, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{11}{\sqrt{113}} = \frac{11\sqrt{113}}{113}$$

103. The normal vectors to the planes are  $\mathbf{n}_1 = \langle -3, 6, 7 \rangle$  and  $\mathbf{n}_2 = \langle 6, -12, -14 \rangle$ . Because  $\mathbf{n}_2 = -2\mathbf{n}_1$ , the planes are parallel. Choose a point in each plane.

$$P(0, -1, 1) \text{ is a point in } -3x + 6y + 7z = 1.$$

$$Q\left(\frac{25}{6}, 0, 0\right) \text{ is a point in } 6x - 12y - 14z = 25.$$

$$\overrightarrow{PQ} = \left\langle \frac{25}{6}, 1, -1 \right\rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

104. The normal vectors to the planes are  $\mathbf{n}_1 = \langle 2, 0, -4 \rangle$  and  $\mathbf{n}_2 = \langle 2, 0, -4 \rangle$ . Because  $\mathbf{n}_1 = \mathbf{n}_2$ , the planes are parallel. Choose a point in each plane.

$$P(2, 0, 0) \text{ is a point in } 2x - 4z = 4.$$

$$Q(5, 0, 0) \text{ is a point in } 2x - 4z = 10.$$

$$\overrightarrow{PQ} = \langle 3, 0, 0 \rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{6}{\sqrt{20}} = \frac{3\sqrt{5}}{5}$$

105.  $\mathbf{u} = \langle 4, 0, -1 \rangle$  is the direction vector for the line.  
 $Q(1, 5, -2)$  is the given point, and  $P(-2, 3, 1)$  is on the line.

$$\overrightarrow{PQ} = \langle 3, 2, -3 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

106.  $\mathbf{u} = \langle 2, 1, 2 \rangle$  is the direction vector for the line.  
 $Q(1, -2, 4)$  is the given point, and  $P(0, -3, 2)$  is a point on the line (let  $t = 0$ ).

$$\overrightarrow{PQ} = \langle 1, 1, 2 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \langle 0, 2, -1 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

107.  $\mathbf{u} = \langle -1, 1, -2 \rangle$  is the direction vector for the line.  
 $Q(-2, 1, 3)$  is the given point, and  $P(1, 2, 0)$  is on the line (let  $t = 0$  in the parametric equations for the line).

$$\overrightarrow{PQ} = \langle -3, -1, 3 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 3 \\ -1 & 1 & -2 \end{vmatrix} = \langle -1, -9, -4 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{1+81+16}}{\sqrt{1+1+4}} = \frac{\sqrt{98}}{6} = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

108.  $\mathbf{u} = \langle 0, 3, 1 \rangle$  is the direction vector for the line.

$Q(4, -1, 5)$  is the given point, and  $P(3, 1, 1)$  is on the line.

$$\overrightarrow{PQ} = \langle 1, -2, 4 \rangle$$

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 0 & 3 & 1 \end{vmatrix} = \langle -14, -1, 3 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{14^2 + 1 + 9}}{\sqrt{9+1}} = \sqrt{\frac{206}{10}} = \sqrt{\frac{103}{5}} = \frac{\sqrt{515}}{5}$$

109. The direction vector for  $L_1$  is  $\mathbf{v}_1 = \langle -1, 2, 1 \rangle$ .

The direction vector for  $L_2$  is  $\mathbf{v}_2 = \langle 3, -6, -3 \rangle$ .

Because  $\mathbf{v}_2 = -3\mathbf{v}_1$ , the lines are parallel.

Let  $Q(2, 3, 4)$  to be a point on  $L_1$  and  $P(0, 1, 4)$  a point on  $L_2$ .  $\overrightarrow{PQ} = \langle 2, 0, 0 \rangle$ .

$\mathbf{u} = \mathbf{v}_2$  is the direction vector for  $L_2$ .

$$\overrightarrow{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 3 & -6 & -3 \end{vmatrix} = \langle -6, 6, -18 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{v}_2\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{36 + 36 + 324}}{\sqrt{9 + 36 + 9}} = \sqrt{\frac{396}{54}} = \sqrt{\frac{22}{3}} = \frac{\sqrt{66}}{3}$$

110. The direction vector for  $L_1$  is  $\mathbf{v}_1 = \langle 6, 9, -12 \rangle$ .

The direction vector for  $L_2$  is  $\mathbf{v}_2 = \langle 4, 6, -8 \rangle$ .

Because  $\mathbf{v}_1 = \frac{3}{2}\mathbf{v}_2$ , the lines are parallel.

Let  $Q(3, -2, 1)$  to be a point on  $L_1$  and  $P(-1, 3, 0)$  a point on  $L_2$ .  $\overrightarrow{PQ} = \langle 4, -5, 1 \rangle$ .

$\mathbf{u} = \mathbf{v}_2$  is the direction vector for  $L_2$ .

$$\overrightarrow{PQ} \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -5 & 1 \\ 4 & 6 & -8 \end{vmatrix} = \langle 34, 36, 44 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{v}_2\|}{\|\mathbf{v}_2\|} = \frac{\sqrt{34^2 + 36^2 + 44^2}}{\sqrt{16 + 36 + 64}} = \frac{\sqrt{4388}}{\sqrt{116}} = \sqrt{\frac{1097}{29}} = \frac{\sqrt{31813}}{29}$$



- 111.** The parametric equations of a line  $L$  parallel to  $\mathbf{v} = \langle a, b, c \rangle$  and passing through the point

$P(x_1, y_1, z_1)$  are

$$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct.$$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

- 112.** The equation of the plane containing  $P(x_1, y_1, z_1)$  and having normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

You need  $\mathbf{n}$  and  $P$  to find the equation.

- 113.** Simultaneously solve the two linear equations representing the planes and substitute the values back into one of the original equations. Then choose a value for  $t$  and form the corresponding parametric equations for the line of intersection.

- 114.**  $x = a$ : plane parallel to  $yz$ -plane containing  $(a, 0, 0)$

$y = b$ : plane parallel to  $xz$ -plane containing  $(0, b, 0)$

$z = c$ : plane parallel to  $xy$ -plane containing  $(0, 0, c)$

- 115.** (a) The planes are parallel if their normal vectors are parallel:

$$\langle a_1, b_1, c_1 \rangle = t \langle a_2, b_2, c_2 \rangle, \quad t \neq 0$$

- (b) The planes are perpendicular if their normal vectors are perpendicular:

$$\langle a_1, b_1, c_1 \rangle \cdot \langle a_2, b_2, c_2 \rangle = 0$$

- 116.** Yes. If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the direction vectors for the lines  $L_1$  and  $L_2$ , then  $\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$  is perpendicular to both  $L_1$  and  $L_2$ .

- 117.** An equation for the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow bcx + acy + abz = abc$$

For example, letting  $y = z = 0$ , the  $x$ -intercept is

$$(a, 0, 0).$$

- 118.** (a) Matches (iii)

(b) Matches (i)

(c) Matches (iv)

(d) Matches (ii)

- 119.** Sphere

$$(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$$

- 120.** Parallel planes

$$4x - 3y + z = 10 \pm 4\|\mathbf{n}\| = 10 \pm 4\sqrt{26}$$

- 121.**  $0.92x - 1.03y + z = 0.02 \Rightarrow z = 0.02 - 0.92x + 1.03y$

(a)

Year	1999	2000	2001	2002	2003	2004	2005
$x$	1.4	1.4	1.4	1.6	1.6	1.7	1.7
$y$	7.3	7.1	7.0	7.0	6.9	6.9	6.9
$z$	6.2	6.1	5.9	5.8	5.6	5.5	5.6
Model $z$	6.25	6.05	5.94	5.76	5.66	5.56	5.56

The approximations are close to the actual values.

- (b) According to the model, if  $x$  and  $z$  decrease, then so will  $y$ . (Answers will vary.)

- 122.** On one side you have the points  $(0, 0, 0)$ ,  $(6, 0, 0)$ , and  $(-1, -1, 8)$ .

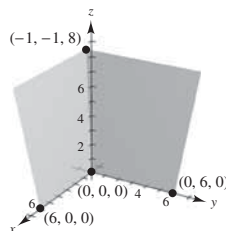
$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 0 \\ -1 & -1 & 8 \end{vmatrix} = -48\mathbf{j} - 6\mathbf{k}$$

On the adjacent side you have the points  $(0, 0, 0)$ ,  $(0, 6, 0)$ , and  $(-1, -1, 8)$ .

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ -1 & -1 & 8 \end{vmatrix} = 48\mathbf{i} + 6\mathbf{k}$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{36}{2340} = \frac{1}{65}$$

$$\theta = \arccos \frac{1}{65} \approx 89.1^\circ$$



123.  $L_1: x_1 = 6 + t; y_1 = 8 - t, z_1 = 3 + t$

$L_2: x_2 = 1 + t, y_2 = 2 + t, z_2 = 2t$

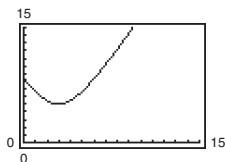
(a) At  $t = 0$ , the first insect is at  $P_1(6, 8, 3)$  and the second insect is at  $P_2(1, 2, 0)$ .

$$\text{Distance} = \sqrt{(6-1)^2 + (8-2)^2 + (3-0)^2} = \sqrt{70} \approx 8.37 \text{ inches}$$

(b)  $\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{5^2 + (6-2t)^2 + (3-t)^2} = \sqrt{5t^2 - 30t + 70}, 0 \leq t \leq 10$

(c) The distance is never zero.

(d) Using a graphing utility, the minimum distance is 5 inches when  $t = 3$  minutes.



124. First find the distance  $D$  from the point  $Q(-3, 2, 4)$  to the plane. Let  $P(4, 0, 0)$  be on the plane.

$\mathbf{n} = \langle 2, 4, -3 \rangle$  is the normal to the plane.

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -7, 2, 4 \rangle \cdot \langle 2, 4, -3 \rangle|}{\sqrt{4 + 16 + 9}} = \frac{|-14 + 8 - 12|}{\sqrt{29}} = \frac{18}{\sqrt{29}} = \frac{18\sqrt{29}}{29}$$

The equation of the sphere with center  $(-3, 2, 4)$  and radius  $18\sqrt{29}/29$  is  $(x+3)^2 + (y-2)^2 + (z-4)^2 = \frac{324}{29}$ .

125. The direction vector  $\mathbf{v}$  of the line is the normal to the plane,  $\mathbf{v} = \langle 3, -1, 4 \rangle$ .

The parametric equations of the line are  
 $x = 5 + 3t, y = 4 - t, z = -3 + 4t$ .

To find the point of intersection, solve for  $t$  in the following equation:

$$3(5 + 3t) - (4 - t) + 4(-3 + 4t) = 7$$

$$26t = 8$$

$$t = \frac{4}{13}$$

Point of intersection:

$$\left(5 + 3\left(\frac{4}{13}\right), 4 - \frac{4}{13}, -3 + 4\left(\frac{4}{13}\right)\right) = \left(\frac{77}{13}, \frac{48}{13}, -\frac{23}{13}\right)$$

126. The normal to the plane,  $\mathbf{n} = \langle 2, -1, -3 \rangle$  is perpendicular to the direction vector  $\mathbf{v} = \langle 2, 4, 0 \rangle$  of the line because  
 $\langle 2, -1, -3 \rangle \cdot \langle 2, 4, 0 \rangle = 0$ .

So, the plane is parallel to the line. To find the distance between them, let  $Q(-2, -1, 4)$  be on the line and

$P(2, 0, 0)$  on the plane.  $\overrightarrow{PQ} = \langle -4, -1, 4 \rangle$ .

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -4, -1, 4 \rangle \cdot \langle 2, -1, -3 \rangle|}{\sqrt{4 + 1 + 9}} = \frac{19}{\sqrt{14}} = \frac{19\sqrt{14}}{14}$$

127. The direction vector of the line  $L$  through  $(1, -3, 1)$  and  $(3, -4, 2)$  is  $\mathbf{v} = \langle 2, -1, 1 \rangle$ .

The parametric equations for  $L$  are  
 $x = 1 + 2t, y = -3 - t, z = 1 + t$ .

Substituting these equations into the equation of the plane gives

$$(1 + 2t) - (-3 - t) + (1 + t) = 2$$

$$4t = -3$$

$$t = -\frac{3}{4}$$

Point of intersection:

$$\left(1 + 2\left(-\frac{3}{4}\right), -3 + \frac{3}{4}, 1 - \frac{3}{4}\right) = \left(-\frac{1}{2}, -\frac{9}{4}, \frac{1}{4}\right)$$

128. The unknown line  $L$  is perpendicular to the normal vector  $\mathbf{n} = \langle 1, 1, 1 \rangle$  of the plane, and perpendicular to the direction vector  $\mathbf{u} = \langle 1, 1, -1 \rangle$ . So, the direction vector of  $L$  is

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \langle -2, 2, 0 \rangle.$$

The parametric equations for  $L$  are  $x = 1 - 2t, y = 2t, z = 2$ .

129. True

130. False. They may be skew lines.

(See Section Project)

131. True

132. False. The lines  $x = t$ ,  $y = 0$ ,  $z = 1$  and  $x = 0$ ,  $y = t$ ,  $z = 1$  are both parallel to the plane  $z = 0$ , but the lines are not parallel.

133. False. Planes  $7x + y - 11z = 5$  and  $5x + 2y - 4z = 1$  are both perpendicular to plane  $2x - 3y + z = 3$ , but are not parallel.

134. True.

## Section 11.6 Surfaces in Space

1. Ellipsoid

Matches graph (c)

2. Hyperboloid of two sheets

Matches graph (e)

3. Hyperboloid of one sheet

Matches graph (f)

4. Elliptic cone

Matches graph (b)

5. Elliptic paraboloid

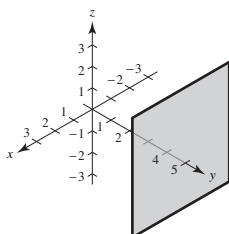
Matches graph (d)

6. Hyperbolic paraboloid

Matches graph (a)

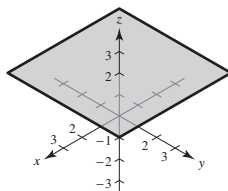
7.  $y = 5$

Plane is parallel to the  $xz$ -plane.



8.  $z = 2$

Plane is parallel to the  $xy$ -plane.



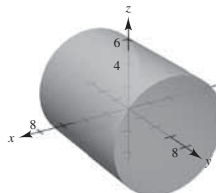
9.  $y^2 + z^2 = 9$

The  $x$ -coordinate is missing so you have a right circular cylinder with rulings parallel to the  $x$ -axis. The generating curve is a circle.



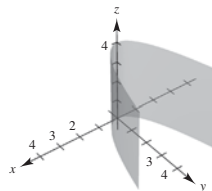
10.  $x^2 + z^2 = 25$

The  $y$ -coordinate is missing so you have a right circular cylinder with rulings parallel to the  $y$ -axis. The generating curve is a circle.



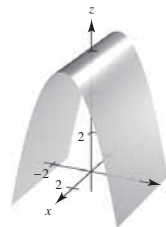
11.  $y = x^2$

The  $z$ -coordinate is missing so you have a parabolic cylinder with rulings parallel to the  $z$ -axis. The generating curve is a parabola.



12.  $y^2 + z = 6$

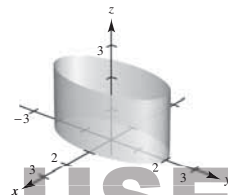
The  $x$ -coordinate is missing so you have a parabolic cylinder with the rulings parallel to the  $x$ -axis. The generating curve is a parabola.



13.  $4x^2 + y^2 = 4$

$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

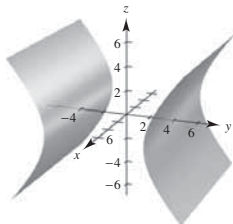
The  $z$ -coordinate is missing so you have an elliptic cylinder with rulings parallel to the  $z$ -axis. The generating curve is an ellipse.



14.  $y^2 - z^2 = 16$

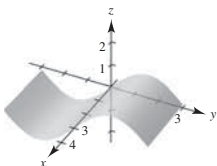
$$\frac{y^2}{16} - \frac{z^2}{16} = 1$$

The  $x$ -coordinate is missing so you have a hyperbolic cylinder with rulings parallel to the  $x$ -axis. The generating curve is a hyperbola.



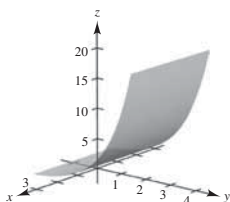
15.  $z = \sin y$

The  $x$ -coordinate is missing so you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is the sine curve.



16.  $z = e^y$

The  $x$ -coordinate is missing so you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is the exponential curve.

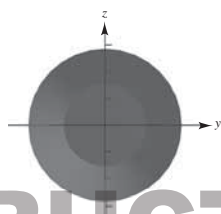


17.  $z = x^2 + y^2$

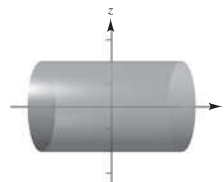
- (a) You are viewing the paraboloid from the  $x$ -axis:  
(20, 0, 0)
- (b) You are viewing the paraboloid from above, but not on the  $z$ -axis: (10, 10, 20)
- (c) You are viewing the paraboloid from the  $z$ -axis:  
(0, 0, 20)
- (d) You are viewing the paraboloid from the  $y$ -axis:  
(0, 20, 0)

18.  $y^2 + z^2 = 4$

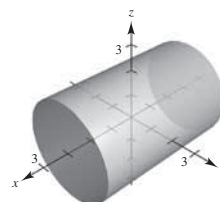
- (a) From (10, 0, 0):



- (b) From (0, 10, 0):



- (c) From (10, 10, 10):



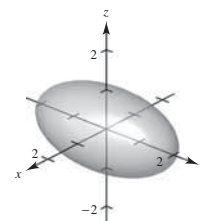
19.  $\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{1} + \frac{y^2}{4} = 1 \text{ ellipse}$$

$$xz\text{-trace: } x^2 + z^2 = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{y^2}{4} + \frac{z^2}{1} = 1 \text{ ellipse}$$



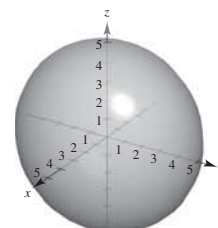
20.  $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{25} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{25} = 1 \text{ ellipse}$$

$$xz\text{-trace: } \frac{x^2}{16} + \frac{z^2}{25} = 1 \text{ ellipse}$$

$$yz\text{-trace: } y^2 + z^2 = 25 \text{ circle}$$



21.  $16x^2 - y^2 + 16z^2 = 4$

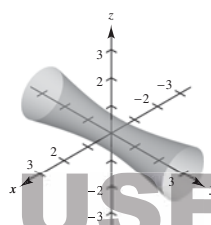
$$4x^2 - \frac{y^2}{4} + 4z^2 = 1$$

Hyperboloid of one sheet

$$xy\text{-trace: } 4x^2 - \frac{y^2}{4} = 1 \text{ hyperbola}$$

$$xz\text{-trace: } 4(x^2 + z^2) = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{-y^2}{4} + 4z^2 = 1 \text{ hyperbola}$$



22.  $-8x^2 + 18y^2 + 18z^2 = 2$

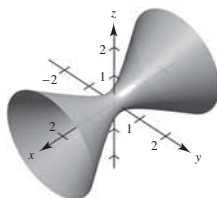
$9y^2 + 9z^2 - 4x^2 = 1$

Hyperboloid of one sheet

$xy$ -trace:  $9y^2 - 4x^2 = 1$  hyperbola

$yz$ -trace:  $9y^2 + 9z^2 = 1$  circle

$xz$ -trace:  $9z^2 - 4x^2 = 1$  hyperbola



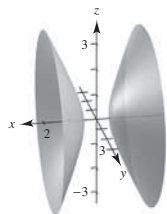
23.  $4x^2 - y^2 - z^2 = 1$

Hyperboloid of two sheets

$xy$ -trace:  $4x^2 - y^2 = 1$  hyperbola

$yz$ -trace: none

$xz$ -trace:  $4x^2 - z^2 = 1$  hyperbola



24.  $z^2 - x^2 - \frac{y^2}{4} = 1$

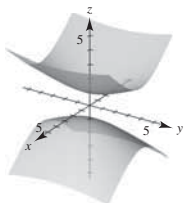
Hyperboloid of two sheets

$xy$ -trace: none

$xz$ -trace:  $z^2 - x^2 = 1$  hyperbola

$yz$ -trace:  $z^2 - \frac{y^2}{4} = 1$  hyperbola

$z = \pm\sqrt{10}: \frac{x^2}{9} + \frac{y^2}{36} = 1$  ellipse



25.  $x^2 - y + z^2 = 0$

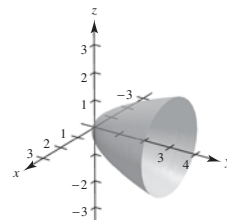
Elliptic paraboloid

$xy$ -trace:  $y = x^2$

$xz$ -trace:  $x^2 + z^2 = 0$ ,  
point  $(0, 0, 0)$

$yz$ -trace:  $y = z^2$

$y = 1: x^2 + z^2 = 1$



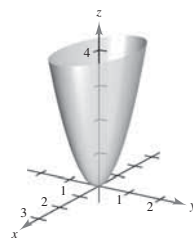
26.  $z = x^2 + 4y^2$

Elliptic paraboloid

$xy$ -trace: point  $(0, 0, 0)$

$xz$ -trace:  $z = x^2$  parabola

$yz$ -trace:  $z = 4y^2$  parabola



27.  $x^2 - y^2 + z = 0$

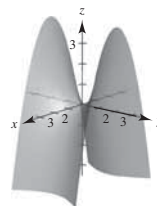
Hyperbolic paraboloid

$xy$ -trace:  $y = \pm x$

$xz$ -trace:  $z = -x^2$

$yz$ -trace:  $z = y^2$

$y = \pm 1: z = 1 - x^2$



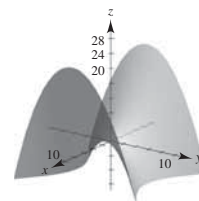
28.  $3z = -y^2 + x^2$

Hyperbolic paraboloid

$xy$ -trace:  $y = \pm x$

$xz$ -trace:  $z = \frac{1}{3}x^2$

$yz$ -trace:  $z = -\frac{1}{3}y^2$



29.  $z^2 = x^2 + \frac{y^2}{9}$

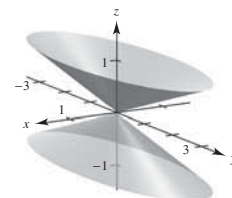
Elliptic cone

$xy$ -trace: point  $(0, 0, 0)$

$xz$ -trace:  $z = \pm x$

$yz$ -trace:  $z = \pm \frac{y}{3}$

When  $z = \pm 1$ ,  $x^2 + \frac{y^2}{9} = 1$  ellipse



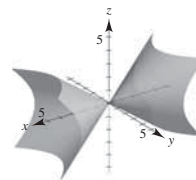
30.  $x^2 = 2y^2 + 2z^2$

Elliptic Cone

$xy$ -trace:  $x = \pm\sqrt{2}y$

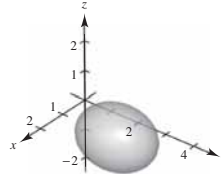
$xz$ -trace:  $x = \pm\sqrt{2}z$

$yz$ -trace: point:  $(0, 0, 0)$



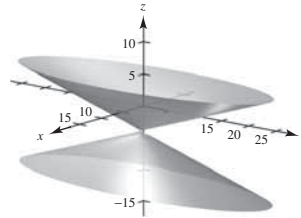
$$\begin{aligned}
 31. \quad & 16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0 \\
 & 16(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 16z^2 = -36 + 16 + 36 \\
 & 16(x - 1)^2 + 9(y - 2)^2 + 16z^2 = 16 \\
 & \frac{(x - 1)^2}{1} + \frac{(y - 2)^2}{16/9} + \frac{z^2}{1} = 1
 \end{aligned}$$

Ellipsoid with center  $(1, 2, 0)$ .

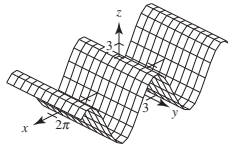


$$\begin{aligned}
 32. \quad & 9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0 \\
 & 9(x^2 - 6x + 9) + (y^2 - 4y + 4) - 9(z^2 + 6z + 9) = 81 - 81 \\
 & 9(x - 3)^2 + (y - 2)^2 - 9(z + 3)^2 = 0
 \end{aligned}$$

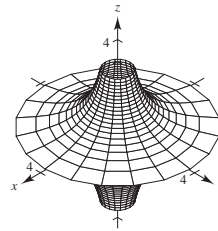
Elliptic cone with center  $(3, 2, -3)$ .



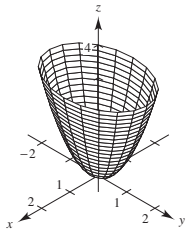
$$33. \quad z = 2 \cos x$$



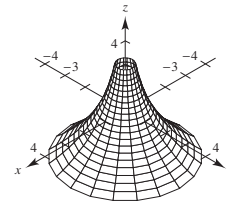
$$\begin{aligned}
 37. \quad & x^2 + y^2 = \left(\frac{2}{z}\right)^2 \\
 & y = \pm \sqrt{\frac{4}{z^2} - x^2}
 \end{aligned}$$



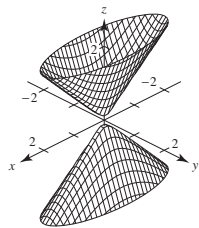
$$34. \quad z = x^2 + 0.5y^2$$



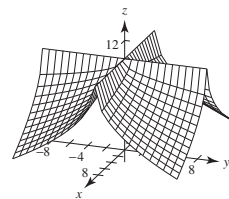
$$\begin{aligned}
 38. \quad & x^2 + y^2 = e^{-z} \\
 & -\ln(x^2 + y^2) = z
 \end{aligned}$$



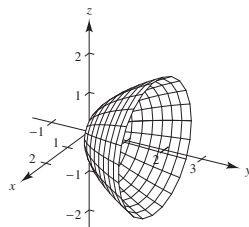
$$\begin{aligned}
 35. \quad & z^2 = x^2 + 7.5y^2 \\
 & z = \pm \sqrt{x^2 + 7.5y^2}
 \end{aligned}$$



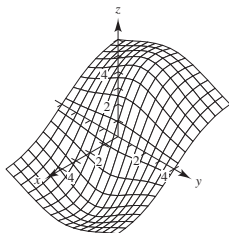
$$39. \quad z = 10 - \sqrt{|xy|}$$



$$\begin{aligned}
 36. \quad & 3.25y = x^2 + z^2 \\
 & z^2 = 3.25y - x^2 \\
 & z = \pm \sqrt{3.25y - x^2}
 \end{aligned}$$



40. 
$$z = \frac{-x}{8 + x^2 + y^2}$$

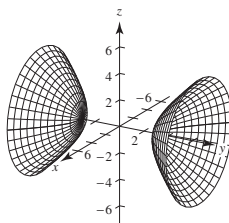


41. 
$$6x^2 - 4y^2 + 6z^2 = -36$$

$$6z^2 = 4y^2 - 6x^2 - 36$$

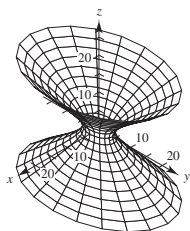
$$3z^2 = 2y^2 - 3x^2 - 18$$

$$z = \pm \frac{1}{\sqrt{3}} \sqrt{2y^2 - 3x^2 - 18}$$



42. 
$$9x^2 + 4y^2 - 8z^2 = 72$$

$$z = \pm \sqrt{\frac{9}{8}x^2 + \frac{1}{2}y^2 - 9}$$

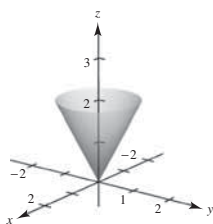


43. 
$$z = 2\sqrt{x^2 + y^2}$$

$$z = 2$$

$$2\sqrt{x^2 + y^2} = 2$$

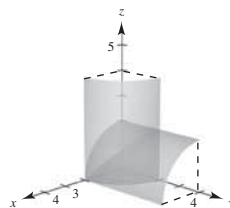
$$x^2 + y^2 = 1$$



44. 
$$z = \sqrt{4 - x^2}$$

$$y = \sqrt{4 - x^2}$$

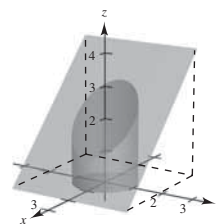
$$x = 0, y = 0, z = 0$$



45. 
$$x^2 + y^2 = 1$$

$$x + z = 2$$

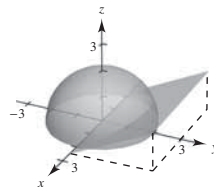
$$z = 0$$



46. 
$$z = \sqrt{4 - x^2 - y^2}$$

$$y = 2z$$

$$z = 0$$



47. 
$$x^2 + z^2 = [r(y)]^2$$
 and  $z = r(y) = \pm 2\sqrt{y}$ ; so,

$$x^2 + z^2 = 4y.$$

48. 
$$x^2 + z^2 = [r(y)]^2$$
 and  $z = r(y) = 3y$ ; so,

$$x^2 + z^2 = 9y^2.$$

49. 
$$x^2 + y^2 = [r(z)]^2$$
 and  $y = r(z) = \frac{z}{2}$ ; so,

$$x^2 + y^2 = \frac{z^2}{4}, 4x^2 + 4y^2 = z^2.$$

50. 
$$y^2 + z^2 = [r(x)]^2$$
 and  $z = r(x) = \frac{1}{2}\sqrt{4 - x^2}$ ; so,

$$y^2 + z^2 = \frac{1}{4}(4 - x^2), x^2 + 4y^2 + 4z^2 = 4.$$

51.  $y^2 + z^2 = [r(x)]^2$  and  $y = r(x) = \frac{2}{x}$ ; so,

$$y^2 + z^2 = \left(\frac{2}{x}\right)^2, y^2 + z^2 = \frac{4}{x^2}.$$

52.  $x^2 + y^2 = [r(z)]^2$  and  $y = r(z) = e^z$ ; so,

$$x^2 + y^2 = e^{2z}.$$

53.  $x^2 + y^2 - 2z = 0$

$$x^2 + y^2 = (\sqrt{2z})^2$$

Equation of generating curve:  $y = \sqrt{2z}$  or  $x = \sqrt{2z}$

54.  $x^2 + z^2 = \cos^2 y$

Equation of generating curve:  $x = \cos y$  or  $z = \cos y$

55. Let  $C$  be a curve in a plane and let  $L$  be a line not in a parallel plane. The set of all lines parallel to  $L$  and intersecting  $C$  is called a cylinder.  $C$  is called the generating curve of the cylinder, and the parallel lines are called rulings.

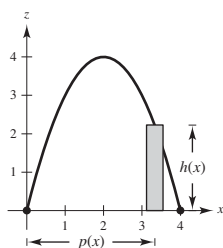
56. The trace of a surface is the intersection of the surface with a plane. You find a trace by setting one variable equal to a constant, such as  $x = 0$  or  $z = 2$ .

57. See pages 814 and 815.

58. In the  $xz$ -plane,  $z = x^2$  is a parabola.

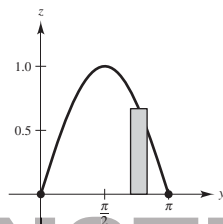
In three-space,  $z = x^2$  is a cylinder.

59.  $V = 2\pi \int_0^4 x(4x - x^2) dx = 2\pi \left[ \frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4 = \frac{218\pi}{3}$



60.  $V = 2\pi \int_0^\pi y \sin y dy$

$$= 2\pi [\sin y - y \cos y]_0^\pi = 2\pi^2$$



61.  $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When  $z = 2$  we have  $2 = \frac{x^2}{2} + \frac{y^2}{4}$ , or

$$1 = \frac{x^2}{4} + \frac{y^2}{8}$$

Major axis:  $2\sqrt{8} = 4\sqrt{2}$

Minor axis:  $2\sqrt{4} = 4$

$$c^2 = a^2 - b^2, c^2 = 4, c = 2$$

Foci:  $(0, \pm 2, 2)$

(b) When  $z = 8$  we have  $8 = \frac{x^2}{2} + \frac{y^2}{4}$ , or

$$1 = \frac{x^2}{16} + \frac{y^2}{32}$$

Major axis:  $2\sqrt{32} = 8\sqrt{2}$

Minor axis:  $2\sqrt{16} = 8$

$$c^2 = 32 - 16 = 16, c = 4$$

Foci:  $(0, \pm 4, 8)$

62.  $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When  $y = 4$  you have  $z = \frac{x^2}{2} + 4$ ,

$$4\left(\frac{1}{2}\right)(z - 4) = x^2.$$

Focus:  $\left(0, 4, \frac{9}{2}\right)$

(b) When  $x = 2$  you have

$$z = 2 + \frac{y^2}{4}, 4(z - 2) = y^2.$$

Focus:  $(2, 0, 3)$

63. If  $(x, y, z)$  is on the surface, then

$$(y + 2)^2 = x^2 + (y - 2)^2 + z^2$$

$$y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 + z^2$$

$$x^2 + z^2 = 8y$$

Elliptic paraboloid

Traces parallel to  $xz$ -plane are circles.

64. If  $(x, y, z)$  is on the surface, then

$$z^2 = x^2 + y^2 + (z - 4)^2$$

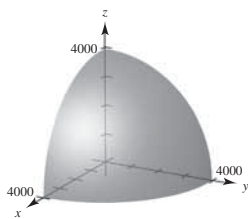
$$z^2 = x^2 + y^2 + z^2 - 8z + 16$$

$$8z = x^2 + y^2 + 16 \Rightarrow z = \frac{x^2}{8} + \frac{y^2}{8} + 2$$

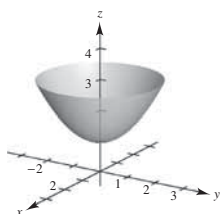
Elliptic paraboloid shifted up 2 units. Traces parallel to  $xy$ -plane are circles.



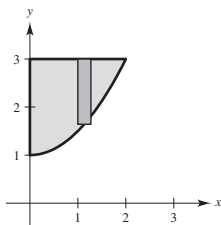
65.  $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3950^2} = 1$



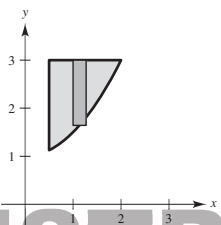
66. (a) 
$$x^2 + y^2 = [r(z)]^2$$
$$= [\sqrt{2(z-1)}]^2$$
$$x^2 + y^2 - 2z + 2 = 0$$



(b) 
$$V = 2\pi \int_0^2 x \left[ 3 - \left( \frac{1}{2}x^2 + 1 \right) \right] dx$$
$$= 2\pi \int_0^2 \left( 2x - \frac{1}{2}x^3 \right) dx$$
$$= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_0^2 = 4\pi \approx 12.6 \text{ cm}^3$$



(c) 
$$V = 2\pi \int_{1/2}^2 x \left[ 3 - \left( \frac{1}{2}x^2 + 1 \right) \right] dx$$
$$= 2\pi \int_{1/2}^2 \left( 2x - \frac{1}{2}x^3 \right) dx$$
$$= 2\pi \left[ x^2 - \frac{x^4}{8} \right]_{1/2}^2$$
$$= 4 - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3$$



67.  $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}, z = bx + ay$

$$bx + ay = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$
$$\frac{1}{a^2} \left( x^2 + a^2bx + \frac{a^4b^2}{4} \right) = \frac{1}{b^2} \left( y^2 - ab^2y + \frac{a^2b^4}{4} \right)$$
$$\frac{\left( x + \frac{a^2b}{2} \right)^2}{a^2} = \frac{\left( y - \frac{ab^2}{2} \right)^2}{b^2}$$
$$y = \pm \frac{b}{a} \left( x + \frac{a^2b}{2} \right) + \frac{ab^2}{2}$$

Letting  $x = at$ , you obtain the two intersecting lines

$$x = at, y = -bt, z = 0 \text{ and } x = at,$$

$$y = bt + ab^2, z = 2abt + a^2b^2.$$

68. Equating twice the first equation with the second equation:

$$2x^2 + 6y^2 - 4z^2 + 4y - 8 = 2x^2 + 6y^2 - 4z^2 - 3x - 2$$

$$4y - 8 = -3x - 2$$

$$3x + 4y = 6, \text{ a plane}$$

69. True. A sphere is a special case of an ellipsoid (centered at origin, for example)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

having  $a = b = c$ .

70. False. For example, the surface  $x^2 + z^2 = e^{-2y}$  can be formed by revolving the graph of  $x = e^{-y}$  about the  $y$ -axis, as the graph of  $z = e^{-y}$  about the  $y$ -axis.

71. False. The trace  $x = 2$  of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \text{ is the point } (2, 0, 0).$$

72. False. Traces perpendicular to the axis are ellipses.

73. The Klein bottle *does not* have both an “inside” and an “outside.” It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

## Section 11.7 Cylindrical and Spherical Coordinates

- 1.
- $(-7, 0, 5)$
- , cylindrical

$$x = r \cos \theta = -7 \cos 0 = -7$$

$$y = r \sin \theta = -7 \sin 0 = 0$$

$$z = 5$$

$$(-7, 0, 5), \text{rectangular}$$

- 2.
- $(2, -\pi, -4)$
- , cylindrical

$$x = r \cos \theta = 2 \cos(-\pi) = -2$$

$$y = r \sin \theta = 2 \sin(-\pi) = 0$$

$$z = -4$$

$$(-2, 0, -4), \text{rectangular}$$

- 3.
- $\left(3, \frac{\pi}{4}, 1\right)$
- , cylindrical

$$x = 3 \cos \frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$y = 3 \sin \frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$z = 1$$

$$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 1\right), \text{rectangular}$$

- 4.
- $\left(6, -\frac{\pi}{4}, 2\right)$
- , cylindrical

$$x = 6 \cos\left(-\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$y = 6 \sin\left(-\frac{\pi}{4}\right) = -3\sqrt{2}$$

$$z = 2$$

$$(3\sqrt{2}, -3\sqrt{2}, 2), \text{rectangular}$$

- 5.
- $\left(4, \frac{7\pi}{6}, 3\right)$
- , cylindrical

$$x = 4 \cos \frac{7\pi}{6} = -2\sqrt{3}$$

$$y = 4 \sin \frac{7\pi}{6} = -2$$

$$z = 3$$

$$(-2\sqrt{3}, -2, 3), \text{rectangular}$$

- 6.
- $\left(-0.5, \frac{4\pi}{3}, 8\right)$
- , cylindrical

$$x = -\frac{1}{2} \cos \frac{4\pi}{3} = \frac{1}{4}$$

$$y = -\frac{1}{2} \sin \frac{4\pi}{3} = \frac{\sqrt{3}}{4}$$

$$z = 8$$

$$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, 8\right), \text{rectangular}$$

- 7.
- $(0, 5, 1)$
- , rectangular

$$r = \sqrt{(0)^2 + (5)^2} = 5$$

$$\theta = \arctan \frac{5}{0} = \frac{\pi}{2}$$

$$z = 1$$

$$\left(5, \frac{\pi}{2}, 1\right), \text{cylindrical}$$

- 8.
- $(2\sqrt{2}, -2\sqrt{2}, 4)$
- , rectangular

$$r = \sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2} = 4$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = -4$$

$$\left(4, -\frac{\pi}{4}, 4\right), \text{cylindrical}$$

- 9.
- $(2, -2, -4)$
- , rectangular

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = -4$$

$$\left(2\sqrt{2}, -\frac{\pi}{4}, -4\right), \text{cylindrical}$$

- 10.
- $(3, -3, 7)$
- , rectangular

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = 7$$

$$\left(3\sqrt{2}, -\frac{\pi}{4}, 7\right), \text{cylindrical}$$

11.  $(1, \sqrt{3}, 4)$ , rectangular

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \arctan \sqrt{3} = \frac{\pi}{3}$$

$$z = 4$$

$$\left(2, \frac{\pi}{3}, 4\right), \text{cylindrical}$$

12.  $(2\sqrt{3}, -2, 6)$ , rectangular

$$r = \sqrt{12 + 4} = 4$$

$$\theta = \arctan\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$z = 6$$

$$\left(4, -\frac{\pi}{6}, 6\right), \text{cylindrical}$$

13.  $z = 4$  is the equation in cylindrical coordinates.  
(plane)

14.  $x = 9$ , rectangular equation

$$r \cos \theta = 9$$

$$r = 9 \sec \theta, \text{cylindrical equation}$$

15.  $x^2 + y^2 + z^2 = 17$ , rectangular equation

$$r^2 + z^2 = 17, \text{cylindrical equation}$$

16.  $z = x^2 + y^2 - 11$ , rectangular equation

$$z = r^2 - 11, \text{cylindrical equation}$$

17.  $y = x^2$ , rectangular equation

$$r \sin \theta = (r \cos \theta)^2$$

$$\sin \theta = r \cos^2 \theta$$

$$r = \sec \theta \cdot \tan \theta, \text{cylindrical equation}$$

18.  $x^2 + y^2 = 8x$ , rectangular equation

$$r^2 = 8r \cos \theta$$

$$r = 8 \cos \theta, \text{cylindrical equation}$$

19.  $y^2 = 10 - z^2$ , rectangular equation

$$(r \sin \theta)^2 = 10 - z^2$$

$$r^2 \sin^2 \theta + z^2 = 10, \text{cylindrical equation}$$

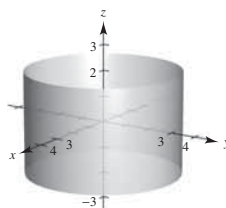
20.  $x^2 + y^2 + z^2 - 3z = 0$ , rectangular equation

$$r^2 + z^2 - 3z = 0, \text{cylindrical equation}$$

21.  $r = 3$

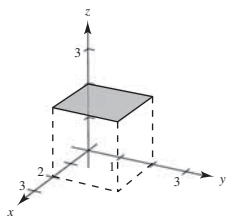
$$\sqrt{x^2 + y^2} = 3$$

$$x^2 + y^2 = 9$$



22.  $z = 2$

Same



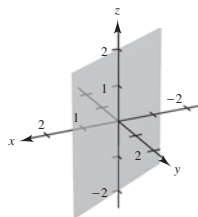
23.  $\theta = \frac{\pi}{6}$

$$\tan \frac{\pi}{6} = \frac{y}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{y}{x}$$

$$x = \sqrt{3}y$$

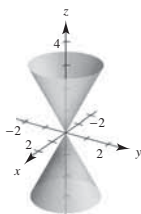
$$x - \sqrt{3}y = 0$$



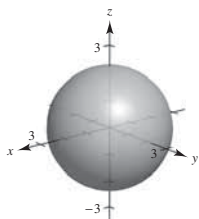
24.  $r = \frac{z}{2}$

$$\sqrt{x^2 + y^2} = \frac{z}{2}$$

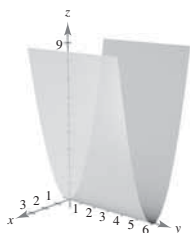
$$x^2 + y^2 - \frac{z^2}{4} = 0$$



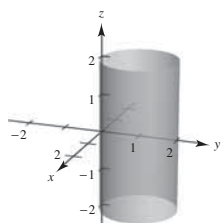
25.  $r^2 + z^2 = 5$   
 $x^2 + y^2 + z^2 = 5$



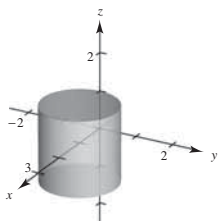
26.  $z = r^2 \cos^2 \theta$   
 $z = x^2$



27.  $r = 2 \sin \theta$   
 $r^2 = 2r \sin \theta$   
 $x^2 + y^2 = 2y$   
 $x^2 + y^2 - 2y = 0$   
 $x^2 + (y - 1)^2 = 1$



28.  $r = 2 \cos \theta$   
 $r^2 = 2r \cos \theta$   
 $x^2 + y^2 = 2x$   
 $x^2 + y^2 - 2x = 0$   
 $(x - 1)^2 + y^2 = 1$



29.  $(4, 0, 0)$ , rectangular  
 $\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$   
 $\tan \theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$   
 $\phi = \arccos 0 = \frac{\pi}{2}$   
 $\left(4, 0, \frac{\pi}{2}\right)$ , spherical

30.  $(-4, 0, 0)$ , rectangular  
 $\rho = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$   
 $\tan \theta = \frac{y}{x} = 0 \Rightarrow \theta = 0$   
 $\theta = \arccos\left(\frac{z}{\rho}\right) = \arccos(0) = \frac{\pi}{2}$   
 $\left(4, 0, \frac{\pi}{2}\right)$ , spherical

31.  $(-2, 2\sqrt{3}, 4)$ , rectangular  
 $\rho = \sqrt{(-2)^2 + (2\sqrt{3})^2 + 4^2} = 4\sqrt{2}$   
 $\tan \theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$   
 $\theta = \frac{2\pi}{3}$   
 $\phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$   
 $\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$ , spherical

32.  $(2, 2, 4\sqrt{2})$ , rectangular  
 $\rho = \sqrt{2^2 + 2^2 + (4\sqrt{2})^2} = 2\sqrt{10}$   
 $\tan \theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$   
 $\phi = \arccos \frac{2}{\sqrt{5}}$   
 $\left(2\sqrt{10}, \frac{\pi}{4}, \arccos \frac{2}{\sqrt{5}}\right)$ , spherical

33.  $(\sqrt{3}, 1, 2\sqrt{3})$ , rectangular  
 $\rho = \sqrt{3 + 1 + 12} = 4$   
 $\tan \theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$   
 $\theta = \frac{\pi}{6}$   
 $\phi = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$   
 $\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right)$ , spherical

34.  $(-1, 2, 1)$ , rectangular

$$\rho = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\tan \theta = \frac{y}{x} = -2 \Rightarrow \theta = \arctan(-2) + \pi$$

$$\phi = \arccos\left(\frac{1}{\sqrt{6}}\right)$$

$$\left(\sqrt{6}, \arctan(-2) + \pi, \arccos \frac{1}{\sqrt{6}}\right), \text{spherical}$$

35.  $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$ , spherical

$$x = 4 \sin \frac{\pi}{4} \cos \frac{\pi}{6} = \sqrt{6}$$

$$y = 4 \sin \frac{\pi}{4} \sin \frac{\pi}{6} = \sqrt{2}$$

$$z = 4 \cos \frac{\pi}{4} = 2\sqrt{2}$$

$$(\sqrt{6}, \sqrt{2}, 2\sqrt{2}), \text{rectangular}$$

36.  $\left(12, \frac{3\pi}{4}, \frac{\pi}{9}\right)$ , spherical

$$x = 12 \sin \frac{\pi}{9} \cos \frac{3\pi}{4} \approx -2.902$$

$$y = 12 \sin \frac{\pi}{9} \sin \frac{3\pi}{4} \approx 2.902$$

$$z = 12 \cos \frac{\pi}{9} \approx 11.276$$

$$(-2.902, 2.902, 11.276), \text{rectangular}$$

37.  $\left(12, -\frac{\pi}{4}, 0\right)$ , spherical

$$x = 12 \sin 0 \cos\left(-\frac{\pi}{4}\right) = 0$$

$$y = 12 \sin 0 \sin\left(-\frac{\pi}{4}\right) = 0$$

$$z = 12 \cos 0 = 12$$

$$(0, 0, 12), \text{rectangular}$$

38.  $\left(9, \frac{\pi}{4}, \pi\right)$ , spherical

$$x = 9 \sin \pi \cos \frac{\pi}{4} = 0$$

$$y = 9 \sin \pi \sin \frac{\pi}{4} = 0$$

$$z = 9 \cos \pi = -9$$

$$(0, 0, -9), \text{rectangular}$$

39.  $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$$

$$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$$

$$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$$

$$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right), \text{rectangular}$$

40.  $\left(6, \pi, \frac{\pi}{2}\right)$ , spherical

$$x = 6 \sin \frac{\pi}{2} \cos \pi = -6$$

$$y = 6 \sin \frac{\pi}{2} \sin \pi = 0$$

$$z = 6 \cos \frac{\pi}{2} = 0$$

$$(-6, 0, 0), \text{rectangular}$$

41.  $y = 2$ , rectangular equation

$$\rho \sin \phi \sin \theta = 2$$

$$\rho = 2 \csc \phi \csc \theta, \text{spherical equation}$$

42.  $z = 6$ , rectangular equation

$$\rho \cos \phi = 6$$

$$\rho = 6 \sec \phi, \text{spherical equation}$$

43.  $x^2 + y^2 + z^2 = 49$ , rectangular equation

$$\rho^2 = 49$$

$$\rho = 7, \text{spherical equation}$$

44.  $x^2 + y^2 - 3z^2 = 0$ , rectangular equation

$$x^2 + y^2 + z^2 = 4z^2$$

$$\rho^2 = 4\rho^2 \cos^2 \phi$$

$$1 = 4 \cos^2 \phi$$

$$\cos \phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}, (\text{cone}) \text{spherical equation}$$

- 45.
- $x^2 + y^2 = 16$
- , rectangular equation

$$\rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \sin^2 \phi \cos^2 \theta = 16$$

$$\rho^2 \sin^2 \phi (\sin^2 \theta + \cos^2 \theta) = 16$$

$$\rho^2 \sin^2 \phi = 16$$

$$\rho \sin \phi = 4$$

$$\rho = 4 \csc \phi, \text{ spherical equation}$$

- 46.
- $x = 13$
- , rectangular equation

$$\rho \sin \phi \cos \theta = 13$$

$$\rho = 13 \csc \phi \sec \theta, \text{ spherical equation}$$

- 47.
- $x^2 + y^2 = 2z^2$
- , rectangular equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi [\cos^2 \theta + \sin^2 \theta] = 2\rho^2 \cos^2 \phi$$

$$\rho^2 \sin^2 \phi = 2\rho^2 \cos^2 \phi$$

$$\frac{\sin^2 \phi}{\cos^2 \phi} = 2$$

$$\tan^2 \phi = 2$$

$$\tan \phi = \pm\sqrt{2}, \text{ spherical equation}$$

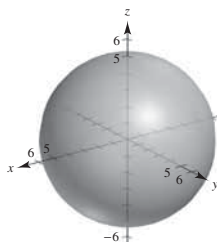
- 48.
- $x^2 + y^2 + z^2 - 9z = 0$
- , rectangular equation

$$\rho^2 - 9\rho \cos \phi = 0$$

$$\rho = 9 \cos \phi, \text{ spherical equation}$$

- 49.
- $\rho = 5$

$$x^2 + y^2 + z^2 = 25$$

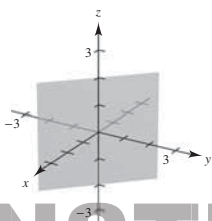


- 50.
- $\theta = \frac{3\pi}{4}$

$$\tan \theta = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$x + y = 0$$



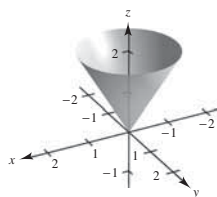
- 51.
- $\phi = \frac{\pi}{6}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$$

$$3x^2 + 3y^2 - z^2 = 0, z \geq 0$$



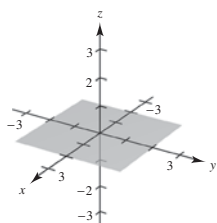
52.  $\phi = \frac{\pi}{2}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = 0$$

xy-plane

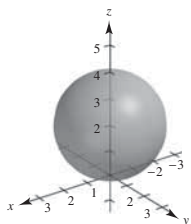


53.  $\rho = 4 \cos \phi$

$$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 4z = 0$$

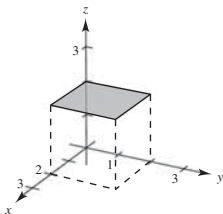
$$x^2 + y^2 + (z - 2)^2 = 4, z \geq 0$$



54.  $\rho = 2 \sec \phi$

$$\rho \cos \phi = 2$$

$$z = 2$$

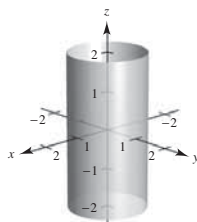


55.  $\rho = \csc \phi$

$$\rho \sin \phi = 1$$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$

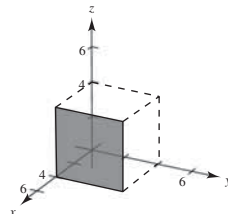


56.  $\rho = 4 \csc \phi \sec \theta$

$$= \frac{4}{\sin \phi \cos \theta}$$

$$\rho \sin \phi \cos \theta = 4$$

$$x = 4$$



57.  $\left(4, \frac{\pi}{4}, 0\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 0^2} = 4$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$$\left(4, \frac{\pi}{4}, \frac{\pi}{2}\right)$$
, spherical

58.  $\left(3, -\frac{\pi}{4}, 0\right)$ , cylindrical

$$\rho = \sqrt{3^2 + 0^2} = 3$$

$$\theta = -\frac{\pi}{4}$$

$$\phi = \arccos\left(\frac{0}{9}\right) = \frac{\pi}{2}$$

$$\left(3, -\frac{\pi}{4}, \frac{\pi}{2}\right)$$
, spherical

59.  $\left(4, \frac{\pi}{2}, 4\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos\left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$$
, spherical

60.  $\left(2, \frac{2\pi}{3}, -2\right)$ , cylindrical

$$\rho = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

$$\left(2\sqrt{2}, \frac{2\pi}{3}, \frac{3\pi}{4}\right), \text{ spherical}$$

61.  $\left(4, -\frac{\pi}{6}, -\frac{\pi}{6}, 6\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos\frac{3}{\sqrt{13}}$$

$$\left(2\sqrt{13}, -\frac{\pi}{6}, \arccos\frac{3}{\sqrt{13}}\right), \text{ spherical}$$

62.  $\left(-4, \frac{\pi}{3}, 4\right)$ , cylindrical

$$\rho = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{3}$$

$$\phi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right), \text{ spherical}$$

63.  $(12, \pi, 5)$ , cylindrical

$$\rho = \sqrt{12^2 + 5^2} = 13$$

$$\theta = \pi$$

$$\phi = \arccos\frac{5}{13}$$

$$\left(13, \pi, \arccos\frac{5}{13}\right), \text{ spherical}$$

64.  $\left(4, \frac{\pi}{2}, 3\right)$ , cylindrical

$$\rho = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos\frac{3}{5}$$

$$\left(5, \frac{\pi}{2}, \arccos\frac{3}{5}\right), \text{ spherical}$$

65.  $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$ , spherical

$$r = 10 \sin \frac{\pi}{2} = 10$$

$$\theta = \frac{\pi}{6}$$

$$z = 10 \cos \frac{\pi}{2} = 0$$

$$\left(10, \frac{\pi}{6}, 0\right), \text{ cylindrical}$$

66.  $\left(4, \frac{\pi}{18}, \frac{\pi}{2}\right)$ , spherical

$$r = 4 \sin \frac{\pi}{2} = 4$$

$$\theta = \frac{\pi}{18}$$

$$z = 4 \cos \frac{\pi}{2} = 0$$

$$\left(4, \frac{\pi}{18}, 0\right), \text{ cylindrical}$$

67.  $\left(36, \pi, \frac{\pi}{2}\right)$ , spherical

$$r = \rho \sin \phi = 36 \sin \frac{\pi}{2} = 36$$

$$\theta = \pi$$

$$z = \rho \cos \phi = 36 \cos \frac{\pi}{2} = 0$$

$$(36, \pi, 0), \text{ cylindrical}$$

68.  $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$ , spherical

$$r = \rho \sin \phi = 18 \sin \frac{\pi}{3} = 9$$

$$\theta = \frac{\pi}{3}$$

$$z = \rho \cos \phi = 18 \cos \frac{\pi}{3} = 9\sqrt{3}$$

$$\left(9, \frac{\pi}{3}, 9\sqrt{3}\right), \text{ cylindrical}$$

69.  $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$ , spherical

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$$

$$\theta = -\frac{\pi}{6}$$

$$z = 6 \cos \frac{\pi}{3} = 3$$

$$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right), \text{ cylindrical}$$



70.  $\left(5, -\frac{5\pi}{6}, \pi\right)$ , spherical

$$r = 5 \sin \pi = 0$$

$$\theta = -\frac{5\pi}{6}$$

$$z = 5 \cos \pi = -5$$

$$\left(0, -\frac{5\pi}{6}, -5\right)$$
, cylindrical

71.  $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$ , spherical

$$r = 8 \sin \frac{\pi}{6} = 4$$

$$\theta = \frac{7\pi}{6}$$

$$z = 8 \cos \frac{\pi}{6} = \frac{8\sqrt{3}}{2}$$

$$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right)$$
, cylindrical

72.  $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$r = 7 \sin \frac{3\pi}{4} = \frac{7\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}$$

$$z = 7 \cos \frac{3\pi}{4} = -\frac{7\sqrt{2}}{2}$$

$$\left(\frac{7\sqrt{2}}{2}, \frac{\pi}{4}, -\frac{7\sqrt{2}}{2}\right)$$
, cylindrical

Rectangular

Cylindrical

Spherical

73. (4, 6, 3)

(7.211, 0.983, 3)

(7.810, 0.983, 1.177)

74. (6, -2, -3)

(6.325, -0.322, -3)

(7.000, -0.322, 2.014)

75. (4.698, 1.710, 8)

$\left(5, \frac{\pi}{9}, 8\right)$

(9.434, 0.349, 0.559)

76. (7.317, -6.816, 6)

(10, -0.75, 6)

(11.662, -0.750, 1.030)

77. (-7.071, 12.247, 14.142)

(14.142, 2.094, 14.142)

$\left(20, \frac{2\pi}{3}, \frac{\pi}{4}\right)$

78. (6.115, 1.561, 4.052)

(6.311, 0.25, 5.052)

(7.5, 0.25, 1)

79. (3, -2, 2)

(3.606, -0.588, 2)

(4.123, -0.588, 1.064)

80.  $(3\sqrt{2}, 3\sqrt{2}, -3)$

(6, 0.785, -3)

(6.708, 0.785, 2.034)

81.  $\left(\frac{5}{2}, \frac{4}{3}, -\frac{3}{2}\right)$

(2.833, 0.490, -1.5)

(3.206, 0.490, 2.058)

82. (0, -5, 4)

(5, -1.571, 4)

(6.403, -1.571, 0.896)

83. (-3.536, 3.536, -5)

$\left(5, \frac{3\pi}{4}, -5\right)$

(7.071, 2.356, 2.356)

84. (-1.732, 1, 3)

$\left(-2, \frac{11\pi}{6}, 3\right)$

(3.606, 2.618, 0.588)

[Note: use the cylindrical coordinate  $\left(2, \frac{5\pi}{6}, 3\right)$ ]

Rectangular

85.  $(2.804, -2.095, 6)$

[**Note:** Use the cylindrical coordinates  $(3.5, 5.642, 6)$ ]

86.  $(2.207, 7.949, -4)$

87.  $(-1.837, 1.837, 1.5)$

88.  $(0, 0, -8)$

89.  $r = 5$

Cylinder

Matches graph (d)

90.  $\theta = \frac{\pi}{4}$

Plane

Matches graph (e)

91.  $\rho = 5$

Sphere

Matches graph (c)

92.  $\phi = \frac{\pi}{4}$

Cone

Matches graph (a)

93.  $r^2 = z, x^2 + y^2 = z$

Paraboloid

Matches graph (f)

94.  $\rho = 4 \sec \phi, z = \rho \cos \phi = 4$

Plane

Matches graph (b)

95. Rectangular to cylindrical:  $r^2 = x^2 + y^2$

$$\tan \theta = \frac{y}{x}$$

$$z = z$$

Cylindrical to rectangular:  $x = r \cos \theta$ 

$$y = r \sin \theta$$

$$z = z$$

96.  $\theta = c$  is a half-plane because of the restriction  $r \geq 0$ .

Cylindrical

$(-3.5, 2.5, 6)$

$(8.25, 1.3, -4)$

$(2.598, 2.356, 1.5)$

$(0, -0.524, -8)$

Spherical

$(6.946, 5.642, 0.528)$

$(9.169, 1.3, 2.022)$

$\left(3, \frac{3\pi}{4}, \frac{\pi}{3}\right)$

$\left(8, -\frac{\pi}{6}, \pi\right)$

97. Rectangular to spherical:  $\rho^2 = x^2 + y^2 + z^2$

$$\tan \theta = \frac{y}{x}$$

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Spherical to rectangular:  $x = \rho \sin \phi \cos \theta$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

98. (a)  $r = a$  Cylinder with  $z$ -axis symmetry

$\theta = b$  Plane perpendicular to  $xy$ -plane

$z = c$  Plane parallel to  $xy$ -plane

(b)  $\rho = a$  Sphere

$\theta = b$  Vertical half-plane

$\phi = c$  Half-cone

99.  $x^2 + y^2 + z^2 = 25$

(a)  $r^2 + z^2 = 25$

(b)  $\rho^2 = 25 \Rightarrow \rho = 5$

100.  $4(x^2 + y^2) = z^2$

(a)  $4r^2 = z^2 \Rightarrow 2r = z$

(b)  $4(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) = \rho^2 \cos^2 \phi$

$$4 \sin^2 \phi = \cos^2 \phi,$$

$$\tan^2 \phi = \frac{1}{4},$$

$$\tan \phi = \frac{1}{2} \Rightarrow \phi = \arctan \frac{1}{2}$$

101.  $x^2 + y^2 + z^2 - 2z = 0$

(a)  $r^2 + z^2 - 2z = 0 \Rightarrow r^2 + (z - 1)^2 = 1$

(b)  $\rho^2 - 2\rho \cos \phi = 0$

$$\rho(\rho - 2 \cos \phi) = 0$$

$$\rho = 2 \cos \phi$$

102.  $x^2 + y^2 = z$

(a)  $r^2 = z$

(b)  $\rho^2 \sin^2 \phi = \rho \cos \phi$

$$\rho \sin^2 \phi = \cos \phi$$

$$\rho = \frac{\cos \phi}{\sin^2 \phi}$$

$$\rho = \csc \phi \cot \phi$$

103.  $x^2 + y^2 = 4y$

(a)  $r^2 = 4r \sin \theta, r = 4 \sin \theta$

(b)  $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta$

$$\rho \sin \phi (\rho \sin \phi - 4 \sin \theta) = 0$$

$$\rho = \frac{4 \sin \theta}{\sin \phi}$$

$$\rho = 4 \sin \theta \csc \phi$$

104.  $x^2 + y^2 = 36$

(a)  $r^2 = 36 \Rightarrow r = 6$

(b)  $\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 36$

$$\rho^2 \sin^2 \phi = 36$$

$$\rho = 6 \csc \phi$$

105.  $x^2 - y^2 = 9$

(a)  $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 9$

$$r^2 = \frac{9}{\cos^2 \theta - \sin^2 \theta}$$

(b)  $\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 9$

$$\rho^2 \sin^2 \phi = \frac{9}{\cos^2 \theta - \sin^2 \theta}$$

$$\rho^2 = \frac{9 \csc^2 \phi}{\cos^2 \theta - \sin^2 \theta}$$

106.  $y = 4$

(a)  $r \sin \theta = 4 \Rightarrow r = 4 \csc \theta$

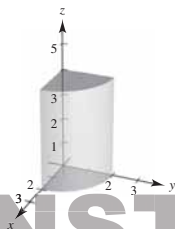
(b)  $\rho \sin \phi \sin \theta = 4,$

$$\rho = 4 \csc \phi \csc \theta$$

107.  $0 \leq \theta \leq \frac{\pi}{2}$

$$0 \leq r \leq 2$$

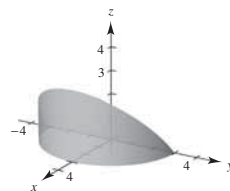
$$0 \leq z \leq 4$$



108.  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$0 \leq r \leq 3$$

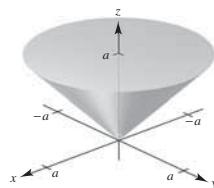
$$0 \leq z \leq r \cos \theta$$



109.  $0 \leq \theta \leq 2\pi$

$$0 \leq r \leq a$$

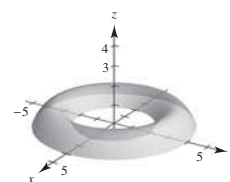
$$r \leq z \leq a$$



110.  $0 \leq \theta \leq 2\pi$

$$2 \leq r \leq 4$$

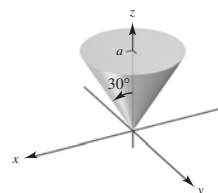
$$z^2 \leq -r^2 + 6r - 8$$



111.  $0 \leq \theta \leq 2\pi$

$$0 \leq \phi \leq \frac{\pi}{6}$$

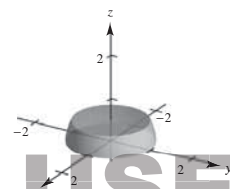
$$0 \leq \rho \leq a \sec \phi$$



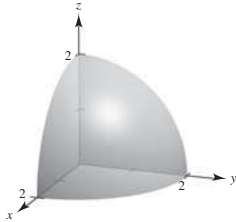
112.  $0 \leq \theta \leq 2\pi$

$$\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$$

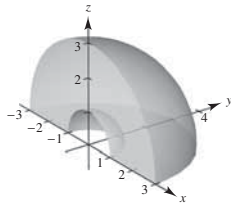
$$0 \leq \rho \leq 1$$



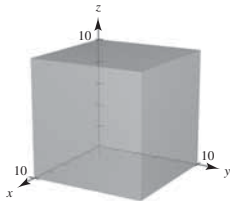
113.  $0 \leq \theta \leq \frac{\pi}{2}$   
 $0 \leq \phi \leq \frac{\pi}{2}$   
 $0 \leq \rho \leq 2$



114.  $0 \leq \theta \leq \pi$   
 $0 \leq \phi \leq \frac{\pi}{2}$   
 $1 \leq \rho \leq 3$



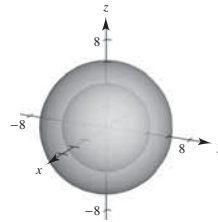
115. Rectangular  
 $0 \leq x \leq 10$   
 $0 \leq y \leq 10$   
 $0 \leq z \leq 10$



116. Cylindrical:  
 $0.75 \leq r \leq 1.25$   
 $0 \leq z \leq 8$



117. Spherical  
 $4 \leq \rho \leq 6$

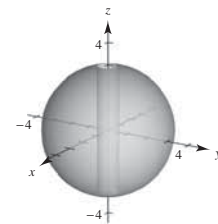


118. Cylindrical

$$\frac{1}{2} \leq r \leq 3$$

$$0 \leq \theta \leq 2\pi$$

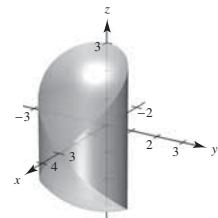
$$-\sqrt{9-r^2} \leq z \leq \sqrt{9-r^2}$$



119. Cylindrical coordinates:

$$r^2 + z^2 \leq 9,$$

$$r \leq 3 \cos \theta, 0 \leq \theta \leq \pi$$

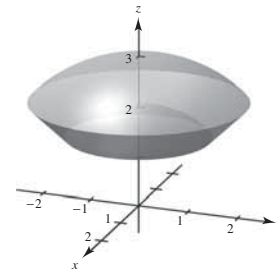


120. Spherical coordinates:

$$\rho \geq 2$$

$$\rho \leq 3$$

$$0 \leq \phi \leq \frac{\pi}{4}$$



121. False.  $r = z \Rightarrow x^2 + y^2 = z^2$  is a cone.

122. True. They both represent spheres of radius 2 centered at the origin.

123. False.  $(r, \theta, z) = (0, 0, 1)$  and  $(r, \theta, z) = (0, \pi, 1)$  represent the same point  $(x, y, z) = (0, 0, 1)$ .

124. True (except for the origin).

125.  $z = \sin \theta, r = 1$

$$z = \sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

The curve of intersection is the ellipse formed by the intersection of the plane  $z = y$  and the cylinder  $r = 1$ .

126.  $\rho = 2 \sec \phi \Rightarrow \rho \cos \phi = 2 \Rightarrow z = 2$  plane  
 $\rho = 4$  sphere

The intersection of the plane and the sphere is a circle.

## Review Exercises for Chapter 11

1.  $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle 4 - 1, 1 - 2 \rangle = \langle 3, -1 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 5 - 1, 4 - 2 \rangle = \langle 4, 2 \rangle$

(b)  $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$

(c)  $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

(d)  $2\mathbf{u} + \mathbf{v} = 2\langle 3, -1 \rangle + \langle 4, 2 \rangle = \langle 10, 0 \rangle$

2.  $P = (-2, -1), Q = (5, -1), R = (2, 4)$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle 7, 0 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 5 \rangle$

(b)  $\mathbf{u} = 7\mathbf{i}$

(c)  $\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$

(d)  $2\mathbf{u} + \mathbf{v} = 2\langle 7, 0 \rangle + \langle 4, 5 \rangle = \langle 18, 5 \rangle$

3.  $\mathbf{v} = \|\mathbf{v}\|(\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$   
 $= 8(\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j})$   
 $= 8\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = 4\mathbf{i} + 4\sqrt{3}\mathbf{j} = \langle 4, 4\sqrt{3} \rangle$

4.  $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j}$   
 $= \frac{1}{2} \cos 225^\circ \mathbf{i} + \frac{1}{2} \sin 225^\circ \mathbf{j}$   
 $= -\frac{\sqrt{2}}{4}\mathbf{i} - \frac{\sqrt{2}}{4}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4} \right\rangle$

5.  $z = 0, y = 4, x = -5: (-5, 4, 0)$

6.  $x = z = 0, y = -7: (0, -7, 0)$

7. Looking down from the positive  $x$ -axis towards the  $yz$ -plane, the point is either in the first quadrant ( $y > 0, z > 0$ ) or in the third quadrant ( $y < 0, z < 0$ ). The  $x$ -coordinate can be any number.

8. Looking towards the  $xy$ -plane from the positive  $z$ -axis. The point is either in the second quadrant ( $x < 0, y > 0$ ) or in the fourth quadrant ( $x > 0, y < 0$ ). The  $z$ -coordinate can be any number.

9.  $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

10. Center:  $\left(\frac{0+4}{2}, \frac{0+6}{2}, \frac{4+0}{2}\right) = (2, 3, 2)$

Radius:

$$\sqrt{(2-0)^2 + (3-0)^2 + (2-4)^2} = \sqrt{4+9+4} = \sqrt{17}$$

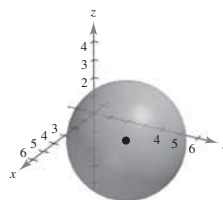
$$(x-2)^2 + (y-3)^2 + (z-2)^2 = 17$$

11.  $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

$$(x-2)^2 + (y-3)^2 + z^2 = 9$$

Center:  $(2, 3, 0)$

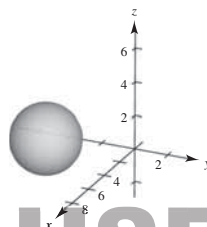
Radius: 3



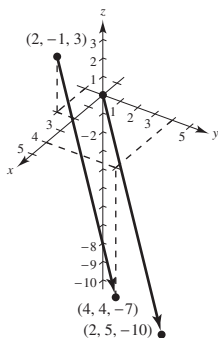
12.  $(x^2 - 10x + 25) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = -34 + 25 + 9 + 4$   
 $(x-5)^2 + (y+3)^2 + (z-2)^2 = 4$

Center:  $(5, -3, 2)$

Radius: 2



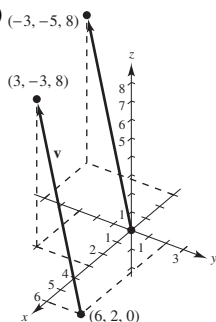
13. (a), (d)



(b)  $\mathbf{v} = \langle 4 - 2, 4 - (-1), -7 - 3 \rangle = \langle 2, 5, -10 \rangle$

(c)  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$

14. (a), (d)



(b)  $\mathbf{v} = \langle 3 - 6, -3 - 2, 8 - 0 \rangle = \langle -3, -5, 8 \rangle$

(c)  $\mathbf{v} = -3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$

15.  $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$

Because  $-2\mathbf{w} = \mathbf{v}$ , the points lie in a straight line.

16.  $\mathbf{v} = \langle 8 - 5, -5 + 4, 5 - 7 \rangle = \langle 3, -1, -2 \rangle$

$\mathbf{w} = \langle 11 - 5, 6 + 4, 3 - 7 \rangle = \langle 6, 10, -4 \rangle$

Because  $\mathbf{v}$  and  $\mathbf{w}$  are not parallel, the points do not lie in a straight line.

17. Unit vector:  $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$

18.  $8 \frac{\langle 6, -3, 2 \rangle}{\sqrt{49}} = \frac{8}{7} \langle 6, -3, 2 \rangle = \left\langle \frac{48}{7}, -\frac{24}{7}, \frac{16}{7} \right\rangle$

19.  $P = \langle 5, 0, 0 \rangle, Q = \langle 4, 4, 0 \rangle, R = \langle 2, 0, 6 \rangle$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle$

(b)  $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c)  $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

20.  $P = \langle 2, -1, 3 \rangle, Q = \langle 0, 5, 1 \rangle, R = \langle 5, 5, 0 \rangle$

(a)  $\mathbf{u} = \overrightarrow{PQ} = \langle -2, 6, -2 \rangle$

$\mathbf{v} = \overrightarrow{PR} = \langle 3, 6, -3 \rangle$

(b)  $\mathbf{u} \cdot \mathbf{v} = (-2)(3) + (6)(6) + (-2)(-3) = 36$

(c)  $\mathbf{v} \cdot \mathbf{v} = 9 + 36 + 9 = 54$

21.  $\mathbf{u} = \langle 7, -2, 3 \rangle, \mathbf{v} = \langle -1, 4, 5 \rangle$

Because  $\mathbf{u} \cdot \mathbf{v} = 0$ , the vectors are orthogonal.

22.  $\mathbf{u} = \langle -4, 3, -6 \rangle, \mathbf{v} = \langle 16, -12, 24 \rangle$

Because  $\mathbf{v} = -4\mathbf{u}$ , the vectors are parallel.

23.  $\mathbf{u} = 5 \left( \cos \frac{3\pi}{4} \mathbf{i} + \sin \frac{3\pi}{4} \mathbf{j} \right) = \frac{5\sqrt{2}}{2} [-\mathbf{i} + \mathbf{j}]$

$\mathbf{v} = 2 \left( \cos \frac{2\pi}{3} \mathbf{i} + \sin \frac{2\pi}{3} \mathbf{j} \right) = -\mathbf{i} + \sqrt{3} \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2} (1 + \sqrt{3})$

$\|\mathbf{u}\| = 5 \quad \|\mathbf{v}\| = 2$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(5\sqrt{2}/2)(1 + \sqrt{3})}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$

$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15^\circ \left[ \text{or, } \frac{3\pi}{4} \cdot \frac{2\pi}{3} = \frac{\pi}{12} \text{ or } 15^\circ \right]$

24.  $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{v} = -\mathbf{i} + 5\mathbf{j}$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{|-6 + 10|}{\sqrt{49} \sqrt{26}} = \frac{4}{7\sqrt{26}}$

$\theta \approx 83.6^\circ$

25.  $\mathbf{u} = \langle 10, -5, 15 \rangle, \mathbf{v} = \langle -2, 1, -3 \rangle$

 $\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$  is parallel to  $\mathbf{v}$  and in the opposite direction.

$\theta = \pi$

26.  $\mathbf{u} = \langle 1, 0, -3 \rangle$

$\mathbf{v} = \langle 2, -2, 1 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -1$

$\|\mathbf{u}\| = \sqrt{10}$

$\|\mathbf{v}\| = 3$

$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1}{3\sqrt{10}}$

$\theta \approx 83.9^\circ$

27. There are many correct answers.

For example:  $\mathbf{v} = \pm\langle 6, -5, 0 \rangle$ .

$$\begin{aligned} 28. W &= \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos \theta = (75)(8) \cos 30^\circ \\ &= 300\sqrt{3} \text{ ft-lb} \end{aligned}$$

In Exercises 29–38,  $\mathbf{u} = \langle 3, -2, 1 \rangle$ ,  $\mathbf{v} = \langle 2, -4, -3 \rangle$ ,  
 $\mathbf{w} = \langle -1, 2, 2 \rangle$ .

$$\begin{aligned} 29. \mathbf{u} \cdot \mathbf{u} &= 3(3) + (-2)(-2) + (1)(1) \\ &= 14 = (\sqrt{14})^2 = \|\mathbf{u}\|^2 \end{aligned}$$

$$\begin{aligned} 30. \cos \theta &= \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{11}{\sqrt{14}\sqrt{29}} \\ \theta &= \arccos\left(\frac{11}{\sqrt{14}\sqrt{29}}\right) \approx 56.9^\circ \end{aligned}$$

$$\begin{aligned} 31. \text{proj}_{\mathbf{u}} \mathbf{w} &= \left( \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u} \\ &= -\frac{5}{14} \langle 3, -2, 1 \rangle \\ &= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle \\ &= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle \end{aligned}$$

$$32. \text{Work} = \mathbf{u} \cdot \mathbf{w} = -3 - 4 + 2 = -5$$

$$\begin{aligned} 37. \text{Area parallelogram} &= \|\mathbf{u} \times \mathbf{v}\| = \|\langle 10, 11, -8 \rangle\| = \sqrt{10^2 + 11^2 + (-8)^2} \quad (\text{See Exercises 34, 36}) \\ &= \sqrt{285} \end{aligned}$$

$$38. \text{Area triangle} = \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2} \sqrt{(-2)^2 + (-1)^2} = \frac{\sqrt{5}}{2} \quad (\text{See Exercise 33})$$

$$39. \mathbf{F} = c(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c \cos 20^\circ & c \sin 20^\circ \end{vmatrix} = -2c \cos 20^\circ \mathbf{i}$$

$$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c \cos 20^\circ$$

$$c = \frac{100}{\cos 20^\circ}$$

$$\mathbf{F} = \frac{100}{\cos 20^\circ} (\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k}) = 100(\mathbf{j} + \tan 20^\circ \mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20^\circ} = 100 \sec 20^\circ \approx 106.4 \text{ lb}$$

$$33. \mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{vmatrix} = -2\mathbf{i} - \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{5}$$

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j}), \text{ unit vector or } \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$$

$$34. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -10\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$$

$$\text{So, } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

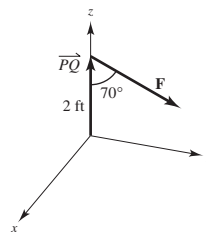
$$35. V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 3, -2, 1 \rangle \cdot \langle -2, -1, 0 \rangle| = |-4| = 4$$

$$\begin{aligned} 36. \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \langle 3, -2, 1 \rangle \times \langle 1, -2, -1 \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} \end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 2 & -4 & -3 \end{vmatrix} = 10\mathbf{i} + 11\mathbf{j} - 8\mathbf{k}$$

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = -6\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

$$(\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$



$$40. V = |\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2(5) = 10$$

$$41. \mathbf{v} = \langle 9 - 3, 11 - 0, 6 - 2 \rangle = \langle 6, 11, 4 \rangle$$

(a) Parametric equations:

$$x = 3 + 6t, y = 11t, z = 2 + 4t$$

$$(b) \text{ Symmetric equations: } \frac{x-3}{6} = \frac{y}{11} = \frac{z-2}{4}$$

$$42. \mathbf{v} = \langle 8 + 1, 10 - 4, 5 - 3 \rangle = \langle 9, 6, 2 \rangle$$

(a) Parametric equations:

$$x = -1 + 9t, y = 4 + 6t, z = 3 + 2t$$

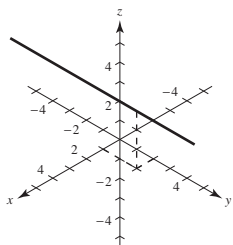
$$(b) \text{ Symmetric equations: } \frac{x+1}{9} = \frac{y-4}{6} = \frac{z-3}{2}$$

$$43. \mathbf{v} = \mathbf{j}$$

$$(a) x = 1, y = 2 + t, z = 3$$

(b) None

(c)

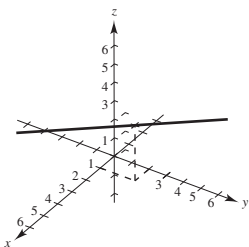


$$44. \text{ Direction numbers: } 1, 1, 1$$

$$(a) x = 1 + t, y = 2 + t, z = 3 + t$$

$$(b) x - 1 = y - 2 = z - 3$$

(c)



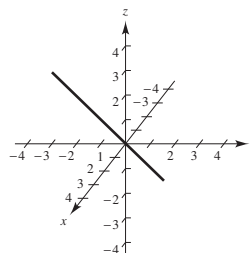
$$45. 3x - 3y - 7z = -4, x - y + 2z = 3$$

Solving simultaneously, you have  $z = 1$ . Substituting  $z = 1$  into the second equation, you have  $y = x - 1$ . Substituting for  $x$  in this equation you obtain two points on the line of intersection,  $(0, -1, 1)$ ,  $(1, 0, 1)$ . The direction vector of the line of intersection is  $\mathbf{v} = \mathbf{i} + \mathbf{j}$ .

$$(a) x = t, y = -1 + t, z = 1$$

$$(b) x = y + 1, z = 1$$

(c)



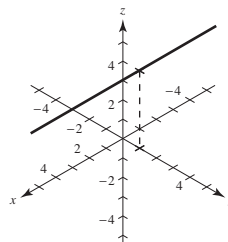
$$46. \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 1 \\ -3 & 1 & 4 \end{vmatrix} = -21\mathbf{i} - 11\mathbf{j} - 13\mathbf{k}$$

Direction numbers: 21, 11, 13

$$(a) x = 21t, y = 1 + 11t, z = 4 + 13t$$

$$(b) \frac{x}{21} = \frac{y-1}{11} = \frac{z-4}{13}$$

(c)



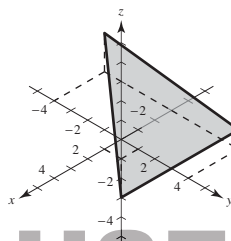
$$47. P = (-3, -4, 2), Q = (-3, 4, 1), R = (1, 1, -2)$$

$$\overrightarrow{PQ} = \langle 0, 8, -1 \rangle, \overrightarrow{PR} = \langle 4, 5, -4 \rangle$$

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = -27\mathbf{i} - 4\mathbf{j} - 32\mathbf{k}$$

$$-27(x + 3) - 4(y + 4) - 32(z - 2) = 0$$

$$27x + 4y + 32z = -33$$

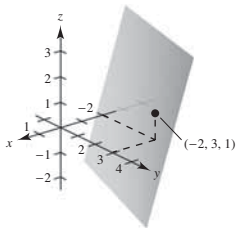




48.  $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$3(x + 2) - 1(y - 3) + 1(z - 1) = 0$$

$$3x - y + z + 8 = 0$$



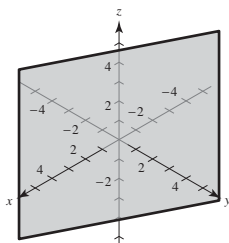
49. The two lines are parallel as they have the same direction numbers,  $-2, 1, 1$ . Therefore, a vector parallel to the plane is  $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ . A point on the first line is  $(1, 0, -1)$  and a point on the second line is  $(-1, 1, 2)$ . The vector  $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$  connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane:  $(x - 1) + 2y = 0$

$$x + 2y = 1$$



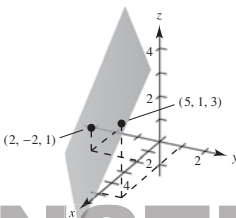
50. Let  $\mathbf{v} = \langle 5 - 2, 1 + 2, 3 - 1 \rangle = \langle 3, 3, 2 \rangle$  be the direction vector for the line through the two points. Let  $\mathbf{n} = \langle 2, 1, -1 \rangle$  be the normal vector to the plane. Then

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \langle -5, 7, -3 \rangle$$

is the normal to the unknown plane.

$$-5(x - 5) + 7(y - 1) - 3(z - 3) = 0$$

$$-5x + 7y - 3z + 27 = 0$$



51.  $Q(1, 0, 2)$  point

$$2x - 3y + 6z = 6$$

A point  $P$  on the plane is  $(3, 0, 0)$ .

$$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$$

$\mathbf{n} = \langle 2, -3, 6 \rangle$  normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

52.  $Q(3, -2, 4)$  point

$$2x - 5y + z = 10$$

A point  $P$  on the plane is  $(5, 0, 0)$ .

$$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$$

$\mathbf{n} = \langle 2, -5, 1 \rangle$  normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

53. The normal vectors to the planes are the same,

$$\mathbf{n} = \langle 5, -3, 1 \rangle.$$

Choose a point in the first plane  $P(0, 0, 2)$ . Choose a point in the second plane,  $Q(0, 0, -3)$ .

$$\overrightarrow{PQ} = \langle 0, 0, -5 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-5|}{\sqrt{35}} = \frac{5}{\sqrt{35}} = \frac{\sqrt{35}}{7}$$

54.  $Q(-5, 1, 3)$  point

$\mathbf{u} = \langle 1, -2, -1 \rangle$  direction vector

$P(1, 3, 5)$  point on line

$$\overrightarrow{PQ} = \langle -6, -2, -2 \rangle$$

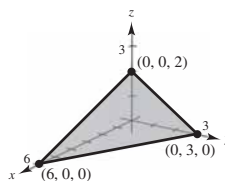
$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \langle -2, -8, 14 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{264}}{\sqrt{6}} = 2\sqrt{11}$$

55.  $x + 2y + 3z = 6$

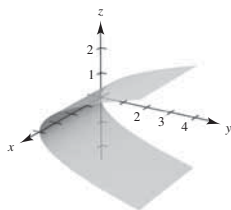
Plane

Intercepts:  $(6, 0, 0), (0, 3, 0), (0, 0, 2)$ ,



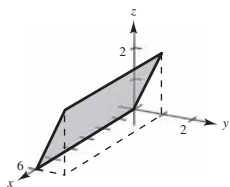
56.  $y = z^2$

Because the  $x$ -coordinate is missing, you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is a parabola in the  $yz$ -coordinate plane.



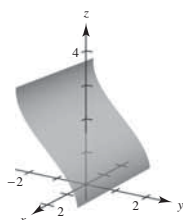
57.  $y = \frac{1}{2}z$

Plane with rulings parallel to the  $x$ -axis.



58.  $y = \cos z$

Because the  $x$ -coordinate is missing, you have a cylindrical surface with rulings parallel to the  $x$ -axis. The generating curve is  $y = \cos z$ .



59.  $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$xz\text{-trace: } \frac{x^2}{16} + z^2 = 1$$

$$yz\text{-trace: } \frac{y^2}{9} + z^2 = 1$$



60.  $16x^2 + 16y^2 - 9z^2 = 0$

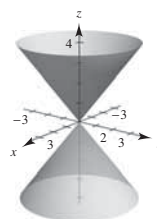
Cone

$xy$ -trace: point  $(0, 0, 0)$

$$xz\text{-trace: } z = \pm \frac{4x}{3}$$

$$yz\text{-trace: } z = \pm \frac{4y}{3}$$

$$z = 4, x^2 + y^2 = 9$$



61.  $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

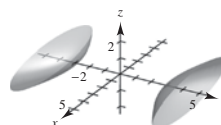
$$\frac{y^2}{9} - \frac{x^2}{16} - z^2 = 1$$

Hyperboloid of two sheets

$$xy\text{-trace: } \frac{y^2}{9} - \frac{x^2}{16} = 1$$

$xz$ -trace: None

$$yz\text{-trace: } \frac{y^2}{9} - z^2 = 1$$



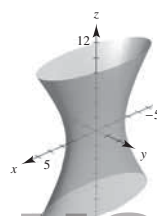
62.  $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{100} = 1$

Hyperboloid of one sheet

$$xy\text{-trace: } \frac{x^2}{25} + \frac{y^2}{4} = 1$$

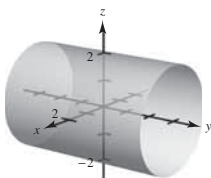
$$xz\text{-trace: } \frac{x^2}{25} - \frac{z^2}{100} = 1$$

$$yz\text{-trace: } \frac{y^2}{4} - \frac{z^2}{100} = 1$$



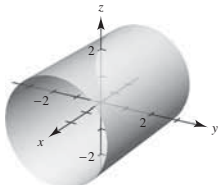
63.  $x^2 + z^2 = 4$ .

Cylinder of radius 2 about y-axis



64.  $y^2 + z^2 = 16$ .

Cylinder of radius 4 about x-axis



65. Let  $y = r(x) = 2\sqrt{x}$  and revolve the curve about the x-axis.

66.  $x^2 + 2y^2 + z^2 = 3y$

$$x^2 + z^2 = 3y - 2y^2$$

Let  $x^2 = 3y - 2y^2$  (Trace in  $xy$ -plane)

Then  $x = \sqrt{3y - 2y^2}$  is a generating curve. Revolve the curve about the y-axis.

67.  $z^2 = 2y$  revolved about y-axis

$$z = \pm\sqrt{2y}$$

$$x^2 + z^2 = [r(y)]^2 = 2y$$

$$x^2 + z^2 = 2y$$

68.  $2x + 3z = 1$  revolved about the x-axis

$$z = \frac{1 - 2x}{3}$$

$$y^2 + z^2 = [r(x)]^2 = \left(\frac{1 - 2x}{3}\right)^2, \text{Cone}$$

69.  $(-2\sqrt{2}, 2\sqrt{2}, 2)$ , rectangular

$$(a) \quad r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4,$$

$$\theta = \arctan(-1) = \frac{3\pi}{4}, z = 2,$$

$$\left(4, \frac{3\pi}{4}, 2\right), \text{cylindrical}$$

$$(b) \quad \rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + (2)^2} = 2\sqrt{5},$$

$$\theta = \frac{3\pi}{4}, \phi = \arccos \frac{2}{2\sqrt{5}} = \arccos \frac{1}{\sqrt{5}},$$

$$\left(2\sqrt{5}, \frac{3\pi}{4}, \arccos \frac{\sqrt{5}}{5}\right), \text{spherical}$$

70.  $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right)$ , rectangular

$$(a) \quad r = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{3}}{2},$$

$$\theta = \arctan \sqrt{3} = \frac{\pi}{3},$$

$$z = \frac{3\sqrt{3}}{2}, \left(\frac{\sqrt{3}}{2}, \frac{\pi}{2}, \frac{3\sqrt{3}}{2}\right), \text{cylindrical}$$

$$(b) \quad \rho = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \frac{\sqrt{30}}{2}, \theta = \frac{\pi}{3},$$

$$\phi = \arccos \frac{3}{\sqrt{10}}, \left(\frac{\sqrt{30}}{2}, \frac{\pi}{3}, \arccos \frac{3}{\sqrt{10}}\right), \text{spherical}$$

71.  $\left(100, -\frac{\pi}{6}, 50\right)$ , cylindrical

$$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos\left(\frac{50}{50\sqrt{5}}\right) = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.4^\circ \text{ or } 1.107$$

$$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4^\circ\right), \text{spherical or } \left(50\sqrt{5}, -\frac{\pi}{6}, 1.1071\right)$$

72.  $\left(81, -\frac{5\pi}{6}, 27\sqrt{3}\right)$ , cylindrical

$$\rho = \sqrt{6561 + 2187} = 54\sqrt{3}$$

$$\theta = -\frac{5\pi}{6}$$

$$\phi = \arccos\left(\frac{27\sqrt{3}}{54\sqrt{3}}\right) = \arccos \frac{1}{2} = \frac{\pi}{3}$$

$$\left(54\sqrt{3}, -\frac{5\pi}{6}, \frac{\pi}{3}\right), \text{spherical}$$

73.  $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4}\right)$ , spherical

$$r^2 = \left(25 \sin\left(\frac{3\pi}{4}\right)\right)^2 \Rightarrow r = \frac{25\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$z = \rho \cos \phi = 25 \cos \frac{3\pi}{4} = -\frac{25\sqrt{2}}{2}$$

$$\left(\frac{25\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2}\right), \text{cylindrical}$$

74.  $\left(12, -\frac{\pi}{2}, \frac{2\pi}{3}\right)$ , spherical

$$r^2 = \left(12 \sin\left(\frac{2\pi}{3}\right)\right)^2 \Rightarrow r = 6\sqrt{3}$$

$$\theta = -\frac{\pi}{2}$$

$$z = \rho \cos \phi = 12 \cos\left(\frac{2\pi}{3}\right) = -6$$

$$\left(6\sqrt{3}, -\frac{\pi}{2}, -6\right), \text{cylindrical}$$

75.  $x^2 - y^2 = 2z$

(a) Cylindrical:

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2z \Rightarrow r^2 \cos 2\theta = 2z$$

(b) Spherical:

$$\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho \cos \phi$$

$$\rho \sin^2 \phi \cos 2\theta - 2 \cos \phi = 0$$

$$\rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$$

76.  $x^2 + y^2 + z^2 = 16$

(a) Cylindrical:  $r^2 + z^2 = 16$

(b) Spherical:  $\rho = 4$

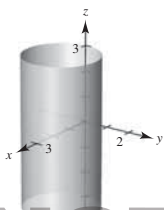
77.  $r = 5 \cos \theta$ , cylindrical equation

$$r^2 = 5r \cos \theta$$

$$x^2 + y^2 = 5x$$

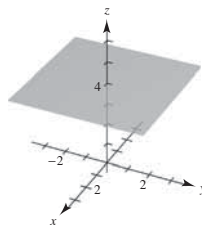
$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2, \text{rectangular equation}$$



78.  $z = 4$ , cylindrical equation

$$z = 4, \text{rectangular equation}$$

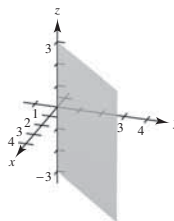


79.  $\theta = \frac{\pi}{4}$ , spherical coordinates

$$\tan \theta = \tan \frac{\pi}{4} = 1$$

$$\frac{y}{x} = 1$$

$$y = x, x \geq 0, \text{rectangular coordinates, half-plane}$$

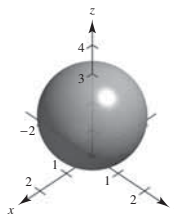


80.  $\rho = 3 \cos \theta$ , spherical coordinates

$$\sqrt{x^2 + y^2 + z^2} = \frac{3z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 3z = 0$$

$$x^2 + y^2 + \left(z - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2, \text{rectangular coordinates, sphere}$$



## Problem Solving for Chapter 11

1.  $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

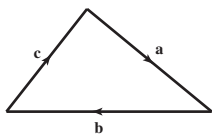
$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$

$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\| \|\mathbf{c}\| \sin A$

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin C$

Then,

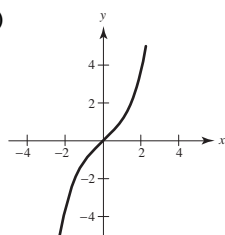
$$\begin{aligned} \frac{\sin A}{\|\mathbf{a}\|} &= \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\sin C}{\|\mathbf{c}\|}. \end{aligned}$$



The other case,  $\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|}$  is similar.

2.  $f(x) = \int_0^x \sqrt{t^4 + 1} dt$

(a)



(b)  $f'(x) = \sqrt{x^4 + 1}$

$f'(0) = 1 = \tan \theta$

$\theta = \frac{\pi}{4}$

$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

(c)  $\pm \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

(d) The line is  $y = x$ :  $x = t, y = t$ .

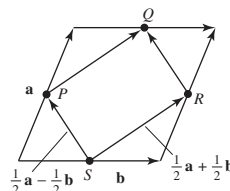
3. Label the figure as indicated.

From the figure, you see that

$$\overrightarrow{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overrightarrow{RQ} \text{ and } \overrightarrow{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overrightarrow{PQ}.$$

Because  $\overrightarrow{SP} = \overrightarrow{RQ}$  and  $\overrightarrow{SR} = \overrightarrow{PQ}$ ,

$PSRQ$  is a parallelogram.



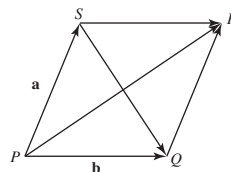
4. Label the figure as indicated.

$\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{SQ} = \mathbf{b} - \mathbf{a}$

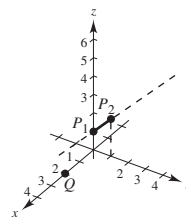
$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 = 0$ , because

$\|\mathbf{a}\| = \|\mathbf{b}\|$  in a rhombus.



5. (a)  $\mathbf{u} = \langle 0, 1, 1 \rangle$  is the direction vector of the line determined by  $P_1$  and  $P_2$ .

$$\begin{aligned} D &= \frac{\|\overrightarrow{P_1Q} \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\|\langle 2, 0, -1 \rangle \times \langle 0, 1, 1 \rangle\|}{\sqrt{2}} \\ &= \frac{\|\langle 1, -2, 2 \rangle\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \end{aligned}$$



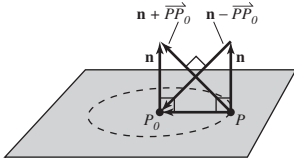
(b) The shortest distance to the line segment

is  $\|\overrightarrow{P_1Q}\| = \|\langle 2, 0, -1 \rangle\| = \sqrt{5}$ .

6.  $(\mathbf{n} + \overrightarrow{PP_0}) \perp (\mathbf{n} - \overrightarrow{PP_0})$

Figure is a square.

So,  $\|\overrightarrow{PP_0}\| = \|\mathbf{n}\|$  and the points  $P$  form a circle of radius  $\|\mathbf{n}\|$  in the plane with center at  $P_0$ .



7. (a)  $V = \pi \int_0^1 (\sqrt{z})^2 dz = \left[ \pi \frac{z^2}{2} \right]_0^1 = \frac{1}{2} \pi$

Note:  $\frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}\pi(1) = \frac{1}{2}\pi$

(b)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$ : (slice at  $z = c$ )

$$\frac{x^2}{(\sqrt{ca})^2} + \frac{y^2}{(\sqrt{cb})^2} = 1$$

At  $z = c$ , figure is ellipse of area

$$\pi(\sqrt{ca})(\sqrt{cb}) = \pi abc.$$

$$V = \int_0^k \pi abc \cdot dc = \left[ \frac{\pi abc^2}{2} \right]_0^k = \frac{\pi abk^2}{2}$$

(c)  $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}(\text{area of base})(\text{height})$

8. (a)  $V = 2 \int_0^r \pi(r^2 - x^2)dx = 2\pi \left[ r^2x - \frac{x^3}{3} \right]_0^r = \frac{4}{3}\pi r^3$

(b) At height  $z = d > 0$ ,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{d^2}{c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{d^2}{c^2} = \frac{c^2 - d^2}{c^2}$$

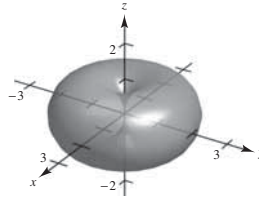
$$\frac{x^2}{\frac{a^2(c^2 - d^2)}{c^2}} + \frac{y^2}{\frac{b^2(c^2 - d^2)}{c^2}} = 1.$$

$$\text{Area} = \pi \sqrt{\left( \frac{a^2(c^2 - d^2)}{c^2} \right) \left( \frac{b^2(c^2 - d^2)}{c^2} \right)} = \frac{\pi ab}{c^2} (c^2 - d^2)$$

$$V = 2 \int_0^c \frac{\pi ab}{c^2} (c^2 - d^2) dd = \frac{2\pi ab}{c^2} \left[ c^2d - \frac{d^3}{3} \right]_0^c = \frac{4}{3}\pi abc$$

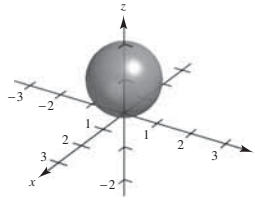
9. (a)  $\rho = 2 \sin \phi$

Torus



(b)  $\rho = 2 \cos \phi$

Sphere



10. (a)  $r = 2 \cos \theta$

Cylinder

(b)  $z = r^2 \cos 2\theta$

$$z^2 = x^2 - y^2$$

Hyperbolic paraboloid

11. From Exercise 66, Section 11.4,

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}.$$

12.  $x = -t + 3, y = \frac{1}{2}t + 1, z = 2t - 1; Q = (4, 3, s)$

(a)  $\mathbf{u} = \langle -2, 1, 4 \rangle$  direction vector for line

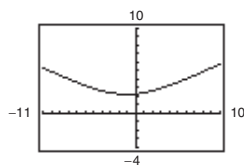
$P = (3, 1, -1)$  point on line

$$\overrightarrow{PQ} = \langle 1, 2, s + 1 \rangle$$

$$\begin{aligned} \overrightarrow{PQ} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & s+1 \\ -2 & 1 & 4 \end{vmatrix} \\ &= (7-s)\mathbf{i} + (-6-2s)\mathbf{j} + 5\mathbf{k} \end{aligned}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(7-s)^2 + (-6-2s)^2 + 25}}{\sqrt{21}}$$

(b)



The minimum is  $D \approx 2.2361$  at  $s = -1$ .

(c) Yes, there are slant asymptotes. Using  $s = x$ , you have

$$D(s) = \frac{1}{\sqrt{21}} \sqrt{5x^2 + 10x + 110} = \frac{\sqrt{5}}{\sqrt{21}} \sqrt{x^2 + 2x + 22} = \frac{\sqrt{5}}{\sqrt{21}} \sqrt{(x+1)^2 + 21} \rightarrow \pm \sqrt{\frac{5}{21}}(x+1)$$

$$y = \pm \frac{\sqrt{105}}{21}(s+1) \text{ slant asymptotes.}$$

13. (a)  $\mathbf{u} = \|\mathbf{u}\|(\cos 0 \mathbf{i} + \sin 0 \mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force  $\mathbf{w} = -\mathbf{j}$

$$\mathbf{T} = \|\mathbf{T}\|(\cos(90^\circ + \theta)\mathbf{i} + \sin(90^\circ + \theta)\mathbf{j})$$

$$= \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

$$\mathbf{0} = \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

$$\|\mathbf{u}\| = \sin \theta \|\mathbf{T}\|$$

$$1 = \cos \theta \|\mathbf{T}\|$$

If  $\theta = 30^\circ$ ,  $\|\mathbf{u}\| = (1/2)\|\mathbf{T}\|$  and  $1 = (\sqrt{3}/2)\|\mathbf{T}\| \Rightarrow \|\mathbf{T}\| = \frac{2}{\sqrt{3}} \approx 1.1547$  lb and  $\|\mathbf{u}\| = \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) \approx 0.5774$  lb

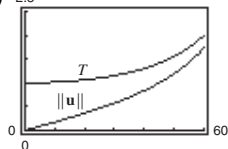
(b) From part (a),  $\|\mathbf{u}\| = \tan \theta$  and  $\|\mathbf{T}\| = \sec \theta$ .

Domain:  $0 \leq \theta \leq 90^\circ$

(c)

$\theta$	$0^\circ$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$
$\mathbf{T}$	1	1.0154	1.0642	1.1547	1.3054	1.5557	2
$\ \mathbf{u}\ $	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321

(d) 2.5



(e) Both are increasing functions.

(f)  $\lim_{\theta \rightarrow \pi/2^-} T = \infty$  and  $\lim_{\theta \rightarrow \pi/2^-} \|\mathbf{u}\| = \infty$ .

Yes. As  $\theta$  increases, both  $T$  and  $\|\mathbf{u}\|$  increase.

14. (a) The tension
- $T$
- is the same in each tow line.

$$\begin{aligned}
 6000\mathbf{i} &= T(\cos 20^\circ + \cos(-20^\circ))\mathbf{i} + T(\sin 20^\circ + \sin(-20^\circ))\mathbf{j} \\
 &= 2T\cos 20^\circ\mathbf{i} \\
 \Rightarrow T &= \frac{6000}{2\cos 20^\circ} \approx 3192.5 \text{ lb}
 \end{aligned}$$

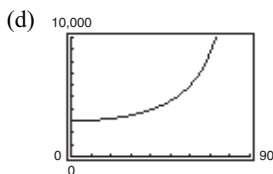
- (b) As in part (a),
- $6000\mathbf{i} = 2T\cos \theta$

$$\Rightarrow T = \frac{3000}{\cos \theta}$$

Domain:  $0 < \theta < 90^\circ$ 

(c)

$\theta$	$10^\circ$	$20^\circ$	$30^\circ$	$40^\circ$	$50^\circ$	$60^\circ$
$T$	3046.3	3192.5	3464.1	3916.2	4667.2	6000.0



- (e) As
- $\theta$
- increases, there is less force applied in the direction of motion.

15. Let
- $\theta = \alpha - \beta$
- , the angle between
- $\mathbf{u}$
- and
- $\mathbf{v}$
- . Then

$$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

For  $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$  and  $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$ ,  $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$  and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\mathbf{k}.$$

So,  $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ .

16. (a) Los Angeles:
- $(4000, -118.24^\circ, 55.95^\circ)$

Rio de Janeiro:  $(4000, -43.23^\circ, 112.90^\circ)$ 

- (b) Los Angeles:
- $x = 4000 \sin(55.95^\circ)\cos(-118.24^\circ)$

Rio de Janeiro:  $x = 4000 \sin(112.90^\circ)\cos(-43.23^\circ)$ 

$$y = 4000 \sin(55.95^\circ)\sin(-118.24^\circ)$$

$$y = 4000 \sin(112.90^\circ)\sin(-43.23^\circ)$$

$$z = 4000 \cos(55.95^\circ)$$

$$z = 4000 \cos(112.90^\circ)$$

$$(x, y, z) \approx (-1568.2, -2919.7, 2239.7)$$

$$(x, y, z) \approx (2684.7, -2523.8, -1556.5)$$

$$(c) \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-1568.2)(2684.7) + (-2919.7)(-2523.8) + (2239.7)(-1556.5)}{(4000)(4000)} \approx -0.02047$$

$$\theta \approx 91.17^\circ \text{ or } 1.59 \text{ radians}$$

- (d)
- $s = r\theta = 4000(1.59) \approx 6360$
- miles



(e) For Boston and Honolulu:

a. Boston:  $(4000, -71.06^\circ, 47.64^\circ)$

Honolulu:  $(4000, -157.86^\circ, 68.69^\circ)$

b. Boston:  $x = 4000 \sin 47.64^\circ \cos(-71.06^\circ)$

Honolulu:  $x = 4000 \sin 68.69^\circ \cos(-157.86^\circ)$

$$y = 4000 \sin 47.64^\circ \sin(-71.06^\circ)$$

$$y = 4000 \sin 68.69^\circ \sin(-157.86^\circ)$$

$$z = 4000 \cos 47.64^\circ$$

$$z = 4000 \cos 68.69^\circ$$

$$(959.4, -2795.7, 2695.1)$$

$$(-3451.7, -1404.4, 1453.7)$$

$$c. \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(959.4)(-3451.7) + (-2795.7)(-1404.4) + (2695.1)(1453.7)}{(4000)(4000)} \approx 0.28329$$

$$\theta \approx 73.54^\circ \text{ or } 1.28 \text{ radians}$$

$$d. s = r\theta = 4000(1.28) \approx 5120 \text{ miles}$$

17. From Theorem 11.13 and Theorem 11.7 (6) you have

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}.$$

18. Assume one of  $a, b, c$ , is not zero, say  $a$ . Choose a point in the first plane such as  $(-d_1/a, 0, 0)$ . The distance between this point and the second plane is

$$D = \frac{|a(-d_1/a) + b(0) + c(0) + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

19.  $x^2 + y^2 = 1$  cylinder

$z = 2y$  plane

Introduce a coordinate system in the plane  $z = 2y$ .

The new  $u$ -axis is the original  $x$ -axis.

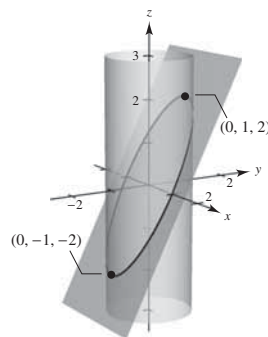
The new  $v$ -axis is the line  $z = 2y, x = 0$ .

Then the intersection of the cylinder and plane satisfies the equation of an ellipse:

$$x^2 + y^2 = 1$$

$$x^2 + \left(\frac{z}{2}\right)^2 = 1$$

$$x^2 + \frac{z^2}{4} = 1 \quad \text{ellipse}$$



20. Essay.