6000

Solutions Manual: Chapter 6

7th Edition

Feedback Control of Dynamic Systems

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Chapter 6

The Frequency-response Design Method

Problems and Solutions for Section 6.1

1. (a) Show that α_0 in Eq. (6.2) is given by

$$\alpha_0 = \left[G(s)\frac{U_0\omega}{s-j\omega}\right]_{s=-j\omega} = -U_0G(-j\omega)\frac{1}{2j}$$

and

$$\alpha_0^* = \left[G(s)\frac{U_0\omega}{s+j\omega}\right]_{s=+j\omega} = U_0G(j\omega)\frac{1}{2j}.$$

(b) By assuming the output can be written as

$$y(t) = \alpha_0 e^{-j\omega t} + \alpha_0^* e^{j\omega t},$$

derive Eqs. (6.4) - (6.6).

Solution:

(a) Eq. (6.2):

$$Y(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_n}{s-p_n} + \frac{\alpha_o}{s+j\omega_o} + \frac{\alpha_o^*}{s-j\omega_o}$$

Multiplying this by $(s+j\omega)$:

$$Y(s)(s+j\omega) = \frac{\alpha_1}{s+a_1}(s+j\omega)+\ldots+\frac{\alpha_n}{s+a_n}(s+j\omega)+\alpha_o+\frac{\alpha_o^*}{s-j\omega}(s+j\omega)$$

$$\Rightarrow \quad \alpha_o = Y(s)(s+j\omega) - \frac{\alpha_1}{s+a_1}(s+j\omega) - \ldots - \frac{\alpha_n}{s+a_n}(s+j\omega) - \frac{\alpha_o^*}{s-j\omega}(s+j\omega)$$

$$\begin{aligned}
\alpha_o &= \alpha_o|_{s=-j_\omega} = \left[Y(s)(s+j\omega) - \frac{\alpha_1}{s+a_1}(s+j\omega) - \ldots - \frac{\alpha_o^*}{s-j\omega}(s+j\omega)\right]_{s=-j_\omega} \\
&= Y(s)(s+j\omega)|_{s=-j_\omega} = G(s)\frac{U_o\omega}{s^2+\omega^2}(s+j\omega)|_{s=-j_\omega} \\
&= G(s)\frac{U_o\omega}{s-j\omega}|_{s=-j_\omega} = -U_oG(-j\omega)\frac{1}{2j}
\end{aligned}$$

Similarly, multiplying Eq. (6.2) by $(s-j\omega)$:

$$\begin{aligned}
Y(s)(s-j\omega) &= \frac{\alpha_1}{s+a_1}(s-j\omega) + \ldots + \frac{\alpha_n}{s+a_n}(s-j\omega) + \frac{\alpha_o}{s+j\omega}(s-j\omega) + \alpha_o^* \\
\alpha_o^* &= \alpha_o^*|_{s=j\omega} = Y(s)(s-j\omega)|_{s=j_\omega} = G(s)\frac{U_o\omega}{s^2+\omega^2}(s-j\omega)|_{s=j_\omega} \\
&= G(s)\frac{U_o\omega}{s+j\omega}|_{s=j_\omega} = U_oG(j\omega)\frac{1}{2j}
\end{aligned}$$

(b)

$$\begin{aligned}
y(t) &= \alpha_o e^{-j\omega t} + \alpha_o^* e^{j\omega t} \\
y(t) &= -U_oG(-j\omega)\frac{1}{2j}e^{-j\omega t} + U_oG(j\omega)\frac{1}{2j}e^{j\omega t} \\
&= U_o\left[\frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j}\right] \\
|G(j\omega)| &= \left\{\mathrm{Re}\left[G(j\omega)\right]^2 + \mathrm{Im}\left[G(j\omega)\right]^2\right\}^{\frac{1}{2}} = A \\
\angle G(j\omega) &= \tan^{-1}\frac{\mathrm{Im}\left[G(j\omega)\right]}{\mathrm{Re}\left[G(j\omega)\right]} = \phi \\
|G(-j\omega)| &= \left\{\mathrm{Re}\left[G(-j\omega)\right]^2 + \mathrm{Im}\left[G(-j\omega)\right]^2\right\}^{\frac{1}{2}} = |G(j\omega)| \\
&= \left\{\mathrm{Re}\left[G(j\omega)\right]^2 + \mathrm{Im}\left[G(j\omega)\right]^2\right\}^{\frac{1}{2}} = A \\
\angle G(-j\omega) &= \tan^{-1}\frac{\mathrm{Im}\left[G(-j\omega)\right]}{\mathrm{Re}\left[G(-j\omega)\right]} = \tan^{-1}\frac{-\mathrm{Im}\left[G(j\omega)\right]}{\mathrm{Re}\left[G(j\omega)\right]} = -\phi \\
&\Rightarrow \quad G(j\omega) = Ae^{j\phi},\ G(-j\omega) = Ae^{-j\phi}
\end{aligned}$$

Thus,

$$\begin{aligned}
y(t) &= U_o\left[\frac{Ae^{j\phi}e^{j\omega t} - Ae^{-j\phi}e^{-j\omega t}}{2j}\right] = U_oA\left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right] \\
y(t) &= U_oA\sin(\omega t+\phi)
\end{aligned}$$

where

$$A = |G(j\omega)|\,,\ \phi = \tan^{-1}\frac{\mathrm{Im}\,[G(j\omega)]}{\mathrm{Re}\,[G(j\omega)]} = \angle G(j\omega)$$

2. (a) Calculate the magnitude and phase of

$$G(s) = \frac{1}{s+10}$$

by hand for $\omega = 1$, 2, 5, 10, 20, 50, and 100 rad/sec.

(b) sketch the asymptotes for $G(s)$ according to the Bode plot rules, and compare these with your computed results from part (a).

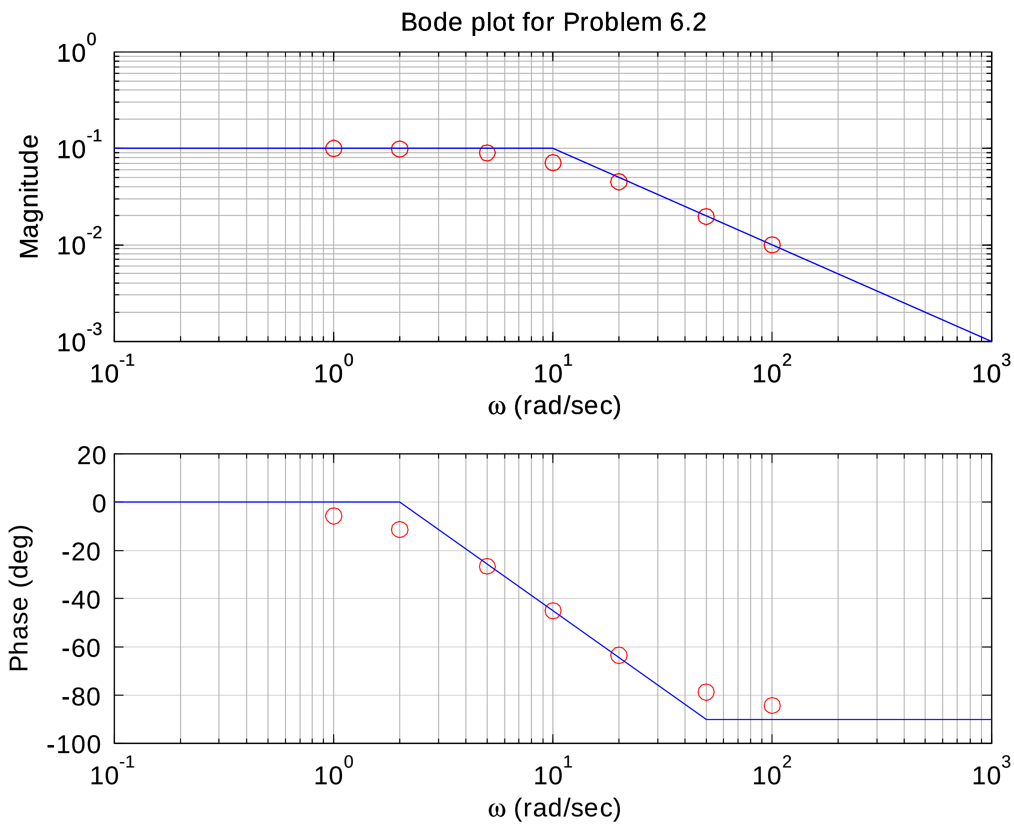
Solution:

(a)

$$\begin{aligned} G(s) &= \frac{1}{s+10},\ G(j\omega) = \frac{1}{10+j\omega} = \frac{10-j\omega}{100+\omega^2} \\ |G(j\omega)| &= \frac{1}{\sqrt{100+\omega^2}},\ \angle G(j\omega) = -\tan^{-1}\frac{\omega}{10} \end{aligned}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
1	0.0995	-5.71
2	0.0981	-11.3
5	0.0894	-26.6
10	0.0707	-45.0
20	0.0447	-63.4
50	0.0196	-78.7
100	0.00995	-84.3

(b) To plot the asymptotes, you first note that $n = 0$, as defined in Section 6.1.1. That signifies that the leftmost portion of the asymptotes will have zero slope. That portion of the asymptotes will be located at the DC gain of the transfer function, which, in this case it can be seen by inspection to be 0.1. So the asymptote starts with a straght horizontal line at 0.1 and that continues until the breakpoint at $\omega = 10$, at which point the asymptote has a slope of $n = -1$ that continues until forever, at least until the edge of the paper. The values computed above by "hand" (at least we hope you didn't cheat) are plotted on the graph below and you see they match quite well except very near the breakpoint, a you should have expected. The Bode plot is :



3. Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{2000}{s(s+200)}$

   (b) $L(s) = \dfrac{100}{s(0.1s+1)(0.5s+1)}$

   (c) $L(s) = \dfrac{1}{s(s+1)(0.02s+1)}$

   (d) $L(s) = \dfrac{1}{(s+1)^2(s+10)^2}$

   (e) $L(s) = \dfrac{10(s+4)}{s(s+1)(s+100)}$

   (f) $L(s) = \dfrac{1000(s+0.1)}{s(s+1)(s+8)^2}$

(g) $L(s) = \dfrac{(s+5)(s+10)}{s(s+1)(s+100)}$

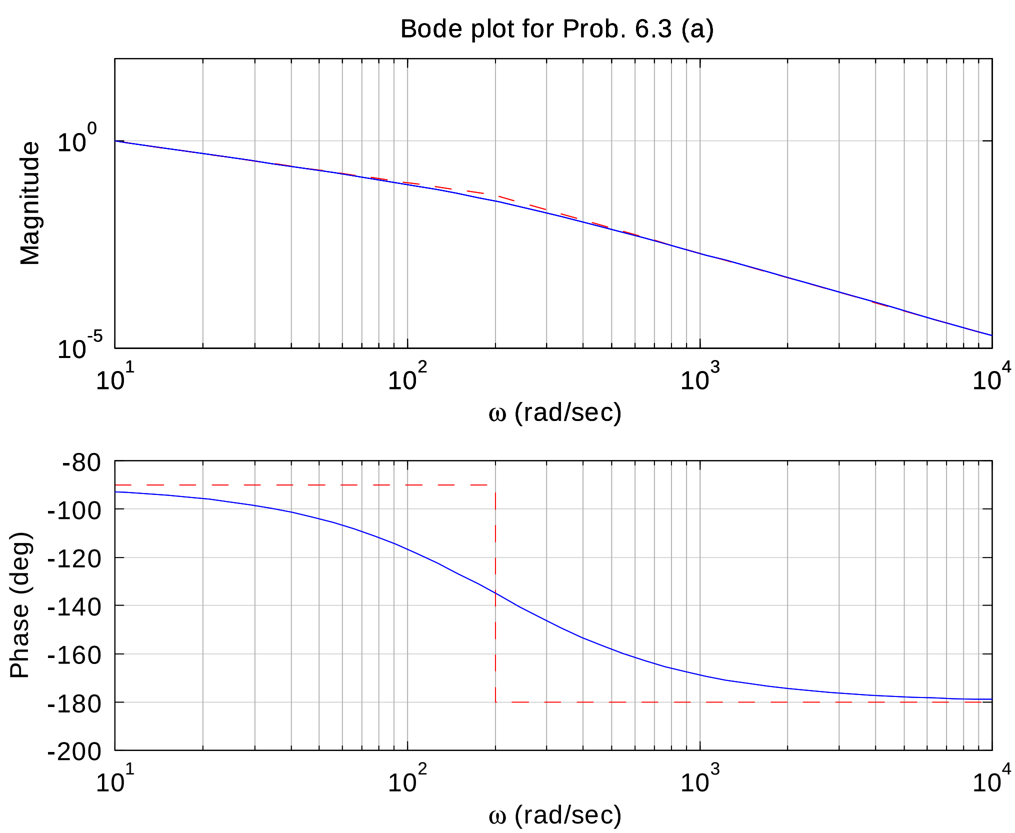
(h) $L(s) = \dfrac{4s(s+10)}{(s+100)(s+500)}$

(i) $L(s) = \dfrac{s}{(s+1)(s+10)(s+50)}$

Solution:

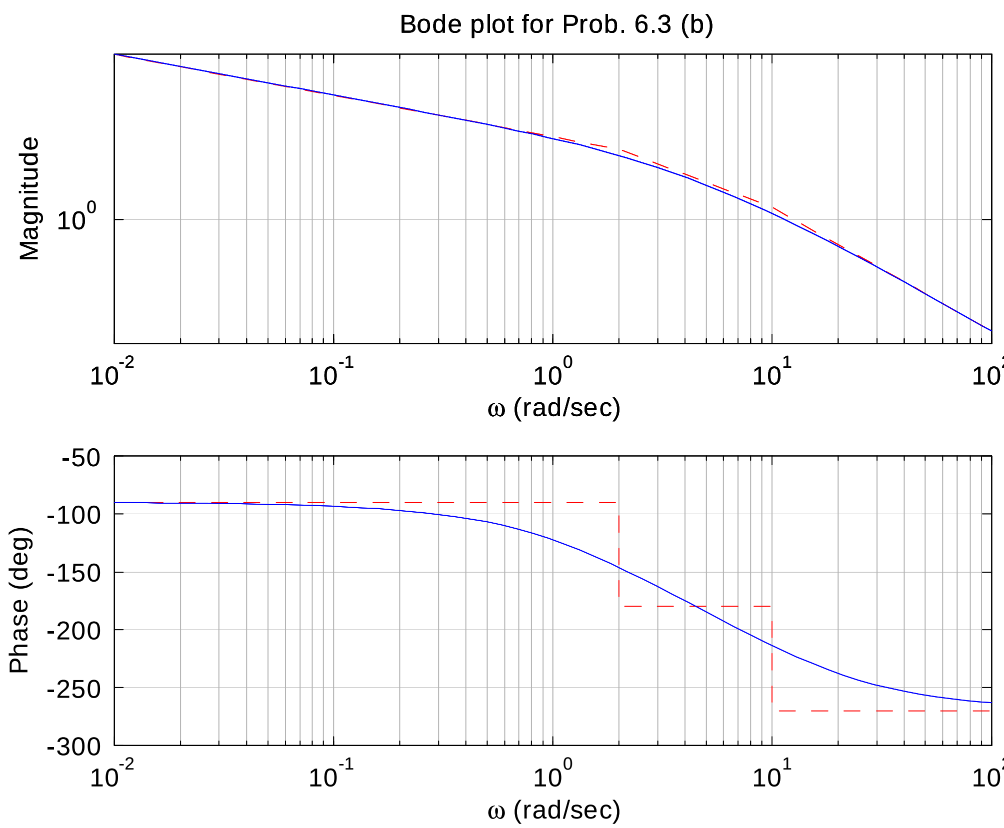
(a) $L(s) = \dfrac{10}{s\left[\frac{s}{200}+1\right]}$

The asymptote will have magnitude $= 10$ at $\omega = 1$ and have a -1 slope at the low frequencies. So the low frequency asymptote will pass through magnitude $= 1$, at $\omega = 10$. At $\omega = 200$, the slope changes to -2, and that slope continues forever. The phase asymptotes are even simpler and shown in the plot below along with the Matlab generated exact curve.



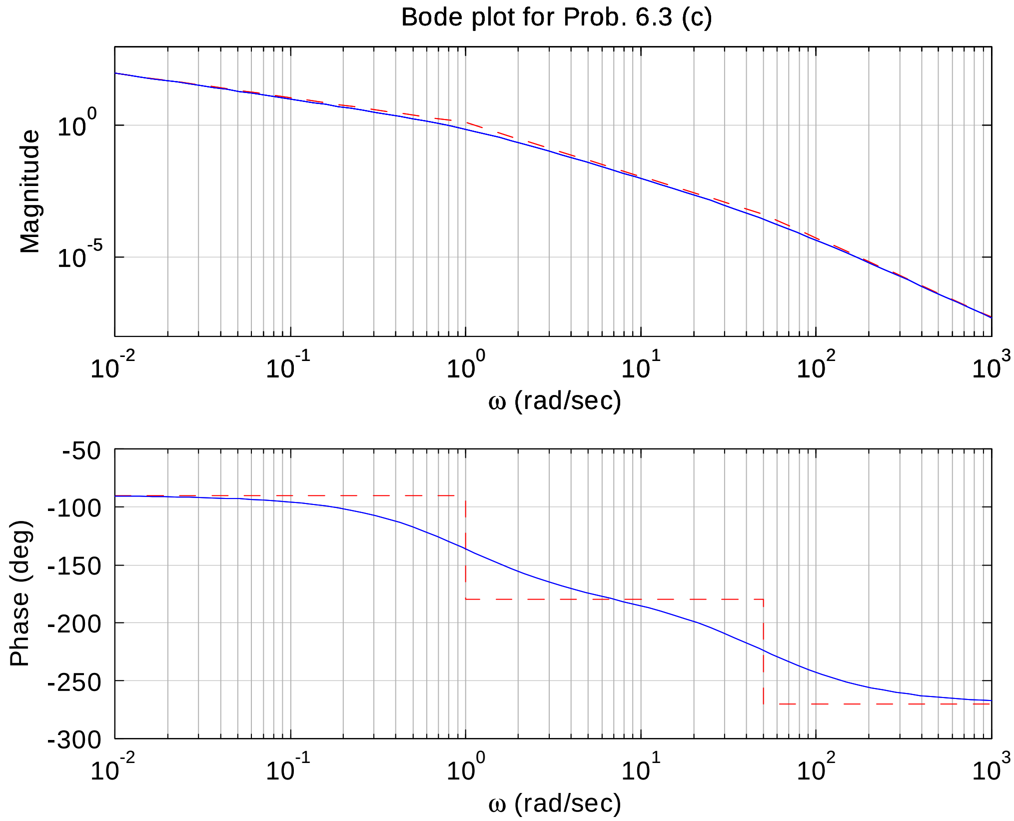
(b) $L(s) = \dfrac{100}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{2}+1\right)}$

Breakpoints are at 2 and 10. The slope starts at n = -1, the changes to -2 at $\omega = 2$, then changes to -3 at $\omega = 10$. The magnitude on the -1 slope goes through 100 at $\omega = 1$.Thus, will be 10,000 at $\omega = 0.01$



(c) $L(s) = \frac{1}{s(s+1)(0.02s+1)}$

Slope starts at n = -1, becomes -2 at $\omega = 1$, then -3 at n = 50. Magnitude = 1 at $\omega = 1$.



(d) $L(s) = \dfrac{\frac{1}{100}}{(s+1)^2\left(\frac{s}{10}+1\right)^2}$

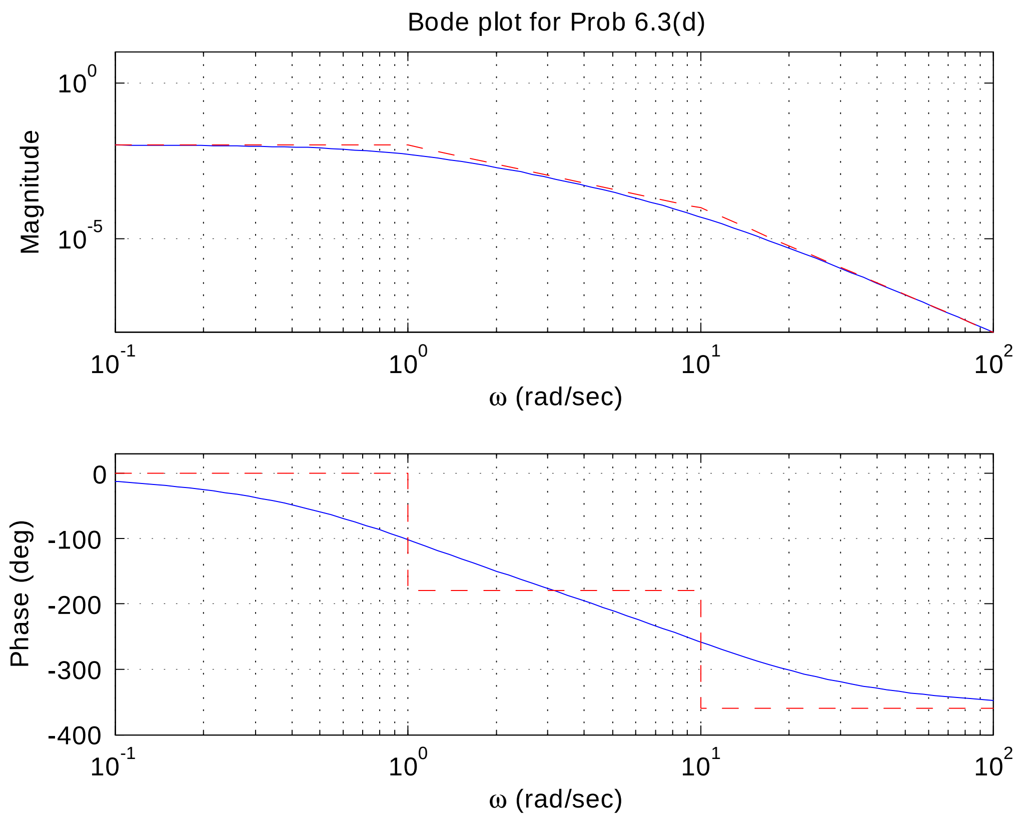
Initially, slope = 0, then becomes n = -2 at $\omega = 1$, then n = -4 at $\omega = 10$.

(e) $L(s) = \dfrac{0.4\left(\frac{s}{4}+1\right)}{s(s+1)(s\frac{s}{100}+1)}$

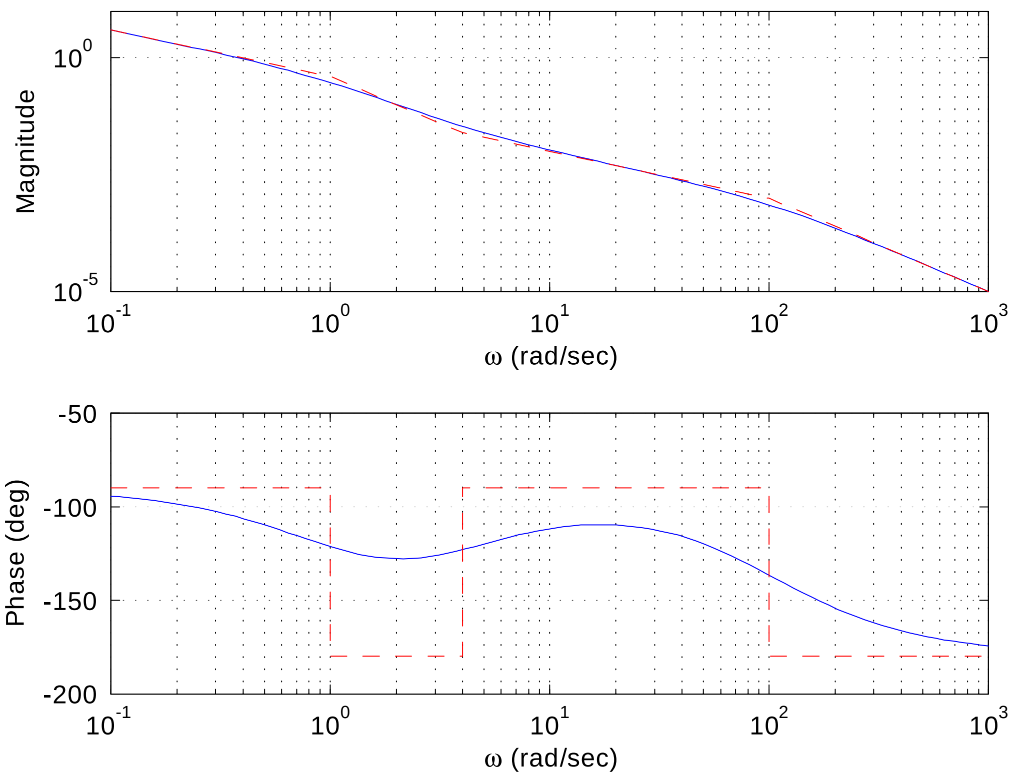
(f) $L(s) = \dfrac{\left(\frac{50}{32}\right)(10s+1)}{s(s+1)\left(\frac{s}{8}+1\right)^2} = \dfrac{(1.56)(10s+1)}{s(s+1)\left(\frac{s}{8}+1\right)^2}$

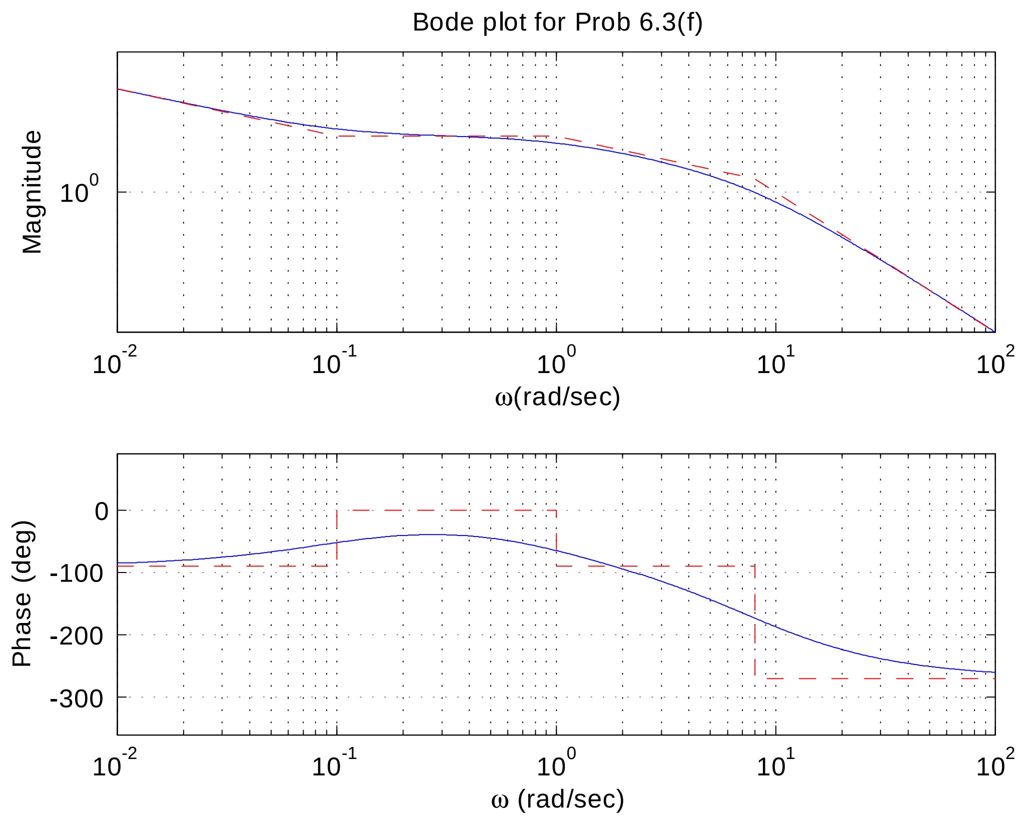
Breakpoints are at $\omega = 0.1, 1, 8$ So, to position the low frequency asymptote, you need to extrapolate the line from $\omega = 1$ backward. The magnitude at $\omega = 1$ would be 1.56, and to position the asymptote at $\omega = 0.1$, we need to multiply by 10 due to the -1 slope Therefore, it is located at a magnitude of 15.6 at $\omega = 0.1$.and will be 156 at $\omega = 0.01$.

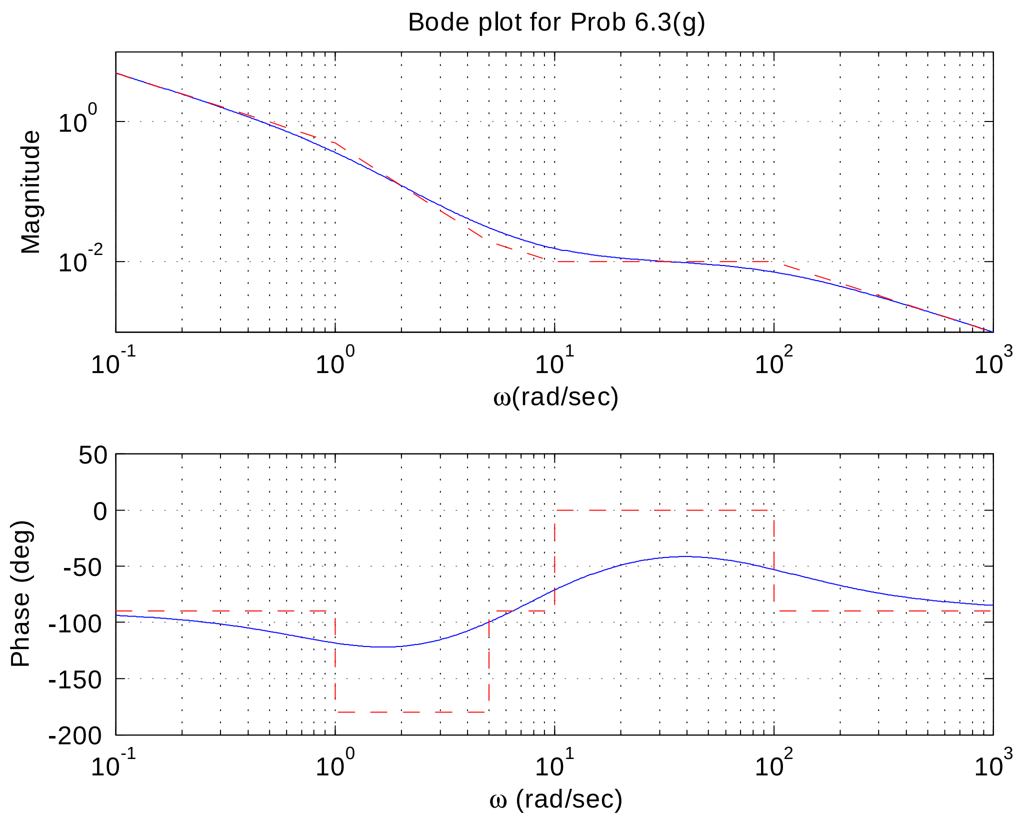
(g) $L(s) = \dfrac{\left(\frac{1}{2}\right)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}{s(s+1)(s+100)}$

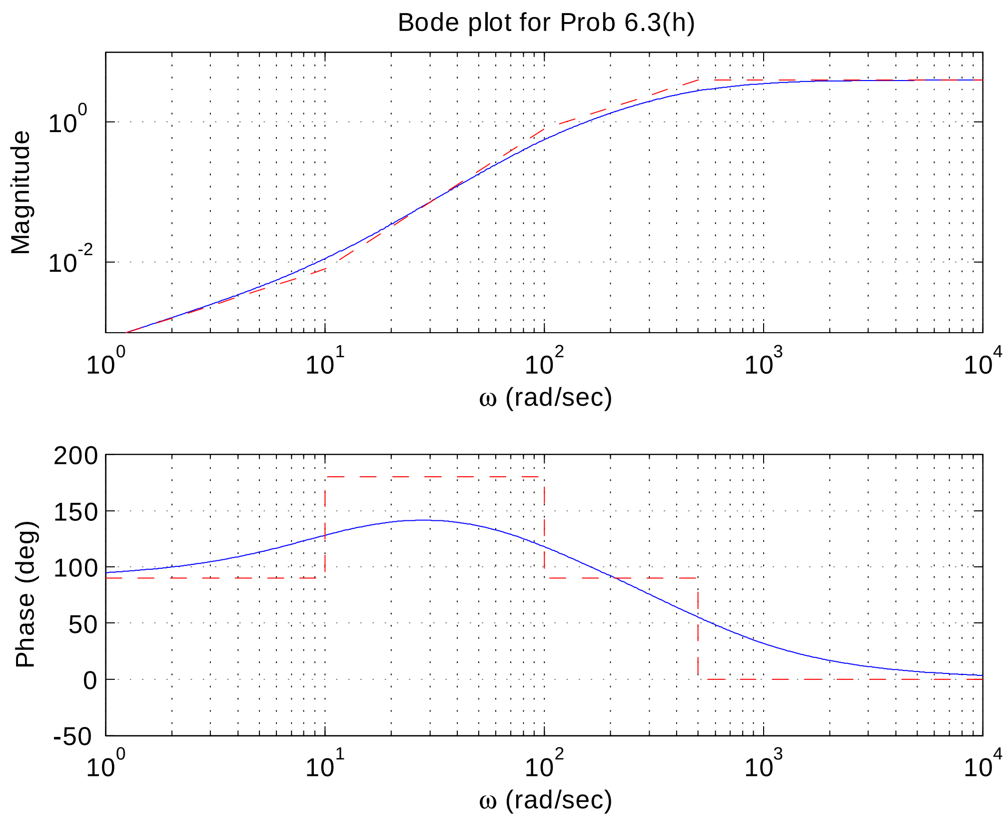

Bode plot for Prob 6.3(d)
Magnitude
10^0
10^{-5}
10^{-1}
10^0
10^1
10^2
ω (rad/sec)
Phase (deg)
0
-100
-200
-300
-400
10^{-1}
10^0
10^1
10^2
ω (rad/sec)

Bode plot for Prob 6.3(e)




Bode plot for Prob 6.3(f)
Magnitude
10^0
10^{-2}
10^{-1}
10^0
10^1
10^2
ω(rad/sec)
Phase (deg)
0
-100
-200
-300
10^{-2}
10^{-1}
10^0
10^1
10^2
ω (rad/sec)


Bode plot for Prob 6.3(g)
Magnitude
10^0
10^-2
10^-1
10^0
10^1
10^2
10^3
ω(rad/sec)
Phase (deg)
50
0
-50
-100
-150
-200
10^-1
10^0
10^1
10^2
10^3
ω (rad/sec)



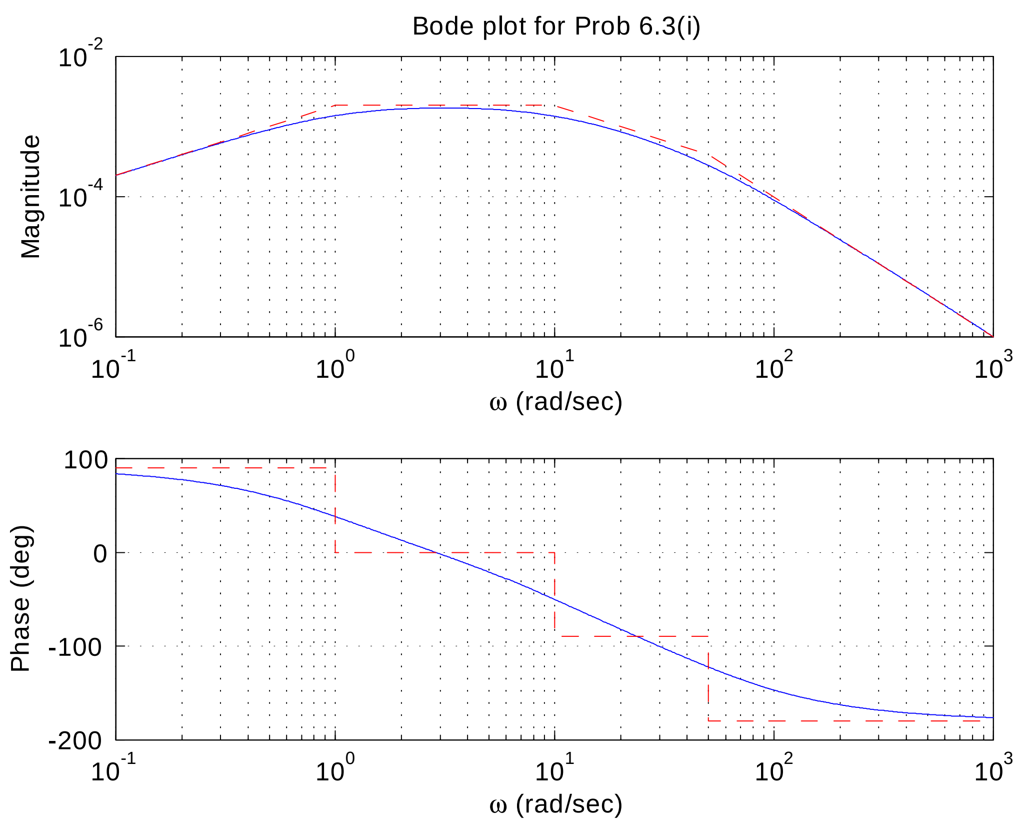
(h) $L(s) = \dfrac{\left(\frac{4}{5000}\right) s \left(\frac{s}{10}+1\right)}{\left(\frac{s}{100}+1\right)\left(\frac{s}{500}+1\right)}$

This one is unusual in that the slope is generally rising with increasing frequency. Finite output at extremely high frequencies doesn't really happen in nature, but hey, this is a theoretical homework problem to check whether you can follow the rules and do what the math tells you to do.

(i) $L(s) = \dfrac{\left(\frac{1}{500}\right) s}{(s+1)\left(\frac{s}{10}+1\right)\left(\frac{s}{50}+1\right)}$

4. *Real poles and zeros.* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s(s+1)(s+5)(s+10)}$



Bode plot for Prob 6.3(i)
Magnitude
10^-2
10^-4
10^-6
10^-1
10^0
10^1
10^2
10^3
ω (rad/sec)
Phase (deg)
100
0
-100
-200
10^-1
10^0
10^1
10^2
10^3
ω (rad/sec)

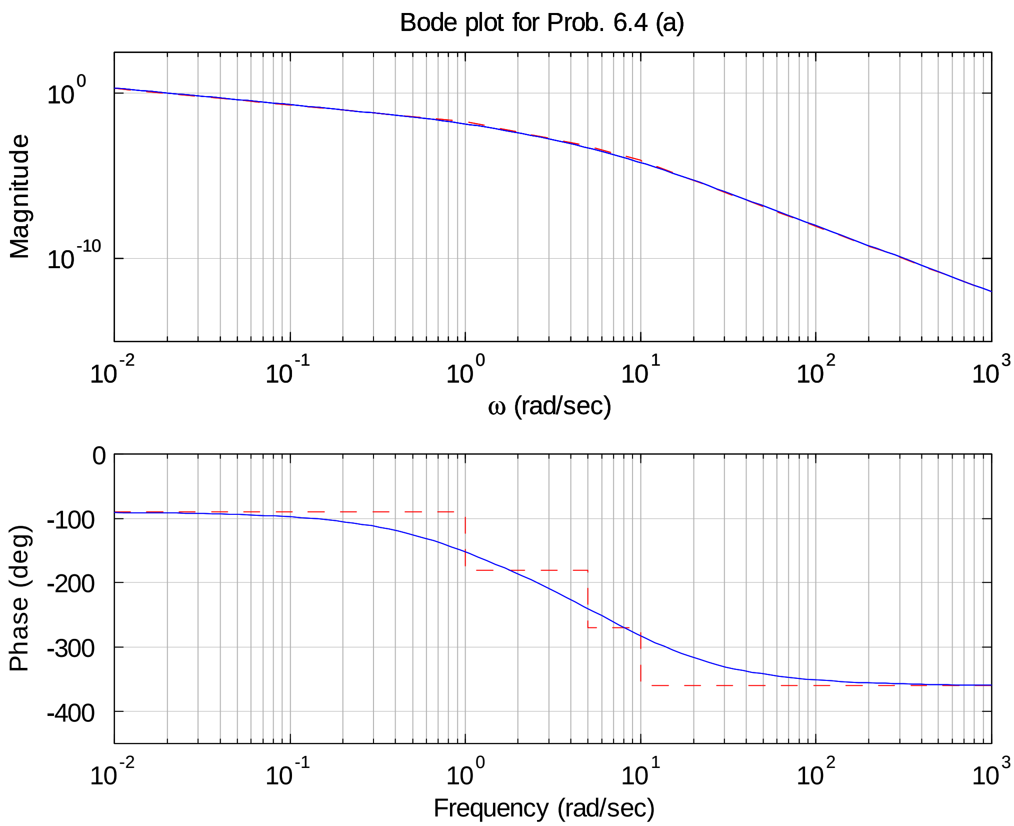
(b) $L(s) = \dfrac{(s+2)}{s(s+1)(s+5)(s+10)}$

(c) $L(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

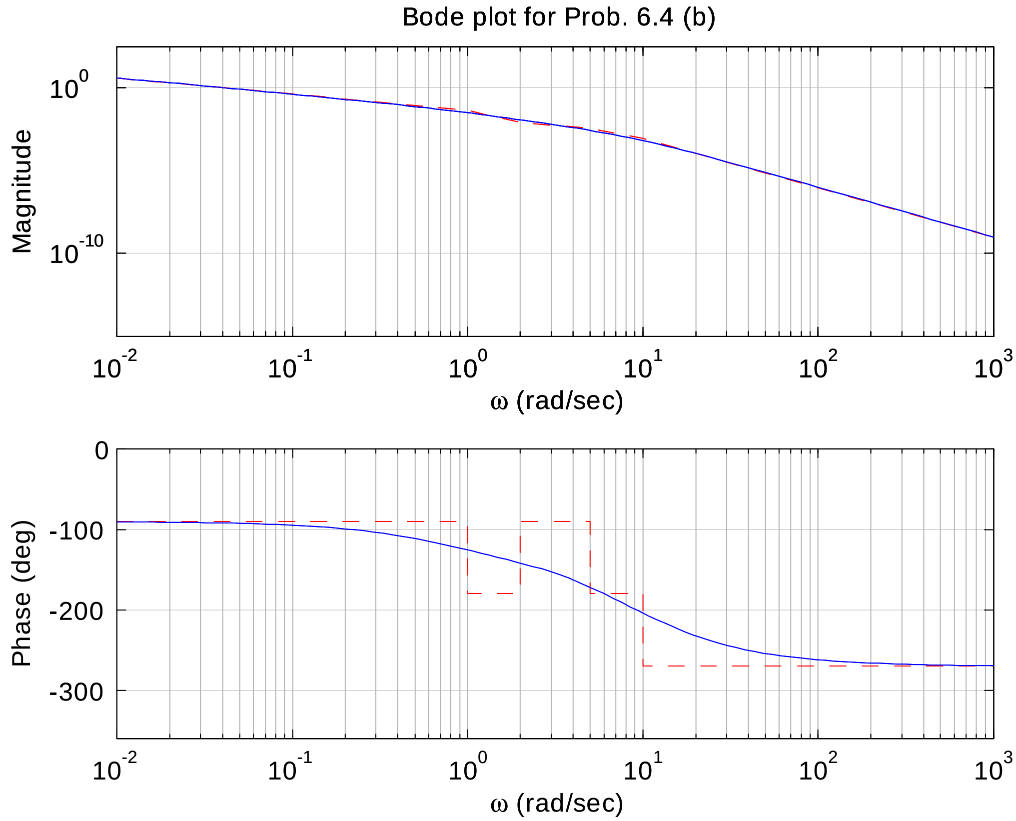
(d) $L(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

Solution:

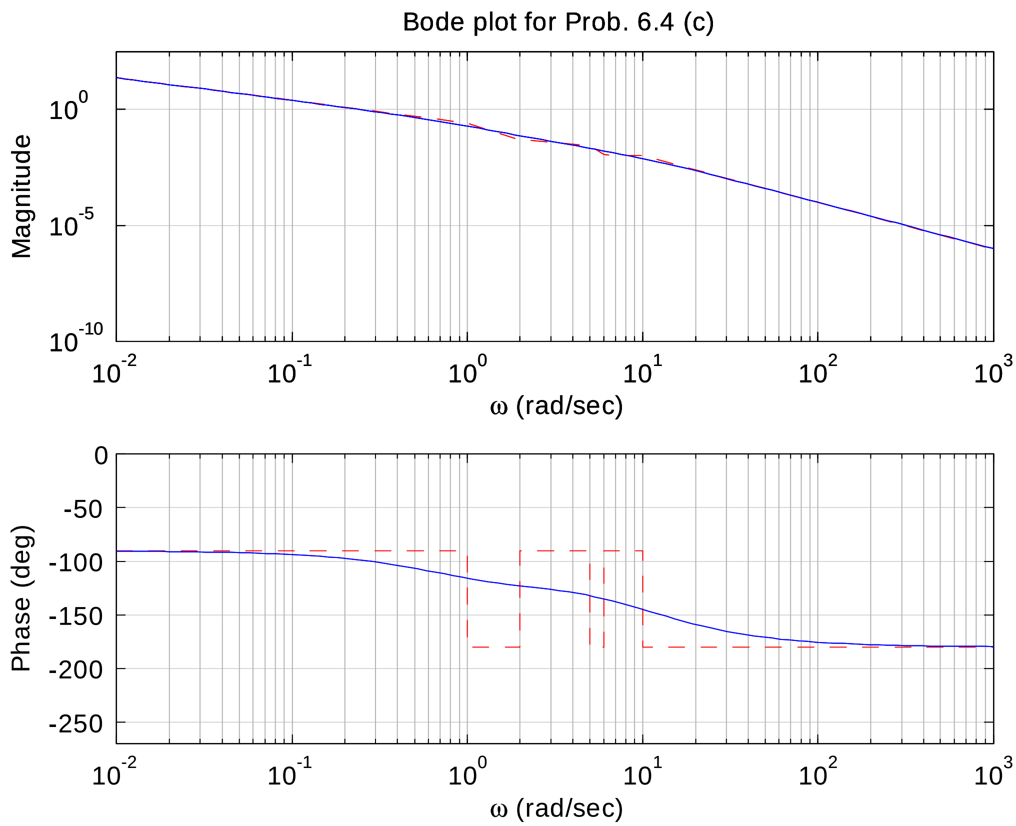
(a) $L(s) = \dfrac{\frac{1}{50}}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



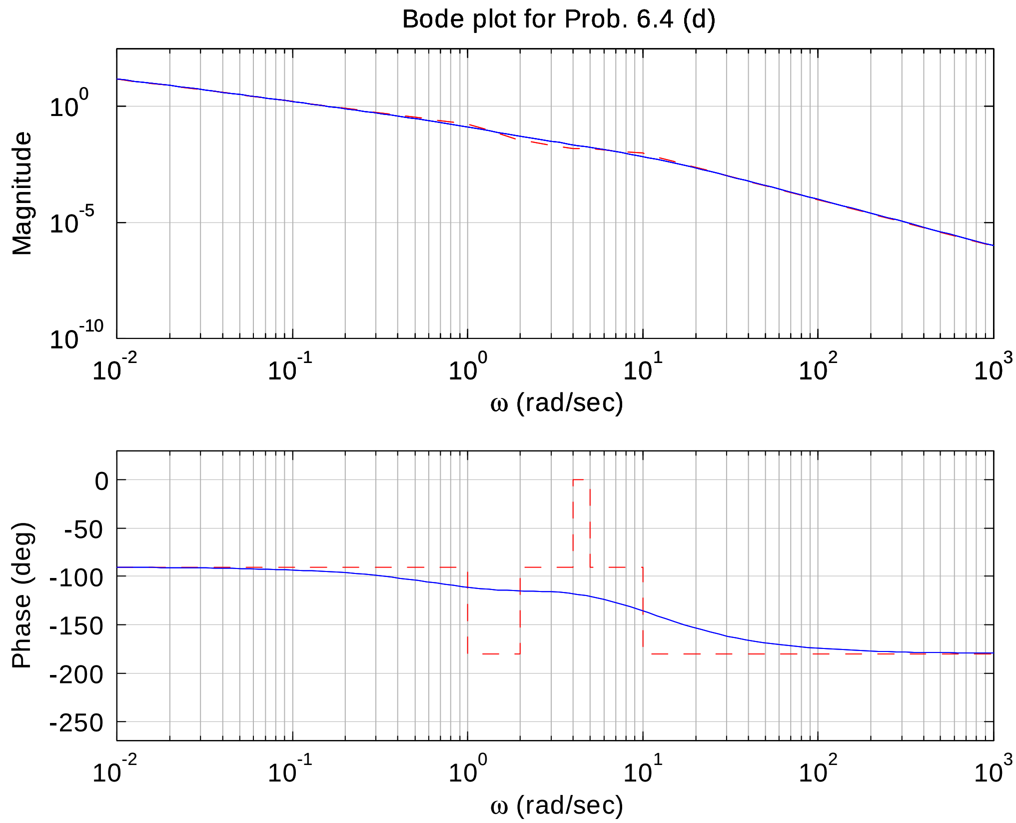
(b) $L(s) = \dfrac{\frac{1}{25}\left(\frac{s}{2}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



(c) $L(s) = \dfrac{\frac{6}{25}\left(\frac{s}{2}+1\right)\left(\frac{s}{6}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



(d) $L(s) = \dfrac{\frac{4}{25}\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



5. *Complex poles and zeros* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions and approximate the transition at the second order break point based on the value of the damping ratio. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2 + 3s + 10}$

(b) $L(s) = \dfrac{1}{s(s^2 + 3s + 10)}$

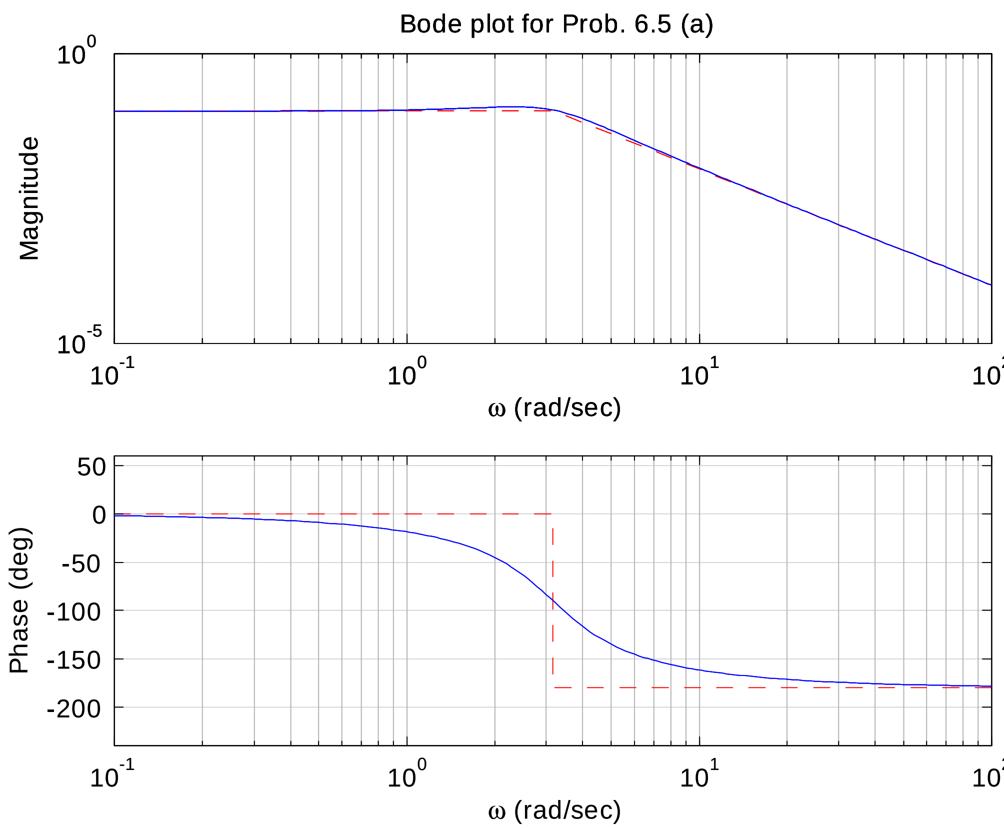
(c) $L(s) = \dfrac{(s^2 + 2s + 8)}{s(s^2 + 2s + 10)}$

(d) $L(s) = \dfrac{(s^2 + 1)}{s(s^2 + 4)}$

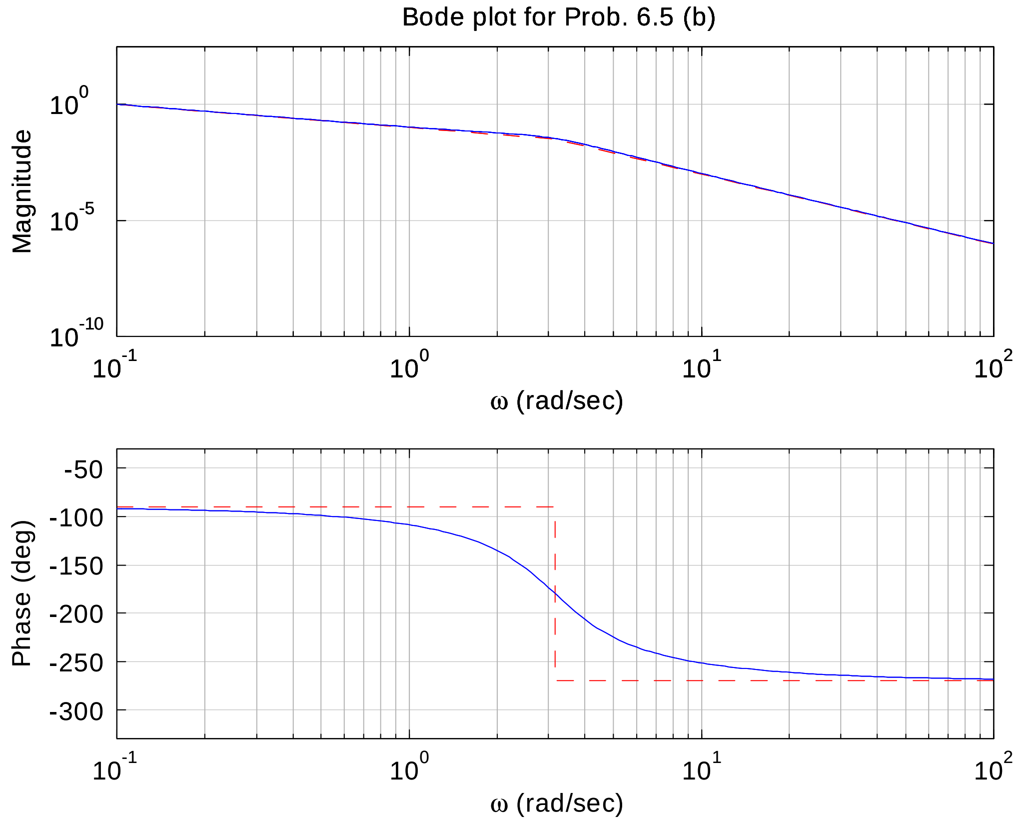
(e) $L(s) = \dfrac{(s^2 + 4)}{s(s^2 + 1)}$

Solution:

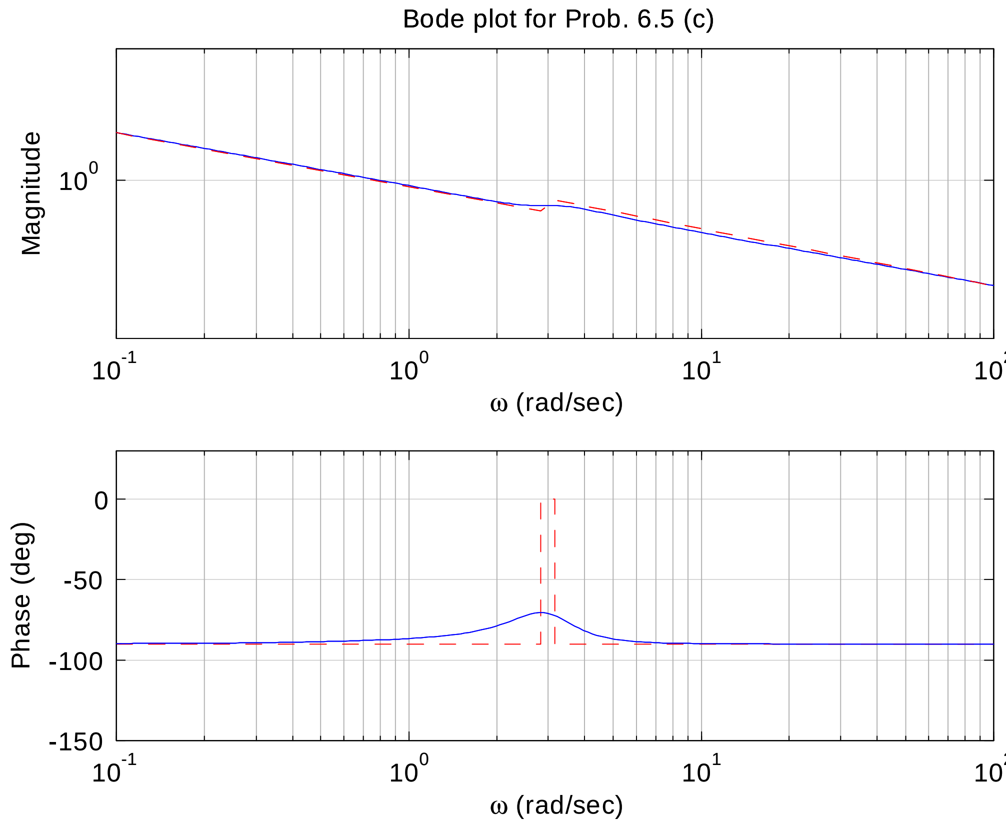
(a) $L(s) = \dfrac{\frac{1}{10}}{\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{3}{10}s + 1}$



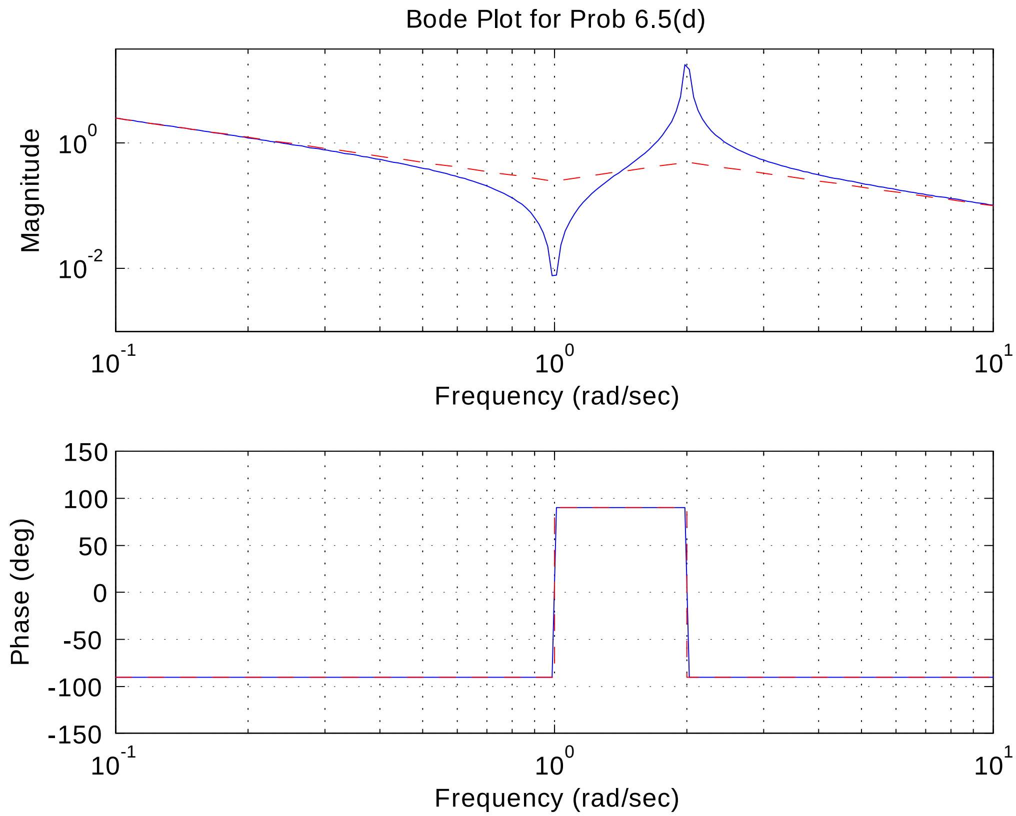
(b) $L(s) = \dfrac{\frac{1}{10}}{s\left[\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{3}{10}s + 1\right]}$



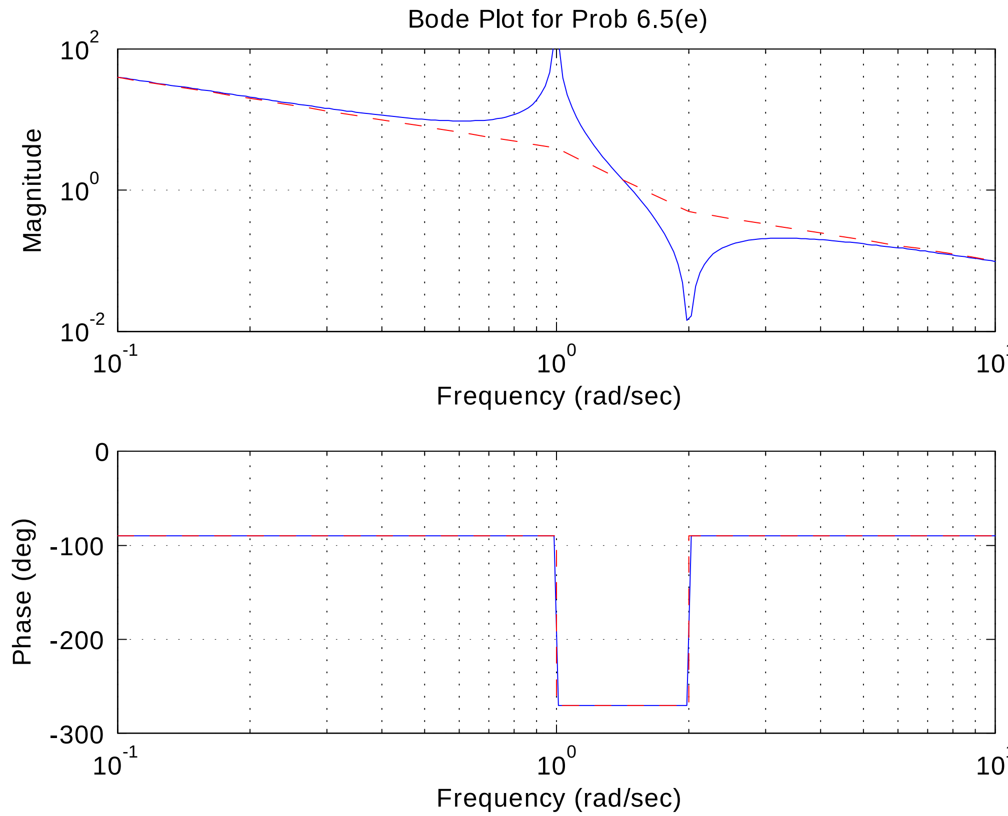
(c) $$L(s) = \frac{\frac{4}{5}\left[\left(\frac{s}{2\sqrt{2}}\right)^2 + \frac{1}{4}s + 1\right]}{s\left[\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{1}{5}s + 1\right]}$$



(d) $L(s) = \dfrac{\frac{1}{4}(s^2+1)}{s\left[\left(\frac{s}{2}\right)^2+1\right]}$



(e) $L(s) = \dfrac{4\left[\left(\frac{s}{2}\right)^2 + 1\right]}{s(s^2+1)}$



6. *Multiple poles at the origin* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2(s+10)}$

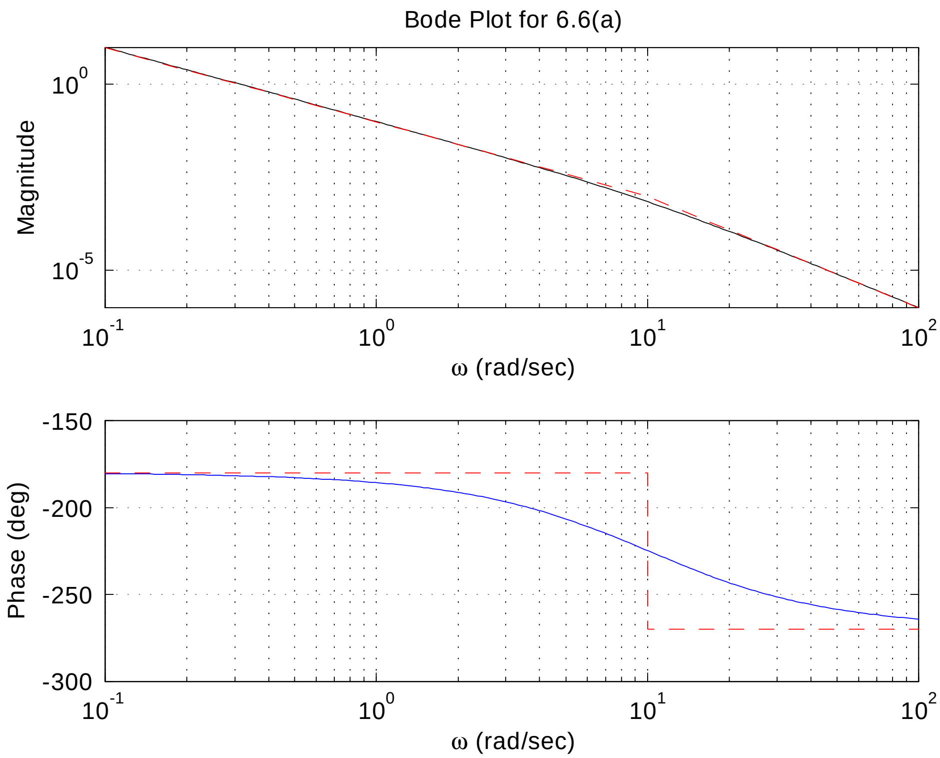
(b) $L(s) = \dfrac{1}{s^3(s+8)}$

(c) $L(s) = \dfrac{1}{s^4(s+10)}$

(d) $L(s) = \dfrac{(s+3)}{s^2(s+10)}$

(e) $L(s) = \dfrac{(s+3)}{s^3(s+5)}$

(f) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)}$



(g) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)^2}$

Solution:

(a) $L(s) = \dfrac{\frac{1}{10}}{s^2\left(\frac{s}{10}+1\right)}$

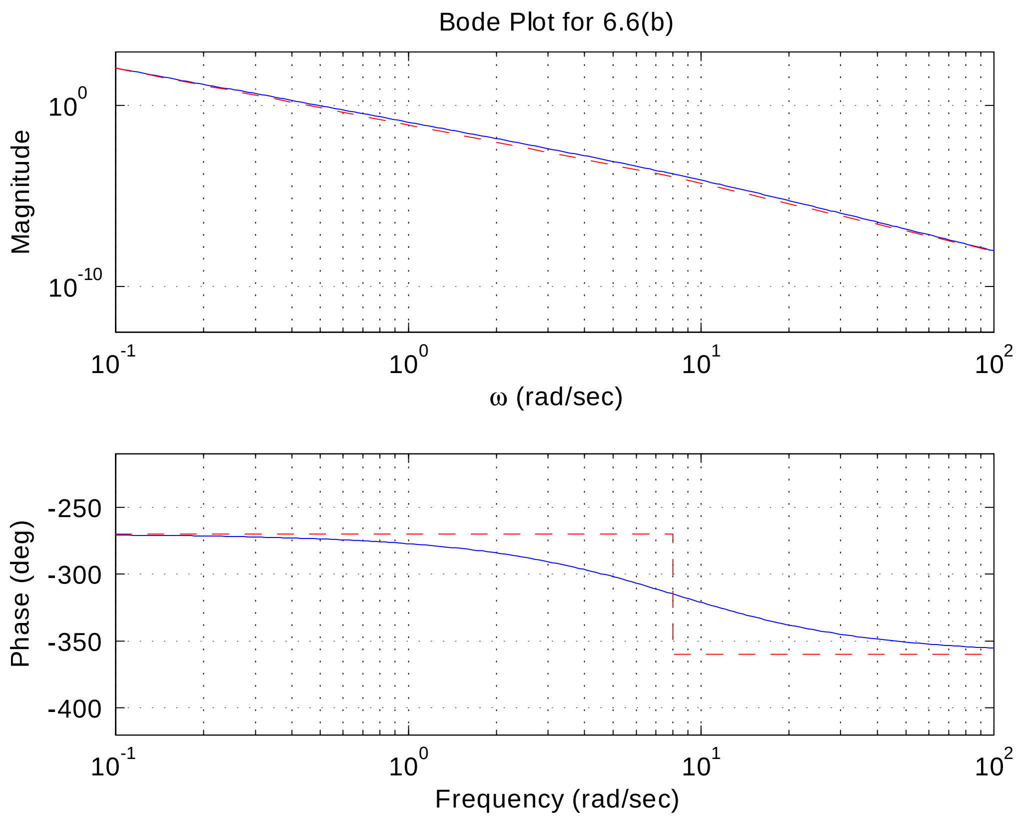
(b) $L(s) = \dfrac{\frac{1}{8}}{s^3\left(\frac{s}{8}+1\right)}$

(c) $L(s) = \dfrac{\frac{1}{10}}{s^4\left(\frac{s}{10}+1\right)}$

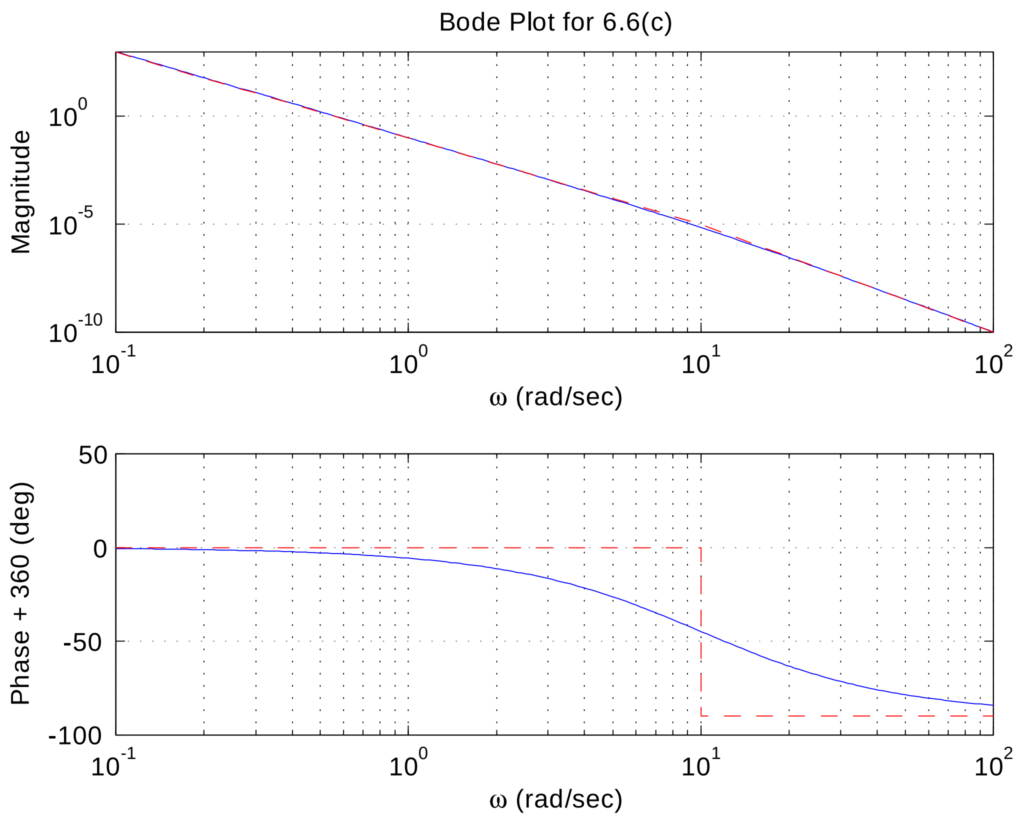
(d) $L(s) = \dfrac{\frac{3}{10}\left(\frac{s}{3}+1\right)}{s^2\left(\frac{s}{10}+1\right)}$

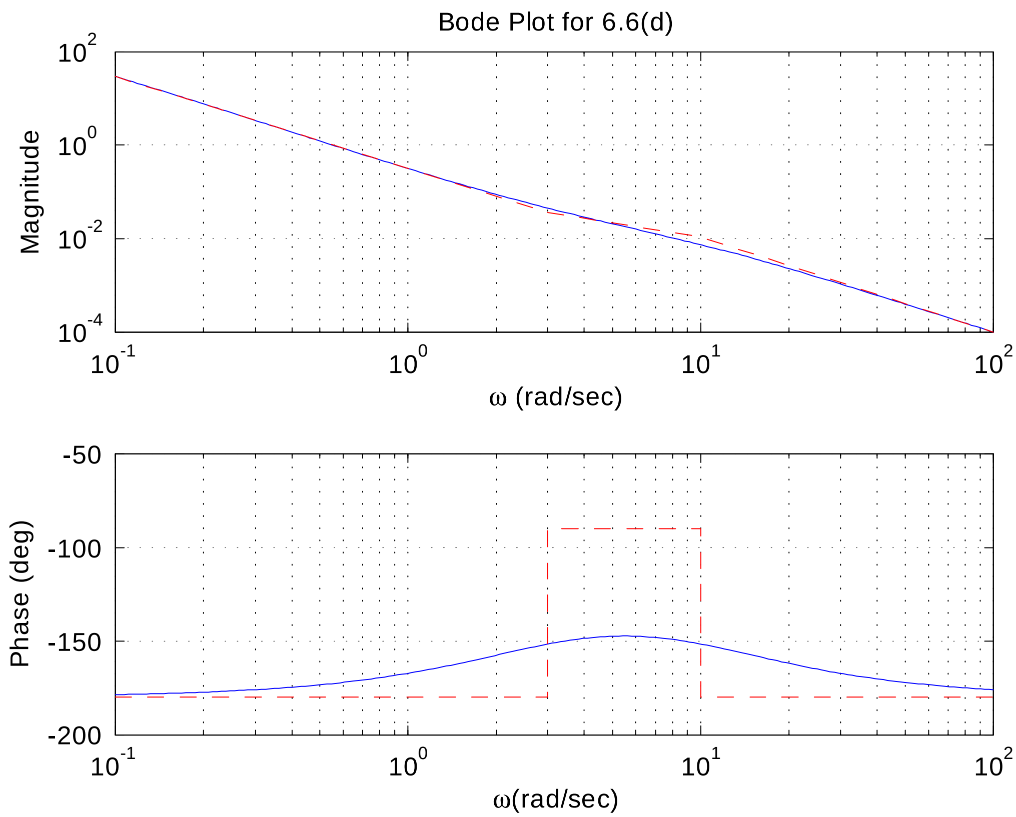
(e) $L(s) = \dfrac{\frac{3}{5}\left(\frac{s}{3}+1\right)}{s^3\left(\frac{s}{5}+1\right)}$

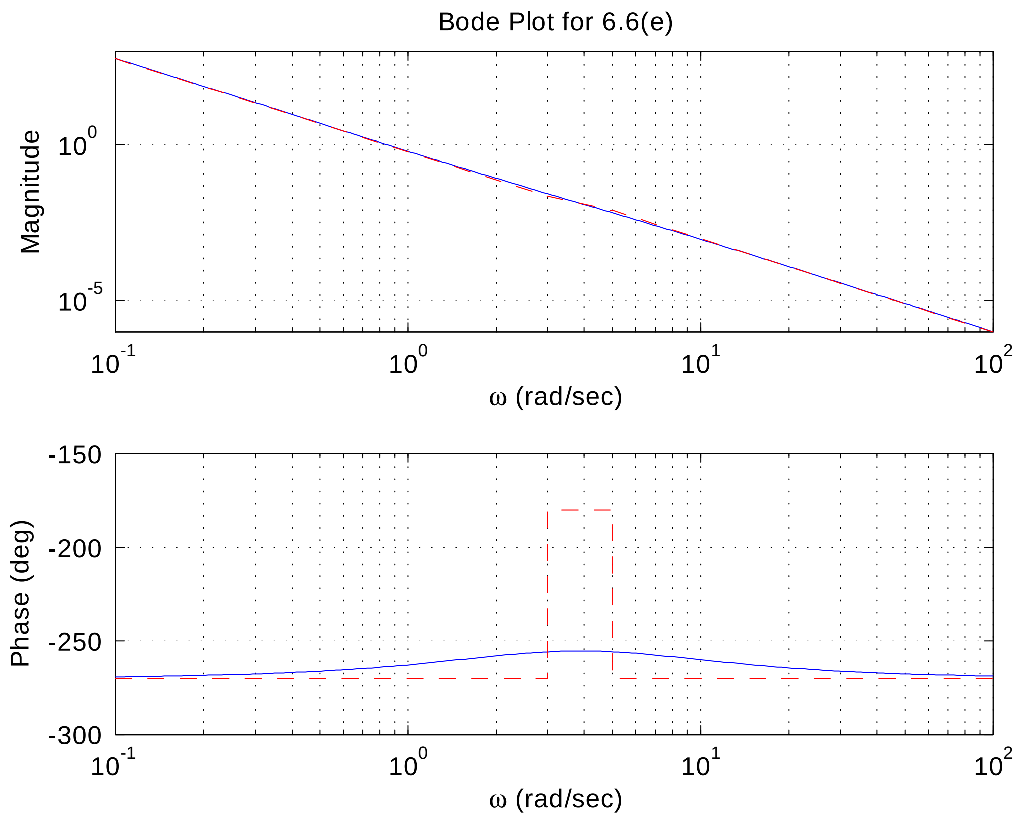
(f) $L(s) = \dfrac{\frac{1}{10}\left(s+1\right)^2}{s^3\left(\frac{s}{10}+1\right)}$

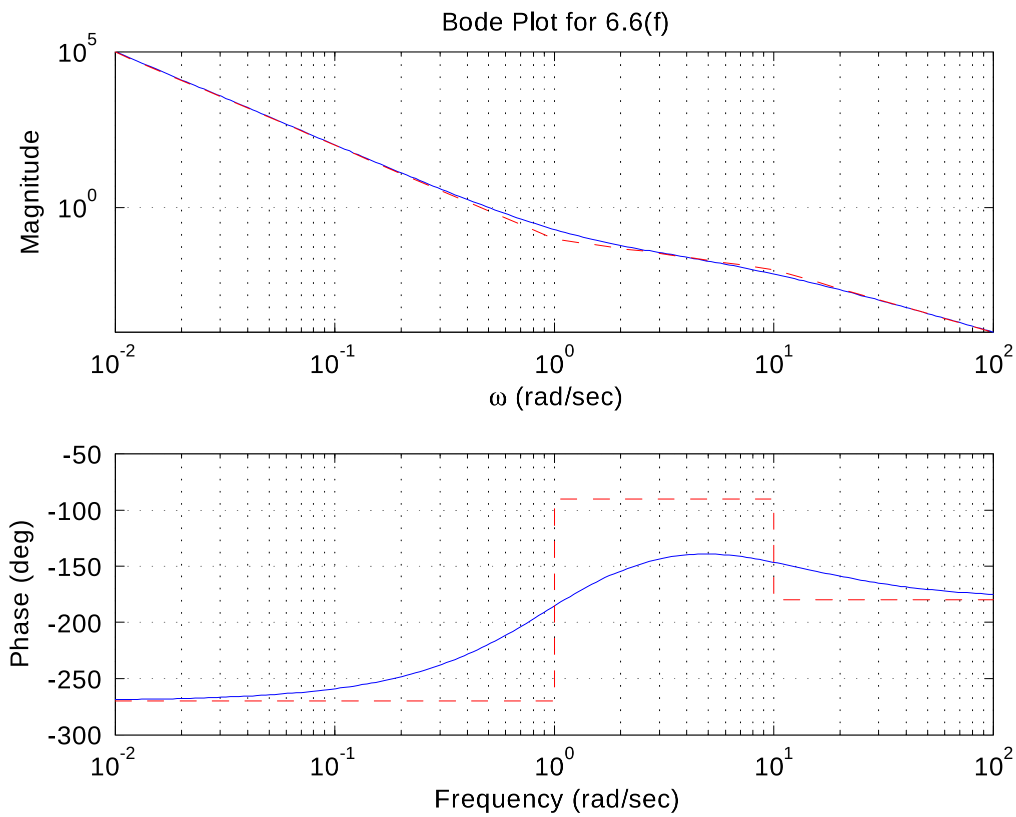


Bode Plot for 6.6(b)
Magnitude
10^{0}
10^{-10}
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
Phase (deg)
-250
-300
-350
-400
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

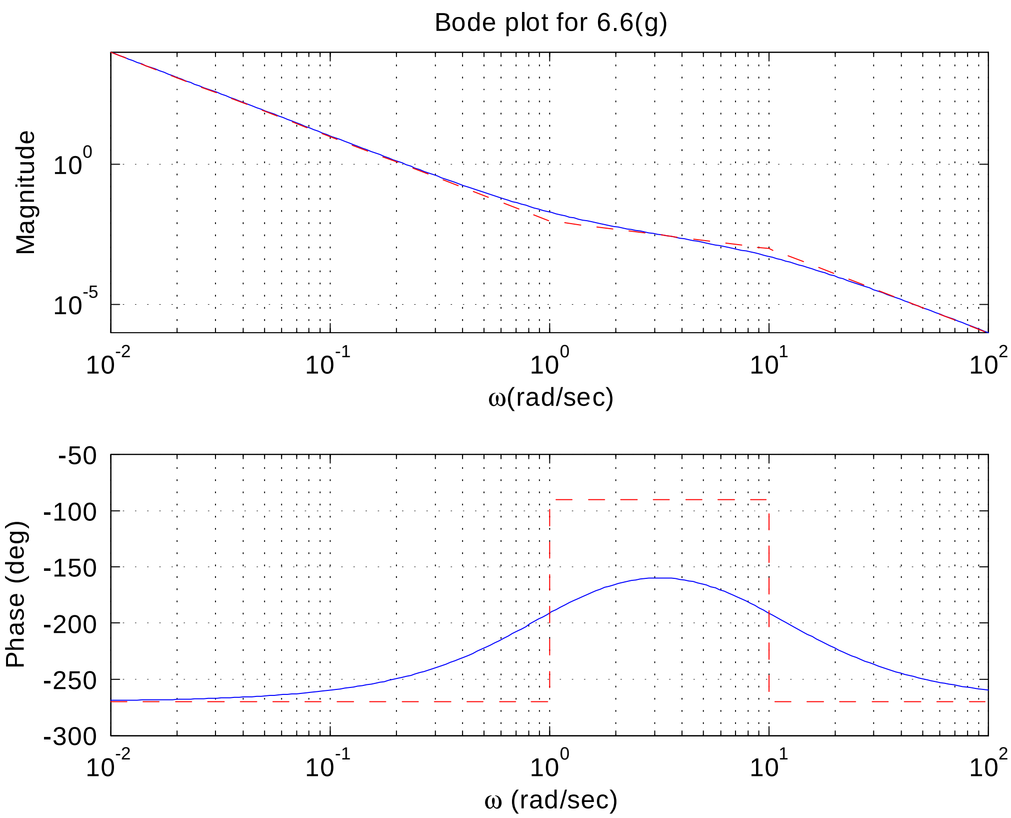

Bode Plot for 6.6(c)
Magnitude
10^{0}
10^{-5}
10^{-10}
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
Phase + 360 (deg)
50
0
-50
-100
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)


Bode Plot for 6.6(d)
Magnitude
10^2
10^0
10^-2
10^-4
10^-1
10^0
10^1
10^2
ω (rad/sec)
Phase (deg)
-50
-100
-150
-200
10^-1
10^0
10^1
10^2
ω(rad/sec)


Bode Plot for 6.6(e)
Magnitude
10^{0}
10^{-5}
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
-150
-200
-250
-300
Phase (deg)
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)



Bode Plot for 6.6(f)
Magnitude
10^5
10^0
10^{-2}
10^{-1}
10^0
10^1
10^2
ω (rad/sec)
Phase (deg)
-50
-100
-150
-200
-250
-300
Frequency (rad/sec)



(g) $L(s) = \dfrac{\frac{1}{100}\,(s+1)^2}{s^3\left(\frac{s}{10}+1\right)^2}$

7. *Mixed real and complex poles* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{(s+2)}{s(s+10)(s^2+2s+2)}$

(b) $L(s) = \dfrac{(s+2)}{s^2(s+10)(s^2+6s+25)}$

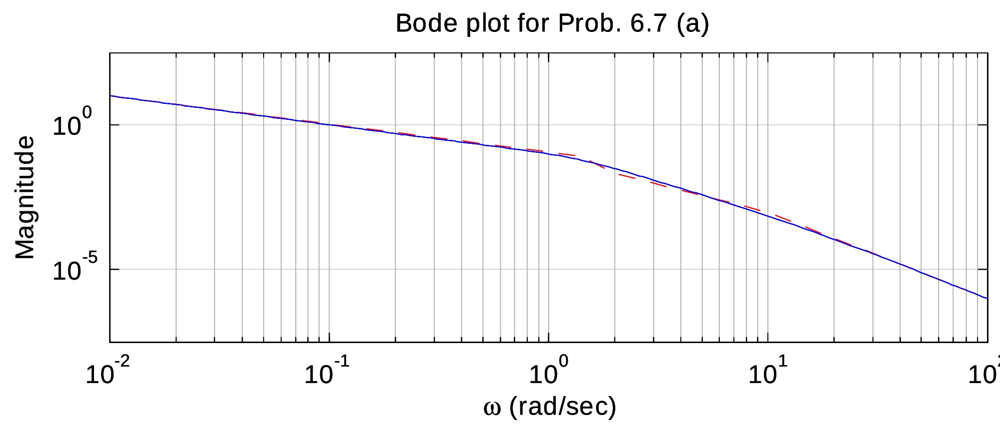
(c) $L(s) = \dfrac{(s+2)^2}{s^2(s+10)(s^2+6s+25)}$

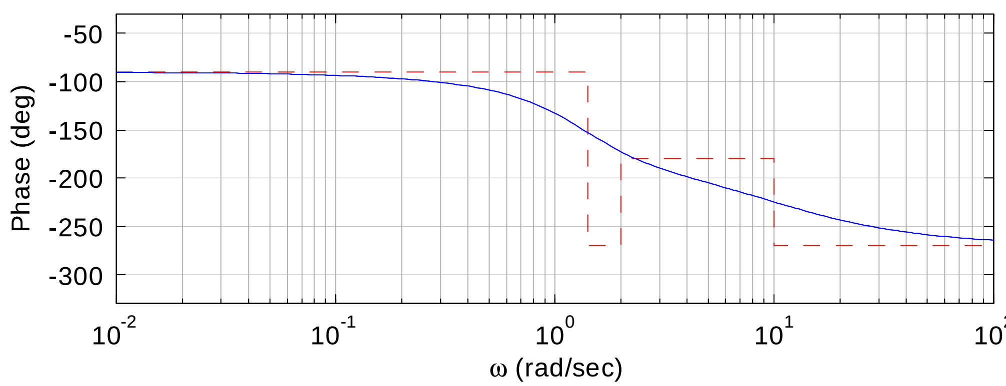
(d) $L(s) = \dfrac{(s+2)(s^2+4s+68)}{s^2(s+10)(s^2+4s+85)}$

(e) $L(s) = \dfrac{[(s+1)^2+1]}{s^2(s+2)(s+3)}$

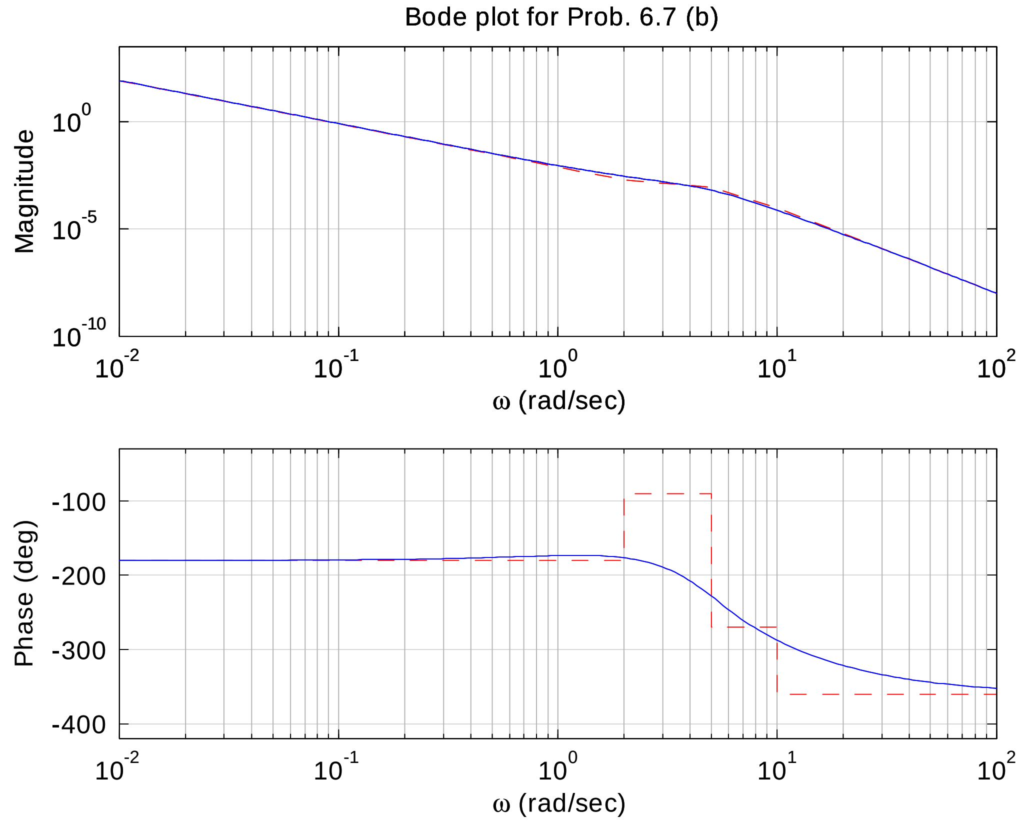
Solution:

(a) $L(s) = \dfrac{\frac{1}{10}\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{\sqrt{2}}\right)^2+s+1\right]}$

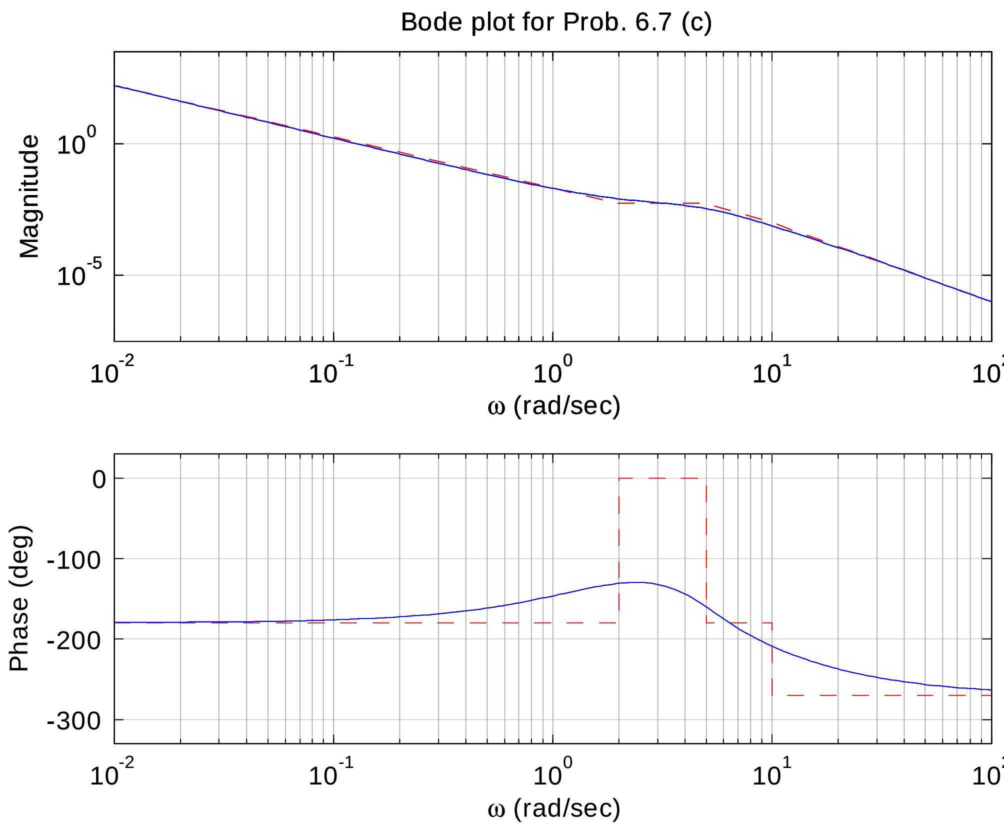




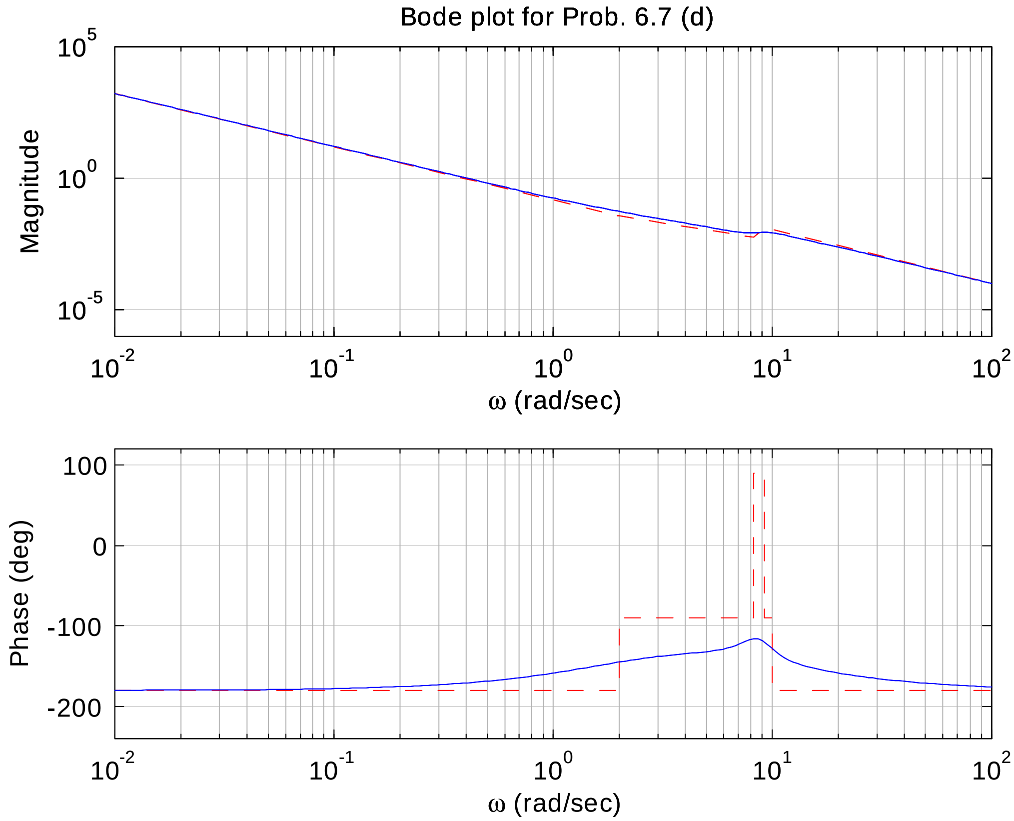
(b) $L(s) = \dfrac{\frac{1}{125}\left(\frac{s}{2}+1\right)}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{5}\right)^2+\frac{6}{25}s+1\right]}$



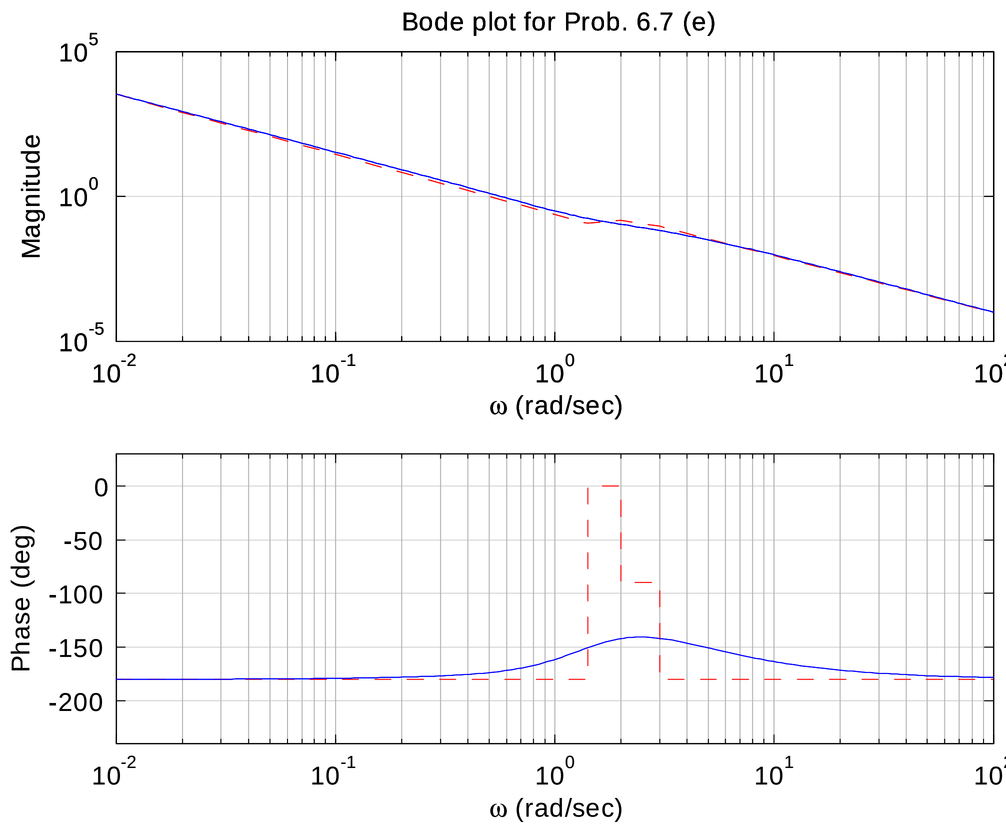
(c) $L(s) = \dfrac{\frac{2}{125}\left(\frac{s}{2}+1\right)^2}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{5}\right)^2+\frac{6}{25}s+1\right]}$



(d) $L(s) = \dfrac{\frac{4}{25}\left(\frac{s}{2}+1\right)\left[\left(\frac{s}{2\sqrt{17}}\right)^2 + \frac{1}{17}s + 1\right]}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{\sqrt{85}}\right)^2 + \frac{4}{85}s + 1\right]}$



(e) $L(s) = \dfrac{\frac{1}{3}\left[\left(\frac{s}{\sqrt{2}}\right)^2 + s + 1\right]}{s^2\left(\frac{s}{2}+1\right)\left(\frac{s}{3}+1\right)}$



8. *Right half plane poles and zeros* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. Make sure the phase asymptotes properly take the RHP singularity into account by sketching the complex plane to see how the $\angle L(s)$ changes as s goes from 0 to $+j\infty$. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{s+2}{s+10}\dfrac{1}{s^2-4}$; The model for a case of magnetic levitation with lead compensation.

(b) $L(s) = \dfrac{s+2}{s(s+10)}\dfrac{1}{(s^2-1)}$; The magnetic levitation system with integral control and lead compensation.

(c) $L(s) = \dfrac{s-1}{s^2}$

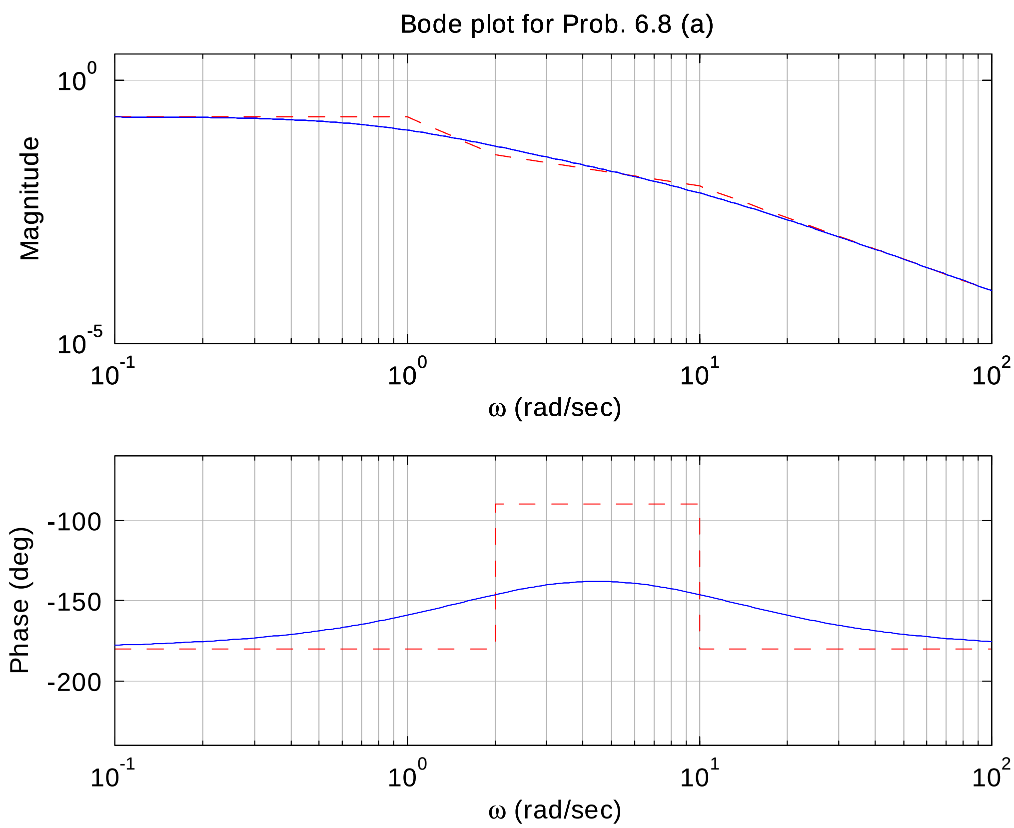
(d) $L(s) = \dfrac{s^2+2s+1}{s(s+20)^2(s^2-2s+2)}$

(e) $L(s) = \dfrac{(s+2)}{s(s-1)(s+6)^2}$

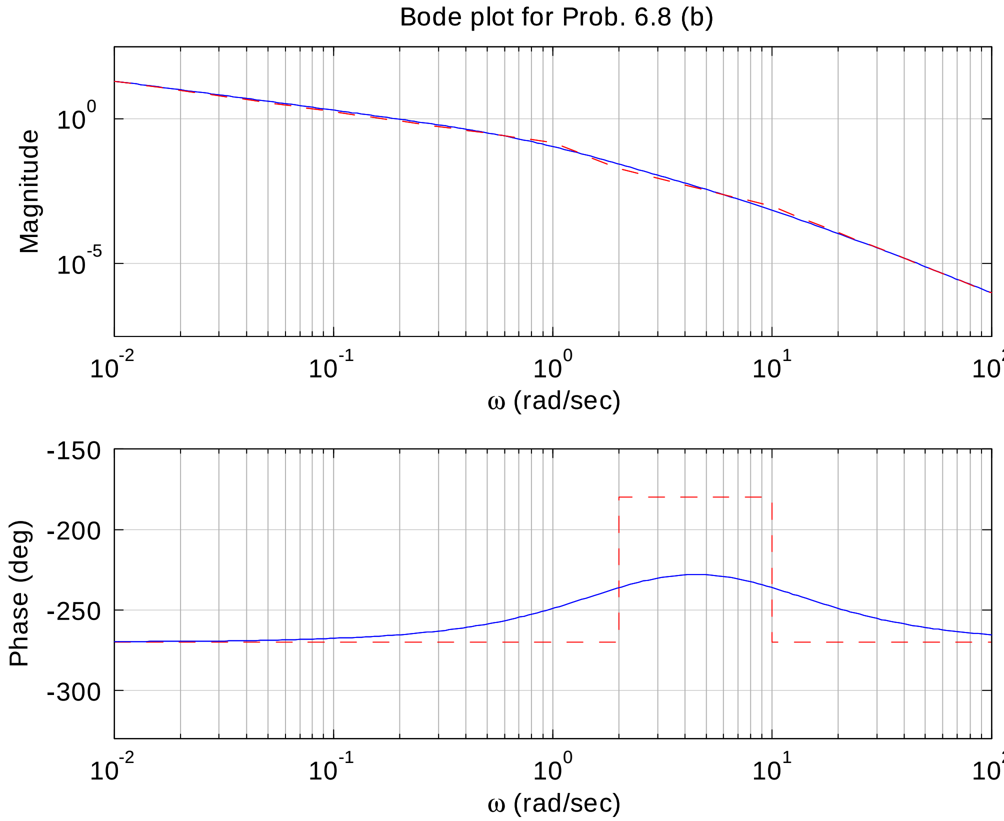
(f) $L(s) = \dfrac{1}{(s-1)[(s+2)^2+3]}$

Solution:

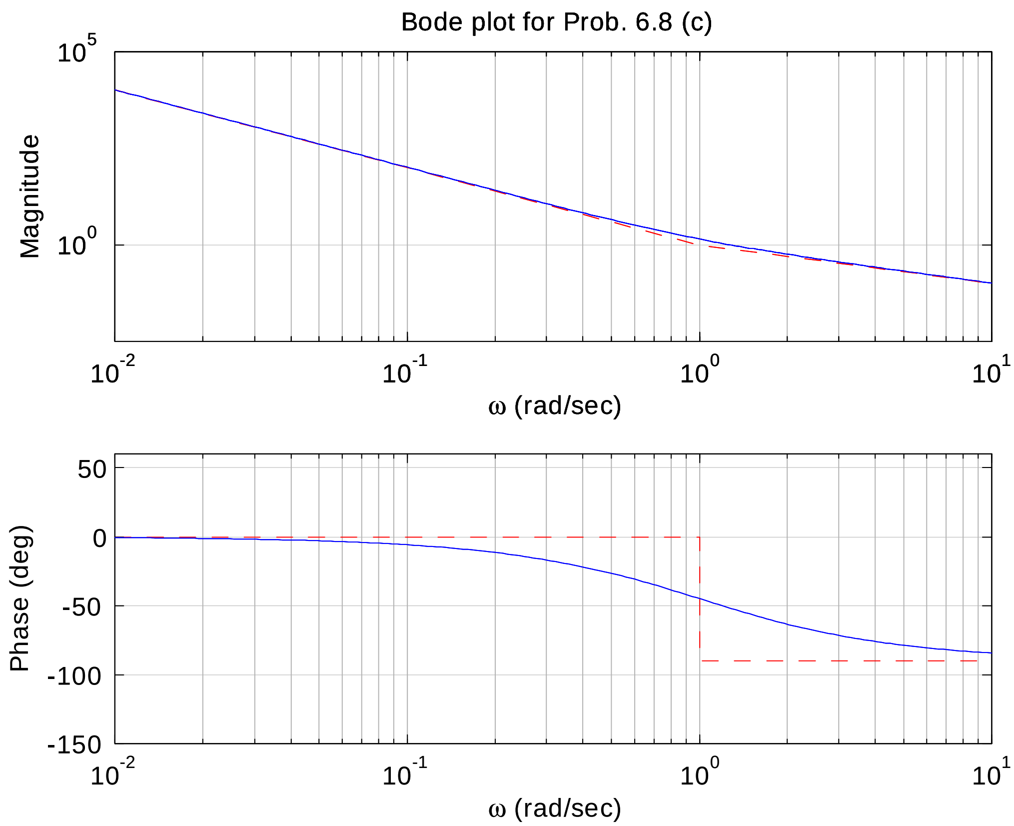
(a) $L(s) = \dfrac{\frac{1}{20}\left(\frac{s}{2}+1\right)}{\left(\frac{s}{10}+1\right)} \dfrac{1}{[(\frac{s}{2})^2-1]}$



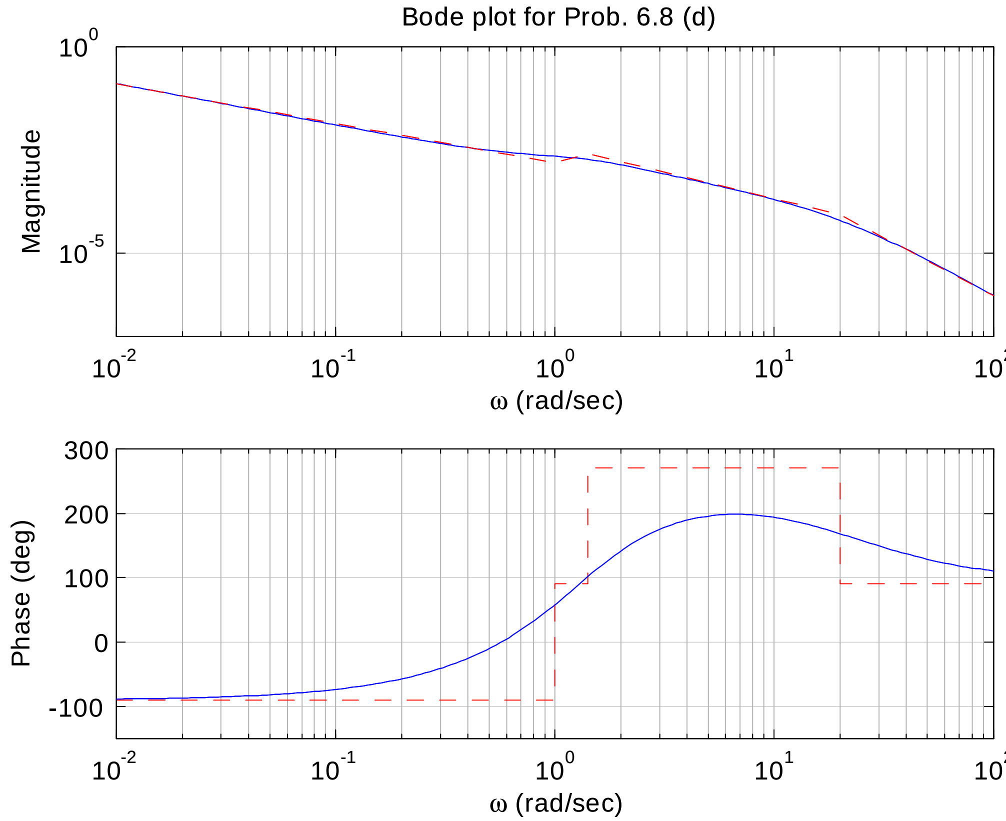
(b) $L(s) = \dfrac{\frac{1}{5}\left(\frac{s}{2}+1\right)}{s(s+10)} \dfrac{1}{s^2-1}$



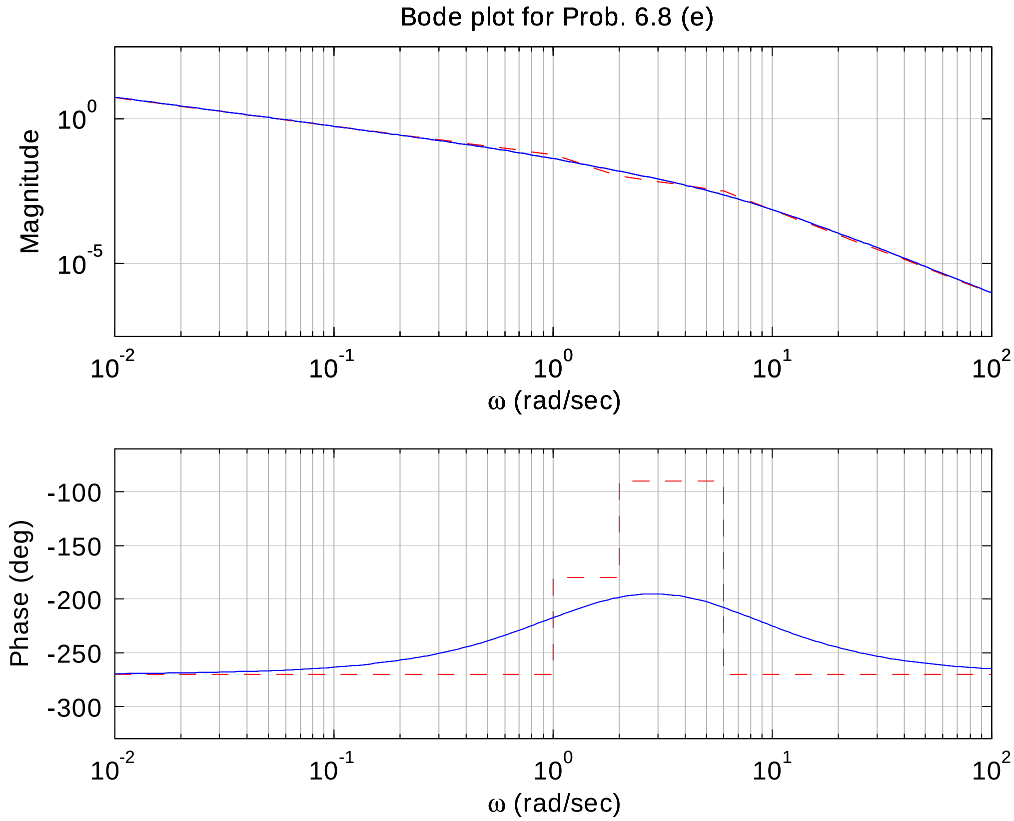
(c) $L(s) = \dfrac{s-1}{s^2}$



(d) $L(s) = \dfrac{\frac{1}{40}(s^2 + 2s + 1)}{s\left(\frac{s}{20} + 1\right)^2 \left[\left(\frac{s}{\sqrt{2}}\right)^2 - s + 1\right]}$

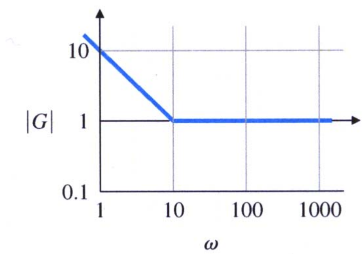


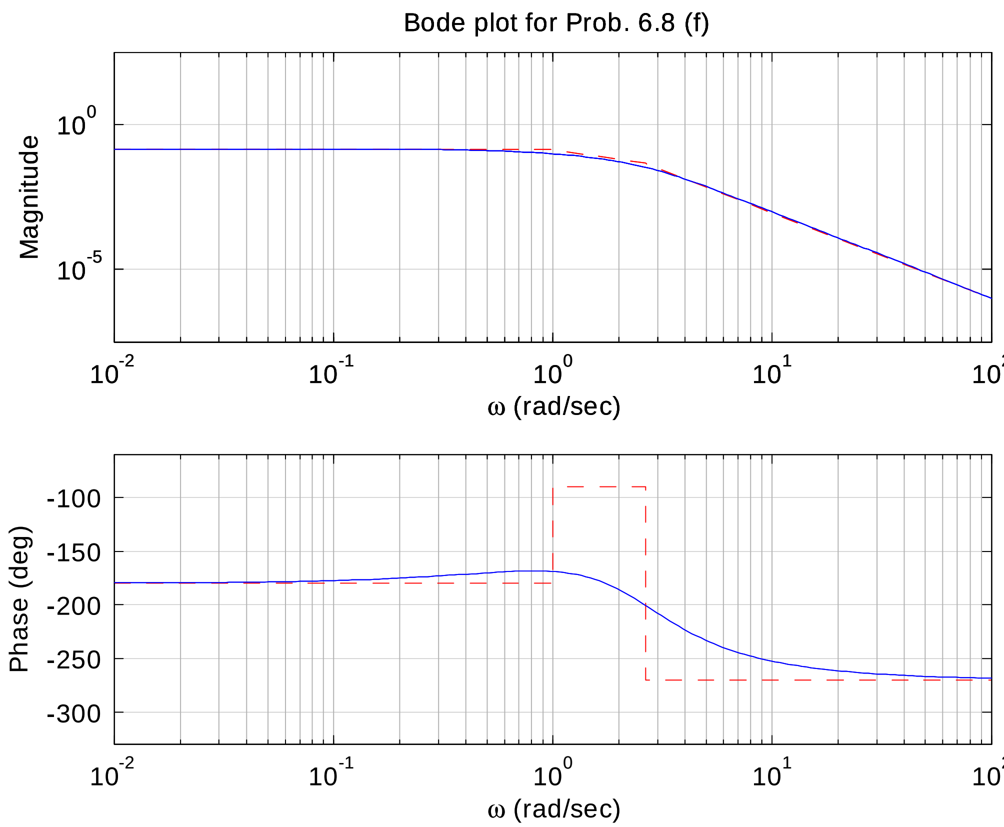
(e) $L(s) = \dfrac{\frac{1}{18}\left(\frac{s}{2}+1\right)}{s(s-1)\left(\frac{s}{6}+1\right)^2}$



(f) $L(s) = \dfrac{\frac{1}{7}}{(s-1)\left[\left(\frac{s}{\sqrt{7}}\right)^2 + \frac{4}{7}s + 1\right]}$

Figure 6.85: Magnitude portion of Bode plot for Problem 9





9. A certain system is represented by the asymptotic Bode diagram shown in Fig. 6.85. Find and sketch the response of this system to a unit step input (assuming zero initial conditions).

Solution:

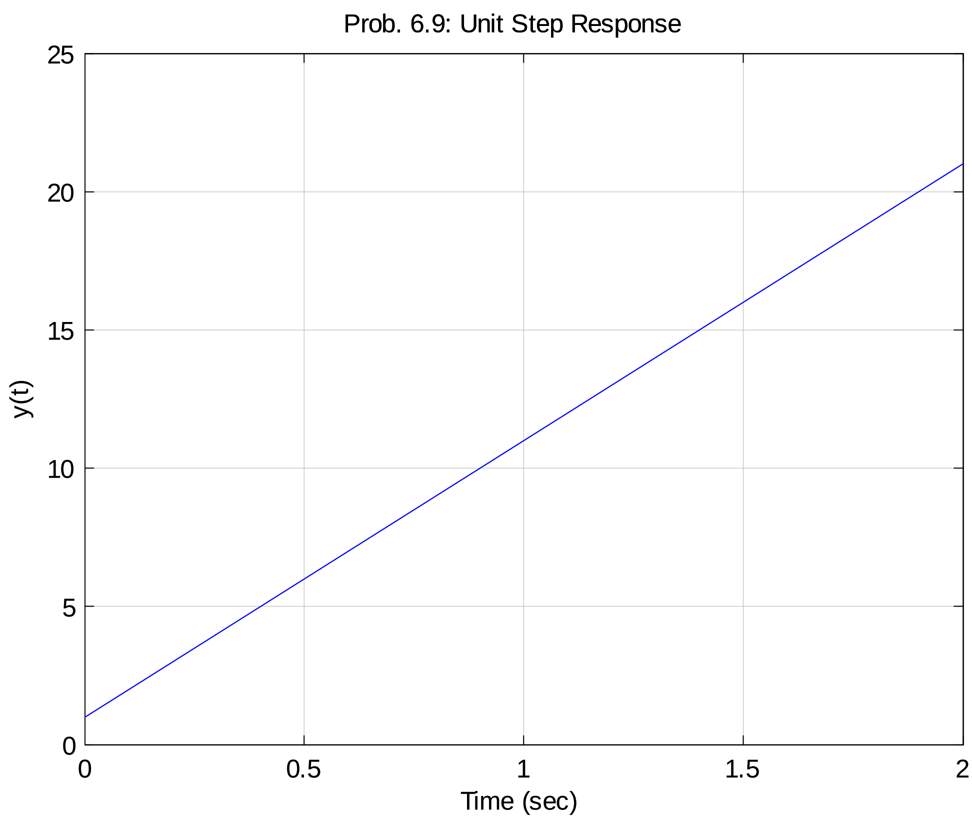
By inspection, the given asymptotic Bode plot is from

Therefore,

$$G(s) = \frac{10(s/10+1)}{s} = \frac{s+10}{s}$$

The response to a unit step input is :

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= \frac{s+10}{s} \times \frac{1}{s} = \frac{1}{s} + \frac{10}{s^2} \\ y(t) &= \mathcal{L}^{-1}\left[Y(s)\right] \\ &= 1(t) + 10t \;\; (t \geq 0) \end{aligned}$$



10. Prove that a magnitude slope of -1 in a Bode plot corresponds to -20 db per decade or -6 db per octave.

Solution:

The definition of db is db $= 20\log|G|$ (1)

Assume slope $= \frac{d(\log|G|)}{d(\log\omega)} = -1$ (2)

$(2) \Longrightarrow \log|G| = -\log\omega + c$ (c is a constant.) (3)

(1) and $(3) \Longrightarrow \text{db} = -20\log\omega + 20c$

Differentiating this,

$$\frac{d\,(\text{db})}{d\,(\log\omega)} = -20$$

Thus, a magnitude slope of -1 corresponds to -20 db per decade.

Similarly,

$$\frac{d\,(\text{db})}{d\,(\log_2\omega)} = \frac{d\,(\text{db})}{d\left(\frac{\log\omega}{\log 2}\right)} \doteqdot -6$$

Thus, a magnitude slope of -1 corresponds to -6 db per octave.

11. A normalized second-order system with a damping ratio $\zeta = 0.5$ and an additional zero is given by

$$G(s) = \frac{s/a+1}{s^2+s+1}.$$

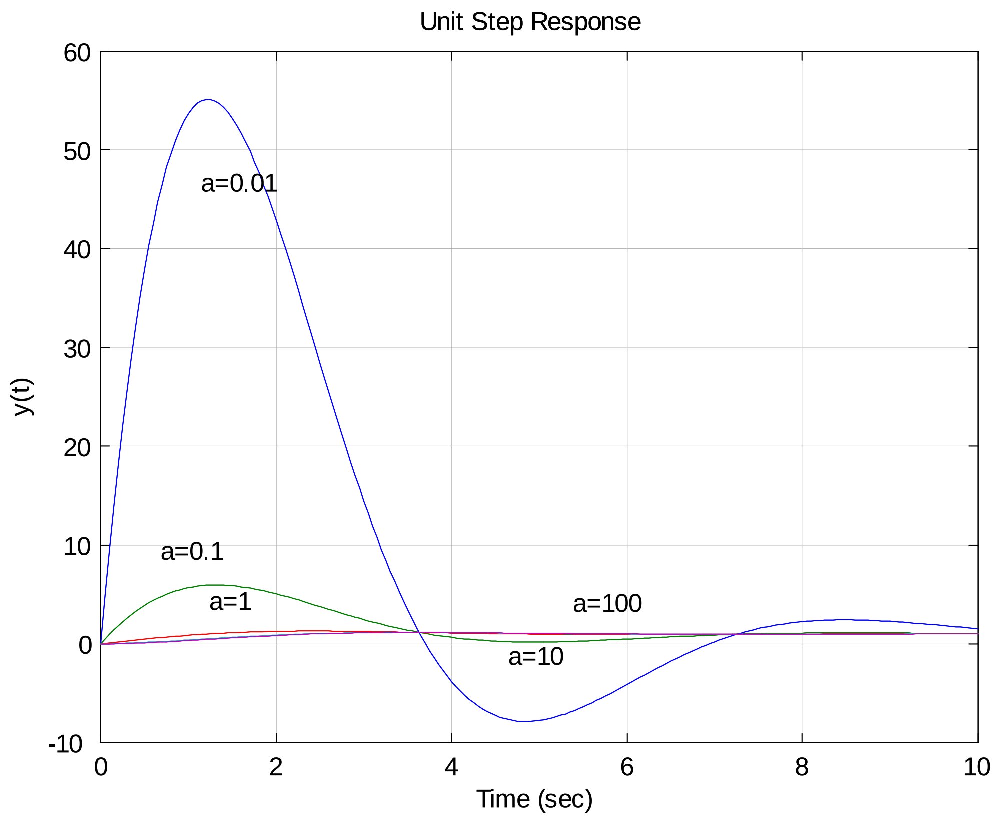
Use MATLAB to compare the M_p from the step response of the system for $a = 0.01, 0.1$, 1, 10, and 100 with the M_r from the frequency response of each case. Is there a correlation between M_r and M_p?

Solution:

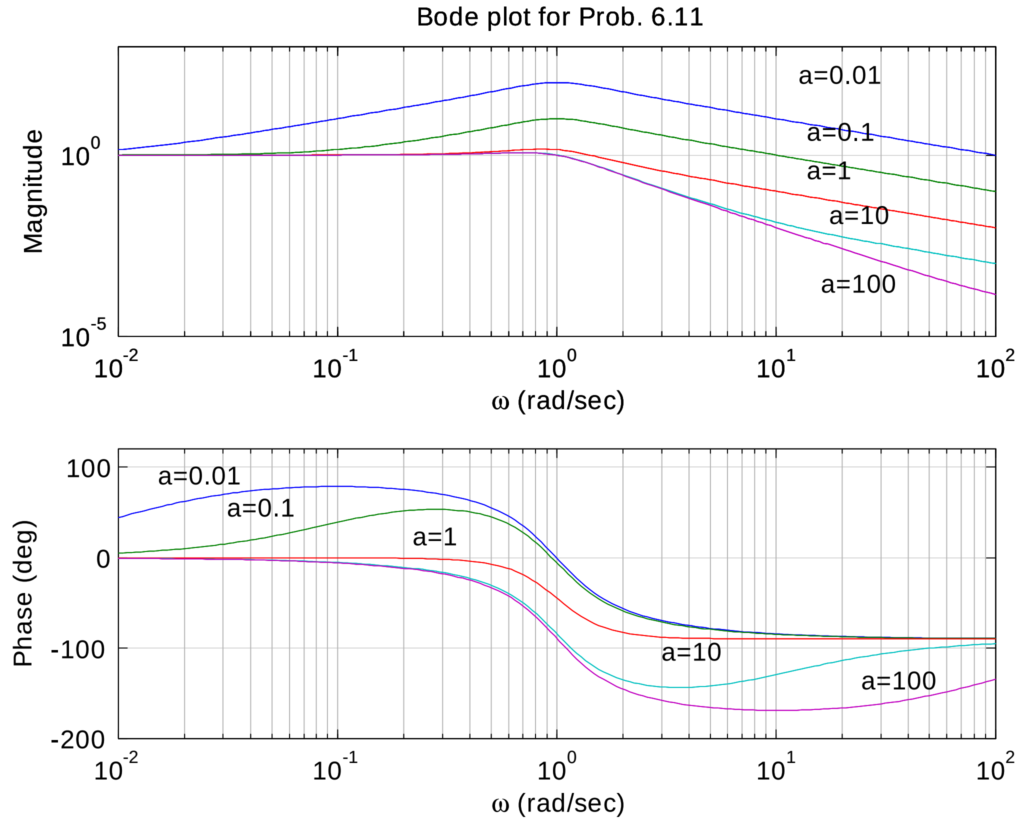
α	Resonant peak, M_r	Overshoot, M_p
0.01	98.8	54.1
0.1	9.93	4.94
1	1.46	0.30
10	1.16	0.16
100	1.15	0.16

As α is reduced, the resonant peak in frequency response increases. This leads us to expect extra peak overshoot in transient response. This effect is significant in case of $\alpha = 0.01, 0.1, 1$, while the resonant peak in frequency response is hardly changed in case of $\alpha = 10$. Thus, we do not have considerable change in peak overshoot in transient response for $\alpha \geq 10$.

The response peak in frequency response and the peak overshoot in transient response are correlated.



Unit Step Response
60
50
40
30
20
10
0
-10
y(t)
0
2
4
6
8
10
Time (sec)
a=0.01
a=0.1
a=1
a=100
a=10



12. A normalized second-order system with $\zeta = 0.5$ and an additional pole is given by.

$$G(s) = \frac{1}{[(s/p)+1](s^2+s+1)}$$

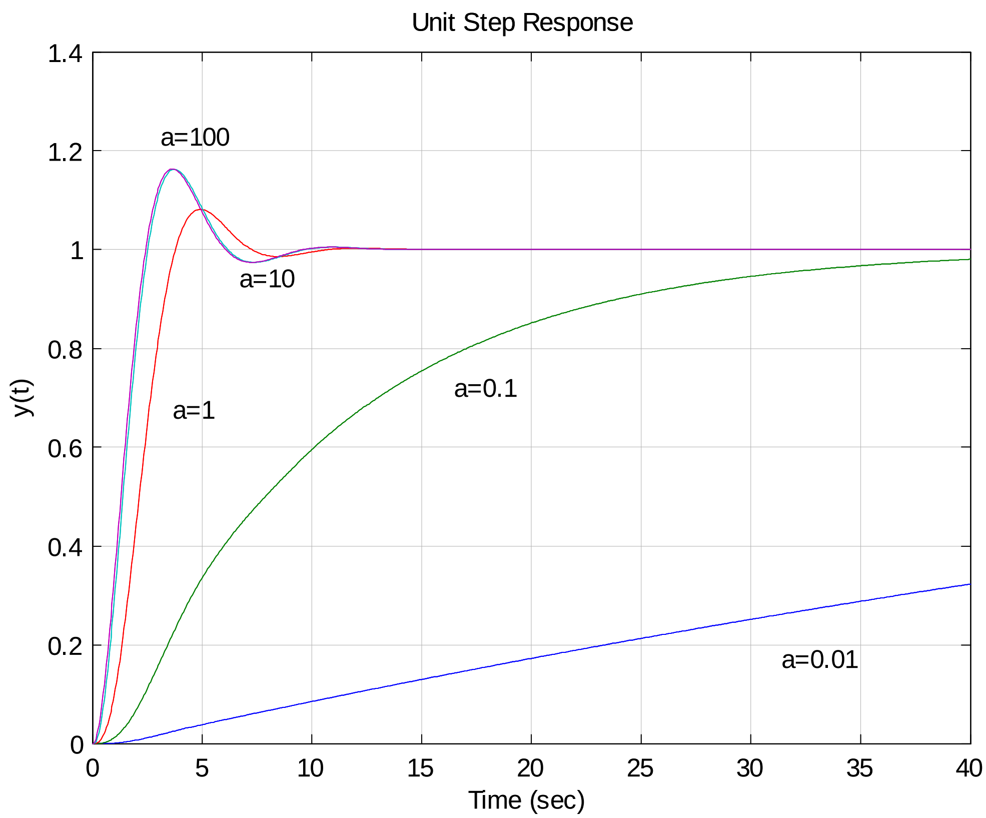
Draw Bode plots with $p = 0.01$, 0.1, 1, 10 and 100. What conclusions can you draw about the effect of an extra pole on the bandwidth compared to the bandwidth for the second-order system with no extra pole?

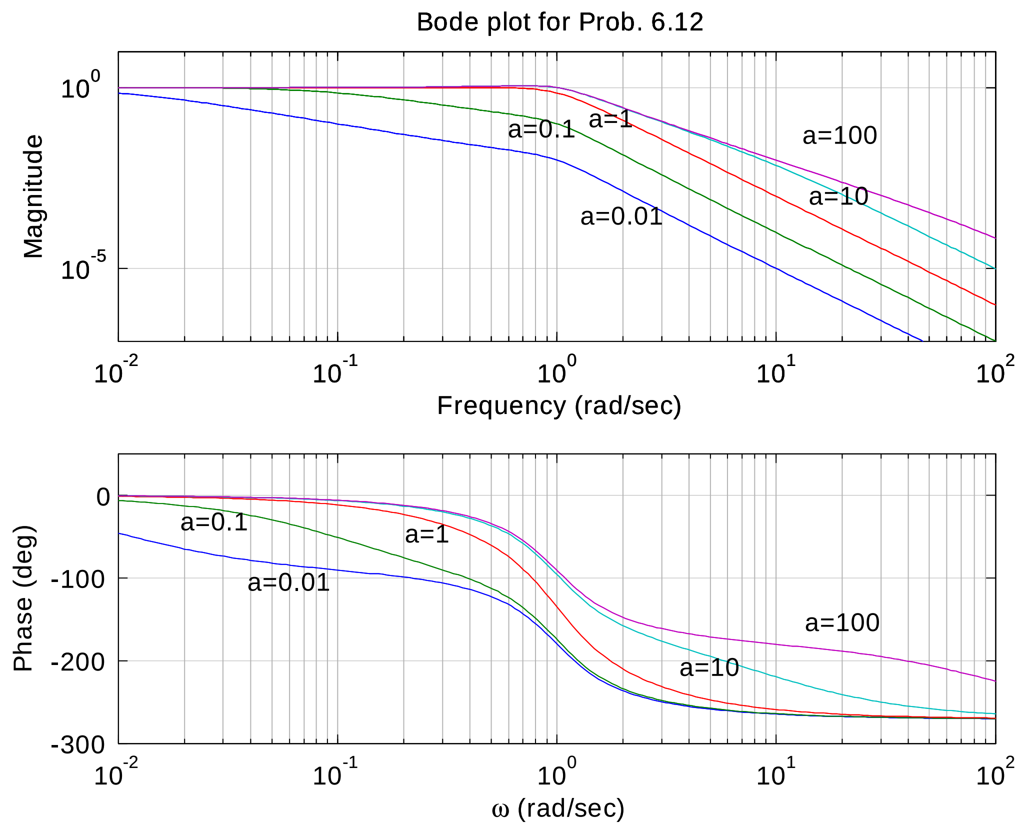
Solution:

p	Additional pole $(-p)$	Bandwidth, ω_{Bw}
0.01	-0.01	0.013
0.1	-0.1	0.11
1	-1	1.0
10	-10	1.5
100	-100	1.7

As p is reduced, the bandwidth decreases. This leads us to expect slower time response and additional rise time. This effect is significant in case of $p = 0.01,\ 0.1, 1$, while the bandwidth is hardly changed in case of $p = 10$. Thus, we do not have considerable change in rise time for $p \geq 10$.

Bandwidth is a measure of the speed of response of a system, such as rise time.





13. For the closed-loop transfer function

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

derive the following expression for the bandwidth ω_{BW} of $T(s)$ in terms of ω_n and ζ:

$$\omega_{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}.$$

Assuming $\omega_n = 1$, plot ω_{BW} for $0 \leq \zeta \leq 1$.

Solution :

The closed-loop transfer function :

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$s = j\omega$,

$$T(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j}$$

$$|T(j\omega)| = \{T(j\omega)T^*(j\omega)\}^{\frac{1}{2}} = \left[\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 \zeta^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}\right]^{\frac{1}{2}}$$

Let $x = \dfrac{\omega_{BW}}{\omega_n}$:

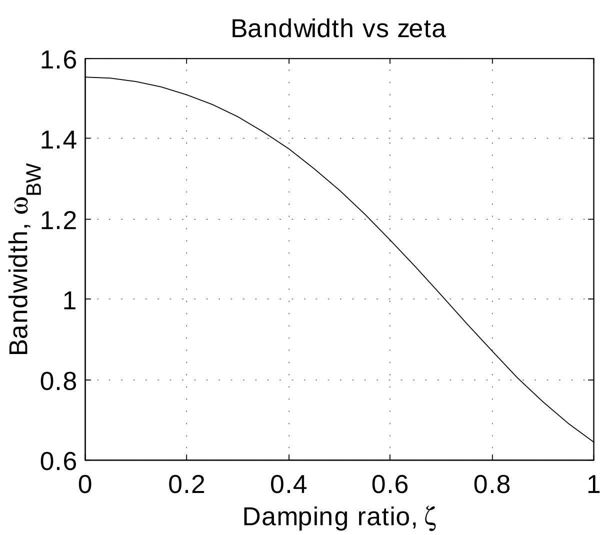
$$|T(j\omega)|_{\omega=\omega_{BW}} = \left[\frac{1}{(1-x^2)^2 + (2\zeta x)^2}\right]^{\frac{1}{2}} = 0.707 = \frac{1}{\sqrt{2}}$$

$$\implies x^4 + (4\zeta^2 - 2)x^2 - 1 = 0$$

$$\implies x = \frac{\omega_{BW}}{\omega_n} = \left[(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2 + 1}\right]^{\frac{1}{2}}$$

$$\implies \omega_{BW} = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}$$

ζ	$x\left(= \dfrac{\omega_{BW}}{\omega_n}\right)$	ω_{BW}
0.2	1.51	$1.51\omega_n$
0.5	1.27	$1.27\omega_n$
0.8	0.87	$0.87\omega_n$



14. Consider the system whose transfer function is

$$G(s) = \frac{A_0\omega_0 s}{Qs^2 + \omega_0 s + \omega_0^2 Q}.$$

This is a model of a tuned circuit with *quality factor* Q. (a) Compute the magnitude and phase of the transfer function analytically, and plot them for $Q = 0.5$, 1, 2, and 5 as a function of the normalized frequency ω/ω_0. (b) Define the bandwidth as the distance between the frequencies on either side of ω_0 where the magnitude drops to 3 db below its value at ω_0 and show that the bandwidth is given by

$$BW = \frac{1}{2\pi}\left(\frac{\omega_0}{Q}\right).$$

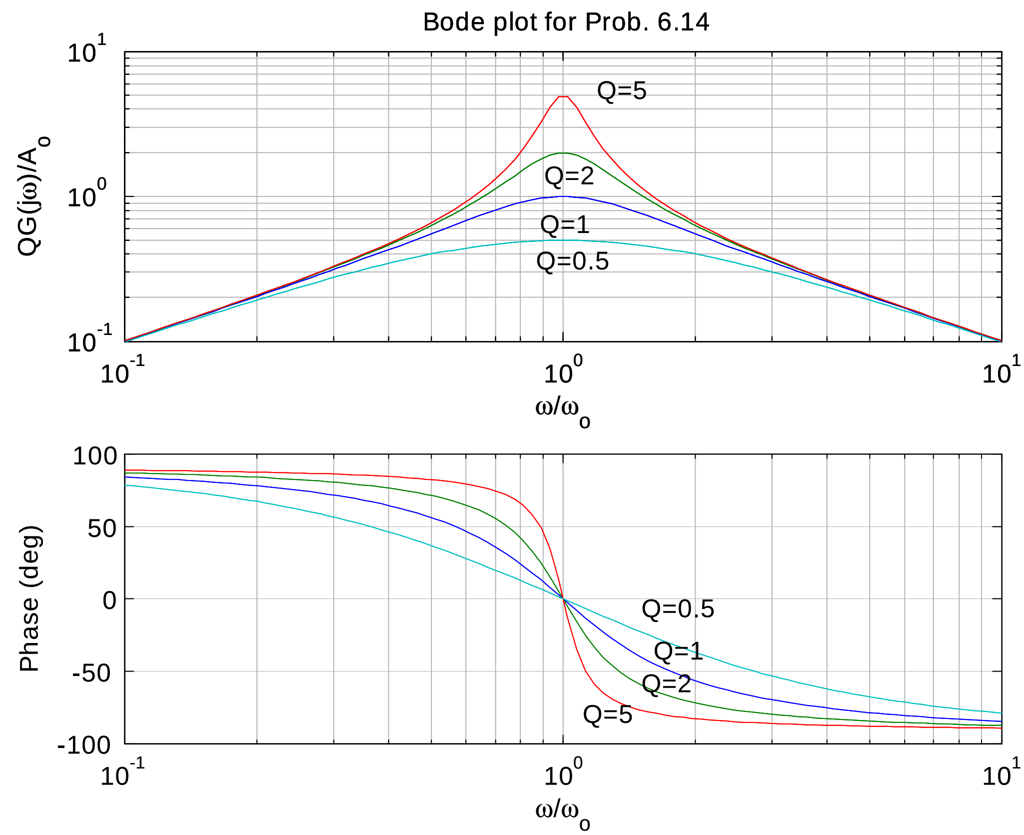
(c) What is the relation between Q and ζ?

Solution :

(a) Let $s = j\omega$,

$$\begin{aligned}
G(j\omega) &= \frac{A_o\omega_o j\omega}{-Q\omega^2 + \omega_o j\omega + \omega_o^2 Q} \\
&= \frac{A_o}{1 + \frac{Q\omega_o^2 - Q\omega^2}{j\omega_o\omega}} \\
|G(j\omega)| &= \frac{A_o}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}} \\
\phi &= -\tan^{-1}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)
\end{aligned}$$

The normalized magnitude $\left(\frac{QG(j\omega)}{A_o}\right)$ and phase are plotted against normalized frequency $\left(\frac{\omega}{\omega_o}\right)$ for different values of Q.



(b) There is symmetry around ω_o. For every frequency $\omega_1 < \omega_o$, there exists a frequency $\omega_2 > \omega_o$ which has the same magnitude

$$|G(j\omega_1)| = |G(j\omega_2)|$$

We have that,

$$\frac{\omega_1}{\omega_o} - \frac{\omega_o}{\omega_1} = -\left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2}\right)$$

which implies $\omega_o^2 = \omega_1\omega_2$. Let $\omega_1 < \omega_o$ and $\omega_2 > \omega_o$ be the two frequencies on either side of ω_o for which the gain drops by 3db from its value of A_o at ω_o.

$$BW = \frac{\omega_2 - \omega_1}{2\pi} = \frac{1}{2\pi}\left(\omega_2 - \frac{\omega_o^2}{\omega_2}\right) \qquad (1)$$

Now ω_2 is found from,

$$\left|\frac{G(j\omega)}{A_o}\right| = \frac{1}{\sqrt{2}}$$

or

$$1 + Q^2 \left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2} \right)^2 = 2$$

which yields

$$Q \left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2} \right) = 1 = \frac{Q}{\omega_o} \left(\omega_2 - \frac{\omega_o^2}{\omega_2} \right) \qquad (2)$$

Comparing (1) and (2) we find,

$$BW = \frac{1}{\sqrt{2}} \left(\frac{Q}{\omega_o} \right)$$

(c)

$$\begin{aligned} G(s) &= \frac{A_0 \omega_0 s}{Qs^2 + \omega_0 s + \omega_0^2 Q} \\ &= \frac{A_0 \omega_0 s}{Q \left(s^2 + \frac{\omega_0}{Q} s + \omega_0^2 \right)} \\ &= \frac{A_0 \omega_0 s}{Q \left(s^2 + 2\zeta \omega_0 s + \omega_0^2 \right)} \end{aligned}$$

Therefore

$$\frac{1}{Q} = 2\zeta$$

15. A DC voltmeter schematic is shown in Fig. 6.86. The pointer is damped so that its maximum overshoot to a step input is 10%.

    (a) What is the undamped natural frequency of the system?

    (b) What is the damped natural frequency of the system?

    (c) Plot the frequency response using MATLAB to determine what input frequency will produce the largest magnitude output?

    (d) Suppose this meter is now used to measure a 1-V AC input with a frequency of 2 rad/sec. What amplitude will the meter indicate after initial transients have died out? What is the phase lag of the output with respect to the input? Use a Bode plot analysis to answer these questions. Use the lsim command in MATLAB to verify your answer in part (d).

    Solution :

    The equation of motion : $I\ddot{\theta} + b\dot{\theta} + k\theta = T = K_m v$, where b is a damping coefficient.

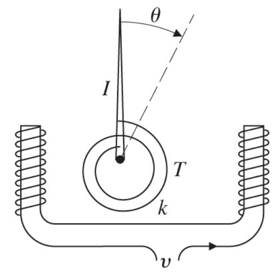


Figure 6.86: Voltmeter schematic

Taking the Laplace transform with zero initial conditions:

$$\Theta(s) = \frac{K_m}{Is^2 + bs + k} V(s) = \frac{\frac{K_m}{I}}{s^2 + 2\zeta\omega_n s + \omega_n^2} V(s)$$

Use $I = 40 \times 10^{-6}$Kg·m^2, $k = 4 \times 10^{-6}$Kg·m^2/s^2, K_m=4×10^{-6}N·m/v

(a) Undamped natural frequency:

$$\omega_n^2 = \frac{k}{I} \Longrightarrow \omega_n = \sqrt{\frac{k}{I}} = 0.316 \text{ rad/sec}$$

(b) Since $M_p = 0.1$ and $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$,

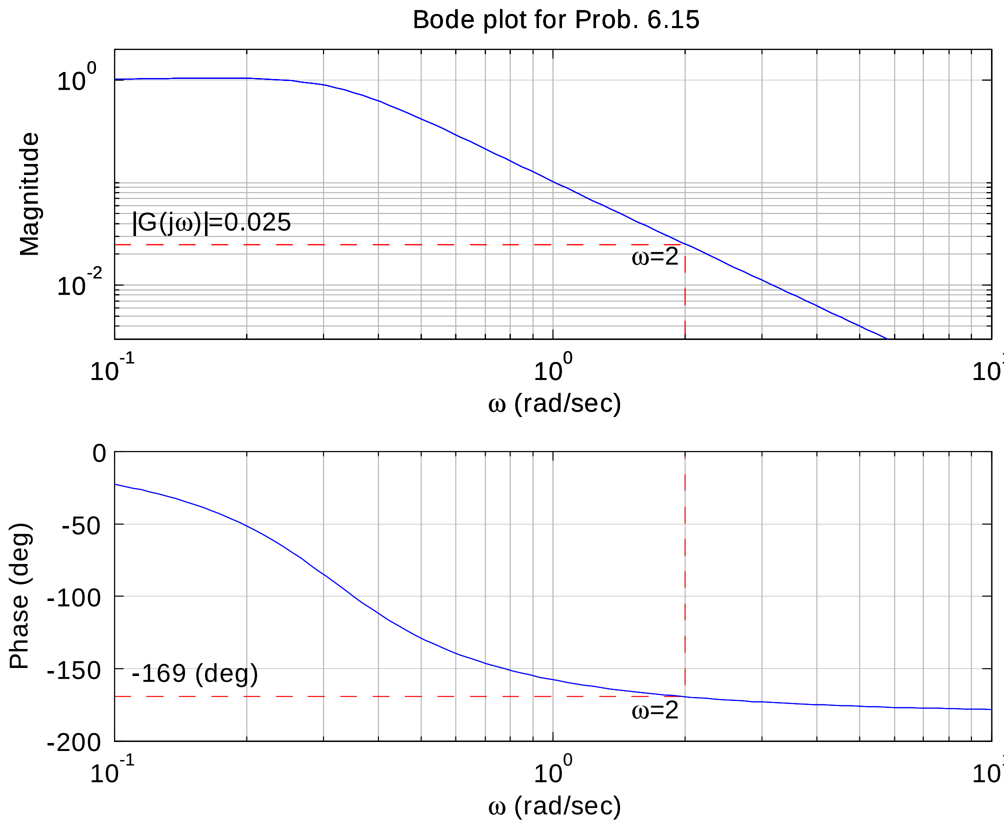
$$\log 0.1 = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \Longrightarrow \zeta = 0.5911 \text{ } (\simeq 0.6 \text{ from Figure 2.44})$$

Damped natural frequency:

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 0.255 \text{ rad/sec}$$

(c)

$$\begin{aligned} T(j\omega) &= \frac{\Theta(j\omega)}{V(j\omega)} = \frac{K_m/I}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} \\ |T(j\omega)| &= \frac{K_m/I}{[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2]^{\frac{1}{2}}} \\ \frac{d\,|T(j\omega)|}{d\omega} &= \left(\frac{K_m}{I}\right) \frac{2\omega\left\{\omega_n^2 - \omega^2 - 2\zeta^2\omega_n^2\right\}}{[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2]^{\frac{3}{2}}} \end{aligned}$$



When $\frac{d|T(j\omega)|}{d\omega} = 0$,

$$\begin{aligned} \omega^2 - (1 - 2\zeta^2)\omega_n^2 &= 0 \\ \omega &= 0.549\omega_n = 0.173 \end{aligned}$$

Alternatively, the peak frequency can be found from the Bode plot:

$$\omega = 0.173 \text{ rad/sec}$$

(d) With $\omega = 2$ rad/sec from the Bode plot:

$$\begin{aligned} \text{Amplitude} &= 0.0252 \text{ rad} \\ \text{Phase} &= -169.1^\circ \end{aligned}$$

Problems and Solutions for Section 6.2

16. Determine the range of K for which the closed-loop systems (see Fig. 6.18) are stable for each of the cases below by making a Bode plot for $K = 1$ and imagining the magnitude plot sliding up or down until instability results. Verify your answers using a very rough sketch of a root-locus plot.

(a) $KG(s) = \dfrac{K(s+3)}{s+30}$

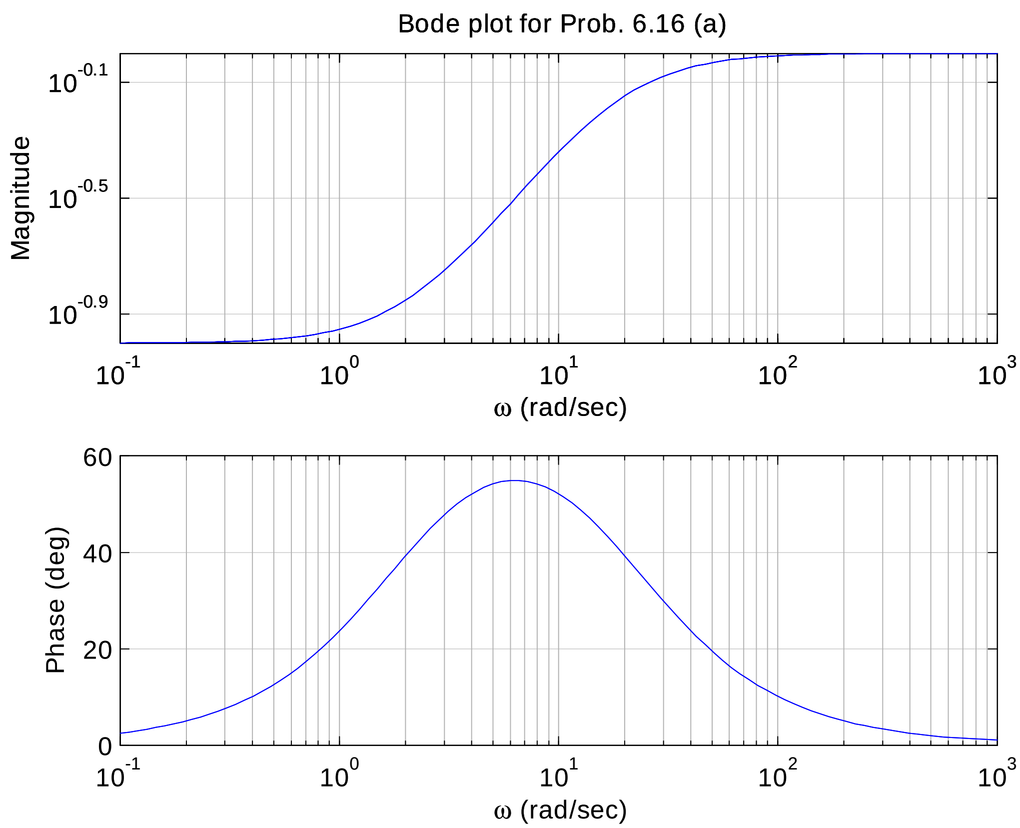
(b) $KG(s) = \dfrac{K}{(s+10)(s+1)^2}$

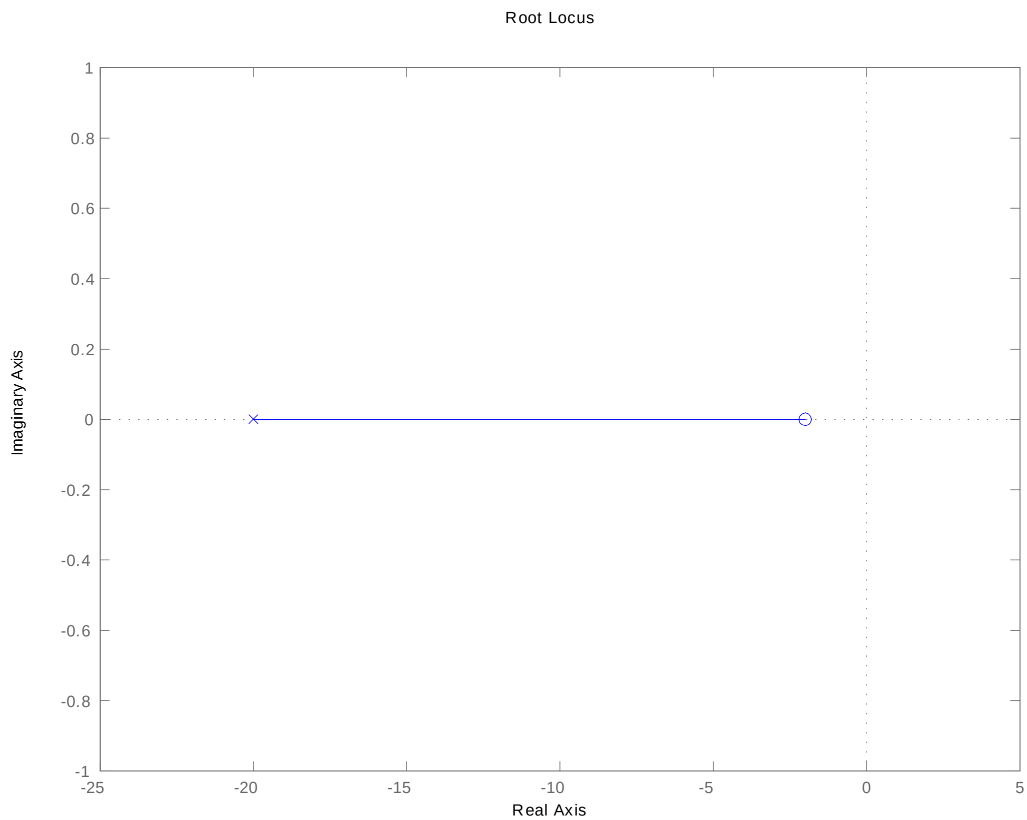
(c) $KG(s) = \dfrac{K(s+10)(s+1)}{(s+100)(s+5)^3}$

Solution :

(a)

$$KG(s) = \frac{K(s+3)}{s+30} = \frac{K}{10}\frac{\left(\frac{s}{3}+1\right)}{\left(\frac{s}{30}+1\right)}$$

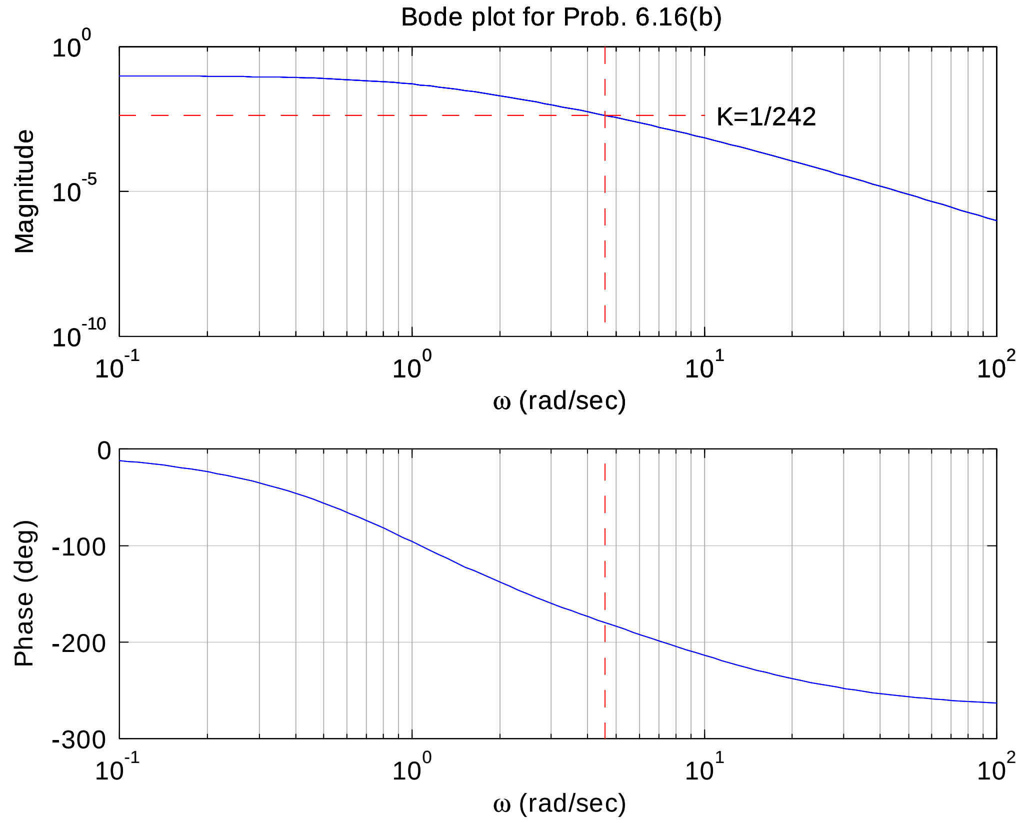




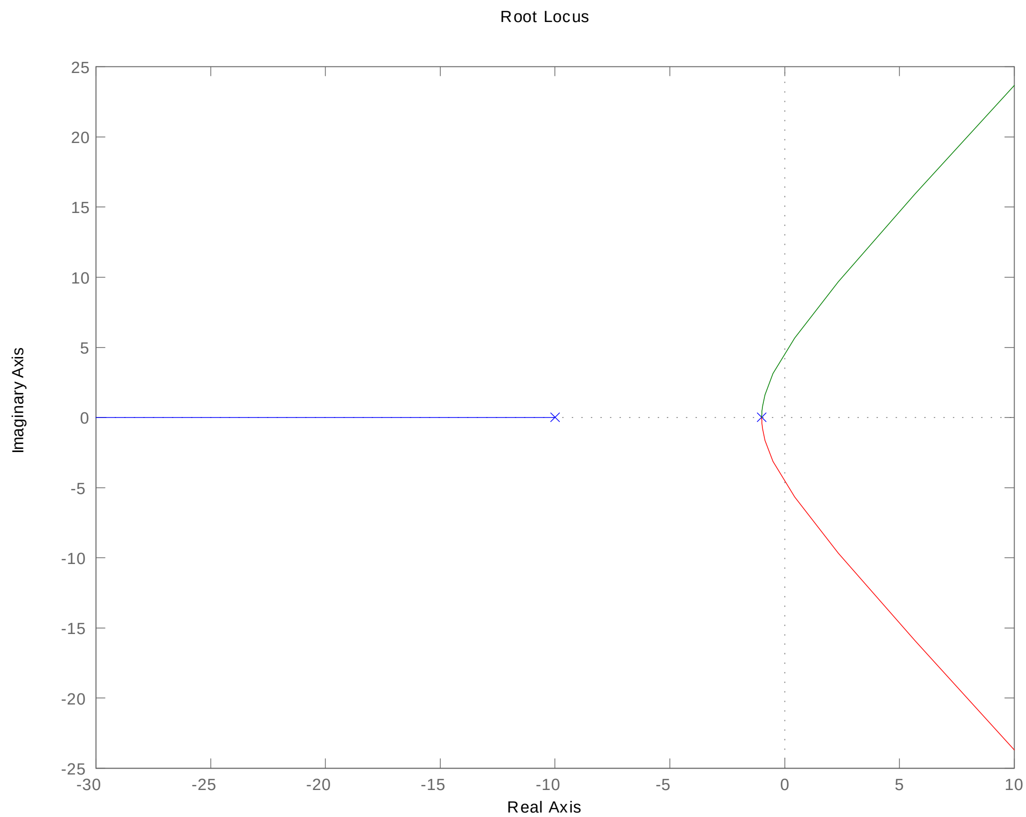
The gain can be raised or lowered on the Bode gain plot and the phase will never be less than -180^o, so the system is stable for any $K > 0$.

(b)

$$KG(s) = \frac{K}{(s+10)(s+1)^2} = \frac{K}{10}\frac{1}{\left(\frac{s}{10}+1\right)(s+1)^2}$$

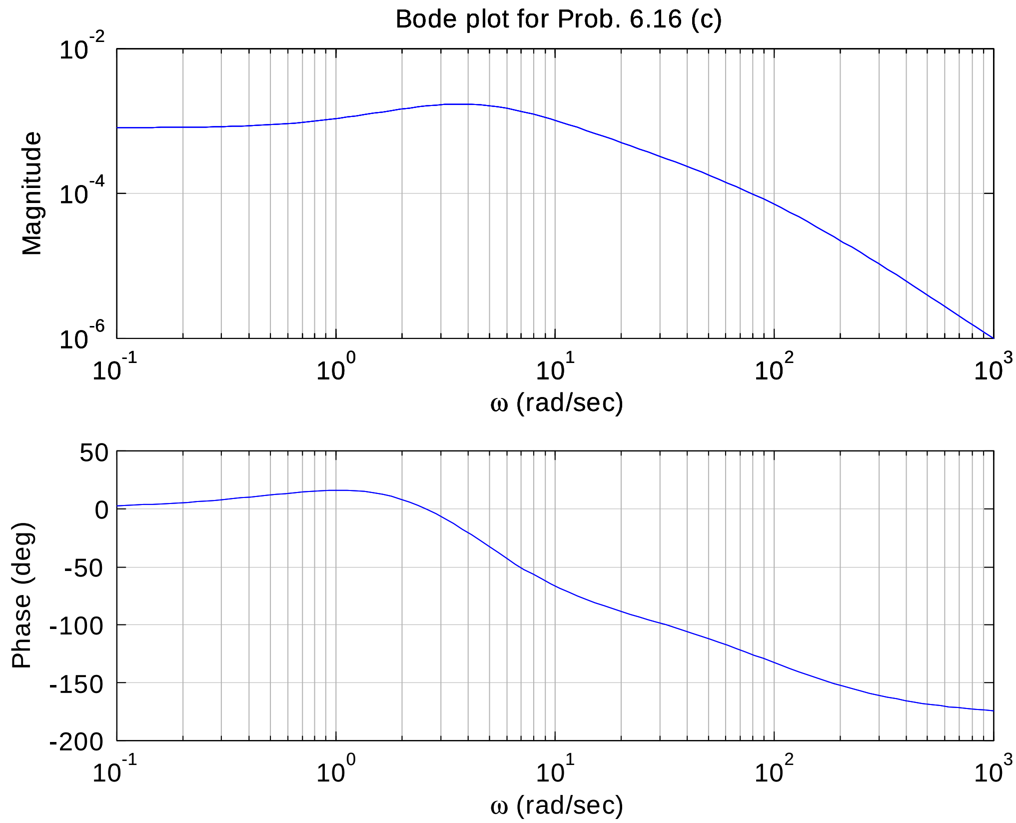


The bode plots show that the gain, K, would equal 242 when the phase crosses 180^o. So, $K < 242$ is Stable and $K > 242$ is Unstable. The phase crosses the 180^o at $\omega = 4.58$ rad/sec. The root locus below verifies the situation.



(c)

$$KG(s) = \frac{K(s+10)(s+1)}{(s+100)(s+5)^3} = \frac{K}{1250}\frac{\left(\frac{s}{10}+1\right)(s+1)}{\left(\frac{s}{100}+1\right)\left(\frac{s}{5}+1\right)^3}$$



The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus.

17. Determine the range of K for which each of the following systems is stable by making a Bode plot for $K = 1$ and imagining the magnitude plot sliding up or down until instability results. Verify your answers using a very rough sketch of a root-locus plot.

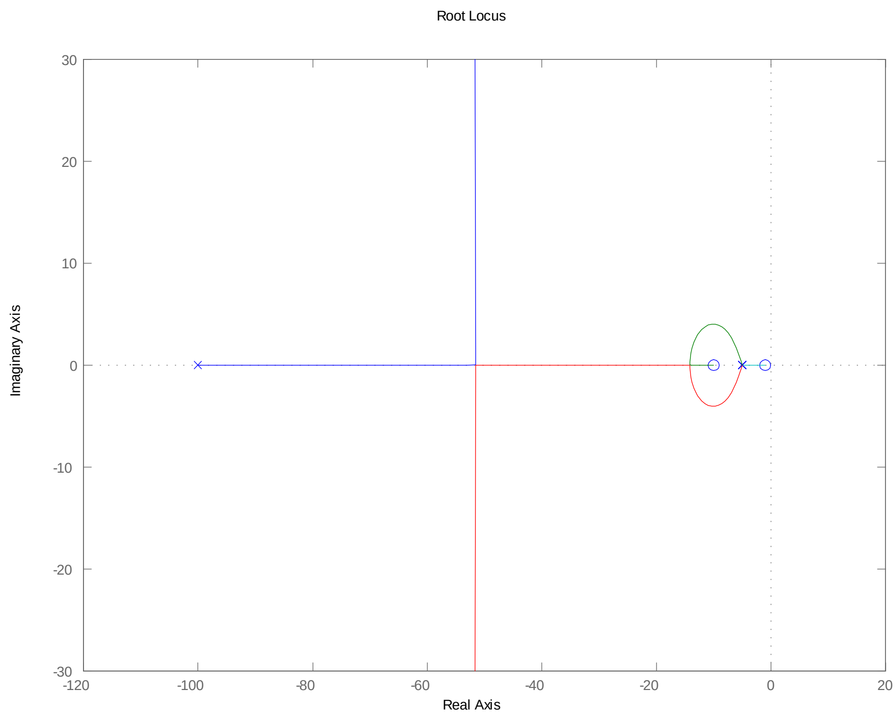
   (a) $KG(s) = \dfrac{K(s+1)}{s(s+10)}$

   (b) $KG(s) = \dfrac{K(s+1)}{s^2(s+10)}$

   (c) $KG(s) = \dfrac{K}{(s+2)(s^2+9)}$

   (d) $KG(s) = \dfrac{K(s+1)^2}{s^3(s+10)}$

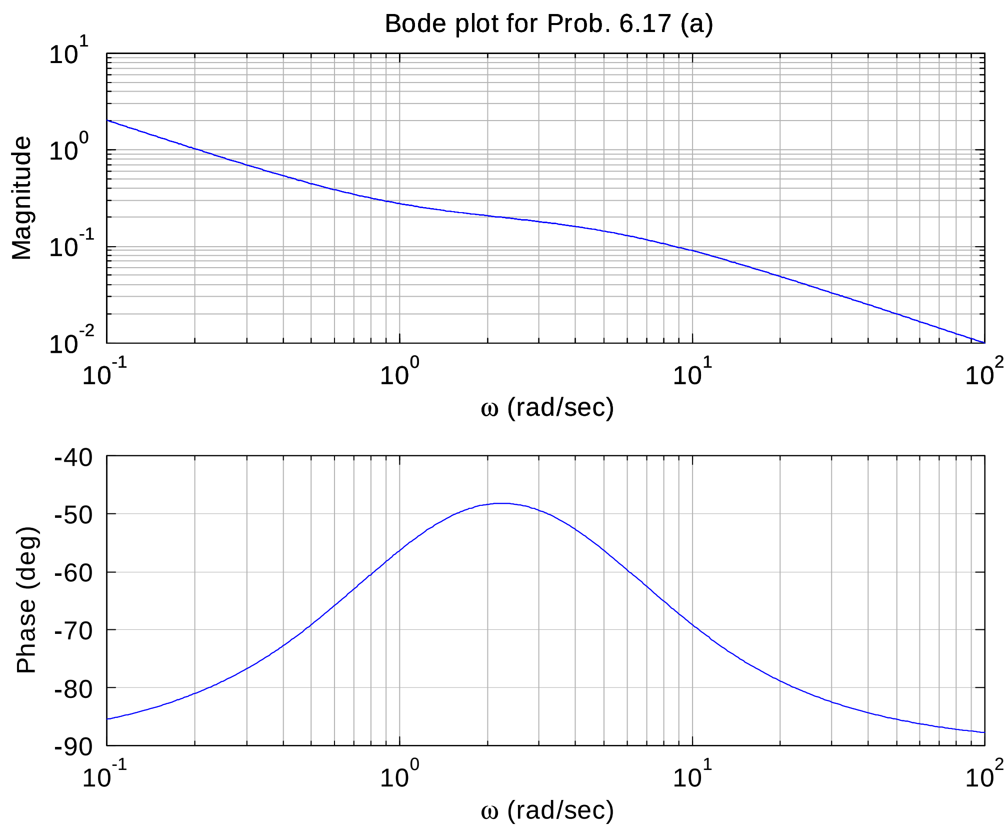
Solution :



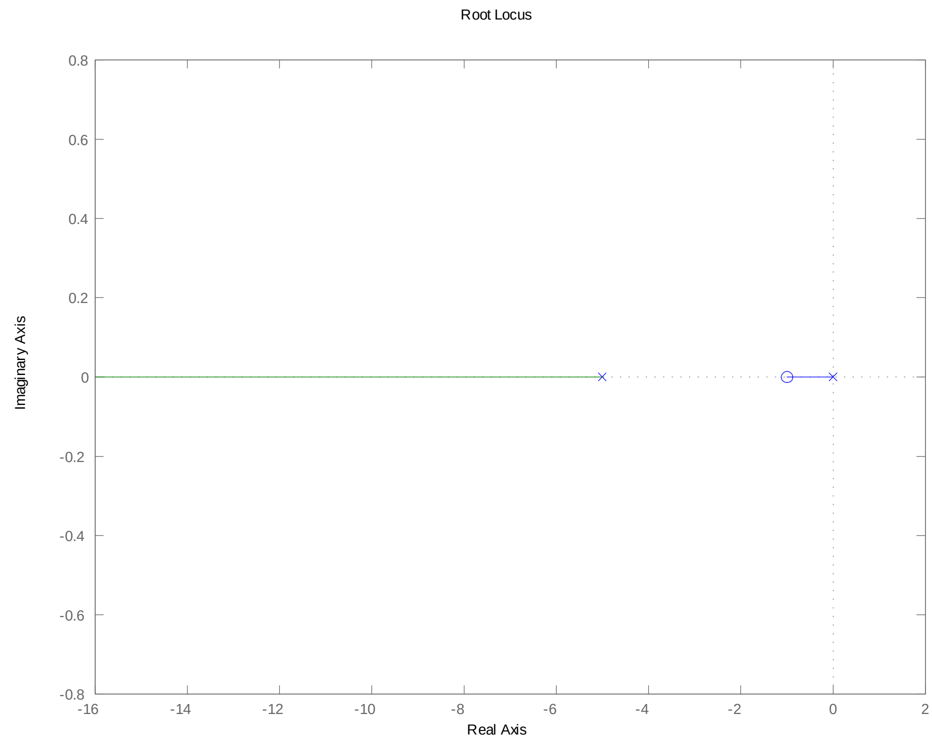
Root Locus
Imaginary Axis
Real Axis
30
20
10
0
-10
-20
-30
-120
-100
-80
-60
-40
-20
0
20

(a)

$$KG(s) = \frac{K(s+1)}{s(s+10)} = \frac{K}{10} \frac{(s+1)}{s\left(\frac{s}{10}+1\right)}$$

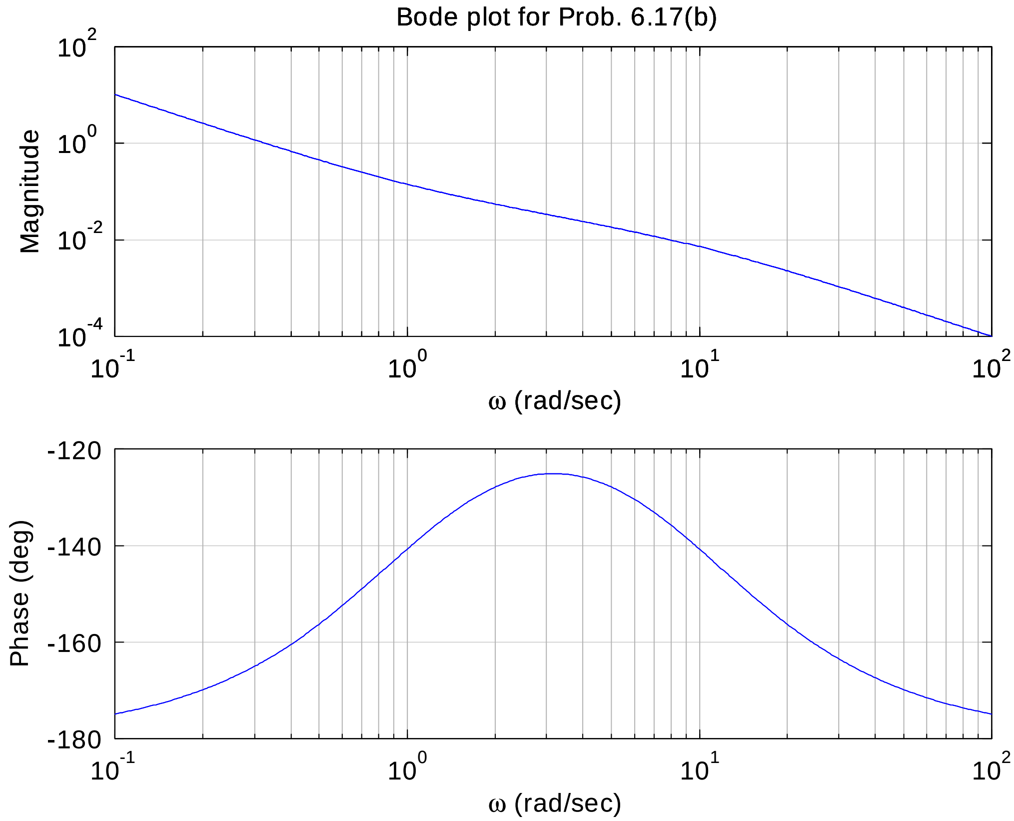


The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus.



(b)

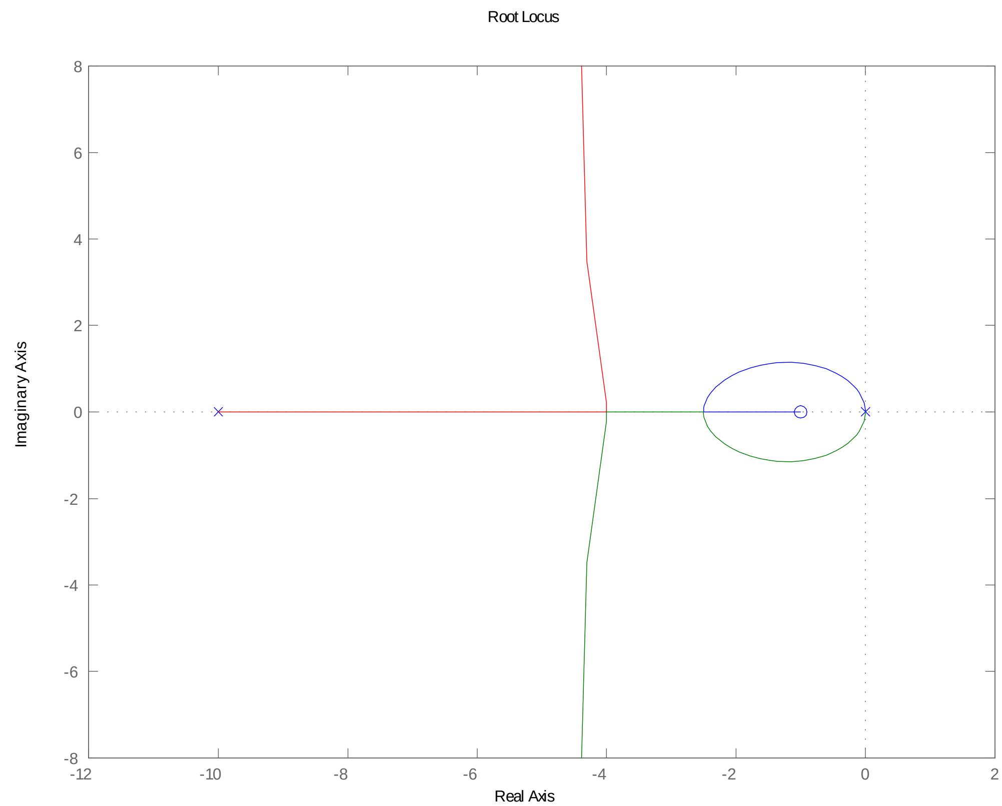
$$KG(s) = \frac{K(s+1)}{s^2(s+10)} = \frac{K}{10}\frac{s+1}{s^2\left(\frac{s}{10}+1\right)}$$

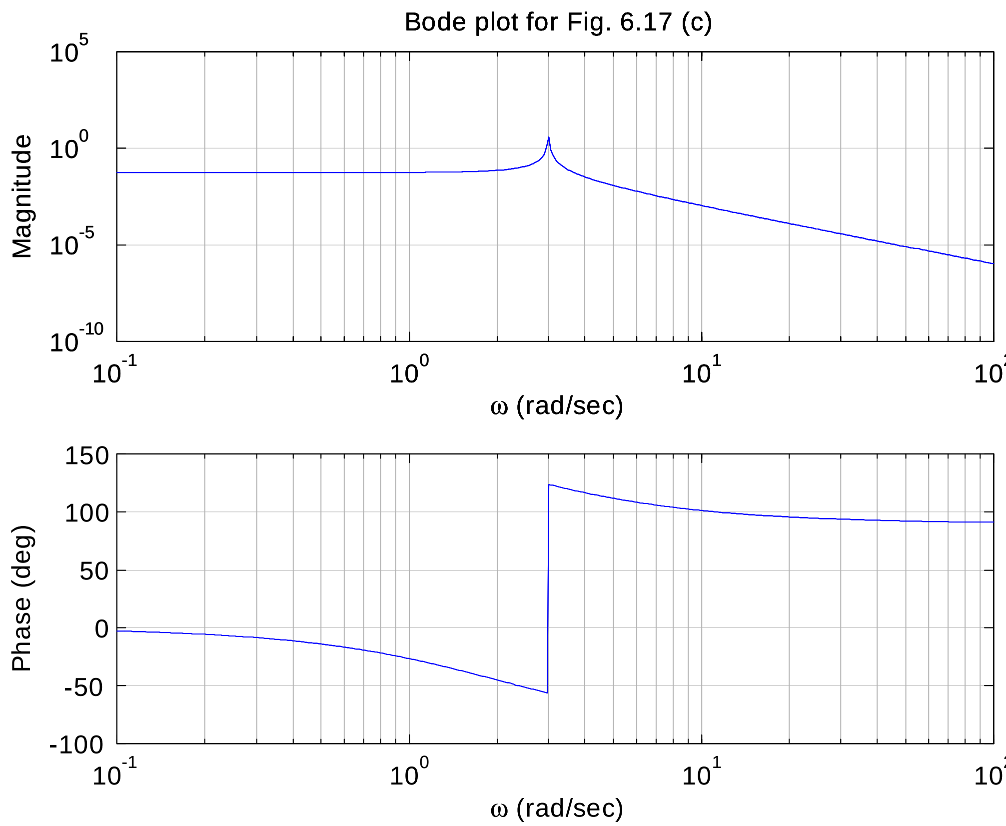


The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus. The system is stable for any $K > 0$.

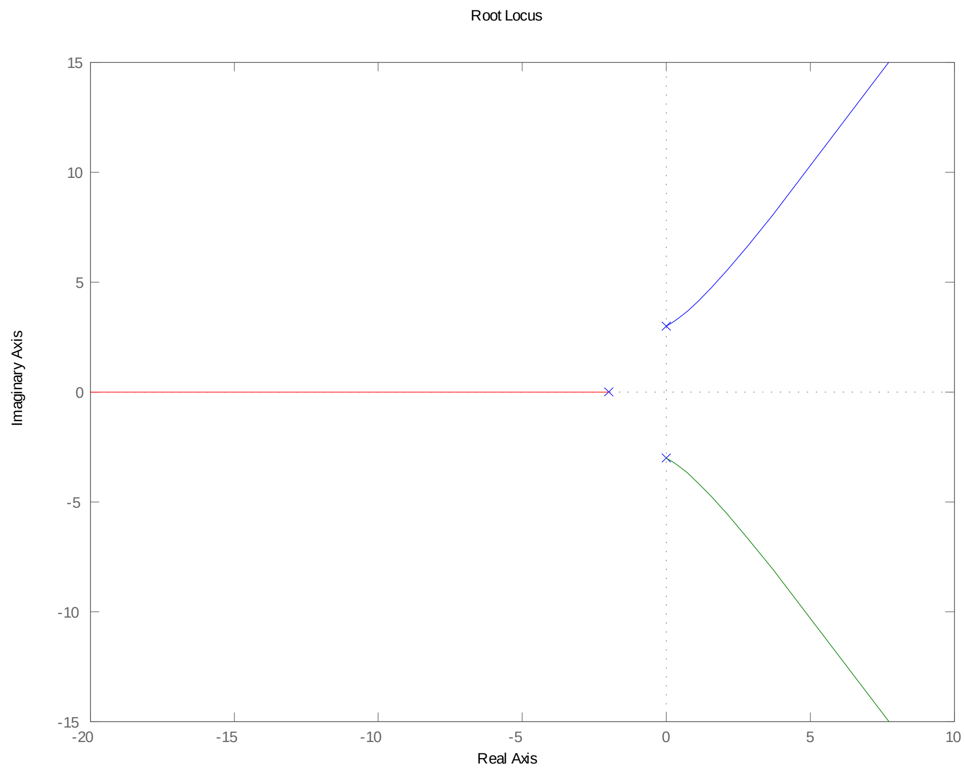
(c)

$$KG(s) = \frac{K}{(s+2)(s^2+9)} = \frac{K}{18}\frac{1}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{9}+1\right)}$$


Root Locus
Imaginary Axis
Real Axis
8
6
4
2
0
-2
-4
-6
-8
-12
-10
-8
-6
-4
-2
0
2

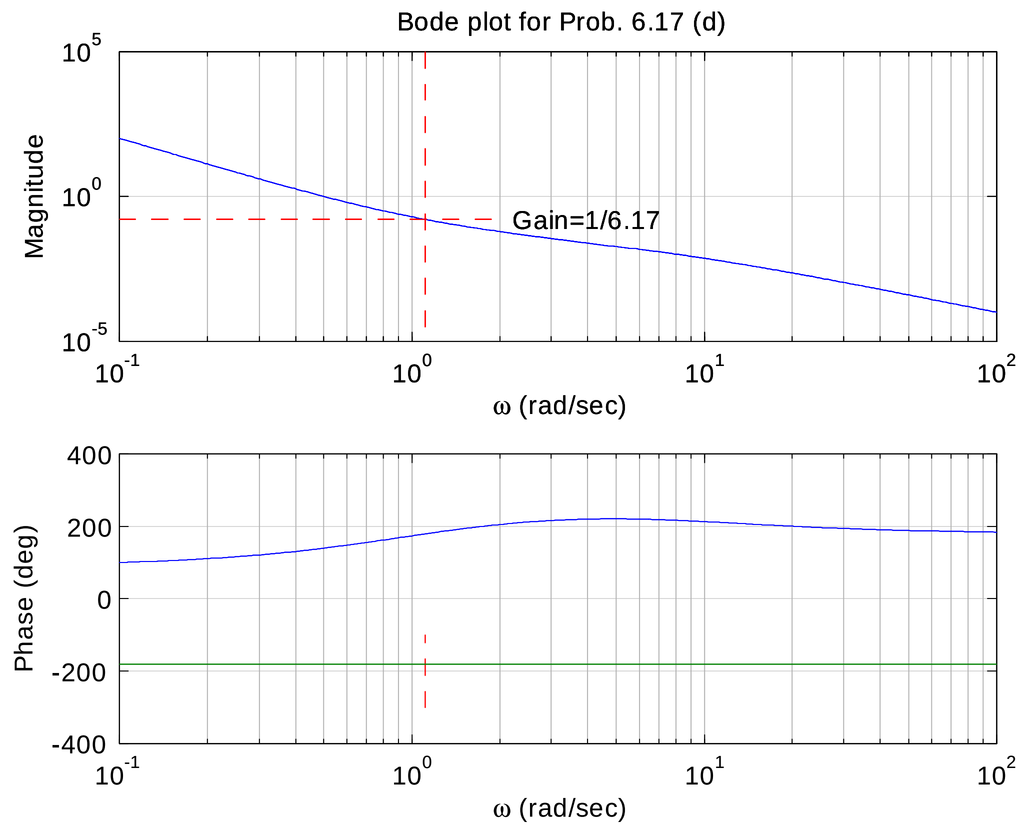


The bode is difficult to read, but the phase really dropped by 180^o at the resonance. (It appears to rise because of the quadrant action in Matlab) Furthermore, there is an infinite magnitude peak of the gain at the resonance because there is zero damping. That means that no matter how much the gain is lowered, the gain will never cross magnitude one when the phase is -180^o. So it can not be made stable for any K. This is much clearer and easier to see in the root locus below.

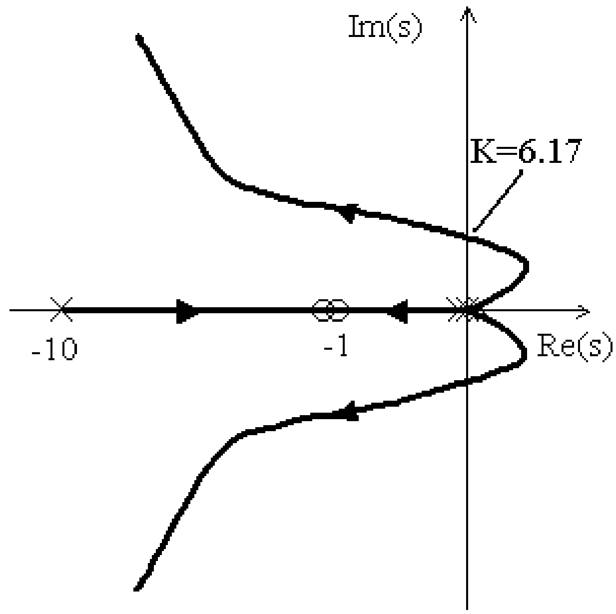


(d)

$$KG(s) = \frac{K(s+1)^2}{s^3(s+10)} = \frac{K}{10}\frac{(s+1)^2}{s^3\left(\frac{s}{10}+1\right)}$$



This is not the normal situation discussed in Section 6.2 where increasing gain leads to instability. Here we see from the root locus that K must be ≥ 6.17 in order for stability. Note that the phase is increasing with frequency here rather than the normal decrease we saw on the previous problems. It's also interesting to note that the margin command in Matlab indicates instability! (which is false.) This problem illustrates that a sketch of the root locus really helps understand what's going on... and that you can't always trust Matlab, or at least that you need good understanding to interpret what Matlab is telling you.



$K < 6.25$: Unstable

$K > 6.25$: Stable

$\omega = 1.12$ rad/sec for $K = 6.17$.

Problems and Solutions for Section 6.3

18. (a) Sketch the Nyquist plot for an open-loop system with transfer function $1/s^2$; that is, sketch

$$\frac{1}{s^2}\,|_{s=C_1},$$

where C_1 is a contour enclosing the entire RHP, as shown in Fig. 6.17. (*Hint*: Assume C_1 takes a small detour around the poles at $s = 0$, as shown in Fig. 6.27.)

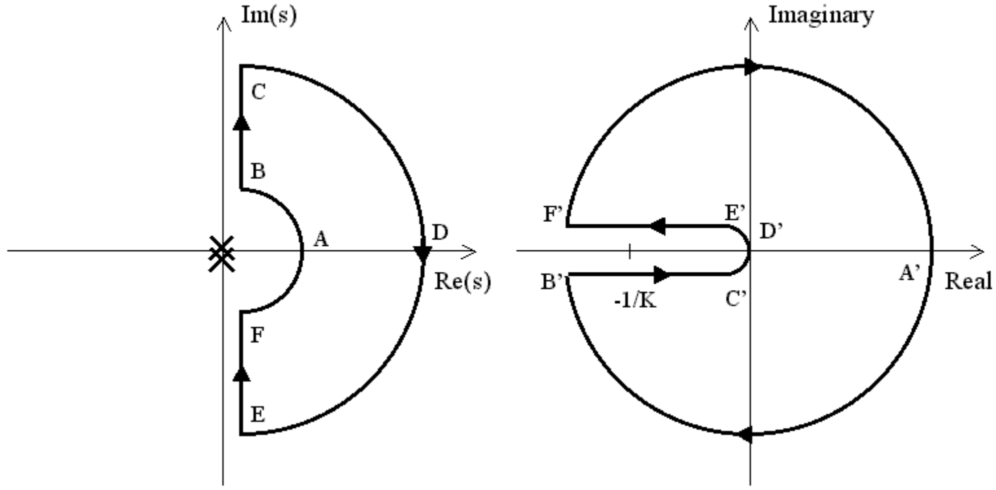
(b) Repeat part (a) for an open-loop system whose transfer function is $G(s) = 1/(s^2 + \omega_0^2)$.

Solution :

(a)

$$G(s) = \frac{1}{s^2}$$

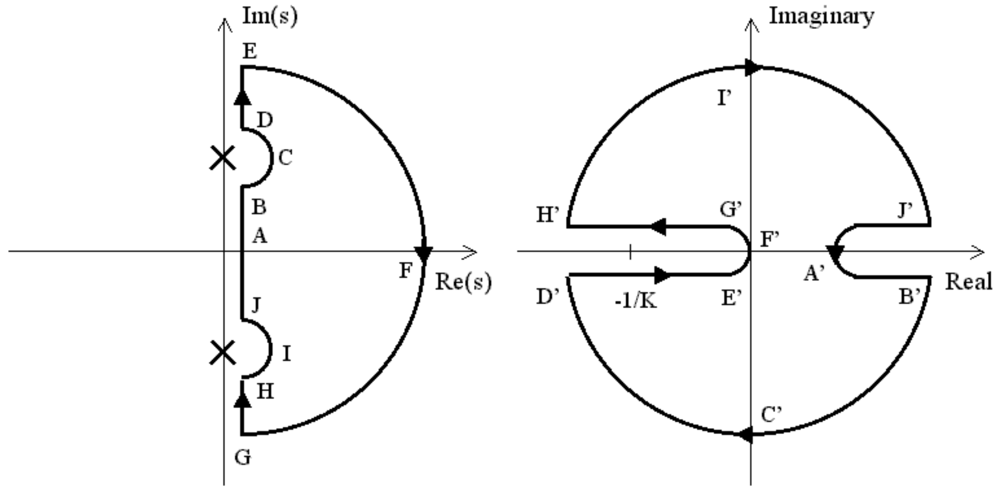
Note that the portion of the Nyquist diagram on the right side below that corresponds to the bode plot is from B' to C'. The large loop from F' to A' to B' arises from the detour around the 2 poles at the origin.



(b)

$$G(s) = \frac{1}{s^2 + \omega_0^2}$$

Note here that the portion of the Nyquist plot coming directly from a Bode plot is the portion from A' to E'. That portion includes a 180^o arc that arose because of the detour around the pole on the imaginary axis.



19. Sketch the Nyquist plot based on the Bode plots for each of the following systems, then compare your result with that obtained using the MATLAB command `nyquist`:

(a) $KG(s) = \dfrac{K(s+2)}{s+10}$

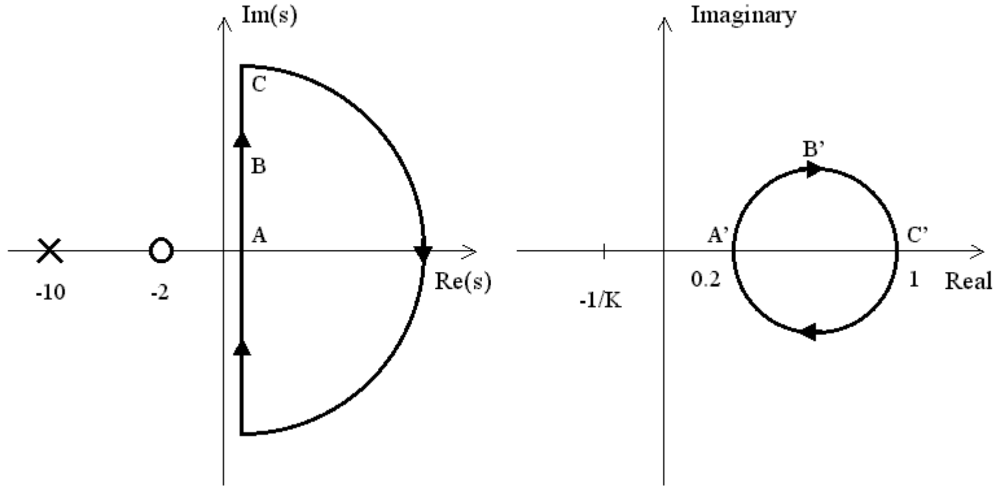
(b) $KG(s) = \dfrac{K}{(s+10)(s+2)^2}$

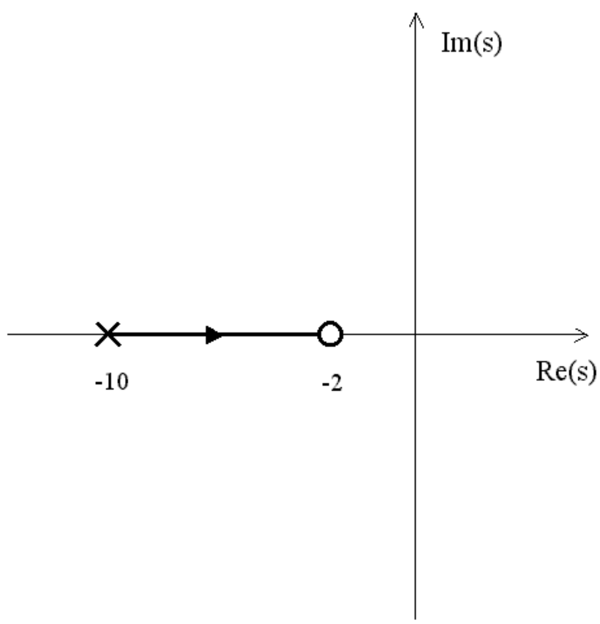
(c) $KG(s) = \dfrac{K(s+10)(s+1)}{(s+100)(s+2)^3}$

(d) Using your plots, estimate the range of K for which each system is stable, and qualitatively verify your result using a rough sketch of a root-locus plot.

Solution :

(a)

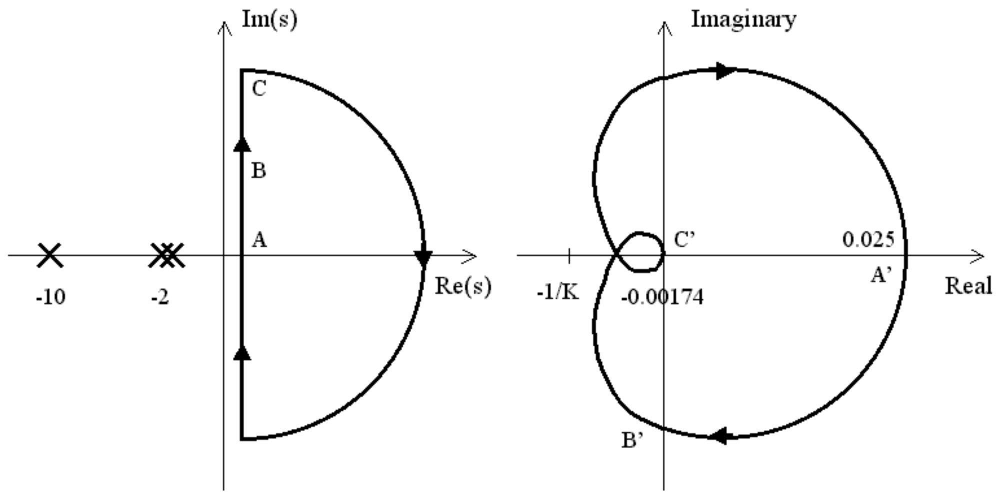




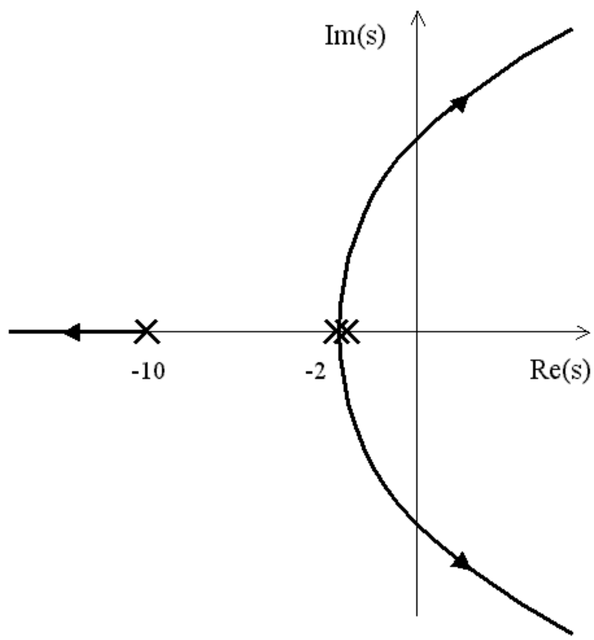
$N = 0, P = 0 \Longrightarrow Z = N + P = 0$

The closed-loop system is stable for any $K > 0$.

(b) The Bode plot shows an initial phase of 0^o hence the Nyquist starts on the positive real axis at A'. The Bode ends with a phase of -270^o hence the Nyquist ends the bottom loop by approaching the origin from the positive imaginary axis (or an angle of -270^o).



The magnitude of the Nyquist plot as it crosses the negative real axis is 0.00174. It will not encircle the $-1/K$ point until $K = 1/0.00174$ $= 576$.

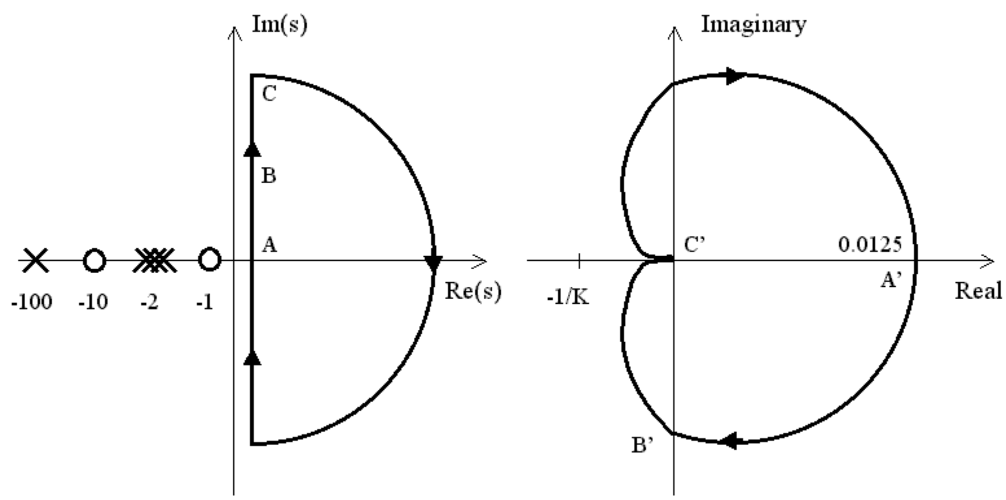


i. $0 < K < 576$
$N = 0, P = 0 \Longrightarrow Z = N + P = 0$
The closed-loop system is stable.

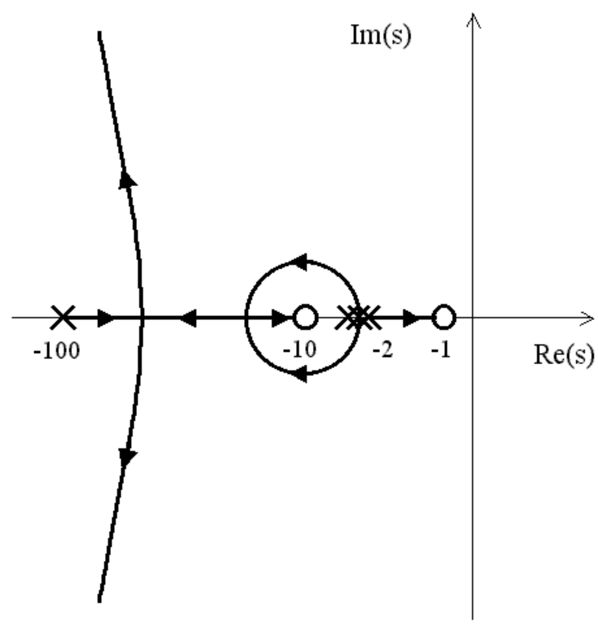
ii. $K > 576$
$N = 2, P = 0 \Longrightarrow Z = N + P = 2$

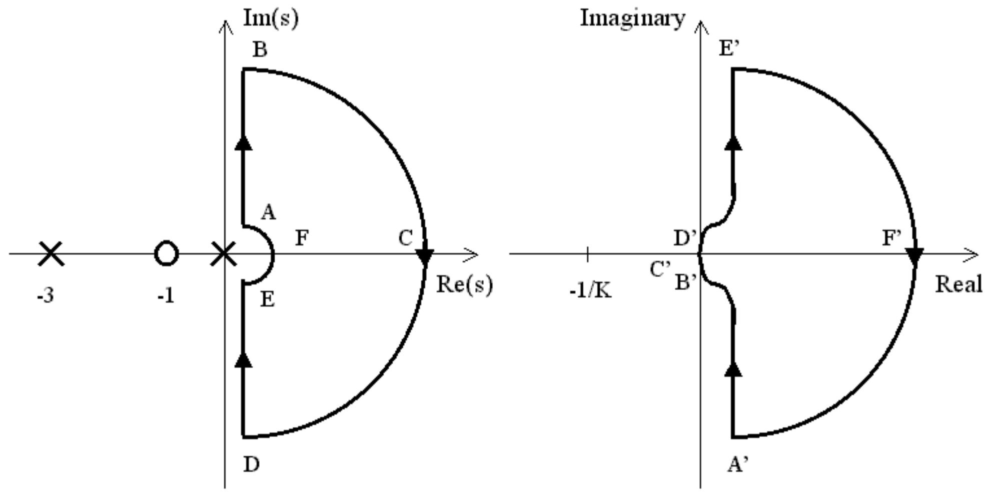
The closed-loop system has two unstable roots as verified by the root locus.

(c) The Bode plot shows an initial phase of 0^o hence the Nyquist starts on the positive real axis at A'. The Bode ends with a phase of -180^o hence the Nyquist ends the bottom loop by approaching the origin from the negative real axis (or an angle of -180^o).



It will never encircle the -1/K point, hence it is always stable. The root locus below confirms that.





$N = 0, P = 0 \Longrightarrow Z = N + P = 0$

The closed-loop system is stable for any $K > 0$.

20. Draw a Nyquist plot for

$$KG(s) = \frac{K(s+1)}{s(s+3)} \tag{1}$$

choosing the contour to be to the right of the singularity on the $j\omega$-axis. and determine the range of K for which the system is stable using the Nyquist Criterion. Then redo the Nyquist plot, this time choosing the contour to be to the left of the singularity on the imaginary axis and again check the range of K for which the system is stable using the Nyquist Criterion. Are the answers the same? Should they be?

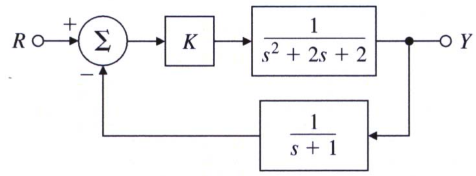
Solution :

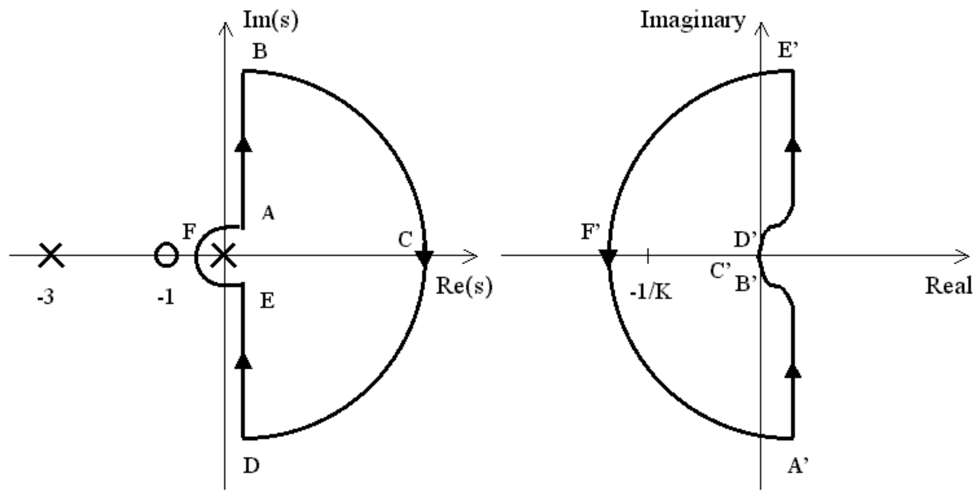
If you choose the contour to the right of the singularity on the origin, the Nyquist plot looks like this :

From the Nyquist plot, the range of K for stability is $-\frac{1}{K} < 0$ ($N = 0, P = 0 \Longrightarrow Z = N + P = 0$). So the system is stable for $K > 0$.

Similarly, in the case with the contour to the left of the singularity on the origin, the Nyquist plot is:

Figure 6.87: Control system for Problem 21





From the Nyquist plot, the range of K for stability is $-\frac{1}{K} < 0$ ($N = -1, P = 1 \Longrightarrow Z = N + P = 0$). So the system is stable for $K > 0$.

The way of choosing the contour around singularity on the $j\omega$-axis does not affect its stability criterion. The results should be the same in either way. However, it is somewhat less cumbersome to pick the contour to the right of a pole on the imaginary axis so that there are no unstable poles within the contour, hence P=0.

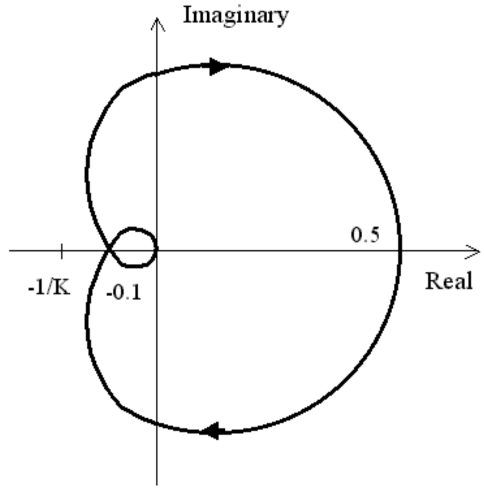
21. Draw the Nyquist plot for the system in Fig. 6.87. Using the Nyquist stability criterion, determine the range of K for which the system is stable. Consider both positive and negative values of K.

Solution :

The characteristic equation:

$$1 + K\frac{1}{(s^2 + 2s + 2)}\frac{1}{(s+1)} = 0$$

$$G(s) = \frac{1}{(s+1)(s^2 + 2s + 2)}$$



For positive K, note that the magnitude of the Nyquist plot as it crosses the negative real axis is 0.1, hence $K < 10$ for stability. For negative K, the entire Nyquist plot is essentially flipped about the imaginary axis, thus the magnitude where it crosses the negative real axis will be 0.5 and the stability limit is that $|K| < 2$ Therefore, the range of K for stability is $-2 < K < 10$.

22. (a) For $\omega = 0.1$ to 100 rad/sec, sketch the phase of the minimum-phase system

$$\left| G(s) = \frac{s+1}{s+10} \right|_{s=j\omega}$$

and the nonminimum-phase system

$$\left| G(s) = -\frac{s-1}{s+10} \right|_{s=j\omega},$$

noting that $\angle(j\omega - 1)$ decreases with ω rather than increasing.

(b) Does a RHP zero affect the relationship between the -1 encirclements on a polar plot and the number of unstable closed-loop roots in Eq. (6.28)?

(c) Sketch the phase of the following unstable system for $\omega = 0.1$ to 100 rad/sec:

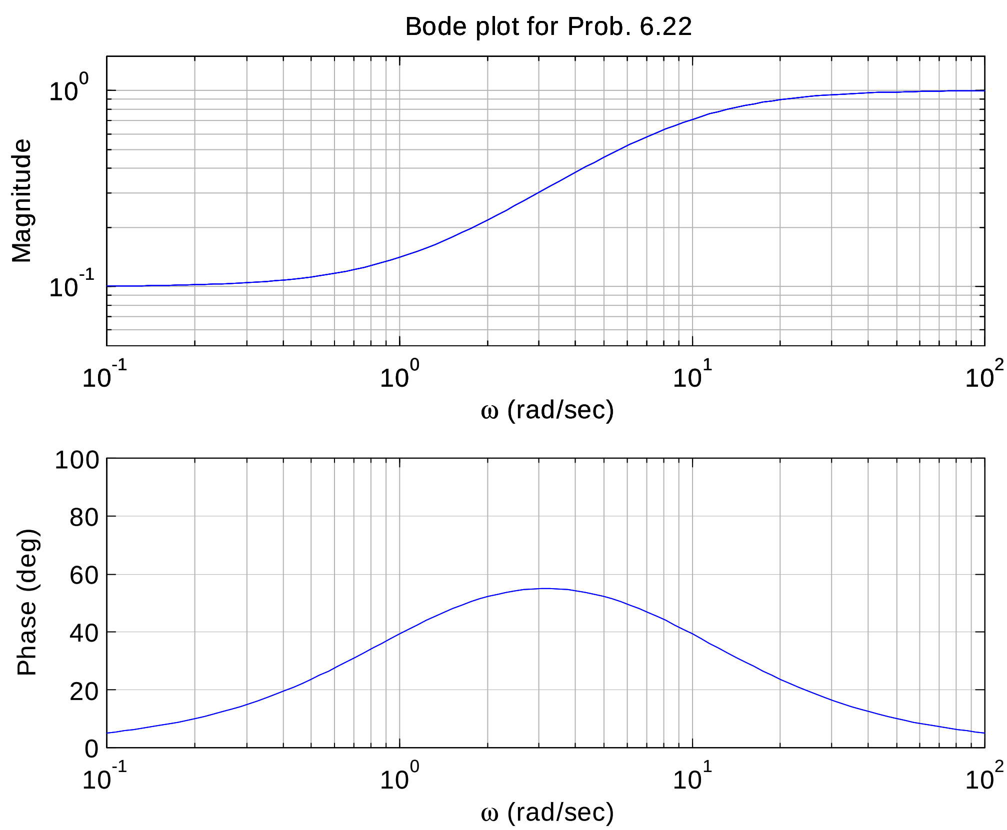
$$G(s) = \left| \frac{s+1}{s-10} \right|_{s=j\omega}.$$

(d) Check the stability of the systems in (a) and (c) using the Nyquist criterion on $KG(s)$. Determine the range of K for which the closed-loop system is stable, and check your results qualitatively using a rough root-locus sketch.

Solution :

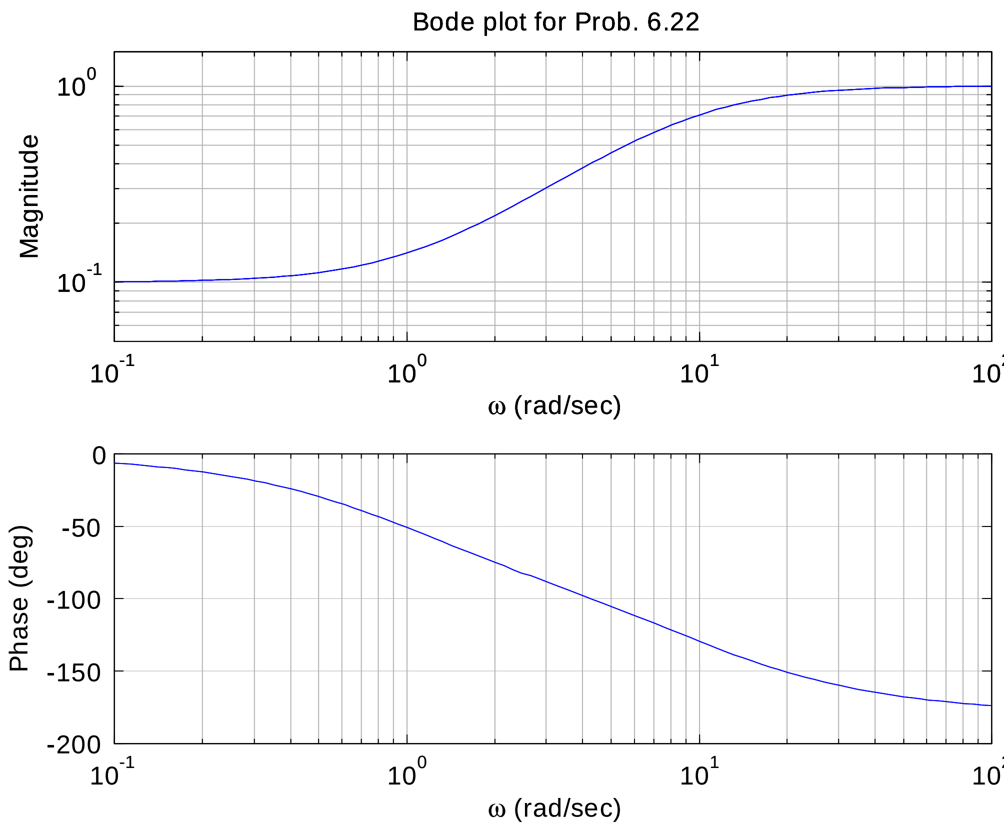
(a) Minimum phase system,

$$G_1(j\omega) = \frac{s+1}{s+10}|_{s=j\omega}$$



Non-minimum phase system,

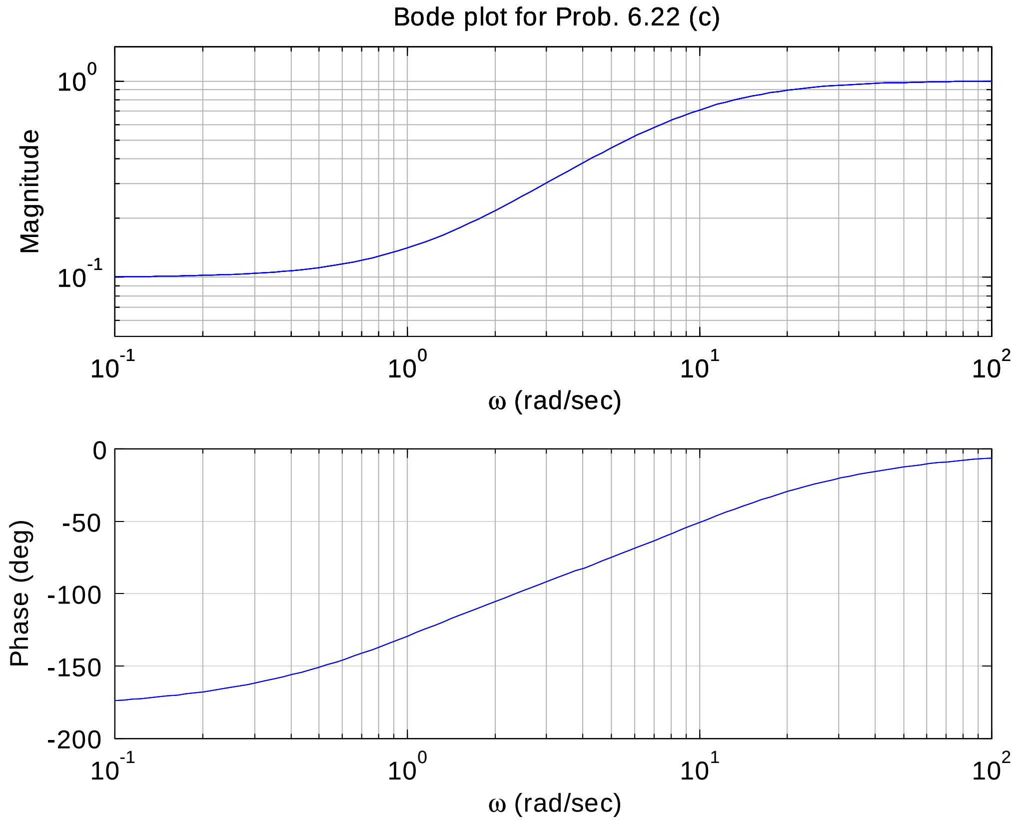
$$G_2(j\omega) = -\frac{s-1}{s+10}|_{s=j\omega}$$



(b) No, a RHP zero doesn't affect the relationship between the -1 encirclements on the Nyquist plot and the number of unstable closed-loop roots in Eq. (6.28).

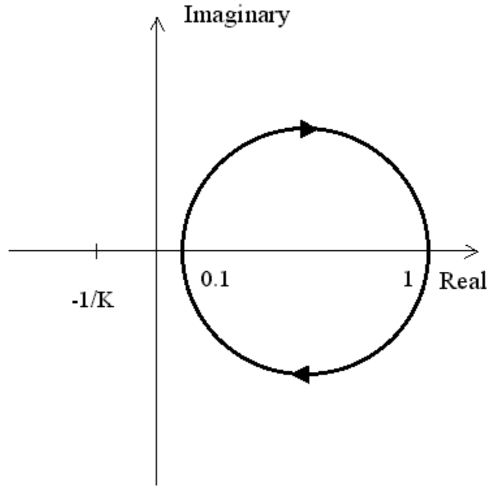
(c) Unstable system:

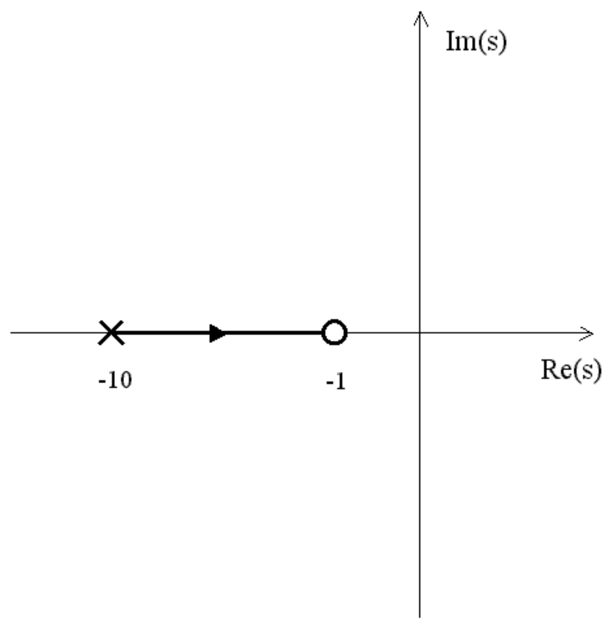
$$G_3(j\omega) = \frac{s+1}{s-10}|_{s=j\omega}$$



(d) Minimum phase system $G_1(j\omega)$:

i. For any $K > 0,\ N = 0,\ P = 0 \Longrightarrow Z = 0 \Longrightarrow$ The system is stable, as verified by the root locus being entirely in the LHP.

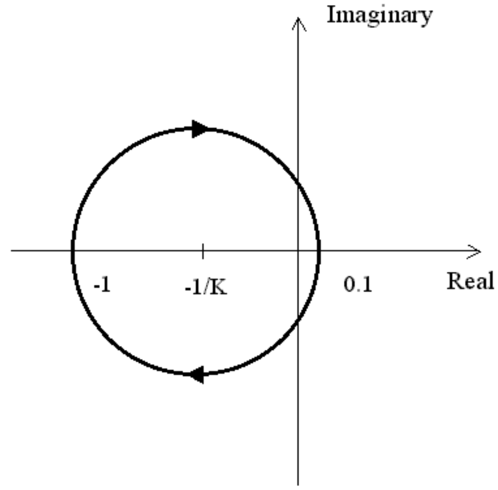


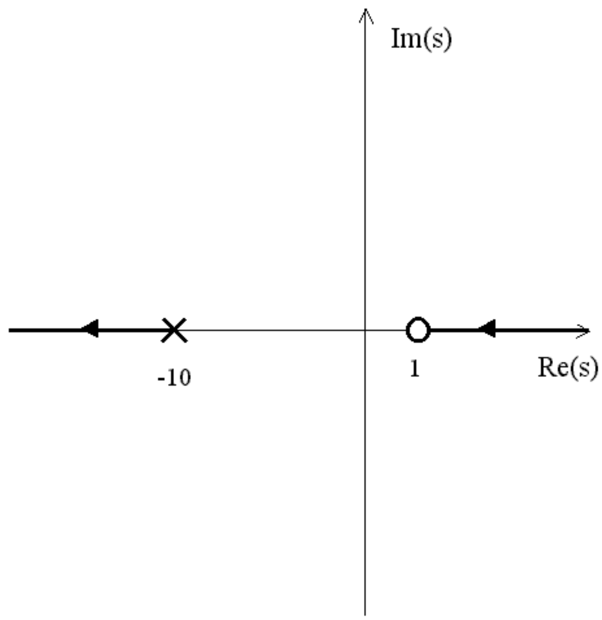


Non-minimum phase system $G_2(j\omega)$: the $-1/K$ point will not be encircled if $K < 1$.

$$\begin{array}{ll} 0 < K < 1 & N = 0,\ P = 0 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \\ 1 < K & N = 1,\ P = 0 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \end{array}$$

This is verified by the Root Locus shown below where the branch of the locus to the left of the pole is from $K < 1$.

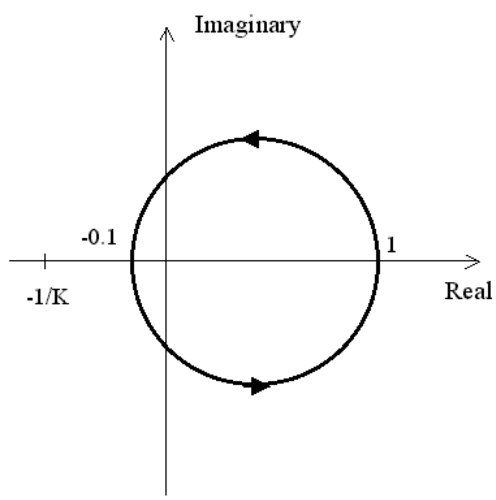


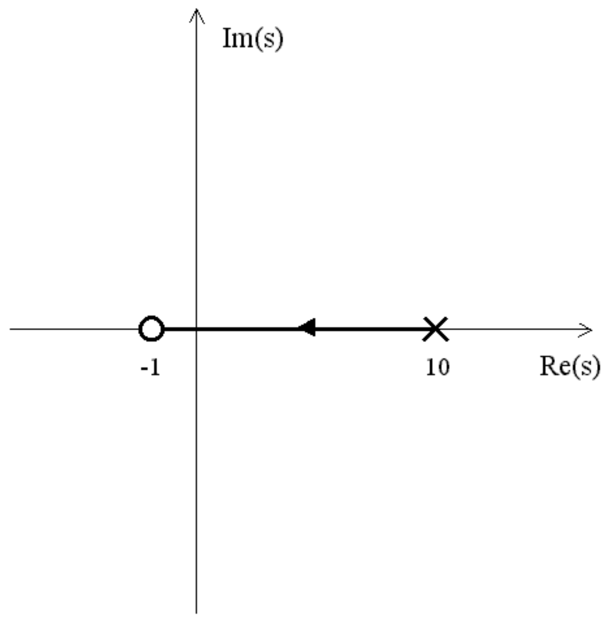


Unstable system $G_3(j\omega)$: The $-1/K$ point will be encircled if $K > 10$, however, $P = 1$, so

$$\begin{array}{rl} 0 < K < 10: & N = 0,\ P = 1 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \\ 10 < K\ : & N = -1,\ P = 1 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \end{array}$$

This is verified by the Root Locus shown below right, where the locus crosses the imaginary axis when $K = 10$, and stays in the LHP for $K > 10$.





23. *Nyquist plots and their classical plane curves:* Determine the Nyquist plot using Matlab for the systems given below with $K = 1$ and verify that the beginning point and end point for the $j\omega > 0$ portion have the correct magnitude and phase:

   (a) the classical curve called Cayley's Sextic, discovered by Maclaurin in 1718
   $$KG(s) = K\frac{1}{(s+1)^3}$$

   (b) the classical curve called the Cissoid, meaning ivy-shaped
   $$KG(s) = K\frac{1}{s(s+1)}$$

   (c) the classical curve called the Folium of Kepler, studied by Kepler in 1609.
   $$KG(s) = K\frac{1}{(s-1)(s+1)^2}$$

   (d) the classical curve called the Folium (not Kepler's)
   $$KG(s) = K\frac{1}{(s-1)(s+2)}$$

   (e) the classical curve called the Nephroid, meaning kidney-shaped.
   $$KG(s) = K\frac{2(s+1)(s^2-4s+1)}{(s-1)^3}$$

(f) the classical curve called Nephroid of Freeth, named after the English mathematician T. J. Freeth.

$$KG(s) = K\frac{(s+1)(s^2+3)}{4(s-1)^3}$$

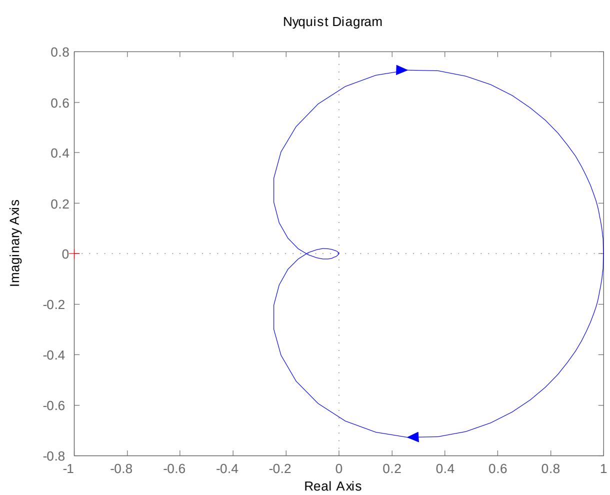
(g) a shifted Nephroid of Freeth

$$KG(s) = K\frac{(s^2+1)}{(s-1)^3}$$

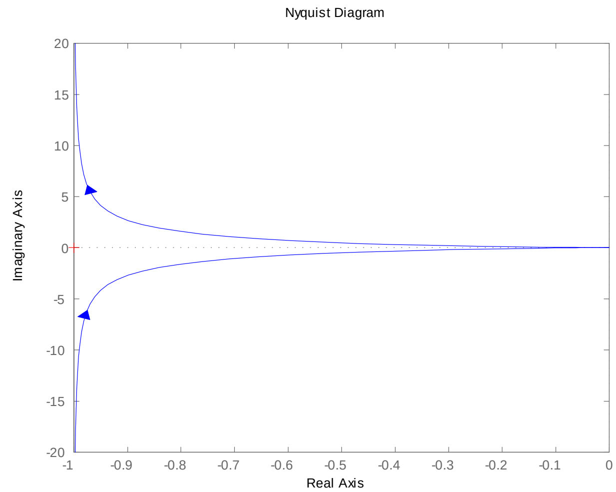
Solution :

These are all accomplished by using Matlab's Nyquist function. All interesting shapes. To check the magnitude and phase for each, plug in $s = 0$ and $s = \inf$ and then compare those values with the beginning and end points on the Nyquist diagrams.

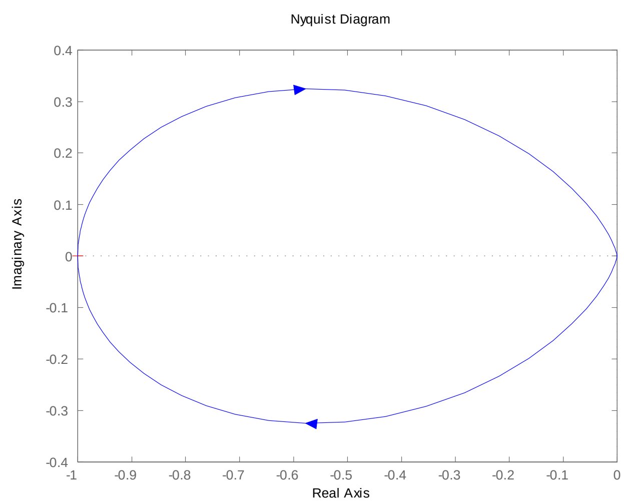
(a)



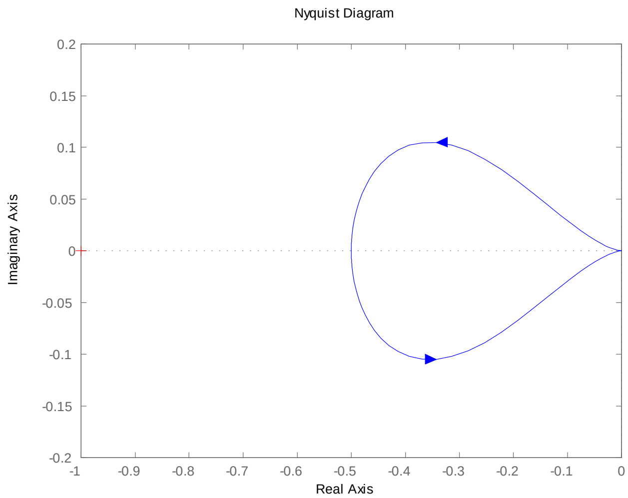
(b)


Nyquist Diagram
Imaginary Axis
Real Axis
20
15
10
5
0
-5
-10
-15
-20
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0

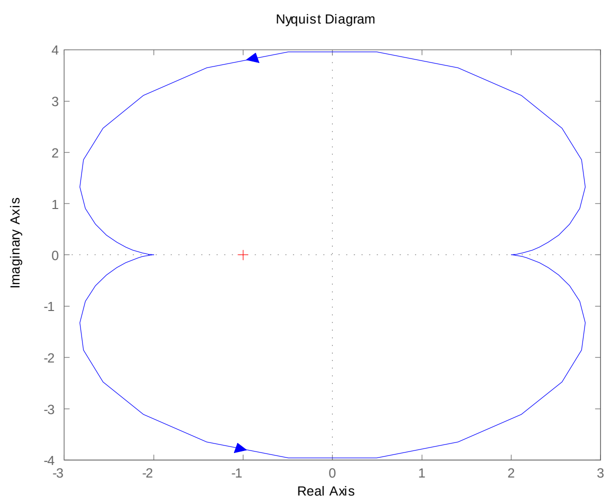
(c)


Nyquist Diagram
Imaginary Axis
Real Axis
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0

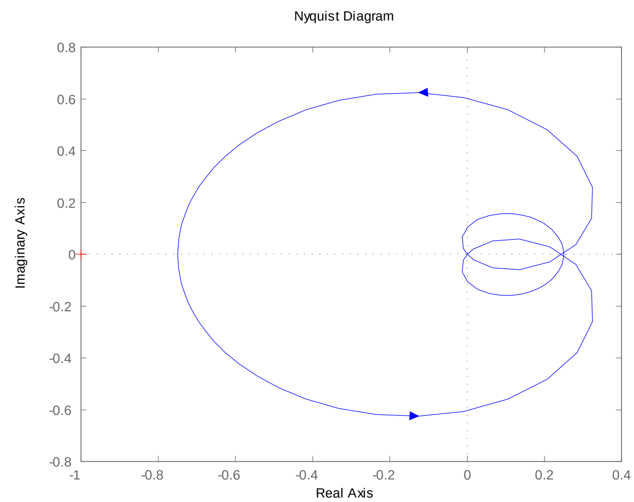
(d)


Nyquist Diagram
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
Imaginary Axis
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
Real Axis

(e)

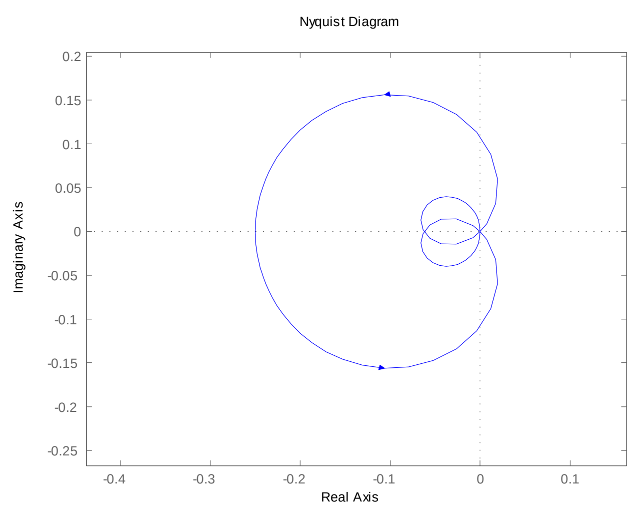

Nyquist Diagram
4
3
2
1
0
-1
-2
-3
-4
Imaginary Axis
-3
-2
-1
0
1
2
3
Real Axis

(f)



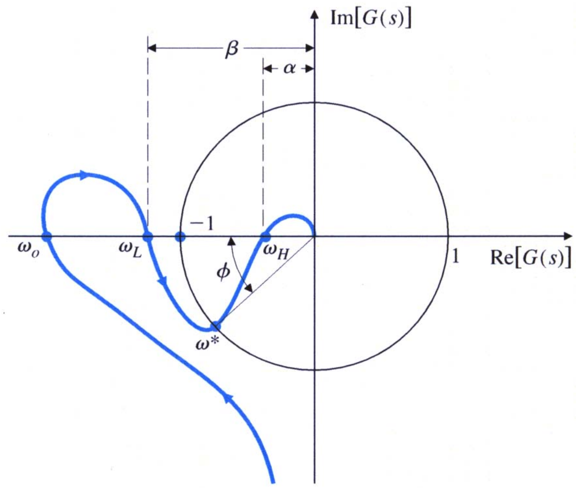
Nyquist Diagram
Imaginary Axis
Real Axis
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

(g)



Nyquist Diagram
Imaginary Axis
Real Axis
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.4
-0.3
-0.2
-0.1
0
0.1

Figure 6.88: Nyquist plot for Problem 24



(a) Problems and Solutions for Section 6.4

24. The Nyquist plot for some actual control systems resembles the one shown in Fig.6.88. What are the gain and phase margin(s) for the system of Fig. 6.88 given that $\alpha = 0.4$, $\beta = 1.3$, and $\phi = 40^o$. Describe what happens to the stability of the system as the gain goes from zero to a very large value. Sketch what the corresponding root locus must look like for such a system. Also sketch what the corresponding Bode plots would look like for the system.

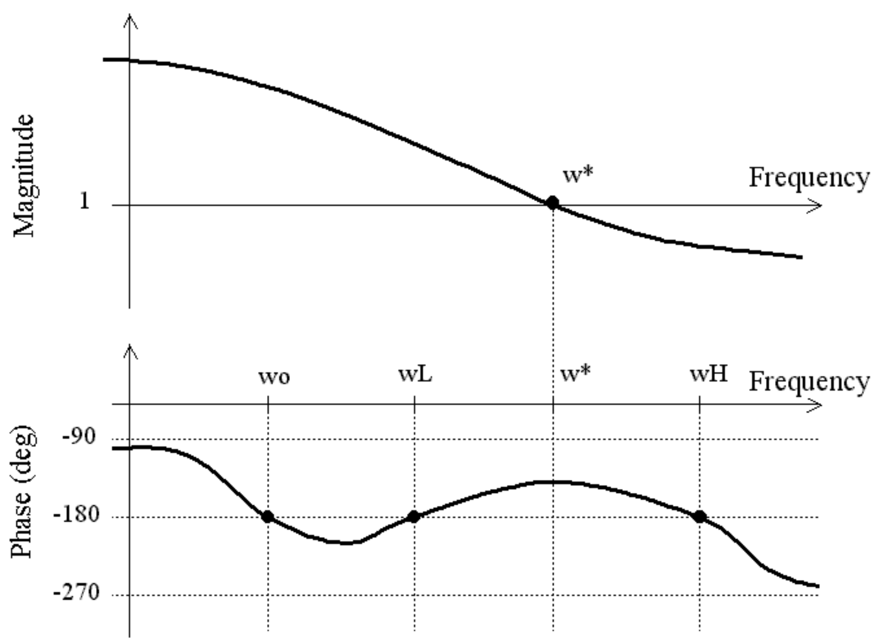
    Solution :

    The phase margin is defined as in Figure 6.34, $PM = \phi\ (\omega = \omega^*)$, but now there are several gain margins! If the system gain is increased (multiplied) by $\frac{1}{|\alpha|}$ or decreased (divided) by $|\beta|$, then the system will go unstable. This is a conditionally stable system. See Figure 6.40 for a typical root locus of a conditionally stable system.

$$\begin{aligned}
\text{gain margin} &= \text{-20 log} \left|\alpha\right|_{dB} (\omega = \omega_H) \\
\text{gain margin} &= +20 \text{ log} \left|\beta\right|_{dB} (\omega = \omega_L)
\end{aligned}$$

For a conditionally stable type of system as in Fig. 6.40, the Bode phase plot crosses -180° twice; however, for this problem we see from the Nyquist plot that it crosses 3 times! For very low values of gain, the entire

Nyquist plot would be shrunk, and the -1 point would occur to the left of the negative real axis crossing at ω_o, so there would be no encirclements and the system would be stable. As the gain increases, the -1 point occurs between ω_o and ω_L so there is an encirclement and the system is unstable. Further increase of the gain causes the -1 point to occur between ω_L and ω_H (as shown in Fig. 6.90) so there is no encirclement and the system is stable. Even more increase in the gain would cause the -1 point to occur between ω_H and the origin where there is an encirclement and the system is unstable. The root locus would look like Fig. 6.40 except that the very low gain portion of the loci would start in the LHP before they loop out into the RHP as in Fig. 6.40. The Bode plot would be vaguely like that drawn below:



25. The Bode plot for

$$G(s) = \frac{100[(s/10) + 1]}{s[(s/1) - 1][(s/100) + 1]}$$

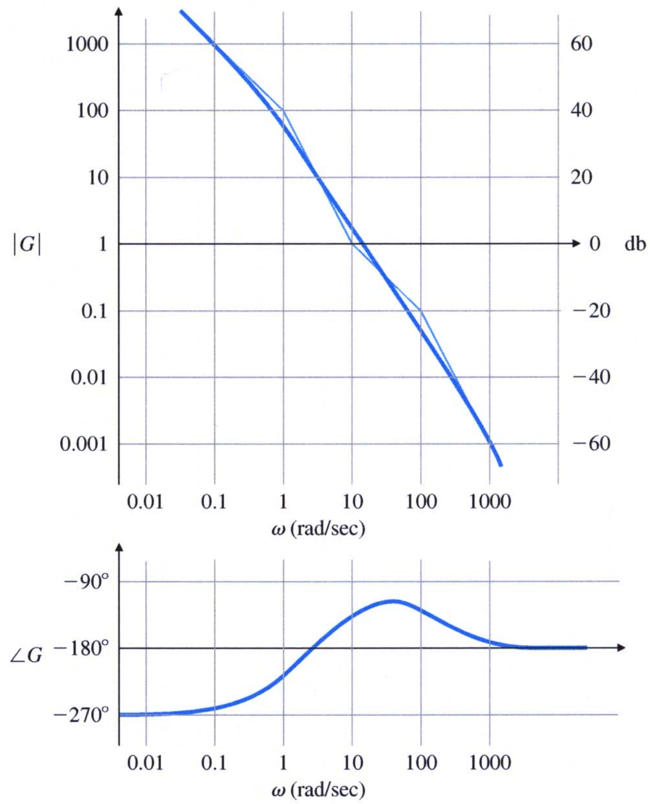
is shown in Fig. 6.89.

(a) Why does the phase start at -270° at the low frequencies?

(b) Sketch the Nyquist plot for $G(s)$.

(c) Is the closed-loop system shown in Fig. 6.89 stable?

Figure 6.89: Bode plot for Problem 25



(d) Will the system be stable if the gain is lowered by a factor of 100? Make a rough sketch of a root locus for the system and qualitatively confirm your answer

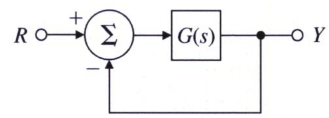
Solution :

(a) From the root locus, the phase at the low frequencies ($\omega = 0+$) is calculated as :

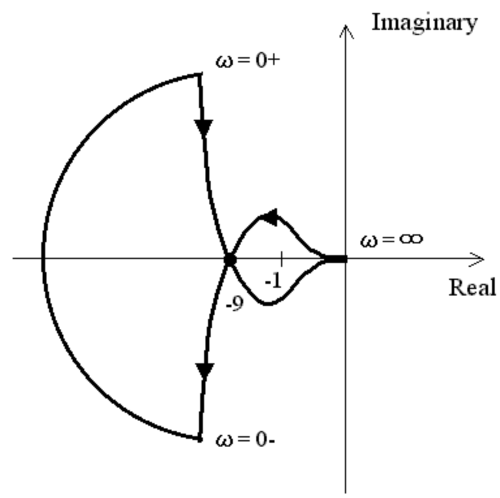
$$
\begin{aligned}
& \text{The phase at the point } \{s = j\omega(\omega = 0+)\} \\
= \; & -180^\circ(\text{pole}: \; s = 1) - 90^\circ(\text{pole}: \; s = 0) + 0^\circ(\text{zero}: \; s = -10) + 0^\circ(\text{pole}: \; s = -100) \\
= \; & -270^\circ
\end{aligned}
$$

Or, more simply, the RHP pole at $s = +1$ causes a -180^o shift from the -90^o that you would expect from a normal system with all the singularities in the LHP.

Figure 6.90: Control system for Problem 26



(b) The Nyquist plot for $G(s)$:



(c) As the Nyquist shows, there is one counter-clockwise encirclement of -1.

$$\Longrightarrow N = -1$$

We have one pole in RHP $\Longrightarrow P = 1$
$Z = N + P = -1 + 1 = 0 \Longrightarrow$ The closed-loop system is stable.

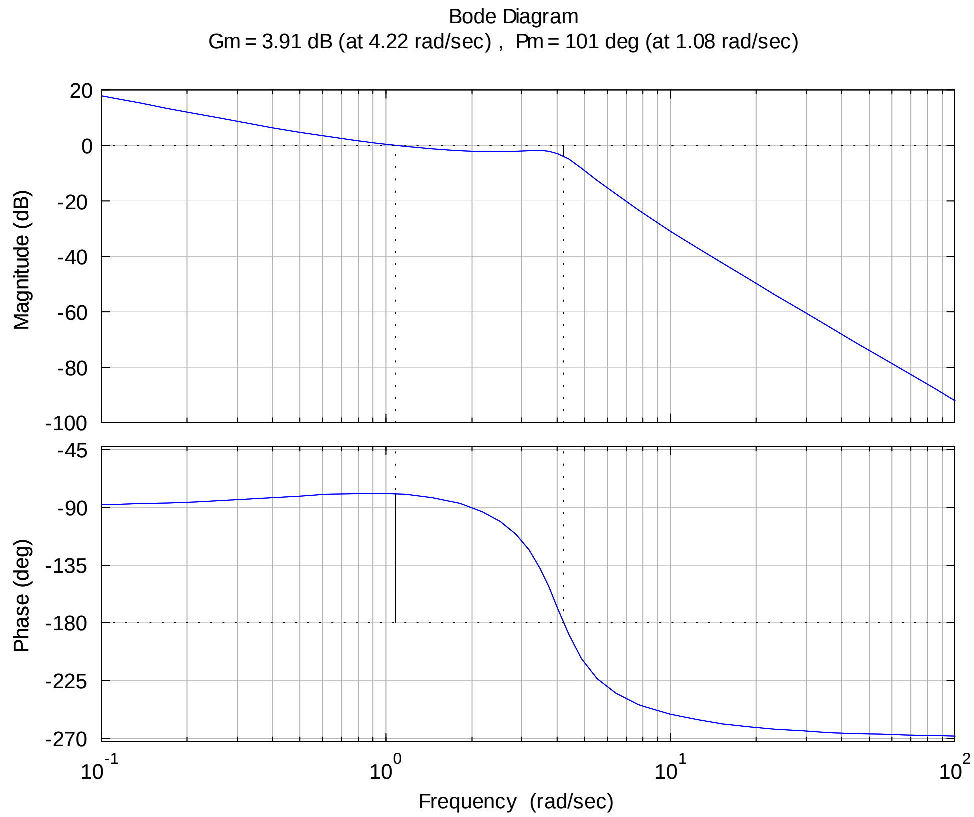
(d) The system goes unstable if the gain is lowered by a factor of 100.

26. Suppose that in Fig. 6.90,

$$G(s) = \frac{25(s+1)}{s(s+2)(s^2+2s+16)}.$$

Use MATLAB's margin to calculate the PM and GM for $G(s)$ and, based on the Bode plots, conclude which margin would provide more useful information to the control designer for this system.

Solution :



From the Bode plot,

$$PM = 101\,\text{deg},\ GM = 3.9\text{db} = 1.57$$

Since both PM and GM are positive, we can say that the closed-loop of this system is stable. But GM is so small that we must be careful not to increase the gain much, which leads the closed-loop system to be unstable. Clearly, the GM is the more important margin for this example.

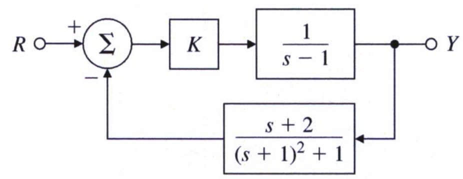
27. Consider the system given in Fig. 6.91.

   (a) Use MATLAB to obtain Bode plots for $K = 1$ and use the plots to estimate the range of K for which the system will be stable.

   (b) Verify the stable range of K by using margin to determine PM for selected values of K.

   (c) Use rlocus and rlocfind to determine the values of K at the stability boundaries.

Figure 6.91: Control system for Problem 27

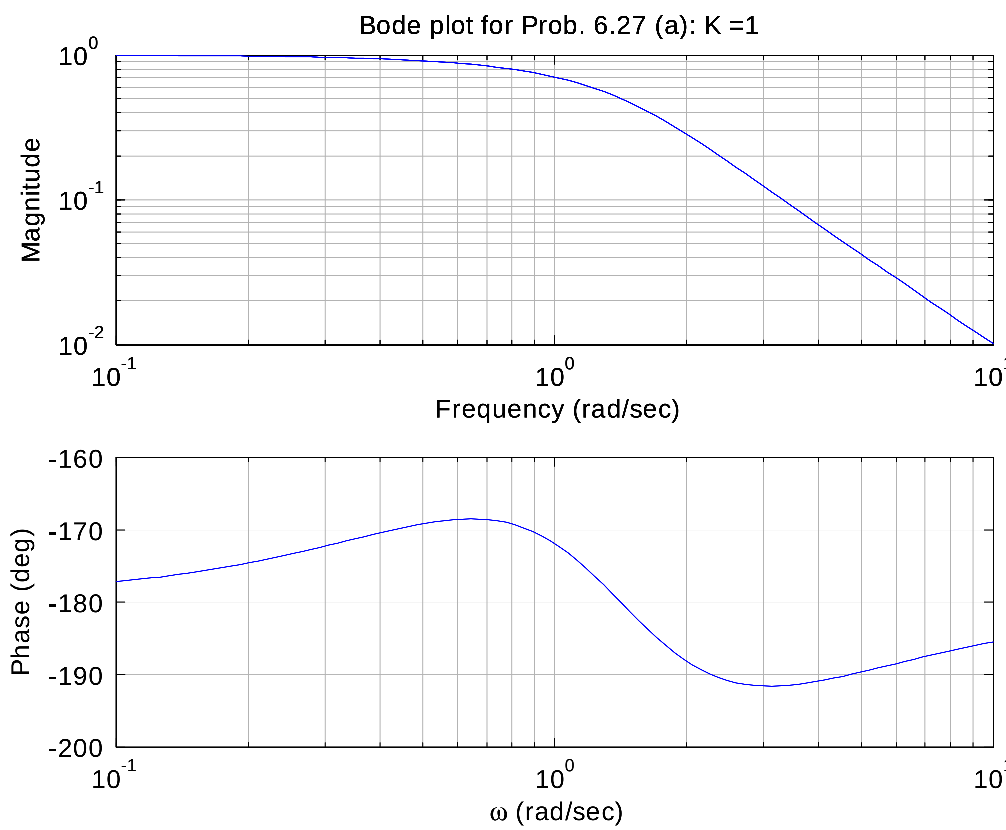


(d) Sketch the Nyquist plot of the system, and use it to verify the number of unstable roots for the unstable ranges of K.

(e) Using Routh's criterion, determine the ranges of K for closed-loop stability of this system.

Solution :

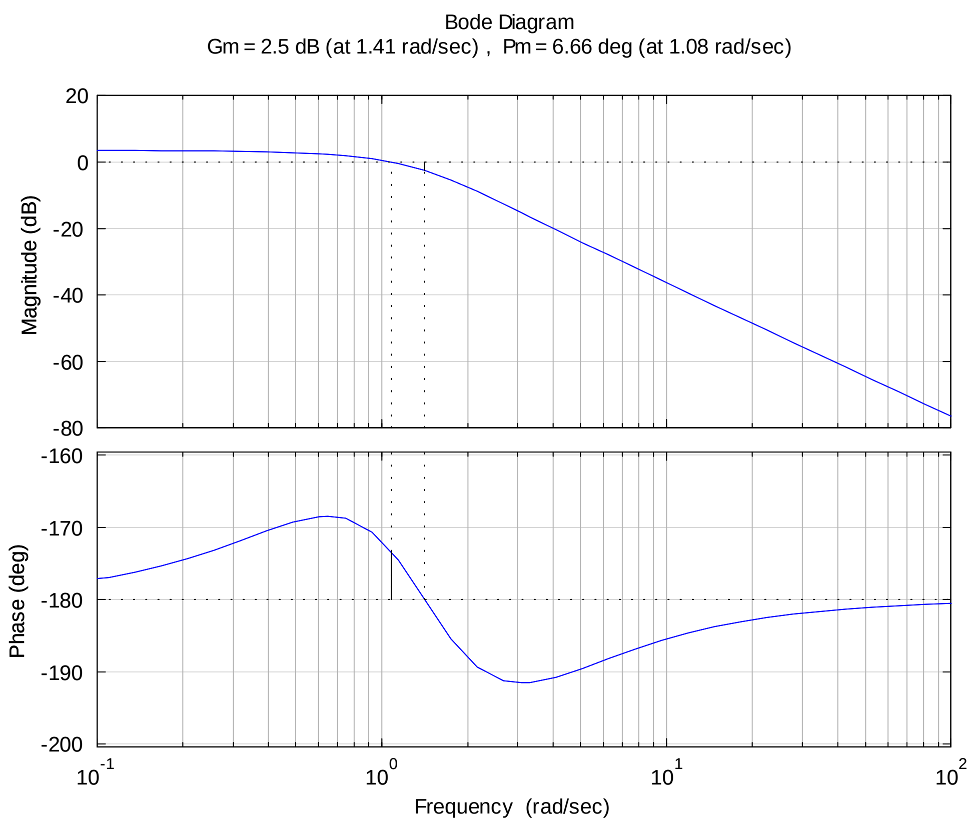
(a) The Bode plot for $K = 1$ is :



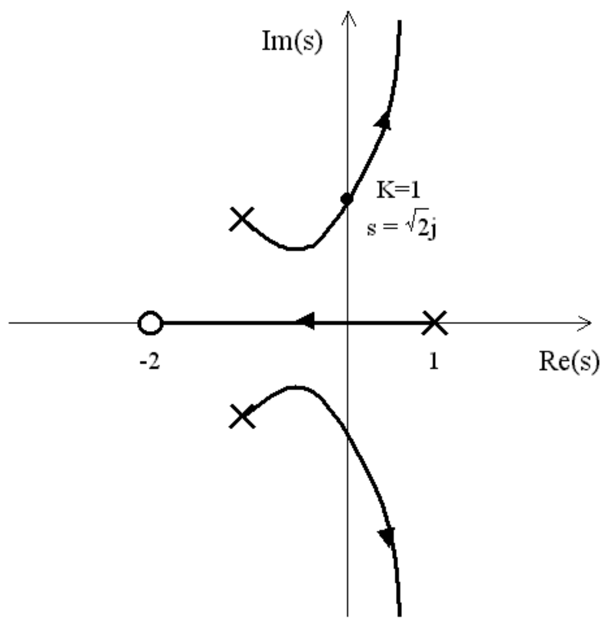
From the Bode plot, the closed-loop system is unstable for $K = 1$. But we can make the closed-system stable with positive GM by increasing the gain K up to the crossover frequency reaches at $\omega = 1.414$ rad/sec ($K = 2$), where the phase plot crosses the $-180°$ line. Therefore :

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$

(b) For example, $PM = 6.66\,\text{deg}$ for $K = 1.5$.



(c) Root locus is :



$j\omega$-crossing :

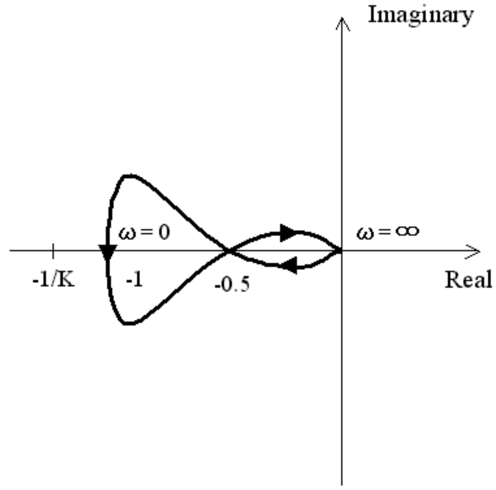
$$1 + K\frac{j\omega + 2}{(j\omega)^3 + (j\omega)^2 - 2} = 0$$

$$\begin{aligned} \omega^2 - 2K + 2 &= 0 \\ \omega(\omega^2 - K) &= 0 \end{aligned}$$

$$K = 2,\ \omega = \pm\sqrt{2},\ \text{or}\ K = 1,\ \omega = 0$$

Therefore,

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$



(d)

i. $0 < K < 1$
$N = 0,\ P = 1 \implies Z = 1$
One unstable closed-loop root.

ii. $1 < K < 2$
$N = -1,\ P = 1 \implies Z = 0$
Stable.

iii. $2 < K$
$N = 1,\ P = 1 \implies Z = 2$
Two unstable closed-loop roots.

(e) The closed-loop transfer function of this system is :

$$\frac{y(s)}{r(s)} = \frac{k\frac{1}{s-1}}{1 + k\frac{1}{s-1} \times \frac{s+2}{(s+1)^2+1}}$$
$$= \frac{K(s^2+2s+2)}{s^3+s^2+Ks+2K-2}$$

So the characteristic equation is :

$$\implies s^3 + s^2 + Ks + 2K - 2 = 0$$

Using the Routh's criterion,

$$\begin{array}{lcc} s^3: & 1 & K \\ s^2: & 1 & 2K-2 \\ s^1: & 2-K & 0 \\ s^0: & 2k=2 & \end{array}$$

For stability,

$$
\begin{aligned}
2-K &> 0 \\
2K-2 &> 0
\end{aligned}
$$

$$
\begin{array}{cc}
\Longrightarrow\ 2 > K > 1 & \\
0 < K < 1 & \text{Unstable} \\
1 < K < 2 & \text{Stable} \\
2 < K & \text{Unstable}
\end{array}
$$

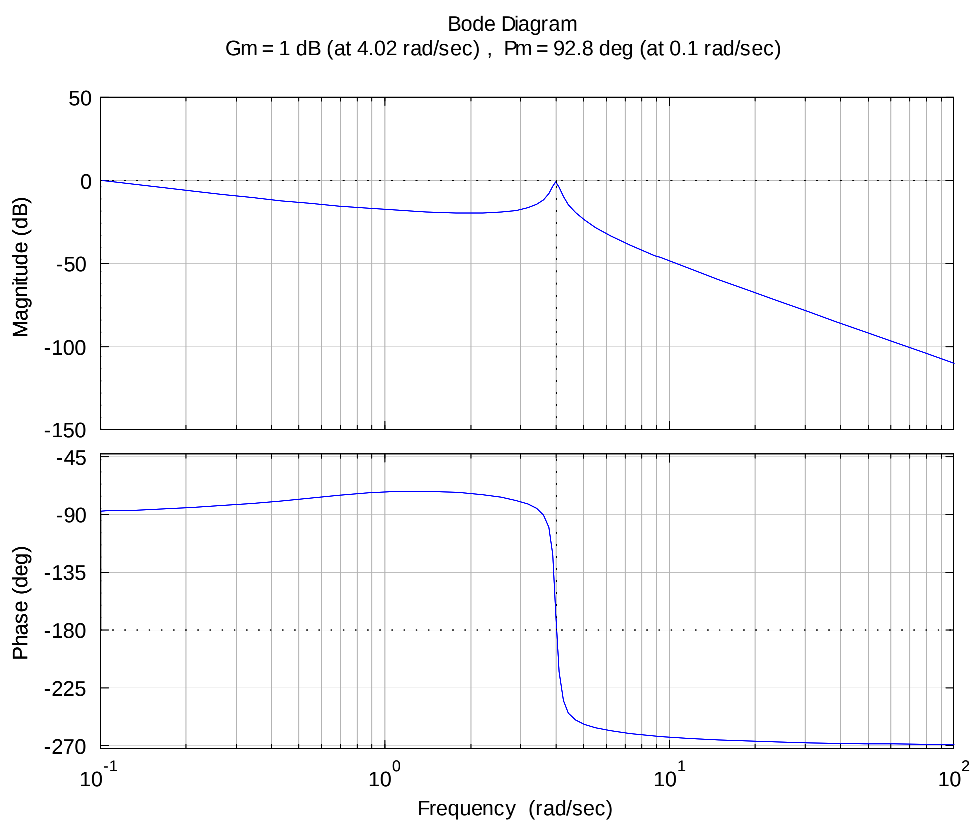
28. Suppose that in Fig. 6.90,

$$G(s) = \frac{3.2(s+1)}{s(s+2)(s^2+0.2s+16)}.$$

Use MATLAB's margin to calculate the PM and GM for $G(s)$ and comment on whether you think this system will have well damped closed-loop roots.

Solution :

MATLAB's margin plot for the given system is :



From the MATLAB margin routine, $PM = 92.8^o$. Based on this result, Fig. 6.36 suggests that the damping will be = 1; that is, the roots will be real. However, closer inspection shows that a very small increase in gain would result in an instability from the resonance leading one to believe that the damping of these roots is very small. Use of MATLAB's damp routine on the closed loop system confirms this where we see that there are two real poles ($\zeta = 1$) and two very lightly damped poles with $\zeta = 0.0027$. This is a good example where one needs to be careful to not use Matlab without thinking.

29. For a given system, show that the ultimate period P_u and the corresponding ultimate gain K_u for the Zeigler-Nichols method can be found using the following:

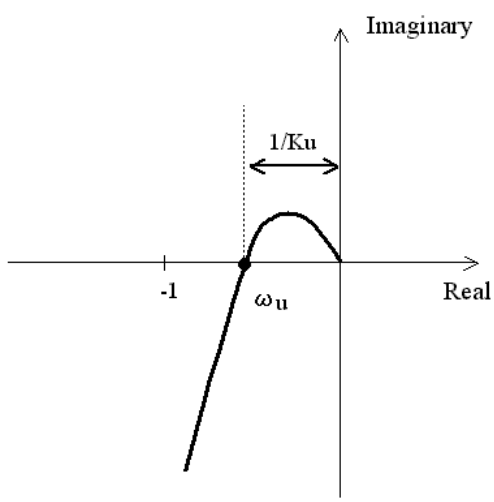
    (a) Nyquist diagram

    (b) Bode plot

    (c) root locus.

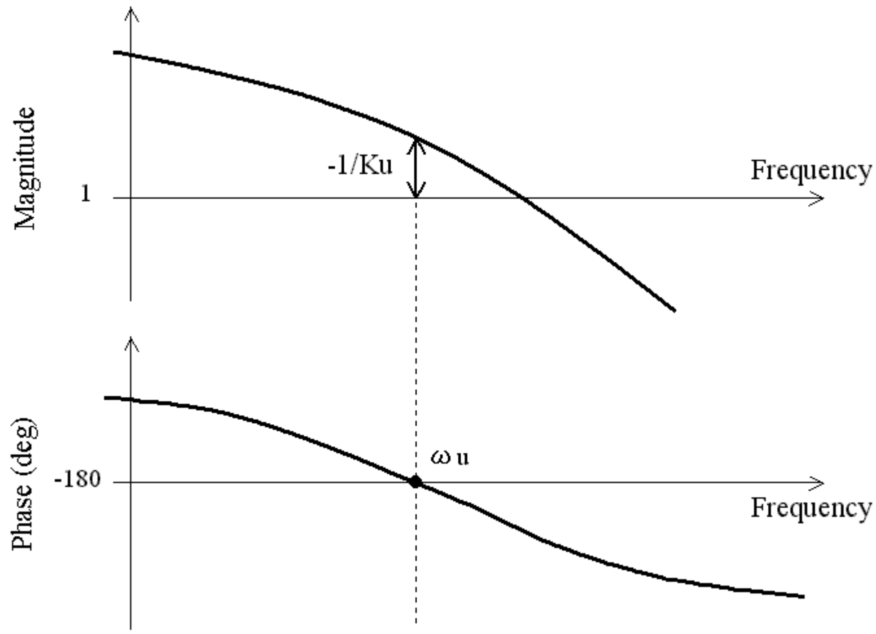
Solution :

(a) See sketch below.



$$P_u = \frac{2\pi}{\omega_u}$$

(b) See sketch below.



$$P_u = \frac{2\pi}{\omega_u}$$

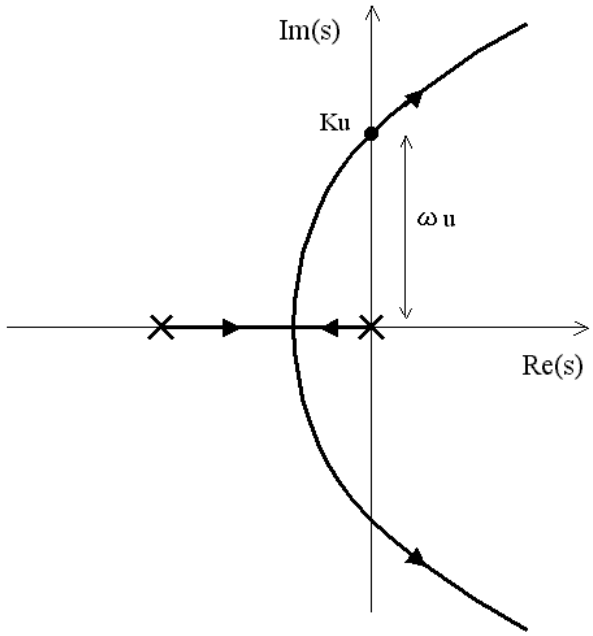
(c)

$$1 + K_u G(j\omega_u) = 0$$

$$1 + K_u \operatorname{Re}[G(j\omega_u)] + K_u j \operatorname{Im}[G(j\omega_u)] = 0$$

$$K_u = -\frac{1}{\operatorname{Re}[G(j\omega_u)]}$$

$$\operatorname{Im}[G(j\omega_u)] = 0; \text{ or } P_u = \frac{2\pi}{\omega_u}$$



30. If a system has the open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

with unity feedback, then the closed-loop transfer function is given by

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Verify the values of the PM shown in Fig. 6.36 for $\zeta = 0.1$, 0.4, and 0.7.

Solution :

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)},\ T(s) = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ	PM from Eq. 6.32	PM from Fig. 6.36	PM from Bode plot
0.1	10°	10°	11.4° ($\omega = 0.99$ rad/sec)
0.4	40°	44°	43.1° ($\omega = 0.85$ rad/sec)
0.7	70°	65°	65.2° ($\omega = 0.65$ rad/sec)

31. Consider the unity feedback system with the open-loop transfer function

$$G(s) = \frac{K}{s(s + 1)[(s^2/25) + 0.4(s/5) + 1]}.$$

(a) Use MATLAB to draw the Bode plots for $G(j\omega)$ assuming $K = 1$.

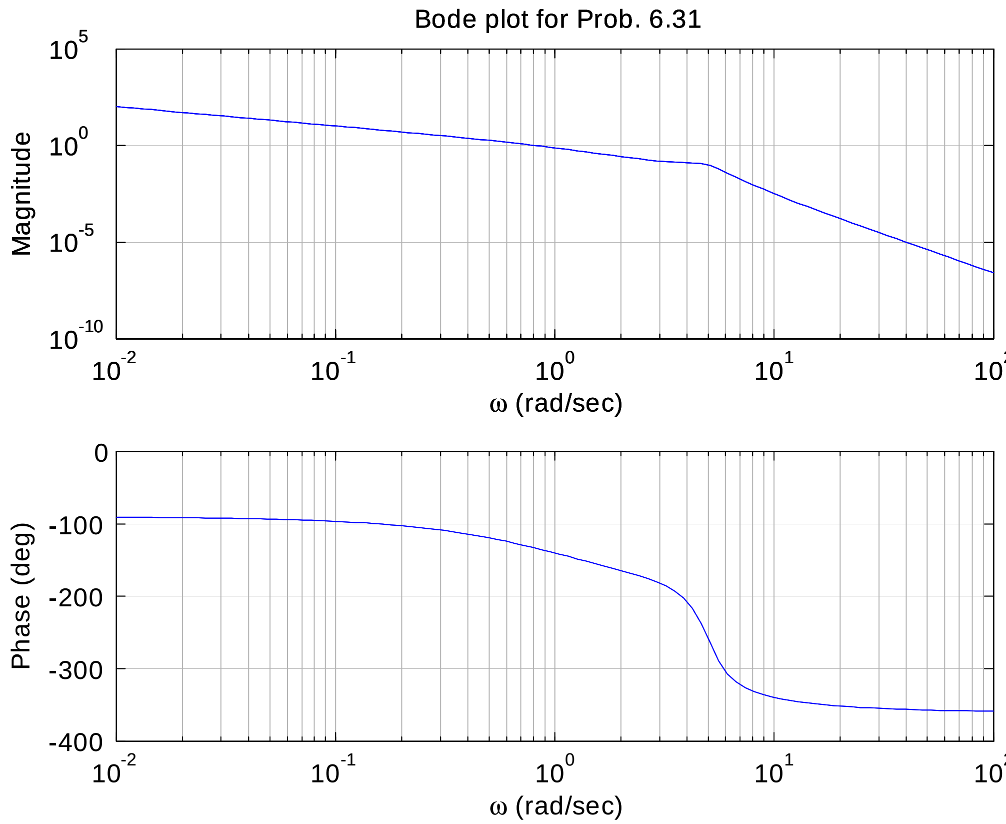
(b) What gain K is required for a PM of 45°? What is the GM for this value of K?

(c) What is K_v when the gain K is set for PM = 45°?

(d) Create a root locus with respect to K, and indicate the roots for a PM of 45°.

Solution :

(a) The Bode plot for $K = 1$ is shown below and we can see from margin that it results in a PM = 48^o.



(b) Although difficult to read the plot above, it is clear that a very slight increase in gain will lower the PM to 45^o, so try $K = 1.1$. The margin routine shows that this yields $PM = 45^o$ and $GM = 15$ db.

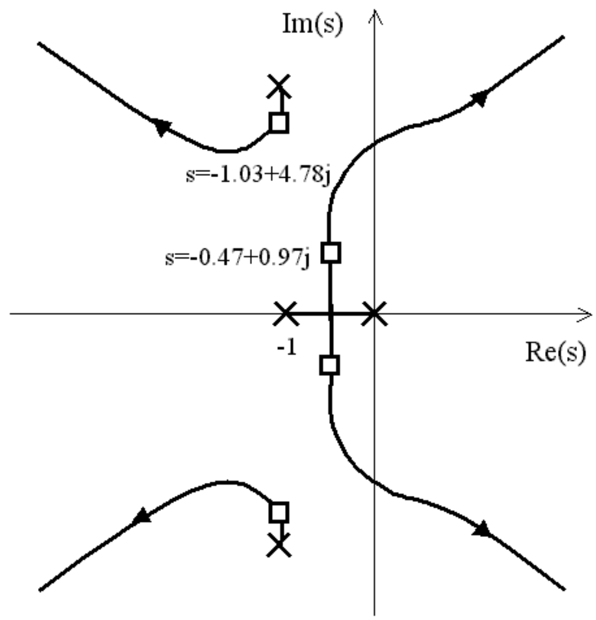
(c) $K_v = \lim_{s \to 0}\{sKG(s)\} = K = 1.1$ when K is set for PM=45°

$$K_v = 1.1$$

(d) The characteristic equation for PM of 45° :

$$1 + \frac{1.1}{s(s+1)\left[\left(\frac{s}{5}\right)^2 + 0.4\left(\frac{s}{5}\right) + 1\right]} = 0$$

$$\implies \quad s^4 + 3s^3 + 27s^2 + 25s + 27.88 = 0$$
$$\implies \quad s = -1.03 \pm j4.78, \ -0.47 \pm j0.97$$



32. For the system depicted in Fig. 6.92(a), the transfer-function blocks are defined by

$$G(s) = \frac{1}{(s+2)^2(s+4)} \quad \text{and} \quad H(s) = \frac{1}{s+1}.$$

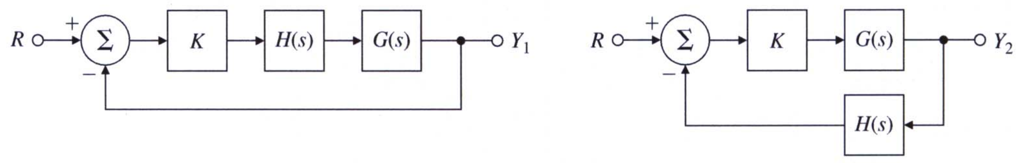
(a) Using rlocus and rlocfind, determine the value of K at the stability boundary.

(b) Using rlocus and rlocfind, determine the value of K that will produce roots with damping corresponding to $\zeta = 0.707$.

(c) What is the gain margin of the system if the gain is set to the value determined in part (b)? Answer this question *without* using any frequency response methods.

(d) Create the Bode plots for the system, and determine the gain margin that results for PM = 65°. What damping ratio would you expect for this PM?

Figure 6.92: Block diagram for Problem 32: (a) unity feedback; (b) $H(s)$ in feedback



(a) (b)

(e) Sketch a root locus for the system shown in Fig. 6.92(b).. How does it differ from the one in part (a)?

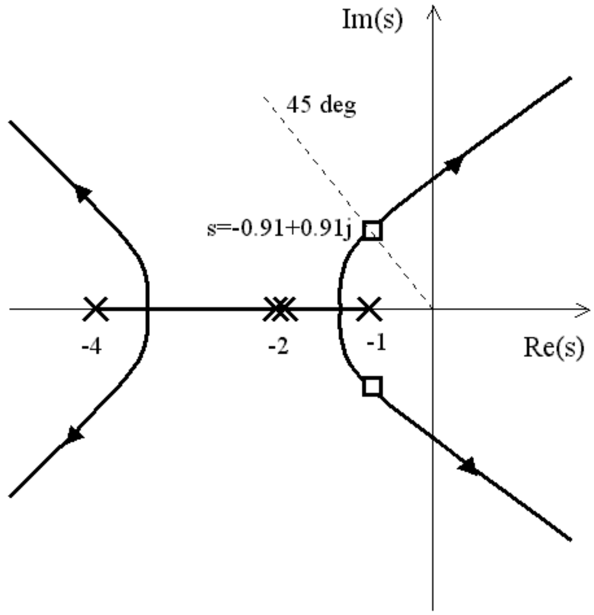
(f) For the systems in Figs. 6.92(a) and (b), how does the transfer function $Y_2(s)/R(s)$ differ from $Y_1(s)/R(s)$? Would you expect the step response to $r(t)$ be different for the two cases?

Solution :

(a) The root locus crosses $j\omega$ axis at $s_0 = j2$.

$$\begin{aligned} K &= \frac{1}{|H(s_0)G(s_0)|}|_{s_0=j2} \\ &= |j2+1|\,|j2+4|\,|j2+2|^2 \end{aligned}$$

$$\Longrightarrow K = 80$$



(b)

$$\zeta = 0.707 \implies 0.707 = \sin\theta \implies \theta = 45^\circ$$

From the root locus given,

$$\begin{aligned} s_1 &= -0.91 + j0.91 \\ K &= \frac{1}{|H(s_1)G(s_1)|}\Big|_{s_1=-0.91+j0.91} \\ &= |0.01 + j0.91|\,|3.09 + j0.91|\,|1.09 + j0.91|^2 \end{aligned}$$

$$\implies K = 5.9$$

(c)

$$GM = \frac{K_a}{K_b} = \frac{80}{5.9} = 13.5$$

(d) From the Root Locus :

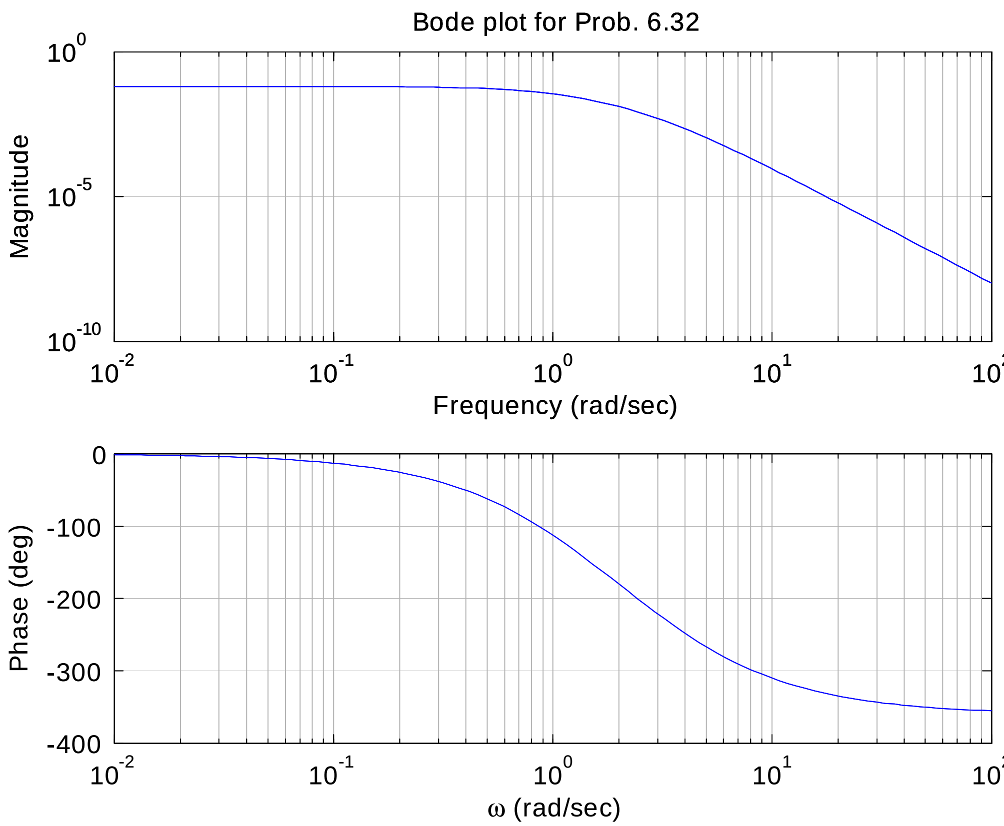
$$G(s)H(s) = \frac{1}{(s+1)(s+2)^2(s+4)}$$

PM=65° when $K = 30$. Instability occurs when $K = 80.0$.

$$\implies GM = 2.67$$

We approximate the damping ratio by $\zeta \simeq \frac{PM}{100}$

$$\zeta \simeq \frac{65}{100} = 0.65$$



(e) The root locus for Fig.6.92(a) is the same as that of Fig.6.92(b).

(f)

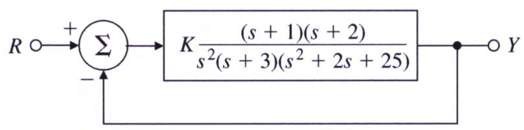
$$\frac{Y_1(s)}{R(s)} = \frac{KG(s)H(s)}{1+KG(s)H(s)} = \frac{K}{(s+1)(s+2)^2(s+4)+K}$$
$$\frac{Y_2(s)}{R(s)} = \frac{KG(s)}{1+KG(s)H(s)} = \frac{K(s+1)}{(s+1)(s+2)^2(s+4)+K}$$

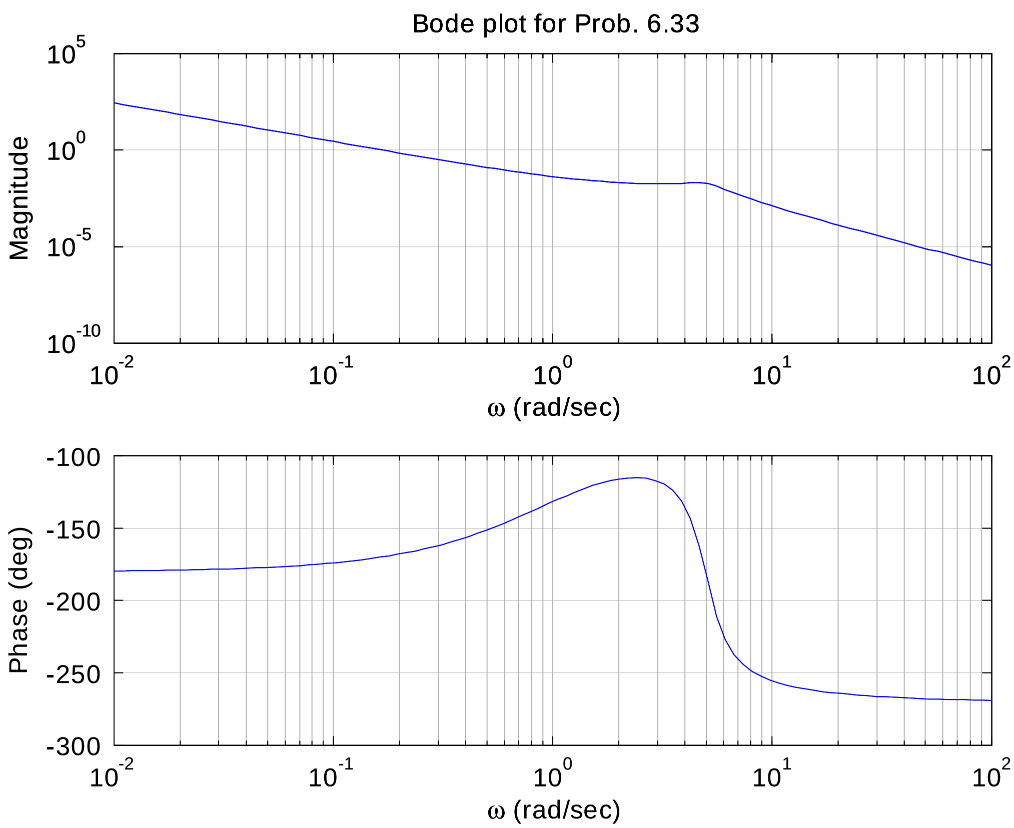
$\frac{Y_1(s)}{R(s)}$ and $\frac{Y_2(s)}{R(s)}$ have the same closed-loop poles. However, $\frac{Y_2(s)}{R(s)}$ has a zero, while $\frac{Y_1(s)}{R(s)}$ doesn't have a zero. We would therefore expect more overshoot from system (b).

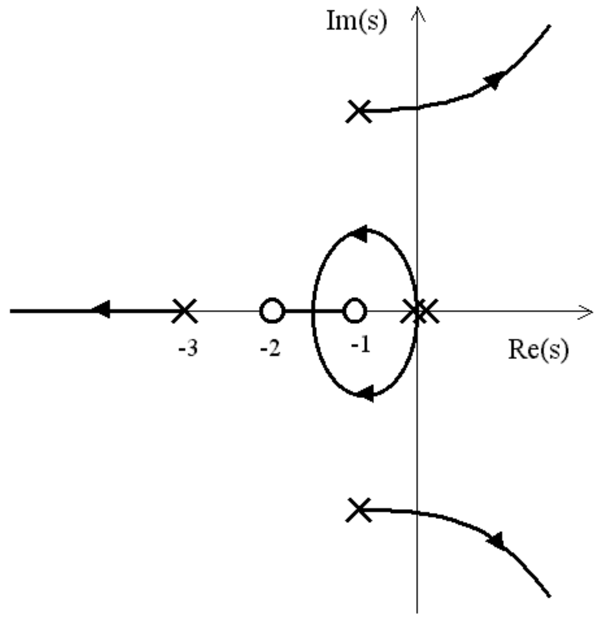
33. For the system shown in Fig. 6.93, use Bode and root-locus plots to determine the gain and frequency at which instability occurs. What gain (or gains) gives a PM of 20°? What is the GM when PM = 20°?

Solution :

Figure 6.93: Control system for Problem 33







The system with $K = 1$ gives,

$$\begin{aligned} GM &= 52 \ (\omega = 5 \text{ rad/sec}) \\ PM &= 10^\circ \ (\omega = 0.165 \text{ rad/sec}) \end{aligned}$$

Therefore, instability occurs at $K_0 = 52$ and $\omega = 5$ rad/sec.

From the Bode plot, a PM of 20° is given by,

$$\begin{aligned} K_1 &= 3.9 \ (\omega = 0.33 \text{ rad/sec}), \quad GM = \frac{52}{3.9} = 13 \\ K_2 &= 49 \ (\omega = 4.6 \text{ rad/sec}), \quad GM = \frac{52}{49} = 1.06 \end{aligned}$$

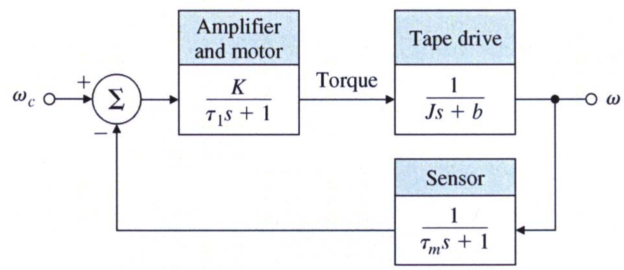
34. A magnetic tape-drive speed-control system is shown in Fig. 6.94. The speed sensor is slow enough that its dynamics must be included. The speed-measurement time constant is $\tau_m = 0.5$ sec; the reel time constant is $\tau_r = J/b = 4$ sec, where $b =$ the output shaft damping constant $=$ 1 N · m · sec; and the motor time constant is $\tau_1 = 1$ sec.

    (a) Determine the gain K required to keep the steady-state speed error to less than 7% of the reference-speed setting.

    (b) Determine the gain and phase margins of the system. Is this a good system design?

Solution :

Figure 6.94: Magnetic tape-drive speed control



(a) From Table 4.1, the error for this Type 1 system is

$$e_{ss} = \frac{1}{1+K}\,|\Omega_c|$$

Since the steady-state speed error is to be less than 7% of the reference speed,

$$\frac{1}{1+K_p} \leq 0.07$$

and for the system in Fig. 6.96 with the numbers plugged in, we see that $K_p = K$. Therefore, $K \geq 13$.

(b)

$$\begin{aligned} |G(s)| &= 0.79 \text{ at } \angle GH = -180^\circ \Longrightarrow GM = \frac{1}{|GH|} = 1.3 \\ \angle GH &= -173^\circ \text{ at } |G(s)| = 1 \Longrightarrow PM = \angle G + 180^\circ = 7^\circ \end{aligned}$$

GM is low $\Longrightarrow$ The system is very close to instability.

PM is low $\Longrightarrow$ The damping ratio is low. $\Longrightarrow$ High overshoot.

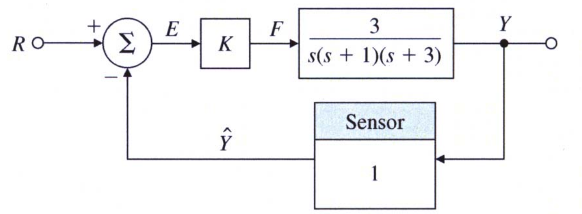
We see that to have a more stable system we have to lower the gain. With small gain, e_{ss} will be higher. Therefore, this is not a good design, and needs compensation.

35. For the system in Fig. 6.95, determine the Nyquist plot and apply the Nyquist criterion

    (a) to determine the range of values of K (positive and negative) for which the system will be stable, and

    (b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

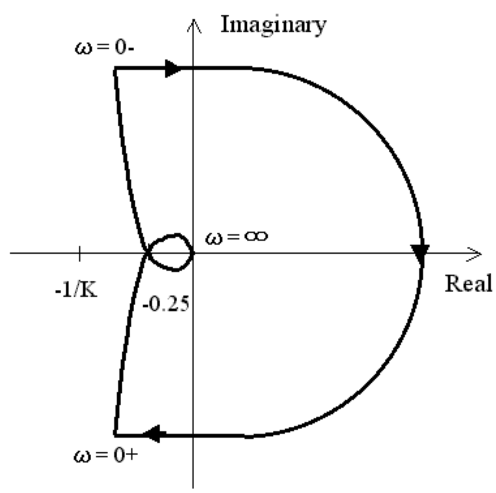
Figure 6.95: Control system for Problems 35, 69, and 70



Solution :

(a) & b.

$$KG(s) = K\frac{3}{s(s+1)(s+3)}$$



From the Nyquist plot above, we see that:

i.

$$-\infty < -\frac{1}{K} < -\frac{1}{4} \implies 0 < K < 4$$

There are no RHP open loop roots, hence $P = 0$ for all cases. For $0 < K < 4$, no encirclements of -1 so $N = 0$,

$$N = 0,\ P = 0 \implies Z = 0$$

The closed-loop system is stable. No roots in RHP.

ii.

$$-\frac{1}{4} < -\frac{1}{K} < 0 \implies 4 < K < \infty$$

Two encirclements of the -1 point, hence

$$N = 2,\ P = 0 \implies Z = 2$$

Two closed-loop roots in RHP.

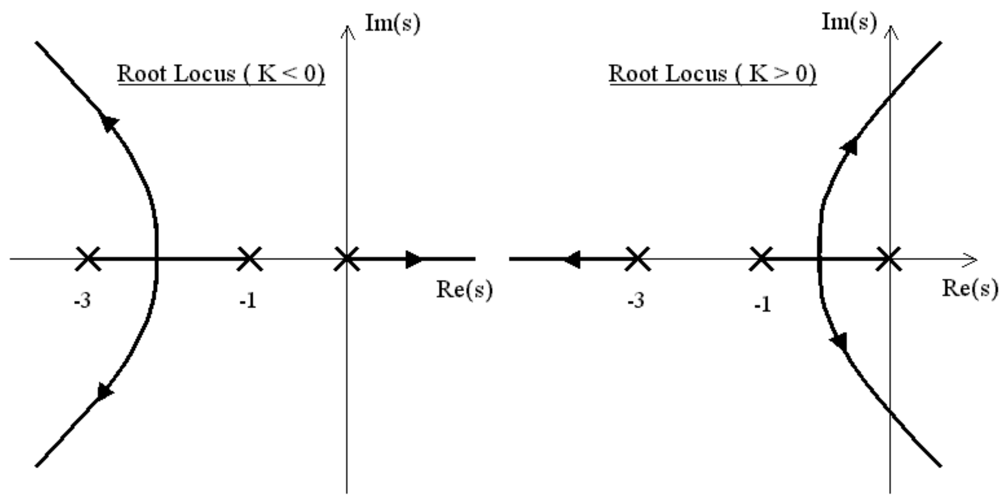
iii.

$$0 < -\frac{1}{K} \implies K < 0$$

$$N = 1,\ P = 0 \implies Z = 1$$

One closed-loop root in RHP.

The root loci below show the same results.



36. For the system shown in Fig. 6.96, determine the Nyquist plot and apply the Nyquist criterion.

(a) to determine the range of values of K (positive and negative) for which the system will be stable, and

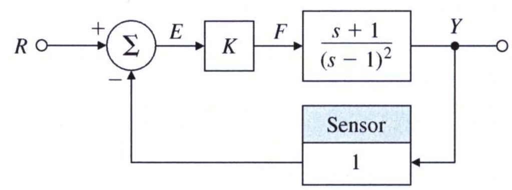
(b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

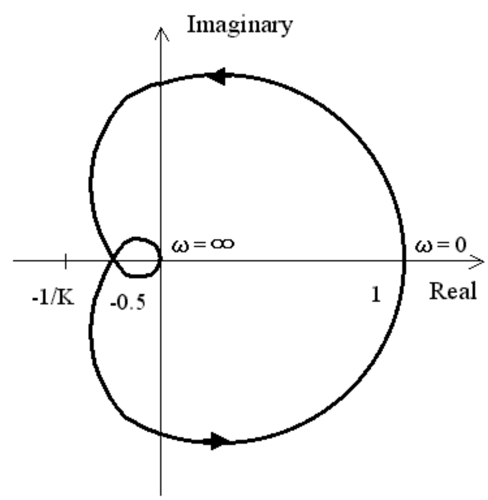
Solution :

(a) & b.

$$KG(s) = K\frac{s+1}{(s-1)^2}$$

Figure 6.96: Control system for Problem 36





From the Nyquist plot we see that:

i.

$$-\infty < -\frac{1}{K} < -\frac{1}{2} \implies 0 < K < 2$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

ii.

$$-\frac{1}{2} < -\frac{1}{K} < 0 \implies 2 < K$$

$$N = -2,\ P = 2 \implies Z = 0$$

The closed-loop system is stable.

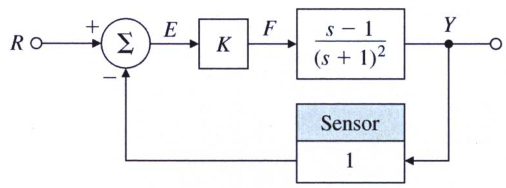
iii.

$$0 < -\frac{1}{K} < 1 \implies K < -1$$

$$N = -1,\ P = 2 \implies Z = 1$$

One closed-loop root in RHP.

Figure 6.97: Control system for Problem 37



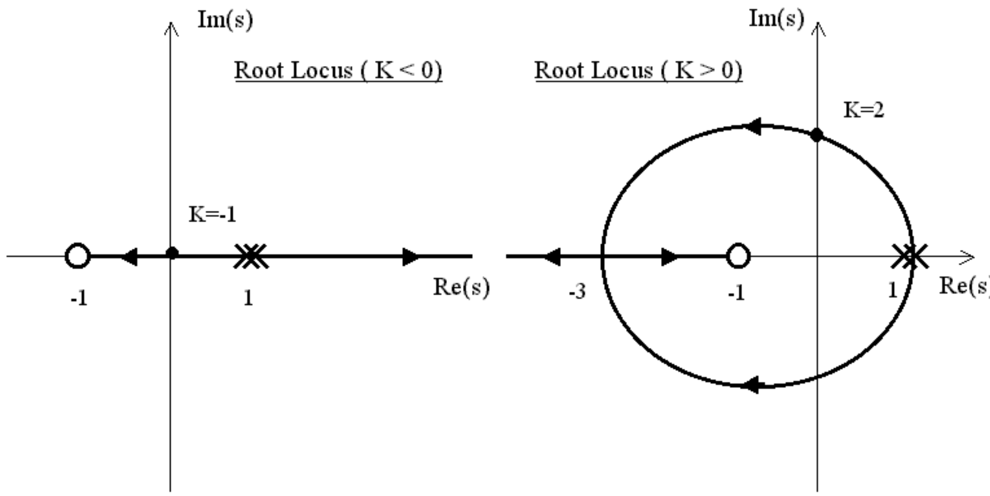
iv.

$$1 < -\frac{1}{K} < \infty \implies -1 < K < 0$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

These results are confirmed by looking at the root loci below:



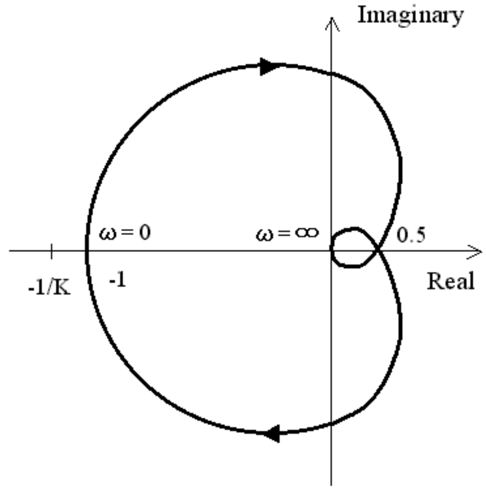
37. For the system shown in Fig. 6.97, determine the Nyquist plot and apply the Nyquist criterion.

    (a) to determine the range of values of K (positive and negative) for which the system will be stable, and

    (b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

Solution :

(a) & b.



$$KG(s) = K\frac{s-1}{(s+1)^2}$$

From the Nyquist plot we see that:

i.

$$-\infty < -\frac{1}{K} < -1 \implies 0 < K < 1$$

$$N = 0,\ P = 0 \implies Z = 0$$

The closed-loop system is stable.

ii.

$$-1 < -\frac{1}{K} < 0 \implies 1 < K$$

$$N = 1,\ P = 0 \implies Z = 1$$

One closed-loop root in RHP.

iii.

$$0 < -\frac{1}{K} < \frac{1}{2} \implies K < -2$$

$$N = 2,\ P = 0 \implies Z = 2$$

Two closed-loop roots in RHP.

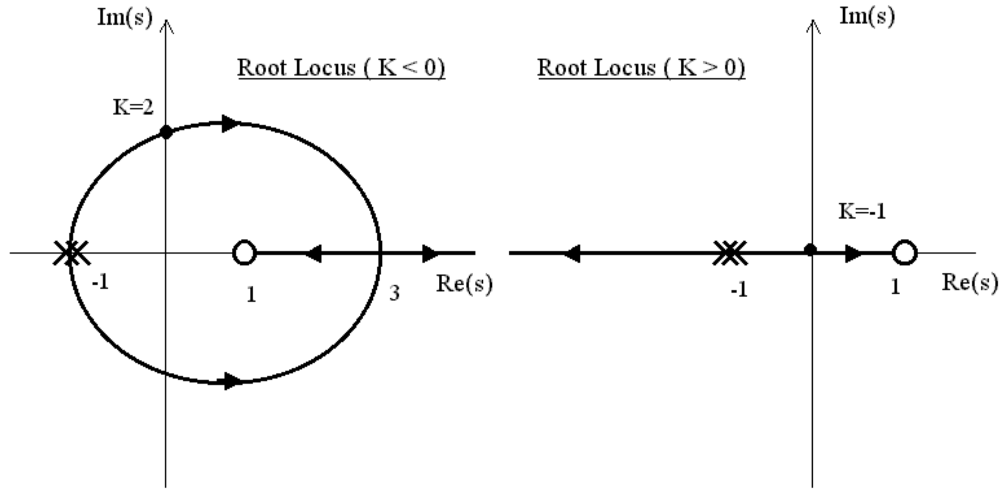
iv.

$$\frac{1}{2} < -\frac{1}{K} \implies -2 < K < 0$$

$$N = 0,\ P = 0 \implies Z = 0$$

The closed-loop system is stable.

These results are confirmed by looking at the root loci below:



38. The Nyquist diagrams for two stable, open-loop systems are sketched in Fig. 6.98. The proposed operating gain is indicated as K_0, and arrows indicate increasing frequency. In each case give a rough estimate of the following quantities for the closed-loop (unity feedback) system:

    (a) phase margin
    (b) damping ratio
    (c) range of gain for stability (if any)
    (d) system type (0, 1, or 2).

    Solution :

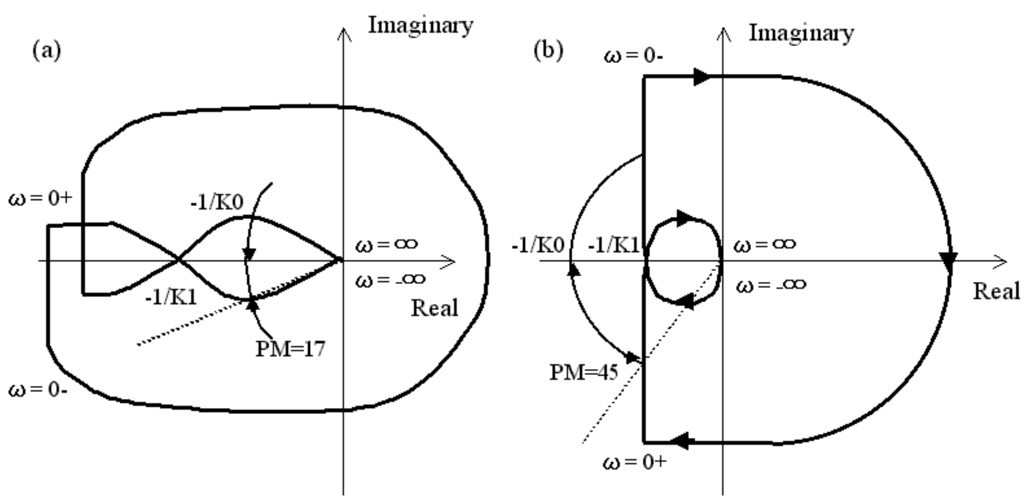
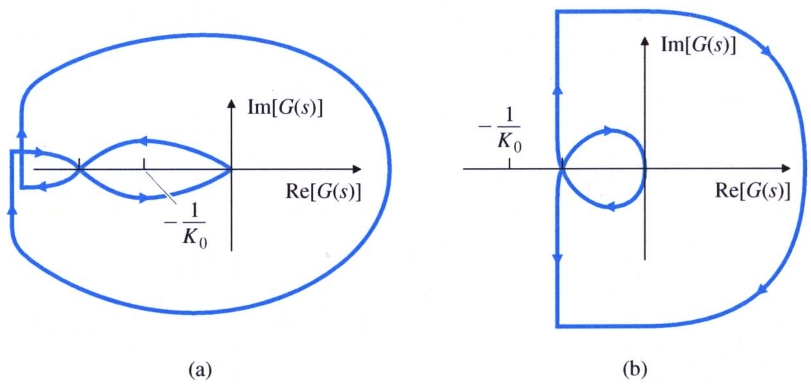


Figure 6.98: Nyquist plots for Problem 38



For both, with $K = K_0$:

$$N = 0,\ P = 0 \implies Z = 0$$

Therefore, the closed-loop system is stable.

	Fig.6.98(a)	Fig.6.98(b)
a. PM	$\simeq 17°$	$\simeq 45°$
b. Damping ratio	$0.17(\simeq \frac{17}{100})$	$0.45(\simeq \frac{45}{100})$

c. To determine the range of gain for stability, call the value of K where the plots cross the negative real axis as K_1. For case (a), $K > K_1$ for stability because gains lower than this amount will cause the -1 point to be encircled. For case (b), $K < K_1$ for stability because gains greater than this amount will cause the -1 point to be encircled.

d. For case (a), the 360^o loop indicates two poles at the origin, hence the system is Type 2. For case (b), the 180^o loop indicates one pole at the origin, hence the system is Type 1.

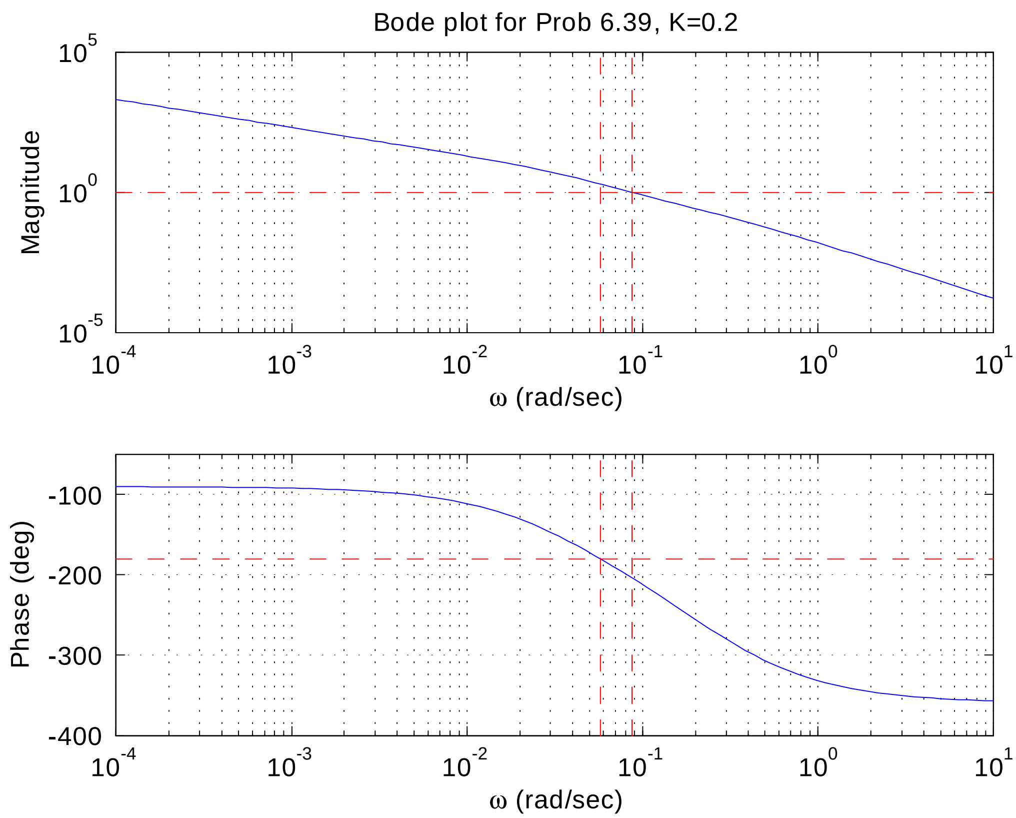
39. The steering dynamics of a ship are represented by the transfer function

$$\frac{V(s)}{\delta_r(s)} = G(s) = \frac{K[-(s/0.142) + 1]}{s(s/0.325 + 1)(s/0.0362 + 1)},$$

where v is the ship's lateral velocity in meters per second, and δ_r is the rudder angle in radians.

(a) Use the MATLAB command `bode` to plot the log magnitude and phase of $G(j\omega)$ for $K = 0.2$

(b) On your plot, indicate the crossover frequency, PM, and GM,



(c) Is the ship steering system stable with $K = 0.2$?

(d) What value of K would yield a PM of 30^o and what would the crossover frequency be?

Solution :

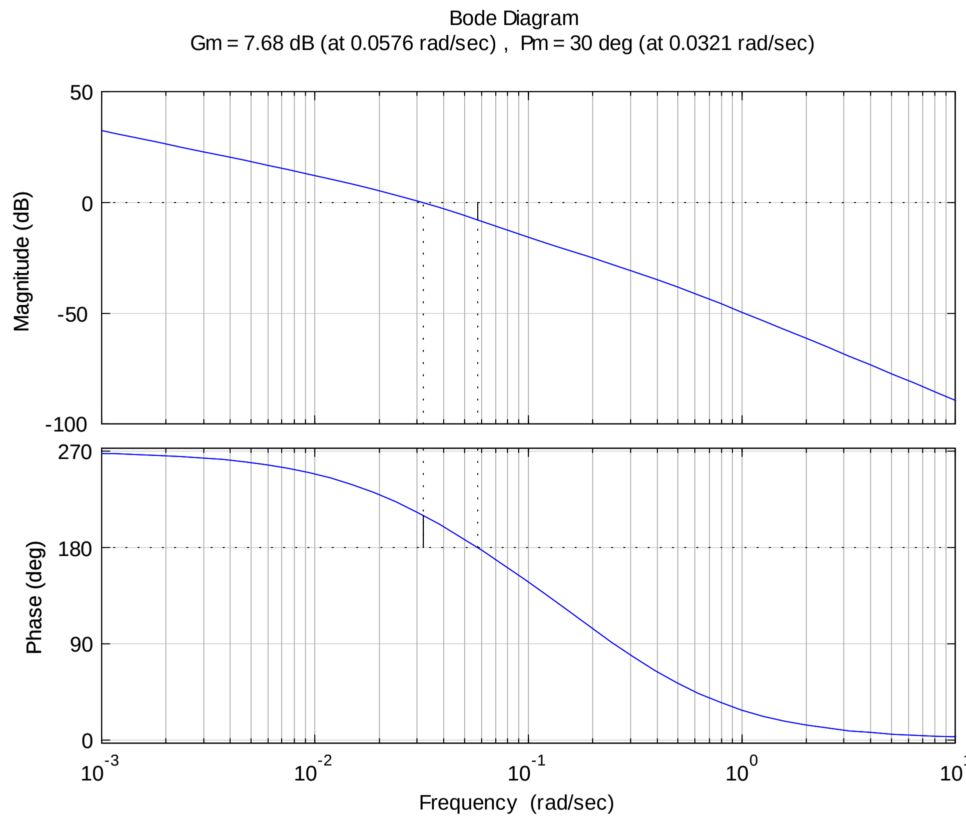
(a) The Bode plot for $K = 0.2$ is :

(b) From the Bode plot above :

$$\begin{aligned} \omega_c &= 0.0870 \text{ rad/sec} \\ PM &= -23.9 \text{ deg} \ \ \text{(unstable)} \\ GM &= 1/1.95 = 0.51 \text{ (unstable)} \end{aligned}$$

(c) Since $PM < 0$, the closed-loop system with $K = 0.2$ is unstable.

(d) From the Bode plot above, we can get better PM by decreasing the gain K. Then we will find that $K = 0.0421$ yields $PM = 30^\circ$ at the crossover frequency $\omega_c = 0.032$ rad/sec. The Bode plot with $K = 0.0421$ is :



and we see that the PM is 30 Deg, as desire. We can see why the decreasing gain produced a better result from the RL, which follows The RL shows that increasing the gain moves the two complex roots from the RHP to the LHP.... barely!

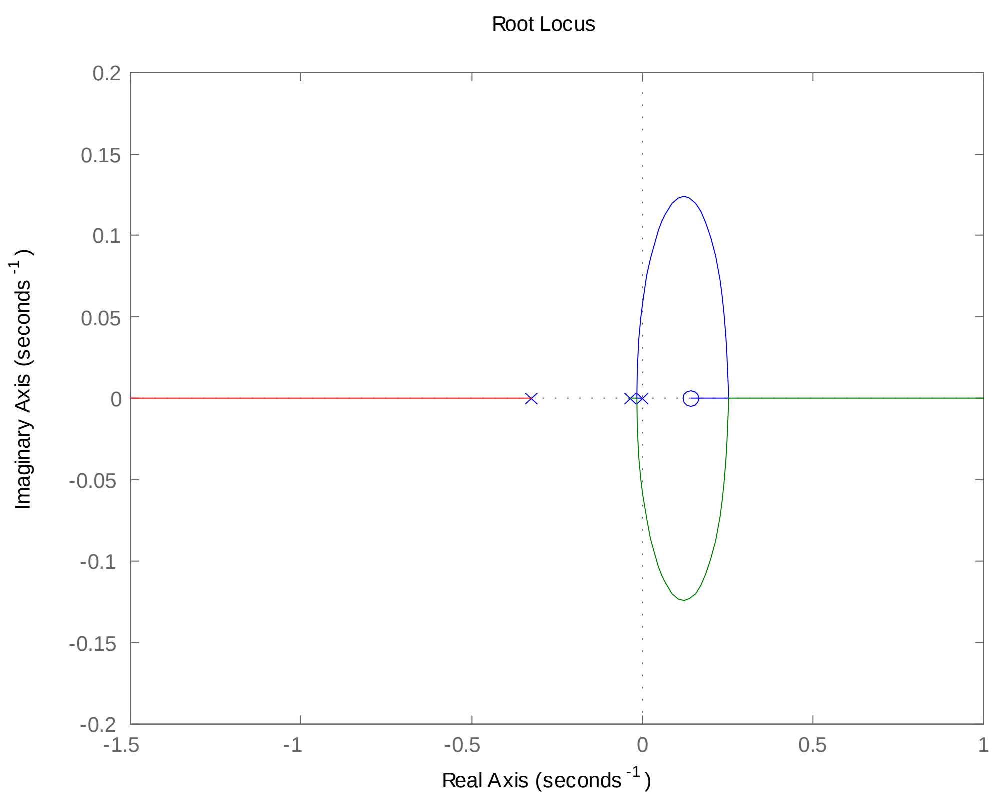
40. For the open-loop system

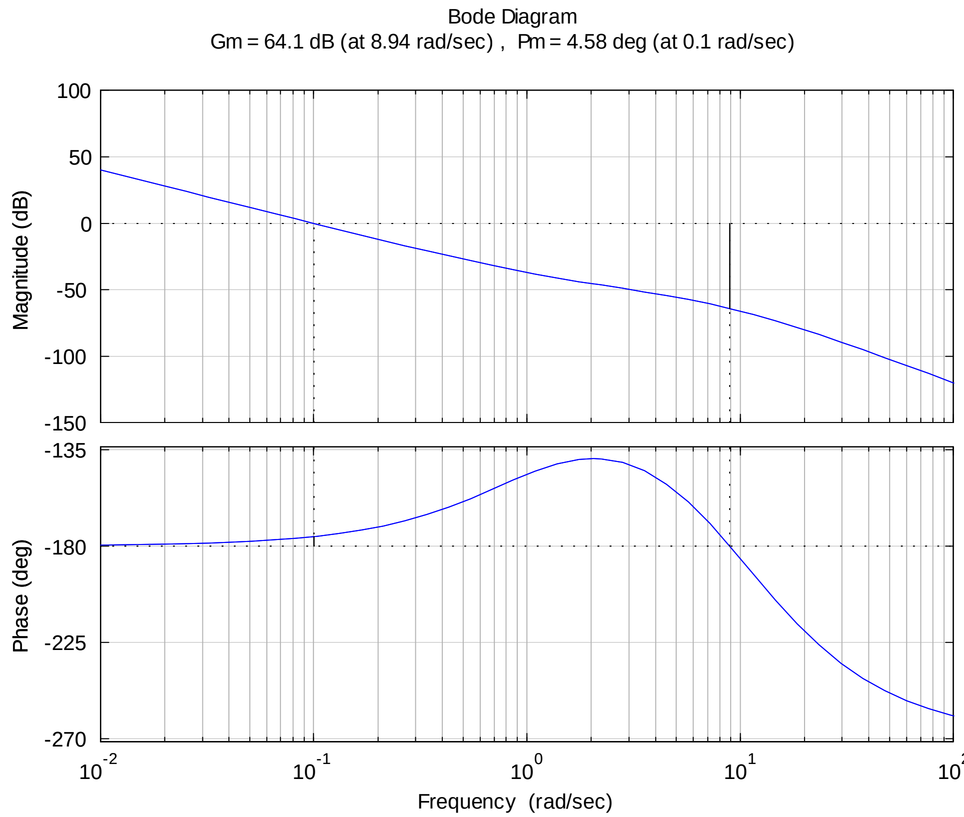
$$KG(s) = \frac{K(s+1)}{s^2(s+10)^2}.$$

Determine the value for K at the stability boundary and the values of K at the points where $PM = 30°$.

Solution :

The bode plot of this system with $K = 1$ is :


Root Locus
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-1.5
-1
-0.5
0
0.5
1
Real Axis (seconds $^{-1}$)
Imaginary Axis (seconds $^{-1}$)



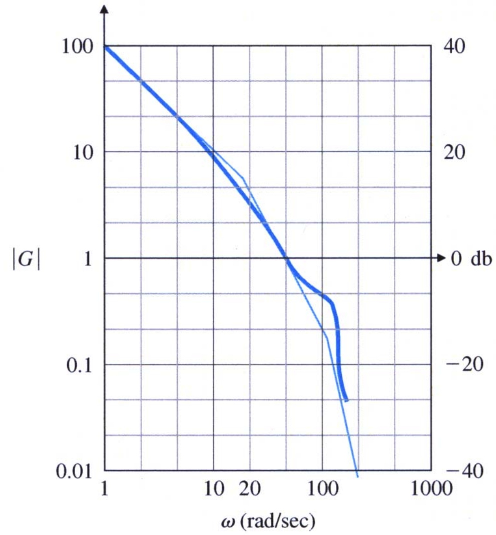
Since $GM = 64.1$ db ($\simeq 1600$), the range of K for stability is :

$$K < 1600$$

From the Bode plot, the magnitude at the frequency with $-150°$ phase is 0.0188 (-34.5 dB) at 0.8282 rad/sec and 0.00198 (-54.1 db) at 4.44 rad/sec. Therefore, the values of K at the points where $PM = 30°$ is :

$$\begin{aligned} K &= \frac{1}{0.0188} = 53.2, \\ K &= \frac{1}{0.00198} = 505 \end{aligned}$$

Figure 6.99: Magnitude frequency response for Problem 41



(a) Problems and Solutions for Section 6.5

41. The frequency response of a plant in a unity feedback configuration is sketched in Fig. 6.99. Assume the plant is open-loop stable and minimum phase.

    (a) What is the velocity constant K_v for the system as drawn?

    (b) What is the damping ratio of the complex poles at $\omega = 100$?

    (c) What is the PM of the system as drawn? (Estimate to within $\pm 10^o$.)

Solution :

(a) From Fig. 6.99,

$$K_v = \lim_{s \to 0} sG = |\text{Low frequency asymptote of } G(j\omega)|_{\omega=1} = 100)$$

(b) Let

$$G_1(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

For the second order system $G_1(s)$,

$$|G_1(j\omega)|_{\omega=1} = \frac{1}{2\zeta} \qquad (1)$$

From Fig. 6.99 :

$$|G_1(j\omega)|_{\omega=100} = \frac{|G(j\omega)|_{\omega=100}}{|\text{Asymptote of } G(j\omega)|_{\omega=100}} \cong \frac{0.4}{0.2} = 2 \qquad (2)$$

From (1) and (2) we have :

$$\frac{1}{2\zeta} = 2 \implies \zeta = 0.25$$

(c) Since the plant is a minimum phase system, we can apply the Bode's approximate gain-phase relationship.

When $|G| = 1$, the slope of $|G|$ curve is $\cong$ -2.

$$\implies \angle G(j\omega) \cong -2 \times 90^\circ \;\; = -180^\circ$$

$$PM \cong \angle G(j\omega) + 180^\circ \;\; = 0^\circ$$

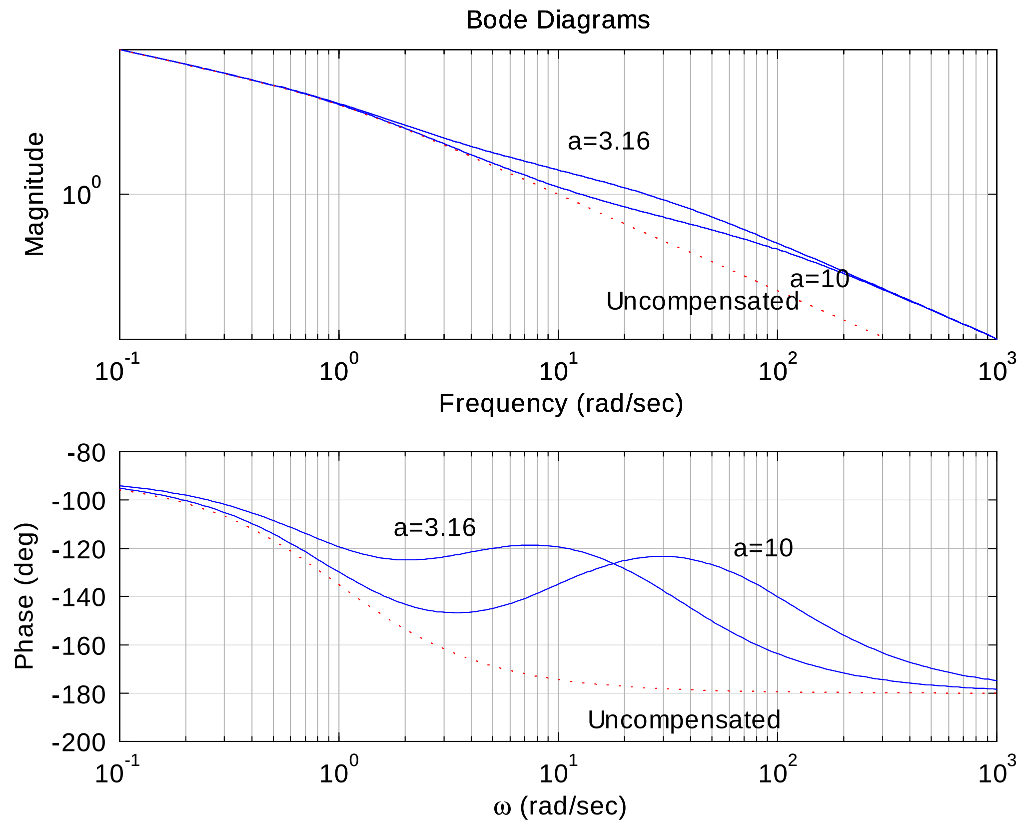
Note : Actual PM by Matlab calculation is 6.4°, so this approximation is within the desired accuracy.

42. For the system

$$G(s) = \frac{100(s/a+1)}{s(s+1)(s/b+1)},$$

where $b = 10a$, find the approximate value of a that will yield the best PM by sketching only candidate values of the frequency response magnitude.

Solution :



Without the zero and pole that contain the a & b terms, the plot of $|G|$ shows a slope of -2 at the $|G| = 1$ crossover at 10 rad/sec. We clearly need to install the zero and pole with the a & b terms somewhere at frequencies greater 1 rad/sec. This will increase the slope from -2 to -1 between the zero and pole. So the problem simplifies to selecting a so that the -1 slope region between the zero and pole brackets the crossover frequency. That scenario will maximize the PM. Referring to the plots above, we see that $3.16 < a < 10$, makes the slope of the asymptote of $|G|$ be -1 at the crossover and represent the two extremes of possibilities for a -1 slope. The maximum PM will occur half way between these extremes on a log scale, or

$$\Longrightarrow\ a = \sqrt{3.16 \times 10} = 5.6$$

Note : Actual PM is as follows :

$$
\begin{aligned}
PM &= 46.8^\circ \text{ for } a = 3.16\ (\omega_c = 25.0 \text{ rad/sec}) \\
PM &= 58.1^\circ \text{ for } a = 5.6\ \ (\omega_c = 17.8 \text{ rad/sec}) \\
PM &= 49.0^\circ \text{ for } a = 10\ (\omega_c = 12.6 \text{ rad/sec})
\end{aligned}
$$

Problem and Solution for Section 6.6

43. For the open-loop system

$$KG(s) = \frac{K(s+1)}{s^2(s+10)^2}.$$

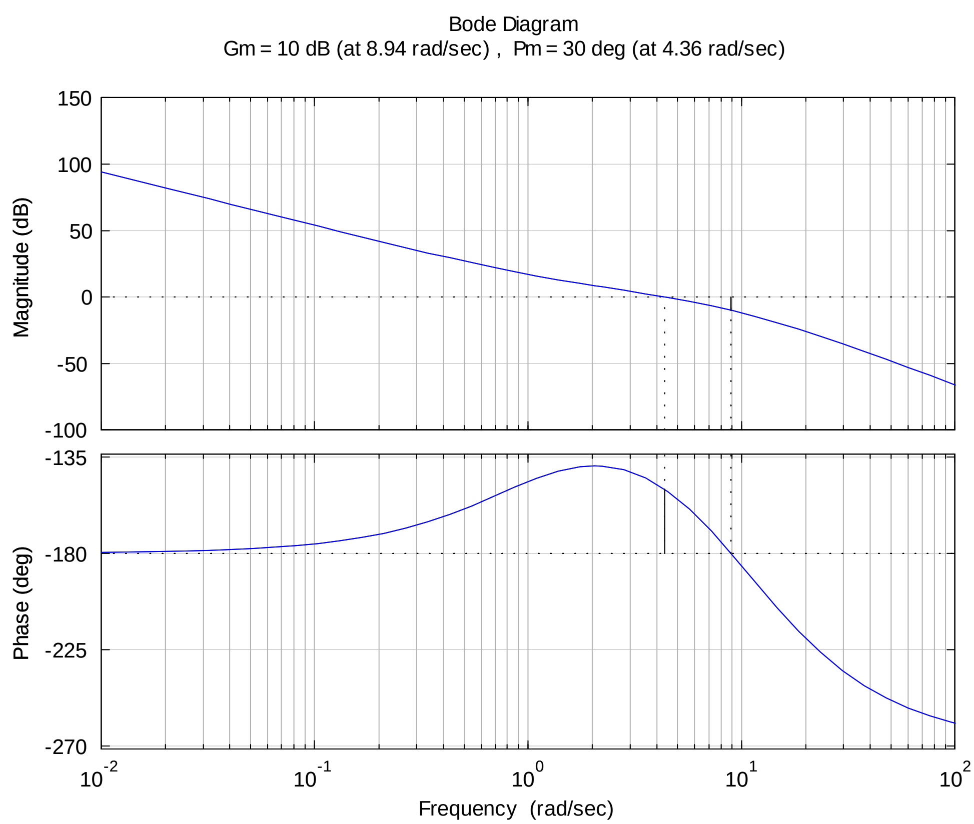
Determine the value for K that will yield PM $\geq 30°$ and the maximum possible closed-loop bandwidth. Use MATLAB to find the bandwidth.

Solution :

From the result of Problem 6.40., the value of K that will yield $PM \geq 30°$ is :

$$53.2 \leq K \leq 505$$

The maximum closed-loop bandwidth will occur with the maximum gain K within the allowable region; therefore, the maximum bandwidth will occur with $K = 505$. The Bode plot of the closed loop system with $K = 505$ is :



Looking at the point with Magnitude 0.707(-3 db), the maximum possible closed-loop bandwidth is :

$$\omega_{BW,\ \max} \simeq 7.7 \text{ rad/sec.}$$

Problems and Solutions for Section 6.7

44. For the lead compensator

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1},$$

where $\alpha < 1$.

(a) Show that the phase of the lead compensator is given by

$$\phi = \tan^{-1}(T_D\omega) - \tan^{-1}(\alpha T_D \omega).$$

(b) Show that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{T_D\sqrt{\alpha}},$$

and that the maximum phase corresponds to

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}.$$

(c) Rewrite your expression for $\omega_{\max}$ to show that the maximum-phase frequency occurs at the geometric mean of the two corner frequencies on a logarithmic scale:

$$\log\omega_{\max} = \frac{1}{2}\left(\log\frac{1}{T_D} + \log\frac{1}{\alpha T_D}\right).$$

(d) To derive the same results in terms of the pole-zero locations, rewrite $D_c(s)$ as

$$D_c(s) = \frac{s+z}{s+p},$$

and then show that the phase is given by

$$\phi = \tan^{-1}\left(\frac{\omega}{|z|}\right) - \tan^{-1}\left(\frac{\omega}{|p|}\right),$$

such that

$$\omega_{\max} = \sqrt{|z||p|}.$$

Hence the frequency at which the phase is maximum is the square root of the product of the pole and zero locations.

Solution :

(a) The frequency response is obtained by letting $s = j\omega$,

$$D_c(j\omega) = K\frac{T_D j\omega + 1}{\alpha T_D j\omega + 1}$$

The phase is given by,$\phi = \tan^{-1}(T_D\omega) - \tan^{-1}(\alpha T_D\omega)$

(b) Using the trigonometric relationship,

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

then

$$\tan(\phi) = \frac{T_D\omega - \alpha T_D\omega}{1 - \alpha T_D^2\omega^2}$$

and since,

$$\sin^2(\phi) = \frac{\tan^2(\phi)}{1 + \tan^2(\phi)}$$

then

$$\sin(\phi) = \sqrt{\frac{\omega^2 T_D^2(1-\alpha)^2}{1 + \alpha^2\omega^4 T_D^4 + (1+\alpha^2)\omega^2 T_D^2}}$$

To determine the frequency at which the phase is a maximum, let us set the derivative with respect to ω equal to zero,

$$\frac{d\sin(\phi)}{d\omega} = 0$$

which leads to

$$2\omega T_D^2(1-\alpha)^2(1 - \alpha\omega^4 T_D^4) = 0$$

The value $\omega = 0$ gives the maximum of the function and setting the second part of the above equation to zero then,

$$\omega^4 = \frac{1}{\alpha^2 T_D^4}$$

or

$$\omega_{\max} = \frac{1}{\sqrt{\alpha}T_D}$$

The maximum phase contribution, that is, the peak of the $\angle D(s)$ curve corresponds to,

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

or

$$\alpha = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}}$$

$$\tan\phi_{\max} = \frac{\omega_{\max}T_D - \alpha\omega_{\max}T_D}{1 + \omega_{\max}^2 T_D^2} = \frac{1-\alpha}{2\sqrt{\alpha}}$$

(c) The maximum frequency occurs midway between the two break frequencies on a logarithmic scale,

$$\begin{aligned}\log\omega_{\max} &= \log\frac{\frac{1}{\sqrt{T_D}}}{\sqrt{\alpha T_D}} \\ &= \log\frac{1}{\sqrt{T_D}} + \log\frac{1}{\sqrt{\alpha T_D}} \\ &= \frac{1}{2}\left(\log\frac{1}{T_D} + \log\frac{1}{\alpha T_D}\right)\end{aligned}$$

as shown in Fig. 6.52.

(d) Alternatively, we may state these results in terms of the pole-zero locations. Rewrite $D_c(s)$ as,

$$D_c(s) = K\frac{(s+z)}{(s+p)}$$

then

$$D_c(j\omega) = K\frac{(j\omega+z)}{(j\omega+p)}$$

and

$$\phi = \tan^{-1}\left(\frac{\omega}{|z|}\right) - \tan^{-1}\left(\frac{\omega}{|p|}\right)$$

or

$$\tan\phi = \frac{\frac{\omega}{|z|} - \frac{\omega}{|p|}}{1 + \frac{\omega}{|z|}\frac{\omega}{|p|}}$$

Setting the derivative of the above equation to zero we find,

$$\left(\frac{\omega}{|z|} - \frac{\omega}{|p|}\right)\left(1 + \frac{\omega^2}{|z|\,|p|}\right) - \frac{2\omega}{|z|\,|p|}\left(\frac{\omega}{|z|} - \frac{\omega}{|p|}\right) = 0$$

and

$$\omega_{\max} = \sqrt{|z|\,|p|}$$

and

$$\log\omega_{\max} = \frac{1}{2}\left(\log|z| + \log|p|\right)$$

Hence the frequency at which the phase is maximum is the square root of the product of the pole and zero locations.

45. For the third-order servo system

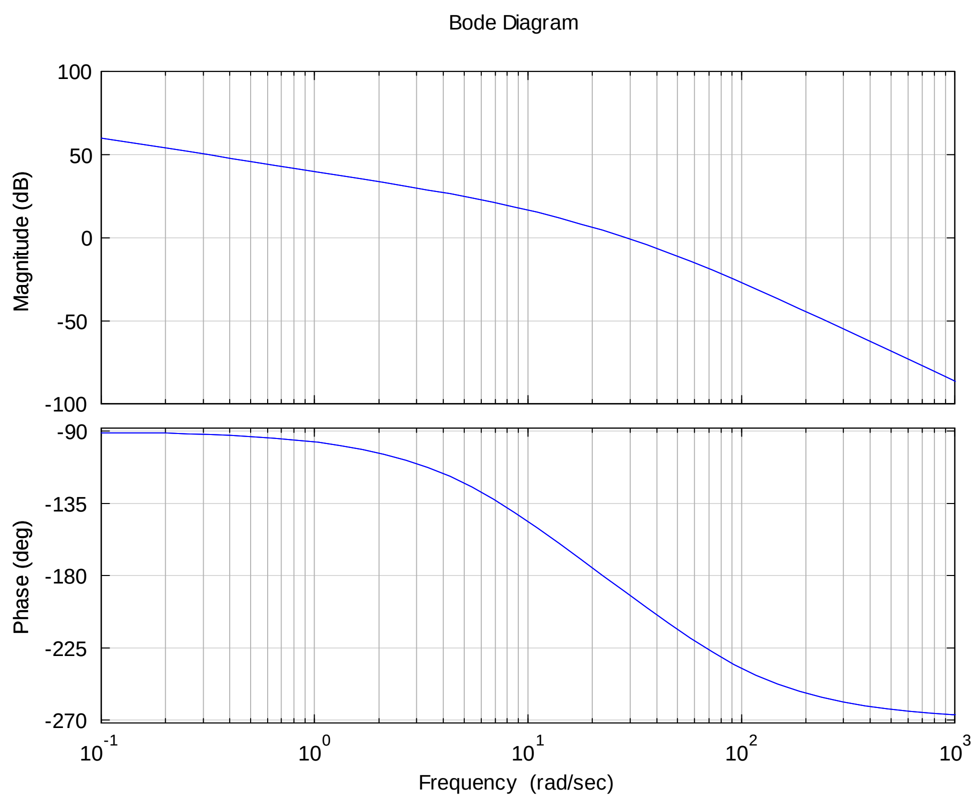
$$G(s) = \frac{50,000}{s(s+10)(s+50)}.$$

Design a lead compensator so that PM $\geq 50°$ and $\omega_{BW} \geq 20$ rad/sec using Bode plot sketches, then verify and refine your design using MATLAB.

Solution :

Let's design the lead compensator so that the system has $PM \geq 50°$ & $\omega_{WB} \simeq \omega_c \geq 20$ rad/sec.

The Bode plot of the given system is :



Start with a lead compensator design with :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1} \ (\alpha < 1)$$

Since the open-loop crossover frequency $\omega_c(\simeq \omega_{BW})$ is already above 20 rad/sec, we are going to just add extra phase around $\omega = \omega_c$ in order to satisfy $PM = 50^\circ$.

Let's add phase lead $\geq 60^\circ$. From Fig. 6.53,

$$\frac{1}{\alpha} \simeq 20 \Longrightarrow \text{choose } \alpha = 0.05$$

To apply maximum phase lead at $\omega = 20$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha} T_D} = 20 \Longrightarrow \frac{1}{T_D} = 4.48, \ \frac{1}{\alpha T_D} = 89.4$$

Therefore by applying the lead compensator with some gain adjustments :

$$D_c(s) = 0.12 \times \frac{\frac{s}{4.5} + 1}{\frac{s}{90} + 1}$$

we get the compensated system with :

$$PM = 65^\circ, \ \omega_c = 22 \text{ rad/sec, so that } \omega_{BW} \gtrsim 25 \text{ rad/sec.}$$

The Bode plot with designed compensator is :

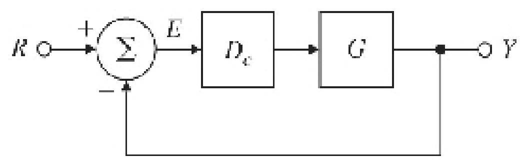
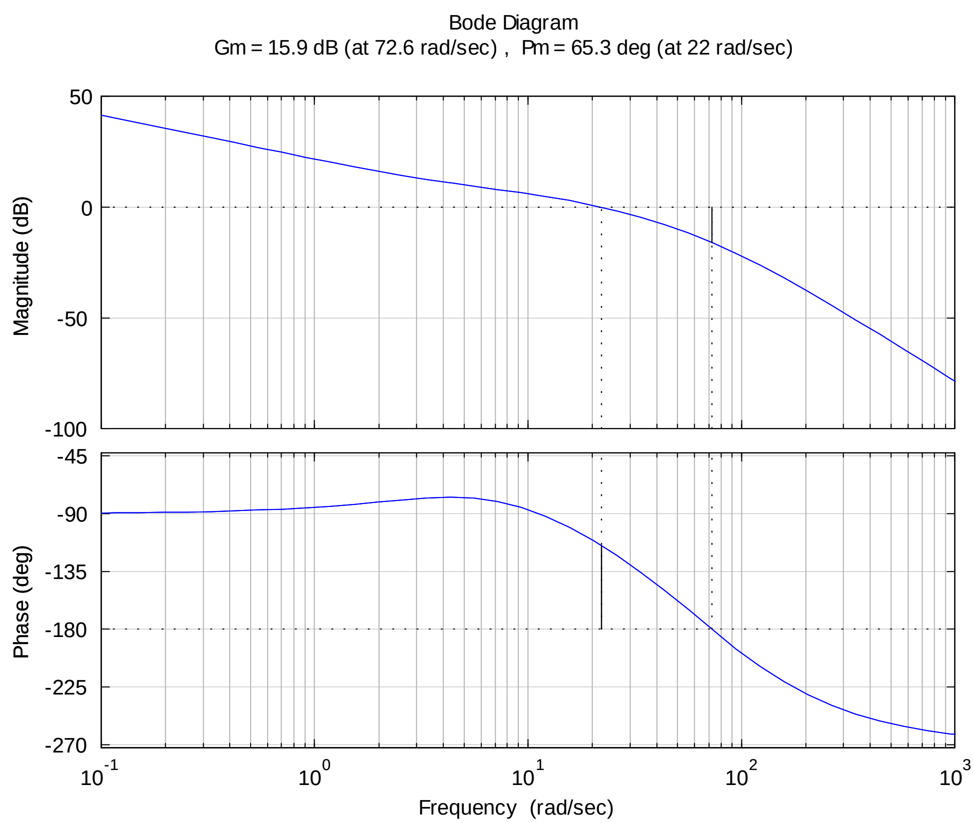


Figure 6.100: Control system for Problem 46



46. For the system shown in Fig. 6.100, suppose that

$$G(s) = \frac{5}{s(s+1)(s/5+1)}.$$

Design a lead compensation $D(s)$ with unity DC gain so that PM $\geq 40^\circ$ using Bode plot sketches, then verify and refine your design using MATLAB. What is the approximate bandwidth of the system?

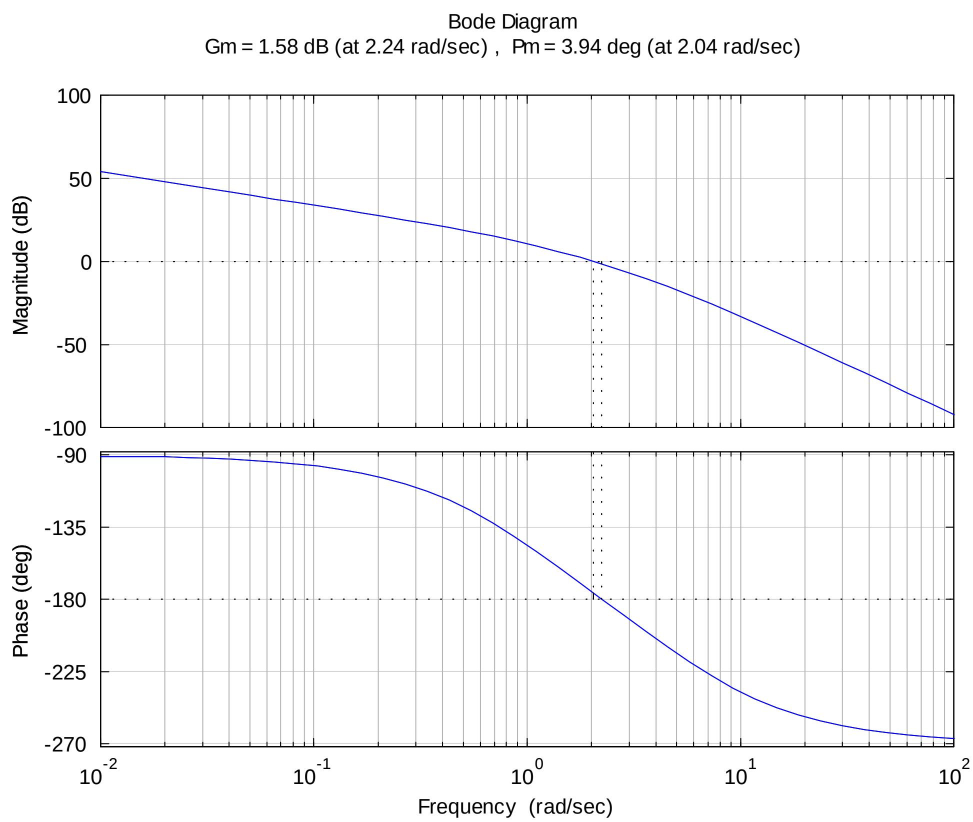
Solution :

Start with a lead compensator design with :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$$

which has unity DC gain with $\alpha < 1$.

The Bode plot of the given system is :



Since $PM = 3.9°$, let's add phase lead $\geq 60°$. From Fig. 6.53,

$$\frac{1}{\alpha} \simeq 20 \Longrightarrow \text{choose } \alpha = 0.05$$

To apply maximum phase lead at $\omega = 10$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha}T} = 10 \Longrightarrow \frac{1}{T} = 2.2,\ \frac{1}{\alpha T} = 45$$

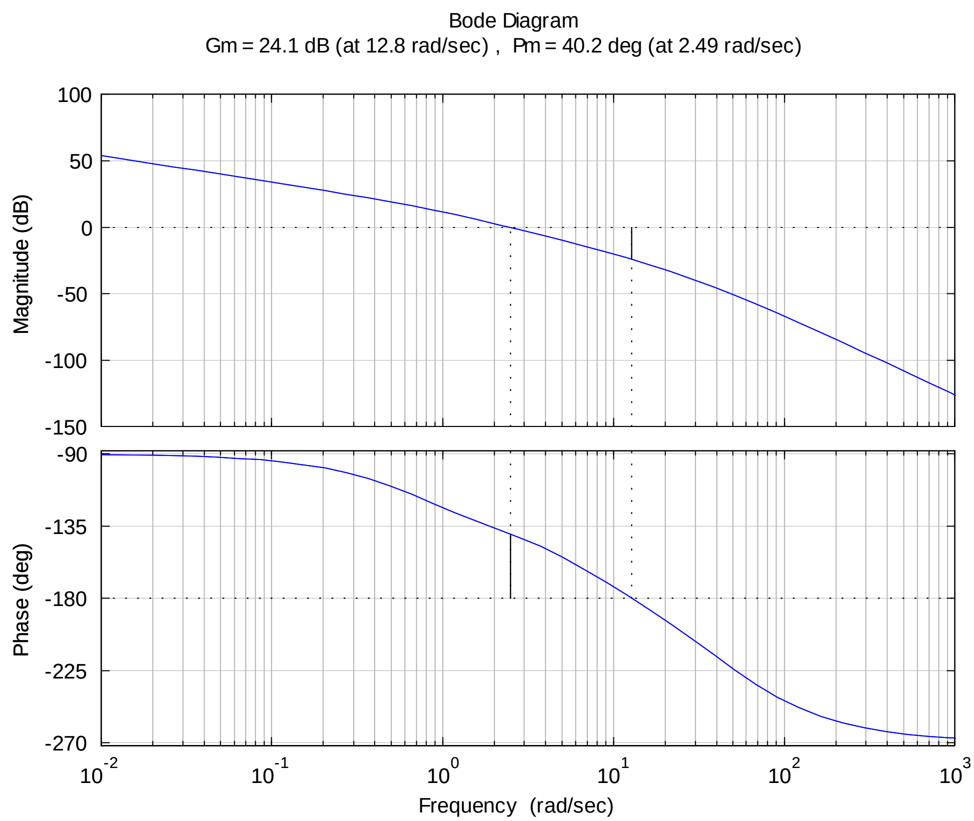
Therefore by applying the lead compensator :

$$D(s) = \frac{\frac{s}{2.2} + 1}{\frac{s}{45} + 1}$$

we get the compensated system with :

$$PM = 40^\circ, \ \omega_c = 2.5$$

The Bode plot with designed compensator is :



From Fig. 6.50, we see that $\omega_{BW} \simeq 2 \times \omega_c \simeq 5$ rad/sec.

47. Derive the transfer function from T_d to θ for the system in Fig. 6.67. Then apply the Final Value Theorem (assuming T_d = constant) to determine whether $\theta(\infty)$ is nonzero for the following two cases:

(a) When $D_c(s)$ has no integral term: $\lim_{s\to 0} D_c(s) = \text{constant}$;

(b) When $D_c(s)$ has an integral term:

$$D_c(s) = \frac{D_c'(s)}{s},$$

where $\lim_{s\to 0} D_c'(s) = \text{constant}$.

Solution :

The transfer function from T_d to θ :

$$\frac{\Theta(s)}{T_d(s)} = \frac{\frac{0.9}{s^2}}{1 + \frac{0.9}{s^2}\frac{2}{s+2}D(s)}$$

where $T_d(s) = |T_d|/s$.

(a) Using the final value theorem :

$$\begin{aligned} \theta(\infty) &= \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\Theta(t) = \lim_{s\to 0}\frac{\frac{0.9}{s^2}}{\frac{s^2(s+2)+1.8D(s)}{s^2(s+2)}}\frac{|T_d|}{s} \\ &= \frac{|T_d|}{\lim\limits_{s\to 0} D_c(s)} = \frac{|T_d|}{\text{constant}} \neq 0 \end{aligned}$$

Therefore, there will be a steady state error in θ for a constant T_d input if there is no integral term in D(s).

(b)

$$\begin{aligned} \theta(\infty) &= \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\Theta(t) = \lim_{s\to 0}\frac{\frac{0.9}{s^2}}{\frac{s^3(s+2)+1.8D'(s)}{s^3(s+2)}}\frac{|T_d|}{s} \\ &= \frac{0}{1.8\lim\limits_{s\to 0} D_c'(s)} = 0 \end{aligned}$$

So when D_c(s) contains an integral term, a constant T_d input will result in a zero steady state error in θ.

48. The inverted pendulum has a transfer function given by Eq. (2.31), which is similar to

$$G(s) = \frac{1}{s^2 - 1}.$$

(a) Design a lead compensator to achieve a PM of 30° using Bode plot sketches, then verify and refine your design using Matlab.

(b) Sketch a root locus and correlate it with the Bode plot of the system.

(c) Could you obtain the frequency response of this system experimentally?

Solution :

(a) Design the lead compensator :

$$D_c(s) = K\frac{T_D s + 1}{\alpha T_D s + 1}$$

such that the compensated system has $PM \simeq 30^\circ$ & $\omega_c \simeq 1$ rad/sec. (Actually, the bandwidth or speed of response was not specified, so any crossover frequency would satisfy the problem statement.)

$$\alpha = \frac{1 - \sin(30^\circ)}{1 + \sin(30^\circ)} = 0.32$$

To apply maximum phase lead at $\omega = 1$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha} T_D} = 1 \Longrightarrow \frac{1}{T_D} = 0.57, \ \frac{1}{\alpha T_D} = 1.77$$

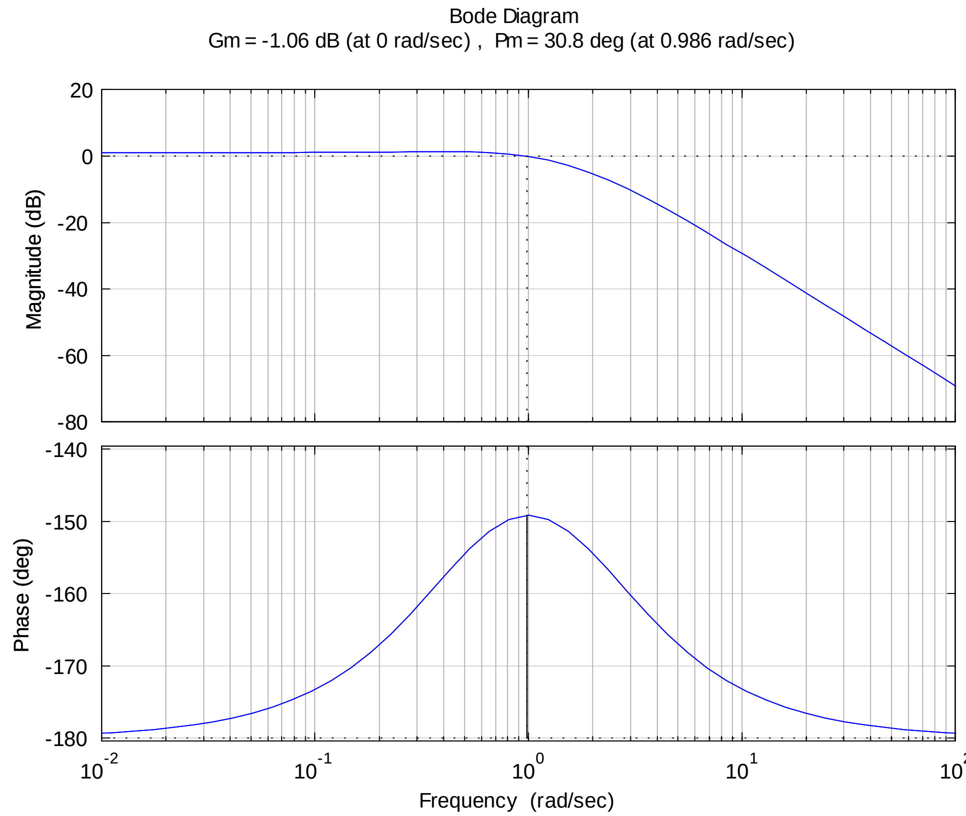
Therefore by applying the lead compensator :

$$D_c(s) = K\frac{\frac{s}{0.57} + 1}{\frac{s}{1.77} + 1}$$

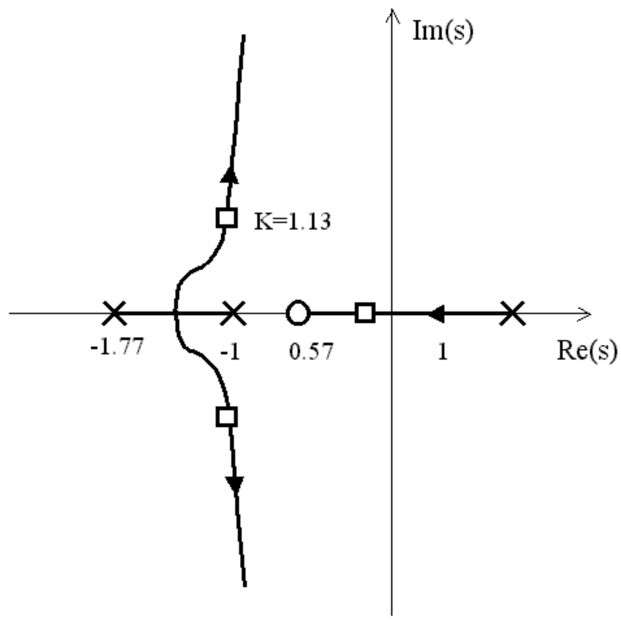
By adjusting the gain K so that the crossover frequency is around 1 rad/sec, $K = 1.13$ results in :

$$PM = 30.8^\circ$$

The Bode plot of compensated system is :



(b) Root Locus of the compensated system is :



and confirms that the system yields all stable roots with reasonable damping. However, it would be a better design if the gain was raised some in order to increase the speed of response of the slow real root. A small decrease in the damping of the complex roots will result.

(c) No, because the sinusoid input will cause the system to blow up because the open loop system is unstable. In fact, the system will "blow up" even without the sinusoid applied. Or, a better description would be that the pendulum will fall over until it hits the table.

49. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s/5+1)(s/50+1)}.$$

(a) Design a lag compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp reference input is less than 0.01.

ii. PM $\geq = 40°$

(b) Verify and refine your design using MATLAB.

Solution :

Let's design the lag compensator :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1},\ \alpha > 1$$

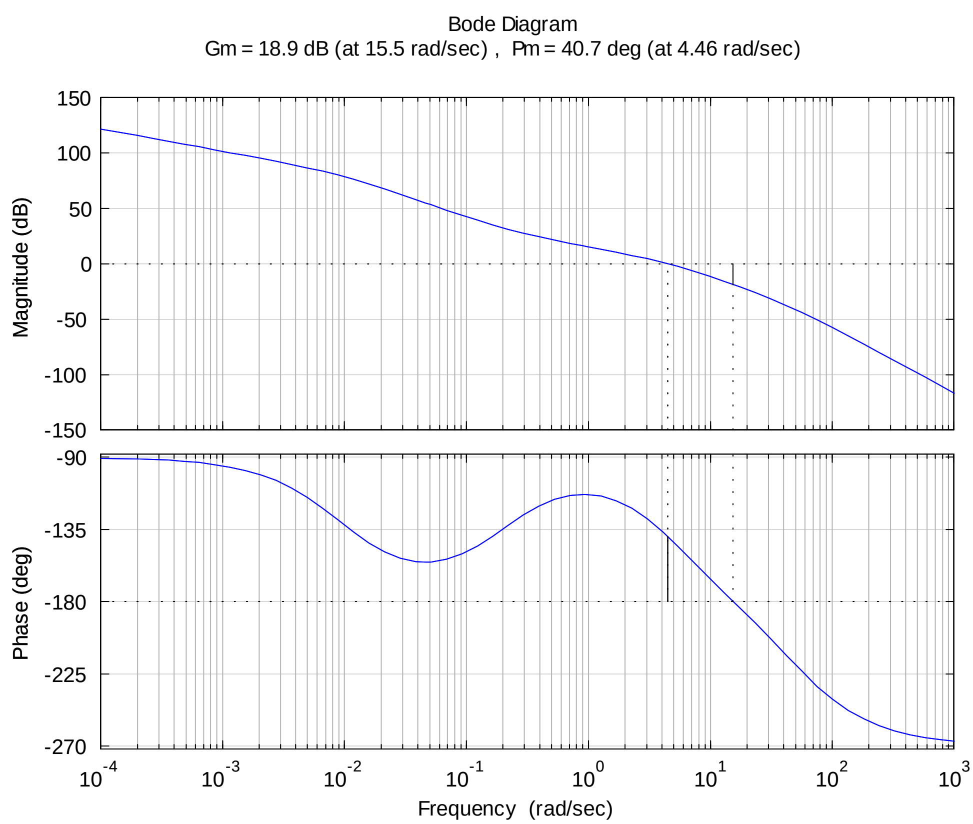
From the first specification,

$$\begin{aligned}
\text{Steady-state error to unit ramp} &= \lim_{s\to 0} s\left|\frac{D_c(s)G(s)}{1+D_c(s)G(s)}\frac{1}{s^2}-\frac{1}{s^2}\right| < 0.01 \\
&\Longrightarrow \frac{1}{K} < 0.01 \\
&\Longrightarrow \text{Choose } K = 150
\end{aligned}$$

Uncompensated, the crossover frequency with $K = 150$ is too high for a good PM. With some trial and error, we find that the lag compensator,

$$D_c(s) = \frac{\frac{s}{0.2}+1}{\frac{s}{0.01}+1}$$

will lower the crossover frequency to $\omega_c \simeq 4.46$ rad/sec where the $PM = 40.7°$.



50. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s/5+1)(s/200+1)}.$$

(a) Design a lead compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp reference input is less than 0.01.

ii. For the dominant closed-loop poles the damping ratio $\zeta \geq 0.4$.

(b) Verify and refine your design using MATLAB including a direct computation of the damping of the dominant closed-loop poles.

Solution :

Let's design the lead compensator :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}, \ \alpha < 1$$

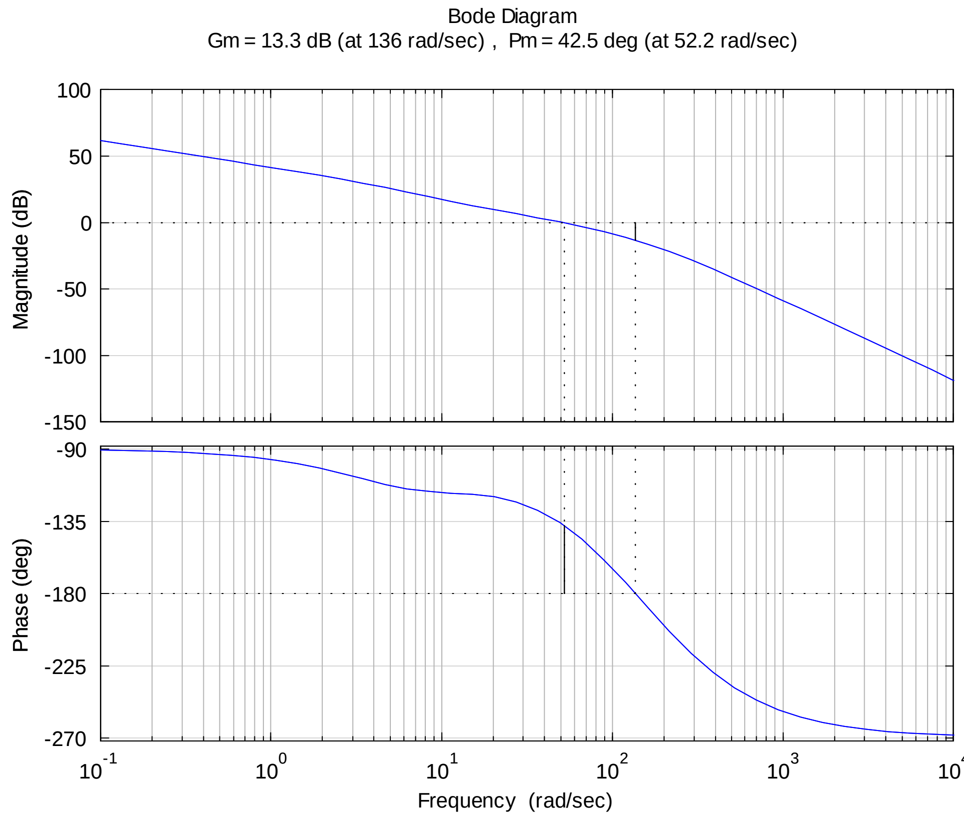
From the first specification,

$$\begin{aligned}
\text{Steady-state error to unit ramp} \quad &= \quad \lim_{s\to 0} s \left| \frac{D_c(s)G(s)}{1+D_c(s)G(s)} \frac{1}{s^2} - \frac{1}{s^2} \right| < 0.01 \\
&\Longrightarrow \quad \frac{1}{K} < 0.01 \\
&\Longrightarrow \quad \text{Choose } K = 150
\end{aligned}$$

From the approximation $\zeta \simeq \dfrac{PM}{100}$, the second specification implies $PM \geq 40$. After trial and error, we find that the compensator,

$$D_c(s) = \frac{\frac{s}{10}+1}{\frac{s}{100}+1}$$

results in a $PM = 42.5°$ and a crossover frequency $\omega_c \simeq 51.2$ rad/sec as shown by the `margin` output:



and the use of `damp` verifies the damping to be $\zeta = 0.42$ for the complex closed-loop roots which exceeds the requirement.

51. A DC motor with negligible armature inductance is to be used in a position control system. Its open-loop transfer function is given by

$$G(s) = \frac{50}{s(s/5 + 1)}.$$

(a) Design a compensator for the motor using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp input is less than 1/200.

ii. The unit step response has an overshoot of less than 20%.

iii. The bandwidth of the compensated system is no less than that of the uncompensated system.

(b) Verify and/or refine your design using MATLAB including a direct computation of the step response overshoot.

Solution :

The first specification implies that a loop gain greater than 200 is required. Since the open loop gain of the plant is 50, a gain from the compensator, K, is required where

$$K > 4 \Longrightarrow \text{ so choose } K = 5$$

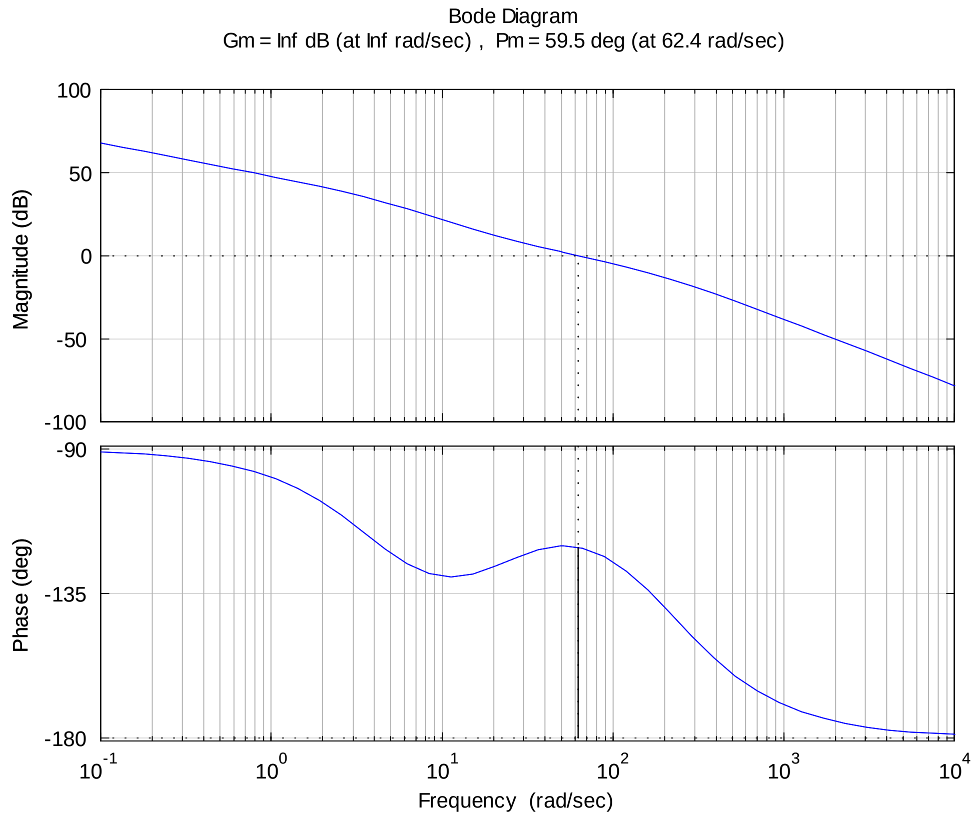
From Figure 3.24, we see that the second specification implies that :

$$\text{Overshoot} < 20\% \Longrightarrow \zeta > 0.5 \Longrightarrow PM > 50^\circ$$

A sketch of the Bode asymptotes of the open loop system with the required loop gain shows a crossover frequency of about 30 rad/sec at a slope of -2; hence, the PM will be quite low. To add phase with no decrease in the crossover frequency, a lead compensator is required. Figure 6.53 shows that a lead ratio of 10:1 will provide about 55^o of phase increase and the asymptote sketch shows that this increase will be centered at the crossover frequency if we select the break points at

$$D_c(s) = \frac{\frac{s}{20} + 1}{\frac{s}{200} + 1}.$$

Use of Matlab's `margin` routine shows that this compensation results in a $PM = 59^\circ$ and a crossover frequency $\omega_c \simeq 60$ rad/sec.



and using the `step` routine on the closed loop system shows the step response to be less than the maximum allowed 20%.

52. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(1 + s/5)(1 + s/20)}.$$

(a) Sketch the system block diagram including input reference commands and sensor noise.

(b) Design a compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp input is less than 0.01.

ii. PM $\geq 45°$

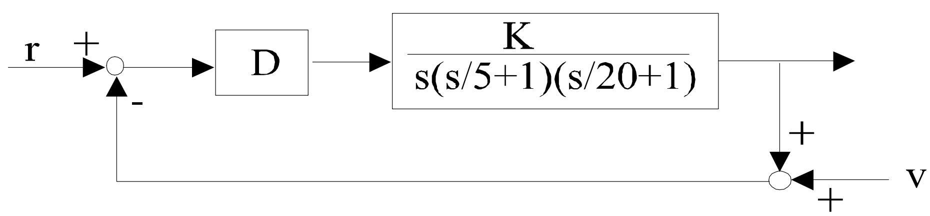
iii. The steady-state error for sinusoidal inputs with $\omega < 0.2$ rad/sec is less than 1/250.

iv. Noise components introduced with the sensor signal at frequencies greater than 200 rad/sec are to be attenuated at the output by at least a factor of 100,.

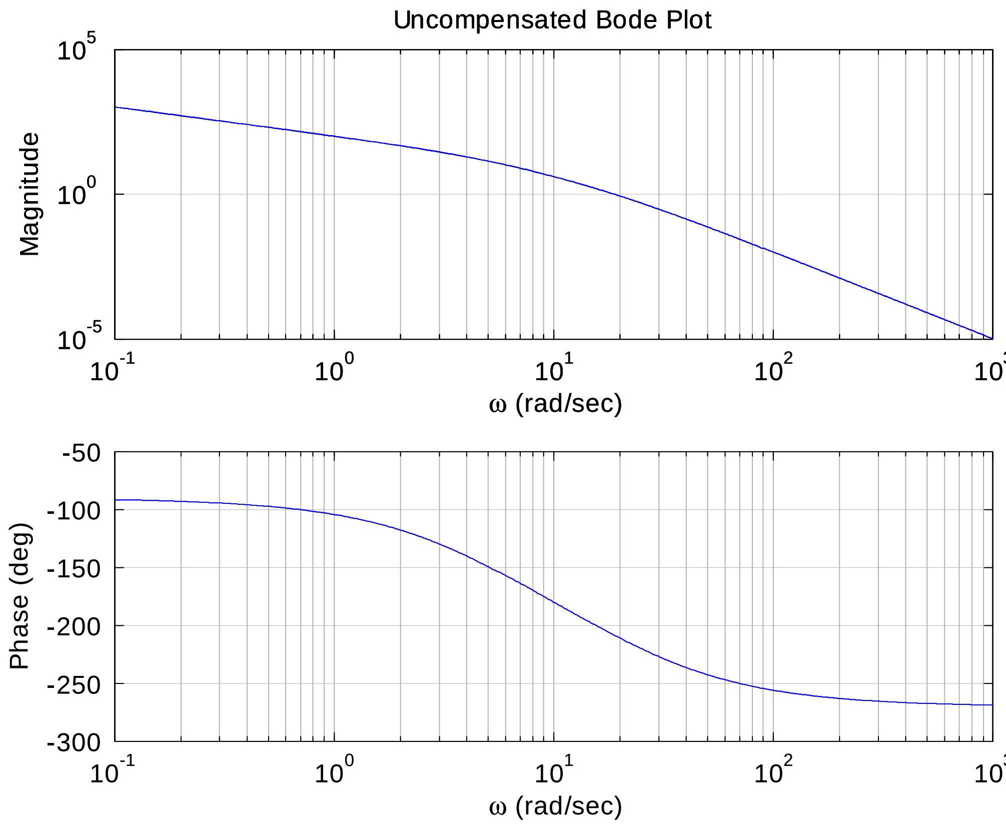
(c) Verify and/or refine your design using MATLAB including a computation of the closed-loop frequency response to verify (iv).

Solution :

a. The block diagram shows the noise, v, entering where the sensor would be:



b. The first specification implies $K_v \geq 100$ and thus $K \geq 100$. The bode plot with $K = 1$ and $D = 1$ below shows that there is a negative PM but all the other specs are met. The easiest way to see this is to hand plot the asymptotes and mark the constraints that the gain must be ≥ 250 at $\omega \leq 0.2$ rad/sec and the gain must be ≤ 0.01 for $\omega \geq 200$ rad/sec.



In fact, the specs are exceeded at the low frequency side, and slightly exceeded on the high frequency side. But it will be difficult to increase the phase at crossover without violating the specs. From a hand plot of the asymptotes, we see that a combination of lead and lag will do the trick. Placing the lag according to

$$D_{lag}(s) = \frac{(s/2+1)}{(s/0.2+1)}$$

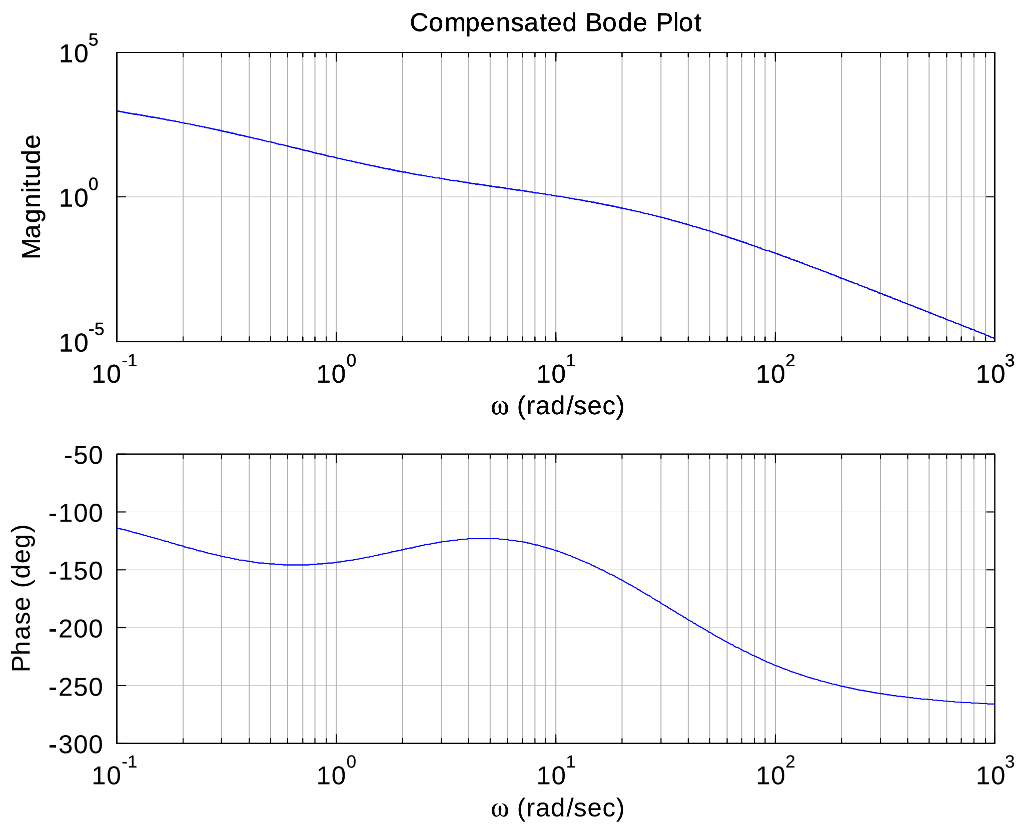
will lower the gain curve at frequencies just prior to crossover so that a -1 slope is more easily achieved at crossover without violating the high frequency constraint. In addition, in order to obtain as much phase at crossover as possible, a lead according to

$$D_{lead}(s) = \frac{(s/5+1)}{(s/50+1)}$$

will preserve the -1 slope from $\omega = 5$ rad/sec to $\omega = 20$ rad/sec which will bracket the crossover frequency and should result in a healthy PM. A look at the Bode plot shows that all specs are met except the $PM = 44$. Perhaps close enough, but a slight increase in lead should do the trick. So our final compensation is

$$D(s) = \frac{(s/2+1)}{(s/0.2+1)} \frac{(s/4+1)}{(s/50+1)}$$

with $K = 100$. This does meet all specs with $PM = 45^o$ exactly, as can be seen by examining the Bode plot below.



53. Consider a type I unity feedback system with

$$G(s) = \frac{K}{s(s+1)}.$$

Design a lead compensator using Bode plot sketches so that $K_v = 20\,\text{sec}^{-1}$ and PM $> 40°$. Use MATLAB to verify and/or refine your design so that it meets the specifications.

Solution :

Use a lead compensation :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1},\ \alpha > 1$$

From the specification, $K_v = 20$ sec^{-1},

$$\begin{aligned} &\Longrightarrow \quad K_v = \lim_{s \to 0} s D_c(s) G(s) = K = 20 \\ &\Longrightarrow \quad K = 20 \end{aligned}$$

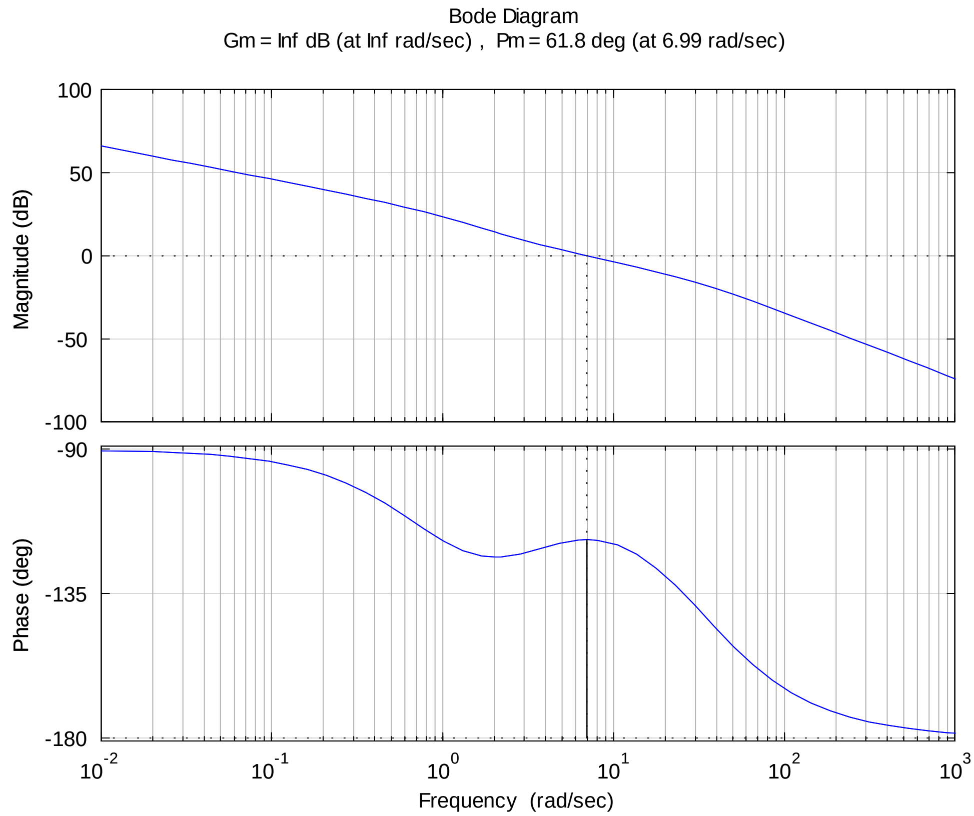
From a hand sketch of the uncompensated Bode plot asymptotes, we see that the slope at crossover is -2, hence the PM will be poor. In fact, an exact computation shows that

$$PM = 12.75 \text{ (at } \omega_c = 4.42 \text{ rad/sec)}$$

Adding a lead compensation

$$D_c(s) = \frac{\frac{s}{3} + 1}{\frac{s}{30} + 1}$$

will provide a -1 slope in the vicinity of crossover and should provide plenty of PM. The Bode plot below verifies that indeed it did and shows that the $PM = 62°$ at a crossover frequency $\cong 7$ rad/sec thus meeting all specs.



54. Consider a satellite-attitude control system with the transfer function

$$G(s) = \frac{0.05(s+25)}{s^2(s^2+0.1s+4)}.$$

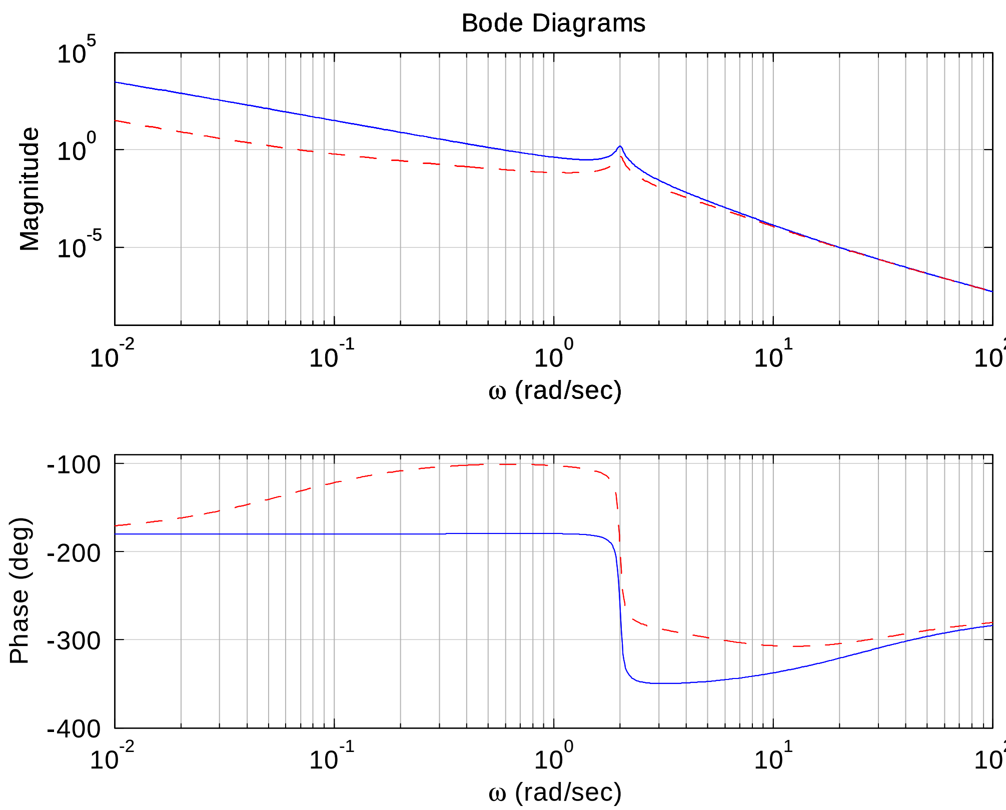
Amplitude-stabilize the system using lead compensation so that GM $\geq$ 2 (6 db), and PM $\geq 45°$, keeping the bandwidth as high as possible with a single lead.

Solution :

The sketch of the uncompensated Bode plot asymptotes shows that the slope at crossover is -2; therefore, a lead compensator will be required in order to have a hope of meeting the PM requirement. Furthermore, the resonant peak needs to be kept below magnitude 1 so that it has no chance of causing an instability (this is amplitude stabilization). This latter requirement means we must lower gain at the resonance. Using the single lead compensator,

$$D_c(s) = \frac{(s+0.06)}{(s+6)}$$

will lower the low frequency gain by a factor of 100, provide a -1 slope at crossover, and will lower the gain some at the resonance. Thus it is a good first cut at a compensation. The MATLAB Bode plot shows the uncompensated and compensated and verifies our intent. Note especially that the resonant peak never crosses magnitude 1 for the compensated (dashed) case.



The MATLAB margin routine shows a $GM = 6.3$ db and $PM = 48°$ thus meeting all specs.

55. In one mode of operation the autopilot of a jet transport is used to control altitude. For the purpose of designing the altitude portion of the autopilot loop, only the long-period airplane dynamics are important. The linearized relationship between altitude and elevator angle for the long-period dynamics is

$$G(s) = \frac{h(s)}{\delta(s)} = \frac{20(s + 0.01)}{s(s^2 + 0.01s + 0.0025)} \frac{\text{ft}}{\text{deg}}.$$

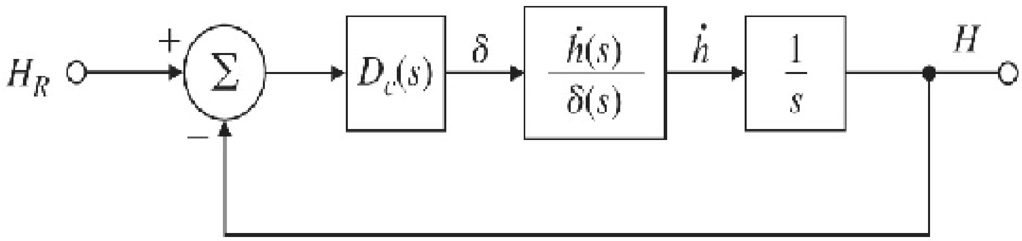


Figure 6.101: Control system for Problem 55

The autopilot receives from the altimeter an electrical signal proportional to altitude. This signal is compared with a command signal (proportional to the altitude selected by the pilot), and the difference provides an error signal. The error signal is processed through compensation, and the result is used to command the elevator actuators. A block diagram of this system is shown in Fig. 6.103. You have been given the task of designing the compensation. Begin by considering a proportional control law $D_c(s) = K$.

(a) Use MATLAB to draw a Bode plot of the open-loop system for $D_c(s) = K = 1$.

(b) What value of K would provide a crossover frequency (i.e., where $|G| = 1$) of 0.16 rad/sec?

(c) For this value of K, would the system be stable if the loop were closed?

(d) What is the PM for this value of K?

(e) Sketch the Nyquist plot of the system, and locate carefully any points where the phase angle is 180° or the magnitude is unity.

(f) Use MATLAB to plot the root locus with respect to K, and locate the roots for your value of K from part (b).

(g) What steady-state error would result if the command was a step change in altitude of 1000 ft?

For parts (h)and (i), assume a compensator of the form

$$D_c(s) = K\frac{T_D s + 1}{\alpha T_D s + 1}.$$

(h) Choose the parameters K, T, and α so that the crossover frequency is 0.16 rad/sec and the PM is greater that 50°. Verify your design by superimposing a Bode plot of $D_c(s)G(s)/K$ on top of the Bode plot you obtained for part (a), and measure the PM directly.

(i) Use MATLAB to plot the root locus with respect to K for the system including the compensator you designed in part (h). Locate the roots for your value of K from part (h).

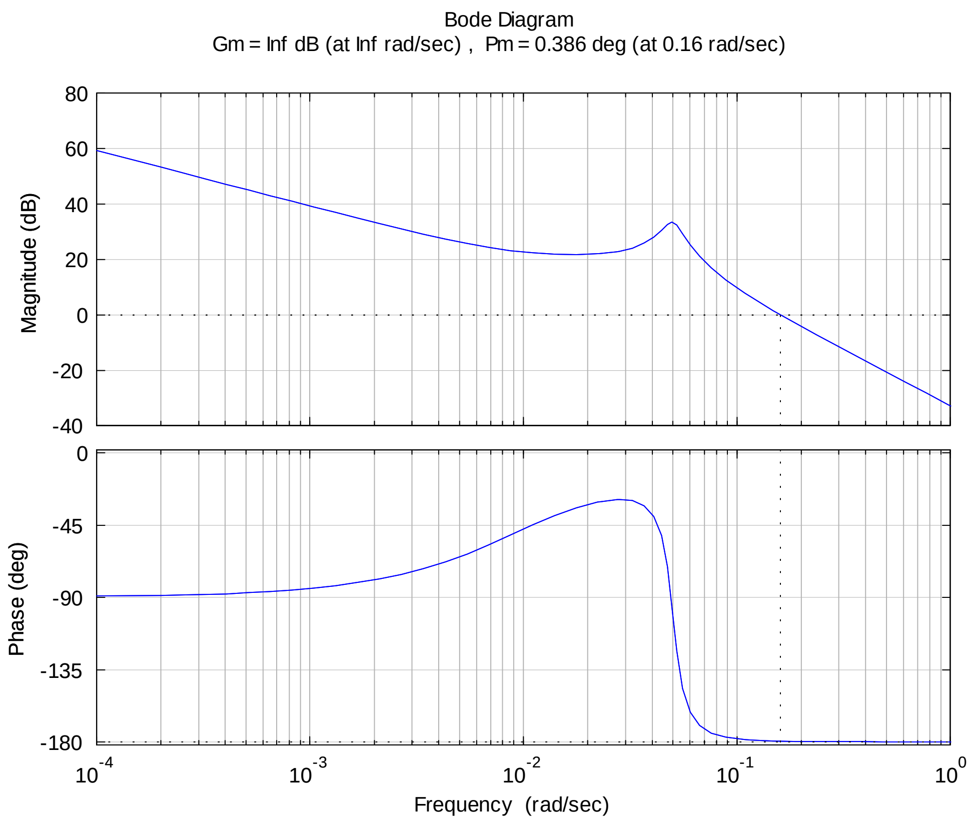
(j) Altitude autopilots also have a mode where the rate of climb is sensed directly and commanded by the pilot.
   i. Sketch the block diagram for this mode,
   ii. define the pertinent $G(s)$,
   iii. design $D(s)$ so that the system has the same crossover frequency as the altitude hold mode and the PM is greater than 50°

Solution :

The plant transfer function :

$$\frac{h(s)}{\delta(s)} = \frac{80\left(\frac{s}{0.01}+1\right)}{s\left\{\left(\frac{s}{0.05}\right)^2 + 2\frac{0.1}{0.05}s + 1\right\}}$$

(a) See the Bode plot :



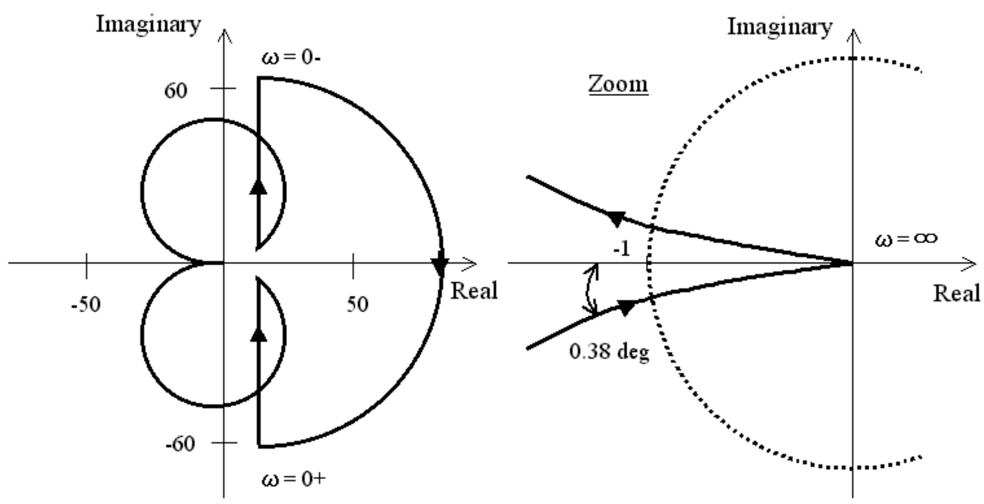
(b) Since $|G| = 865$ at $\omega = 0.16$,

$$K = \frac{1}{|G|}|_{\omega=0.16} = 0.0012$$

(c) The system would be stable, but poorly damped.

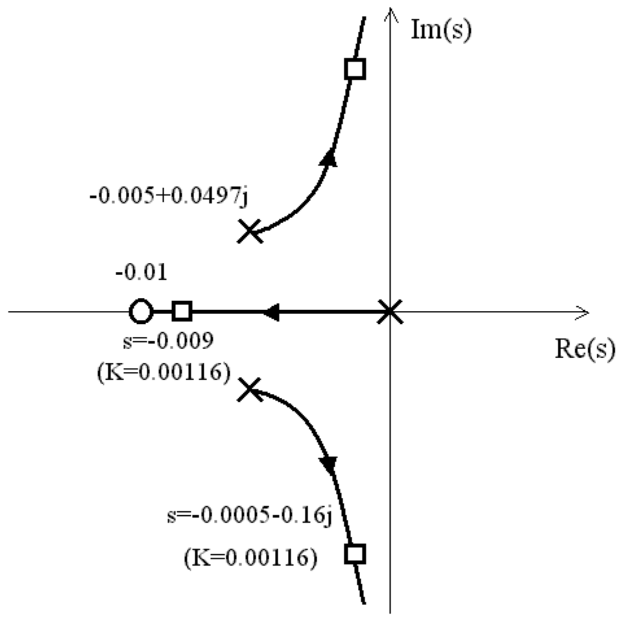
(d) $PM = 0.39°$

(e) The Nyquist plot for $D_c(j\omega)G(j\omega)$:



The phase angle never quite reaches $-180°$.

(f) See the Root locus :



The closed-loop roots for $K = 0.0012$ are :

$$s = -0.009,\ -0.005 \pm j0.16$$

(g) The steady-state error e_∞ :

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{1}{1+K\dfrac{h(s)}{\delta(s)}} \frac{1000}{s} \\ &= 0 \end{aligned}$$

as it should be for this Type 1 system.

(h) Phase margin of the plant :

$$PM = 0.39^\circ \ (\omega_c = 0.16 \text{ rad/sec})$$

Necessary phase lead and $\dfrac{1}{\alpha}$:

$$\text{necessary phase lead} = 50^\circ - 0.39^\circ \ \simeq 50^\circ$$

From Fig. 6.54 :

$$\Longrightarrow \frac{1}{\alpha} = 8$$

Set the maximum phase lead frequency at ω_c :

$$\omega = \frac{1}{\sqrt{\alpha} T_D} = \omega_c = 0.16 \Longrightarrow T_D = 18$$

so the compensation is

$$D(s) = K\frac{18s+1}{2.2s+1}$$

For a gain K, we want $|D_c(j\omega_c)G(j\omega_c)| = 1$ at $\omega = \omega_c = 0.16$. So evaluate via Matlab

$$\left|\frac{D_c(j\omega_c)G(j\omega_c)}{K}\right|_{\omega_c=0.16} \quad \text{and find it} \quad = \quad 2.5\times 10^3$$

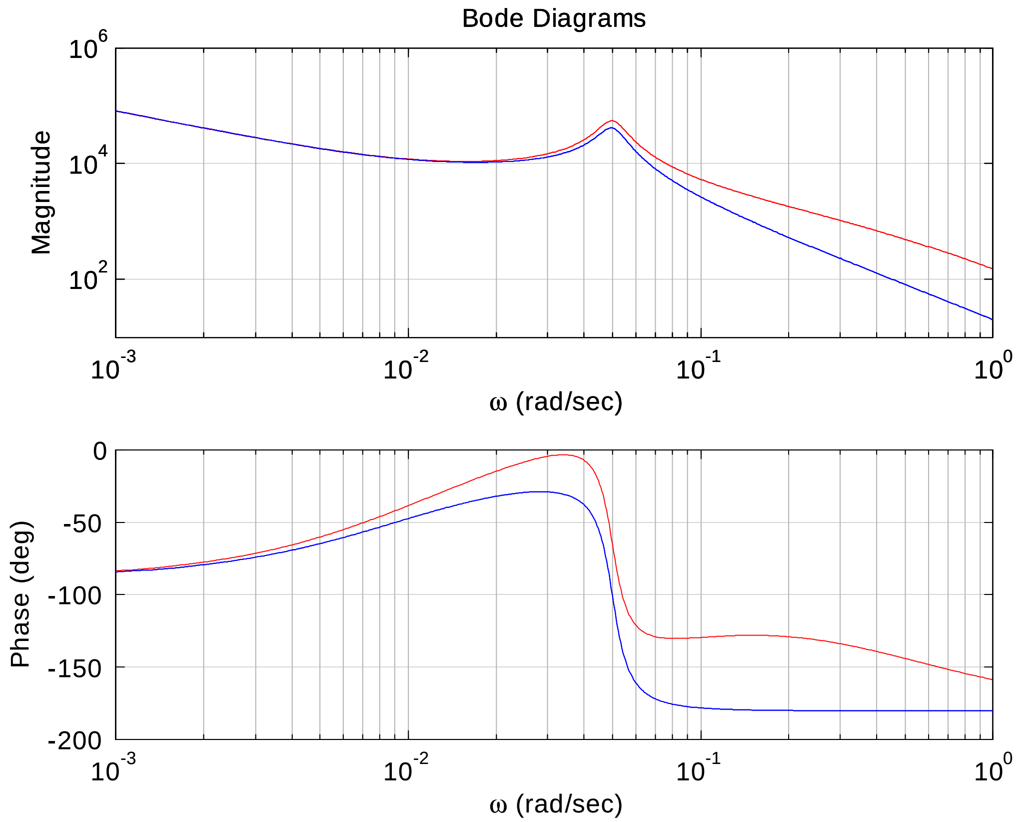
$$\Longrightarrow \quad K = \frac{1}{2.5\times 10^3} = 4.0\times 10^{-4}$$

Therefore the compensation is :

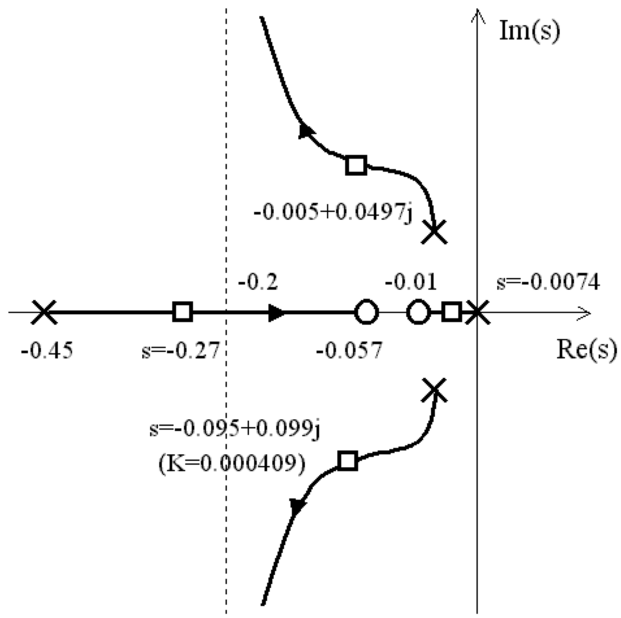
$$D_c(s) = 4.0\times 10^{-4}\frac{18s+1}{2.2s+1}$$

which results in the Phase margin :

$$PM = 52^\circ \ (\omega_c = 0.16 \text{ rad/sec})$$



(i) See the Root locus :

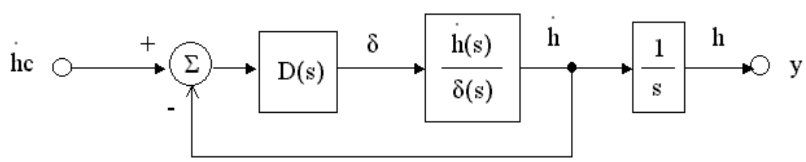


The closed-loop roots for $K = 4.0 \times 10^{-4}$ are :

$$s = -0.27,\ -0.0074,\ -0.095 \pm j0.099$$

(j) In this case, the reference input and the feedback parameter are the rate of climb.

i. The block diagram for this mode is :



ii. Define $G(s)$ as :

$$G(s) = \frac{\dot{h}(s)}{\delta(s)} = \frac{80\left(\frac{s}{0.01}+1\right)}{s\left(\frac{s}{0.05}\right)^2 + 2\frac{0.1}{0.05}s + 1}$$

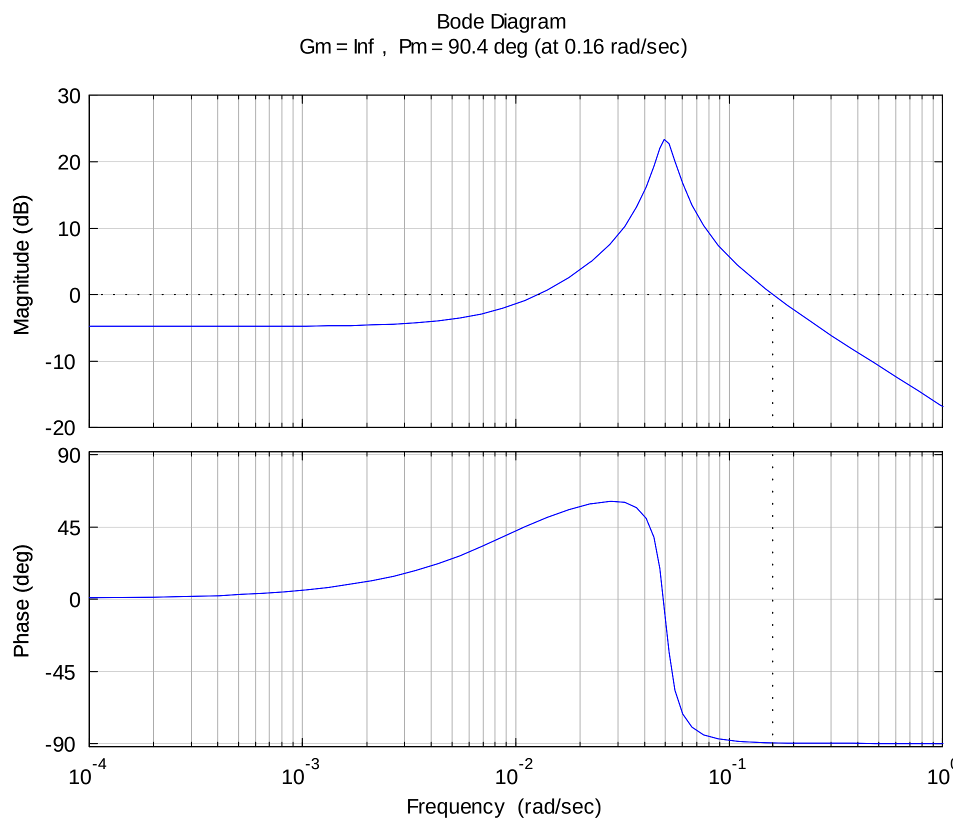
iii. By evaluating the gain of $G(s)$ at $\omega = \omega_c = 0.16$, and setting K equal to its inverse, we see that proportional feedback :

$$D_c(s) = K = 0.0072$$

satisfies the given specifications by providing:

$$PM = 90^\circ \; (\omega_c = 0.16 \text{ rad/sec})$$

The Bode plot of the compensated system is :



56. For a system with open-loop transfer function $_c$

$$G(s) = \frac{10}{s[(s/1.4)+1][(s/3)+1]},$$

design a lag compensator with unity DC gain so that PM $\geq 40^\circ$. What is the approximate bandwidth of this system?

Solution :

Lag compensation design :

Use

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$$

K=1 so that DC gain of $D_c(s) = 1$.

(a) Find the stability margins of the plant without compensation by plotting the Bode, find that:

$$\begin{aligned} PM &= -20^\circ\ (\omega_c = 3.0 \text{ rad/sec}) \\ GM &= 0.44\ (\omega = 2.05 \text{ rad/sec}) \end{aligned}$$

(b) The lag compensation needs to lower the crossover frequency so that a $PM \simeq 40^\circ$ will result, so we see from the uncompensated Bode that we need the crossover at about

$$\Longrightarrow \omega_{c,new} = 0.81$$

where

$$|G(j\omega_c)| = 10.4$$

so the lag needs to lower the gain at $\omega_{c,new}$ from 10.4 to 1.

(c) Pick the zero breakpoint of the lag to avoid influencing the phase at $\omega = \omega_{c,new}$ by picking it a factor of 20 below the crossover, so

$$\frac{1}{T_D} = \frac{\omega_{c,new}}{20}$$

$$\Longrightarrow T_D = 25$$

(d) Choose α :
Since $D_c(j\omega) \cong \frac{1}{\alpha}$ for $\omega >> \frac{1}{T}$, let

$$\frac{1}{\alpha} = \frac{1}{|G(j\omega_{c,new})|}$$

$$\alpha = |G(j\omega_{c,new})| = 10.4$$

(e) Compensation :

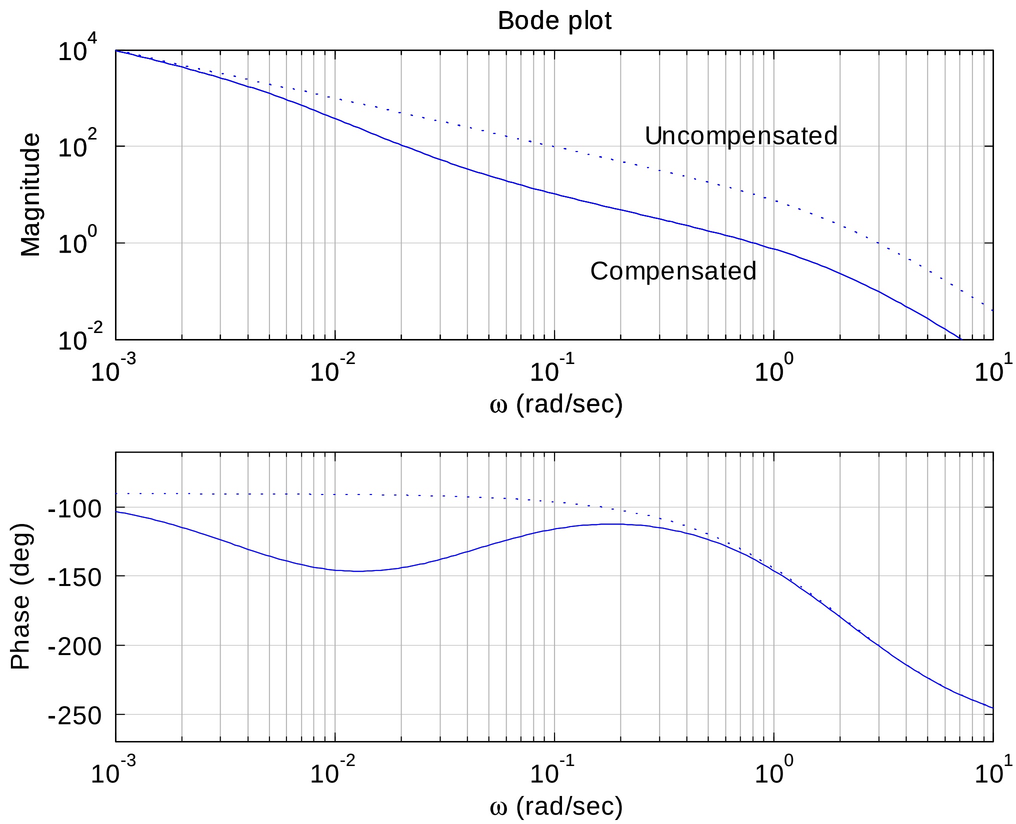
$$D_c(s) = \frac{\frac{s}{0.04} + 1}{\frac{s}{0.0038} + 1}$$

(f) Stability margins of the compensated system :

$$\begin{aligned} PM &= 42^\circ\ (\omega_c = 0.8 \text{ rad/sec}) \\ GM &= 4.4\ (\omega = 2.0 \text{ rad/sec}) \end{aligned}$$

Approximate bandwidth ω_{BW}:

$$PM \cong 42^\circ \Longrightarrow \omega_{BW} \cong 2\omega_c = 1.6 \text{ (rad/sec)}$$



57. For the ship-steering system in Problem 39,

   (a) Design a compensator that meets the following specifications:
      i. velocity constant $K_v = 2$,
      ii. PM $\geq 50°$,
      iii. unconditional stability (PM > 0 for all $\omega \leq \omega_c$, the crossover frequency).

   (b) For your final design, draw a root locus with respect to K, and indicate the location of the closed-loop poles.

Solution :

The transfer function of the ship steering is

$$\frac{V(s)}{\delta_r(s)} = G(s) = \frac{K[-(s/0.142)+1]}{s(s/0.325+1)(s/0.0362)+1)}.$$

(a) Since the velocity constant, K_v must be 2, we require that $K = 2$.

i. The phase margin of the uncompensated ship is

$$PM = -111^\circ \ (\omega_c = 0.363 \text{ rad/sec})$$

which means it would be impossible to stabilize this system with one lead compensation, since the maximum phase increase would be 90^o. There is no specification leading to maintaining a high bandwidth, so the use of lag compensation appears to be the best choice. So we use a lag compensation:

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$$

ii. The crossover frequency which provides $PM \simeq 50^\circ$ is obtained by looking at the uncompensated Bode plot below, where we see that the crossover frequency needs to be lowered to

$$\omega_{c,new} = 0.017,$$

where the uncompensated gain is

$$|G(j\omega_{c,new})| = 107$$

iii. Keep the zero of the lag a factor of 20 below the crossover to keep the phase lag from the compensation from fouling up the PM, so we find:

$$\begin{aligned} \frac{1}{T_D} &= \frac{\omega_{c,new}}{20} \\ &\Longrightarrow \quad T_D = 1.2 \times 10^3 \end{aligned}$$

iv. Choose α so that the gain reduction is achieved at crossover :

$$\alpha = |G(j\omega_{c,new})| = 107$$

$(D(j\omega) \simeq \dfrac{1}{\alpha}$ for $\omega \gg \dfrac{1}{T})$

v. So the compensation is :

$$D_c(s) = \frac{1200s + 1}{12.6s + 1} = \frac{\frac{s}{0.0008} + 1}{\frac{s}{0.08} + 1}$$

vi. Stability margins of the compensated system :

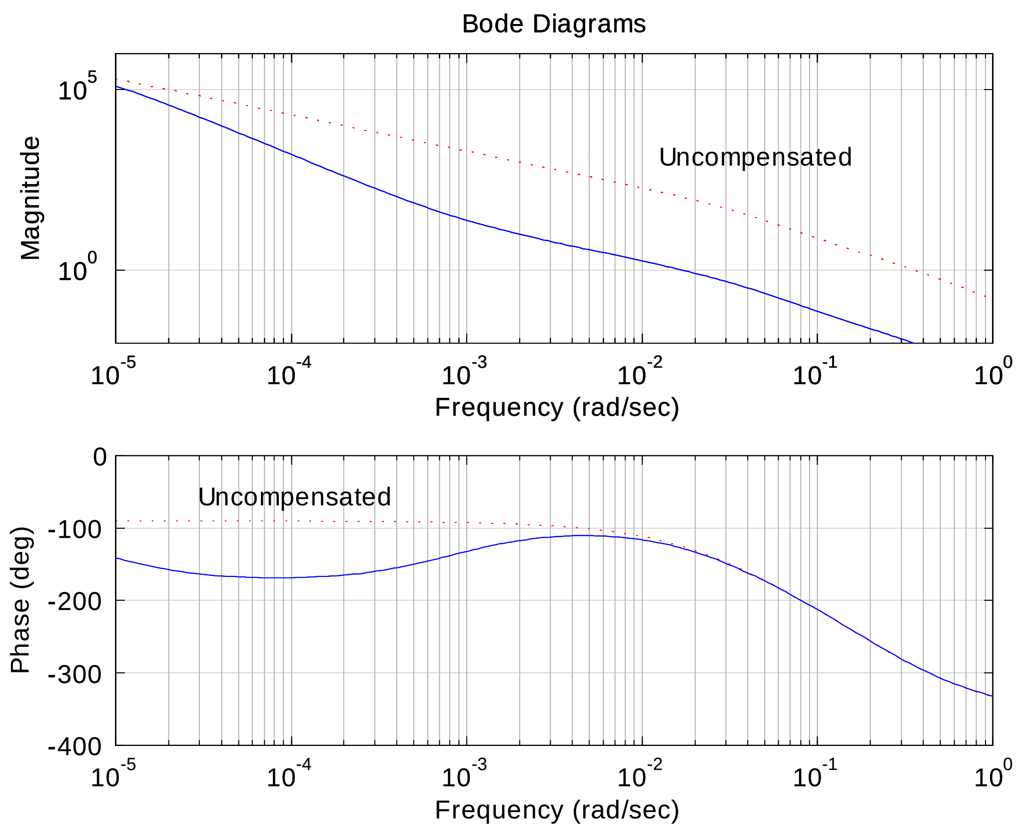
$$\begin{aligned} PM &= 52.1^\circ \ (\omega_c = 0.017 \text{ rad/sec}) \\ GM &= 5.32 \ (\omega = 0.057 \text{ rad/sec}) \end{aligned}$$

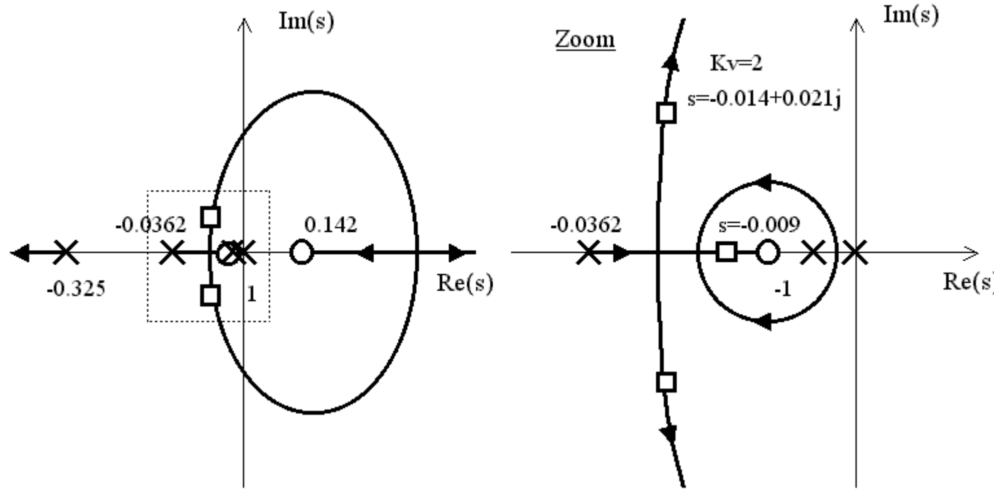
and the system is unconditionally stable since the phase > 0 for all $\omega < \omega_c$.as can be seen by the plot below.

(b) See the root locus. (Note that this is a zero degree root locus.)

The closed-loop roots for $K = 2$ are :

$$s = -0.33,\ -0.0009,\ -0.014 \pm j0.021$$





58. For a unity feedback system with

$$G(s) = \frac{1}{s(\frac{s}{20}+1)(\frac{s^2}{100^2}+0.5\frac{s}{100}+1)} \tag{2}$$

(a) A lead compensator is introduced with $\alpha = 1/5$ and a zero at $1/T_D = 20$. How must the gain be changed to obtain crossover at $\omega_c = 31.6$ rad/sec, and what is the resulting value of K_v?

(b) With the lead compensator in place, what is the required value of K for a lag compensator that will readjust the gain to a K_v value of 100?

(c) Place the pole of the lag compensator at 3.16 rad/sec, and determine the zero location that will maintain the crossover frequency at $\omega_c = 31.6$ rad/sec. Plot the compensated frequency response on the same graph.

(d) Determine the PM of the compensated design.

Solution :

(a) From a sketch of the asymptotes with the lead compensation (with $K_1 = 1$) :

$$D_1(s) = K_1 \frac{\frac{s}{20}+1}{\frac{s}{100}+1}$$

in place, we see that the slope is -1 from zero frequency to $\omega = 100$ rad/sec. Therefore, to obtain crossover at $\omega_c = 31.6$ rad/sec, the gain $K_1 = 31.6$ is required. Therefore,

$$K_v = 31.6$$

(b) To increase K_v to be 100, we need an additional gain of 3.16 from the lag compensator at very low frequencies to yield $K_v = 100$.

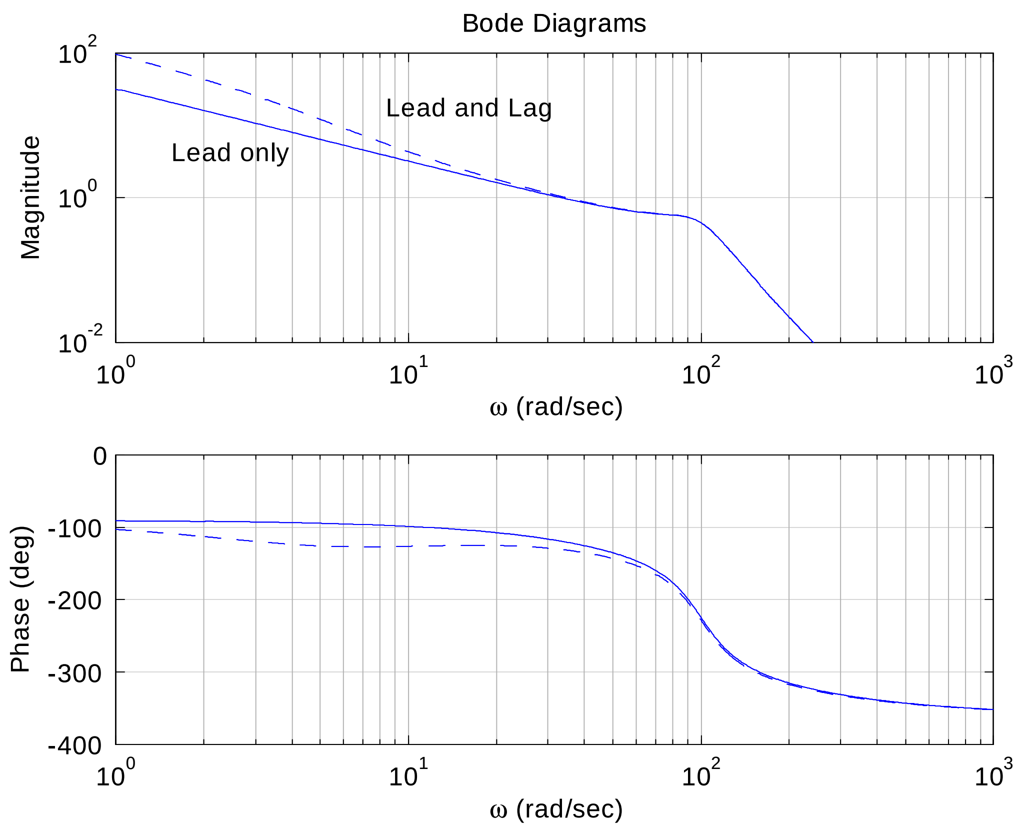
(c) For a low frequency gain increase of 3.16, and the pole at 3.16 rad/sec, the zero needs to be at 10 in order to maintain the crossover at $\omega_c = 31.6$ rad/sec. So the lag compensator is

$$D_2(s) = 3.16\frac{\frac{s}{10}+1}{\frac{s}{3.16}+1}$$

and

$$D_1(s)D_2(s) = 100\frac{\frac{s}{20}+1}{\frac{s}{100}+1}\,\frac{\frac{s}{10}+1}{\frac{s}{3.16}+1}$$

The Bode plots of the system before and after adding the lag compensation are



(d) By using the margin routine from MATLAB, we see that

$$PM = 49^\circ \ (\omega_c = 34.5 \text{ deg/sec})$$

59. Golden Nugget Airlines had great success with their free bar near the tail of the airplane. (See Problem 5.40) However, when they purchased a much larger airplane to handle the passenger demand, they discovered that there was some flexibility in the fuselage that caused a lot of unpleasant yawing motion at the rear of the airplane when in turbulence and was causing the revelers to spill their drinks. The approximate transfer function for the dutch roll mode (See Section 10.3.1) is

$$\frac{r(s)}{\delta_r(s)} = \frac{8.75(4s^2 + 0.4s + 1)}{(s/0.01 + 1)(s^2 + 0.24s + 1)}$$

where r is the airplane's yaw rate and δ_r is the rudder angle. In performing a Finite Element Analysis (FEA) of the fuselage structure and adding those dynamics to the dutch roll motion, they found that the transfer function needed additional terms that reflected the fuselage lateral bending that occurred due to excitation from the rudder and turbulence. The revised transfer function is

$$\frac{r(s)}{\delta_r(s)} = \frac{8.75(4s^2 + 0.4s + 1)}{(s/0.01 + 1)(s^2 + 0.24s + 1)} \cdot \frac{1}{(\frac{s^2}{\omega_b^2} + 2\zeta\frac{s}{\omega_b} + 1)}$$

where ω_b is the frequency of the bending mode ($= 10$ rad/sec) and ζ is the bending mode damping ratio ($= 0.02$). Most swept wing airplanes have a "yaw damper" which essentially feeds back yaw rate measured by a rate gyro to the rudder with a simple proportional control law. For the new Golden Nugget airplane, the proportional feedback gain, $K = 1$, where

$$\delta_r(s) = -Kr(s). \qquad (3)$$

(a) Make a Bode plot of the open-loop system, determine the PM and GM for the nominal design, and plot the step response and Bode magnitude of the closed-loop system. What is the frequency of the lightly damped mode that is causing the difficulty?

(b) Investigate remedies to quiet down the oscillations, but maintain the same low frequency gain in order not to affect the quality of the dutch roll damping provided by the yaw rate feedback. Specifically, investigate one at a time:

   i. increasing the damping of the bending mode from $\zeta = 0.02$ to $\zeta = 0.04$. (Would require adding energy absorbing material in the fuselage structure)

   ii. increasing the frequency of the bending mode from $\omega_b = 10$ rad/sec to $\omega_b = 20$ rad/sec. (Would require stronger and heavier structural elements)

iii. adding a low pass filter in the feedback, that is, replace K in Eq. (3) with $KD_c(s)$ where

$$D_c(s) = \frac{1}{s/\tau_p + 1}. \tag{4}$$

Pick τ_p so that the objectionable features of the bending mode are reduced while maintaining the PM $\geq 60^o$.

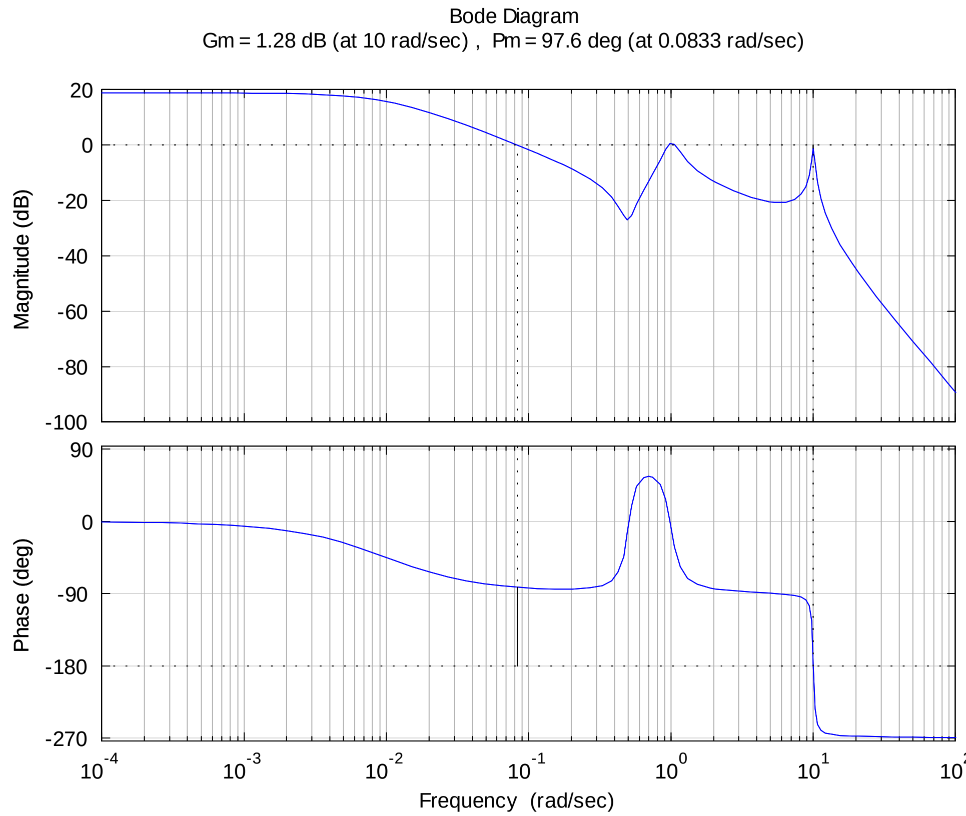
iv. adding a notch filter as described in Section 5.4.3. Pick the frequency of the notch zero to be at ω_b with a damping of $\zeta = 0.04$ and pick the denominator poles to be $(s/100 + 1)^2$ keeping the DC gain of the filter = 1.

(c) Investigate the sensitivity of the two compensated designs above (iii and iv) by determining the effect of a reduction in the bending mode frequency of -10%. Specifically, re-examine the two designs by tabulating the GM, PM, closed loop bending mode damping ratio and resonant peak amplitude, and qualitatively describe the differences in the step response.

(d) What do you recommend to Golden Nugget to help their customers quit spilling their drinks? (Telling them to get back in their seats is not an acceptable answer for this problem! Make the recommendation in terms of improvements to the yaw damper.)

Solution :

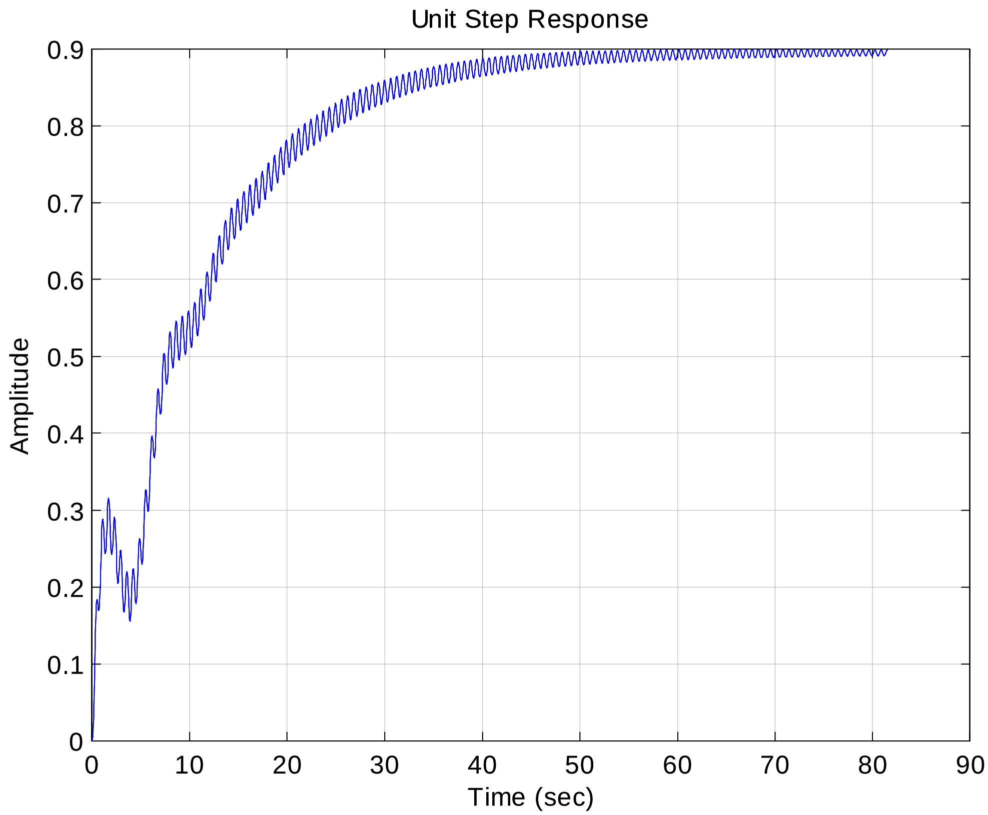
(a) The Bode plot of the open-loop system is :



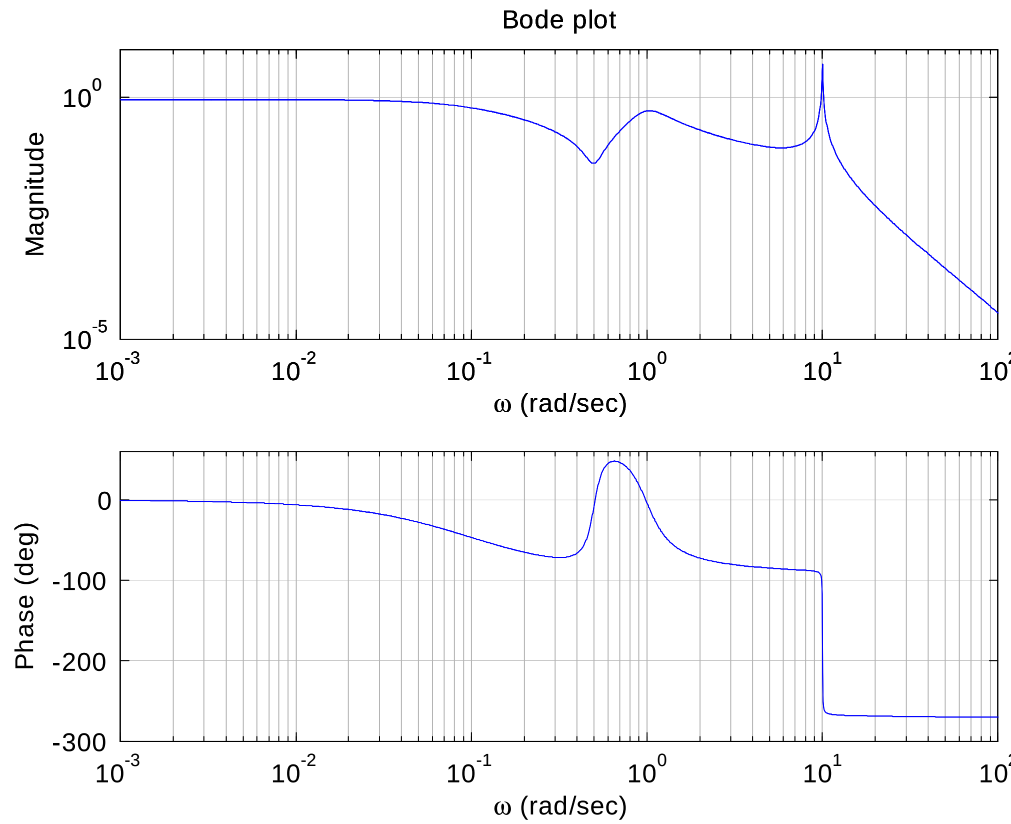
$$PM = 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec})$$
$$GM = 1.28 \ (\omega = 10.0 \text{ rad/sec})$$

The low GM is caused by the resonance being close to instability.

The closed-loop system unit step response is :



The Bode plot of the closed-loop system is :

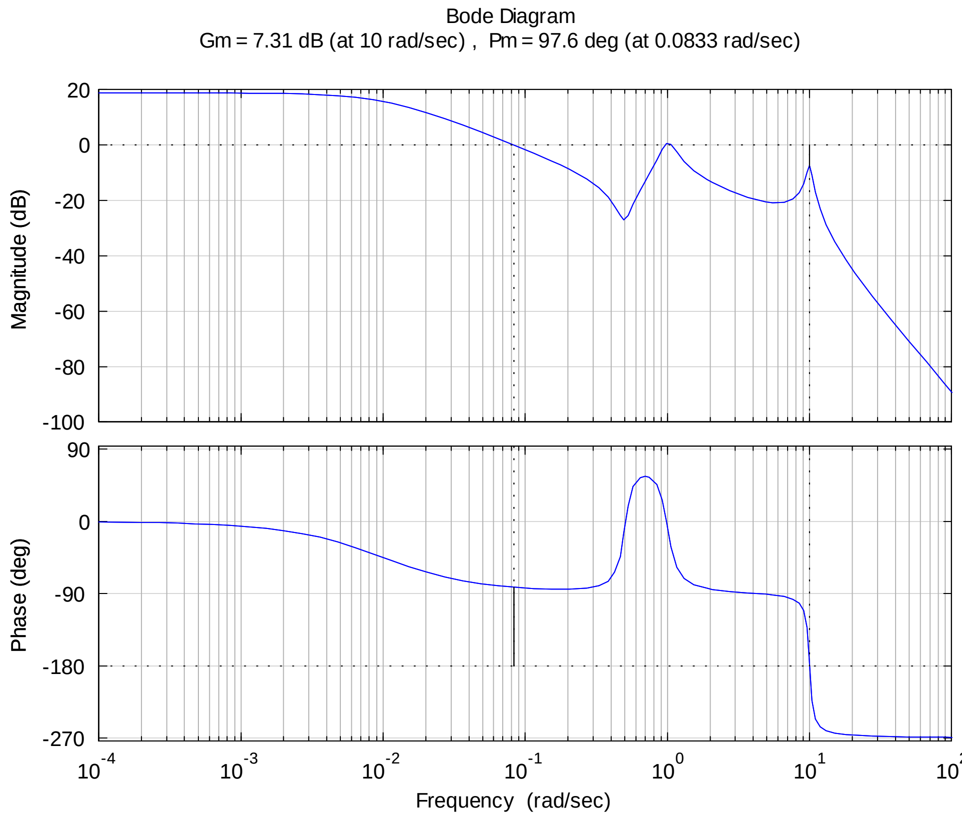


From the Bode plot of the closed -loop system, the frequency of the lightly damped mode is :

$$\omega \simeq 10 \text{ rad/sec}$$

and this is borne out by the step response that shows a lightly damped oscillation at 1.6 Hz or 10 rad/sec.

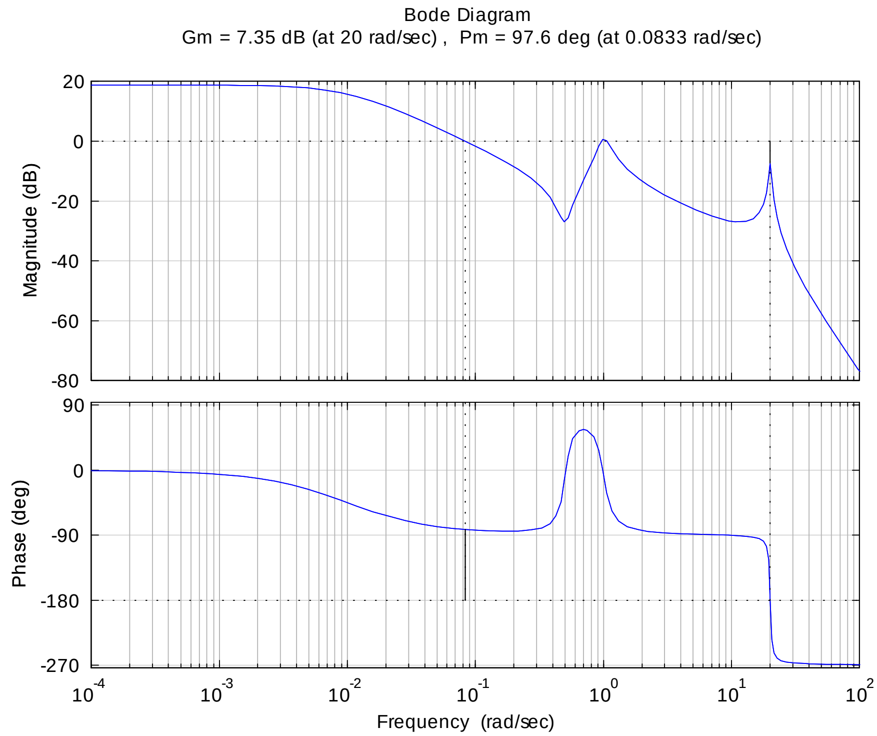
i. The Bode plot of the system with the bending mode damping increased from $\zeta = 0.02$ to $\zeta = 0.04$ is :



$$
\begin{aligned}
PM &= 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 7.31 \ (\omega = 10.0 \text{ rad/sec})
\end{aligned}
$$

and we see that the GM has increased considerably because the resonant peak is well below magnitude 1; thus the system will be much better behaved.

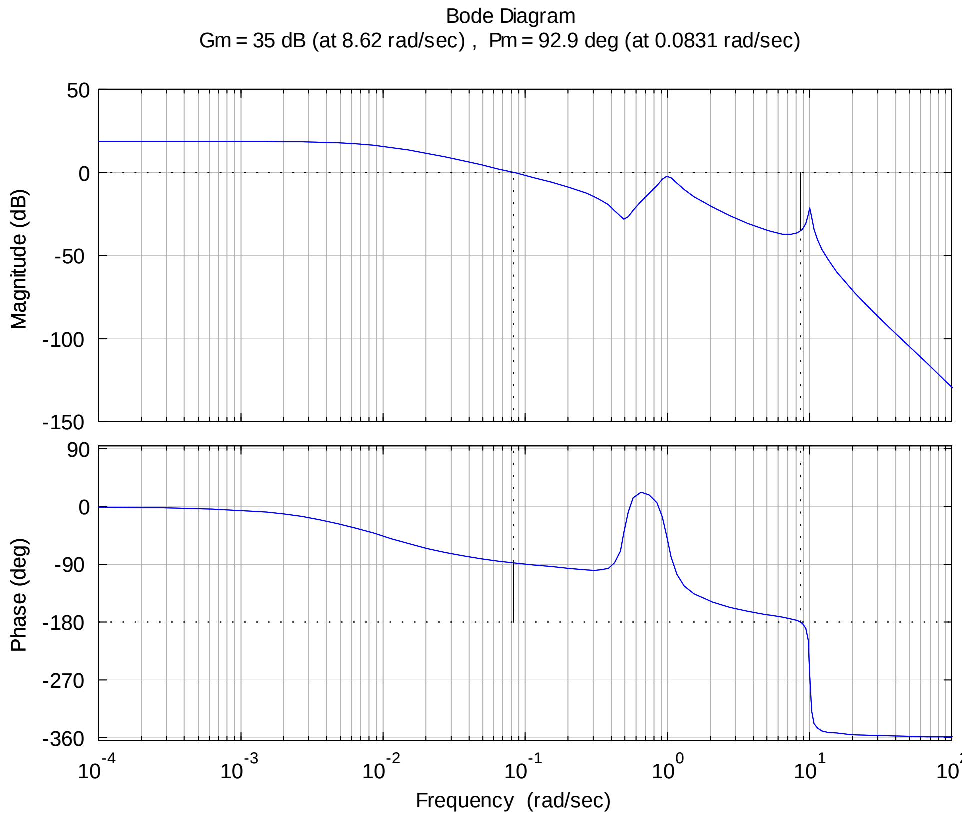
ii. The Bode plot of this system (ω_b = 10 rad/sec $\Longrightarrow \omega_b$ = 20 rad/sec) is :



$$
\begin{aligned}
PM &= 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 7.34 \ (\omega = 20.0 \text{ rad/sec})
\end{aligned}
$$

and again, the GM is much improved and the resonant peak is significantly reduced from magnitude 1.

iii. By picking up $\tau_p = 1$, the Bode plot of the system with the low pass filter is :



$$PM = 92.9^\circ\ (\omega = 0.0831 \text{ rad/sec})$$
$$GM = 34.97\ (\omega = 8.62 \text{ rad/sec})$$

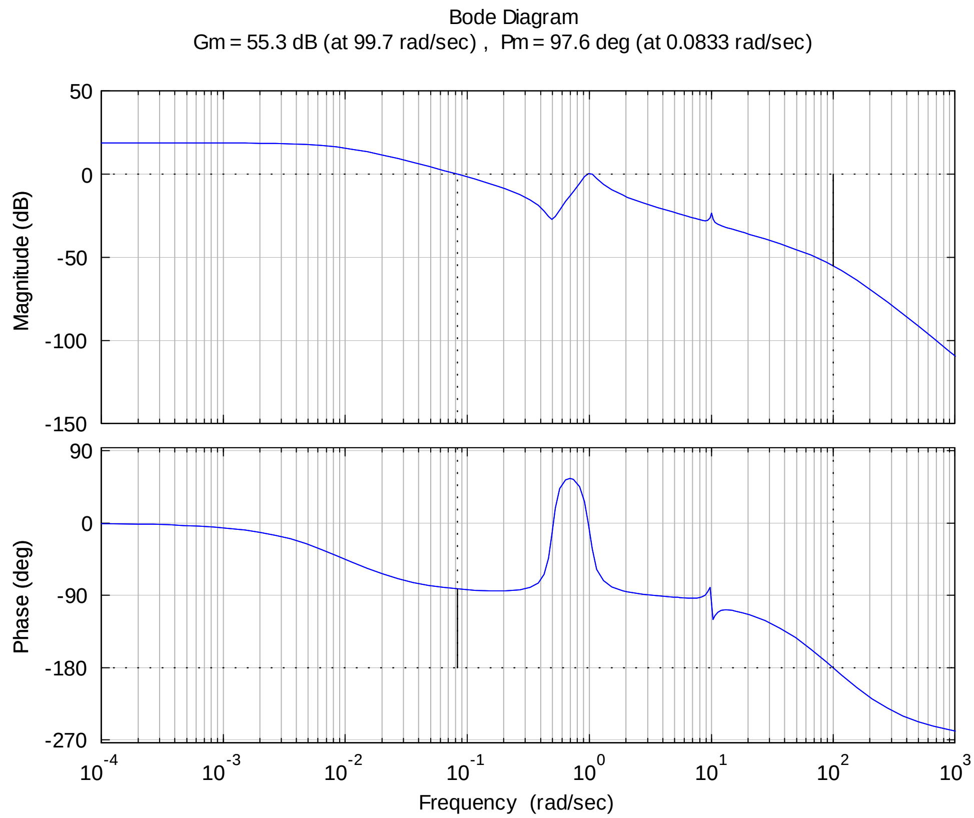
which are healthy margins and the resonant peak is, again, well below magnitude 1.

iv. The Bode plot of the system with the given notch filter is :

$$PM = 97.6^\circ\ (\omega = 0.0833 \text{ rad/sec})$$
$$GM = 55.3\ (\omega = 99.7 \text{ rad/sec})$$

which are the healthiest margins of all the designs since the notch filter has essentially canceled the bending mode resonant peak.

(b) Generally, the notch filter is very sensitive to where to place the notch zeros in order to reduce the lightly damped resonant peak. So if you want to use the notch filter, you must have a good estimation of the location of the bending mode poles and the poles must remain at that location for all aircraft conditions. On the other hand, the low pass filter is relatively robust to where to place its break point.

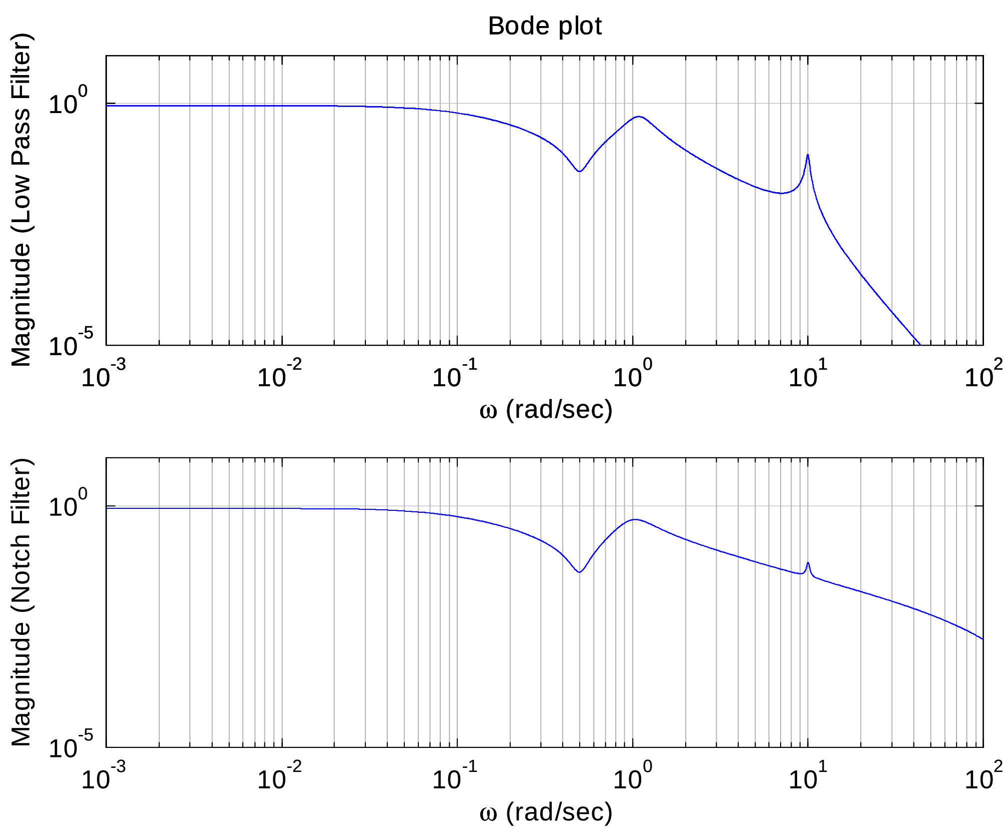


Bode Diagram
Gm = 55.3 dB (at 99.7 rad/sec) , Pm = 97.6 deg (at 0.0833 rad/sec)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
90
0
-90
-180
-270
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Evaluation of the margins with the bending mode frequency lowered by 10% will show a drastic reduction in the margins for the notch filter and very little reduction for the low pass filter.

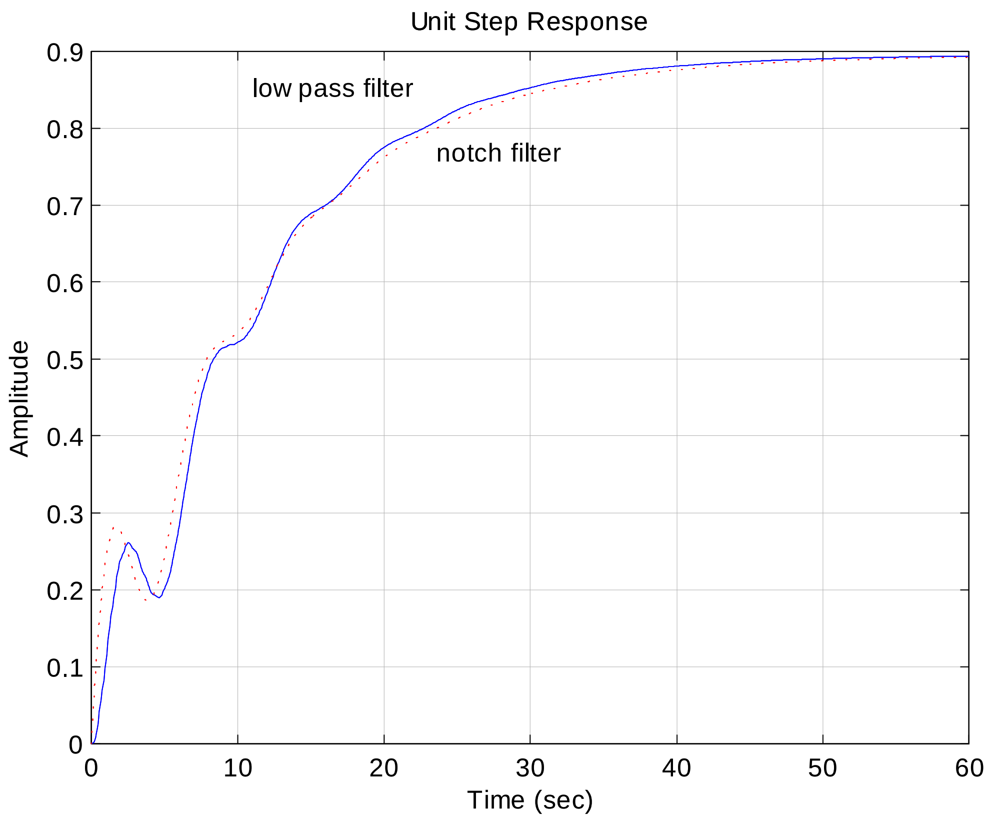
	Low Pass Filter	Notch Filter
GM	34.97 ($\omega = 8.62$ rad/sec)	55.3 ($\omega = 99.7$ rad/sec)
PM	92.9° ($\omega = 0.0831$ rad/sec)	97.6° ($\omega = 0.0833$ rad/sec)
Closed-loop bending mode damping ratio	$\simeq 0.02$	$\simeq 0.04$
Resonant peak	0.087	0.068

The magnitude plots of the closed-loop systems are :



The closed-loop step responses are :

(c) While increasing the natural damping of the system would be the best solution, it might be difficult and expensive to carry out. Likewise, increasing the frequency typically is expensive and makes the structure heavier, not a good idea in an aircraft. Of the remaining


Unit Step Response
low pass filter
notch filter
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
10
20
30
40
50
60

two options, it is a better design to use a low pass filter because of its reduced sensitivity to mismatches in the bending mode frequency. Therefore, the best recommendation would be to use the low pass filter.

60. Consider a system with the open-loop transfer function (loop gain)

$$G(s) = \frac{1}{s(s+1)(s/10+1)}.$$

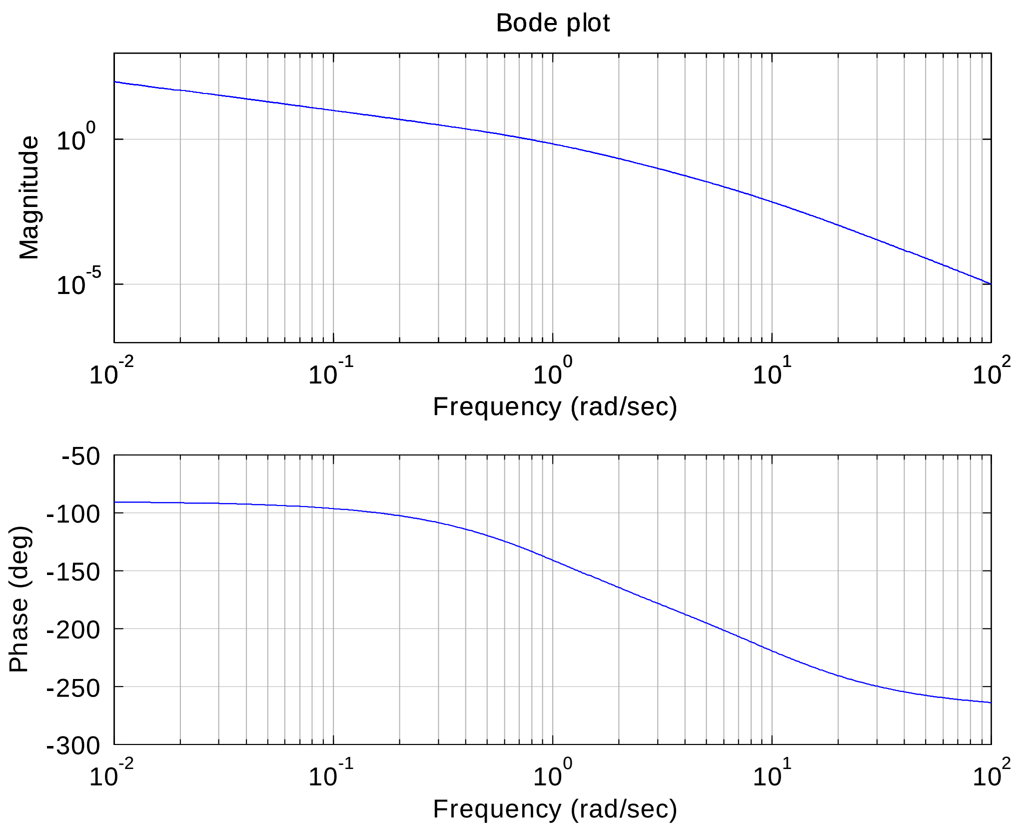
(a) Create the Bode plot for the system, and find GM and PM.

(b) Compute the sensitivity function and plot its magnitude frequency response.

(c) Compute the Vector Margin (VM).

Solution :

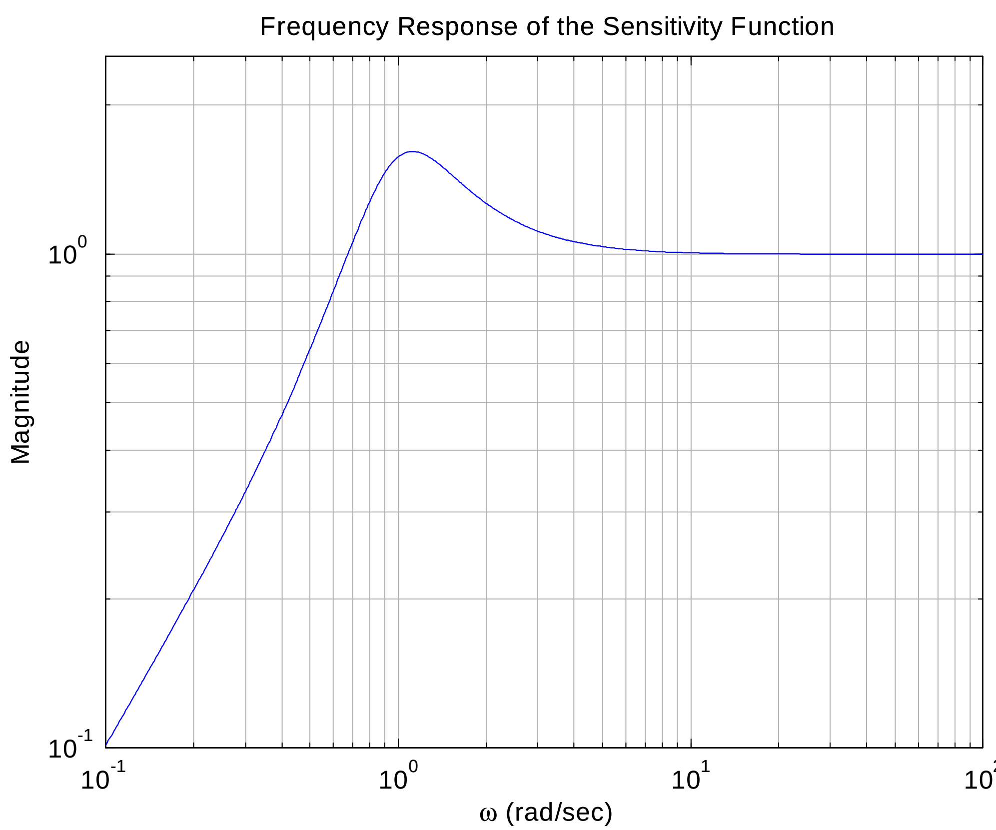
(a) The Bode plot is :



(b) Sensitivity function is :

$$S(s) = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{1}{s(s+1)(\frac{s}{10}+1)}}$$

The magnitude frequency response of this sensitivity function is :



(c) Vector Margin is defined as :

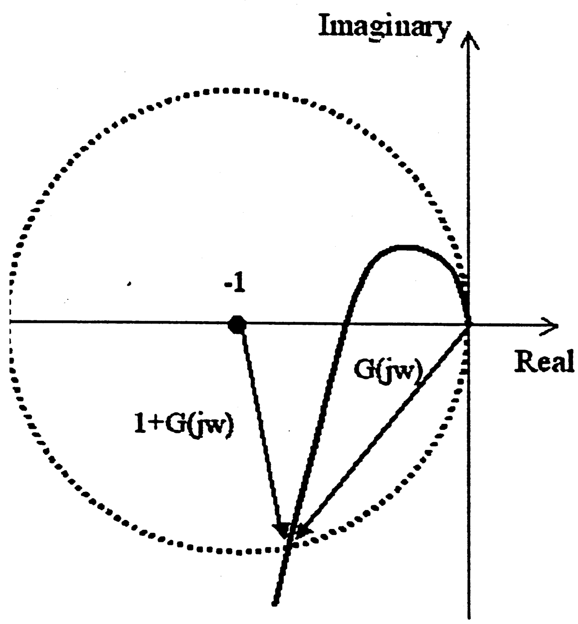
$$VM = \min_{\omega} \frac{1}{|s(j\omega)|} = \frac{1}{1.61} = 0.62$$

61. Prove that the sensitivity function $S(s)$ has magnitude greater than 1 inside a circle with a radius of 1 centered at the -1 point. What does

this imply about the shape of the Nyquist plot if closed-loop control is to outperform open-loop control at all frequencies?

Solution :

$$S(s) = \frac{1}{1 + G(s)}$$



Inside the unit circle, $|1 + G(s)| < 1$ which implies $|S(s)| > 1$.

Outside the unit circle, $|1 + G(s)| > 1$ which implies $|S(s)| < 1$.

On the unit circle, $|1 + G(s)| = 1$ which means $|S(s)| = 1$.

If the closed-loop control is going to outperform open-loop control then $|S(s)| \leq 1$ for all s. This means that the Nyquist plot must lie outside the circle of radius one centered at -1.

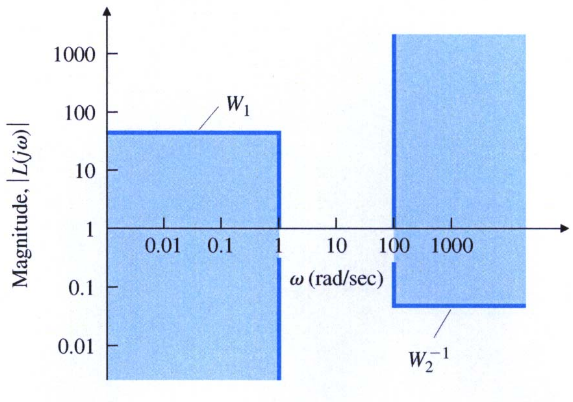
62. Consider the system in Fig. 6.100 with the plant transfer function

$$G(s) = \frac{10}{s(s/10 + 1)}.$$

We wish to design a compensator $D(s)$ that satisfies the following design specifications:

(a) i. $K_v = 100$,
   ii. PM $\geq 45°$,
   iii. sinusoidal inputs of up to 1 rad/sec to be reproduced with $\leq 2\%$ error,

Figure 6.102: Control system constraints for Problem 62



iv. sinusoidal inputs with a frequency of greater than 100 rad/sec to be attenuated at the output to $\leq 5\%$ of their input value.

(b) Create the Bode plot of $G(s)$, choosing the open-loop gain so that $K_v = 100$.

(c) Show that a *sufficient* condition for meeting the specification on sinusoidal inputs is that the magnitude plot lies outside the shaded regions in Fig. 6.102. Recall that

$$\frac{Y}{R} = \frac{KG}{1+KG} \quad \text{and} \quad \frac{E}{R} = \frac{1}{1+KG}.$$

(d) Explain why introducing a lead network alone cannot meet the design specifications.

(e) Explain why a lag network alone cannot meet the design specifications.

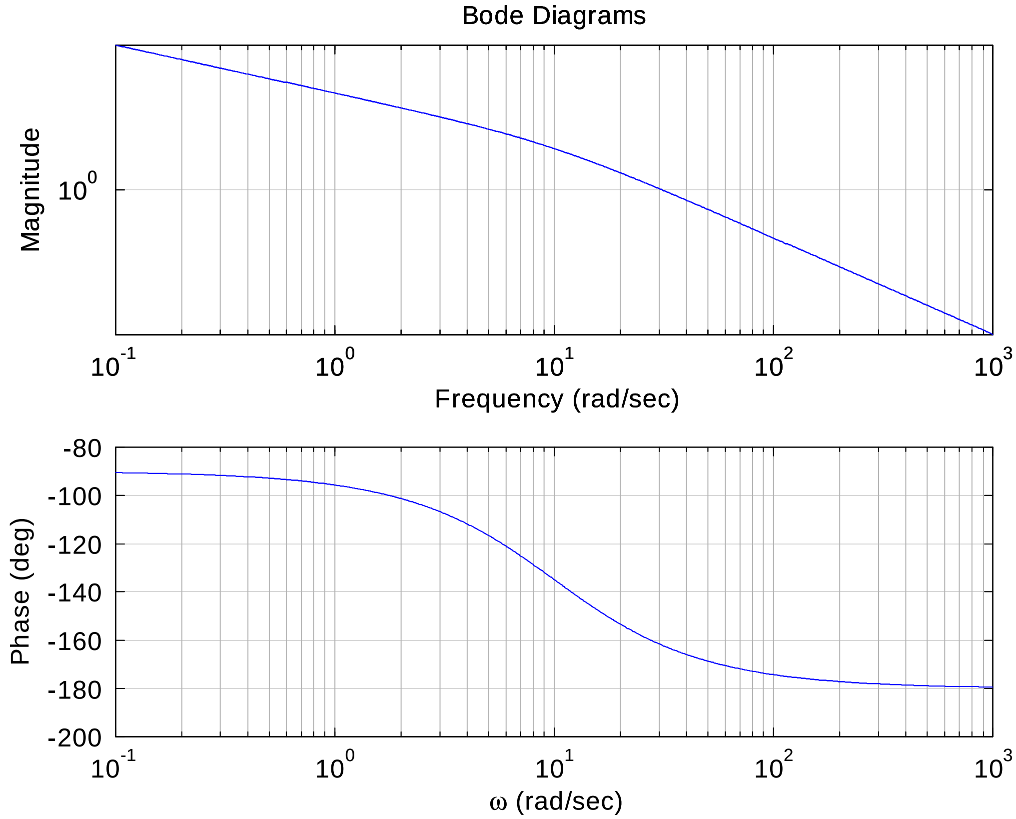
(f) Develop a full design using a lead-lag compensator that meets all the design specifications, without altering the previously chosen low frequency open-loop gain.

Solution :

(a) To satisfy the given velocity constant K_v,

$$\begin{aligned} K_v &= \lim_{s \to 0} sKG(s) = 10K = 100 \\ &\Longrightarrow \quad K = 10 \end{aligned}$$

(b) The Bode plot of $G(s)$ with the open-loop gain $K = 10$ is :



(c) From the 3rd specification,

$$\left|\frac{E}{R}\right| = \left|\frac{1}{1+KG}\right| < 0.02 \ (2\%)$$
$$\Longrightarrow \ |KG| > 49 \ (\text{at } \omega < 1 \text{ rad/sec})$$

From the 4th specification,

$$\left|\frac{Y}{R}\right| = \left|\frac{KG}{1+KG}\right| < 0.05 \ (5\%)$$
$$\Longrightarrow \ |KG| < 0.0526 \ (\text{at } \omega > 100 \text{ rad/sec})$$

which agree with the figure.

(d) A lead compensator may provide a sufficient PM, but it increases the gain at high frequency so that it violates the specification above.

(e) A lag compensator could satisfy the PM specification by lowering the crossover frequency, but it would violate the low frequency specification, W_1.

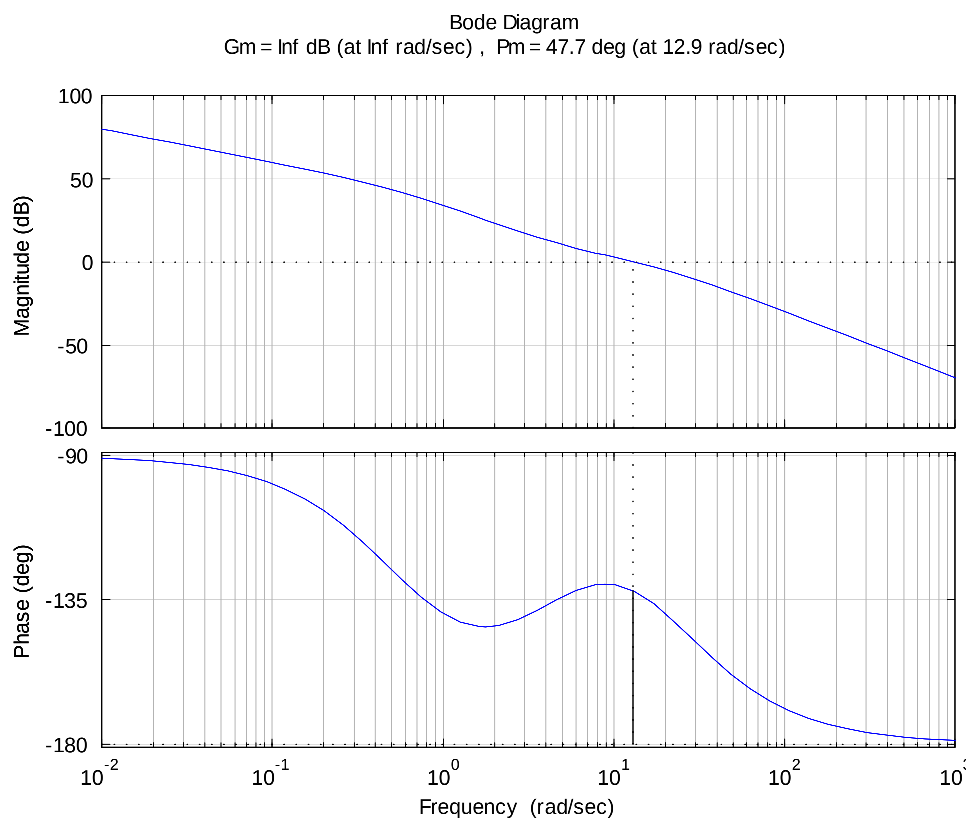
(f) One possible lead-lag compensator is :

$$D_c(s) = 100\frac{\frac{s}{8.52}+1}{\frac{s}{22.36}+1}\frac{\frac{s}{4.47}+1}{\frac{s}{0.568}+1}$$

which meets all the specification :

$$\begin{aligned} K_v &= 100 \\ PM &= 47.7^\circ \text{ (at } \omega_c = 12.9 \text{ rad/sec)} \\ |KG| &= 50.45 \text{ (at } \omega = 1 \text{ rad/sec)} > 49 \\ |KG| &= 0.032 \text{ (at } \omega = 100 \text{ rad/sec)} < 0.0526 \end{aligned}$$

The Bode plot of the compensated open-loop system $D_c(s)G(s)$ is :

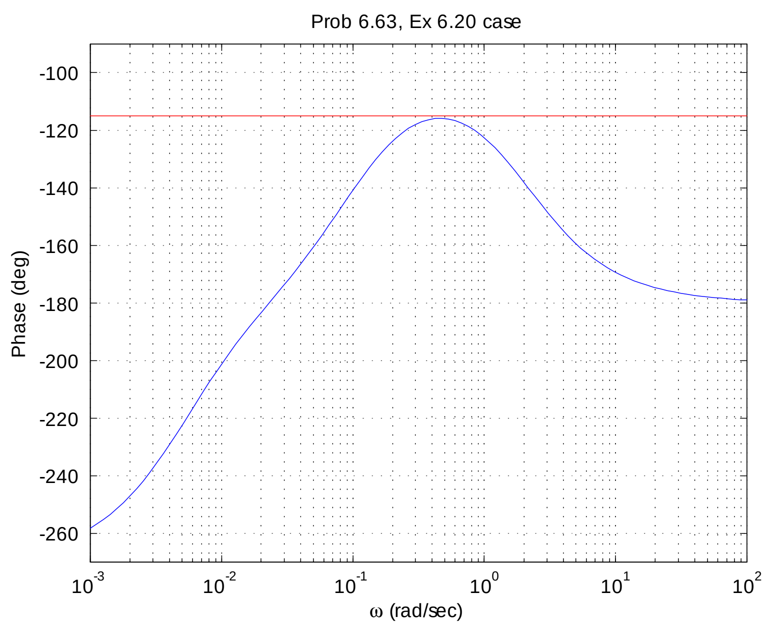


63. For Example 6.20, redo the design by selecting $1/T_D = 0.05$ and then determining the highest possible value of $1/T_I$ that will meet the PM

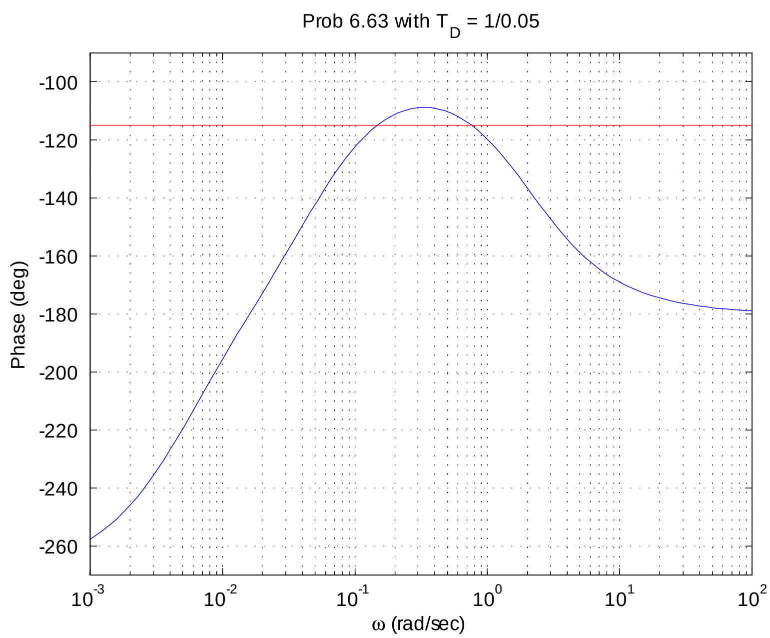
requirement. Then examine the improvement, if any, in the response to a step disturbance torque.

Solution:

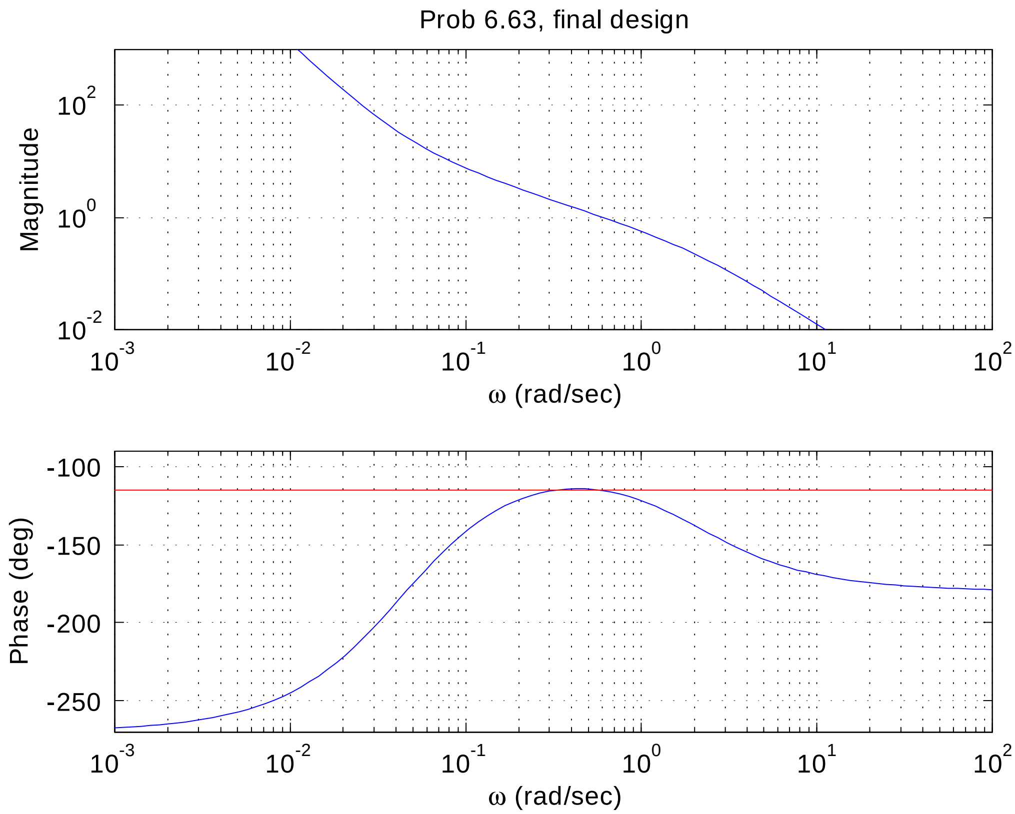
From Example 6:20, note that $1/T_D = 0.1$ was selected for the case where the phase curve just touched 115 Deg, the minimum requirement to achieve a PM = 65 deg. By increasing T_D so that $1/T_D = 0.05$, we should get more damping, and there should be more leeway in selecting a faster integration term. Let's first repeat the phase curve of Fig. 6.68.



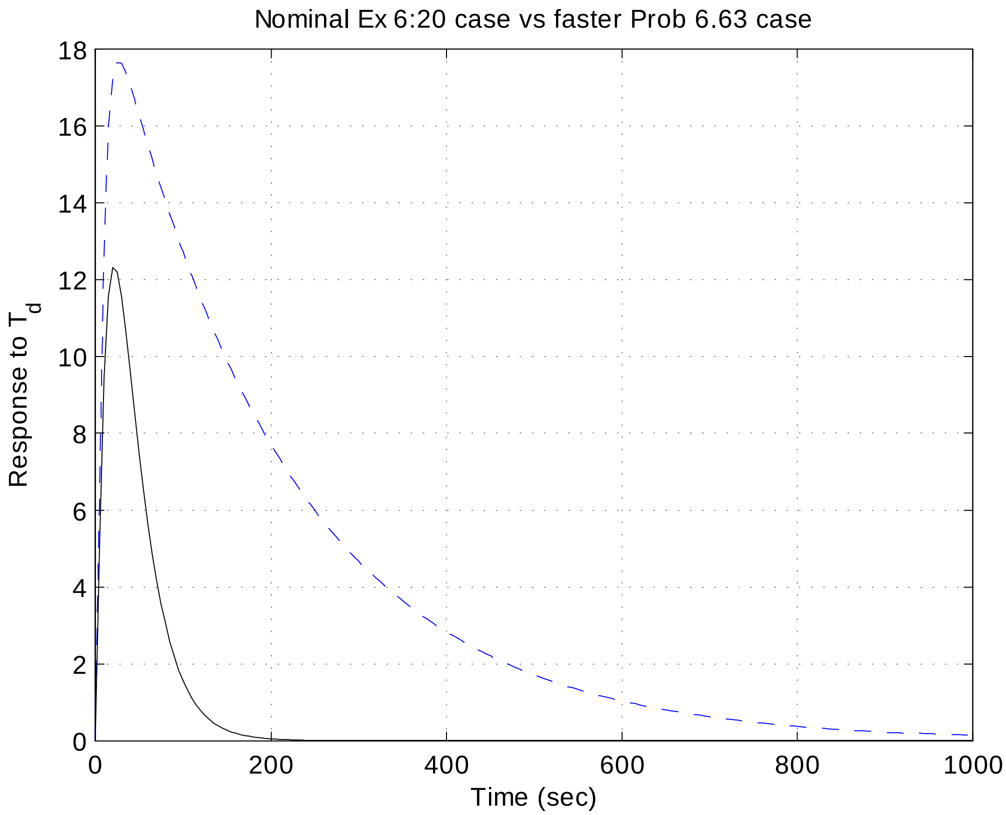
Now, let's change the derivative term for more damping as specified by the problem statement, i.e., $1/T_D = 0.05$, and repeat the phase plot



and, indeed, we see that we can speed up the integral term until the phase just kisses the required -115 deg line again. (Once that's done, we'll adjust the gain so crossover occurs at the desired spot). So, use trial and error and we discover that $1/T_I = 0.04$ (almost a factor of 10 increase !) will produce a phase curve that just touches the required phase line. Then we find that the gain, K, that produces the gain $= 1$ crossover at the peak of the phase curve is $K = 0.035$. The resulting Bode plot for this case is



and from the Matlab margin command, we find that the PM requirement is still met, and the crossover frequency has increased from 0.45 r/sec to 0.61 r/sec. More importantly, the resulting increased speed of the integral term has signifantly improved the response to a disturbance torque. The response of the output to a step disturbance torque is shown below for the original case from Example 6:20 and the modified design here with $1/T_I = 0.04$, $1/T_D = 0.05$, and $K = 0.035$. The original design is shown with dashed lines, while the modified design is the solid curve.



Problems and Solutions for Section 6.8

64. Assume that the system

$$G(s) = \frac{e^{-T_d s}}{s+10},$$

has a 0.2-sec time delay ($T_d = 0.2$ sec). While maintaining a phase margin $\geq 40°$, find the maximum possible bandwidth using the following:

(a) One lead-compensator section

$$D_c(s) = K\frac{s+a}{s+b},$$

where $b/a = 100$;

(b) Two lead-compensator sections

$$D_c(s) = K\left(\frac{s+a}{s+b}\right)^2,$$

where $b/a = 10$.

(c) Comment on the statement in the text about the limitations on the bandwidth imposed by a delay.

Solution :

(a) One lead section :

With $b/a = 100$, the lead compensator can add the maximum phase lead :

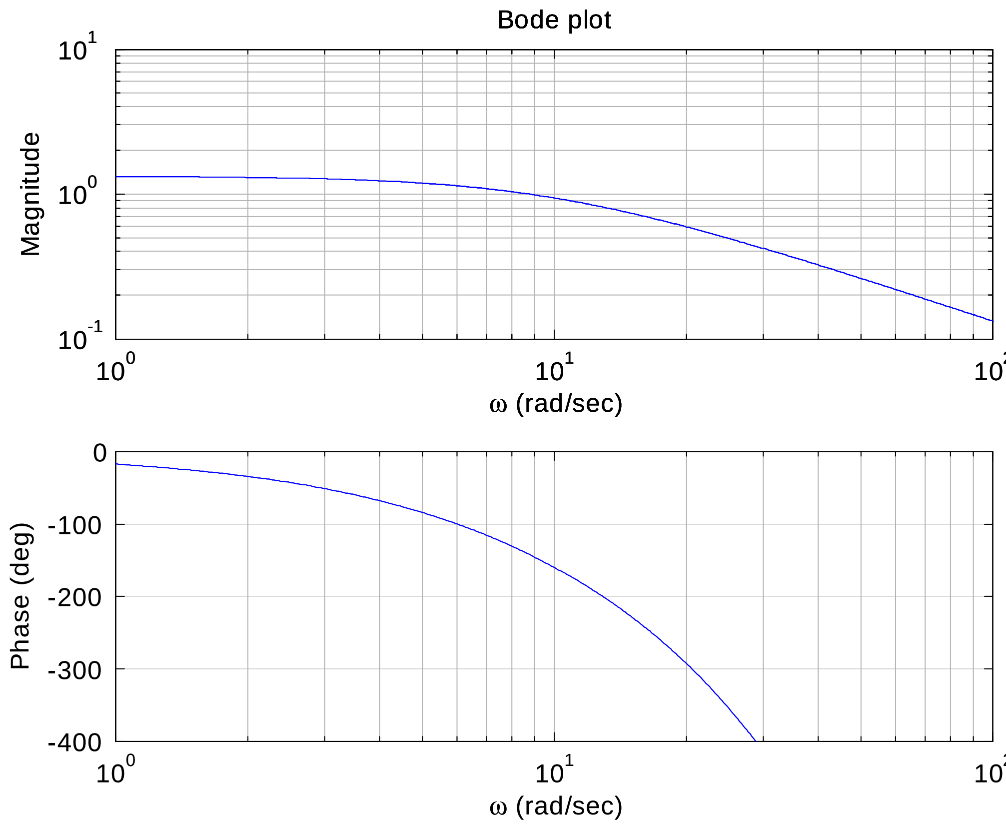
$$\begin{aligned} \phi_{\max} &= \sin^{-1}\frac{1-\frac{a}{b}}{1+\frac{a}{b}} \\ &= 78.6 \text{ deg (at } \omega = 10a \text{ rad/sec)} \end{aligned}$$

By trial and error, a good compensator is :

$$\begin{aligned} K &= 1202,\ a = 15 \Longrightarrow D_a(s) = 1202\frac{s+15}{s+1500} \\ PM &= 40^\circ \text{ (at } \omega_c = 11.1 \text{ rad/sec)} \end{aligned}$$

The Bode plot is shown below. Note that the phase is adjusted for the time delay by subtracting ωT_d at each frequency point while there is no effect on the magnitude. For reference, the figures also include the case of proportional control, which results in :

$$K = 13.3,\ PM = 40^\circ \text{ (at } \omega_c = 8.6 \text{ rad/sec)}$$



(b) Two lead sections :

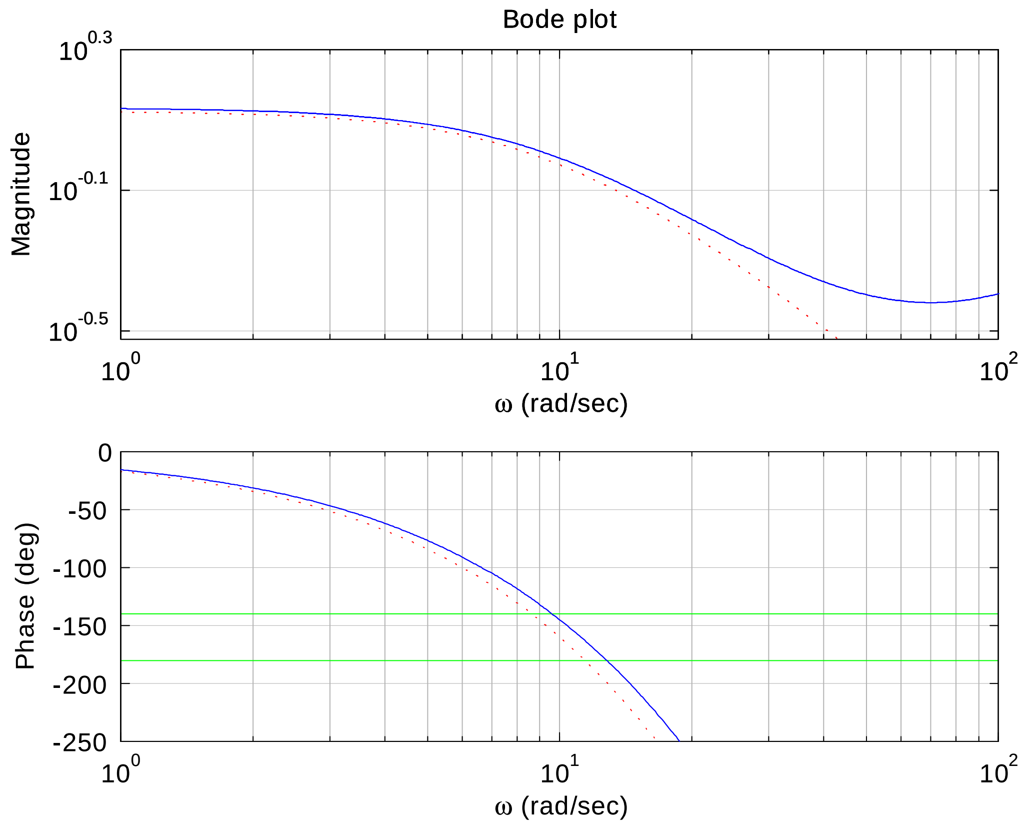
With $b/a = 10$, the lead compensator can add the maximum phase lead :

$$\begin{aligned} \phi_{\max} &= \sin^{-1}\frac{1-\frac{a}{b}}{1+\frac{a}{b}} \\ &= 54.9 \text{ deg (at } \omega = \sqrt{10}a \text{ rad/sec)} \end{aligned}$$

By trial and error, one of the possible compensators is :

$$\begin{aligned} K &= 1359,\ a = 70 \Longrightarrow D_b(s) = 1359\frac{(s+70)^2}{(s+700)^2} \\ PM &= 40^\circ \text{ (at } \omega_c = 9.6 \text{ rad/sec)} \end{aligned}$$

The Bode plot is shown below.



(c) The statement in the text is that it should be difficult to stabilize a system with time delay at crossover frequencies, $\omega_c \gtrsim 3/T_d$. This problem confirms this limit, as the best crossover frequency achieved was $\omega_c = 9.6$ rad/sec whereas $3/T_d = 15$ rad/sec. Since the bandwidth is approximately twice the crossover frequency, the limitations imposed on the bandwidth by the time delay is verified.

65. Determine the range of K for which the following systems are stable:

(a) $G(s) = K\frac{e^{-4s}}{s}$

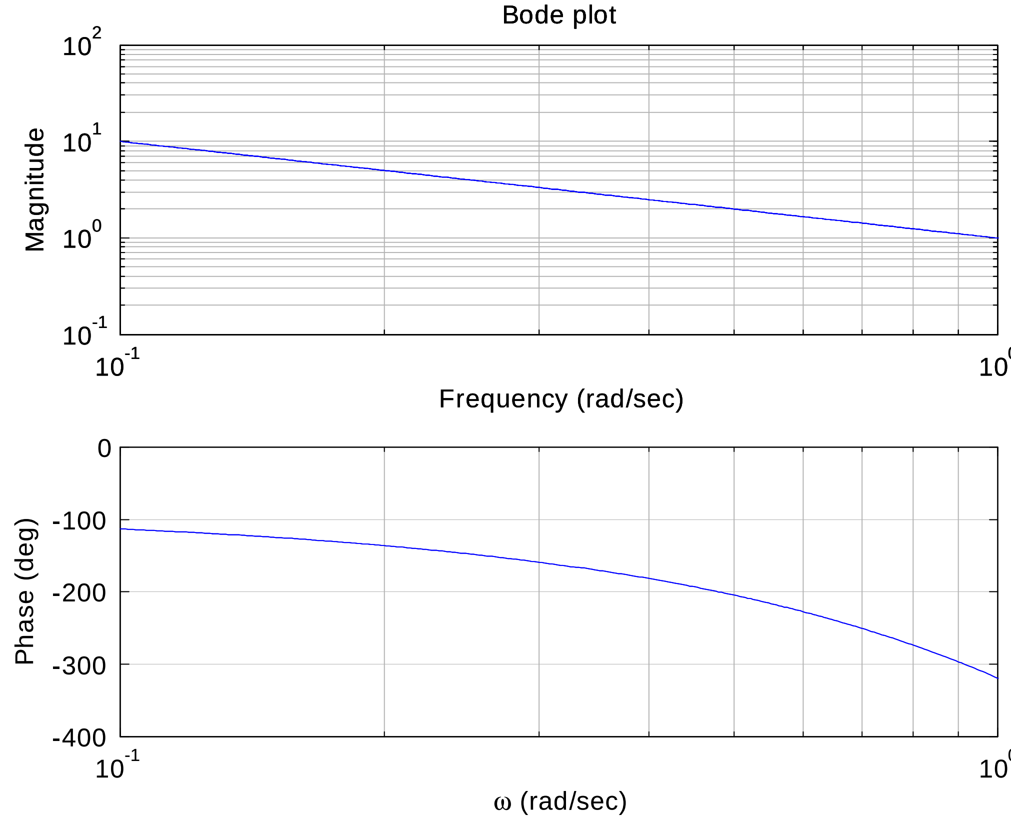
(b) $G(s) = K\frac{e^{-s}}{s(s+2)}$

Solution :

(a)

$$\left|\frac{G(j\omega)}{K}\right| = 2.54, \text{ when } \angle\frac{G(j\omega)}{K} = -180^\circ$$

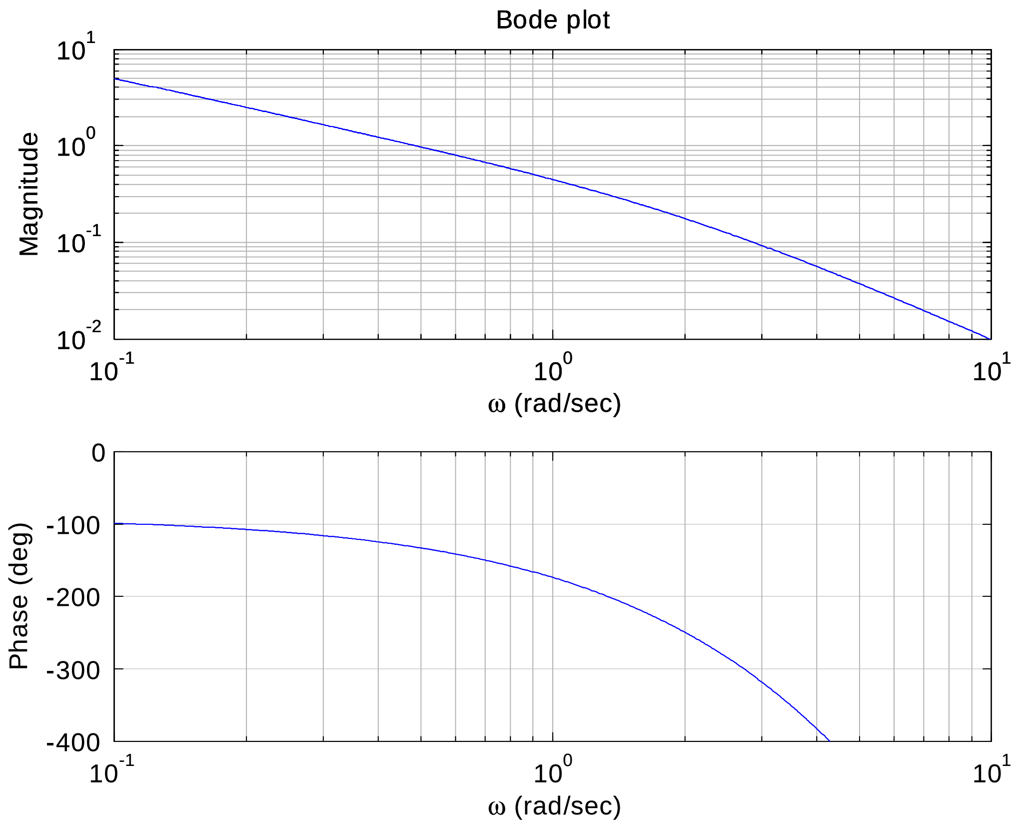
$$\text{range of stability}: \; 0 < K < \frac{1}{2.54}$$



(b)

$$\left|\frac{G(j\omega)}{K}\right| = 0.409 = \frac{1}{2.45}, \text{ when } \angle\frac{G(j\omega)}{K} = -180^\circ$$

$$\text{range of stability : } 0 < K < 2.45$$



66. Consider the heat exchanger of Example 2.16 with the open-loop transfer function

$$G(s) = \frac{e^{-5s}}{(10s+1)(60s+1)}.$$

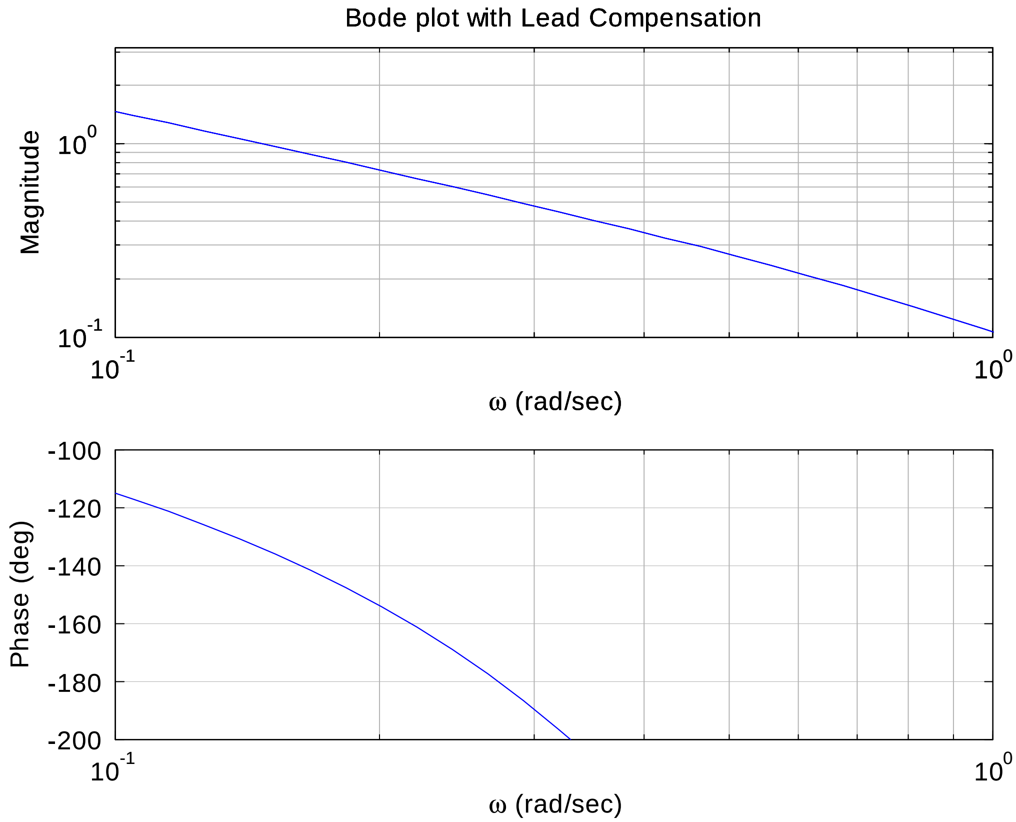
(a) Design a lead compensator that yields PM $\geq 45°$ and the maximum possible closed-loop bandwidth.

(b) Design a PI compensator that yields PM $\geq 45°$ and the maximum possible closed-loop bandwidth.

Solution :

(a) First, make sure that the phase calculation includes the time delay lag of $-T_d\omega = -5\omega$. A convenient placement of the lead zero is at $\omega = 0.1$ because that will preserve the -1 slope until the lead pole. We then raise the gain until the specified PM is obtained in order to maximize the crossover frequency. The resulting lead compensator,

$$D_c(s) = \frac{90(s+0.1)}{(s+1)}$$

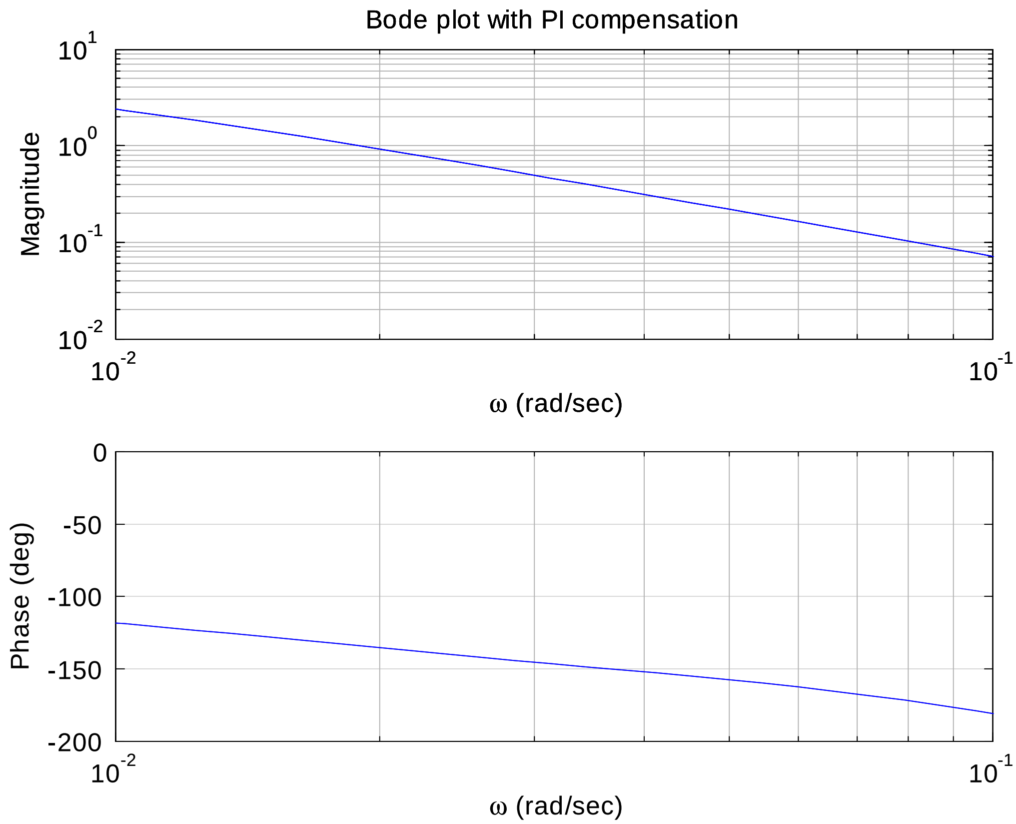


yields $PM = 46°$ as seen by the Bode below.Also note that the crossover frequency, $\omega_c = 0.15$ rad/sec, which can be read approximately from the plot above, and verified by using the margin command in MATLAB with the phase adjusted by the time delay lag.

(b) The brealpoint of the PI compensator needs to be kept well below 0.1 in order to maintain a positive phase margin at as high a crossover frequency as possible. In Table 4.1, Zeigler-Nichols suggest a breakpoint at $\omega = 1/17$, so we will select a PI of the form :

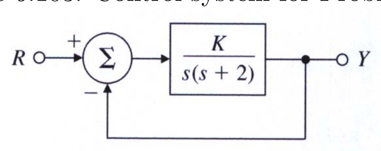
$$D_c(s) = K\left(1 + \frac{1}{20s}\right)$$

and select the gain so that the PM specification is met. For $K = 0.55$ the phase margin is 46° as shown by the Bode below:



Note with this compensation that $\omega_c = 0.02$ rad/sec, which is considerably lower than that yielded by the lead compensation.

Figure 6.103: Control system for Problem 67



Problems and Solutions for Section 6.9

67. A feedback control system is shown in Fig.6.103. The closed-loop system is specified to have an overshoot of less than 30% to a step input.

(a) Determine the corresponding PM specification in the frequency domain and the corresponding closed-loop resonant peak value M_r. (See Fig. 6.37)

(b) From Bode plots of the system, determine the maximum value of K that satisfies the PM specification.

(c) Plot the data from the Bode plots (adjusted by the K obtained in part (b)) on a copy of the Nichols chart in Fig. 6.81 and determine the resonant peak magnitude M_r. Compare that with the approximate value obtained in part (a).

(d) Use the Nichols chart to determine the resonant peak frequency ω_r and the closed-loop bandwidth.

Solution :

(a) From Fig. 6.37 :

$$M_p \leq 0.3 \Longrightarrow PM \geq 40^o \Longrightarrow M_r \leq 1.5$$

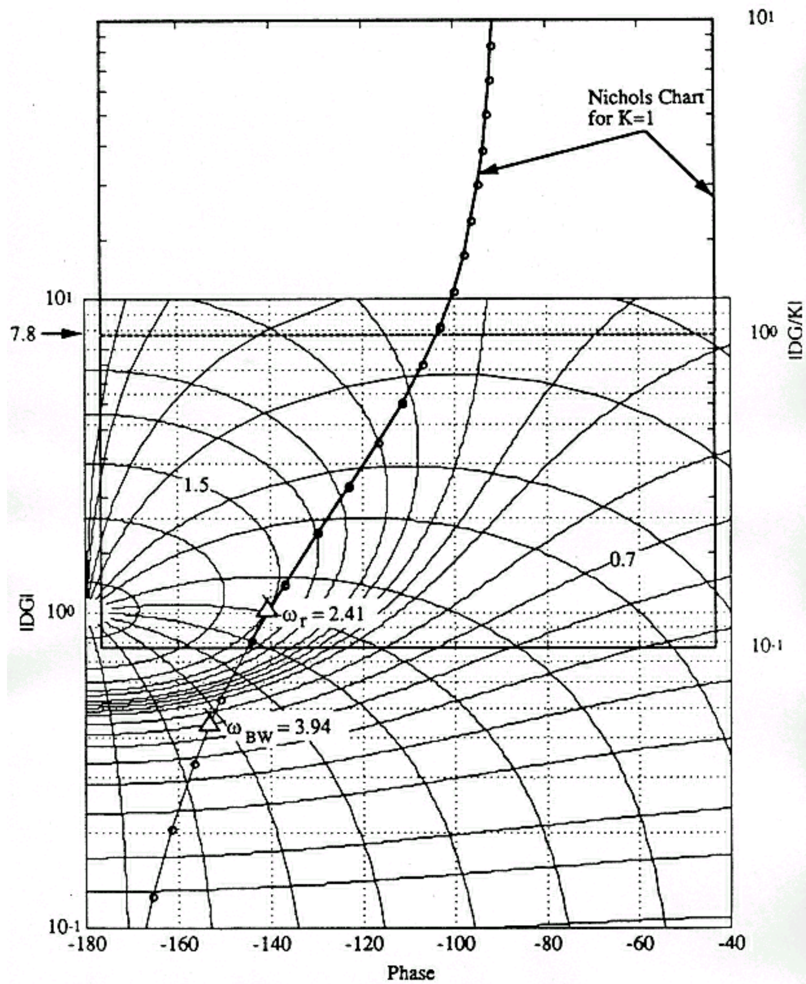
$$\text{resonant peak : } M_r \leq 1.5$$

(b) A sketch of the asymptotes of the open loop Bode shows that a PM of $\cong 40^o$ is obtained when $K = 8$. A Matlab plot of the Bode can be used to refine this and yields

$$K = 7.81$$

for $PM = 40^o$.

(c) The Nichols chart below shows that $M_r = 1.5$ which agrees exactly with the prediction from Fig. 6.37:



(d) The corresponding frequency where the curve is tangent to $M_r = 1.5$ is:

$$\omega_r = 2.41 \text{ rad/sec}$$

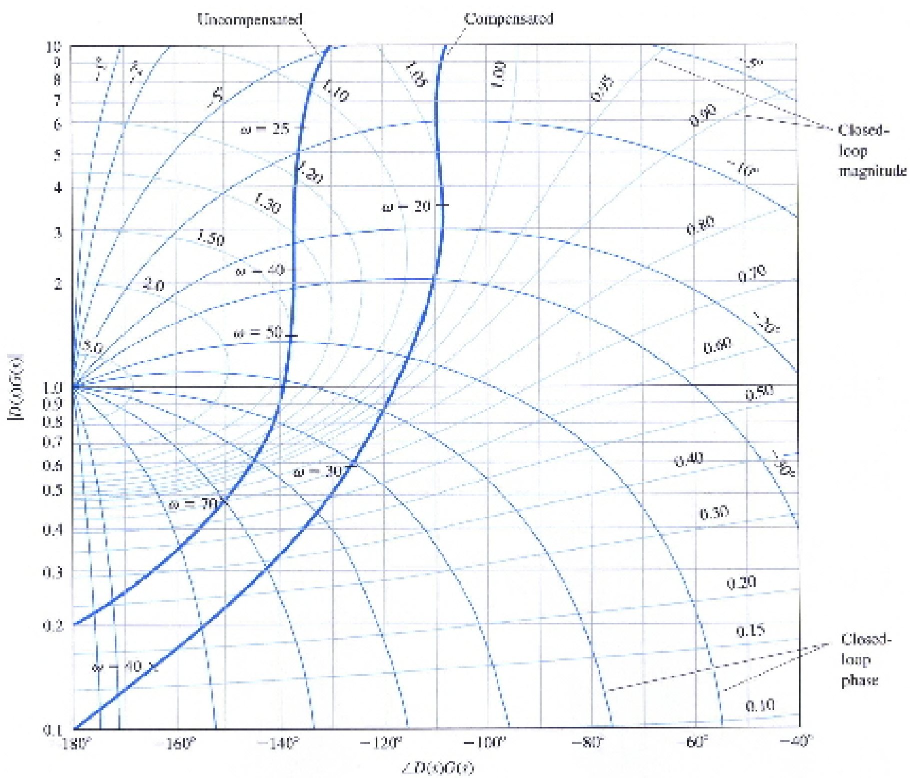
as can be determined by noting the frequency from the Bode plot that corresponds to the point on the Nichols chart.

The bandwidth ω_{BW} is determined by where the curve crosses the closed-loop magnitude of 0.7 and noting the frequency from the Bode plot that corresponds to the point on the Nichols chart

$$\omega_{BW} = 3.94 \text{ rad/sec}$$

68. The Nichols plot of an uncompensated and a compensated system are shown in Fig. 6.104.

Figure 6.104: Nichols plot for Problem 68



(a) What are the resonance peaks of each system?
(b) What are the PM and GM of each system?
(c) What are the bandwidths of each system?
(d) What type of compensation is used?

Solution :

(a) Resonant peak :

Uncompensated system	:	Resonant peak = 1.5 (ω_r = 50 rad/sec)
Compensated system	:	Resonant peak = 1..05 (ω_r = 20 rad/sec)

(b) PM, GM :

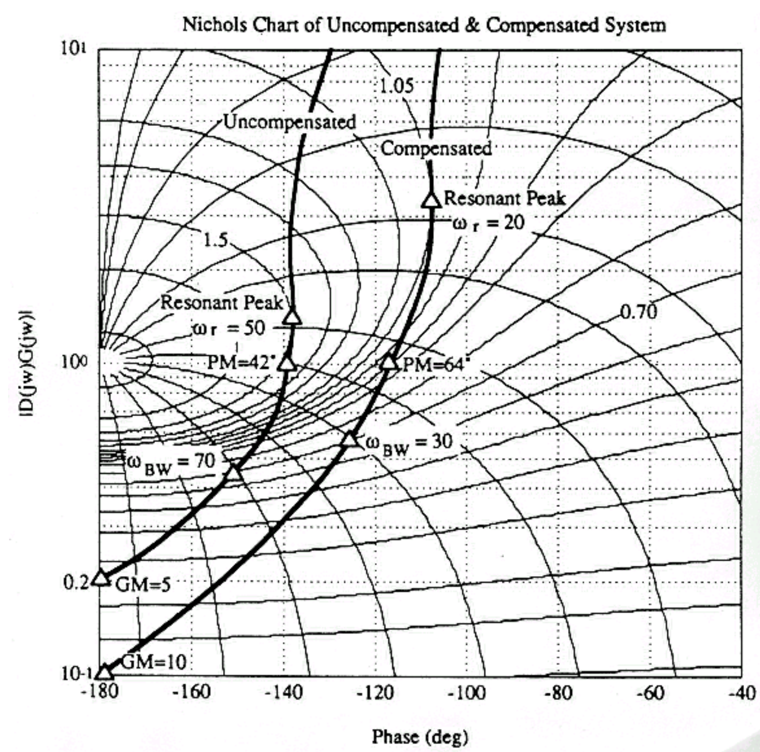
$$\text{Uncompensated system} \quad : \quad PM = 42^\circ,\ GM = \frac{1}{0.2} = 5$$

$$\text{Compensated system} \quad : \quad PM = 64^\circ,\ GM = \frac{1}{0.1} = 10$$

(c) Bandwidth :

Uncompensated system	:	Bandwidth = 70 rad/sec
Compensated system	:	Bandwidth = 30 rad/sec

(d) Lag compensation is used, since the bandwidth is reduced.



69. Consider the system shown in Fig. 6.95.

    (a) Construct an inverse Nyquist plot of $[Y(j\omega)/E(j\omega)]^{-1}$. (See Appendix W6.9.2)

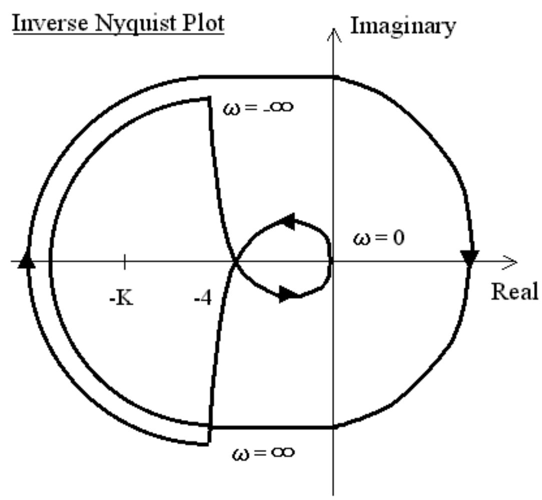
    (b) Show how the value of K for neutral stability can be read directly from the inverse Nyquist plot.

    (c) For $K = 4$, 2, and 1, determine the gain and phase margins.

    (d) Construct a root-locus plot for the system, and identify corresponding points in the two plots. To what damping ratios ζ do the GM and PM of part (c) correspond?

Solution :

(a) See the inverse Nyquist plot.



(b) Let

$$G(j\omega) = \frac{Y(j\omega)}{E(j\omega)}$$

The characteristic equation with $s = j\omega$:

$$1 + K_u G(j\omega) = 0$$

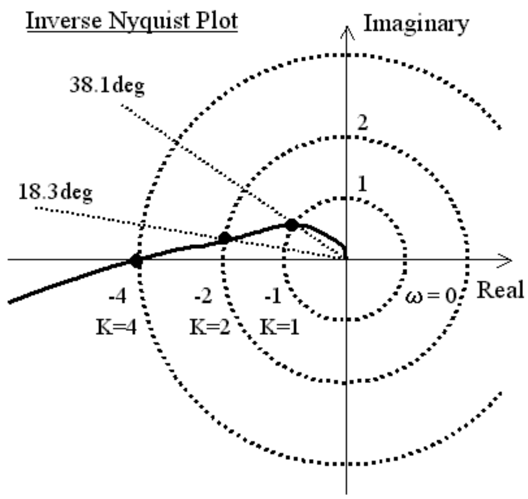
$$\Longrightarrow G^{-1} = -K_u$$

From the inverse Nyquist plot,

$$-K_u = -4 \Longrightarrow K_u = 4$$

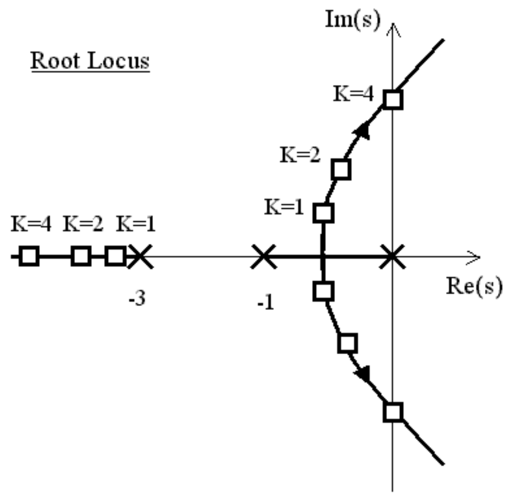
(c)

K	GM	PM
4	$\frac{-4}{-4} = 1$	$0°$
2	$\frac{-4}{-2} = 2$	$18.3°$
1	$\frac{-4}{-1} = 4$	$38.1°$



(d)

K	closed-loop poles	ζ
4	-4 $\pm j1.73$	0
2	-3.63 $-0.19 \pm j1.27$	0.14
1	-3.37 $-0.31 \pm j0.89$	0.33



70. An unstable plant has the transfer function

$$\frac{Y(s)}{F(s)} = \frac{s+1}{(s-1)^2}.$$

A simple control loop is to be closed around it, in the same manner as the block diagram in Fig. 6.95.

(a) Construct an inverse Nyquist plot of Y/F. (See Appendix W6.9.2)

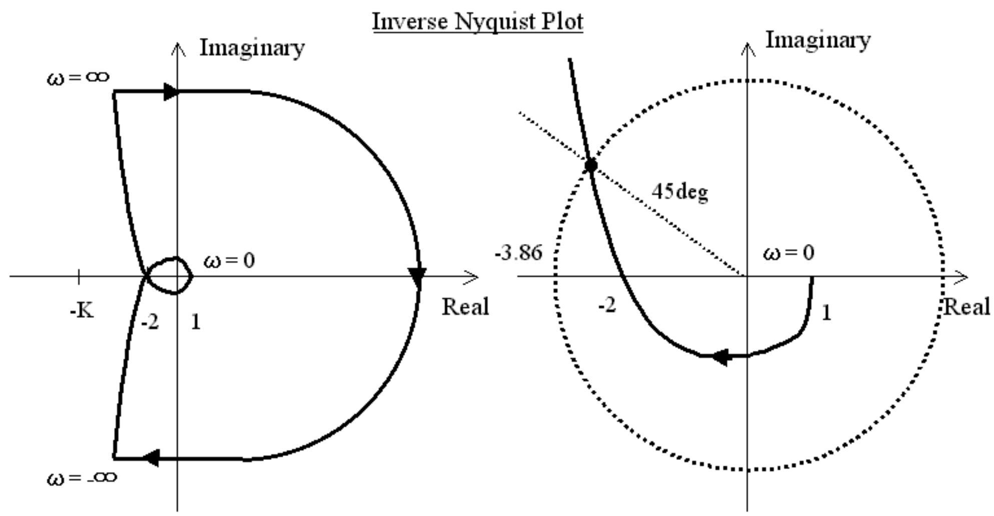
(b) Choose a value of K to provide a PM of 45°. What is the corresponding GM?

(c) What can you infer from your plot about the stability of the system when $K < 0$?

(d) Construct a root-locus plot for the system, and identity corresponding points in the two plots. In this case, to what value of ζ does PM = 45° correspond?

Solution :

(a) The plots are :



(b) From the inverse Nyquist plot, $K = 3.86$ provides a phase margin of $45°$.

Since $K = 2$ gives $\angle G(j\omega)^{-1} = 180°$,

$$GM = \frac{2}{3.86} = 0.518$$

Note that GM is less than 1, but the system with $K = 3.86$ is stable.

$$K = 3.86,\ GM = 0.518$$

(c) We can apply stability criteria to the inverse Nyquist plot as follows :

Let

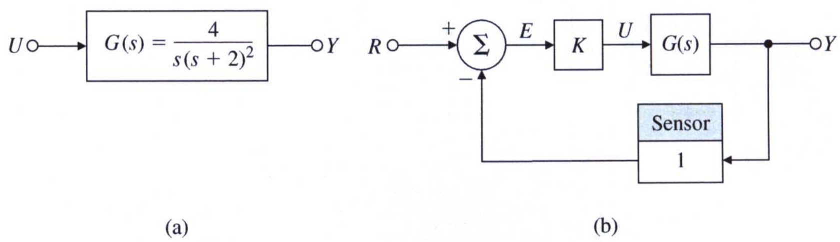
$$\begin{aligned} N &= \text{Net number of clockwise encirclement of } -K \\ P &= \text{Number of poles of } G^{-1} \text{ in RHP} \\ (&= \text{Number of zeros of } G \text{ in RHP)} \\ Z &= \text{Number of closed-loop system roots in RHP} \end{aligned}$$

Then,

$-K < -2$	$K > 2 \Longrightarrow N = 0, P = 0$ $\Longrightarrow Z = 0 \Longrightarrow$ Stable
$-2 < -K < 1$	$-1 > K > 2 \Longrightarrow N = 2, P = 0$ $\Longrightarrow Z = 2 \Longrightarrow$ Two unstable closed-loop roots
$-K > 1$	$K < -1 \Longrightarrow N = 1, P = 0$ $\Longrightarrow Z = 1 \Longrightarrow$ One unstable closed-loop root

Then, we can infer from the inverse Nyquist plot the stability situation when K is negative. In summary, when K is negative, there are either one or two unstable roots, and the system is always unstable.

Figure 6.105: Control system for Problem 71

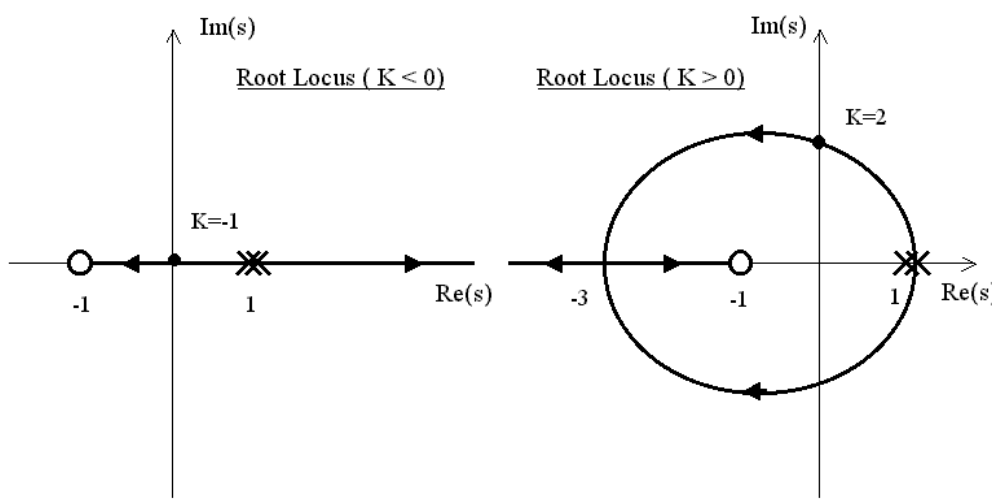


(d) The stability situation seen in the root locus plot agrees with that obtained from the inverse Nyquist plot.
They show :

$K > 2$	Stable
$-1 < K < 2$	Two unstable closed-loop roots
$K < -1$	One unstable closed-loop root

For the phase margin 45°,

$$\begin{aligned} \text{closed-loop roots} &= -0.932 \pm 1.999j \\ \zeta &= 0.423 \end{aligned}$$



71. Consider the system shown in Fig. 6.105(a).

(a) Construct a Bode plot for the system.

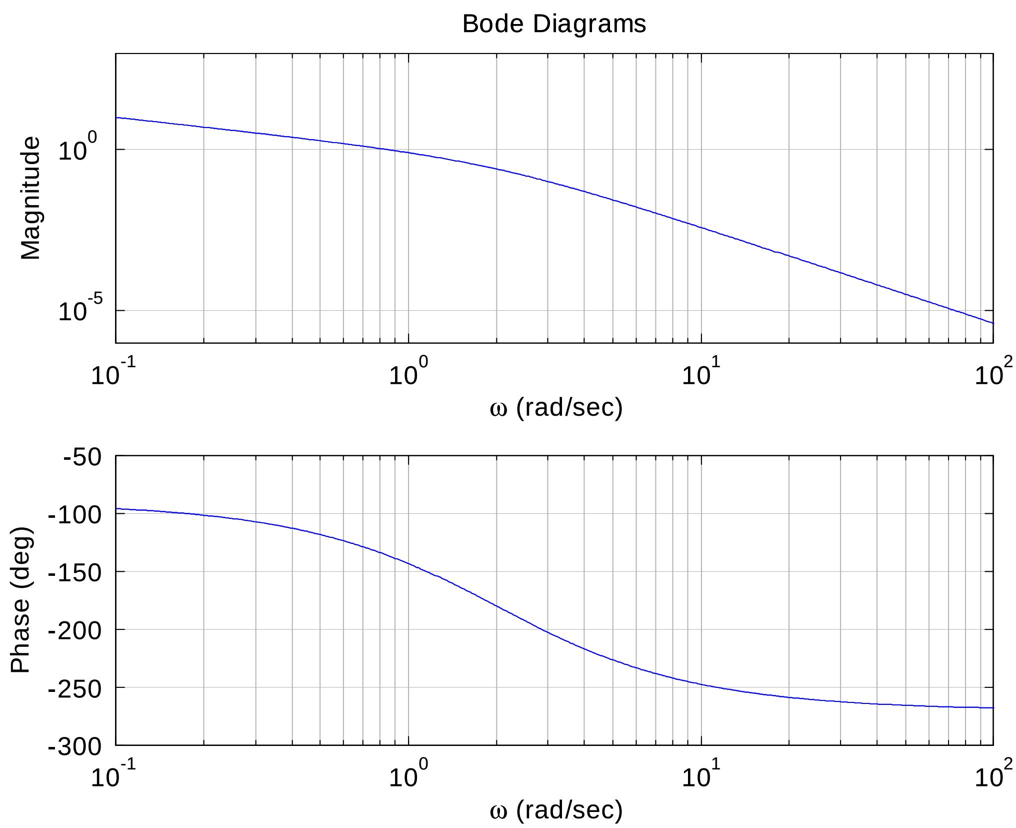
(b) Use your Bode plot to sketch an inverse Nyquist plot. (See Appendix W6.9.2)

(c) Consider closing a control loop around $G(s)$, as shown in Fig. 6.105(b). Using the inverse Nyquist plot as a guide, read from your Bode plot the values of GM and PM when $K = 0.7$, 1.0, 1.4, and 2. What value of K yields PM $= 30°$?

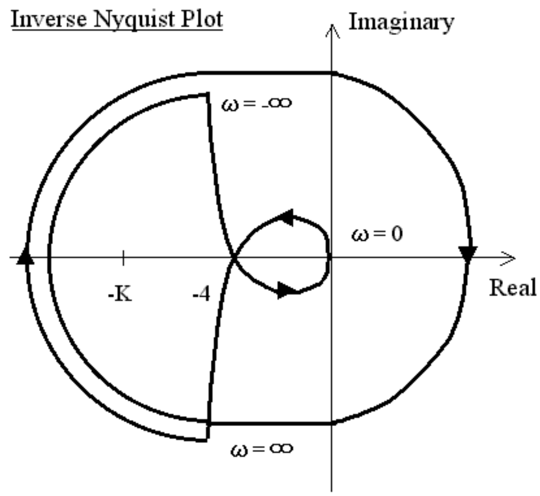
(d) Construct a root-locus plot, and label the same values of K on the locus. To what value of ζ does each pair of PM/GM values correspond? Compare the ζ vs PM with the rough approximation in Fig. 6.36

Solution :

(a) The figure follows, with K = 1. :



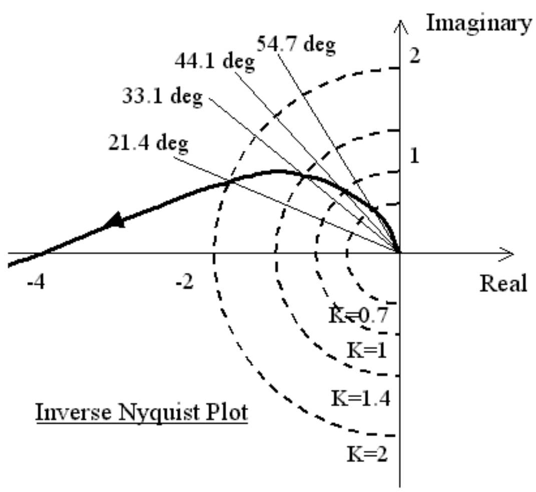
(b) The Inverse Nyquist is:

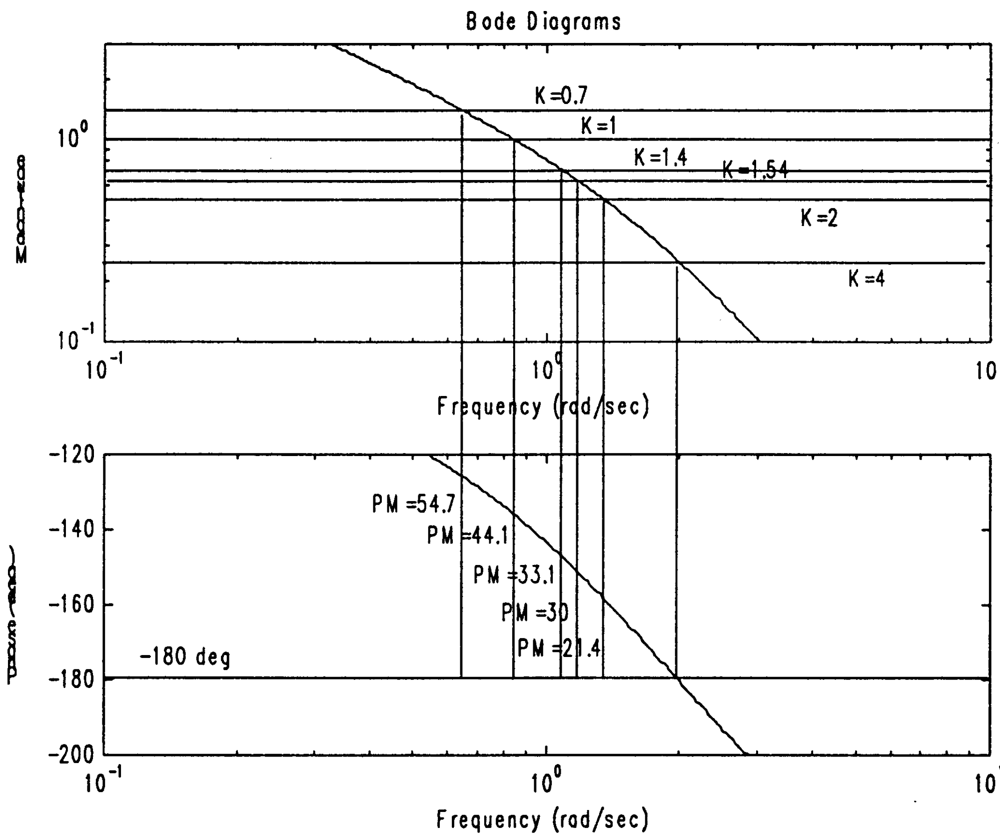


(c) With guidance from the Inverse Nyquist, we can use Matlab's `margin` command to find:ng

K	GM	PM
$K = 0.7$	$5.71\ (\omega = 2.00)$	$54.7°\ (\omega_c = 0.64)$
$K = 1$	$4.00\ (\omega = 2.00)$	$44.1°\ (\omega_c = 0.85)$
$K = 1.4$	$2.86\ (\omega = 2.00)$	$33.1°\ (\omega_c = 1.08)$
$K = 2$	$2.00\ (\omega = 2.00)$	$21.4°\ (\omega_c = 1.36)$
For $PM = 30°$ $K = 1.54$	$2.60\ (\omega = 2.00)$	$30.0°\ (\omega_c = 1.15)$

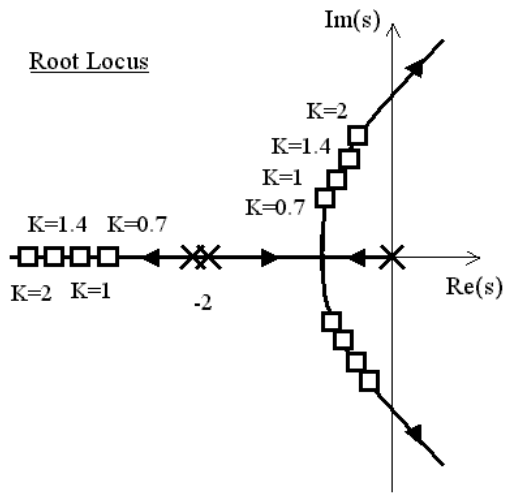
and then looking at the Bode, we can interpolate to fgure out what K would yield the desired 30 Deg PM, as seen below:




Bode Diagrams
K =0.7
K =1
K =1.4
K =1.54
K =2
K =4
Frequency (rad/sec)
PM =54.7
PM =44.1
PM =33.1
PM =30
PM =21.4
-180 deg
Frequency (rad/sec)

(d) Now we take a look at the RL, but expediting the detailed value of ζ by using Matlab's `damp` with the various values of K in the feeedback loop.

K	closed-loop roots	ζ
$K = 0.7$	-2.97 $-0.51 \pm 0.82j$	0.53
$K = 1$	-3.13 $-0.43 \pm 1.04j$	0.38
$K = 1.4$	-3.30 $-0.35 \pm 1.25j$	0.27
$K = 2$	-3.51 $-0.25 \pm 1.49j$	0.16



K	PM	ζ
$K = 0.7$	55	0.53
$K = 1$	44	0.38
$K = 1.4$	33	0.27
$K = 2$	21	0.16

The comparison is not as good as in Fig. 6.36, but it is not far off.