CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1–12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$ so

$$y_1'' = -4\cos 2x = -4y_1 \quad \text{and} \quad y_2'' = -4\sin 2x = -4y_2.$$

Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$ so

$$y_1'' = 9e^{3x} = 9y_1 \quad \text{and} \quad y_2'' = 9e^{-3x} = 9y_2.$$

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$ so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, and $y_2' = \cos x + 2\sin 2x$, $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 \;=\; (-\cos x + 4\cos 2x) + (\cos x - \cos 2x) \;=\; 3\cos 2x$$

and

$$y_2'' + y_2 \;=\; (-\sin x + 4\cos 2x) + (\sin x - \cos 2x) \;=\; 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$ then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x\,y' + 4y \;=\; x^2\left(6x^{-4}\right) + 5x\left(-2x^{-3}\right) + 4\left(x^{-2}\right) \;=\; 0.$$

If $y = y_2 = x^{-2}\ln x$ then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x\,y' + 4y \;&=\; x^2\left(-5x^{-4} + 6x^{-4}\ln x\right) + 5x\left(x^{-3} - 2x^{-3}\ln x\right) + 4\left(x^{-2}\ln x\right) \\ &=\; \left(-5x^{-2} + 5x^{-2}\right) + \left(6x^{-2} - 10x^{-2} + 4x^{-2}\right)\ln x \;=\; 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3r\,e^{rx} \;=\; 2\,e^{rx}$ that simplifies to $3r = 2$. Thus $r = 2/3$.

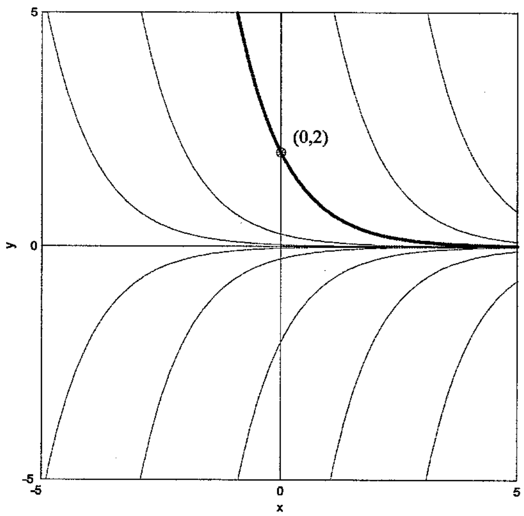
14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} \;=\; e^{rx}$ that simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y \;=\; 0$ gives the equation $r^2 e^{rx} + r\,e^{rx} - 2\,e^{rx} = 0$ that simplifies to $r^2 + r - 2 \;=\; (r+2)(r-1) \;=\; 0$. Thus $r = -2$ or $r = 1$.

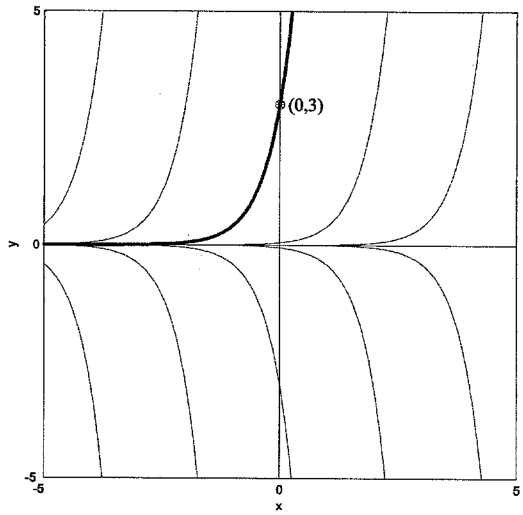
16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y \;=\; 0$ gives the equation $3r^2 e^{rx} + 3r\,e^{rx} - 4\,e^{rx} = 0$ that simplifies to $3r^2 + 3r - 4 \;=\; 0$. The quadratic formula then gives the solutions $r \;=\; \left(-3 \pm \sqrt{57}\right)/6$.

The verifications of the suggested solutions in Problems 17–26 are similar to those in Problems 1–12. We illustrate the determination of the value of C only in some typical cases. However, we illustrate typical solution curves for each of these problems.

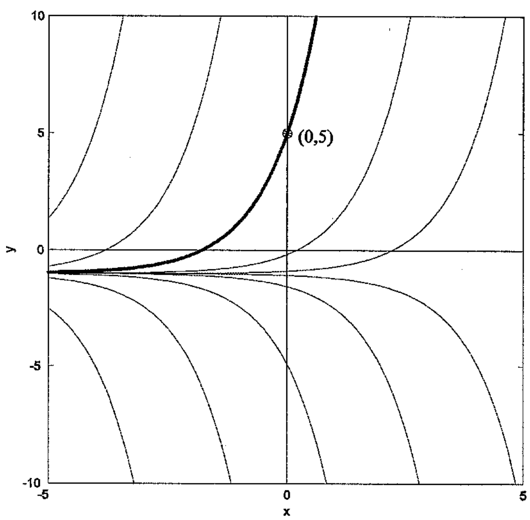
17. $C = 2$

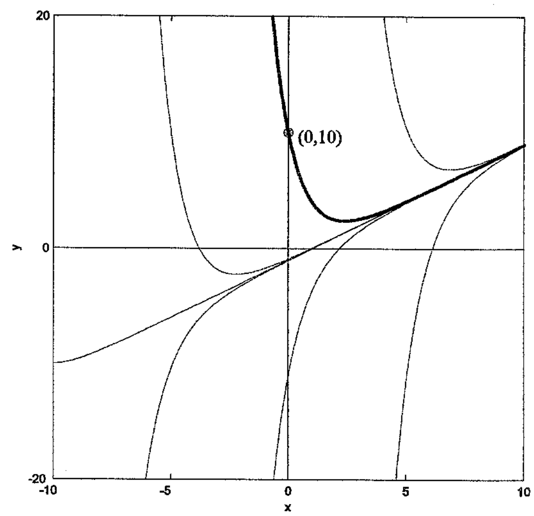


18. $C = 3$



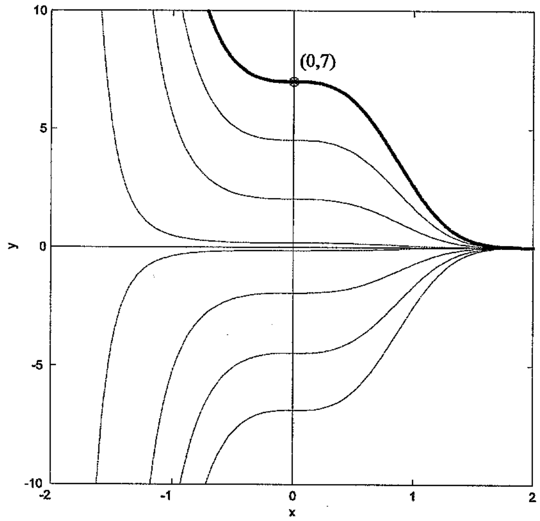
19. If $y(x) = Ce^{x} - 1$ then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$. The figure is on the left below.

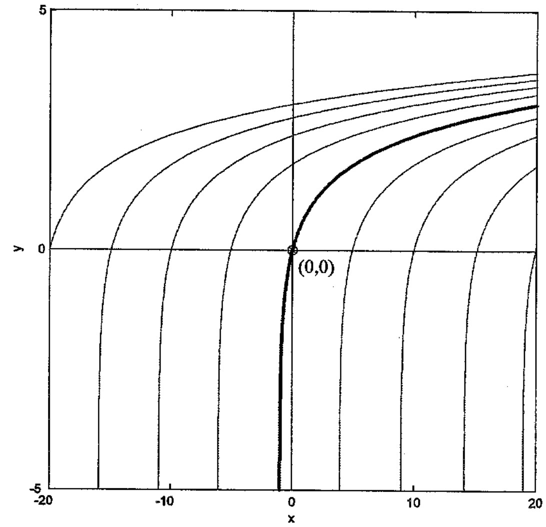




20. If $y(x) = Ce^{-x} + x - 1$ then $y(0) = 10$ gives $C - 1 = 10$, so $C = 11$. The figure is on the right above.

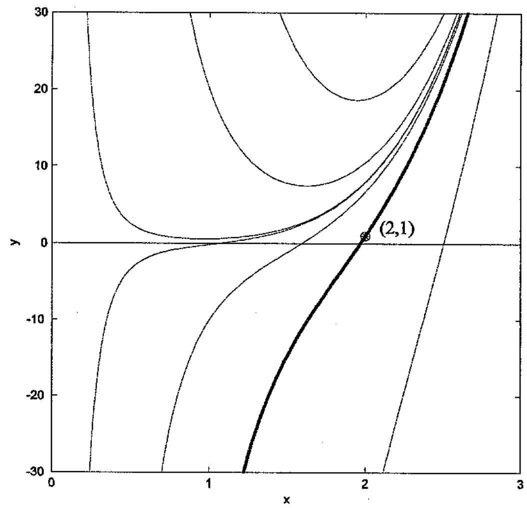
21. $C = 7$. The figure is on the left at the top of the next page.

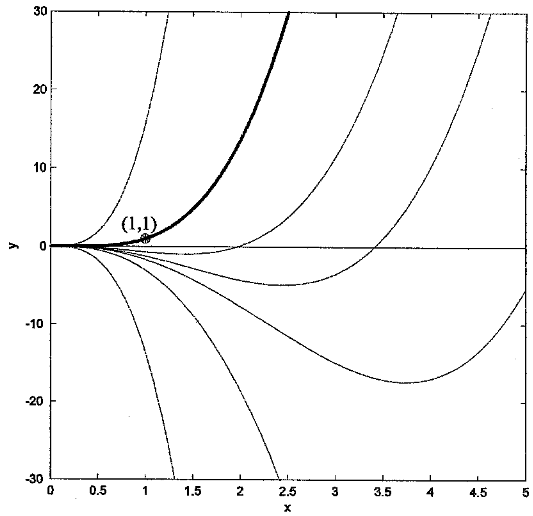




22. If $y(x) = \ln(x+C)$ then $y(0)=0$ gives $\ln C = 0$, so $C = 1$. The figure is on the right above.

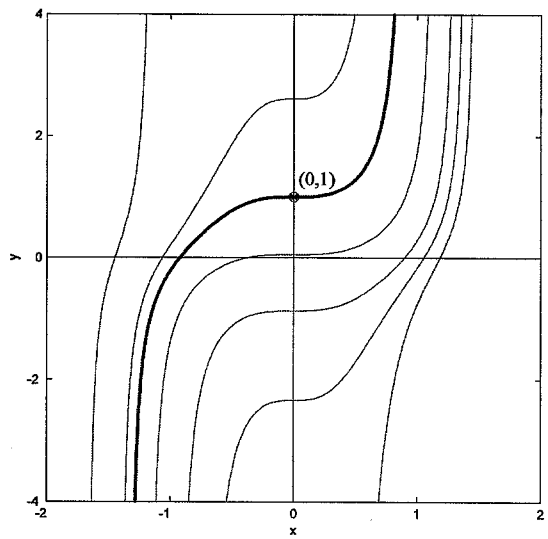
23. If $y(x) = \frac{1}{4}x^5 + Cx^{-2}$ then $y(2)=1$ gives the equation $\frac{1}{4}\cdot 32 + C\cdot\frac{1}{8} = 1$ with solution $C = -56$. The figure is on the left below.



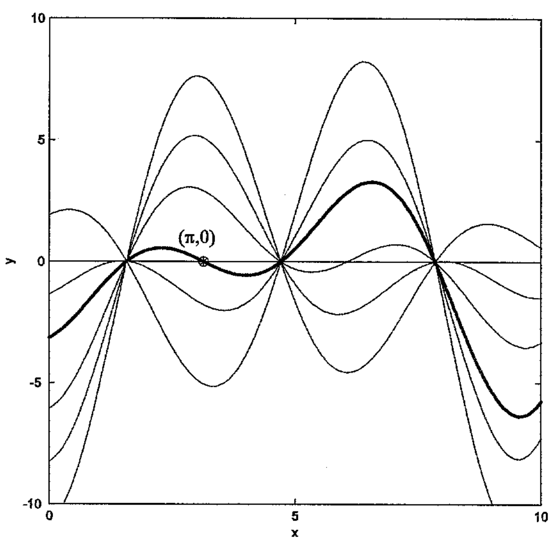


24. $C = 17$. The figure is on the right above.

25. If $y(x) = \tan(x^2 + C)$ then $y(0) = 1$ gives the equation $\tan C = 1$. Hence one value of C is $C = \pi/4$ (as is this value plus any integral multiple of π).



26. Substitution of $x = \pi$ and $y = 0$ into $y = (x + C)\cos x$ yields the equation $0 = (\pi + C)(-1)$, so $C = -\pi$.



27. $y' = x + y$

28. The slope of the line through (x, y) and $(x/2, 0)$ is $y' = (y-0)/(x-x/2) = 2y/x$, so the differential equation is $x y' = 2y$.

29. If $m = y'$ is the slope of the tangent line and m' is the slope of the normal line at (x, y), then the relation $m m' = -1$ yields $m' = 1/y' = (y-1)/(x-0)$. Solution for y' then gives the differential equation $(1-y)y' = x$.

30. Here $m = y'$ and $m' = D_x(x^2 + k) = 2x$, so the orthogonality relation $m m' = -1$ gives the differential equation $2x y' = -1$.

31. The slope of the line through (x, y) and $(-y, x)$ is $y' = (x-y)/(-y-x)$, so the differential equation is $(x+y)y' = y-x$.

In Problems 32–36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative, the word "is" with the = sign, the phrase "proportional to" with k, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt = k\sqrt{P}$

33. $dv/dt = k v^2$

34. $dv/dt = k(250 - v)$

35. $dN/dt = k(P - N)$

36. $dN/dt = k N(P - N)$

37. The second derivative of any linear function is zero, so we spot the two solutions $y(x) \equiv 1$ or $y(x) = x$ of the differential equation $y'' = 0$.

38. A function whose derivative equals itself, and hence a solution of the differential equation $y' = y$ is $y(x) = e^x$.

39. We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2. The choice $k = 1$ balances the equation, and provides the solution $y(x) = x^2$.

40. If y is a constant, then $y' \equiv 0$ so the differential equation reduces to $y^2 = 1$. This gives the two constant-valued solutions $y(x) \equiv 1$ and $y(x) \equiv -1$.

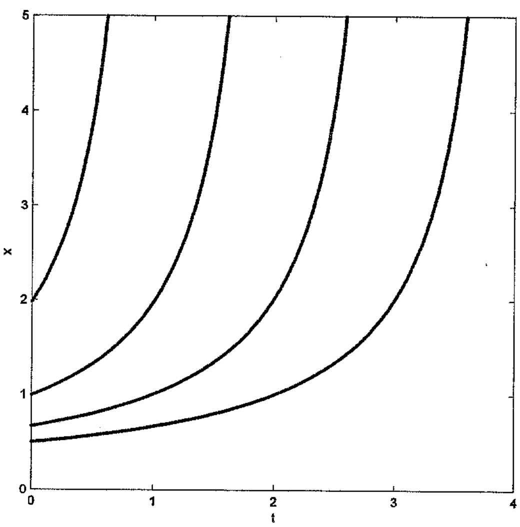
41. We reason that if $y = ke^x$, then each term in the differential equation is a multiple of e^x. The choice $k = \frac{1}{2}$ balances the equation, and provides the solution $y(x) = \frac{1}{2}e^x$.

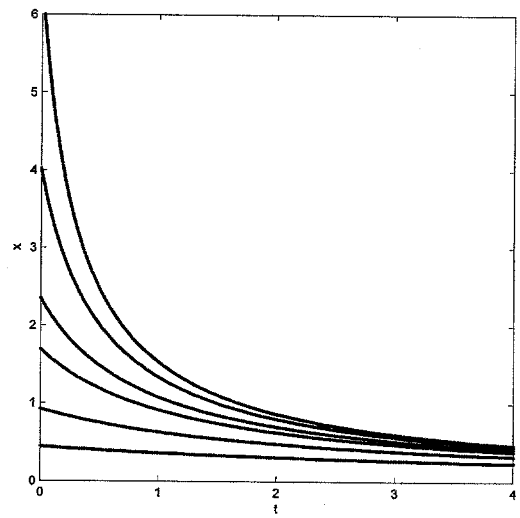
42. Two functions, each equaling the negative of its own second derivative, are the two solutions $y(x) = \cos x$ and $y(x) = \sin x$ of the differential equation $y'' = -y$.

43. **(a)** We need only substitute $x(t) = 1/(C - kt)$ in both sides of the differential equation $x' = kx^2$ for a routine verification.

(b) The zero-valued function $x(t) \equiv 0$ obviously satisfies the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** The figure on the left below shows typical graphs of solutions of the differential equation $x' = \frac{1}{2}x^2$.





(b) The figure on the right above shows typical graphs of solutions of the differential equation $x' = -\frac{1}{2}x^2$. We see that — whereas the graphs with $k = \frac{1}{2}$ appear to "diverge to infinity" — each solution with $k = -\frac{1}{2}$ appears to approach 0 as $t \to \infty$. Indeed, we see from the Problem 43(a) solution $x(t) = 1/(C - \frac{1}{2}t)$ that $x(t) \to \infty$ as $t \to 2C$. However, with $k = -\frac{1}{2}$ it is clear from the resulting solution $x(t) = 1/(C + \frac{1}{2}t)$ that $x(t)$ remains bounded on any finite interval, but $x(t) \to 0$ as $t \to +\infty$.

45. Substitution of $P' = 1$ and $P = 10$ into the differential equation $P' = kP^2$ gives $k = \frac{1}{100}$, so Problem 43(a) yields a solution of the form $P(t) = 1/(C - t/100)$. The initial condition $P(0) = 2$ now yields $C = \frac{1}{2}$, so we get the solution

$$P(t) = \frac{1}{\dfrac{1}{2} - \dfrac{t}{100}} = \frac{100}{50-t}.$$

We now find readily that $P = 100$ when $t = 49$, and that $P = 1000$ when $t = 49.9$. It appears that P grows without bound (and thus "explodes") as t approaches 50.

46. Substitution of $v' = -1$ and $v = 5$ into the differential equation $v' = kv^2$ gives $k = -\frac{1}{25}$, so Problem 43(a) yields a solution of the form $v(t) = 1/(C + t/25)$. The initial condition $v(0) = 10$ now yields $C = \frac{1}{10}$, so we get the solution

$$v(t) = \frac{1}{\dfrac{1}{10} + \dfrac{t}{25}} = \frac{50}{5+2t}.$$

We now find readily that $v = 1$ when $t = 22.5$, and that $v = 0.1$ when $t = 247.5$. It appears that v approaches 0 as t increases without bound. Thus the boat gradually slows, but never comes to a "full stop" in a finite period of time.

47. **(a)** $y(10) = 10$ yields $10 = 1/(C-10)$, so $C = 101/10$.

(b) There is no such value of C, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually (in Fig. 1.1.8 of the text) that one and only one solution curve passes through each point (a, b) of the *xy*-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

48. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $xy' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \leq 0$ and $x \geq 0$, respectively).

(c) If $a \geq 0$ (for instance), choose C_+ fixed so that $C_+ a^4 = b$. Then the function

$$y(x) = \begin{cases} C_- x^4 & \text{if } x \leq 0, \\ C_+ x^4 & \text{if } x \geq 0 \end{cases}$$

satisfies the given differential equation for every real number value of C_-.

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x+1$ yields $y(x) = \int (2x+1)\,dx = x^2 + x + C$. Then substitution of $x = 0$, $y = 3$ gives $3 = 0 + 0 + C = C$, so $y(x) = x^2 + x + 3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \tfrac{1}{3}(x-2)^3 + C$. Then substitution of $x = 2$, $y = 1$ gives $1 = 0 + C = C$, so $y(x) = \tfrac{1}{3}(x-2)^3 + 1$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \tfrac{2}{3}x^{3/2} + C$. Then substitution of $x = 4$, $y = 0$ gives $0 = \tfrac{16}{3} + C$, so $y(x) = \tfrac{2}{3}(x^{3/2} - 8)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}\,dx = -1/x + C$. Then substitution of $x = 1$, $y = 5$ gives $5 = -1 + C$, so $y(x) = -1/x + 6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2} + C$. Then substitution of $x = 2$, $y = -1$ gives $-1 = 2 \cdot 2 + C$, so $y(x) = 2\sqrt{x+2} - 5$.

6. Integration of $y' = x(x^2+9)^{1/2}$ yields $y(x) = \int x(x^2+9)^{1/2}\,dx = \tfrac{1}{3}(x^2+9)^{3/2} + C$. Then substitution of $x = -4$, $y = 0$ gives $0 = \tfrac{1}{3}(5)^3 + C$, so
$y(x) = \tfrac{1}{3}\left[(x^2+9)^{3/2} - 125\right]$.

7. Integration of $y' = 10/(x^2+1)$ yields $y(x) = \int 10/(x^2+1)\,dx = 10\tan^{-1}x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 10 \cdot 0 + C$, so $y(x) = 10\tan^{-1}x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \tfrac{1}{2}\sin 2x + C$. Then substitution of $x = 0$, $y = 1$ gives $1 = 0 + C$, so $y(x) = \tfrac{1}{2}\sin 2x + 1$.

9. Integration of $y' = 1/\sqrt{1-x^2}$ yields $y(x) = \int 1/\sqrt{1-x^2}\,dx = \sin^{-1}x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 0 + C$, so $y(x) = \sin^{-1}x$.

10. Integration of $y' = xe^{-x}$ yields

$$y(x) = \int xe^{-x}\,dx = \int ue^{u}\,du = (u-1)e^{u} = -(x+1)e^{-x} + C$$

(when we substitute $u = -x$ and apply Formula #46 inside the back cover of the textbook). Then substitution of $x = 0,\ y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2.$

11. If $a(t) = 50$ then $v(t) = \int 50\,dt = 50t + v_0 = 50t + 10.$ Hence

$$x(t) = \int (50t + 10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$ then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15.$ Hence

$$x(t) = \int (-20t - 15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$ then $v(t) = \int 3t\,dt = \tfrac{3}{2}t^2 + v_0 = \tfrac{3}{2}t^2 + 5.$ Hence

$$x(t) = \int (\tfrac{3}{2}t^2 + 5)\,dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t + 1$ then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7.$ Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 4(t+3)^2$. then $v(t) = \int 4(t+3)^2\,dt = \tfrac{4}{3}(t+3)^3 + C = \tfrac{4}{3}(t+3)^3 - 37$ (taking $C = -37$ so that $v(0) = -1$). Hence

$$x(t) = \int \left[\tfrac{4}{3}(t+3)^3 - 37\right]dt = \tfrac{1}{3}(t+3)^4 - 37t + C = \tfrac{1}{3}(t+3)^4 - 37t - 26.$$

16. If $a(t) = 1/\sqrt{t+4}$ then $v(t) = \int 1/\sqrt{t+4}\,dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int (2\sqrt{t+4} - 5)\,dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

17. If $a(t) = (t+1)^{-3}$ then $v(t) = \int (t+1)^{-3}\,dt = -\frac{1}{2}(t+1)^{-2} + C = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$ (taking $C = \frac{1}{2}$ so that $v(0) = 0$). Hence

$$x(t) = \int \left[-\tfrac{1}{2}(t+1)^{-2} + \tfrac{1}{2}\right] dt = \tfrac{1}{2}(t+1)^{-1} + \tfrac{1}{2}t + C = \tfrac{1}{2}\left[(t+1)^{-1} + t - 1\right]$$

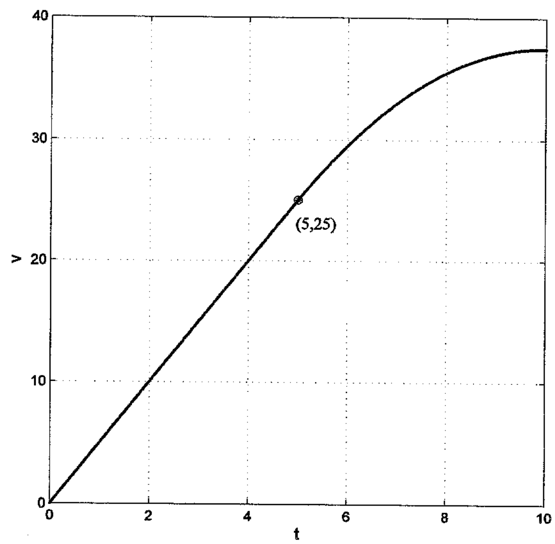
(taking $C = -\frac{1}{2}$ so that $x(0) = 0$).

18. If $a(t) = 50\sin 5t$ then $v(t) = \int 50\sin 5t\,dt = -10\cos 5t + C = -10\cos 5t$ (taking $C = 0$ so that $v(0) = -10$). Hence

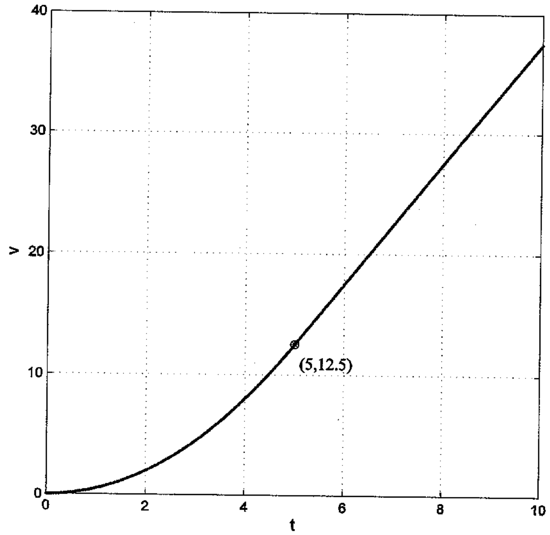
$$x(t) = \int (-10\cos 5t)\,dt = -2\sin 5t + C = -2\sin 5t + 10$$

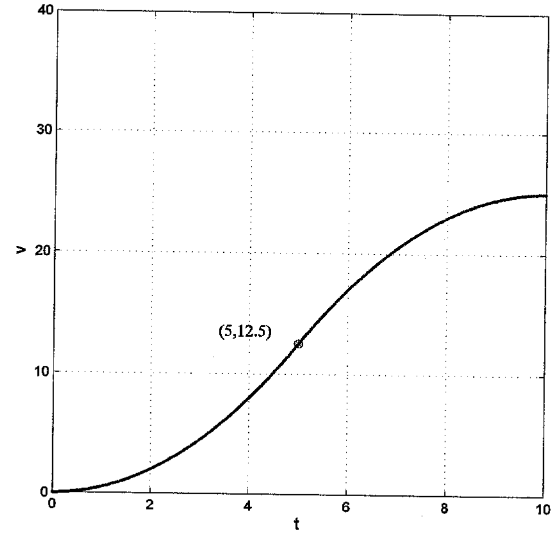
(taking $C = -10$ so that $x(0) = 8$).

19. Note that $v(t) = 5$ for $0 \le t \le 5$ and that $v(t) = 10 - t$ for $5 \le t \le 10$. Hence $x(t) = 5t + C_1$ for $0 \le t \le 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ for $5 \le t \le 10$. Now $C_1 = 0$ because $x(0) = 0$, and continuity of $x(t)$ requires that $x(t) = 5t$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -\frac{25}{2}$, and we get the following graph.



20. Note that $v(t)=t$ for $0\le t\le 5$ and that $v(t)=5$ for $5\le t\le 10$. Hence $x(t)=\frac{1}{2}t^2+C_1$ for $0\le t\le 5$ and $x(t)=5t+C_2$ for $5\le t\le 10$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=5t+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, and we get the graph on the left below.



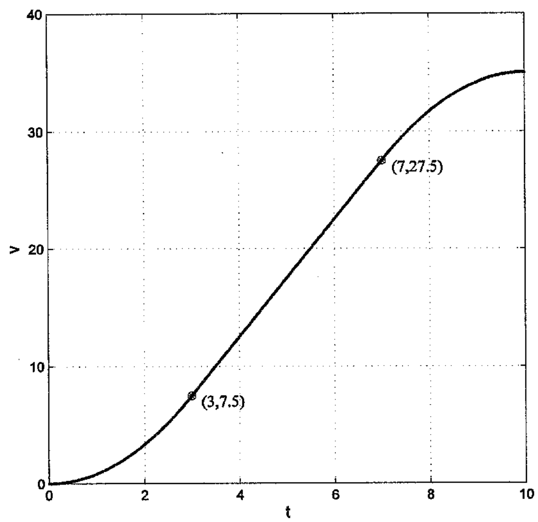


21. Note that $v(t)=t$ for $0\le t\le 5$ and that $v(t)=10-t$ for $5\le t\le 10$. Hence $x(t)=\frac{1}{2}t^2+C_1$ for $0\le t\le 5$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ for $5\le t\le 10$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ agree when $t=5$. This implies that $C_2=-25$, and we get the graph on the right above.

22. For $0\le t\le 3$: $v(t)=\frac{5}{3}t$ so $x(t)=\frac{5}{6}t^2+C_1$. Now $C_1=0$ because $x(0)=0$, so $x(t)=\frac{5}{6}t^2$ on this first interval, and its right endpoint value is $x(3)=7\frac{1}{2}$.

For $3\le t\le 7$: $v(t)=5$ so $x(t)=5t+C_2$. Now $x(3)=7\frac{1}{2}$ implies that $C_2=-7\frac{1}{2}$, so $x(t)=5t-7\frac{1}{2}$ on this second interval, where its right endpoint value is $x(7)=27\frac{1}{2}$.

For $7\le t\le 10$: $v-5=-\frac{5}{3}(t-7)$, so $v(t)=-\frac{5}{3}t+\frac{50}{3}$. Hence $x(t)=-\frac{5}{6}t^2+\frac{50}{3}t+C_3$, and $x(7)=27\frac{1}{2}$ implies that $C_3=-\frac{290}{6}$. Finally, $x(t)=\frac{1}{6}(-5t^2+100t-290)$ on this third interval, and we get the graph at the top of the next page.



23. $v = -9.8t + 49$, so the ball reaches its maximum height ($v = 0$) after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

24. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground ($y = 0$) when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

25. $a = -10\text{ m/s}^2$ and $v_0 = 100\text{ km/h} \approx 27.78\text{ m/s}$, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, $t \approx 2.78$, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

26. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20,\ 20.61$. Hence the projectile is in the air 20.61 seconds.

27. $a = -9.8$ m/s2 so $v = -9.8\,t - 10$ and

$$y = -4.9\,t^2 - 10\,t + y_0.$$

The ball hits the ground when $y = 0$ and

$$v = -9.8\,t - 10 = -60,$$

so $t \approx 5.10$ s. Hence

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57\text{ m}.$$

28. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground ($y = 0$) when $t \approx 4.77$ sec, with velocity $v = v(4.77) \approx -192.64$ ft/sec, an impact speed of about 131 mph.

29. Integration of $dv/dt = 0.12\,t^3 + 0.6\,t$, $v(0) = 0$ gives $v(t) = 0.3\,t^2 + 0.04\,t^3$. Hence $v(10) = 70$. Then integration of $dx/dt = 0.3\,t^2 + 0.04\,t^3$, $x(0) = 0$ gives $x(t) = 0.1\,t^3 + 0.04\,t^4$, so $x(10) = 200$. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/sec.

30. Taking $x_0 = 0$ and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$v = -at + 88,$$

and $v = 0$ yields $t = 88/a$. Substituting this value of t and $x = 176$ in

$$x = -at^2/2 + 88t,$$

we solve for $a = 22$ ft/sec^2. Hence the car skids for $t = 88/22 = 4$ sec.

31. If $a = -20$ m/sec^2 and $x_0 = 0$ then the car's velocity and position at time t are given by

$$v = -20t + v_0, \quad x = -10\,t^2 + v_0 t.$$

It stops when $v = 0$ (so $v_0 = 20t$), and hence when

$$x = 75 = -10\,t^2 + (20t)t = 10\,t^2.$$

Thus $t = \sqrt{7.5}$ sec so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/sec} \approx 197 \text{ km/hr}.$$

32. Starting with $x_0 = 0$ and $v_0 = 50$ km/h $= 5\times10^4$ m/h, we find by the method of Problem 30 that the car's deceleration is $a = (25/3)\times10^7$ m/h^2. Then, starting with $x_0 = 0$ and $v_0 = 100$ km/h $= 10^5$ m/h, we substitute $t = v_0/a$ into

$$x = -\tfrac{1}{2}at^2 + v_0 t$$

and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

33. If $v_0 = 0$ and $y_0 = 20$ then

$$v = -at \text{ and } y = -\tfrac{1}{2}at^2 + 20.$$

Substitution of $t = 2$, $y = 0$ yields $a = 10$ ft/sec^2. If $v_0 = 0$ and

$y_0 = 200$ then

$$v = -10t \text{ and } y = -5t^2 + 200.$$

Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}$ sec and $v = -20\sqrt{10} \approx -63.25$ ft/sec.

34. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of t and $y = 144$ in

$$y = -16t^2 + v_0 t,$$

we solve for $v_0 = 96$ ft/sec as the initial speed with which the person can throw a ball straight upward.

On Planet Gzyx: From Problem 27, the surface gravitational acceleration on planet Gzyx is $a = 10$ ft/sec^2, so

$$v = -10t + 96 \quad \text{and} \quad y = -5t^2 + 96t.$$

Therefore $v = 0$ yields $t = 9.6$ sec, and thence $y_{max} = y(9.6) = 460.8$ ft is the height a ball will reach if its initial velocity is 96 ft/sec.

35. If $v_0 = 0$ and $y_0 = h$ then the stone's velocity and height are given by

$$v = -gt, \quad y = -0.5\,gt^2 + h.$$

Hence $y = 0$ when $t = \sqrt{2h/g}$ so

$$v = -g\sqrt{2h/g} = -\sqrt{2gh}.$$

36. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12$ ft/sec to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

37. We use units of miles and hours. If $x_0 = v_0 = 0$ then the car's velocity and position after t hours are given by

$$v = at, \quad x = \tfrac{1}{2}t^2.$$

Since $v = 60$ when $t = 5/6$, the velocity equation yields $a = 72$ mi/hr^2. Hence the distance traveled by 12:50 pm is

$$x = (0.5)(72)(5/6)^2 = 25 \text{ miles.}$$

38. Again we have

$$v = at, \qquad x = \tfrac{1}{2}t^2.$$

But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields

$$35 = (0.5)(60/\mathrm{t})(t^2) = 30t,$$

whence $t = 7/6$ hr, that is, 1:10 p.m.

39. Integration of $y' = (9/v_S)(1-4x^2)$ yields

$$y = (3/v_S)(3x - 4x^3) + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer's trajectory is

$$y(x) = (3/v_S)(3x - 4x^3 + 1).$$

Substitution of $y(1/2) = 1$ now gives $v_S = 6$ mph.

40. Integration of $y' = 3(1-16x^4)$ yields

$$y = 3x - (48/5)x^5 + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer's trajectory is

$$y(x) = (1/5)(15x - 48x^5 + 6),$$

so his downstream drift is $y(1/2) = 2.4$ miles.

41. The bomb equations are $a = -32$, $v = -32$, and $s_B = s = -16t^2 + 800$, with $t = 0$ at the instant the bomb is dropped. The projectile is fired at time $t = 2$, so its corresponding equations are $a = -32$, $v = -32(t-2) + v_0$, and

$$s_P = s = -16(t-2)^2 + v_0(t-2)$$

for $t \geq 2$ (the arbitrary constant vanishing because $s_P(2) = 0$). Now the condition $s_B(t) = -16t^2 + 800 = 400$ gives $t = 5$, and then the requirement that $s_P(5) = 400$ also yields $v_0 = 544/3 \approx 181.33$ ft/s for the projectile's needed initial velocity.

42. Let $x(t)$ be the (positive) altitude (in miles) of the spacecraft at time t (hours), with $t = 0$ corresponding to the time at which the its retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time t. Then $v_0 = -1000$ and $x_0 = x(0)$ is unknown. But the (constant) acceleration is $a = +20000$, so

$$v(t) = 20000t - 1000 \quad \text{and} \quad x(t) = 10000t^2 - 1000t + x_0.$$

Now $v(t) = 20000t - 1000 = 0$ (soft touchdown) when $t = \frac{1}{20}$ hr (that is, after exactly 3 minutes of descent. Finally, the condition

$$0 = x(\tfrac{1}{20}) = 10000(\tfrac{1}{20})^2 - 1000(\tfrac{1}{20}) + x_0$$

yields $x_0 = 25$ miles for the altitude at which the retrorockets should be fired.

43. The velocity and position functions for the spacecraft are $v_S(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_P(t) = \frac{1}{10}c = 3\times10^7$ and $x_P(t) = 3\times10^7 t$. The condition that $x_S = x_P$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3\times10^7 t$, whence

$$\begin{aligned} t &= \frac{3\times10^7}{0.0049} \approx 6.12245\times10^9 \text{ sec} \\ &\approx \frac{6.12245\times10^9}{(3600)(24)(365.25)} \approx 194 \text{ years.} \end{aligned}$$

Since the projectile is traveling at $\frac{1}{10}$ the speed of light, it has then traveled a distance of about 19.4 light years, which is about 1.8367×10^{17} meters.

44. Let $a > 0$ denote the constant deceleration of the car when braking, and take $x_0 = 0$ for the cars position at time $t = 0$ when the brakes are applied. In the police experiment with $v_0 = 25$ ft/s, the distance the car travels in t seconds is given by

$$x(t) = -\frac{1}{2}at^2 + \frac{88}{60}\cdot 25t$$

(with the factor $\frac{88}{60}$ used to convert the velocity units from mi/hr to ft/s). When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$ we find that $a = \frac{1210}{81} \approx 14.94$ ft/s^2. With this value of the deceleration and the (as yet) unknown velocity v_0 of the car involved in the accident, its position function is

$$x(t) = -\frac{1}{2}\cdot\frac{1210}{81}t^2 + v_0 t.$$

The simultaneous equations $x(t) = 210$ and $x'(t) = 0$ finally yield $v_0 = \frac{110}{9}\sqrt{42} \approx 79.21$ ft/s, almost exactly 54 miles per hour.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

The instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. It may help some students to simplify the statement of the existence-uniqueness theorem as follows:

Suppose that the function $f(x,y)$ and the partial derivative $\partial f/\partial y$ are both continuous in some neighborhood of the point (a, b). Then the initial value problem

$$\frac{dy}{dx} = f(x,y), \qquad y(a) = b$$

has a unique solution in some neighborhood of the point a.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Instead, we provide some details of the construction of the figure for the Problem 1 answer, and then include without further comment the similarly constructed figures for Problems 2 through 9.

1. The following sequence of *Mathematica* commands generates the slope field and the solution curves through the given points. Begin with the differential equation $dy/dx = f(x,y)$ where

```
f[x_, y_] := -y - Sin[x]
```

Then set up the viewing window

```
a = -3; b = 3; c = -3; d = 3;
```

The components (u,v) of unit vectors corresponding to the short slope field line segments are given by

```
u[x_, y_] := 1/Sqrt[1 + f[x, y]^2]
v[x_, y_] := f[x, y]/Sqrt[1 + f[x, y]^2]
```

The slope field is then constructed by the commands

```
Needs["Graphics`PlotField`"]
dfield = PlotVectorField[{u[x, y], v[x, y]}, {x, a, b}, {y, c, d},
         HeadWidth -> 0, HeadLength -> 0, PlotPoints -> 19,
         PlotRange -> {{a, b}, {c, d}}, Axes -> True, Frame -> True,
         FrameLabel -> {"x", "y"}, AspectRatio -> 1];
```

The original curve shown in Fig. 1.3.12 of the text (and its initial point not shown there) are plotted by the commands

```
x0 = -1.9; y0 = 0;
point0 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
               y[x], {x, a, b}];
soln[[1,1,2]];
curve0 = Plot[soln[[1,1,2]], {x, a, b},
           PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

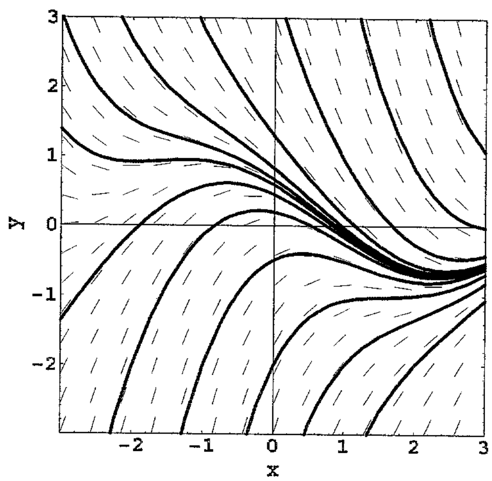
The *Mathematica* **NDSolve** command carries out an approximate numerical solution of the given differential equation. Numerical solution techniques are discussed in Sections 2.4–2.6 of the textbook.

The coordinates of the 12 points are marked in Fig. 1.3.12 in the textbook. For instance the 7th point is (–2.5, 1). It and the corresponding solution curve are plotted by the commands

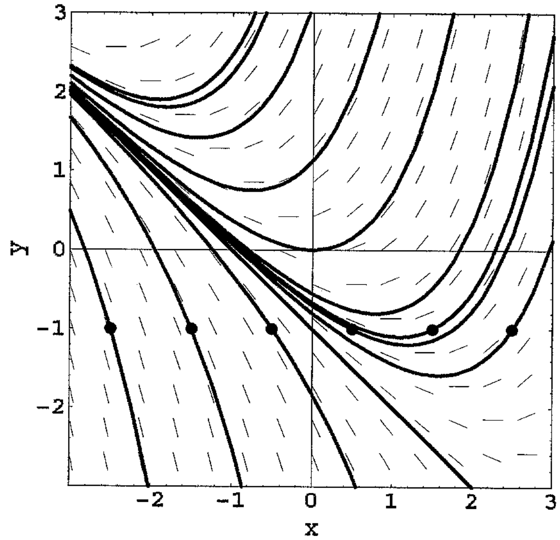
```
x0 = -2.5; y0 = 1;
point7 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
               y[x], {x, a, b}];
soln[[1,1,2]];
curve7 = Plot[soln[[1,1,2]], {x, a, b},
           PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

Finally, the desired figure is assembled by the *Mathematica* command

```
Show[ dfield, point0,curve0,
      point1,curve1, point2,curve2, point3,curve3,
      point4,curve4, point5,curve5, point6,curve6,
      point7,curve7, point8,curve8, point9,curve9,
      point10,curve10, point11,curve11, point12,curve12];
```



2.



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y 0
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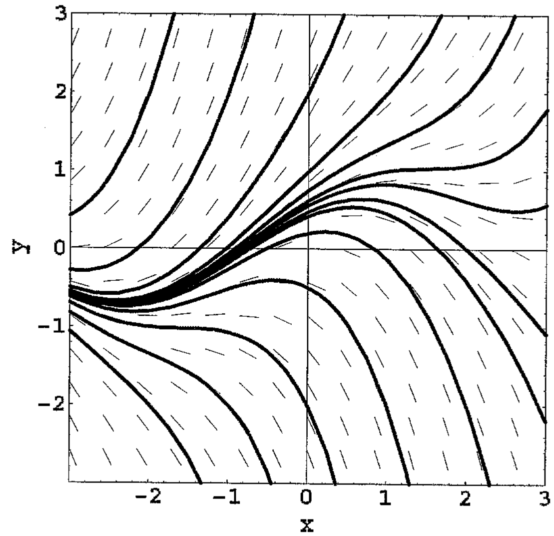
x

3.



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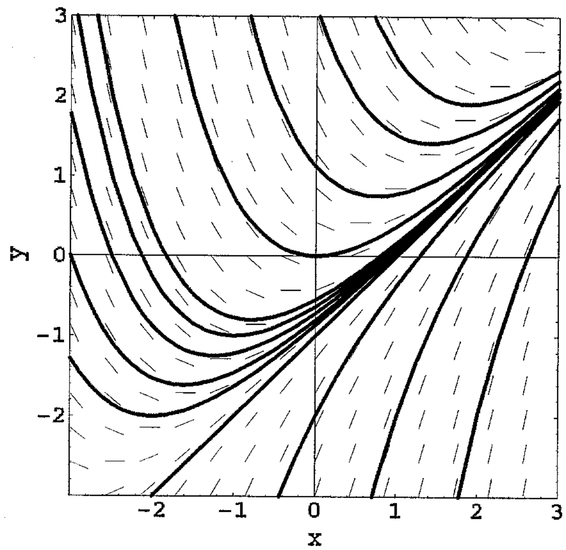
x

4.



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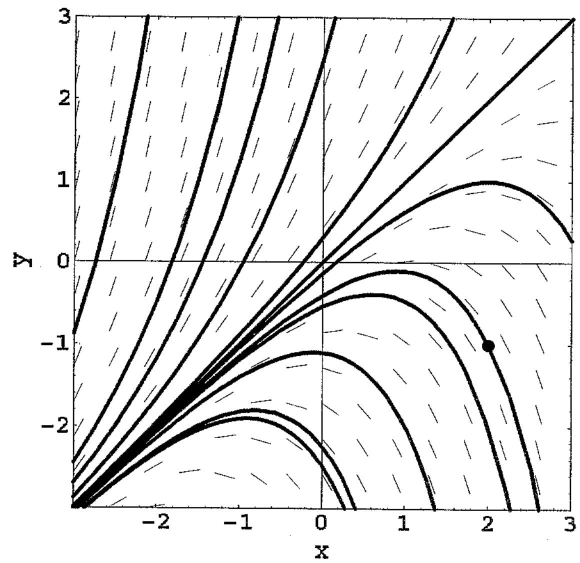
x

5.



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-1
0
1
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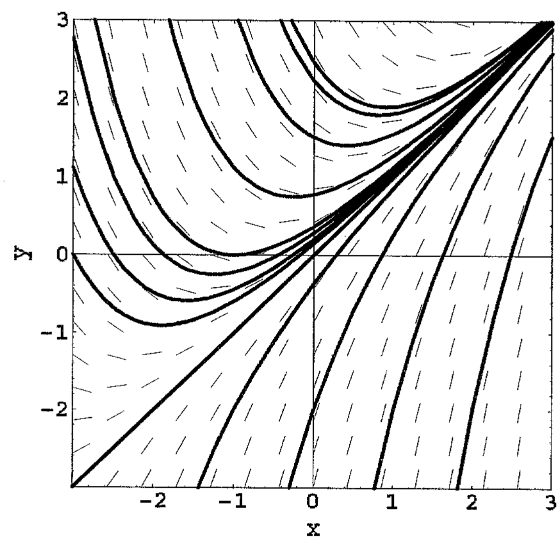
3
x

6.



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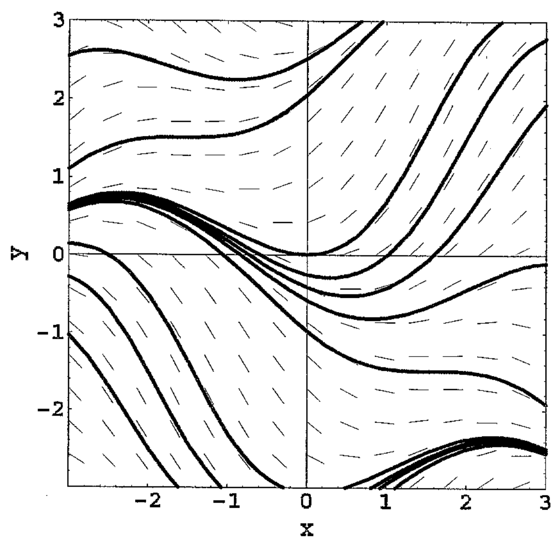
-2
-2
-1
0
1
2
3
x

7.



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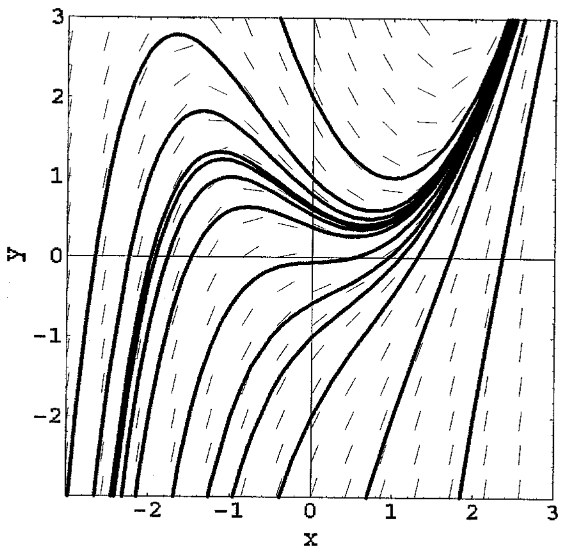
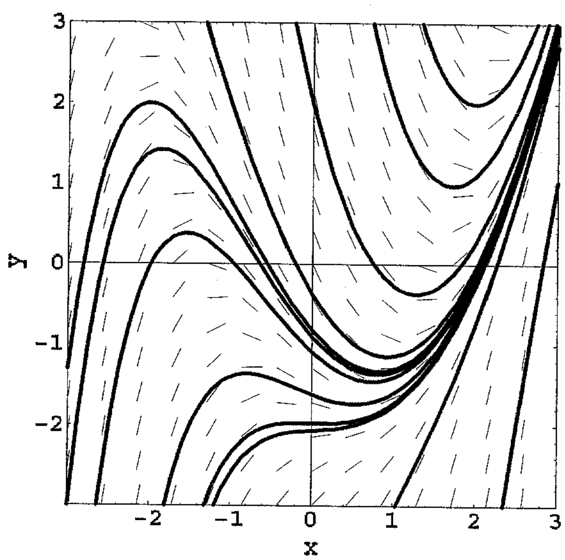
y 0
-1
-2
-2
-1
0
1
2
3
x

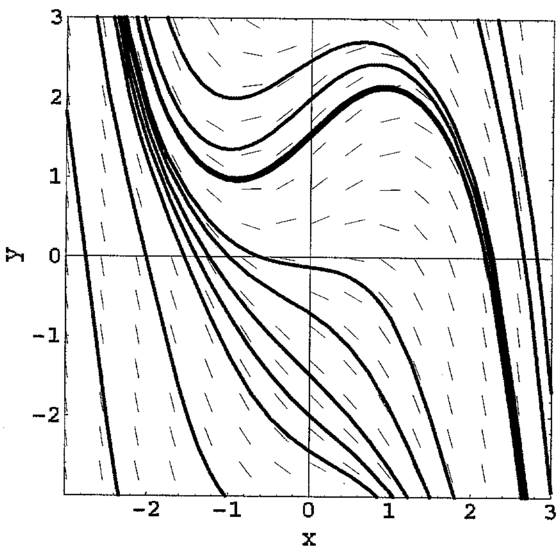
8.



9.



10.



11. Because both $f(x,y) = 2x^2y^2$ and $\partial f/\partial y = 4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x = 1$.

12. Both $f(x,y) = x \ln y$ and $\partial f/\partial y = x/y$ are continuous in a neighborhood of $(1, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

13. Both $f(x,y) = y^{1/3}$ and $\partial f/\partial y = (1/3)y^{-2/3}$ are continuous near $(0, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

14. $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0, 0)$, but $\partial f/\partial y = (1/3)y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x = 0$.

15. $f(x,y) = (x - y)^{1/2}$ is not continuous at $(2, 2)$ because it is not even defined if $y > x$. Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point $x = 2$.

16. $f(x,y) = (x - y)^{1/2}$ and $\partial f/\partial y = -(1/2)(x - y)^{-1/2}$ are continuous in a neighborhood of $(2, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 2$.

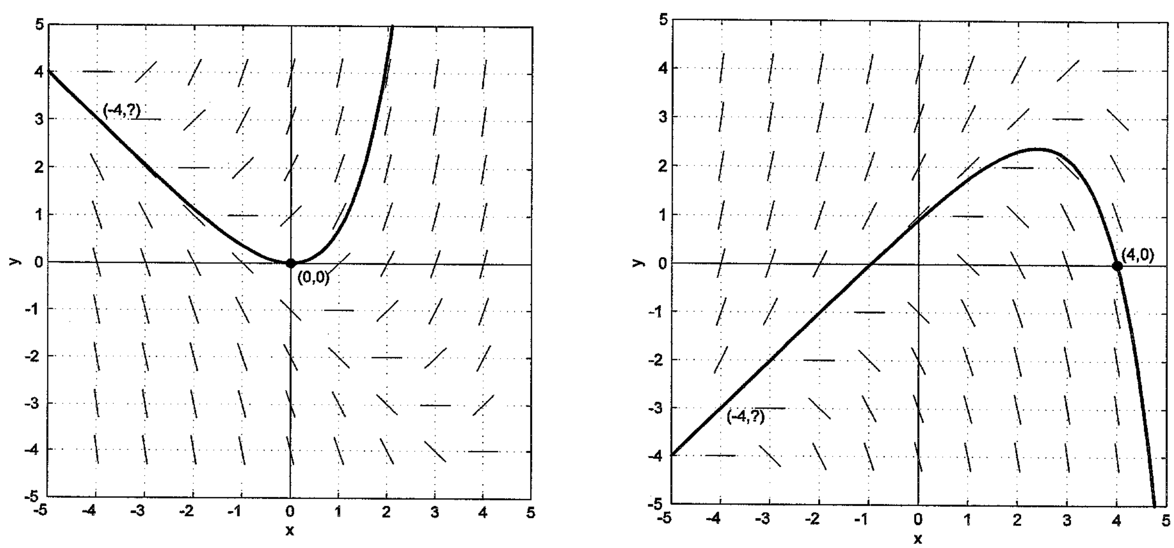
17. Both $f(x,y) = (x-1)/y$ and $\partial f/\partial y = -(x-1)/y^2$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

18. Neither $f(x,y) = (x-1)/y$ nor $\partial f/\partial y = -(x-1)/y^2$ is continuous near $(1, 0)$, so the existence-uniqueness theorem guarantees nothing.

19. Both $f(x,y) = \ln(1 + y^2)$ and $\partial f/\partial y = 2y/(1 + y^2)$ are continuous near $(0, 0)$, so the theorem guarantees the existence of a unique solution near $x = 0$.

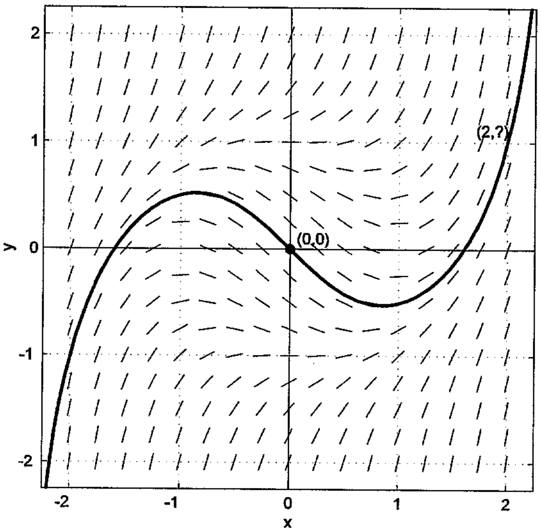
20. Both $f(x,y) = x^2 - y^2$ and $\partial f/\partial y = -2y$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

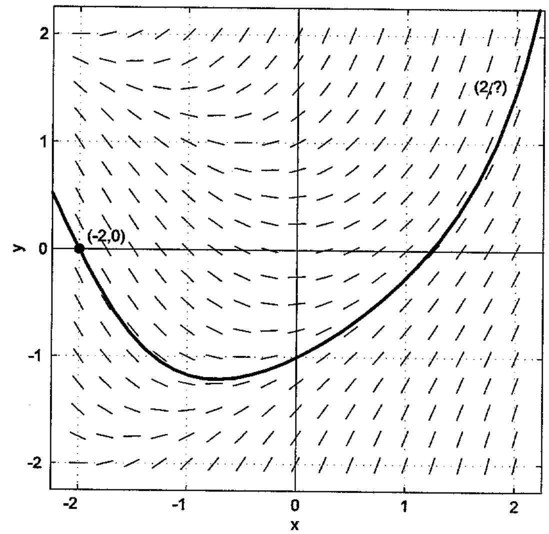
21. The curve in the figure on the left below can be constructed using the commands illustrated in Problem 1 above. Tracing this solution curve, we see that $y(-4) \approx 3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx 3.0183$.



22. Tracing the curve in the figure on the right at the bottom of the preceding page, we see that $y(-4) \approx -3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx -3.0017$.

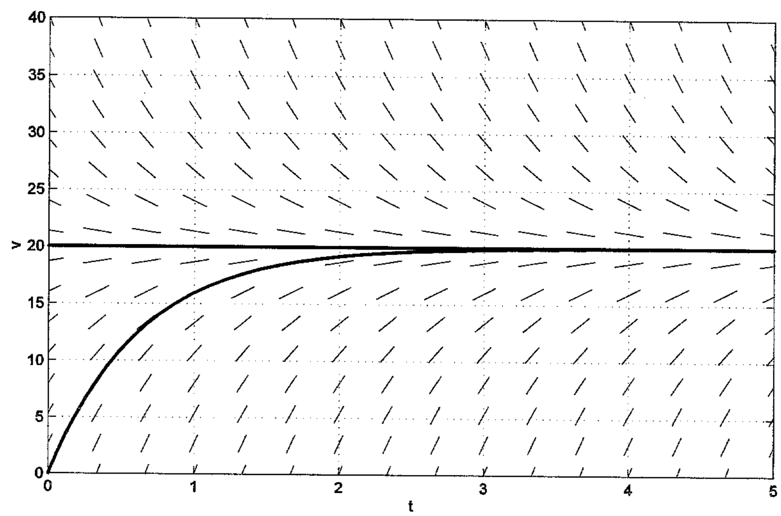
23. Tracing the curve in figure on the left below, we see that $y(2) \approx 1$. A more accurate approximation is $y(2) \approx 1.0044$.



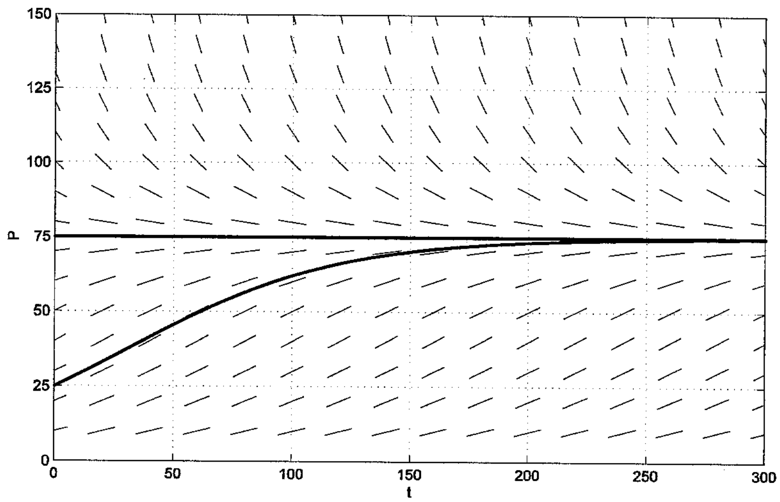


24. Tracing the curve in the figure on the right above, we see that $y(2) \approx 1.5$. A more accurate approximation is $y(2) \approx 1.4633$.

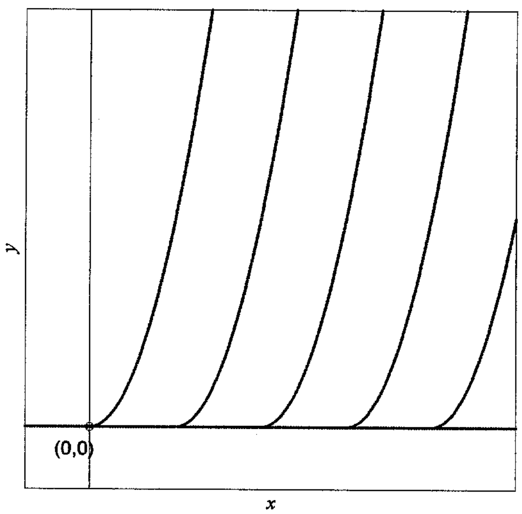
25. The figure below indicates a limiting velocity of 20 ft/sec — about the same as jumping off a $6\frac{1}{4}$-foot wall, and hence quite survivable. Tracing the curve suggests that $v(t) = 19$ ft/sec when t is a bit less than 2 seconds. An exact solution gives $t \approx 1.8723$ then.

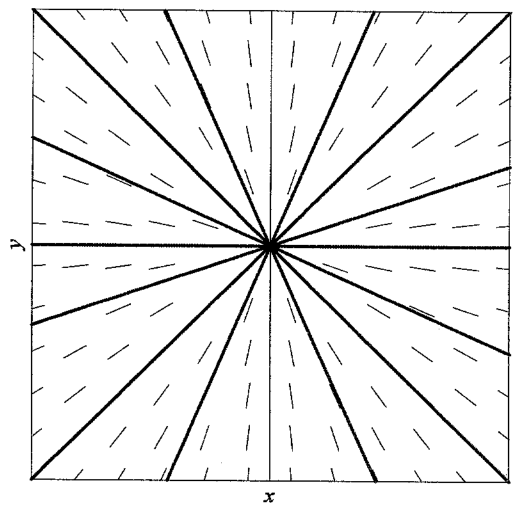


26. The figure below suggests that there are 40 deer after about 60 months; a more accurate value is $t \approx 61.61$. And it's pretty clear that the limiting population is 75 deer.



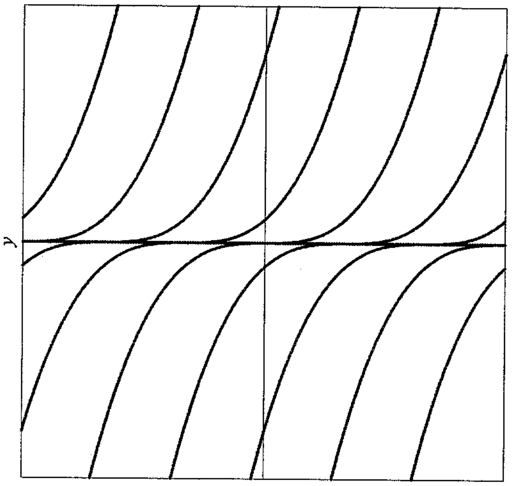
27. If $b < 0$ then the initial value problem $y' = 2\sqrt{y},\ y(0) = b$ has no solution, because the square root of a negative number would be involved. If $b > 0$ we get a unique solution curve through $(0, b)$ defined for all x by following a parabola — in the figure on the left below — down (and leftward) to the x-axis and then following the x-axis to the left. But starting at (0,0) we can follow the positive x-axis to the point $(c, 0)$ and then branching off on the parabola $y = (x - c)^2$. This gives infinitely many different solutions if $b = 0$.

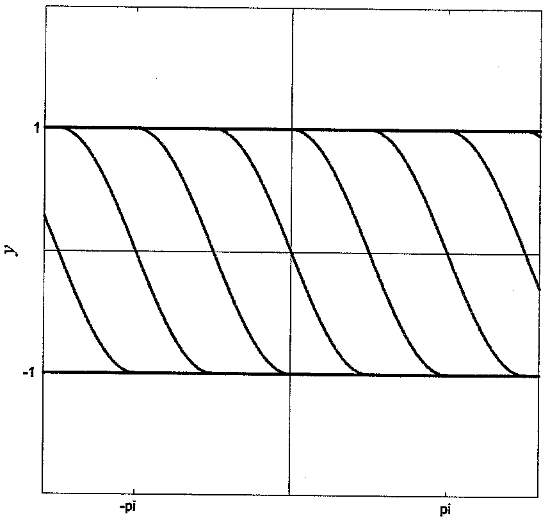




28. The figure on the right above makes it clear initial value problem $xy' = y,\ y(a) = b$ has a unique solution off the y-axis where $a \neq 0$; infinitely many solutions through the origin where $a = b = 0$; no solution if $a = 0$ but $b \neq 0$ (so the point (a, b) lies on the positive or negative y-axis).

29. Looking at the figure on the left below, we see that we can start at the point (a,b) and follow a branch of a cubic up or down to the x-axis, then follow the x-axis an arbitrary distance before branching off (down or up) on another cubic. This gives infinitely many solutions of the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ that are defined for all x. However, if $b \neq 0$ there is only a single cubic $y = (x-c)^3$ passing through (a,b), so the solution is unique near $x = a$.





30. The function $y(x) = \cos(x-c)$, with $y'(x) = -\sin(x-c)$, satisfies the differential equation $y' = -\sqrt{1-y^2}$ on the interval $c < x < c+\pi$ where $\sin(x-c) > 0$, so it follows that

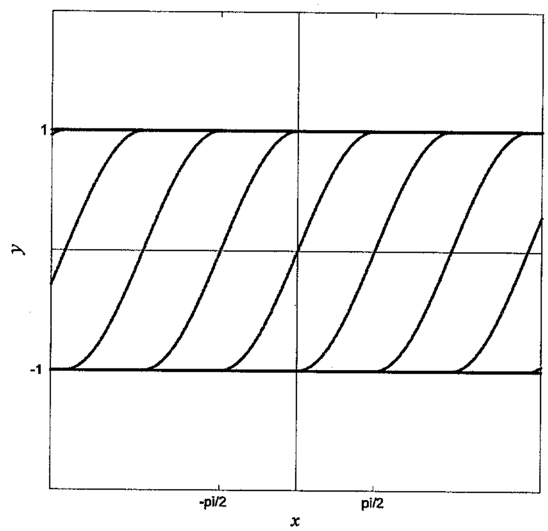
$$-\sqrt{1-y^2} = -\sqrt{1-\cos^2(x-c)} = -\sqrt{\sin^2(x-c)} = -\sin(x-c) = y.$$

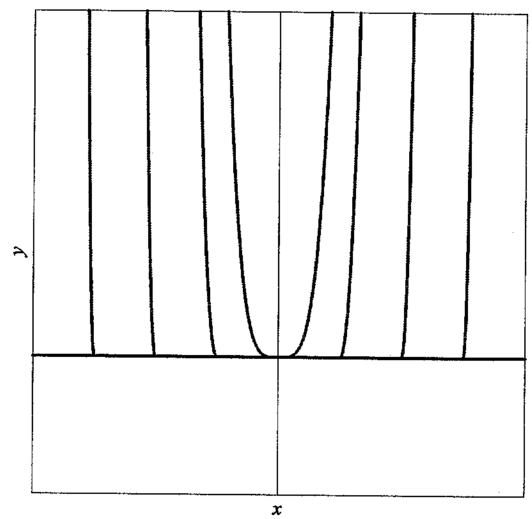
If $|b| > 1$ then the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$ then there is only one curve of the form $y = \cos(x-c)$ through the point (a,b); this give a unique solution. But if $b = \pm 1$ then we can combine a left ray of the line $y = +1$, a cosine curve from the line $y = +1$ to the line $y = -1$, and then a right ray of the line $y = -1$. Looking at the figure on the right above, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a, \pm 1)$.

31. The function $y(x) = \sin(x-c)$, with $y'(x) = \cos(x-c)$, satisfies the differential equation $y' = \sqrt{1-y^2}$ on the interval $c - \pi/2 < x < c + \pi/2$ where $\cos(x-c) > 0$, so it follows that

$$\sqrt{1-y^2} = \sqrt{1-\sin^2(x-c)} = \sqrt{\cos^2(x-c)} = -\sin(x-c) = y.$$

If $|b| > 1$ then the initial value problem $y' = \sqrt{1-y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$ then there is only one curve of the form $y = \sin(x-c)$ through the point (a,b); this give a unique solution. But if $b = \pm 1$ then we can combine a left ray of the line $y = -1$, a sine curve from the line $y = -1$ to the line $y = +1$, and then a right ray of the line $y = +1$. Looking at the figure on the left below, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a, \pm 1)$.





32. Looking at the figure on the right above, we see that we can piece together a "left half" of a quartic for x negative, an interval along the x-axis, and a "right half" of a quartic curve for x positive. This makes it clear that the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has infinitely many solutions (defined for all x) if $b \geq 0$; there is no solution if $b < 0$ because this would involve the square root of a negative number.

33. Looking at the figure provided in the answers section of the textbook, it suffices to observe that, among the pictured curves $y = x/(cx-1)$ for all possible values of c,

- there is a unique one of these curves through any point not on either coordinate axis;
- there is no such curve through any point on the y-axis other than the origin; and
- there are infinitely many such curves through the origin (0,0).

But in addition we have the constant-valued solution $y(x) \equiv 0$ that "covers" the x-axis. It follows that the given differential equation has near (a,b)

- a unique solution if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many different solutions if $a = b = 0$.

34. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = y - x + 1,\ \ y(-1) = -1.2$ is $y(x) = x - 0.2\,e^{x+1}$, whence $y(1) \approx -0.4778$. With the same differential equation but with initial condition $y(-1) = -0.8$ the solution is $y(x) = x + 0.2\,e^{x+1}$, whence $y(1) \approx 2.4778$.

(b) Similarly, the solution of the initial value problem $y' = y - x + 1,\ \ y(-3) = -3.01$ is $y(x) = x - 0.01\,e^{x+3}$, whence $y(3) \approx -1.0343$. With the same differential equation but with initial condition $y(-3) = -2.99$ the solution is $y(x) = x + 0.01\,e^{x+3}$, whence $y(3) \approx 7.0343$. Thus close initial values $y(-3) = -3 \pm 0.01$ yield $y(3)$ values that are far apart.

35. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = x - y + 1,\ \ y(-3) = -0.2$ is $y(x) = x + 2.8\,e^{-x-3}$, whence $y(2) \approx 2.0189$. With the same differential equation but with initial condition $y(-3) = +0.2$ the solution is $y(x) = x + 3.2\,e^{-x-3}$, whence $y(2) \approx 2.0216$.

(b) Similarly, the solution of the initial value problem $y' = x - y + 1,\ \ y(-3) = -0.5$ is $y(x) = x + 2.5\,e^{-x-3}$, whence $y(2) \approx 2.0189$. With the same differential equation but with initial condition $y(-3) = +0.5$ the solution is $y(x) = x + 3.5\,e^{-x-3}$, whence $y(2) \approx 2.0236$. Thus the initial values $y(-3) = \pm 0.5$ that are not close both yield $y(2) \approx 2.02$.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

1. $$\int \frac{dy}{y} = -\int 2x\,dx; \quad \ln y = -x^2 + c; \quad y(x) = e^{-x^2+c} = C\,e^{-x^2}$$

2. $$\int \frac{dy}{y^2} = -\int 2x\,dx; \quad -\frac{1}{y} = -x^2 - C; \quad y(x) = \frac{1}{x^2 + C}$$

3. $\displaystyle\int \frac{dy}{y} = \int \sin x\,dx; \quad \ln y = -\cos x + c; \quad y(x) = e^{-\cos x + c} = Ce^{-\cos x}$

4. $\displaystyle\int \frac{dy}{y} = \int \frac{4\,dx}{1+x}; \quad \ln y = 4\ln(1+x) + \ln C; \quad y(x) = C(1+x)^4$

5. $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}; \quad \sin^{-1} y = \sqrt{x} + C; \quad y(x) = \sin\left(\sqrt{x} + C\right)$

6. $\displaystyle\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx; \quad 2\sqrt{y} = 2x^{3/2} + 2C; \quad y(x) = \left(x^{3/2} + C\right)^2$

7. $\displaystyle\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx; \quad \tfrac{3}{2}y^{2/3} = 3x^{4/3} + \tfrac{3}{2}C; \quad y(x) = \left(2x^{4/3} + C\right)^{3/2}$

8. $\displaystyle\int \cos y\,dy = \int 2x\,dx; \quad \sin y = x^2 + C; \quad y(x) = \sin^{-1}\left(x^2 + C\right)$

9. $\displaystyle\int \frac{dy}{y} = \int \frac{2\,dx}{1-x^2} = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$ (partial fractions)

$\displaystyle\ln y = \ln(1+x) - \ln(1-x) + \ln C; \quad y(x) = C\frac{1+x}{1-x}$

10. $\displaystyle\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}; \quad -\frac{1}{1+y} = -\frac{1}{1+x} - C = -\frac{1+C(1+x)}{1+x}$

$\displaystyle 1+y = \frac{1+x}{1+C(1+x)}; \quad y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{x - C(1+x)}{1+C(1+x)}$

11. $\displaystyle\int \frac{dy}{y^3} = \int x\,dx; \quad -\frac{1}{2y^2} = \frac{x^2}{2} - \frac{C}{2}; \quad y(x) = \left(C - x^2\right)^{-1/2}$

12. $\displaystyle\int \frac{y\,dy}{y^2+1} = \int x\,dx; \quad \tfrac{1}{2}\ln\left(y^2+1\right) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln C; \quad y^2 + 1 = Ce^{x^2}$

13. $\displaystyle\int \frac{y^3\,dy}{y^4+1} = \int \cos x\,dx; \quad \tfrac{1}{4}\ln\left(y^4+1\right) = \sin x + C$

14. $\displaystyle\int \left(1+\sqrt{y}\right)dy = \int \left(1+\sqrt{x}\right)dx; \quad y + \tfrac{2}{3}y^{3/2} = x + \tfrac{2}{3}x^{3/2} + C$

15. $\int\left(\frac{2}{y^2}-\frac{1}{y^4}\right)dy = \int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx; \quad -\frac{2}{y}+\frac{1}{3y^3} = \ln|x|+\frac{1}{x}+C$

16. $\int\frac{\sin y\,dy}{\cos y} = \int\frac{x\,dx}{1+x^2}; \quad -\ln(\cos x) = \tfrac{1}{2}\ln\left(1+x^2\right) \ +\ln C$

$\sec y = C\sqrt{1+x^2}; \quad y(x) = \sec^{-1}\left(C\sqrt{1+x^2}\right)$

17. $y' = 1+x+y+xy = (1+x)(1+y)$

$\int\frac{dy}{1+y} = \int(1+x)\,dx; \quad \ln|1+y| = x+\tfrac{1}{2}x^2+C$

18. $x^2y' = 1-x^2+y^2-x^2y^2 = (1-x^2)(1+y^2)$

$\int\frac{dy}{1+y^2} = \int\left(\frac{1}{x^2}-1\right)dx; \quad \tan^{-1}y = -\frac{1}{x}-x+C; \quad y(x) = \tan\left(C-\frac{1}{x}-x\right)$

19. $\int\frac{dy}{y} = \int e^x\,dx; \quad \ln y = e^x+\ln C; \quad y(x) = C\exp(e^x)$

$y(0)=2e$ implies $C=2$ so $y(x) = 2\exp(e^x)$.

20. $\int\frac{dy}{1+y^2} = \int 3x^2\,dx; \quad \tan^{-1}y = x^3+C; \quad y(x) = \tan\left(x^3+C\right)$

$y(0)=1$ implies $C=\tan^{-1}1=\pi/4$ so $y(x) = \tan\left(x^3+\pi/4\right)$.

21. $\int 2y\,dy = \int\frac{x\,dx}{\sqrt{x^2-16}}; \quad y^2 = \sqrt{x^2-16}+C$

$y(5)=2$ implies $C=1$ so $y^2 = 1+\sqrt{x^2-16}$.

22. $\int\frac{dy}{y} = \int\left(4x^3-1\right)dx; \quad \ln y = x^4-x+\ln C; \quad y(x) = C\exp(x^4-x)$

$y(1)=-3$ implies $C=-3$ so $y(x) = -3\exp(x^4-x)$.

23. $\int\frac{dy}{2y-1} = \int dx; \quad \tfrac{1}{2}\ln(2y-1) = x+\tfrac{1}{2}\ln C; \quad 2y-1 = Ce^{2x}$

$y(1)=1$ implies $C=e^{-2}$ so $y(x) = \tfrac{1}{2}\left(1+e^{2x-2}\right)$.

24. $\displaystyle\int \frac{dy}{y} = \int \frac{\cos x\,dx}{\sin x}; \quad \ln y = \ln(\sin x) + \ln C; \quad y(x) = C\sin x$

$y(\frac{\pi}{2}) = \frac{\pi}{2}$ implies $C = \frac{\pi}{2}$ so $y(x) = \frac{\pi}{2}\sin x$.

25. $\displaystyle\int \frac{dy}{y} = \int \left(\frac{1}{x} + 2x\right); \quad \ln y = \ln x + x^2 + \ln C; \quad y(x) = C\,x\exp(x^2)$

$y(1) = 1$ implies $C = e^{-1}$ so $y(x) = x\exp(x^2 - 1)$.

26. $\displaystyle\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right); \quad -\frac{1}{y} = x^2 + x^3 + C; \quad y(x) = \frac{-1}{x^2 + x^3 + C}$

$y(1) = -1$ implies $C = -1$ so $y(x) = \dfrac{1}{1 - x^2 - x^3}$.

27. $\displaystyle\int e^y\,dy = \int 6e^{2x}\,dx; \quad e^y = 3e^{2x} + C; \quad y(x) = \ln\left(3e^{2x} + C\right)$

$y(0) = 0$ implies $C = -2$ so $y(x) = \ln\left(3e^{2x} - 2\right)$.

28. $\displaystyle\int \sec^2 y\,dy = \int \frac{dx}{2\sqrt{x}}; \quad \tan y = \sqrt{x} + C; \quad y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$

$y(4) = \frac{\pi}{4}$ implies $C = -1$ so $y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)$.

29. (a) Separation of variables gives the general solution

$$\int\left(-\frac{1}{y^2}\right)dy = -\int x\,dx; \quad \frac{1}{y} = -x + C; \quad y(x) = -\frac{1}{x - C}.$$

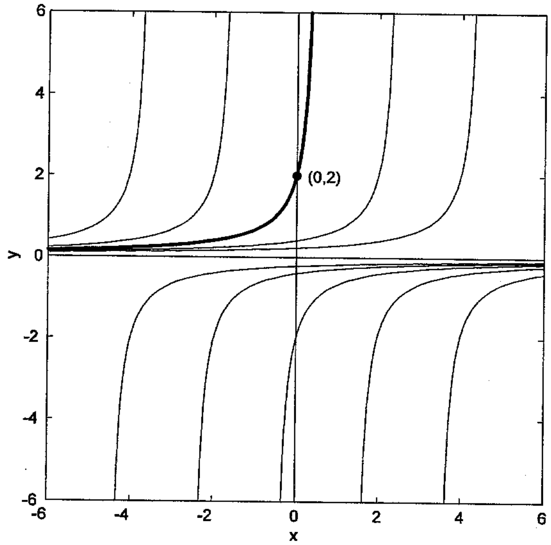
(b) Inspection yields the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

(c) In the figure at the top of the next page we see that there is a unique solution curve through every point in the *xy*-plane.

30. When we take square roots on both sides of the differential equation and separate variables, we get

$$\int \frac{dy}{2\sqrt{y}} = \int dx; \quad \sqrt{y} = x - C; \quad y(x) = (x - C)^2.$$

This general solution provides the parabolas illustrated in Fig. 1.4.5 in the textbook.



Problem 29 Figure

Observe that $y(x)$ is always nonnegative, consistent with both the square root and the original differential equation. We spot also the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

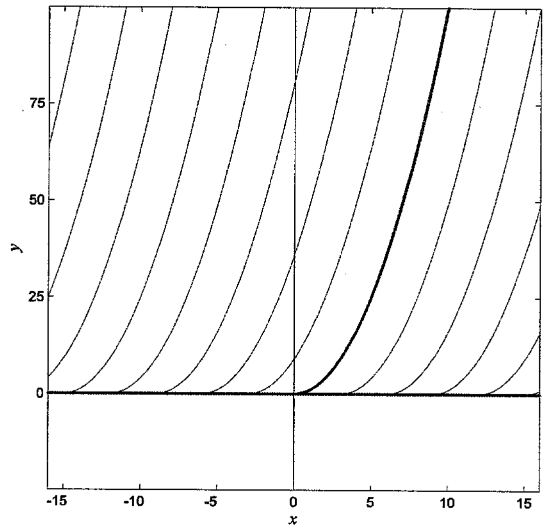
(a) Looking at Fig. 1.4.5, we see immediately that the differential equation $(y')^2 = 4y$ has no solution curve through the point (a,b) if $b < 0$.

(b) But if $b \geq 0$ we obviously can combine branches of parabolas with segments along the x-axis to form infinitely many solution curves through (a,b).

(c) Finally, if $b > 0$ then on a interval containing (a,b) there are exactly *two* solution curves through this point, corresponding to the two indicated parabolas through (a,b), one ascending and one descending from left to right.

31. The formal separation-of-variables process is the same as in Problem 30 where, indeed, we started by taking square roots in $(y')^2 = 4y$ to get the differential equation $y' = 2\sqrt{y}$. But whereas y' can be either positive or negative in the original equation, the latter equation requires that y' be *nonnegative*. This means that only the *right half* of each parabola $y = (x-C)^2$ qualifies as a solution curve. Inspecting the figure at the top of the next page, we therefore see that through the point (a,b) there passes:

(a) No solution curve if $b < 0$,
(b) A unique solution curve if $b > 0$,
(c) Infinitely many solution curves if $b = 0$, because in this case we can pick any $c > a$ and define the solution $y(x) = 0$ if $x \leq c$, $y(x) = (x-c)^2$ if $x \geq c$.

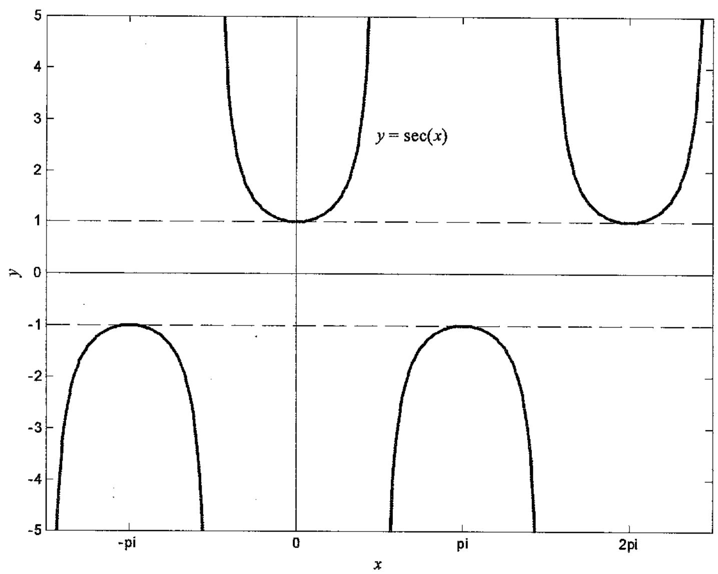


Problem 31 Figure

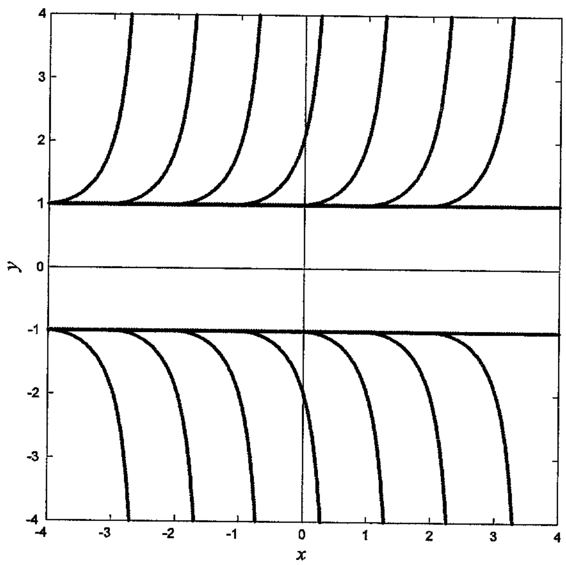
32. Separation of variables gives

$$x = \int \frac{dy}{y\sqrt{y^2 - 1}} = \sec^{-1}|y| + C$$

if $|y| > 1$, so the general solution has the form $y(x) = \pm\sec(x - C)$. But the original differential equation $y' = y\sqrt{y^2 - 1}$ implies that $y' > 0$ if $y > 1$, while $y' < 0$ if $y < -1$. Consequently, only the *right halves* of translated branches of the curve $y = \sec x$ (figure below) qualify as general solution curves. This explains the plotted



general solution curves we see in the figure below. In addition, we spot the two singular solutions $y(x) \equiv 1$ and $y(x) \equiv -1$. It follows upon inspection of this figure that the initial value problem $y' = y\sqrt{y^2 - 1}$, $y(a) = b$ has a unique solution if $|b| > 1$ and has no solution if $|b| < 1$. But if $b = 1$ (and similarly if $b = -1$) then we can pick any $c > a$ and define the solution $y(x) = 1$ if $x \le c$, $y(x) = |\sec(x - c)|$ if $c \le x < c + \frac{\pi}{2}$. So we see that if $b = \pm 1$, then the initial value problem $y' = y\sqrt{y^2 - 1}$, $y(a) = b$ has infinitely many solutions.



33. The population growth rate is $k = \ln(30000/25000)/10 \approx 0.01823$, so the population of the city t years after 1960 is given by $P(t) = 25000e^{0.01823t}$. The expected year 2000 population is then $P(40) = 25000e^{0.01823\times 40} \approx 51840$.

34. The population growth rate is $k = \ln(6)/10 \approx 0.17918$, so the population after t hours is given by $P(t) = P_0 e^{0.17918t}$. To find how long it takes for the population to double, we therefore need only solve the equation $2P_0 = P_0 e^{0.17918t}$ for $t = (\ln 2)/0.17918 \approx 3.87$ hours.

35. As in the textbook discussion of radioactive decay, the number of ^{14}C atoms after t years is given by $N(t) = N_0 e^{-0.0001216\,t}$. Hence we need only solve the equation $\frac{1}{6}N_0 = N_0 e^{-0.0001216\,t}$ for $t = (\ln 6)/0.0001216 \approx 14735$ years to find the age of the skull.

36. As in Problem 35, the number of ^{14}C atoms after t years is given by $N(t) = 5.0\times10^{10}\,e^{-0.0001216\,t}$. Hence we need only solve the equation $4.6\times10^{10} = 5.0\times10^{10}\,e^{-0.0001216\,t}$ for the age $t = \left(\ln(5.0/4.6)\right)/0.0001216 \approx 686$ years of the relic. Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

37. The amount in the account after t years is given by $A(t) = 5000\,e^{0.08t}$. Hence the amount in the account after 18 years is given by $A(18) = 5000\,e^{0.08\times18} \approx 21{,}103.48$ dollars.

38. When the book has been overdue for t years, the fine owed is given in dollars by $A(t) = 0.30\,e^{0.05t}$. Hence the amount owed after 100 years is given by $A(100) = 0.30\,e^{0.05\times100} \approx 44.52$ dollars.

39. To find the decay rate of this drug in the dog's blood stream, we solve the equation $\frac{1}{2} = e^{-5k}$ (half-life 5 hours) for $k = (\ln 2)/5 \approx 0.13863$. Thus the amount in the dog's bloodstream after t hours is given by $A(t) = A_0\,e^{-0.13863t}$. We therefore solve the equation $A(1) = A_0\,e^{-0.13863} = 50\times45 = 2250$ for $A_0 \approx 2585$ mg, the amount to anesthetize the dog properly.

40. To find the decay rate of radioactive cobalt, we solve the equation $\frac{1}{2} = e^{-5.27k}$ (half-life 5.27 years) for $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after t years is given by $A(t) = A_0\,e^{-0.13153t}$. We therefore solve the equation $A(t) = A_0\,e^{-0.13153t} = 0.01A_0$ for $t = (\ln 100)/0.13153 \approx 35.01$ and find that it will be about 35 years until the region is again inhabitable.

41. Taking $t = 0$ when the body was formed and $t = T$ now, the amount $Q(t)$ of ^{238}U in the body at time t (in years) is given by $Q(t) = Q_0e^{-kt}$, where $k = (\ln 2)/(4.51\times10^9)$. The given information tells us that

$$\frac{Q(T)}{Q_0 - Q(T)} = 0.9.$$

After substituting $Q(T) = Q_0e^{-kT}$, we solve readily for $e^{kT} = 19/9$, so $T = (1/k)\ln(19/9) \approx 4.86\times10^9$. Thus the body was formed approximately 4.86 billion years ago.

42. Taking $t = 0$ when the rock contained only potassium and $t = T$ now, the amount $Q(t)$ of potassium in the rock at time t (in years) is given by $Q(t) = Q_0e^{-kt}$, where $k = (\ln 2)/(1.28\times10^9)$. The given information tells us that the amount $A(t)$ of argon at time t is

$$A(t) = \tfrac{1}{9}[Q_0 - Q(t)]$$

and also that $A(T) = Q(T)$. Thus

$$Q_0 - Q(T) = 9\,Q(T).$$

After substituting $Q(T) = Q_0 e^{-kT}$ we readily solve for

$$T = (\ln 10/\ln 2)(1.28\times 10^9) \approx 4.25\times 10^9.$$

Thus the age of the rock is about 1.25 billion years.

43. Because $A = 0$ the differential equation reduces to $T' = kT$, so $T(t) = 25e^{-kt}$. The fact that $T(20) = 15$ yields $k = (1/20)\ln(5/3)$, and finally we solve

$$5 = 25e^{-kt} \qquad \text{for} \qquad t = (\ln 5)/k \approx 63 \text{ min.}$$

44. The amount of sugar remaining undissolved after t minutes is given by $A(t) = A_0 e^{-kt}$; we find the value of k by solving the equation $A(1) = A_0 e^{-k} = 0.75A_0$ for $k = -\ln 0.75 \approx 0.28768$. To find how long it takes for half the sugar to dissolve, we solve the equation $A(t) = A_0 e^{-kt} = \frac{1}{2}A_0$ for $t = (\ln 2)/0.28768 \approx 2.41$ minutes.

45. **(a)** The light intensity at a depth of x meters is given by $I(x) = I_0 e^{-1.4x}$. We solve the equation $I(x) = I_0 e^{-1.4x} = \frac{1}{2}I_0$ for $x = (\ln 2)/1.4 \approx 0.495$ meters.

(b) At depth 10 meters the intensity is $I(10) = I_0 e^{-1.4\times 10} \approx (8.32\times 10^{-7})\,I_0$.

(c) We solve the equation $I(x) = I_0 e^{-1.4x} = 0.01I_0$ for $x = (\ln 100)/1.4 \approx 3.29$ meters.

46. **(a)** The pressure at an altitude of x miles is given by $p(x) = 29.92\,e^{-0.2x}$. Hence the pressure at altitude 10000 ft is $p(10000/5280) \approx 20.49$ inches, and the pressure at altitude 30000 ft is $p(30000/5280) \approx 9.60$ inches.

(b) To find the altitude where $p = 15$ in., we solve the equation $29.92\,e^{-0.2x} = 15$ for $x = (\ln 29.92/15)/0.2 \approx 3.452$ miles $\approx 18{,}200$ ft.

47. If $N(t)$ denotes the number of people (in thousands) who have heard the rumor after t days, then the initial value problem is

$$N' = k(100 - N), \quad N(0) = 0$$

and we are given that $N(7) = 10$. When we separate variables ($dN/(100-N) = k\,dt$) and integrate, we get $\ln(100-N) = -kt + C$, and the initial condition $N(0) = 0$ gives $C = \ln 100$. Then $100 - N = 100e^{-kt}$, so $N(t) = 100\left(1-e^{-kt}\right)$. We substitute $t = 7$, $N = 10$ and solve for the value $k = \ln(100/90)/7 \approx 0.01505$. Finally, 50 thousand people have heard the rumor after $t = (\ln 2)/k \approx 46.05$ days.

48. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, at time t (in billions of years after the creation of the universe). Then $N_8(t) = N_0 e^{-kt}$ and $N_5(t) = N_0 e^{-ct}$, where N_0 is the initial number of atoms of each isotope. Also, $k = (\ln 2)/4.51$ and $c = (\ln 2)/0.71$ from the given half-lives. We divide the equations for N_8 and N_5 and find that when t has the value corresponding to "now",

$$e^{(c-k)t} = \frac{N_8}{N_5} = 137.7.$$

Finally we solve this last equation for $t = (\ln 137.7)/(c-k) \approx 5.99$. Thus we get an estimate of about 6 billion years for the age of the universe.

49. The cake's temperature will be 100° after 66 min 40 sec; this problem is just like Example 6 in the text.

50. (a) $A(t) = 10e^{kt}$. Also $30 = A(\tfrac{15}{2}) = 10e^{15k/2}$, so so

$$e^{15k/2} = 3; \quad k = \frac{2}{15}\ln 3 = \ln\left(3^{2/15}\right).$$

Therefore $A(t) = 10(e^k)^t = 10\cdot 3^{2t/15}$.

(b) After 5 years we have $A(5) = 10\cdot 3^{2/3} \approx 20.80$ pu.

(c) $A(t) = 100$ when $A(t) = 10\cdot 3^{2t/15}$; $t = \dfrac{15}{2}\cdot\dfrac{\ln(10)}{\ln(3)} \approx 15.72$ years.

51. (a) $A(t) = 15e^{-kt}$; $10 = A(5) = 15e^{-kt}$, so

$$\frac{3}{2} = e^{kt}; \quad k = \frac{1}{5}\ln\frac{3}{2}.$$

Therefore

$$A(t) = 15\exp\left(-\frac{t}{5}\ln\frac{3}{2}\right) = 15\cdot\left(\frac{3}{2}\right)^{-t/5} = 15\cdot\left(\frac{2}{3}\right)^{t/5}.$$

(b) After 8 months we have

$$A(8) = 15 \cdot \left(\frac{2}{3}\right)^{8/5} \approx 7.84 \text{ su.}$$

(c) $A(t) = 1$ when

$$A(t) = 15 \cdot \left(\frac{2}{3}\right)^{t/5} = 1; \quad t = 5 \cdot \frac{\ln(\frac{1}{15})}{\ln(\frac{2}{3})} \approx 33.3944.$$

Thus it will be safe to return after about 33.4 months.

52. If $L(t)$ denotes the number of human language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives $k = \frac{1}{6000}\ln\frac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that $L(T) = e^{kT} = 3300$, so $T = \frac{1}{k}\ln 3300 = \frac{6000 \ln 3300}{\ln(3/2)} \approx 119887.18$. This result suggests that the original human language was spoken about 120 thousand years ago.

53. If $L(t)$ denotes the number of Native America language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives $k = \frac{1}{6000}\ln\frac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that $L(T) = e^{kT} = 150$, so $T = \frac{1}{k}\ln 150 = \frac{6000 \ln 150}{\ln(3/2)} \approx 74146.48$. This result suggests that the ancestors of today's Native Americans first arrived in the western hemisphere about 74 thousand years ago.

54. With $A(y)$ constant, Equation (19) in the text takes the form

$$\frac{dy}{dt} = k\sqrt{y}$$

We readily solve this equation for $2\sqrt{y} = kt + C$. The condition $y(0) = 9$ yields $C = 6$, and then $y(1) = 4$ yields $k = 2$. Thus the depth at time t (in hours) is $y(t) = (3-t)^2$, and hence it takes 3 hours for the tank to empty.

55. With $A = \pi(3)^2$ and $a = \pi(1/12)^2$, and taking $g = 32$ ft/sec^2, Equation (20) reduces to $162\,y' = -\sqrt{y}$. The solution such that $y = 9$ when $t = 0$ is given by $324\sqrt{y} = -t + 972$. Hence $y = 0$ when $t = 972$ sec $= 16$ min 12 sec.

56. The radius of the cross-section of the cone at height y is proportional to y, so $A(y)$ is proportional to y^2. Therefore Equation (20) takes the form

$$y^2 y' = -k\sqrt{y},$$

and a general solution is given by

$$2y^{5/2} = -5kt + C.$$

The initial condition $y(0) = 16$ yields $C = 2048$, and then $y(1) = 9$ implies that $5k = 1562$. Hence $y = 0$ when

$$t = C/5k = 2048/1562 \approx 1.31 \text{ hr}.$$

57. The solution of $y' = -k\sqrt{y}$ is given by

$$2\sqrt{y} = -kt + C.$$

The initial condition $y(0) = h$ (the height of the cylinder) yields $C = 2\sqrt{h}$. Then substitution of $t = T$, $y = 0$ gives $k = (2\sqrt{h})/T$. It follows that

$$y = h(1 - t/T)^2.$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h(1 - t/T)^2 = V_0(1 - t/T)^2.$$

58. Since $x = y^{3/4}$, the cross-sectional area is $A(y) = \pi x^2 = \pi y^{3/2}$. Hence the general equation $A(y)\,y' = -a\sqrt{2gy}$ reduces to the differential equation $yy' = -k$ with general solution

$$(1/2)y^2 = -kt + C.$$

The initial condition $y(0) = 12$ gives $C = 72$, and then $y(1) = 6$ yields $k = 54$. Upon solving for y we find that the depth at time t is

$$y(t) = \sqrt{144 - 108t}.$$

Hence the tank is empty after $t = 144/108$ hr, that is, at 1:20 p.m.

59. **(a)** Since $x^2 = by$, the cross-sectional area is $A(y) = \pi x^2 = \pi by$. Hence the equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation

$$y^{1/2}y' = -k = -(a/\pi b)\sqrt{2g}$$

with the general solution

$$(2/3)y^{3/2} = -kt + C.$$

The initial condition $y(0) = 4$ gives $C = 16/3$, and then $y(1) = 1$ yields $k = 14/3$. It follows that the depth at time t is

$$y(t) = (8 - 7t)^{2/3}.$$

(b) The tank is empty after $t = 8/7$ hr, that is, at 1:08:34 p.m.

(c) We see above that $k = (a/\pi b)\sqrt{2g} = 14/3$. Substitution of $a = \pi r^2$, $b = 1$, $g = (32)(3600)^2$ ft/hr^2 yields $r = (1/60)\sqrt{7/12}$ ft ≈ 0.15 in for the radius of the bottom-hole.

60. With $g = 32$ ft/sec^2 and $a = \pi(1/12)^2$, Equation (24) simplifies to

$$A(y)\frac{dy}{dt} = -\frac{\pi}{18}\sqrt{y}.$$

If z denotes the distance from the center of the cylinder down to the fluid surface, then $y = 3 - z$ and $A(y) = 10(9 - z^2)^{1/2}$. Hence the equation above becomes

$$10(9-z^2)^{1/2}\frac{dz}{dt} = \frac{\pi}{18}(3-z)^{1/2},$$
$$180(3+z)^{1/2}dz = \pi\, dt,$$

and integration yields

$$120(3+z)^{1/2} = \pi t + C.$$

Now $z = 0$ when $t = 0$, so $C = 120(3)^{3/2}$. The tank is empty when $z = 3$ (that is, when $y = 0$) and thus after

$$t = (120/\pi)(6^{3/2} - 3^{3/2}) \approx 362.90 \text{ sec.}$$

It therefore takes about 6 min 3 sec for the fluid to drain completely.

61. $A(y) = \pi(8y - y^2)$ as in Example 7 in the text, but now $a = \pi/144$ in Equation (24), so the initial value problem is

$$18(8y - y^2)y' = -\sqrt{y}, \qquad y(0) = 8.$$

We seek the value of t when $y = 0$. The answer is $t \approx 869$ sec $=$ 14 min 29 sec.

62. The cross-sectional area function for the tank is $A = \pi(1-y^2)$ and the area of the bottom-hole is $a = 10^{-4}\pi$, so Eq. (24) in the text gives the initial value problem

$$\pi(1-y^2)\frac{dy}{dt} = -10^{-4}\pi\sqrt{2\times 9.8y}, \quad y(0) = 1.$$

Simplification gives

$$\left(y^{-1/2}-y^{3/2}\right)\frac{dy}{dt} = -1.4\times 10^{-4}\sqrt{10}$$

so integration yields

$$2y^{1/2}-\frac{2}{5}y^{5/2} = -1.4\times 10^{-4}\sqrt{10}\,t+C.$$

The initial condition $y(0)=1$ implies that $C = 2 - 2/5 = 8/5$, so $y=0$ after $t = (8/5)/(1.4\times 10^{-4}\sqrt{10}) \approx 3614$ seconds. Thus the tank is empty at about 14 seconds after 2 pm.

63. **(a)** As in Example 8, the initial value problem is

$$\pi(8y-y^2)\frac{dy}{dt} = -\pi k\sqrt{y}, \qquad y(0)=4$$

where $k = 0.6r^2\sqrt{2g} = 4.8r^2$. Integrating and applying the initial condition just in the Example 8 solution in the text, we find that

$$\frac{16}{3}y^{3/2}-\frac{2}{5}y^{5/2} = -kt+\frac{448}{15}.$$

When we substitute $y = 2$ (ft) and $t = 1800$ (sec, that is, 30 min), we find that $k \approx 0.009469$. Finally, $y = 0$ when

$$t = \frac{448}{15k} \approx 3154 \text{ sec} = 53 \text{ min } 34 \text{ sec}.$$

Thus the tank is empty at 1:53:34 pm.

(b) The radius of the bottom-hole is

$$r = \sqrt{k/4.8} \approx 0.04442 \text{ ft} \approx 0.53 \text{ in, thus about a half inch.}$$

64. The given rate of fall of the water level is $dy/dt = -4$ in/hr $= -(1/10800)$ ft/sec. With $A = \pi x^2$ and $a = \pi r^2$, Equation (24) is

$$(\pi x^2)(1/10800) = -(\pi r^2)\sqrt{2gy} = -8\pi r^2\sqrt{y}.$$

Hence the curve is of the form $y = kx^4$, and in order that it pass through $(1, 4)$ we must have $k = 4$. Comparing $\sqrt{y} = 2x^2$ with the equation above, we see that

$$(8r^2)(10800) = 1/2,$$

so the radius of the bottom hole is $r = 1/(240\sqrt{3})$ ft $\approx 1/35$ in.

65. Let $t = 0$ at the time of death. Then the solution of the initial value problem

$$T' = k(70 - T), \qquad T(0) = 98.6$$

is

$$T(t) = 70 + 28.6e^{-kt}.$$

If $t = a$ at 12 noon, then we know that

$$T(t) = 70 + 28.6e^{-ka} = 80,$$

$$T(a+1) = 70 + 28.6e^{-k(a+1)} = 75.$$

Hence

$$28.6e^{-ka} = 10 \quad \text{and} \quad 28.6e^{-ka}e^{-k} = 5.$$

It follows that $e^{-k} = 1/2$, so $k = \ln 2$. Finally the first of the previous two equations yields

$$a = (\ln 2.86)/(\ln 2) \approx 1.516 \text{ hr} \approx 1 \text{ hr } 31 \text{ min},$$

so the death occurred at 10:29 a.m.

66. Let $t = 0$ when it began to snow, and $t = t_0$ at 7:00 a.m. Let x denote distance along the road, with $x = 0$ where the snowplow begins at 7:00 a.m. If $y = ct$ is the snow depth at time t, w is the width of the road, and $v = dx/dt$ is the plow's velocity, then "plowing at a constant rate" means that the product wyv is constant. Hence our differential equation is of the form

$$k\frac{dx}{dt} = \frac{1}{t}.$$

The solution with $x = 0$ when $t = t_0$ is

$$t = t_0 e^{kx}.$$

We are given that $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, so it follows that

$$t_0 + 1 = t_0 e^{2k} \qquad \text{and} \qquad t_0 + 3 = t_0 e^{4k}.$$

Elimination of t_0 yields the equation

$$e^{4k} - 3e^{2k} + 2 = (e^{2k} - 1)(e^{2k} - 2) = 0,$$

so it follows (since $k > 0$) that $e^{2k} = 2$. Hence $t_0 + 1 = 2t_0$, so $t_0 = 1$. Thus it began to snow at 6 a.m.

67. We still have $t = t_0 e^{kx}$, but now the given information yields the conditions

$$t_0 + 1 = t_0 e^{4k} \qquad \text{and} \qquad t_0 + 2 = t_0 e^{7k}$$

at 8 a.m. and 9 a.m., respectively. Elimination of t_0 gives the equation

$$2e^{4k} - e^{7k} - 1 = 0,$$

which we solve numerically for $k = 0.08276$. Using this value, we finally solve one of the preceding pair of equations for $t_0 = 2.5483$ hr $\approx$ 2 hr 33 min. Thus it began to snow at 4:27 a.m.

68. (a) Note first that if θ denotes the angle between the tangent line and the horizontal, then $\alpha = \frac{\pi}{2} - \theta$ so $\cot\alpha = \cot(\frac{\pi}{2} - \theta) = \tan\theta = y'(x)$. It follows that

$$\sin\alpha = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \frac{1}{\sqrt{1 + \cot^2\alpha}} = \frac{1}{\sqrt{1 + y'(x)^2}}.$$

Therefore the mechanical condition $(\sin\alpha)/v =$ constant (positive) with $v = \sqrt{2gy}$ translates to

$$\frac{1}{\sqrt{2gy}\sqrt{1 + (y')^2}} = \text{constant}, \quad \text{so} \quad y[1 + (y')^2] = 2a$$

for some positive constant a. We readily solve the latter equation for the differential equation

$$y' = \frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}.$$

(b) The substitution $y = 2a\sin^2 t,\ dy = 4a\sin t\cos t\,dt$ now gives

$$4a\sin t\cos t\,dt = \sqrt{\frac{2a - 2a\sin^2 t}{2a\sin^2 t}}\,dx = \frac{\cos t}{\sin t}\,dx,$$
$$dx = 4a\sin^2 t\,dt.$$

Integration now gives

$$x = \int 4a\sin^2 t\,dt = 2a\int(1-\cos 2t)\,dt$$
$$= 2a(t - \tfrac{1}{2}\sin 2t) + C = a(2t - \sin 2t) + C,$$

and we recall that $y = 2a\sin^2 t = a(1-\cos 2t)$. The requirement that $x = 0$ when $t = 0$ implies that $C = 0$. Finally, the substitution $\theta = 2t$ (nothing to do with the previously mentioned angle θ of inclination from the horizontal) yields the desired parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta)$$

of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis. [See Example 5 in Section 9.4 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition (Upper Saddle River, NJ: Prentice Hall, 2008).]

69. Substitution of $v = dy/dx$ in the differential equation for $y = y(x)$ gives

$$a\frac{dv}{dx} = \sqrt{1+v^2},$$

and separation of variables then yields

$$\int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{a}; \quad \sinh^{-1} v = \frac{x}{a} + C_1; \quad \frac{dy}{dx} = \sinh\left(\frac{x}{a} + C_1\right).$$

The fact that $y'(0) = 0$ implies that $C_1 = 0$, so it follows that

$$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right); \qquad y(x) = a\cosh\left(\frac{x}{a}\right) + C.$$

Of course the (vertical) position of the x-axis can be adjusted so that $C = 0$, and the units in which T and ρ are measured may be adjusted so that $a = 1$. In essence, then the shape of the hanging cable is the hyperbolic cosine graph $y = \cosh x$.

SECTION 1.5

LINEAR FIRST-ORDER EQUATIONS

1. $\rho = \exp\left(\int 1\,dx\right) = e^x; \quad D_x\left(y \cdot e^x\right) = 2e^x; \quad y \cdot e^x = 2e^x + C; \quad y(x) \;=\; 2 + Ce^{-x}$

$y(0) = 0$ implies $C = -2$ so $y(x) \;=\; 2 - 2e^{-x}$

2. $\rho = \exp\left(\int (-2)\,dx\right) = e^{-2x}; \quad D_x\left(y \cdot e^{-2x}\right) = 3; \quad y \cdot e^{-2x} = 3x + C; \quad y(x) \;=\; (3x + C)e^{2x}$

$y(0) = 0$ implies $C = 0$ so $y(x) \;=\; 3x\,e^{2x}$

3. $\rho = \exp\left(\int 3\,dx\right) = e^{3x}; \quad D_x\left(y \cdot e^{3x}\right) = 2x; \quad y \cdot e^{3x} = x^2 + C; \quad y(x) \;=\; (x^2 + C)e^{-3x}$

4. $\rho = \exp\left(\int (-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y \cdot e^{-x^2}\right) = 1; \quad y \cdot e^{-x^2} = x + C; \quad y(x) \;=\; (x + C)e^{x^2}$

5. $\rho = \exp\left(\int (2/x)\,dx\right) = e^{2\ln x} = x^2; \quad D_x\left(y \cdot x^2\right) = 3x^2; \quad y \cdot x^2 = x^3 + C$

$y(x) \;=\; x + C/x^2; \quad y(1) = 5$ implies $C = 4$ so $y(x) \;=\; x + 4/x^2$

6. $\rho = \exp\left(\int (5/x)\,dx\right) = e^{5\ln x} = x^5; \quad D_x\left(y \cdot x^5\right) = 7x^6; \quad y \cdot x^5 = x^7 + C$

$y(x) \;=\; x^2 + C/x^5; \quad y(2) = 5$ implies $C = 32$ so $y(x) \;=\; x^2 + 32/x^5$

7. $\rho = \exp\left(\int (1/2x)\,dx\right) = e^{(\ln x)/2} = \sqrt{x}; \quad D_x\left(y \cdot \sqrt{x}\right) = 5; \quad y \cdot \sqrt{x} = 5x + C$

$y(x) \;=\; 5\sqrt{x} + C/\sqrt{x}$

8. $\rho = \exp\left(\int (1/3x)\,dx\right) = e^{(\ln x)/3} = \sqrt[3]{x}; \quad D_x\left(y \cdot \sqrt[3]{x}\right) = 4\sqrt[3]{x}; \quad y \cdot \sqrt[3]{x} = 3x^{4/3} + C$

$y(x) \;=\; 3x + Cx^{-1/3}$

9. $\rho = \exp\left(\int (-1/x)\,dx\right) = e^{-\ln x} = 1/x; \quad D_x\left(y \cdot 1/x\right) = 1/x; \quad y \cdot 1/x = \ln x + C$

$y(x) \;=\; x\ln x + Cx; \quad y(1) = 7$ implies $C = 7$ so $y(x) \;=\; x\ln x + 7x$

10. $\rho = \exp\left(\int (-3/2x)\,dx\right) = e^{(-3\ln x)/2} = x^{-3/2};$

$D_x\left(y \cdot x^{-3/2}\right) = 9x^{1/2}/2; \quad y \cdot x^{-3/2} = 3x^{3/2} + C; \qquad y(x) \;=\; 3x^3 + Cx^{3/2}$

11. $\rho = \exp\left(\int (1/x - 3)\,dx\right) = e^{\ln x - 3x} = x e^{-3x}$; $D_x\left(y \cdot x e^{-3x}\right) = 0$; $y \cdot x e^{-3x} = C$

$y(x) = C x^{-1} e^{3x}$; $y(1) = 0$ implies $C = 0$ so $y(x) \equiv 0$ (constant)

12. $\rho = \exp\left(\int (3/x)\,dx\right) = e^{3\ln x} = x^3$; $D_x\left(y \cdot x^3\right) = 2x^7$; $y \cdot x^3 = \frac{1}{4}x^8 + C$

$y(x) = \frac{1}{4}x^5 + C x^{-3}$; $y(2) = 1$ implies $C = -56$ so $y(x) = \frac{1}{4}x^5 - 56x^{-3}$

13. $\rho = \exp\left(\int 1\,dx\right) = e^x$; $D_x\left(y \cdot e^x\right) = e^{2x}$; $y \cdot e^x = \frac{1}{2}e^{2x} + C$

$y(x) = \frac{1}{2}e^x + C e^{-x}$; $y(0) = 1$ implies $C = \frac{1}{2}$ so $y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

14. $\rho = \exp\left(\int (-3/x)\,dx\right) = e^{-3\ln x} = x^{-3}$; $D_x\left(y \cdot x^{-3}\right) = x^{-1}$; $y \cdot x^{-3} = \ln x + C$

$y(x) = x^3 \ln x + C x^3$; $y(1) = 10$ implies $C = 10$ so $y(x) = x^3 \ln x + 10x^3$

15. $\rho = \exp\left(\int 2x\,dx\right) = e^{x^2}$; $D_x\left(y \cdot e^{x^2}\right) = x e^{x^2}$; $y \cdot e^{x^2} = \frac{1}{2}e^{x^2} + C$

$y(x) = \frac{1}{2} + C e^{-x^2}$; $y(0) = -2$ implies $C = -\frac{5}{2}$ so $y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}$

16. $\rho = \exp\left(\int \cos x\,dx\right) = e^{\sin x}$; $D_x\left(y \cdot e^{\sin x}\right) = e^{\sin x}\cos x$; $y \cdot e^{\sin x} = e^{\sin x} + C$

$y(x) = 1 + C e^{-\sin x}$; $y(\pi) = 2$ implies $C = 1$ so $y(x) = 1 + e^{-\sin x}$

17. $\rho = \exp\left(\int 1/(1+x)\,dx\right) = e^{\ln(1+x)} = 1 + x$; $D_x\left(y \cdot (1+x)\right) = \cos x$; $y \cdot (1+x) = \sin x + C$

$y(x) = \dfrac{C + \sin x}{1 + x}$; $y(0) = 1$ implies $C = 1$ so $y(x) = \dfrac{1 + \sin x}{1 + x}$

18. $\rho = \exp\left(\int (-2/x)\,dx\right) = e^{-2\ln x} = x^{-2}$; $D_x\left(y \cdot x^{-2}\right) = \cos x$; $y \cdot x^{-2} = \sin x + C$

$y(x) = x^2\left(\sin x + C\right)$

19. $\rho = \exp\left(\int \cot x\,dx\right) = e^{\ln(\sin x)} = \sin x$; $D_x\left(y \cdot \sin x\right) = \sin x \cos x$

$y \cdot \sin x = \frac{1}{2}\sin^2 x + C$; $y(x) = \frac{1}{2}\sin x + C \csc x$

20. $\rho = \exp\left(\int(-1-x)\,dx\right) = e^{-x-x^2/2};\quad D_x\left(y\cdot e^{-x-x^2/2}\right) = (1+x)e^{-x-x^2/2}$

$y\cdot e^{-x-x^2/2} = -e^{-x-x^2/2} + C;\quad y(x) \;=\; -1 + C\,e^{x+x^2/2}$

$y(0)=0$ implies $C=1$ so $y(x) \;=\; -1 + e^{x+x^2/2}$

21. $\rho = \exp\left(\int(-3/x)\,dx\right) = e^{-3\ln x} = x^{-3};\quad D_x\left(y\cdot x^{-3}\right) = \cos x;\quad y\cdot x^{-3} = \sin x + C$

$y(x) \;=\; x^3\sin x + C\,x^3;\quad y(2\pi)=0$ implies $C=0$ so $y(x) \;=\; x^3\sin x$

22. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2};\quad D_x\left(y\cdot e^{-x^2}\right) = 3x^2;\quad y\cdot e^{-x^2} = x^3 + C$

$y(x) \;=\; \left(x^3 + C\right)e^{+x^2};\quad y(0)=5$ implies $C=5$ so $y(x) \;=\; \left(x^3+5\right)e^{+x^2}$

23. $\rho = \exp\left(\int(2-3/x)\,dx\right) = e^{2x-3\ln x} = x^{-3}e^{2x};\quad D_x\left(y\cdot x^{-3}e^{2x}\right) = 4\,e^{2x}$

$y\cdot x^{-3}e^{2x} = 2\,e^{2x} + C;\quad y(x) \;=\; 2x^3 + C\,x^3e^{-2x}$

24. $\rho = \exp\left(\int 3x/(x^2+4)\,dx\right) = e^{3\ln(x^2+4)/2} = (x^2+4)^{3/2};\quad D_x\left(y\cdot(x^2+4)^{3/2}\right) = x(x^2+4)^{1/2}$

$y\cdot(x^2+4)^{3/2} = \tfrac{1}{3}(x^2+4)^{3/2} + C;\quad y(x) \;=\; \tfrac{1}{3} + C(x^2+4)^{-3/2}$

$y(0)=1$ implies $C = \tfrac{16}{3}$ so $y(x) \;=\; \tfrac{1}{3}\left[1 + 16(x^2+4)^{-3/2}\right]$

25. First we calculate

$$\int\frac{3x^3\,dx}{x^2+1} \;=\; \int\left[3x - \frac{3x}{x^2+1}\right]dx \;=\; \frac{3}{2}\left[x^2 - \ln(x^2+1)\right].$$

It follows that $\rho = (x^2+1)^{-3/2}\exp(3x^2/2)$ and thence that

$$\begin{aligned} D_x\left(y\cdot(x^2+1)^{-3/2}\exp(3x^2/2)\right) \;&=\; 6x(x^2+4)^{-5/2}, \\ y\cdot(x^2+1)^{-3/2}\exp(3x^2/2) \;&=\; -2(x^2+4)^{-3/2} + C, \\ y(x) \;&=\; -2\exp(3x^2/2) + C(x^2+1)^{3/2}\exp(-3x^2/2). \end{aligned}$$

Finally, $y(0) = 1$ implies that $C = 3$ so the desired particular solution is

$$y(x) \;=\; -2\exp(3x^2/2) + 3(x^2+1)^{3/2}\exp(-3x^2/2).$$

26. With $x' = dx/dy$, the differential equation is $y^3x' + 4y^2x = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int (4/y)\,dy\right) = e^{4\ln y} = y^4; \quad D_y\left(x \cdot y^4\right) = y$$

$$x \cdot y^4 = \frac{1}{2}y^2 + C; \quad x(y) = \frac{1}{2y^2} + \frac{C}{y^4}$$

27. With $x' = dx/dy$, the differential equation is $x' - x = ye^y$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int (-1)\,dy\right) = e^{-y}; \quad D_y\left(x \cdot e^{-y}\right) = y$$

$$x \cdot e^{-y} = \tfrac{1}{2}y^2 + C; \quad x(y) = \left(\tfrac{1}{2}y^2 + C\right)e^y$$

28. With $x' = dx/dy$, the differential equation is $(1+y^2)x' - 2yx = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int (-2y/(1+y^2)\,dy\right) = e^{-\ln(y^2+1)} = (1+y^2)^{-1}$$

$$D_y\left(x \cdot (1+y^2)^{-1}\right) = (1+y^2)^{-2}$$

An integral table (or trigonometric substitution) now yields

$$\frac{x}{1+y^2} = \int \frac{dy}{\left(1+y^2\right)^2} = \frac{1}{2}\left(\frac{y}{1+y^2} + \tan^{-1} y + C\right)$$

$$x(y) = \tfrac{1}{2}\left[y + \left(1+y^2\right)\left(\tan^{-1} y + C\right)\right]$$

29. $$\rho = \exp\left(\int (-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y \cdot e^{-x^2}\right) = e^{-x^2}; \quad y \cdot e^{-x^2} = C + \int_0^x e^{-t^2}\,dt$$

$$y(x) = e^{x^2}\left(C + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)\right)$$

30. After division of the given equation by $2x$, multiplication by the integrating factor $\rho = x^{-1/2}$ yields

$$x^{-1/2}y' - \tfrac{1}{2}x^{-3/2}y = x^{-1/2}\cos x,$$

$$D_x\left(x^{-1/2}y\right) = x^{-1/2}\cos x,$$

$$x^{-1/2}y = C + \int_1^x t^{-1/2}\cos t\,dt.$$

The initial condition $y(1) = 0$ implies that $C = 0$, so the desired particular solution is

$$y(x) = x^{1/2} \int_1^x t^{-1/2} \cos t \, dt.$$

31. **(a)** $y_c' = Ce^{-\int P\,dx}(-P) = -P\,y_c$, so $y_c' + P\,y_c = 0$.

(b) $y_p' = (-P)e^{-\int P\,dx} \cdot \left[\int \left(Q e^{\int P\,dx}\right) dx\right] + e^{-\int P\,dx} \cdot Q e^{\int P\,dx} = -Py_p + Q$

32. **(a)** If $y = A\cos x + B\sin x$ then

$$y' + y = (A+B)\cos x + (B-A)\sin x = 2\sin x$$

provided that $A = -1$ and $B = 1$. These coefficient values give the particular solution $y_p(x) = \sin x - \cos x$.

(b) The general solution of the equation $y' + y = 0$ is $y(x) = Ce^{-x}$ so addition to the particular solution found in part (a) gives $y(x) = Ce^{-x} + \sin x - \cos x$.

(c) The initial condition $y(0) = 1$ implies that $C = 2$, so the desired particular solution is $y(x) = 2e^{-x} + \sin x - \cos x$.

33. The amount $x(t)$ of salt (in kg) after t seconds satisfies the differential equation $x' = -x/200$, so $x(t) = 100e^{-t/200}$. Hence we need only solve the equation $10 = 100e^{-t/200}$ for $t = 461$ sec $= 7$ min 41 sec (approximately).

34. Let $x(t)$ denote the amount of pollutants in the lake after t days, measured in millions of cubic feet (mft^3). The volume of the lake is 8000 mft^3, and the initial amount $x(0)$ of pollutants is $x_0 = (0.25\%)(8000) = 20$ mft^3. We want to know when $x(t) = (0.10\%)(8000) = 8$ mft^3. We set up the differential equation in infinitesimal form by writing

$$dx = [\text{in}] - [\text{out}] = (0.0005)(500)\,dt - \frac{x}{8000} \cdot 500\,dt,$$

which simplifies to

$$\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}, \quad \text{or} \quad \frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}.$$

Using the integrating factor $\rho = e^{t/16}$, we readily derive the solution $x(t) = 4 + 16e^{-t/16}$ for which $x(0) = 20$. Finally, we find that $x = 8$ when $t = 16\ln 4 \approx 22.2$ days.

35. The only difference from the Example 4 solution in the textbook is that $V = 1640$ km^3 and $r = 410$ km^3/yr for Lake Ontario, so the time required is

$$t = \frac{V}{r}\ln 4 = 4\ln 4 \approx 5.5452 \text{ years.}$$

36. **(a)** The volume of brine in the tank after t min is $V(t) = 60 - t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t}, \qquad x(0) = 0.$$

The solution is

$$x(t) = (60-t) - \frac{(60-t)^3}{3600}.$$

(b) The maximum amount ever in the tank is $40/\sqrt{3} \approx 23.09$ lb. This occurs after $t = 60 - 20\sqrt{3} \approx 25/36$ min.

37. The volume of brine in the tank after t min is $V(t) = 100 + 2t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 5 - \frac{3x}{100+2t}, \qquad x(0) = 50.$$

The integrating factor $\rho(t) = (100 + 2t)^{3/2}$ leads to the solution

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{3/2}}.$$

such that $x(0) = 50$. The tank is full after $t = 150$ min, at which time $x(150) = 393.75$ lb.

38. **(a)** $dx/dt = -x/20$ and $x(0) = 50$ so $x(t) = 50e^{-t/20}$.

(b) The solution of the linear differential equation

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200} = \frac{5}{2}e^{-t/20} - \frac{1}{40}y$$

with $y(0) = 50$ is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

(c) The maximum value of y occurs when

$$y'(t) = -\frac{15}{4}e^{-t/40} + 5e^{-t/20} = -\frac{5}{4}e^{-t/40}\left(3-4e^{-t/40}\right) = 0.$$

We find that $y_{max} = 56.25$ lb when $t = 40\ln(4/3) \approx 11.51$ min.

39. **(a)** The initial value problem

$$\frac{dx}{dt} = -\frac{x}{10}, \qquad x(0) = 100$$

for Tank 1 has solution $x(t) = 100e^{-t/10}$. Then the initial value problem

$$\frac{dy}{dt} = \frac{x}{10} - \frac{y}{10} = 10e^{-t/10} - \frac{y}{10}, \qquad y(0) = 0$$

for Tank 2 has solution $y(t) = 10te^{-t/10}$.

(b) The maximum value of y occurs when

$$y'(t) = 10e^{-t/10} - te^{-t/10} = 0$$

and thus when $t = 10$. We find that $y_{max} = y(10) = 100e^{-1} \approx 36.79$ gal.

40. **(b)** Assuming inductively that $x_n = t^n e^{-t/2}/\left(n!2^n\right)$, the equation for x_{n+1} is

$$\frac{dx_{n+1}}{dt} = \frac{1}{2}x_n - \frac{1}{2}x_{n+1} = \frac{t^n e^{-t/2}}{n!\,2^{n+1}} - \frac{1}{2}x_{n+1}.$$

We easily solve this first–order equation with $x_{n+1}(0) = 0$ and find that

$$x_{n+1} = \frac{t^{n+1}e^{-t/2}}{(n+1)!\,2^{n+1}},$$

thereby completing the proof by induction.

41. **(a)** $A'(t) = 0.06A + 0.12S = 0.06A + 3.6e^{0.05t}$

(b) The solution with $A(0) = 0$ is

$$A(t) = 360(e^{0.06t} - e^{0.05t}),$$

so $A(40) \approx 1308.283$ thousand dollars.

42. The mass of the hailstone at time t is $m = (4/3)\pi r^3 = (4/3)\pi k^3 t^3$. Then the equation $d(mv)/dt = mg$ simplifies to

$$tv' + 3v = gt.$$

The solution satisfying the initial condition $v(0) = 0$ is $v(t) = gt/4$, so $v'(t) = g/4$.

43. The solution of the initial value problem $y' = x - y,\ \ y(-5) = y_0$ is

$$y(x) = x - 1 + (y_0 + 6)e^{-x-5}.$$

Substituting $x = 5$, we therefore solve the equation $4 + (y_0 + 6)e^{-10} = y_1$ with $y_1 = 3.998,\ 3.999,\ 4,\ 4.001,\ 4.002$ for the desired initial values $y_0 = -50.0529,\ -28.0265,\ -6.0000,\ 16.0265,\ 38.0529$, respectively.

44. The solution of the initial value problem $y' = x + y,\ \ y(-5) = y_0$ is

$$y(x) = -x - 1 + (y_0 - 4)e^{x+5}.$$

Substituting $x = 5$, we therefore solve the equation $-6 + (y_0 - 4)e^{10} = y_1$ with $y_1 = -10,\ -5,\ 0,\ 5,\ 10$ for the desired initial values $y_0 = 3.99982,\ 4.00005,\ 4.00027,\ 4.00050,\ 4.00073$, respectively.

45. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_o = 10$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_o - (r/V)x$, we get the equation

$$\frac{dx}{dt} = 2 - \frac{1}{10}x$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we readily find that the solution with $x(0) = 0$ is

$$x(t) = 20\left(1 - e^{-t/10}\right).$$

Then we find that $x = 10$ when $t = 10\ln 2 \approx 6.93$ months, and observe finally that, as expected, $x(t) \to 20$ as $t \to \infty$.

46. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_0 = 10(1+\cos t)$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_0 - (r/V)x$, we get the equation

$$\frac{dx}{dt} = 2(1+\cos t) - \frac{1}{10}x, \quad \text{that is,} \quad \frac{dx}{dt} + \frac{1}{10}x = 2(1+\cos t)$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we get

$$x \cdot e^{t/10} = \int (2e^{t/10} + 2e^{t/10}\cos t)\,dt$$

$$= 20e^{t/10} + 2 \cdot \frac{e^{t/10}}{(\frac{1}{10})^2 + 1^2}\left(\frac{1}{10}\cos t + \sin t\right) + C.$$

When we impose the condition $x(0) = 0$, we get the desired particular solution

$$x(t) = \frac{20}{101}\left(101 - 102e^{-t/10} + \cos t + 10\sin t\right).$$

In order to determine when $x = 10$, we need to solve numerically. For instance, we can use the *Mathematica* commands

```
x = (20/101)(101 - 102 Exp[-t/10] + Cos[t] + 10 Sin[t]);
FindRoot[ x == 10, {t,7} ]

{t -> 6.474591767017537}
```

and find that this occurs after about 6.47 months. Finally, as $t \to \infty$ we observe that $x(t)$ approaches the function $20 + \frac{20}{101}(\cos t + 10\sin t)$ that does, indeed, oscillate about the equilibrium solution $x(t) \equiv 20$.

SECTION 1.6

SUBSTITUTION METHODS AND EXACT EQUATIONS

It is traditional for every elementary differential equations text to include the particular types of equations that are found in this section. However, no one of them is vitally important solely in its own right. Their main purpose (at this point in the course) is to familiarize students with the technique of transforming a differential equation by substitution. The subsection on airplane flight trajectories (together with Problems 56–59) is included as an application, but is optional material and may be omitted if the instructor desires.

The differential equations in Problems 1–15 are homogeneous, so we make the substitutions

$$v = \frac{y}{x}, \qquad y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

For each problem we give the differential equation in x, $v(x)$, and $v' = dv/dx$ that results, together with the principal steps in its solution.

1. $x(v+1)v' = -(v^2+2v-1)$; $\displaystyle\int \frac{2(v+1)\,dv}{v^2+2v-1} = -\int 2x\,dx$; $\ln(v^2+2v-1) = -2\ln x + \ln C$

$x^2(v^2+2v-1) = C$; $y^2+2xy-x^2 = C$

2. $2xvv' = 1$; $\displaystyle\int 2v\,dv = \int \frac{dx}{x}$; $v^2 = \ln x + C$; $y^2 = x^2(\ln x + C)$

3. $xv' = 2\sqrt{v}$; $\displaystyle\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$; $\sqrt{v} = \ln x + C$; $y = x(\ln x + C)^2$

4. $x(v-1)v' = -(v^2+1)$; $\displaystyle\int \frac{2(1-v)\,dv}{v^2+1} = \int \frac{2\,dx}{x}$; $2\tan^{-1} v - \ln(v^2+1) = 2\ln x + C$

$2\tan^{-1}(y/x) - \ln(y^2/x^2+1) = 2\ln x + C$

5. $x(v+1)v' = -2v^2$; $\displaystyle\int \left(\frac{1}{v}+\frac{1}{v^2}\right)dv = -\int \frac{2\,dx}{x}$; $\ln v - \dfrac{1}{v} = -2\ln x + C$

$\ln y - \ln x - \dfrac{x}{y} = -2\ln x + C$; $\ln(xy) = C + \dfrac{x}{y}$

6. $x(2v+1)v' = -2v^2$; $\displaystyle\int \left(\frac{2}{v}+\frac{1}{v^2}\right)dv = -\int \frac{2\,dx}{x}$; $\ln v^2 - \dfrac{1}{v} = -2\ln x + C$

$2\ln y - 2\ln x - \dfrac{x}{y} = -2\ln x + C$; $2y\ln y = x + Cy$

7. $xv^2v' = 1$; $\displaystyle\int 3v^2\,dv = \int \frac{3\,dx}{x}$; $v^3 = 3\ln x + C$; $y^3 = x^3(3\ln x + C)$

8. $xv' = e^v$; $\displaystyle -\int e^{-v}\,dv = -\int \frac{dx}{x}$; $e^{-v} = -\ln x + C$; $-v = \ln(C - \ln x)$

$y = -x\ln(C - \ln x)$

9. $xv' = v^2;\quad -\int \frac{dv}{v^2} = -\int \frac{dx}{x};\quad \frac{1}{v} = -\ln x + C;\quad x = y(C - \ln x)$

10. $xvv' = 2v^2 + 1;\quad \int \frac{4v\,dv}{2v^2+1} = \int \frac{4\,dx}{x};\quad \ln\left(2v^2+1\right) = 4\ln x + \ln C$

$2y^2/x^2 + 1 = Cx^4;\quad x^2 + 2y^2 = Cx^6$

11. $x\left(1-v^2\right)v' = v + v^3;\quad \int \frac{1-v^2}{v^3+v}dv = \int \frac{dx}{x};\quad \int\left(\frac{1}{v} - \frac{2v}{v^2+1}\right)dv = \int \frac{dx}{x}$

$\ln v - \ln\left(v^2+1\right) = \ln x + \ln C;\quad v = Cx\left(v^2+1\right);\quad y = C\left(x^2+y^2\right)$

12. $xvv' = \sqrt{v^2+4};\quad \int \frac{v\,dv}{\sqrt{v^2+4}} = \int \frac{dx}{x};\quad \sqrt{v^2+4} = \ln x + C$

$v^2 + 4 = \left(\ln x + C\right)^2;\quad 4x^2 + y^2 = x^2\left(\ln x + C\right)^2$

13. $xv' = \sqrt{v^2+1};\quad \int \frac{dv}{\sqrt{v^2+1}} = \int \frac{dx}{x};\quad \ln\left(v + \sqrt{v^2+1}\right) = \ln x + \ln C$

$v + \sqrt{v^2+1} = Cx;\quad y + \sqrt{x^2+y^2} = Cx^2$

14. $xvv' = \sqrt{1+v^2} - \left(1+v^2\right)$

$$\ln x = \int \frac{v\,dv}{\sqrt{1+v^2} - (1+v^2)}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}\left(1-\sqrt{u}\right)} \qquad (u = 1+v^2)$$

$$= -\int \frac{dw}{w} = -\ln w + \ln C$$

with $w = 1 - \sqrt{u}$. Back-substitution and simplification finally yields the implicit solution $x - \sqrt{x^2+y^2} = C$.

15. $x(v+1)v' = -2\left(v^2+2v\right);\quad \int \frac{2(v+1)\,dv}{v^2+2v} = -\int \frac{4\,dx}{x};\quad \ln\left(v^2+2v\right) = -4\ln x + \ln C$

$v^2 + 2v = C/x^4;\quad x^2y^2 + 2x^3y = C$

16. The substitution $v = x + y + 1$ leads to

$$x = \int \frac{dv}{1+\sqrt{v}} = \int \frac{2u\,du}{1+u} \qquad (v = u^2)$$
$$= 2u - 2\ln(1+u) + C$$
$$x = 2\sqrt{x+y+1} - 2\ln(1+\sqrt{x+y+1}) + C$$

17. $v = 4x + y;\quad v' = v^2 + 4;\quad x = \int \frac{dv}{v^2+4} = \frac{1}{2}\tan^{-1}\frac{v}{2} + \frac{C}{2}$

$v = 2\tan(2x - C);\quad y = 2\tan(2x - C) - 4x$

18. $v = x + y;\quad v v' = v + 1;\quad x = \int \frac{v\,dv}{v+1} = \int \left(1 - \frac{1}{v+1}\right) dv = v - \ln(v+1) - C$

$y = \ln(x + y + 1) + C.$

Problems 19–25 are Bernoulli equations. For each, we indicate the appropriate substitution as specified in Equation (10) of this section, the resulting linear differential equation in v, its integrating factor ρ, and finally the resulting solution of the original Bernoulli equation.

19. $v = y^{-2};\quad v' - 4v/x = -10/x^2;\quad \rho = 1/x^4;\quad y^2 = x/\left(Cx^5 + 2\right)$

20. $v = y^3;\quad v' + 6xv = 18x;\quad \rho = e^{3x^2};\quad y^3 = 3 + Ce^{-3x^2}$

21. $v = y^{-2};\quad v' + 2v = -2;\quad \rho = e^{2x};\quad y^2 = 1/\left(Ce^{-2x} - 1\right)$

22. $v = y^{-3};\quad v' - 6v/x = -15/x^2;\quad \rho = x^{-6};\quad y^3 = 7x/\left(7Cx^7 + 15\right)$

23. $v = y^{-1/3};\quad v' - 2v/x = -1;\quad \rho = x^{-2};\quad y = \left(x + Cx^2\right)^{-3}$

24. $v = y^{-2};\quad v' + 2v = e^{-2x}/x;\quad \rho = e^{2x};\quad y^2 = e^{2x}/(C + \ln x)$

25. $v = y^3;\quad v' + 3v/x = 3/\sqrt{1+x^4};\quad \rho = x^3;\quad y^3 = \left(C + 3\sqrt{1+x^4}\right)/\left(2x^3\right)$

26. The substitution $v = y^3$ yields the linear equation $v' + v = e^{-x}$ with integrating factor $\rho = e^x$. Solution: $y^3 = e^{-x}(x + C)$

27. The substitution $v = y^3$ yields the linear equation $x v' - v = 3x^4$ with integrating factor $\rho = 1/x$. Solution: $y = (x^4 + Cx)^{1/3}$

28. The substitution $v = e^y$ yields the linear equation $x\,v' - 2v = 2x^3e^{2x}$ with integrating factor $\rho = 1/x^2$. Solution: $y = \ln(C\,x^2 + x^2e^{2x})$

29. The substitution $v = \sin y$ yields the homogeneous equation $2xv\,v' = 4x^2 + v^2$. Solution: $\sin^2 y = 4x^2 - C\,x$

30. First we multiply each side of the given equation by e^y. Then the substitution $v = e^y$ gives the homogeneous equation $(x + v)\,v' = x - v$ of Problem 1 above. Solution: $x^2 - 2x\,e^y - e^{2y} = C$

Each of the differential equations in Problems 31–42 is of the form $M\,dx + N\,dy = 0$, and the exactness condition $\partial M / \partial y = \partial N / \partial x$ is routine to verify. For each problem we give the principal steps in the calculation corresponding to the method of Example 9 in this section.

31. $F = \int(2x+3y)\,dx = x^2+3xy+g(y);\quad F_y = 3x+g'(y) = 3x+2y = N$

$g'(y) = 2y;\quad g(y) = y^2;\qquad x^2+3xy+y^2 = C$

32. $F = \int(4x-y)\,dx = 2x^2-xy+g(y);\quad F_y = -x+g'(y) = 6y-x = N$

$g'(y) = 6y;\quad g(y) = 3y^2;\qquad x^2-xy+3y^2 = C$

33. $F = \int(3x^2+2y^2)\,dx = x^3+xy^2+g(y);\quad F_y = 4xy+g'(y) = 4xy+6y^2 = N$

$g'(y) = 6y^2;\quad g(y) = 2y^3;\qquad x^3+2xy^2+2y^3 = C$

34. $F = \int(2xy^2+3x^2)\,dx = x^3+x^2y^2+g(y);\quad F_y = 2x^2y+g'(y) = 2x^2y+4y^3 = N$

$g'(y) = 4y^3;\quad g(y) = y^4;\qquad x^3+x^2y^2+y^4 = C$

35. $F = \int(x^3+y/x)\,dx = \frac{1}{4}x^4+y\ln x+g(y);\quad F_y = \ln x+g'(y) = y^2+\ln x = N$

$g'(y) = y^2;\quad g(y) = \frac{1}{3}y^3;\qquad \frac{1}{4}x^3+\frac{1}{3}y^2+y\ln x = C$

36. $F = \int(1+ye^{xy})\,dx = x+e^{xy}+g(y);\quad F_y = xe^{xy}+g'(y) = 2y+xe^{xy} = N$

$g'(y) = 2y;\quad g(y) = y^2;\qquad x+e^{xy}+y^2 = C$

37. $F = \int(\cos x+\ln y)\,dx = \sin x+x\ln y+g(y);\quad F_y = x/y+g'(y) = x/y+e^y = N$

$g'(y) = e^y;\quad g(y) = e^y;\qquad \sin x+x\ln y+e^y = C$

38. $F = \int (x + \tan^{-1} y)\,dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + g(y); \quad F_y = \dfrac{x}{1+y^2} + g'(y) = \dfrac{x+y}{1+y^2} = N$

$g'(y) = \dfrac{y}{1+y^2}; \quad g(y) = \tfrac{1}{2}\ln(1+y^2); \qquad \tfrac{1}{2}x^2 + x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) = C$

39. $F = \int (3x^2y^3 + y^4)\,dx = x^3y^3 + x\,y^4 + g(y);$

$F_y = 3x^3y^2 + 4xy^3 + g'(y) = 3x^3y^2 + y^4 + 4xy^3 = N$

$g'(y) = y^4; \quad g(y) = \tfrac{1}{5}y^5; \qquad x^3y^3 + xy^4 + \tfrac{1}{5}y^5 = C$

40. $F = \int (e^x \sin y + \tan y)\,dx = e^x \sin y + x\tan y + g(y);$

$F_y = e^x \cos y + x\sec^2 y + g'(y) = e^x \cos y + x\sec^2 y = N$

$g'(y) = 0; \quad g(y) = 0; \qquad e^x \sin y + x\tan y = C$

41. $F = \int \left(\dfrac{2x}{y} - \dfrac{3y^2}{x^4}\right)dx = \dfrac{x^2}{y} + \dfrac{y^2}{x^3} + g(y);$

$F_y = -\dfrac{x^2}{y^2} + \dfrac{2y}{x^3} + g'(y) = -\dfrac{x^2}{y^2} + \dfrac{2y}{x^3} + \dfrac{1}{\sqrt{y}} = N$

$g'(y) = \dfrac{1}{\sqrt{y}}; \quad g(y) = 2\sqrt{y}; \qquad \dfrac{x^2}{y} + \dfrac{y^2}{x^3} + 2\sqrt{y} = C$

42. $F = \int \left(y^{-2/3} - \dfrac{3}{2}x^{-5/2}y\right)dx = x\,y^{-2/3} + x^{-3/2}y + g(y);$

$F_y = -\dfrac{2}{3}x\,y^{-5/3} + x^{-3/2} + g'(y) = x^{-3/2} - \dfrac{2}{3}x\,y^{-5/3} = N$

$g'(y) = 0; \quad g(y) = 0; \qquad x\,y^{-2/3} + x^{-3/2}y = C$

43. The substitution $y' = p,\ y'' = p'$ in $xy'' = y'$ yields

$$\begin{aligned}
xp' &= p, \qquad \text{(separable)}\\
\int \frac{dp}{p} &= \int \frac{dx}{x} \quad \Rightarrow \quad \ln p = \ln x + \ln C,\\
y' &= p = Cx,\\
y(x) &= \tfrac{1}{2}Cx^2 + B = Ax^2 + B.
\end{aligned}$$

44. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = 0$ yields

$$
\begin{aligned}
ypp' + p^2 &= 0 \quad \Rightarrow \quad yp' = -p, \qquad \text{(separable)} \\
\int \frac{dp}{p} &= -\int \frac{dy}{y} \quad \Rightarrow \quad \ln p = -\ln y + \ln C, \\
p &= C/y \quad \Rightarrow \quad x = \int \frac{1}{p}\,dy = \int \frac{y}{C}\,dy \\
x(y) &= \frac{y^2}{2C} + B = Ay^2 + B.
\end{aligned}
$$

45. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' + 4y = 0$ yields

$$
\begin{aligned}
pp' + 4y &= 0, \qquad \text{(separable)} \\
\int p\,dp &= -\int 4y\,dy \quad \Rightarrow \quad \tfrac{1}{2}p^2 = -2y^2 + C, \\
p^2 &= -4y^2 + 2C = 4\left(\tfrac{1}{2}C - y^2\right), \\
x &= \int \frac{1}{p}\,dy = \int \frac{dy}{2\sqrt{k^2 - y^2}} = \frac{1}{2}\sin^{-1}\frac{y}{k} + D, \\
y(x) &= k\sin[2x - 2D] = k(\sin 2x \cos 2D - \cos 2x \sin 2D), \\
y(x) &= A\cos 2x + B\sin 2x.
\end{aligned}
$$

46. The substitution $y' = p,\ y'' = p'$ in $xy'' + y' = 4x$ yields

$$
\begin{aligned}
xp' + p &= 4x, \qquad \text{(linear in } p\text{)} \\
D_x[x \cdot p] &= 4x \quad \Rightarrow \quad x \cdot p = 2x^2 + A, \\
p &= \frac{dy}{dx} = 2x + \frac{A}{x}, \\
y(x) &= x^2 + A\ln x + B.
\end{aligned}
$$

47. The substitution $y' = p,\ y'' = p'$ in $y'' = (y')^2$ yields

$$
\begin{aligned}
p' &= p^2, \qquad \text{(separable)} \\
\int \frac{dp}{p^2} &= \int x\,dx \quad \Rightarrow \quad -\frac{1}{p} = x + B, \\
\frac{dy}{dx} &= -\frac{1}{x + B}, \\
y(x) &= A - \ln|x + A|.
\end{aligned}
$$

48. The substitution $y' = p,\ y'' = p'$ in $x^2y'' + 3xy' = 2$ yields

$$\begin{aligned} x^2p' + 3xp &= 2 \quad \Rightarrow \quad p' + \frac{3}{p}p = \frac{2}{x^2}, \quad \text{(linear in } p) \\ D_x[x^3 \cdot p] &= 2x \quad \Rightarrow \quad x^3 \cdot p = x^2 + C, \\ \frac{dy}{dx} &= \frac{1}{x} + \frac{C}{x^3}, \\ y(x) &= \ln x + \frac{A}{x^2} + B. \end{aligned}$$

49. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = yy'$ yields

$$\begin{aligned} yp\,p' + p^2 &= yp \quad \Rightarrow \quad y\,p' + p = y \quad \text{(linear in } p), \\ D_y[y \cdot p] &= y, \\ yp &= \frac{1}{2}y^2 + \frac{1}{2}C \quad \Rightarrow \quad p = \frac{y^2 + C}{2y}, \\ x &= \int \frac{1}{p}dy = \int \frac{2y\,dy}{y^2 + C} = \ln\left(y^2 + C\right) - \ln B, \\ y^2 + C &= Be^x \quad \Rightarrow \quad y(x) = \pm\left(A + Be^x\right)^{1/2}. \end{aligned}$$

50. The substitution $y' = p,\ y'' = p'$ in $y'' = (x + y')^2$ gives $p' = (x + p)^2$, and then the substitution $v = x + p,\ p' = v' - 1$ yields

$$\begin{aligned} v' - 1 &= v^2 \quad \Rightarrow \quad \frac{dv}{dx} = 1 + v^2, \\ \int \frac{dv}{1 + v^2} &= \int dx \quad \Rightarrow \tan^{-1} v = x + A, \\ x + y' &= v = \tan(x + A) \quad \Rightarrow \quad \frac{dy}{dx} = \tan(x + A) - x, \\ y(x) &= \ln\left|\sec(x + A)\right| - \tfrac{1}{2}x^2 + B. \end{aligned}$$

51. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2y(y')^3$ yields

$$\begin{aligned} p\,p' &= 2yp^3 \quad \Rightarrow \quad \int \frac{dp}{p^2} = \int 2y\,dy \quad \Rightarrow -\frac{1}{p} = y^2 + C, \\ x &= \int \frac{1}{p}dy = -\frac{1}{3}y^3 - Cx + D, \\ y^3 + 3x + Ay + B &= 0 \end{aligned}$$

52. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y^3 y'' = 1$ yields

$$y^3 p\,p' = 1 \quad\Rightarrow\quad \int p\,dp = \int \frac{dy}{y^3} \quad\Rightarrow\quad \frac{1}{2}p^2 = -\frac{1}{2y^2} + \frac{A}{2},$$

$$p^2 = \frac{Ay^2 - 1}{y^2} \quad\Rightarrow\quad x = \int \frac{1}{p}\,dy = \int \frac{y\,dy}{\sqrt{Ay^2 - 1}},$$

$$x = \frac{1}{A}\sqrt{Ay^2 - 1} + C \quad\Rightarrow\quad Ax + B = \sqrt{Ay^2 - 1},$$

$$Ay^2 - (Ax + B)^2 = 1.$$

53. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2yy'$ yields

$$p\,p' = 2yp \quad\Rightarrow\quad \int dp = \int 2y\,dy \quad\Rightarrow\quad p = y^2 + A^2,$$

$$x = \int \frac{1}{p}\,dy = \int \frac{dy}{y^2 + A^2} = \frac{1}{A}\tan^{-1}\frac{y}{A} + C,$$

$$\tan^{-1}\frac{y}{A} = A(x - C) \quad\Rightarrow\quad \frac{y}{A} = \tan(Ax - AC),$$

$$y(x) = A\tan(Ax + B).$$

54. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' = 3\left(y'\right)^2$ yields

$$yp\,p' = 3p^2 \quad\Rightarrow\quad \int \frac{dp}{p} = \int \frac{3\,dy}{y}$$

$$\ln p = 3\ln y + \ln C \quad\Rightarrow\quad p = Cy^3,$$

$$x = \int \frac{1}{p}\,dy = \int \frac{dy}{Cy^3} = -\frac{1}{2Cy^2} + B,$$

$$Ay^2(B - x) = 1.$$

55. The substitution $v = ax + by + c,\ y = (v - ax - c)/b$ in $y' = F(ax + by + c)$ yields the separable differential equation $(dv/dx - a)/b = F(v)$, that is, $dv/dx = a + b\,F(v)$.

56. If $v = y^{1-n}$ then $y = v^{1/(1-n)}$ so $y' = v^{n/(1-n)} v'/(1-n)$. Hence the given Bernoulli equation transforms to

$$\frac{v^{n/(1-n)}}{1-n}\frac{dv}{dx} + P(x)\,v^{1/(1-n)} = Q(x)\,v^{n/(1-n)}.$$

Multiplication by $(1-n)/v^{n/(1-n)}$ then yields the linear differential equation $v' + (1-n)Pv = (1-n)Qv$.

57. If $v = \ln y$ then $y = e^v$ so $y' = e^v v'$. Hence the given equation transforms to $e^v v' + P(x)\,e^v = Q(x)\,v\,e^v$. Cancellation of the factor e^v then yields the linear differential equation $v' - Q(x)\,v = P(x)$.

58. The substitution $v = \ln y$, $y = e^v$, $y' = e^v\,v'$ yields the linear equation $x\,v' + 2\,v = 4x^2$ with integrating factor $\rho = x^2$. Solution: $y = \exp(x^2 + C/x^2)$

59. The substitution $x = u - 1$, $y = v - 2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{u-v}{u+v}.$$

The substitution $v = pu$ leads to

$$\ln u = -\int \frac{(p+1)\,dp}{(p^2+2p-1)} = -\frac{1}{2}\left[\ln\left(p^2+2p-1\right) - \ln C\right].$$

We thus obtain the implicit solution

$$u^2\left(p^2+2p-1\right) = C$$

$$u^2\left(\frac{v^2}{u^2} + 2\frac{v}{u} - 1\right) = v^2 + 2uv - u^2 = C$$

$$(y+2)^2 + 2(x+1)(y+2) - (x+1)^2 = C$$

$$y^2 + 2xy - x^2 + 2x + 6y = C.$$

60. The substitution $x = u - 3$, $y = v - 2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{-u+2v}{4u-3v}.$$

The substitution $v = pu$ leads to

$$\ln u = \int \frac{(4-3p)\,dp}{(3p+1)(p-1)} = \frac{1}{4}\int\left(\frac{1}{p-1} - \frac{15}{3p+1}\right)dp$$

$$= \frac{1}{4}\left[\ln(p-1) - 5\ln(3p+1) + \ln C\right].$$

We thus obtain the implicit solution

$$\begin{aligned}
u^4 &= \frac{C(p-1)}{(3p+1)^5} = \frac{C(v/u-1)}{(3v/u+1)^5} = \frac{Cu^4(v-u)}{(3v+u)^5} \\
(3v+u)^5 &= C(v-u) \\
(x+3y+3)^5 &= C(y-x-5).
\end{aligned}$$

61. The substitution $v = x-y$ yields the separable equation $v' = 1-\sin v$. With the aid of the identity

$$\frac{1}{1-\sin v} = \frac{1+\sin v}{\cos^2 v} = \sec^2 v + \sec v \tan v$$

we obtain the solution

$$x = \tan(x-y) + \sec(x-y) + C.$$

62. The substitution $y = vx$ in the given homogeneous differential equation yields the separable equation $x\left(2v^3-1\right)v' = -\left(v^4+v\right)$ that we solve as follows:

$$\begin{aligned}
\int \frac{2v^3-1}{v^4+v}\,dv &= -\int \frac{dx}{x} \\
\int \left(\frac{2v-1}{v^2-v+1} - \frac{1}{v} + \frac{1}{v+1}\right) dv &= -\int \frac{dx}{x} \qquad \text{(partial fractions)} \\
\ln(v^2-v+1) - \ln v + \ln(v+1) &= -\ln x + \ln C \\
x(v^2-v+1)(v+1) &= Cv \\
(y^2-xy+x^2)(x+y) &= Cxy \\
x^3+y^3 &= Cxy
\end{aligned}$$

63. If we substitute $y = y_1 + 1/v$, $y' = y_1' - v'/v^2$ (primes denoting differentiation with respect to x) into the Riccati equation $y' = Ay^2 + By + C$ and use the fact that $y_1' = Ay_1^2 + By_1 + C$, then we immediately get the linear differential equation $v' + (B + 2Ay_1)v = -A$.

In Problems 64 and 65 we outline the application of the method of Problem 63 to the given Riccati equation.

64. The substitution $y = x + 1/v$ yields the linear equation $v' - 2xv = 1$ with integrating factor $\rho = e^{-x^2}$. In Problem 29 of Section 1.5 we saw that the general solution of this linear equation is $v(x) = e^{x^2}\left[C+\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]$ in terms of the *error function* erf(x) introduced there. Hence the general solution of our Riccati equation is given by $y(x) = x+e^{-x^2}\left[C+\frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]^{-1}$.

65. The substitution $y = x+1/v$ yields the trivial linear equation $v' = -1$ with immediate solution $v(x) = C-x$. Hence the general solution of our Riccati equation is given by $y(x) = x+1/(C-x)$.

66. The substitution $y' = C$ in the Clairaut equation immediately yields the general solution $y = Cx + g(C)$.

67. Clearly the line $y = Cx - C^2/4$ and the tangent line at $(C/2, C^2/4)$ to the parabola $y = x^2$ both have slope C.

68. $\ln\left(v+\sqrt{1+v^2}\right) = -k\ln x + k\ln a = \ln\left(x/a\right)^{-k}$

$$v+\sqrt{1+v^2} = \left(x/a\right)^{-k}$$

$$\left[\left(x/a\right)^{-k}-v\right]^2 = 1+v^2$$

$$\left(x/a\right)^{-2k}-2v\left(x/a\right)^{-k}+v^2 = 1+v^2$$

$$v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-2k}-1\right]/\left(\frac{x}{a}\right)^{-k} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k}-\left(\frac{x}{a}\right)^{k}\right]$$

69. With $a = 100$ and $k = 1/10$, Equation (19) in the text is

$$y = 50[(x/100)^{9/10} - (x/100)^{11/10}].$$

The equation $y'(x) = 0$ then yields

$$(x/100)^{1/10} = (9/11)^{1/2},$$

so it follows that

$$y_{\max} = 50[(9/11)^{9/2} - (9/11)^{11/2}] \approx 3.68 \text{ mi.}$$

70. With $k = w/v_0 = 10/500 = 1/10$, Eq. (16) in the text gives

$$\ln\left(v+\sqrt{1+v^2}\right) = -\frac{1}{10}\ln x + C$$

where $v = y/x$. Substitution of $x = 200, y = 150, v = 3/4$ yields $C = \ln\left(2 \cdot 200^{1/10}\right)$, thence

$$\ln\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = -\frac{1}{10}\ln x + \ln\left(2 \cdot 200^{1/10}\right),$$

which — after exponentiation and then multiplication of the resulting equation by x — simplifies as desired to $y + \sqrt{x^2 + y^2} = 2\left(200x^9\right)^{1/10}$. If $x = 0$ then this equation yields $y = 0$, thereby verifying that the airplane reaches the airport at the origin.

71. **(a)** With $a = 100$ and $k = w/v_0 = 2/4 = 1/2$, the solution given by equation (19) in the textbook is $y(x) = 50[(x/100)^{1/2} - (x/100)^{3/2}]$. The fact that $y(0) = 0$ means that this trajectory goes through the origin where the tree is located.

(b) With $k = 4/4 = 1$ the solution is $y(x) = 50[1 - (x/100)^2]$ and we see that the swimmer hits the bank at a distance $y(0) = 50$ north of the tree.

(c) With $k = 6/4 = 3/2$ the solution is $y(x) = 50[(x/100)^{-1/2} - (x/100)^{5/2}]$. This trajectory is asymptotic to the positive x-axis, so we see that the swimmer never reaches the west bank of the river.

72. The substitution $y' = p,\ y'' = p'$ in $ry'' = [1 + (y')^2]^{3/2}$ yields

$$rp' = (1 + p^2)^{3/2} \quad \Rightarrow \quad \int \frac{r\,dp}{(1 + p^2)^{3/2}} = \int dx.$$

Now integral formula #52 in the back of our favorite calculus textbook gives

$$\frac{rp}{\sqrt{1 + p^2}} = x - a \quad \Rightarrow \quad r^2 p^2 = (1 + p^2)(x - a)^2,$$

and we solve readily for

$$p^2 = \frac{(x - a)^2}{r^2 - (x - a)^2} \quad \Rightarrow \quad \frac{dy}{dx} = p = \frac{x - a}{\sqrt{r^2 - (x - a)^2}},$$

whence

$$y = \int \frac{(x - a)\,dx}{\sqrt{r^2 - (x - a)^2}} = -\sqrt{r^2 - (x - a)^2} + b,$$

which finally gives $(x - a)^2 + (y - b)^2 = r^2$ as desired.

SECTION 1.7

POPULATION MODELS

Section 1.7 introduces the first of the two major classes of mathematical models studied in the textbook, and is a prerequisite to the discussion of equilibrium solutions and stability in Section 7.1.

In Problems 1–8 we outline the derivation of the desired particular solution, and then sketch some typical solution curves.

1. Noting that $x > 1$ because $x(0) = 2$, we write

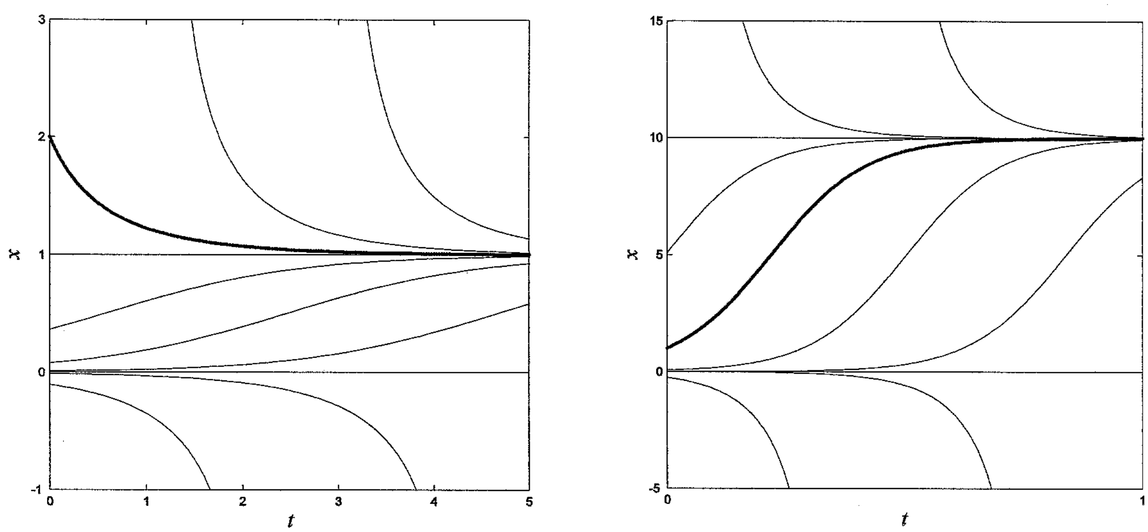
$$\int \frac{dx}{x(1-x)} = \int 1\,dt; \qquad \int\left(\frac{1}{x} - \frac{1}{x-1}\right)dx = \int 1\,dt$$

$$\ln x - \ln(x-1) = t + \ln C; \qquad \frac{x}{x-1} = Ce^t$$

$$x(0) = 2 \text{ implies } C = 2; \qquad x = 2(x-1)e^t$$

$$x(t) = \frac{2e^t}{2e^t - 1} = \frac{2}{2-e^{-t}}.$$

Typical solution curves are shown in the figure on the left below.



2. Noting that $x < 10$ because $x(0) = 1$, we write

$$\int \frac{dx}{x(10-x)} = \int 1\,dt; \qquad \int\left(\frac{1}{x} + \frac{1}{10-x}\right)dx = \int 10\,dt$$

$$\ln x - \ln(10-x) = 10t + \ln C; \qquad \frac{x}{10-x} = Ce^{10t}$$

$$x(0)=1 \text{ implies } C = \frac{1}{9}; \qquad 9x = (10-x)e^{10t}$$

$$x(t) = \frac{10e^{10t}}{9+e^{10t}} = \frac{10}{1+9e^{-10t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

3. Noting that $x > 1$ because $x(0) = 3$, we write

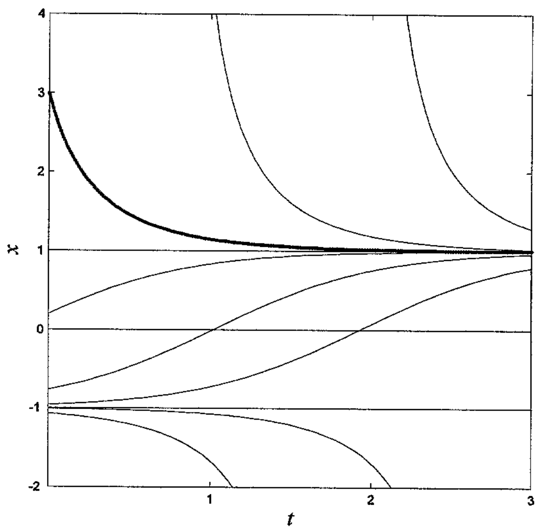
$$\int \frac{dx}{(1+x)(1-x)} = \int 1\, dt; \qquad \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \int (-2)\, dt$$

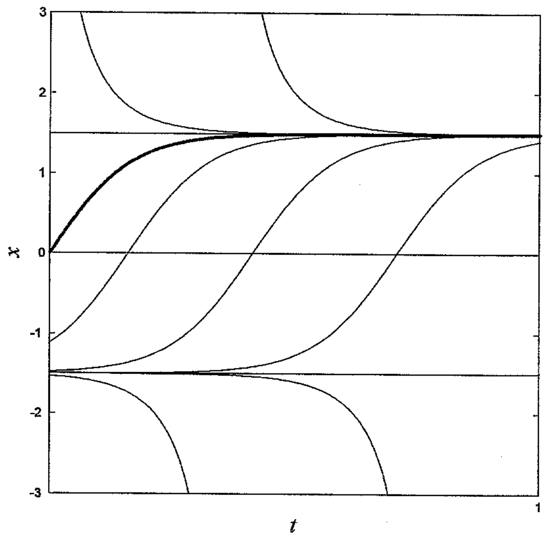
$$\ln(x-1) - \ln(x+1) = -2t + \ln C; \qquad \frac{x-1}{x+1} = Ce^{-2t}$$

$$x(0)=3 \text{ implies } C = \frac{1}{2}; \qquad 2(x-1) = (x+1)e^{-2t}$$

$$x(t) = \frac{2+e^{-2t}}{2-e^{-2t}} = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

Typical solution curves are shown in the figure on the left below.





4. Noting that $|x| < \frac{3}{2}$ because $x(0) = 0$, we write

$$\int \frac{dx}{(3+2x)(3-2x)} = \int 1\,dt; \qquad \int\left(\frac{1}{3+2x}+\frac{1}{3-2x}\right)dx = \int 6\,dt$$

$$\frac{1}{2}\ln(3+2x)-\frac{1}{2}\ln(3-2x) = 6t+\frac{1}{2}\ln C; \qquad \frac{3+2x}{3-2x} = Ce^{12t}$$

$$x(0)=0 \text{ implies } C=1; \qquad 3+2x = (3-2x)e^{12t}$$

$$x(t) = \frac{3e^{12t}-3}{2e^{12t}+2} = \frac{3\left(e^{12t}-1\right)}{2\left(e^{12t}+1\right)}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

5. Noting that $x>5$ because $x(0)=8$, we write

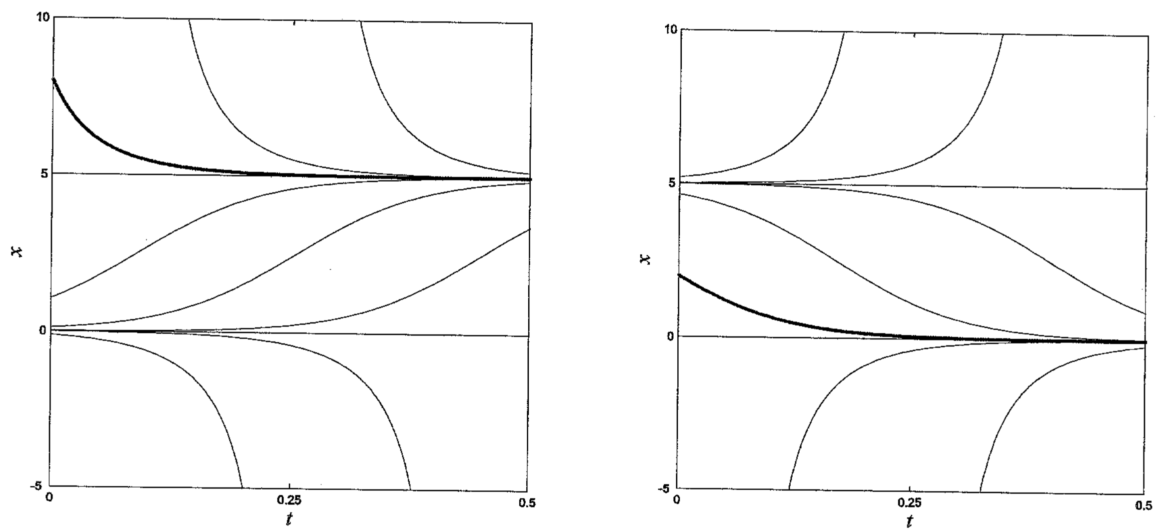
$$\int \frac{dx}{x(x-5)} = \int(-3)\,dt; \qquad \int\left(\frac{1}{x}-\frac{1}{x-5}\right)dx = \int 15\,dt$$

$$\ln x-\ln(x-5) = 15t+\ln C; \qquad \frac{x}{x-5} = Ce^{15t}$$

$$x(0)=8 \text{ implies } C=8/3; \qquad 3x = 8(x-5)e^{15t}$$

$$x(t) = \frac{-40e^{15t}}{3-8e^{15t}} = \frac{40}{8-3e^{-15t}}.$$

Typical solution curves are shown in the figure on the left below.



6. Noting that $x<5$ because $x(0)=2$, we write

$$\int \frac{dx}{x(5-x)} = \int (-3)\,dt; \qquad \int \left(\frac{1}{x}+\frac{1}{5-x}\right)dx = \int (-15)\,dt$$

$$\ln x - \ln(5-x) = -15t + \ln C; \qquad \frac{x}{5-x} = Ce^{-15t}$$

$$x(0)=2 \text{ implies } C = 2/3; \qquad 3x = 2(5-x)e^{-15t}$$

$$x(t) = \frac{10e^{-15t}}{3+2e^{-15t}} = \frac{10}{2+3e^{15t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

7. Noting that $x > 7$ because $x(0)=11$, we write

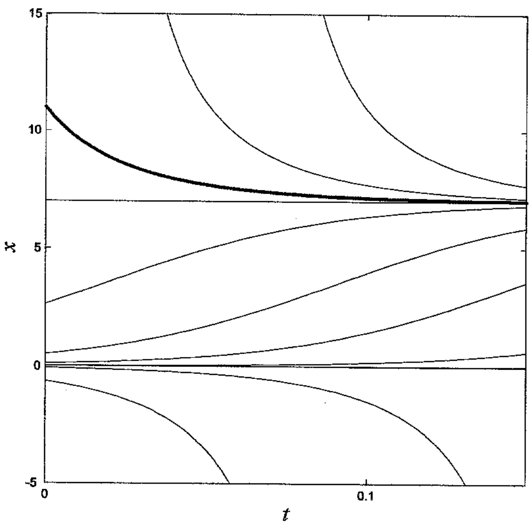
$$\int \frac{dx}{x(x-7)} = \int (-4)\,dt; \qquad \int \left(\frac{1}{x}-\frac{1}{x-7}\right)dx = \int 28\,dt$$

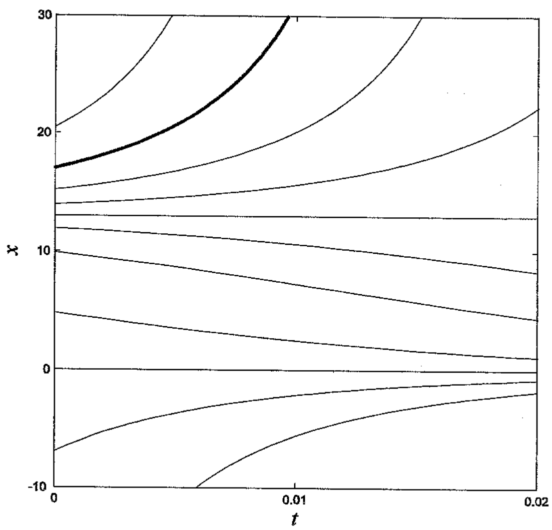
$$\ln x - \ln(x-7) = 28t + \ln C; \qquad \frac{x}{x-7} = Ce^{28t}$$

$$x(0)=11 \text{ implies } C = 11/4; \qquad 4x = 11(x-17)e^{28t}$$

$$x(t) = \frac{-77e^{28t}}{4-11e^{28t}} = \frac{77}{11-4e^{-28t}}.$$

Typical solution curves are shown in the figure on the left below.





8. Noting that $x > 13$ because $x(0)=17$, we write

$$\int \frac{dx}{x(x-13)} = \int 7\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-13}\right) dx = \int (-91)\,dt$$

$$\ln x - \ln(x-13) = -91t + \ln C; \qquad \frac{x}{x-13} = C e^{-91t}$$

$$x(0) = 17 \text{ implies } C = 17/4; \qquad 4x = 17(x-13)e^{-91t}$$

$$x(t) = \frac{-221 e^{-91t}}{4 - 17 e^{-91t}} = \frac{221}{17 - 4e^{91t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

9. Substitution of $P(0) = 100$ and $P'(0) = 20$ into $P' = k\sqrt{P}$ yields $k = 2$, so the differential equation is $P' = 2\sqrt{P}$. Separation of variables and integration, $\int dP/2\sqrt{P} = \int dt$, gives $\sqrt{P} = t + C$. Then $P(0) = 100$ implies $C = 10$, so $P(t) = (t+10)^2$. Hence the number of rabbits after one year is $P(12) = 484$.

10. Given $P' = -\delta P = -(k/\sqrt{P})P = -k\sqrt{P}$, separation of variables and integration as in Problem 9 yields $2\sqrt{P} = -kt + C$. The initial condition $P(0) = 900$ gives $C = 60$, and then the condition $P(6) = 441$ implies that $k = 3$. Therefore $2\sqrt{P} = -3t + 60$, so $P = 0$ after $t = 20$ weeks.

11. **(a)** Starting with $dP/dt = k\sqrt{P}$, $dP/dt = k\sqrt{P}$, we separate the variables and integrate to get $P(t) = (kt/2 + C)^2$. Clearly $P(0) = P_0$ implies $C = \sqrt{P_0}$.

(b) If $P(t) = (kt/2 + 10)^2$, then $P(6) = 169$ implies that $k = 1$. Hence $P(t) = (t/2 + 10)^2$, so there are 256 fish after 12 months.

12. Solution of the equation $P' = kP^2$ by separation of variables and integration,

$$\int \frac{dP}{P^2} = \int k\,dt; \quad -\frac{1}{P} = kt - C,$$

gives $P(t) = 1/(C - kt)$. Now $P(0) = 12$ implies that $C = 1/12$, so now $P(t) = 12/(1-12kt)$. Then $P(10) = 24$ implies that $k = 1/240$, so finally $P(t) = 240/(20-t)$. Hence $P = 48$ when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$.

13. **(a)** If the birth and death rates both are proportional to P^2 and $\beta > \delta$, then Eq. (1) in this section gives $P' = kP^2$ with k positive. Separating variables and integrating as in Problem 12, we find that $P(t) = 1/(C - kt)$. The initial condition $P(0) = P_0$ then gives $C = 1/P_0$, so $P(t) = 1/(1/P_0 - kt) = P_0/(1 - kP_0 t)$.

(b) If $P_0 = 6$ then $P(t) = 6/(1 - 6kt)$. Now the fact that $P(10) = 9$ implies that $k = 180$, so $P(t) = 6/(1 - t/30) = 180/(30 - t)$. Hence it is clear that $P \to \infty$ as $t \to 30$ (doomsday).

14. Now $dP/dt = -kP^2$ with $k > 0$, and separation of variables yields $P(t) = 1/(kt + C)$. Clearly $C = 1/P_0$ as in Problem 13, so $P(t) = P_0/(1 + kP_0 t)$. Therefore it is clear that $P(t) \to 0$ as $t \to \infty$, so the population dies out in the long run.

15. If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0 P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 16 and 17) that $a = B_0/P_0$ and $b = D_0/P_0^2 = k$.

16. The relations in Problem 15 give $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120 + 40e^{-t/15})$. We find that $P = 0.95M$ after about 27.69 months.

17. The relations in Problem 15 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240 - 60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months.

18. If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0 P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 19 and 20) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$.

19. The relations in Problem 18 give $k = 1/1000$ and $M = 90$. The solution is $P(t) = 9000/(100 - 10e^{9t/100})$. We find that $P = 10M$ after about 24.41 months.

20. The relations in Problem 18 give $k = 1/1100$ and $M = 120$. The solution is $P(t) = 13200/(110 + 10e^{6t/55})$. We find that $P = 0.1M$ after about 42.12 months.

21. Starting with the differential equation $dP/dt = kP(200 - P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 200k\,dt;$$

$$\ln \frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0) = 100$ gives $C = 1$, and $P'(0) = 1$ implies that $1 = k \cdot 100(200-100)$, so we find that $k = 1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t) = 200/\left(1+e^{-t/50}\right)$. Finally, $P(60) = 200/\left(1+e^{-6/5}\right) \approx 153.7$ million.

22. We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100-50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50 \times 100}{50 + (100-50)e^{-0.04t}},$$

and we solve this equation for $t \approx 34.66$. Thus the rumor will have spread to 80% of the population in a little less than 35 days.

23. **(a)** $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50 + 150e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

24. The differential equation for $N(t)$ is $N'(t) = kN(15-N)$. When we substitute $N(0) = 5$ (thousands) and $N'(0) = 0.5$ (thousands/day) we find that $k = 0.01$. With N in place of P, this is the logistic equation in Eq. (3) of the text, so its solution is given by Equation (7):

$$N(t) = \frac{15 \times 5}{5 + 10 \exp[-(0.01)(15)t]} = \frac{15}{1 + 2\,e^{-0.15t}}.$$

Upon substituting $N = 10$ on the left, we solve for $t = (\ln 4)/(0.15) \approx 9.24$ days.

25. Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M - 25.00) = 3/8, \qquad 47.54k(M - 47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Equation (4) gives the population function

$$P(t) = \frac{2500}{25 + 75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A. D.

26. The differential equation for $P(t)$ is

$$P'(t) = 0.001P^2 - \delta P.$$

When we substitute $P(0) = 100$ and $P'(0) = 8$ we find that $\delta = 0.02$, so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P - 20).$$

We separate variables and integrate, noting that $P > 20$ because $P_0 = 100$:

$$\int \frac{dP}{P(P-20)} = \int 0.001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-20} - \frac{1}{P} \right) dP = \int 0.02\,dt;$$

$$\ln \frac{P-20}{P} = \frac{1}{50}t + \ln C \quad \Rightarrow \quad \frac{P-20}{P} = Ce^{t/50}.$$

Now $P(0) = 100$ gives $C = 4/5$, hence

$$5(P - 20) = 4P\,e^{t/50} \quad \Rightarrow \quad P(t) = \frac{100}{5 - 4e^{t/50}}.$$

It follows readily that $P = 200$ when $t = 50 \ln(9/8) \approx 5.89$ months.

27. We are given that

$$P' = kP^2 - 0.01P,$$

When we substitute $P(0) = 200$ and $P'(0) = 2$ we find that $k = 0.0001$, so

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P-100).$$

We separate variables and integrate, noting that $P > 100$ because $P_0 = 200$:

$$\int \frac{dP}{P(P-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-100} - \frac{1}{P}\right) dP = \int 0.01\,dt;$$

$$\ln\frac{P-100}{P} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{P-100}{P} = Ce^{t/100}.$$

Now $P(0) = 100$ gives $C = 1/2$, hence

$$2(P-100) = Pe^{t/100} \quad \Rightarrow \quad P(t) = \frac{200}{2 - e^{t/100}}.$$

(a) $P = 1000$ when $t = 100\ln(9/5) \approx 58.78.$

(b) $P \to \infty$ as $t \to 100\ln 2 \approx 69.31.$

28. Our alligator population satisfies the equation

$$\frac{dP}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x-100).$$

With x in place of P, this is the same differential equation as in Problem 27, but now we use absolute values to allow both possibilities $x < 100$ and $x > 100$:

$$\int \frac{dx}{x(x-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{x-100} - \frac{1}{x}\right) dP = \int 0.01\,dt;$$

$$\ln\frac{|x-100|}{x} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{|x-100|}{x} = Ce^{t/100}. \qquad (*)$$

(a) If $x(0) = 25$ then $x < 100$ and $|x-100| = 100 - x$, so (*) gives $C = 3$ and hence

$$100 - x = 3xe^{t/100} \quad \Rightarrow \quad x(t) = \frac{100}{1 + 3e^{t/100}}.$$

We therefore see that $x(t) \to 0$ as $t \to \infty$.

(b) But if $x(0) = 150$ then $x > 100$ and $|x-100| = x-100$, so (*) gives $C = 1/3$ and hence

$$3(x-100) = xe^{t/100} \quad \Rightarrow \quad x(t) = \frac{300}{3-e^{t/100}}.$$

Now $x(t) \to +\infty$ as $t \to (100\ln 3)^-$, so doomsday occurs after about 109.86 months.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 = 0.0001489P(210.544 - P)$$

where $k = 0.0001489$ and $P = 210.544$. With $P_0 = 3.9$ also, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{(3.9) + (210.544 - 3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9 + 206.644e^{-0.03135t}}.$$

(a) This solution gives $P(140) \approx 127.008$, fairly close to the actual 1930 U.S. census population of 123.2 million.

(b) The limiting population as $t \to \infty$ is $821.122/3.9 = 210.544$ million.

(c) Since the actual U.S. population in 200 was about 281 million — already exceeding the maximum population predicted by the logistic equation — we see that that this model did *not* continue to hold throughout the 20th century.

30. The equation is separable, so we have

$$\int \frac{dP}{P} = \int \beta_0 e^{-\alpha t}\,dt, \quad \text{so} \quad \ln P = -\frac{\beta_0}{\alpha}e^{-\alpha t} + C.$$

The initial condition $P(0) = P_0$ gives $C = \ln P_0 + \beta_0/\alpha$, so

$$P(t) = P_0 \exp\left[\frac{\beta_0}{\alpha}\left(1 - e^{-\alpha t}\right)\right].$$

31. If we substitute $P(0) = 10^6$ and $P'(0) = 3\times 10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t}P,$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 30 is

$$P(t) = P_0 \exp[(0.3/\alpha)(1 - e^{-\alpha t})].$$

The fact that $P(6) = 2P_0$ now yields the equation

$$f(\alpha) = (0.3)(1 - e^{-6\alpha}) - \alpha \ln 2 = 0$$

for α. We apply Newton's iterative formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

with $f'(\alpha) = 1.8e^{-6\alpha} - \ln 2$ and initial guess $\alpha_0 = 1$, and find that $\alpha \approx 0.3915$. Therefore the limiting cell population as $t \to \infty$ is

$$P_0 \exp(\beta_0 / \alpha) = 10^6 \exp(0.3/0.3915) \approx 2.15 \times 10^6.$$

Thus the tumor does not grow much further after 6 months.

32. We separate the variables in the logistic equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(M-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int kM\,dt;$$

$$\ln \frac{P}{|M-P|} = kMt + \ln C \quad \Rightarrow \quad \frac{P}{|M-P|} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|M-P| = M-P$, so substitution of $t = 0,\ P = P_0$ in (*) gives $C = P_0/(M-P_0)$. It follows that

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|M-P| = P-M$, so substitution of $t = 0,\ P = P_0$ in (*) gives $C = P_0/(P_0 - M)$, and it follows that

$$\frac{P}{P-M} = \frac{P_0}{P_0 - M} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(M - P_0)P = (M-P)P_0 e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0 e^{kMt}}{(M-P_0) + P_0 e^{kMt}},$$

which gives the desired result upon division of numerator and denominator by e^{kMt}.

33. (a) We separate the variables in the extinction-explosion equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(P-M)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P-M} - \frac{1}{P}\right) dP = \int kM\,dt;$$

$$\ln\frac{|P-M|}{P} = kMt + \ln C \quad \Rightarrow \quad \frac{|P-M|}{P} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|P-M| = M-P$, so substitution of $t = 0,\ P = P_0$ in (*) gives $C = (M-P_0)/P_0$. It follows that

$$\frac{M-P}{P} = \frac{M-P_0}{P_0} e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|P-M| = P-M$, so substitution of $t = 0,\ P = P_0$ in (*) gives $C = (P_0 - M)/P_0$, and it follows that

$$\frac{P-M}{P} = \frac{P_0-M}{P_0} e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(P-M)P_0 = (P_0-M)P\,e^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0}{P_0 + (M-P_0)\,e^{kMt}}.$$

(b) If $P_0 < M$ then the coefficient $M - P_0$ is positive and the denominator increases without bound, so $P(t) \to 0$ as $t \to \infty$. But if $P_0 > M$, then the denominator $P_0 - (P_0 - M)\,e^{kMt}$ approaches zero — so $P(t) \to +\infty$ — as t approaches the value $(1/kM)\ln[P_0/(P_0 - M)] > 0$ from the left.

34. Differentiation of both sides of the logistic equation $P' = kP\cdot(M-P)$ yields

$$\begin{aligned} P'' &= \frac{dP'}{dP}\cdot\frac{dP}{dt} \\ &= [k\cdot(M-P) + kP\cdot(-1)]\cdot kP(M-P) \\ &= k[M-2P]\cdot kP(M-P) = 2k^2P(M-\tfrac{1}{2}P)(M-P) \end{aligned}$$

as desired. The conclusions that $P'' > 0$ if $0 < P < \frac{1}{2}M$, that $P'' = 0$ if $P = \frac{1}{2}M$, and that $P'' < 0$ if $\frac{1}{2}M < P < M$ are then immediate. Thus it follows that each of the curves for which $P_0 < M$ has an inflection point where it crosses the horizontal line $P = \frac{1}{2}M$.

35. Any way you look at it, you should see that, the larger the parameter $k > 0$ is, the faster the logistic population $P(t)$ approaches its limiting population M.

36. With $x = e^{-50kM}$, $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$, Eqs. (7) in the text take the form

$$\frac{P_0 M}{P_0 + (M - P_0)x} = P_1, \quad \frac{P_0 M}{P_0 + (M - P_0)x^2} = P_2$$

from which we get

$$P_0 + (M - P_0)x = P_0 M / P_1, \quad P_0 + (M - P_0)x^2 = P_0 M / P_2$$

$$x = \frac{P_0(M - P_1)}{P_1(M - P_0)}, \quad x^2 = \frac{P_0(M - P_2)}{P_2(M - P_0)} \qquad \text{(i)}$$

$$\frac{P_0^2(M - P_1)^2}{P_1^2(M - P_0)^2} = \frac{P_0(M - P_2)}{P_2(M - P_0)}$$

$$P_0 P_2 (M - P_1)^2 = P_1^2 (M - P_0)(M - P_2)$$

$$P_0 P_2 M^2 - 2P_0 P_1 P_2 M + P_0 P_1^2 P_2 = P_1^2 M^2 - P_1^2 (P_0 + P_2) M + P_0 P_1^2 P_2$$

We cancel the final terms on the two sides of this last equation and solve for

$$M = \frac{P_1(2P_0 P_2 - P_0 P_1 - P_1 P_2)}{P_0 P_2 - P_1^2}. \qquad \text{(ii)}$$

Substitution of the given values $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$ now gives $M = 188.121$. The first equation in (i) and $x = \exp(-kMt_1)$ yield

$$k = -\frac{1}{Mt_1} \ln \frac{P_0(M - P_1)}{P_1(M - P_0)}. \qquad \text{(iii)}$$

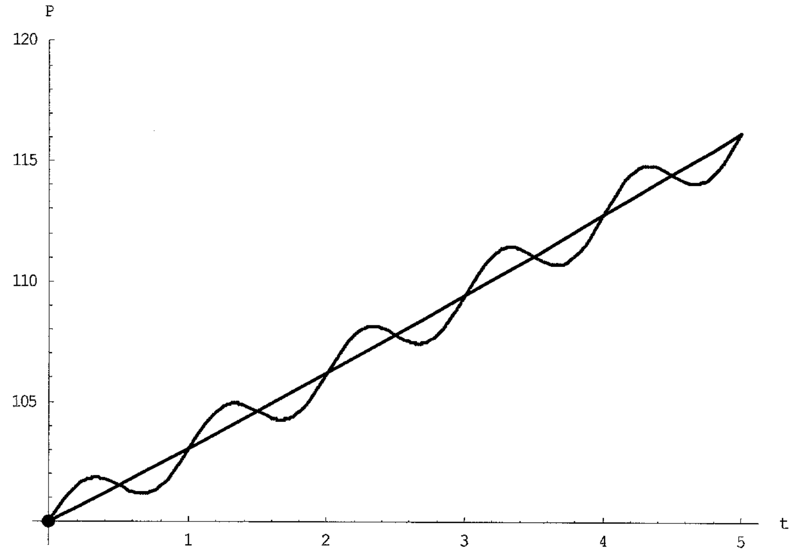
Now substitution of $t_1 = 50$ and our numerical values of M, P_0, P_1, P_2 gives $k = 0.000167716$. Finally, substitution of these values of k and M (and P_0) in the logistic solution (4) gives the logistic model of Eq. (8) in the text.

In Problems 37 and 38 we give just the values of k and M calculated using Eqs. (ii) and (iii) in Problem 36 above, the resulting logistic solution, and the predicted year 2000 population.

37. $k = 0.0000668717$ and $M = 338.027$, so $P(t) = \dfrac{25761.7}{76.212 + 261.815 e^{-0.0226045 t}}$,

predicting $P = 192.525$ in the year 2000.

38. $k = 0.000146679$ and $M = 208.250$, so $P(t) = \dfrac{4829.73}{23.192 + 185.058\,e^{-0.0305458t}}$, predicting $P = 248.856$ in the year 2000.



39. We readily separate the variables and integrate:

$$\int \frac{dP}{P} = \int (k + b\cos 2\pi t)\,dt \quad \Rightarrow \quad \ln P = kt + \frac{b}{2\pi}\sin 2\pi t + \ln C.$$

Clearly $C = P_0$, so we find that $P(t) = P_0 \exp\left(kt + \dfrac{b}{2\pi}\sin 2\pi t\right)$. The colored curve in the figure above shows the graph that results with the typical numerical values $P_0 = 100$, $k = 0.03$, and $b = 0.06$. It oscillates about the black curve which represents natural growth with P_0 and $k = 0.03$. We see that the two agree at the end of each full year.

SECTION 1.8

ACCELERATION-VELOCITY MODELS

This section consists of three essentially independent subsections that can be studied separately: resistance proportional to velocity, resistance proportional to velocity-squared, and inverse-square gravitational acceleration.

1. Equation: $v' = k(250 - v), \quad v(0) = 0, \quad v(10) = 100$

Solution: $\displaystyle\int \frac{(-1)\,dv}{250 - v} = -\int k\,dt; \quad \ln(250 - v) = -kt + \ln C,$

$v(0) = 0$ implies $C = 250; \quad v(t) = 250(1 - e^{-kt})$

$v(10) = 100$ implies $k = \frac{1}{10}\ln(250/150) \approx 0.0511;$

Answer: $v = 200$ when $t = -(\ln 50/250)/k \approx 31.5$ sec

2. Equation: $v' = -kv, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $x'(t) = v(t) = v_0 e^{-kt}; \quad x(t) = -(v_0/k)e^{-kt} + C$

$C = x_0 + (v_0/k)e^{-kt}; \quad x(t) = x_0 + (v_0/k)\left(1 - e^{-kt}\right)$

Answer: $\displaystyle\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x_0 + (v_0/k)\left(1 - e^{-kt}\right)\right] = x_0 + (v_0/k)$

3. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $v(t) = 40\,e^{-kt}$ with $k = (1/10)\ln 2$

$x(t) = (40/k)(1 - e^{-kt})$

Answer: $\displaystyle x(\infty) = \lim_{t\to\infty}(40/k)(1 - e^{-kt}) = 40/k = 400/\ln 2 \approx 577$ ft

4. Equation: $v' = -kv^2, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $\displaystyle -\int \frac{dv}{v^2} = \int k\,dt; \quad \frac{1}{v} = kt + C; \quad C = \frac{1}{v_0}$

$\displaystyle x'(t) = v(t) = \frac{v_0}{1 + v_0 kt}; \quad x(t) = \frac{1}{k}\ln\left(1 + v_0 kt\right) + x_0$

$x(t) \to \infty$ as $x(t) \to \infty$

5. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $\displaystyle v = \frac{40}{1 + 40kt}$ (as in Problem 3)

$v(10) = 20$ implies $40k = 1/10$, so $\displaystyle v(t) = \frac{400}{10 + t}$

$x(t) = 400\ln[(10 + t)/10]$

Answer: $x(60) = 400\ln 7 \approx 778$ ft

6. Equation: $v' = -kv^{3/2}, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{2v^{3/2}} = \int \frac{k\,dt}{2}; \quad \frac{1}{\sqrt{v}} = \frac{kt}{2} + C; \quad C = \frac{1}{\sqrt{v_0}}$

$$x'(t) = v(t) = \frac{4v_0}{\left(2 + kt\sqrt{v_0}\right)^2}; \quad x(t) = -\frac{4\sqrt{v_0}}{k\left(2 + kt\sqrt{v_0}\right)} + C$$

$$C = x_0 + \frac{2\sqrt{v_0}}{k}; \quad x(t) = x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{2 + kt\sqrt{v_0}}\right)$$

$$x(\infty) = x_0 + 2\sqrt{v_0}/k$$

7. Equation: $v' = 10 - 0.1v, \quad x(0) = v(0) = 0$

(a) $\int \frac{-0.1\,dv}{10 - 0.1v} = \int (-0.1)\,dt; \quad \ln(10 - 0.1v) = -t/10 + \ln C$

$$v(0) = 0 \text{ implies } C = 10; \quad \ln[(10 - 0.1v)/10] = -t/10$$

$$v(t) = 100(1 - e^{-t/10}); \quad v(\infty) = 100 \text{ ft/sec (limiting velocity)}$$

(b) $x(t) = 100t - 1000(1 - e^{-t/10})$

$v = 90$ ft/sec when $t = 23.0259$ sec and $x = 1402.59$ ft

8. Equation: $v' = 10 - 0.001v^2, \quad x(0) = v(0) = 0$

(a) $\int \frac{0.01\,dv}{1 - 0.0001v^2} = \int \frac{dt}{10}; \quad \tanh^{-1}\frac{v}{100} = \frac{t}{10} + C$

$$v(0) = 0 \text{ implies } C = 0 \text{ so } v(t) = 100\tanh(t/10)$$

$$v(\infty) = \lim_{t\to\infty} 100\tanh(t/10) = 100 \lim_{t\to\infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100 \text{ ft/sec}$$

(b) $x(t) = 1000\ln(\cosh t/10)$

$v = 90$ ft/sec when $t = 14.7222$ sec and $x = 830.366$ ft

9. The solution of the initial value problem

$$1000\,v' = 5000 - 100\,v, \qquad v(0) = 0$$

is

$$v(t) = 50(1 - e^{-t/10}).$$

Hence, as $t \to \infty$, we see that $v(t)$ approaches $v_{\max} = 50$ ft/sec ≈ 34 mph.

10. Before opening parachute:

$$\begin{aligned} v' &= -32 - 0.15v, \quad v(0) = 0, \quad y(0) = 10000 \\ v(t) &= 213.333(e^{-0.15t} - 1), \qquad v(20) = -202.712 \text{ ft/sec} \\ y(t) &= 11422.2 - 1422.22e^{-0.15ty} - 213.333t, \quad y(20) = 7084.75 \text{ ft} \end{aligned}$$

After opening parachute:

$$\begin{aligned} v' &= -32 - 1.5v, \quad v(0) = -202.712, \quad y(0) = 7084.75 \\ v(t) &= -21.3333 - 181.379e^{-1.5t} \\ y(t) &= 6964.83 + 120.919e^{-1.5t} - 21.3333t, \\ y &= 0 \text{ when } t = 326.476 \end{aligned}$$

Thus she opens her parachute after 20 sec at a height of 7085 feet, and the total time of descent is $20 + 326.476 = 346.476$ sec, about 5 minutes and 46.5 seconds. Her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper's terminal velocity was 100 mph = 440/3 ft/sec, then Equation (7) in the text yields $\rho = 12/55$. Then we find by solving Equation (9) numerically with $y_0 = 1200$ and $v_0 = 0$ that $y = 0$ when $t \approx 12.5$ sec. Thus the newspaper account is inaccurate.

12. With $m = 640/32 = 20$ slugs, $W = 640$ lb, $B = (8)(62.5) = 500$ lb, and $F_R = -v$ lb (F_R is upward when $v < 0$), the differential equation is

$$20\,v'(t) = -640 + 500 - v = -140 - v.$$

Its solution with $v(0) = 0$ is

$$v(t) = 140\left(e^{-0.05t} - 1\right),$$

and integration with $y(0) = 0$ yields

$$y(t) = 2800\left(e^{-0.05t} - 1\right) - 140t.$$

Using these equations we find that $t = 20\ln(28/13) \approx 15.35$ sec when $v = -75$ ft/sec, and that $y(15.35) \approx -648.31$ ft. Thus the maximum safe depth is just under 650 ft.

Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. To solve the initial value problem $v' = -9.8 - 0.0011v^2$, $v(0) = 49$ we write

$$\int \frac{dv}{9.8 + 0.0011v^2} = -\int dt; \quad \int \frac{0.010595\,dv}{1 + (0.010595v)^2} = -\int 0.103827\,dt$$

$$\tan^{-1}(0.010595v) = -0.103827t + C; \quad v(0) = 49 \text{ implies } C = 0.478854$$

$$v(t) = 94.3841\tan(0.478854 - 0.103827t)$$

Integration with $y(0) = 0$ gives

$$y(t) = 108.468 + 909.052\ \ln(\cos(0.478854 - 0.103827t)).$$

We solve $v(0) = 0$ for $t = 4.612$, and then calculate $y(4.612) = 108.468$.

18. We solve the initial value problem $v' = -9.8 + 0.0011v^2$, $v(0) = 0$ much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

$$v(t) = -94.3841\tanh(0.103827t),$$

and then integration with $y(0) = 108.47$ gives

$$y(t) = 108.47 - 909.052\ \ln(\cosh(0.103827t)).$$

We solve $y(0) = 0$ for $t = \cosh^{-1}(\exp(108.47/909.052))/0.103.827 \approx 4.7992$, and then calculate $v(4.7992) = -43.489$.

19. Equation: $v' = 4 - (1/400)v^2, \quad v(0) = 0$

Solution: $$\int \frac{dv}{4 - (1/400)v^2} = \int dt; \quad \int \frac{(1/40)\,dv}{1 - (v/40)^2} = \int \frac{1}{10}\,dt$$

$$\tanh^{-1}(v/40) = t/10 + C; \quad C = 0; \quad v(t) = 40\tanh(t/10)$$

Answer: $v(10) \approx 30.46$ ft/sec, $v(\infty) = 40$ ft/sec

20. Equation: $v' = -32 - (1/800)v^2, \quad v(0) = 160, \quad y(0) = 0$

Solution: $$\int \frac{dv}{32 + (1/800)v^2} = -\int dt; \quad \int \frac{(1/160)\,dv}{1 + (v/160)^2} = -\int \frac{1}{5}\,dt;$$

$$\tan^{-1}(v/160) = -t/5 + C; \quad v(0) = 160 \text{ implies } C = \pi/4$$

$$v(t) = 160\tan\left(\frac{\pi}{4}-\frac{t}{5}\right)$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4}-\frac{t}{5}\right)\right)+400\ln 2$$

We solve $v(t) = 0$ for $t = 3.92699$ and then calculate $y(3.92699) = 277.26$ ft.

21. Equation: $v' = -g-\rho v^2, \quad v(0) = v_0, \quad y(0) = 0$

Solution: $$\int\frac{dv}{g+\rho v^2} = -\int dt; \qquad \int\frac{\sqrt{\rho/g}\,dv}{1+\left(\sqrt{\rho/g}\,v\right)^2} = -\int\sqrt{g\rho}\,dt;$$

$$\tan^{-1}\left(\sqrt{\rho/g}\,v\right) = -\sqrt{g\rho}\,t+C; \qquad v(0)=v_0 \text{ implies } C=\tan^{-1}\left(\sqrt{\rho/g}\,v_0\right)$$

$$v(t) = -\sqrt{\frac{g}{\rho}}\tan\left(t\sqrt{g\rho}-\tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)$$

We solve $v(t) = 0$ for $t = \dfrac{1}{\sqrt{g\rho}}\tan^{-1}\left(v_0\sqrt{\dfrac{\rho}{g}}\right)$ and substitute in Eq. (17) for $y(t)$:

$$y_{\max} = \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}-\tan^{-1}v_0\sqrt{\rho/g}\right)}{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}\right)}\right|$$

$$= \frac{1}{\rho}\ln\left(\sec\left(\tan^{-1}v_0\sqrt{\rho/g}\right)\right) = \frac{1}{\rho}\ln\sqrt{1+\frac{\rho v_0^2}{g}}$$

$$y_{\max} = \frac{1}{2\rho}\ln\left(1+\frac{\rho v_0^2}{g}\right)$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem $v' = -32+0.075v^2, \quad v(0)=0$ is

$$v(t) = -20.666\tanh(1.54919t),$$

so the terminal speed is 20.666 ft/sec. Then a further integration with $y(0)=0$ gives

$$y(t) = 10000-13.333\ln(\cosh(1.54919t)).$$

We solve $y(0)=0$ for $t=484.57$. Thus the descent takes about 8 min 5 sec.

23. Before opening parachute:

$$\begin{aligned} v' &= -32+0.00075v^2, \quad v(0)=0, \quad y(0)=10000 \\ v(t) &= -206.559\tanh(0.154919t) \qquad v(30) = -206.521 \text{ ft/sec} \\ y(t) &= 10000-1333.33\ln(\cosh(0.154919t)), \quad y(30) = 4727.30 \text{ ft} \end{aligned}$$

After opening parachute:

$$\begin{aligned} v' &= -32+0.075v^2, \quad v(0)=-206.521, \quad y(0)=4727.30 \\ v(t) &= -20.6559\tanh(1.54919t+0.00519595) \\ y(t) &= 4727.30-13.3333\ln(\cosh(1.54919t+0.00519595)) \\ y &= 0 \text{ when } \quad t=229.304 \end{aligned}$$

Thus she opens her parachute after 30 sec at a height of 4727 feet, and the total time of descent is $30+229.304=259.304$ sec, about 4 minutes and 19.3 seconds.

24. Let M denote the mass of the Earth. Then

(a) $\sqrt{2GM/R} = c$ implies $R = 0.884\times10^{-3}$ meters, about 0.88 cm;

(b) $\sqrt{2G(329320M)/R} = c$ implies $R = 2.91\times10^{3}$ meters, about 2.91 kilometers.

25. **(a)** The rocket's apex occurs when $v=0$. We get the desired formula when we set $v=0$ in Eq. (23),

$$v^2 = v_0^2+2GM\left(\frac{1}{r}-\frac{1}{R}\right),$$

and solve for r.

(b) We substitute $v=0$, $r=R+10^5$ ($100 \text{ km}=10^5$ m) and the mks values $G=6.6726\times10^{-11}$, $M = 5.975\times10^{24}$, $R = 6.378\times10^6$ in Eq. (23) and solve for $v_0=1389.21$ m/s ≈ 1.389 km/s.

(c) When we substitute $v_0=(9/10)\sqrt{2GM/R}$ in the formula derived in part (a), we find that $r_{\text{max}}=100R/19$.

26. By an elementary computation (as in Section 1.2) we find that an initial velocity of $v_0=16$ ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let r denote the ratio of the radius of the asteroid to the radius $R=3960$ miles of the earth, so that

$$r = \frac{1.5}{3960} = \frac{1}{2640}.$$

Then the mass and radius of the asteroid are given by

$$M_a = r^3 M \quad \text{and} \quad R_a = rR$$

in terms of the mass M and radius R of the earth. Hence the escape velocity from the asteroid's surface is given by

$$v_a = \sqrt{\frac{2GM_a}{R_a}} = \sqrt{\frac{2G\cdot r^3 M}{rR_a}} = r\sqrt{\frac{2GM}{R}} = r\,v_0$$

in terms of the escape velocity v_0 from the earth's surface. Hence $v_a \approx 36680/2640$ ≈ 13.9 ft/sec. Since the escape velocity from this asteroid is thus less than the initial velocity of 16 ft/sec that your legs can provide, you can indeed jump right off this asteroid into space.

27. **(a)** Substitution of $v_0^2 = 2GM/R = k^2/R$ in Eq. (23) of the textbook gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r}} = \frac{k}{\sqrt{r}}.$$

We separate variables and proceed to integrate:

$$\int \sqrt{r}\,dr = \int k\,dt \quad \Rightarrow \quad \frac{2}{3}r^{3/2} = kt + \frac{2}{3}R^{3/2}$$

(using the fact that $r = R$ when $t = 0$). We solve for $r(t) = \left(\tfrac{3}{2}kt + R^{3/2}\right)^{2/3}$ and note that $r(t) \to \infty$ as $t \to \infty$.

(b) If $v_0 > 2GM/R$ then Eq. (23) gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r} + \left(v_0^2 - \frac{2GM}{R}\right)} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Therefore, at every instant in its ascent, the upward velocity of the projectile in this part is greater than the velocity at the same instant of the projectile of part (a). It's as though the projectile of part (a) is the fox, and the projectile of this part is a rabbit that runs faster. Since the fox goes to infinity, so does the faster rabbit.

28. **(a)** Integration of gives

$$\frac{1}{2}v^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

and we solve for

$$\frac{dr}{dt} = v = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

taking the negative square root because $v < 0$ in descent. Hence

$$t = -\sqrt{\frac{r_0}{2GM}} \int \sqrt{\frac{r}{r_0 - r}}\, dr \qquad (r = r_0 \cos^2\theta)$$

$$= \sqrt{r_0/2GM} \int 2r_0 \cos^2\theta \, d\theta$$

$$= \frac{r_0^{3/2}}{\sqrt{2GM}} (\theta + \sin\theta\cos\theta)$$

$$t = \sqrt{\frac{r_0}{2GM}} \left(\sqrt{rr_0 - r^2} + r_0 \cos^{-1}\sqrt{\frac{r}{r_0}} \right)$$

(b) Substitution of $G = 6.6726 \times 10^{-11}$, $M = 5.975 \times 10^{24}$ kg, $r = R = 6.378 \times 10^6$ m, and $r_0 = R + 10^6$ yields $t = 510.504$, that is, about $8\frac{1}{2}$ minutes for the descent to the surface of the earth. (Recall that we are ignoring air resistance.)

(c) Substitution of the same numeral values along with $v_0 = 0$ in the original differential equation of part (a) yields $v = -4116.42$ m/s ≈ -4.116 km/s for the velocity at impact with the earth's surface where $r = R$.

29. Integration of $v\dfrac{dv}{dy} = -\dfrac{GM}{(y+R)^2}$, $y(0) = 0$, $v(0) = v_0$ gives

$$\frac{1}{2}v^2 = \frac{GM}{y+R} - \frac{GM}{R} + \frac{1}{2}v_0^2$$

which simplifies to the desired formula for v^2. Then substitution of $G = 6.6726 \times 10^{-11}$, $M = 5.975 \times 10^{24}$ kg, $R = 6.378 \times 10^6$ m $v = 0$, and $v_0 = 1$ yields an equation that we easily solve for $y = 51427.3$, that is, about 51.427 km.

30. When we integrate

$$v\frac{dv}{dr} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, \qquad r(0) = R, \quad r'(0) = v_0$$

in the usual way and solve for v, we get

$$v = \sqrt{\frac{2GM_e}{r} - \frac{2GM_e}{R} - \frac{2GM_m}{r-S} + \frac{2GM_m}{R-S} + v_0^2}.$$

The earth and moon attractions balance at the point where the right-hand side in the acceleration equation vanishes, which is when

$$r = \frac{\sqrt{M_e}\,S}{\sqrt{M_e} - \sqrt{M_m}}.$$

If we substitute this value of r, $M_m = 7.35 \times 10^{22}$ kg, $S = 384.4 \times 10^6$, and the usual values of the other constants involved, then set $v = 0$ (to just reach the balancing point), we can solve the resulting equation for $v_0 = 11{,}109$ m/s. Note that this is only 71 m/s less than the earth escape velocity of 11,180 m/s, so the moon really doesn't help much.

CHAPTER 1 Review Problems

The main objective of this set of review problems is practice in the identification of the different types of first-order differential equations discussed in this chapter. In each of Problems 1–36 we identify the type of the given equation and indicate an appropriate method of solution.

1. If we write the equation in the form $y' - (3/x)y = x^2$ we see that it is *linear* with integrating factor $\rho = x^{-3}$. The method of Section 1.5 then yields the general solution $y = x^3(C + \ln x)$.

2. We write this equation in the *separable* form $y'/y^2 = (x+3)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(3 - Cx - x\ln x)$.

3. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - \ln x)$.

4. We note that $D_y\left(2xy^3 + e^x\right) = D_x\left(3x^2y^2 + \sin y\right) = 6xy^2$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $x^2y^3 + e^x - \cos y = C$.

5. We write this equation in the *separable* form $y'/y^2 = (2x-3)/x^4$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = C\exp[(1-x)/x^3]$.

6. We write this equation in the *separable* form $y'/y^2 = (1-2x)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(1 + Cx + 2x\ln x)$.

7. If we write the equation in the form $y' + (2/x)y = 1/x^3$ we see that it is *linear* with integrating factor $\rho = x^2$. The method of Section 1.5 then yields the general solution $y = x^{-2}(C + \ln x)$.

8. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = 3Cx/(C - x^3)$.

9. If we write the equation in the form $y' + (2/x)y = 6x\sqrt{y}$ we see that it is a *Bernoulli equation* with $n = 1/2$. The substitution $v = y^{-1/2}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (x^2 + C/x)^2$.

10. We write this equation in the *separable* form $y'/(1+y^2) = 1+x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = \tan(C + x + x^3/3)$.

11. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - 3\ln x)$.

12. We note that $D_y\left(6xy^3 + 2y^4\right) = D_x\left(9x^2y^2 + 8xy^3\right) = 18xy^2 + 8y^3$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $3x^2y^3 + 2xy^4 = C$.

13. We write this equation in the *separable* form $y'/y^2 = 5x^4 - 4x$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = 1/(C + 2x^2 - x^5)$.

14. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = x^2/(C + 2\ln x)$.

15. This is a *linear* differential equation with integrating factor $\rho = e^{3x}$. The method of Section 1.5 yields the general solution $y = (x^3 + C)e^{-3x}$.

16. The substitution $v = y - x,\ y = v + x,\ y' = v' + 1$ gives the separable equation $v' + 1 = (y-x)^2 = v^2$ in the new dependent variable v. The resulting implicit general solution of the original equation is $y - x - 1 = C\,e^{2x}(y - x + 1)$.

17. We note that $D_y\left(e^x + y e^{xy}\right) = D_x\left(e^y + x e^{xy}\right) = e^{xy} + xy\,e^{xy}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $e^x + e^y + e^{xy} = C$.

18. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = Cx^2(x^2 - y^2)$.

19. We write this equation in the *separable* form $y'/y^2 = \left(2 - 3x^5\right)/x^3$. Then separation of variables and integration as in Section 1.4 yields the general solution

$y = x^2 / (x^5 + Cx^2 + 1)$.

20. If we write the equation in the form $y' + (3/x)y = 3x^{-5/2}$ we see that it is *linear* with integrating factor $\rho = x^3$. The method of Section 1.5 then yields the general solution $y = 2x^{-3/2} + Cx^{-3}$.

21. If we write the equation in the form $y' + \big(1/(x+1)\big)y = 1/(x^2-1)$ we see that it is *linear* with integrating factor $\rho = x+1$. The method of Section then 1.5 yields the general solution $y = [C + \ln(x-1)] / (x+1)$.

22. If we write the equation in the form $y' - (6/x)y = 12x^3y^{2/3}$ we see that it is a *Bernoulli equation* with $n = 1/3$. The substitution $v = y^{-2/3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (2x^4 + Cx^2)^3$.

23. We note that $D_y\left(e^y + y\cos x\right) = D_x\left(x e^y + \sin x\right) = e^y + \cos x$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $x e^y + y \sin x = C$.

24. We write this equation in the *separable* form $y'/y^2 = \left(1-9x^2\right)/x^{3/2}$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x^{1/2} / (6x^2 + Cx^{1/2} + 2)$.

25. If we write the equation in the form $y' + \big(2/(x+1)\big)y = 3$ we see that it is *linear* with integrating factor $\rho = (x+1)^2$. The method of Section 1.5 then yields the general solution $y = x + 1 + C\,(x+1)^{-2}$.

26. We note that $D_y\left(9x^{1/2}y^{4/3} - 12x^{1/5}y^{3/2}\right) = D_x\left(8x^{3/2}y^{1/3} - 15x^{6/5}y^{1/2}\right) =$ $12x^{1/2}y^{1/3} - 18x^{1/5}y^{1/2}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $6x^{3/2}y^{4/3} - 10x^{6/5}y^{3/2} = C$.

27. If we write the equation in the form $y' + (1/x)y = -x^2y^4/3$ we see that it is a *Bernoulli equation* with $n = 4$. The substitution $v = y^{-3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = x^{-1}(C + \ln x)^{-1/3}$.

28. If we write the equation in the form $y' + (1/x)y = 2e^{2x}/x$ we see that it is *linear* with integrating factor $\rho = x$. The method of Section 1.5 then yields the general solution $y = x^{-1}(C + e^{2x})$.

29. If we write the equation in the form $y' + (1/(2x+1))y = (2x+1)^{1/2}$ we see that it is *linear* with integrating factor $\rho = (2x+1)^{1/2}$. The method of Section 1.5 then yields the general solution $y = (x^2 + x + C)(2x+1)^{-1/2}$.

30. The substitution $v = x + y,\ y = v - x,\ y' = v' - 1$ gives the separable equation $v' - 1 = \sqrt{v}$ in the new dependent variable v. The resulting implicit general solution of the original equation is $x = 2(x+y)^{1/2} - 2\ln[1 + (x+y)^{1/2}] + C$.

31. $dy/(y+7) = 3x^2 dx$ is separable; $y' + 3x^2 y = 21x^2$ is linear.

32. $dy/(y^2-1) = x\,dx$ is separable; $y' + xy = xy^3$ is a Bernoulli equation with $n = 3$.

33. $(3x^2+2y^2)\,dx + 4xy\,dy = 0$ is exact; $y' = -\frac{1}{4}(3x/y + 2y/x)$ is homogeneous.

34. $(x+3y)\,dx + (3x-y)\,dy = 0$ is exact; $y' = \dfrac{1+3y/x}{y/x-3}$ is homogeneous.

35. $dy/(y+1) = 2x\,dx/(x^2+1)$ is separable; $y' - (2x/(x^2+1))y = 2x/(x^2+1)$ is linear.

36. $dy/(\sqrt{y} - y) = \cot x\,dx$ is separable; $y' + (\cot x)y = (\cot x)\sqrt{y}$ is a Bernoulli equation with $n = 1/2$.