CHAPTER 1

1.1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_y + 8\mathbf{a}_z + 16\mathbf{a}_x + 14\mathbf{a}_y - 4\mathbf{a}_z = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26, 10, 4)}{|(26, 10, 4)|} = \underline{(0.92, 0.36, 0.14)}$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5, 0, 0) + (8, 7, -2) - (-30, 12, -24) = (43, -5, 22)$, and $|(43, -5, 22)| = \underline{48.6}$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$:

$|(-10, 4, -8)||(16, 14, -4)|(-2, 11, -10) = (13.4)(21.6)(-2, 11, -10)$
$= \underline{(-580.5, 3193, -2902)}$

1.2. Vector $\mathbf{A}$ extends from the origin to (1,2,3) and vector $\mathbf{B}$ from the origin to (2,3,-2).

a) Find the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$: First

$$\mathbf{A} - \mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z) - (2\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z) = (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)$$

whose magnitude is $|\mathbf{A} - \mathbf{B}| = [(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z) \cdot (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)]^{1/2} = \sqrt{1 + 1 + 25} = 3\sqrt{3} = 5.20$. The unit vector is therefore

$$\mathbf{a}_{AB} = \underline{(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)/5.20}$$

b) find the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of $\mathbf{A}$ and $\mathbf{B}$:

The midpoint is located at

$$P_{mp} = [1 + (2 - 1)/2,\ 2 + (3 - 2)/2,\ 3 + (-2 - 3)/2)] = (1.5, 2.5, 0.5)$$

The unit vector is then

$$\mathbf{a}_{mp} = \frac{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)}{\sqrt{(1.5)^2 + (2.5)^2 + (0.5)^2}} = \underline{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)/2.96}$$

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or
$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$
Expanding, obtain
$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$
or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64 - 176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(-\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y:

A unit vector tangent to this circle in the general increasing y direction is $\mathbf{t} = -\mathbf{a}_\phi$. Its x and y components are $\mathbf{t}_x = -\mathbf{a}_\phi \cdot \mathbf{a}_x = \sin\phi$, and $\mathbf{t}_y = -\mathbf{a}_\phi \cdot \mathbf{a}_y = -\cos\phi$. At the point $(-\sqrt{3}, 1)$, $\phi = 150°$, and so $\mathbf{t} = \sin 150° \mathbf{a}_x - \cos 150° \mathbf{a}_y = \underline{0.5(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)}$.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1, 2, -1) = \underline{(48, 36, 18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \underline{(-0.26, 0.39, 0.88)}$$

c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \underline{(0.59, 0.20, -0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2 + 2), 18z^2)|$, or $10 = |(4xy, 2x^2 + 4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.6. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of:

a) the dot product: First, $\mathbf{A} \cdot \mathbf{B} = 2 - 3 + 6 = 5 = AB\cos\theta$, where $A = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$, and where $B = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Therefore $\cos\theta = 5/14$, so that $\theta = \underline{69.1°}$.

b) the cross product: Begin with

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 1 & 3 \\ 1 & -3 & 2 \end{vmatrix} = 11\mathbf{a}_x - \mathbf{a}_y - 7\mathbf{a}_z$$

and then $|\mathbf{A} \times \mathbf{B}| = \sqrt{11^2 + 1^2 + 7^2} = \sqrt{171}$. So now, with $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta = \sqrt{171}$, find $\theta = \sin^{-1}\left(\sqrt{171}/14\right) = \underline{69.1°}$

1.7. Given the vector field $\mathbf{E} = 4zy^2\cos 2x\mathbf{a}_x + 2zy\sin 2x\mathbf{a}_y + y^2\sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy\sin 2x = 0$, the surfaces are 1) the plane $\underline{z = 0}$, with $|x| < 2$, $|y| < 2$; 2) the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$; 3) the plane $\underline{x = 0}$, with $|y| < 2$, $|z| < 2$; 4) the plane $\underline{x = \pi/2}$, with $|y| < 2$, $|z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy\sin 2x = y^2\sin 2x$, or on the plane $\underline{2z = y}$, with $|x| < 2$, $|y| < 2$, $|z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2\cos 2x = zy\sin 2x = y^2\sin 2x = 0$. This condition is met on the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$.

1.8. Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

We use the relation $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \mathbf{n}$. With the given vectors we find

$$\mathbf{A} \times \mathbf{B} = 14\mathbf{a}_y + 7\mathbf{a}_z = 7\sqrt{5}\underbrace{\left[\frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\right]}_{\pm\mathbf{n}} = \sqrt{9+4+16}\sqrt{4+1+4}\,\sin\theta\,\mathbf{n}$$

where $\mathbf{n}$ is identified as shown; we see that $\mathbf{n}$ can be positive or negative, as $\sin\theta$ can be positive or negative. This apparent sign ambiguity is not the real problem, however, as we really want the magnitude of the angle anyway. Choosing the positive sign, we are left with $\sin\theta = 7\sqrt{5}/(\sqrt{29}\sqrt{9}) = 0.969$. *Two* values of θ (75.7° and 104.3°) satisfy this equation, and hence the real ambiguity.

In using the dot product, we find $\mathbf{A} \cdot \mathbf{B} = 6 - 2 - 8 = -4 = |\mathbf{A}||\mathbf{B}| \cos\theta = 3\sqrt{29}\cos\theta$, or $\cos\theta = -4/(3\sqrt{29}) = -0.248 \Rightarrow \theta = -75.7°$. Again, the minus sign is not important, as we care only about the angle magnitude. The main point is that *only one* θ value results when using the dot product, so no ambiguity.

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2 + y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53°}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2 + y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2 + 49} \times 7\,dz dx = \int_0^4 \frac{350}{x^2 + 49} dx$$

$$= 350 \times \frac{1}{7}\left[\tan^{-1}\left(\frac{4}{7}\right) - 0\right] = \underline{26}$$

1.10. By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners, and passes through the center of the cube:

Assuming a side length, b, two diagonal vectors would be $\mathbf{A} = b(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = b(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$. Now use $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, or $b^2(1 - 1 + 1) = (\sqrt{3}b)(\sqrt{3}b)\cos\theta \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \underline{70.53°}$. This result (in magnitude) is the same for *any* two diagonal vectors.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

The two points can be written as vectors from the origin:

$$\mathbf{A}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \quad \text{and} \quad \mathbf{A}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

The desired vector will now be the difference:

$$\mathbf{A}_{12} = \mathbf{A}_2 - \mathbf{A}_1 = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

whose magnitude is

$$|\mathbf{A}_{12}| = \sqrt{\mathbf{A}_{12} \cdot \mathbf{A}_{12}} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. Given that $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, where the three vectors represent line segments and extend from a common origin,

a) must the three vectors be coplanar?

In terms of the components, the vector sum will be

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (A_x + B_x + C_x)\mathbf{a}_x + (A_y + B_y + C_y)\mathbf{a}_y + (A_z + B_z + C_z)\mathbf{a}_z$$

which we require to be zero. Suppose the coordinate system is configured so that vectors **A** and **B** lie in the x-y plane; in this case $A_z = B_z = 0$. Then C_z has to be zero in order for the three vectors to sum to zero. Therefore, the three vectors must be coplanar.

b) If $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$, are the four vectors coplanar?

The vector sum is now

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = (A_x + B_x + C_x + D_x)\mathbf{a}_x + (A_y + B_y + C_y + D_y)\mathbf{a}_y + (A_z + B_z + C_z + D_z)\mathbf{a}_z$$

Now, for example, if **A** and **B** lie in the x-y plane, **C** and **D** need not, as long as $C_z + D_z = 0$. So the four vectors need not be coplanar to have a zero sum.

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{(5, 25, 55)}{60.6} = \underline{(0.08, 0.41, 0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$: $\mathbf{r}_1 - \mathbf{r}_2 = (9, -4, 1)$ and $\mathbf{r}_2 - \mathbf{r}_3 = (-2, 5, -6)$. So $\mathbf{r}_1 - \mathbf{r}_2 \times \mathbf{r}_2 - \mathbf{r}_3 = (19, 52, 37)$. Then

$$\mathbf{a}_p = \frac{(19, 52, 37)}{|(19, 52, 37)|} = \frac{(19, 52, 37)}{66.6} = \underline{(0.29, 0.78, 0.56)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area} = \frac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| = \underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area} = \frac{1}{2}|(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_2 - \mathbf{r}_3)| = \frac{1}{2}|(-9, 4, -1) \times (-2, 5, -6)| = \underline{33.3}$$

1.16. If $\mathbf{A}$ represents a vector one unit in length directed due east, $\mathbf{B}$ represents a vector three units in length directed due north, and $\mathbf{A}+\mathbf{B} = 2\mathbf{C} - \mathbf{D}$ and $2\mathbf{A} - \mathbf{B} = \mathbf{C} + 2\mathbf{D}$, determine the length and direction of $\mathbf{C}$. (difficulty 1)

Take north as the positive y direction, and then east as the positive x direction. Then we may write

$$\mathbf{A}+\mathbf{B} = \mathbf{a}_x + 3\mathbf{a}_y = 2\mathbf{C} - \mathbf{D}$$

and

$$2\mathbf{A}-\mathbf{B} = 2\mathbf{a}_x - 3\mathbf{a}_y = \mathbf{C} + 2\mathbf{D}$$

Multiplying the first equation by 2, and then adding the result to the second equation eliminates $\mathbf{D}$, and we get

$$4\mathbf{a}_x + 3\mathbf{a}_y = 5\mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

The length of $\mathbf{C}$ is $|\mathbf{C}| = \left[(4/5)^2 + (3/5)^2\right]^{1/2} = \underline{1}$

$\mathbf{C}$ lies in the x-y plane at angle from due north (the y axis) given by $\alpha = \tan^{-1}(4/3) = 53.1°$ (or $36.9°$ from the x axis). For those having nautical leanings, this is very close to the compass point NE$\frac{3}{4}$E (not required).

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2}[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)] = (0.095, 0.516, 0.207)$$

Finally,

$$\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{|(0.095, 0.516, 0.207)|} = \underline{(0.168, 0.915, 0.367)}$$

1.18. A certain vector field is given as $\mathbf{G} = (y+1)\mathbf{a}_x + x\mathbf{a}_y$. a) Determine $\mathbf{G}$ at the point (3,-2,4): $\mathbf{G}(3,-2,4) = \underline{-\mathbf{a}_x + 3\mathbf{a}_y}$.

b) obtain a unit vector defining the direction of $\mathbf{G}$ at (3,-2,4). $|\mathbf{G}(3,-2,4)| = [1+3^2]^{1/2} = \sqrt{10}$. So the unit vector is

$$\mathbf{a}_G(3,-2,4) = \frac{-\mathbf{a}_x + 3\mathbf{a}_y}{\sqrt{10}}$$

1.19. a) Express the field $\mathbf{D} = (x^2+y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables: Have $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $x^2 + y^2 = \rho^2$. Therefore

$$\mathbf{D} = \frac{1}{\rho}(\cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y)$$

Then

$$D_\rho = \mathbf{D}\cdot\mathbf{a}_\rho = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\rho) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\rho)\right] = \frac{1}{\rho}\left[\cos^2\phi + \sin^2\phi\right] = \frac{1}{\rho}$$

and

$$D_\phi = \mathbf{D}\cdot\mathbf{a}_\phi = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)\right] = \frac{1}{\rho}\left[\cos\phi(-\sin\phi) + \sin\phi\cos\phi\right] = 0$$

Therefore

$$\underline{\mathbf{D} = \frac{1}{\rho}\mathbf{a}_\rho}$$

b) Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, $\underline{\mathbf{D} = 0.5\mathbf{a}_\rho}$. To express this in cartesian, we use

$$\mathbf{D} = 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_x)\mathbf{a}_x + 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_y)\mathbf{a}_y = 0.5\cos 36^\circ\mathbf{a}_x + 0.5\sin 36^\circ\mathbf{a}_y = \underline{0.41\mathbf{a}_x + 0.29\mathbf{a}_y}$$

1.20. If the three sides of a triangle are represented by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, all directed counter-clockwise, show that $|\mathbf{C}|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ and expand the product to obtain the law of cosines.

With the vectors drawn as described above, we find that $\mathbf{C} = -(\mathbf{A}+\mathbf{B})$ and so $|\mathbf{C}|^2 = C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ So far so good. Now if we expand the product, obtain

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = A^2 + B^2 + 2\mathbf{A}\cdot\mathbf{B}$$

where $\mathbf{A}\cdot\mathbf{B} = AB\cos(180^\circ - \alpha) = -AB\cos\alpha$ where α is the interior angle at the junction of $\mathbf{A}$ and $\mathbf{B}$. Using this, we have $C^2 = A^2 + B^2 - 2AB\cos\alpha$, which is the law of cosines.

1.21. Express in cylindrical components:

a) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$:
$C(3, 2, -7) \rightarrow C(\rho = 3.61, \phi = 33.7°, z = -7)$ and
$D(-1, -4, 2) \rightarrow D(\rho = 4.12, \phi = -104.0°, z = 2)$.
Now $\mathbf{R}_{CD} = (-4, -6, 9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4, 6, -9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1, -4, 2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1, 4, -2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction.

a) Using spherical components, write an expression for the velocity field, $\mathbf{v}$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perperdicular distance from the rotation axis. With clockwise rotation, we obtain

$$\mathbf{v}(r, \theta) = \underline{\Omega r \sin\theta \, \mathbf{a}_\phi \ (r < a)}$$

b) Convert to rectangular components:

From here, the problem is the same as part c in Problem 1.20, except the rotation direction is reversed. The answer is $\mathbf{v}(x, y) = \underline{\Omega\,[-y\,\mathbf{a}_x + x\,\mathbf{a}_y]}$, where $(x^2 + y^2 + z^2)^{1/2} < a$.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2 \int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \; + \int_3^{4.5} \int_{100°}^{130°} 3 \, d\phi \, dz$$
$$+ \int_3^{4.5} \int_{100°}^{130°} 5 \, d\phi \, dz \; + \; 2 \int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

1.23c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \; + \; 4 \times 2 \; + \; 2 \times \left[\frac{30^\circ}{360^\circ} \times 2\pi \times 3 \; + \; \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100^\circ$, $z = 3$) and B($\rho = 5$, $\phi = 130^\circ$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2 with the x axis respectively.

a) Express each vector in rectangular components; Have $\mathbf{a}_1 = A_{x1}\mathbf{a}_x + A_{y1}\mathbf{a}_y$, so that $A_{x1} = \mathbf{a}_1 \cdot \mathbf{a}_x = \cos\phi_1$. Then, $A_{y1} = \mathbf{a}_1 \cdot \mathbf{a}_y = \cos(90 - \phi_1) = \sin\phi_1$. Therefore,

$$\mathbf{a}_1 = \cos\phi_1 \, \mathbf{a}_x + \sin\phi_1 \, \mathbf{a}_y \quad \text{and similarly,} \quad \mathbf{a}_2 = \cos\phi_2 \, \mathbf{a}_x + \sin\phi_2 \, \mathbf{a}_y$$

b) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2$: From the definition of the dot product,

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{a}_2 &= (1)(1)\cos(\phi_1 - \phi_2) \\ &= (\cos\phi_1 \, \mathbf{a}_x + \sin\phi_1 \, \mathbf{a}_y) \cdot (\cos\phi_2 \, \mathbf{a}_x + \sin\phi_2 \, \mathbf{a}_y) = \cos\phi_1 \, \cos\phi_2 + \sin\phi_1 \, \sin\phi_2 \end{aligned}$$

c) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2 \cos\phi_1 - \cos\phi_2 \sin\phi_1$: From the definition of the cross product, and since $\mathbf{a}_1$ and $\mathbf{a}_2$ both lie in the x-y plane,

$$\begin{aligned} \mathbf{a}_1 \times \mathbf{a}_2 = (1)(1)\sin(\phi_1 - \phi_2) \, \mathbf{a}_z &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \cos\phi_1 & \sin\phi_1 & 0 \\ \cos\phi_2 & \sin\phi_2 & 0 \end{vmatrix} \\ &= [\sin\phi_2 \cos\phi_1 - \cos\phi_2 \sin\phi_1] \; \mathbf{a}_z \end{aligned}$$

thus verified.

1.25. Given point $P(r = 0.8, \, \theta = 30^\circ, \, \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2} \left(\cos\phi \, \mathbf{a}_r + \frac{\sin\phi}{\sin\theta} \, \mathbf{a}_\phi \right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.
b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.
c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. Express the uniform vector field, $\mathbf{F} = 5\,\mathbf{a}_x$ in

a) cylindrical components: $F_\rho = 5\,\mathbf{a}_x \cdot \mathbf{a}_\rho = 5\cos\phi$, and $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(\rho,\phi) = 5(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)}$.

b) spherical components: $F_r = 5\,\mathbf{a}_x \cdot \mathbf{a}_r = 5\sin\theta\cos\phi$; $F_\theta = 5\,\mathbf{a}_x \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi$; $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(r,\theta,\phi) = 5\left[\sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi\right]}$.

1.27. The surfaces $r = 2$ and 4, $\theta = 30°$ and $50°$, and $\phi = 20°$ and $60°$ identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20°}^{60°}\int_{30°}^{50°}\int_{2}^{4} r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20°}^{60°}\int_{30°}^{50°}(4^2+2^2)\sin\theta\,d\theta\,d\phi + \int_{2}^{4}\int_{20°}^{60°} r(\sin 30° + \sin 50°)\,dr\,d\phi$$
$$+\,2\int_{30°}^{50°}\int_{2}^{4} r\,dr\,d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_{2}^{4} dr \;+\; 2\int_{30°}^{50°}(4+2)\,d\theta + \int_{20°}^{60°}(4\sin 50° + 4\sin 30° + 2\sin 50° + 2\sin 30°)\,d\phi$$
$$= \underline{17.49}$$

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50°, \phi = 20°)$ to $B(r = 4, \theta = 30°, \phi = 60°)$ or

$$A(x = 2\sin 50°\cos 20°, y = 2\sin 50°\sin 20°, z = 2\cos 50°)$$

to

$$B(x = 4\sin 30°\cos 60°, y = 4\sin 30°\sin 60°, z = 4\cos 30°)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

1.28. State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when

a) $\mathbf{A}\cdot\mathbf{a}_x = \mathbf{B}\cdot\mathbf{a}_x$: For this to be true, both $\mathbf{A}$ and $\mathbf{B}$ must be oriented at the same angle, θ, from the x axis. But this would allow either vector to lie anywhere along a conical surface of angle θ about the x axis. Therefore, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

b) $\mathbf{A}\times\mathbf{a}_x = \mathbf{B}\times\mathbf{a}_x$: This is a more restrictive condition because the cross product gives a vector. For both cross products to lie in the same direction, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{a}_x$ must be coplanar. But if $\mathbf{A}$ lies at angle θ to the x axis, $\mathbf{B}$ could lie at θ *or* at $180° - \theta$ to give the same cross product. So again, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

1.28c) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$ *and* $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$: In this case, we need to satisfy both requirements in parts *a* and *b* – that is, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{a}_x$ must be coplanar, *and* $\mathbf{A}$ and $\mathbf{B}$ must lie at the same angle, θ, to $\mathbf{a}_x$. With coplanar vectors, this latter condition might imply that both $+\theta$ and $-\theta$ would therefore work. But the negative angle reverses the direction of the cross product direction. Therefore both vectors must lie in the same plane *and* lie at the same angle to x; i.e., $\mathbf{A}$ *must* be equal to $\mathbf{B}$.

d) $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$ *and* $\mathbf{A} \times \mathbf{C} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is any vector except $\mathbf{C} = 0$: This is just the general case of part *c*. Since we can orient our coordinate system in any manner we choose, we can arrange it so that the x axis coincides with the direction of vector $\mathbf{C}$. Thus all the arguments of part *c* apply, and again we conclude that $\mathbf{A}$ *must* be equal to $\mathbf{B}$.

1.29. Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

a) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad: Use

$$\mathbf{a}_x = (\mathbf{a}_x \cdot \mathbf{a}_r)\mathbf{a}_r + (\mathbf{a}_x \cdot \mathbf{a}_\theta)\mathbf{a}_\theta + (\mathbf{a}_x \cdot \mathbf{a}_\phi)\mathbf{a}_\phi =$$
$$\sin(1)\cos(0.8)\mathbf{a}_r + \cos(1)\cos(0.8)\mathbf{a}_\theta + (-\sin(0.8))\mathbf{a}_\phi = \underline{0.59\mathbf{a}_r + 0.38\mathbf{a}_\theta - 0.72\mathbf{a}_\phi}$$

b) $x = 3$, $y = 2$, $z = -1$: First, transform the point to spherical coordinates. Have $r = \sqrt{14}$, $\theta = \cos^{-1}(-1/\sqrt{14}) = 105.5^\circ$, and $\phi = \tan^{-1}(2/3) = 33.7^\circ$. Then

$$\mathbf{a}_x = \sin(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_r + \cos(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_\theta + (-\sin(33.7^\circ))\mathbf{a}_\phi$$
$$= \underline{0.80\mathbf{a}_r - 0.22\mathbf{a}_\theta - 0.55\mathbf{a}_\phi}$$

c) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$: Again, convert the point to spherical coordinates. $r = \sqrt{\rho^2 + z^2} = \sqrt{8.5}$, $\theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59.0^\circ$, and $\phi = 0.7\,\text{rad} = 40.1^\circ$. Now

$$\mathbf{a}_x = \sin(59^\circ)\cos(40.1^\circ)\mathbf{a}_r + \cos(59^\circ)\cos(40.1^\circ)\mathbf{a}_\theta + (-\sin(40.1^\circ))\mathbf{a}_\phi$$
$$= \underline{0.66\mathbf{a}_r + 0.39\mathbf{a}_\theta - 0.64\mathbf{a}_\phi}$$

1.30. Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x, y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

a) Determine the location and magnitude of the maximum tailwind encountered: Tailwind would be x-directed, and so we look at the x component only. Over the flight range, this function maximizes at a value of $0.86/(1 + 0.5y^2)$ at $x = 10$ (at the end of the trip). It reaches a local minimum of $0.50/(1 + 0.5y^2)$ at $x = 4$, and has another local maximum of $0.66/(1 + 0.5y^2)$ at the trip start, $x = 0$.

b) Repeat for headwind: The x component is always positive, and so therefore no headwind exists over the travel range.

c) Repeat for crosswind: Crosswind will be found from the y component, which is seen to maximize over the flight range at a value of $0.4/(1 + 0.5y^2)$ at the trip start ($x = 0$).

d) Would more favorable tailwinds be available at some other latitude? If so, where? Minimizing the denominator accomplishes this; in particular, the lattitude associated with $y = 0$ gives the strongest tailwind.