

Solutions Manual  
for  
Digital Communications, 5th Edition  
(Chapter 2) <sup>1</sup>

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### Problem 2.1

a.

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{t-a} da$$

Hence :

$$\begin{aligned} -\hat{x}(-t) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{-t-a} da \\ &= -\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{x(-b)}{-t+b} (-db) \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{t+b} db \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{t-b} db = \hat{x}(t) \end{aligned}$$

where we have made the change of variables :  $b = -a$  and used the relationship :  $x(b) = x(-b)$ .

b. In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

c.  $x(t) = \cos \omega_0 t$ , so its Fourier transform is :  $X(f) = \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)]$ ,  $f_0 = 2\pi\omega_0$ . Exploiting the phase-shifting property (2-1-4) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2} [-j\delta(f - f_0) + j\delta(f + f_0)] = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] = F^{-1} \{\sin 2\pi f_0 t\}$$

Hence,  $\hat{x}(t) = \sin \omega_0 t$ .

d. In a similar way to part (c) :

$$\begin{aligned} x(t) = \sin \omega_0 t &\Rightarrow X(f) = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] \Rightarrow \hat{X}(f) = \frac{1}{2} [-\delta(f - f_0) - \delta(f + f_0)] \\ &\Rightarrow \hat{X}(f) = -\frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] = -F^{-1} \{\cos 2\pi f_0 t\} \Rightarrow \hat{x}(t) = -\cos \omega_0 t \end{aligned}$$

e. The positive frequency content of the new signal will be :  $(-j)(-j)X(f) = -X(f)$ ,  $f > 0$ , while the negative frequency content will be :  $j \cdot jX(f) = -X(f)$ ,  $f < 0$ . Hence, since  $\hat{\hat{X}}(f) = -X(f)$ , we have :  $\hat{\hat{x}}(t) = -x(t)$ .

f. Since the magnitude response of the Hilbert transformer is characterized by :  $|H(f)| = 1$ , we have that :  $|\hat{X}(f)| = |H(f)||X(f)| = |X(f)|$ . Hence :

$$\int_{-\infty}^{\infty} |\hat{X}(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t) dt = \int_{-\infty}^{\infty} x^2(t) dt$$

**g.** From parts (a) and (b) above, we note that if  $x(t)$  is even,  $\hat{x}(t)$  is odd and vice-versa. Therefore,  $x(t)\hat{x}(t)$  is always odd and hence :  $\int_{-\infty}^{\infty} x(t)\hat{x}(t) dt = 0$ .

## Problem 2.2

1. Using relations

$$\begin{aligned} X(f) &= \frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0) \\ Y(f) &= \frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0) \end{aligned}$$

and Parseval's relation, we have

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)y(t) dt &= \int_{-\infty}^{\infty} X(f)Y^*(f) df \\ &= \int_{-\infty}^{\infty} \left[ \frac{1}{2}X_l(f - f_0) + \frac{1}{2}X_l(-f - f_0) \right] \left[ \frac{1}{2}Y_l(f - f_0) + \frac{1}{2}Y_l(-f - f_0) \right]^* df \\ &= \frac{1}{4} \int_{-\infty}^{\infty} X_l(f - f_0)Y_l^*(f - f_0) df + \frac{1}{4} \int_{-\infty}^{\infty} X_l(-f - f_0)Y_l^*(-f - f_0) df \\ &= \frac{1}{4} \int_{-\infty}^{\infty} X_l(u)Y_l^*(u) du + \frac{1}{4} \int_{-\infty}^{\infty} X_l^*(v)Y_l(v) dv \\ &= \frac{1}{2} \operatorname{Re} \left[ \int_{-\infty}^{\infty} X_l(f)Y_l^*(f) df \right] \\ &= \frac{1}{2} \operatorname{Re} \left[ \int_{-\infty}^{\infty} x_l(t)y_l^*(t) dt \right] \end{aligned}$$

where we have used the fact that since  $X_l(f - f_0)$  and  $Y_l(-f - f_0)$  do not overlap,  $X_l(f - f_0)Y_l(-f - f_0) = 0$  and similarly  $X_l(-f - f_0)Y_l(f - f_0) = 0$ .

2. Putting  $y(t) = x(t)$  we get the desired result from the result of part 1.

### Problem 2.3

A well-known result in estimation theory based on the minimum mean-squared-error criterion states that the minimum of  $\mathcal{E}_e$  is obtained when the error is orthogonal to each of the functions in the series expansion. Hence :

$$\int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^K s_k f_k(t) \right] f_n^*(t) dt = 0, \quad n = 1, 2, \dots, K \quad (1)$$

since the functions  $\{f_n(t)\}$  are orthonormal, only the term with  $k = n$  will remain in the sum, so :

$$\int_{-\infty}^{\infty} s(t) f_n^*(t) dt - s_n = 0, \quad n = 1, 2, \dots, K$$

or:

$$s_n = \int_{-\infty}^{\infty} s(t) f_n^*(t) dt \quad n = 1, 2, \dots, K$$

The corresponding residual error  $\mathcal{E}_e$  is :

$$\begin{aligned} \mathcal{E}_{\min} &= \int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^K s_k f_k(t) \right] \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* dt \\ &= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty} \sum_{k=1}^K s_k f_k(t) s^*(t) dt - \sum_{n=1}^K s_n^* \int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^K s_k f_k(t) \right] f_n^*(t) dt \\ &= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty} \sum_{k=1}^K s_k f_k(t) s^*(t) dt \\ &= \mathcal{E}_s - \sum_{k=1}^K |s_k|^2 \end{aligned}$$

where we have exploited relationship (1) to go from the second to the third step in the above calculation.

Note : Relationship (1) can also be obtained by simple differentiation of the residual error with respect to the coefficients  $\{s_n\}$ . Since  $s_n$  is, in general, complex-valued  $s_n = a_n + jb_n$  we have to differentiate with respect to both real and imaginary parts :

$$\begin{aligned} \frac{d}{da_n} \mathcal{E}_e &= \frac{d}{da_n} \int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^K s_k f_k(t) \right] \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* dt = 0 \\ \Rightarrow - \int_{-\infty}^{\infty} a_n f_n(t) \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right]^* + a_n^* f_n^*(t) \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right] dt &= 0 \\ \Rightarrow -2a_n \int_{-\infty}^{\infty} \text{Re} \left\{ f_n^*(t) \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right] \right\} dt &= 0 \\ \Rightarrow \int_{-\infty}^{\infty} \text{Re} \left\{ f_n^*(t) \left[ s(t) - \sum_{n=1}^K s_n f_n(t) \right] \right\} dt &= 0, \quad n = 1, 2, \dots, K \end{aligned}$$

where we have exploited the identity :  $(x + x^*) = 2\text{Re}\{x\}$ . Differentiation of  $\mathcal{E}_e$  with respect to  $b_n$  will give the corresponding relationship for the imaginary part; combining the two we get (1).

### Problem 2.4

The procedure is very similar to the one for the real-valued signals described in the book (pages 33-37). The only difference is that the projections should conform to the complex-valued vector space :

$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1^*(t) dt$$

and, in general for the  $k$ -th function :

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i^*(t) dt, \quad i = 1, 2, \dots, k-1$$

### Problem 2.5

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \begin{cases} -1/\sqrt{3}, & 0 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t) g_4(t) dt = -1/\sqrt{3} \Rightarrow g'_3(t) = s_3(t) - c_{43} g_4(t) = \begin{cases} 2/3, & 0 \leq t \leq 2 \\ -4/3, & 2 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

Hence :

$$g_3(t) = \frac{g'_3(t)}{\sqrt{E_3}} = \begin{cases} 1/\sqrt{6}, & 0 \leq t \leq 2 \\ -2/\sqrt{6}, & 2 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

where  $E_3$  denotes the energy of  $g'_3(t)$  :  $E_3 = \int_0^3 (g'_3(t))^2 dt = 8/3$ .

For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t) g_4(t) dt = 0 \quad \text{and} \quad c_{32} = \int_{-\infty}^{\infty} s_2(t) g_3(t) dt = 0$$

Hence :

$$g_2'(t) = s_2(t) - c_{42}g_4(t) - c_{32}g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g_2'(t)}{\sqrt{\mathcal{E}_2}} = \begin{cases} 1/\sqrt{2}, & 0 \leq t \leq 1 \\ -1/\sqrt{2}, & 1 \leq t \leq 2 \\ 0, & \text{o.w} \end{cases}$$

where :  $\mathcal{E}_2 = \int_0^2 (s_2(t))^2 dt = 2$ .

Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t)g_4(t)dt = -2/\sqrt{3}, \quad c_{31} = \int_{-\infty}^{\infty} s_1(t)g_3(t)dt = 2/\sqrt{6}, \quad c_{21} = 0$$

Hence :

$$g_1'(t) = s_1(t) - c_{41}g_4(t) - c_{31}g_3(t) - c_{21}g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions  $(g_2(t), g_3(t), g_4(t))$  the basis representation of the signals is :

$$\begin{aligned} \mathbf{s}_4 &= (0, 0, \sqrt{3}) \Rightarrow \mathcal{E}_4 = 3 \\ \mathbf{s}_3 &= (0, \sqrt{8/3}, -1/\sqrt{3}) \Rightarrow \mathcal{E}_3 = 3 \\ \mathbf{s}_2 &= (\sqrt{2}, 0, 0) \Rightarrow \mathcal{E}_2 = 2 \\ \mathbf{s}_1 &= (2/\sqrt{6}, -2/\sqrt{3}, 0) \Rightarrow \mathcal{E}_1 = 2 \end{aligned}$$

## Problem 2.6

Consider the set of signals  $\tilde{\phi}_{nl}(t) = j\phi_{nl}(t)$ ,  $1 \leq n \leq N$ , then by definition of lowpass equivalent signals and by Equations 2.2-49 and 2.2-54, we see that  $\phi_n(t)$ 's are  $\sqrt{2}$  times the lowpass equivalents of  $\phi_{nl}(t)$ 's and  $\tilde{\phi}_n(t)$ 's are  $\sqrt{2}$  times the lowpass equivalents of  $\tilde{\phi}_{nl}(t)$ 's. We also note that since  $\phi_n(t)$ 's have unit energy,  $\langle \phi_{nl}(t), \tilde{\phi}_{nl}(t) \rangle = \langle \phi_{nl}(t), j\phi_{nl}(t) \rangle = -j$  and since the inner product is pure imaginary, we conclude that  $\phi_n(t)$  and  $\tilde{\phi}_n(t)$  are orthogonal. Using the orthonormality of the set  $\phi_{nl}(t)$ , we have

$$\langle \phi_{nl}(t), -j\phi_{ml}(t) \rangle = j\delta_{mn}$$

and using the result of problem 2.2 we have

$$\langle \phi_n(t), \tilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n, m$$

We also have

$$\langle \phi_n(t), \phi_m(t) \rangle = 0 \quad \text{for all } n \neq m$$

and

$$\langle \tilde{\phi}_n(t), \tilde{\phi}_m(t) \rangle = 0 \quad \text{for all } n \neq m$$

Using the fact that the energy in lowpass equivalent signal is twice the energy in the bandpass signal we conclude that the energy in  $\phi_n(t)$ 's and  $\tilde{\phi}_n(t)$ 's is unity and hence the set of  $2N$  signals  $\{\phi_n(t), \tilde{\phi}_n(t)\}$  constitute an orthonormal set. The fact that this orthonormal set is sufficient for expansion of bandpass signals follows from Equation 2.2-57.

### Problem 2.7

Let  $x(t) = m(t) \cos 2\pi f_0 t$  where  $m(t)$  is real and lowpass with bandwidth less than  $f_0$ . Then  $\mathcal{F}[\hat{x}(t)] = -j \operatorname{sgn}(f) \left[ \frac{1}{2} M(f - f_0) + \frac{1}{2} M(f + f_0) \right]$  and hence  $\mathcal{F}[\hat{x}(t)] = -\frac{j}{2} M(f - f_0) + \frac{j}{2} M(f + f_0)$  where we have used that fact that  $M(f - f_0) = 0$  for  $f < 0$  and  $M(f + f_0) = 0$  for  $f > 0$ . This shows that  $\hat{x}(t) = m(t) \sin 2\pi f_0 t$ . Similarly we can show that Hilbert transform of  $m(t) \sin 2\pi f_0 t$  is  $-m(t) \cos 2\pi f_0 t$ . From above and Equation 2.2-54 we have

$$\mathcal{H}[\phi_n(t)] = \sqrt{2} \phi_{ni}(t) \sin 2\pi f_0 t + \sqrt{2} \phi_{nq}(t) \cos 2\pi f_0 t = -\tilde{\phi}_n(t)$$

### Problem 2.8

For real-valued signals the correlation coefficients are given by :  $\rho_{km} = \frac{1}{\sqrt{\mathcal{E}_k \mathcal{E}_m}} \int_{-\infty}^{\infty} s_k(t) s_m(t) dt$  and the Euclidean distances by :  $d_{km}^{(e)} = \{ \mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k \mathcal{E}_m} \rho_{km} \}^{1/2}$ . For the signals in this problem :

$$\mathcal{E}_1 = 2, \mathcal{E}_2 = 2, \mathcal{E}_3 = 3, \mathcal{E}_4 = 3$$

$$\rho_{12} = 0 \quad \rho_{13} = \frac{2}{\sqrt{6}} \quad \rho_{14} = -\frac{2}{\sqrt{6}}$$

$$\rho_{23} = 0 \quad \rho_{24} = 0$$

$$\rho_{34} = -\frac{1}{3}$$

and:

$$\begin{aligned} d_{12}^{(e)} &= 2 & d_{13}^{(e)} &= \sqrt{2 + 3 - 2\sqrt{6} \frac{2}{\sqrt{6}}} = 1 & d_{14}^{(e)} &= \sqrt{2 + 3 + 2\sqrt{6} \frac{2}{\sqrt{6}}} = 3 \\ d_{23}^{(e)} &= \sqrt{2 + 3} = \sqrt{5} & d_{24}^{(e)} &= \sqrt{5} \\ d_{34}^{(e)} &= \sqrt{3 + 3 + 2 * 3 \frac{1}{3}} = 2\sqrt{2} \end{aligned}$$

### Problem 2.9

We know from Fourier transform properties that if a signal  $x(t)$  is real-valued then its Fourier transform satisfies :  $X(-f) = X^*(f)$  (Hermitian property). Hence the condition under which  $s_l(t)$  is real-valued is :  $S_l(-f) = S_l^*(f)$  or going back to the bandpass signal  $s(t)$  (using 2-1-5):

$$S_+(f_c - f) = S_+^*(f_c + f)$$

The last condition shows that in order to have a real-valued lowpass signal  $s_l(t)$ , the positive frequency content of the corresponding bandpass signal must exhibit hermitian symmetry around the center frequency  $f_c$ . In general, bandpass signals do not satisfy this property (they have Hermitian symmetry around  $f = 0$ ), hence, the lowpass equivalent is generally complex-valued.

### Problem 2.10

a. To show that the waveforms  $f_n(t)$ ,  $n = 1, \dots, 3$  are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t)f_n(t)dt = 0, \quad m \neq n$$

Clearly:

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \int_0^4 f_1(t)f_2(t)dt \\ &= \int_0^2 f_1(t)f_2(t)dt + \int_2^4 f_1(t)f_2(t)dt \\ &= \frac{1}{4} \int_0^2 dt - \frac{1}{4} \int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4 - 2) \\ &= 0 \end{aligned}$$

Similarly:

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} f_1(t)f_3(t)dt = \int_0^4 f_1(t)f_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt - \frac{1}{4} \int_2^3 dt + \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$

and :

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} f_2(t)f_3(t)dt = \int_0^4 f_2(t)f_3(t)dt \\ &= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt + \frac{1}{4} \int_2^3 dt - \frac{1}{4} \int_3^4 dt \\ &= 0 \end{aligned}$$



Thus, the signals  $f_n(t)$  are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1, 2, 3$$

Hence, they are orthonormal.

**b.** We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)f_n(t)dt, \quad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t)f_1(t)dt = -\frac{1}{2} \int_0^1 dt + \frac{1}{2} \int_1^2 dt - \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t)f_2(t)dt = \frac{1}{2} \int_0^4 x(t)dt = 0 \\ x_3 &= \int_0^4 x(t)f_3(t)dt = -\frac{1}{2} \int_0^1 dt - \frac{1}{2} \int_1^2 dt + \frac{1}{2} \int_2^3 dt + \frac{1}{2} \int_3^4 dt = 0 \end{aligned}$$

As it is observed,  $x(t)$  is orthogonal to the signal waveforms  $f_n(t)$ ,  $n = 1, 2, 3$  and thus it can not be represented as a linear combination of these functions.

### Problem 2.11

**a.** As an orthonormal set of basis functions we consider the set

$$\begin{aligned} f_1(t) &= \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} & f_2(t) &= \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \\ f_3(t) &= \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases} & f_4(t) &= \begin{cases} 1 & 3 \leq t < 4 \\ 0 & \text{o.w} \end{cases} \end{aligned}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

b. The representation vectors are

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix}\end{aligned}$$

c. The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left| \begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix} \right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned}d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix} \right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix} \right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix} \right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix} \right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix} \right|^2} = \sqrt{19}\end{aligned}$$

Thus, the minimum distance between any pair of vectors is  $d_{\min} = \sqrt{5}$ .

### Problem 2.12

As a set of orthonormal functions we consider the waveforms

$$f_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} \quad f_2(t) = \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \quad f_3(t) = \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}\end{aligned}$$

Note that  $s_3(t) = s_2(t) - s_1(t)$  and that the dimensionality of the waveforms is 3.

### Problem 2.13

1.  $P(E_2) = P(R2, R3, R4) = 3/7$ .
2.  $P(E_3|E_2) = \frac{P(E_3E_2)}{P(E_2)} = \frac{P(R2)}{3/7} = \frac{1}{3}$ .
3. Here  $E_4 = \{R2, R4, B2, R1, B1\}$  and  $P(E_2|E_4E_3) = \frac{P(E_2E_3E_4)}{P(E_3E_4)} = \frac{P(R2)}{P(R2, B2, R1, B1)} = \frac{1}{4}$ .
4.  $E_5 = \{R2, R4, B2\}$ . We have  $P(E_3E_5) = P(R2, B2) = \frac{2}{7}$  and  $P(E_3) = P(R1, R2, B1, B2) = \frac{4}{7}$  and  $P(E_5) = \frac{3}{7}$ . Obviously  $P(E_3E_5) \neq P(E_3)P(E_5)$  and the events are not independent.

### Problem 2.14

1.  $P(R) = P(A)P(R|A) + P(B)P(R|B) + P(C)P(R|C) = 0.2 \times 0.05 + 0.3 \times 0.1 + 0.5 \times 0.15 = 0.01 + 0.03 + 0.075 = 0.115$ .
2.  $P(A|R) = \frac{P(A)P(R|A)}{P(R)} = \frac{0.01}{0.115} \approx 0.087$ .

### Problem 2.15

The relationship holds for  $n = 2$  (2-1-34) :  $p(x_1, x_2) = p(x_2|x_1)p(x_1)$

Suppose it holds for  $n = k$ , i.e :  $p(x_1, x_2, \dots, x_k) = p(x_k|x_{k-1}, \dots, x_1)p(x_{k-1}|x_{k-2}, \dots, x_1) \dots p(x_1)$

Then for  $n = k + 1$  :

$$\begin{aligned} p(x_1, x_2, \dots, x_k, x_{k+1}) &= p(x_{k+1}|x_k, x_{k-1}, \dots, x_1)p(x_k, x_{k-1}, \dots, x_1) \\ &= p(x_{k+1}|x_k, x_{k-1}, \dots, x_1)p(x_k|x_{k-1}, \dots, x_1)p(x_{k-1}|x_{k-2}, \dots, x_1) \dots p(x_1) \end{aligned}$$

Hence the relationship holds for  $n = k + 1$ , and by induction it holds for any  $n$ .

**Problem 2.16**

1. Let  $T$  and  $R$  denote channel input and outputs respectively. Using Bayes rule we have

$$\begin{aligned} p(T = 0|R = A) &= \frac{p(T = 0)p(R = A|T = 0)}{p(T = 0)p(R = A|T = 0) + p(T = 1)p(R = A|T = 1)} \\ &= \frac{0.4 \times \frac{1}{6}}{0.4 \times \frac{1}{6} + 0.6 \times \frac{1}{3}} \\ &= \frac{1}{4} \end{aligned}$$

and therefore  $p(T = 1|R = A) = \frac{3}{4}$ , obviously if  $R = A$  is observed, the best decision would be to declare that a 1 was sent, i.e.,  $T = 1$ , because  $T = 1$  is more probable than  $T = 0$ . Similarly it can be verified that  $p(T = 0|R = B) = \frac{4}{7}$  and  $p(T = 0|R = C) = \frac{1}{4}$ . Therefore, when the output is B, the best decision is 0 and when the output is C, the best decision is  $T = 1$ . Therefore the decision function  $d$  can be defined as

$$d(R) = \begin{cases} 1, & R = A \text{ or } C \\ 0, & R = B \end{cases}$$

This is the optimal decision scheme.

2. Here we know that a 0 is transmitted, therefore we are looking for  $p(\text{error}|T = 0)$ , this is the probability that the receiver declares a 1 was sent when actually a 0 was transmitted. Since by the decision method described in part 1 the receiver declares that a 1 was sent when  $R = A$  or  $R = C$ , therefore,  $p(\text{error}|T = 0) = p(R = A|T = 0) + p(R = C|T = 0) = \frac{1}{3}$ .
3. We have  $p(\text{error}|T = 0) = \frac{1}{3}$ , and  $p(\text{error}|T = 1) = p(R = B|T = 1) = \frac{1}{3}$ . Therefore, by the total probability theorem

$$\begin{aligned} p(\text{error}) &= p(T = 0)p(\text{error}|T = 0) + p(T = 1)p(\text{error}|T = 1) \\ &= 0.4 \times \frac{1}{3} + 0.6 \times \frac{1}{3} \\ &= \frac{1}{3} \end{aligned}$$

**Problem 2.17**

Following the same procedure as in example 2-1-1, we prove :

$$p_Y(y) = \frac{1}{|a|} p_X\left(\frac{y-b}{a}\right)$$

**Problem 2.18**

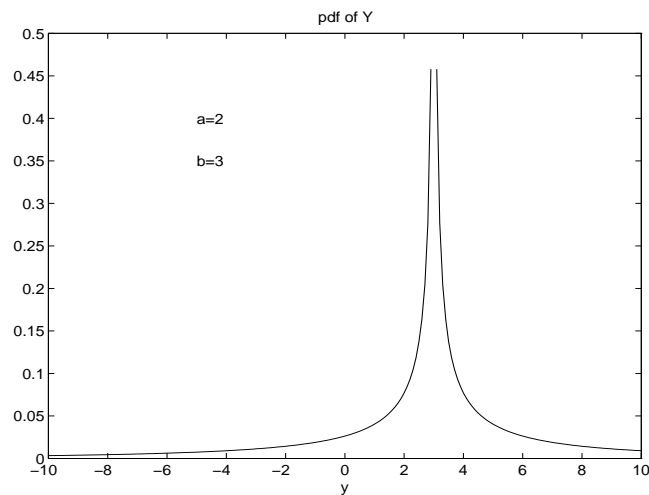
Relationship (2-1-44) gives :

$$p_Y(y) = \frac{1}{3a[(y-b)/a]^{2/3}} p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

$X$  is a gaussian r.v. with zero mean and unit variance :  $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Hence :

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}[(y-b)/a]^{2/3}} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$

**Problem 2.19**

1) The random variable  $X$  is Gaussian with zero mean and variance  $\sigma^2 = 10^{-8}$ . Thus  $p(X > x) = Q(\frac{x}{\sigma})$  and

$$p(X > 10^{-4}) = Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159$$

$$p(X > 4 \times 10^{-4}) = Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5}$$

$$p(-2 \times 10^{-4} < X \leq 10^{-4}) = 1 - Q(1) - Q(2) = .8182$$

2)

$$p(X > 10^{-4} | X > 0) = \frac{p(X > 10^{-4}, X > 0)}{p(X > 0)} = \frac{p(X > 10^{-4})}{p(X > 0)} = \frac{.159}{.5} = .318$$

**Problem 2.20**

1)  $y = g(x) = ax^2$ . Assume without loss of generality that  $a > 0$ . Then, if  $y < 0$  the equation  $y = ax^2$  has no real solutions and  $f_Y(y) = 0$ . If  $y > 0$  there are two solutions to the system, namely  $x_{1,2} = \sqrt{y/a}$ . Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation  $y = g(x)$  has no solutions if  $y < -b$ . Thus  $F_Y(y)$  and  $f_Y(y)$  are zero for  $y < -b$ . If  $-b \leq y \leq b$ , then for a fixed  $y$ ,  $g(x) < y$  if  $x < y$ ; hence  $F_Y(y) = F_X(y)$ . If  $y > b$  then  $g(x) \leq b < y$  for every  $x$ ; hence  $F_Y(y) = 1$ . At the points  $y = \pm b$ ,  $F_Y(y)$  is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of  $y = g(x)$  is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned} p(Y=b) &= p(X < 0) = F_X(0) = \frac{1}{2} \\ p(Y=a) &= p(X > 0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus  $F_Y(y)$  is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable  $y = g(x)$  takes the values  $y_n = x_n$  with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus,  $F_Y(y)$  is a staircase function with  $F_Y(y) = 0$  if  $y < x_1$  and  $F_Y(y) = 1$  if  $y > x_N$ . The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^N [F_X(a_{i+1}) - F_X(a_i)] \delta(y - x_i) \\ &= \sum_{i=1}^N \left[ Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right) \right] \delta(y - x_i) \end{aligned}$$

### Problem 2.21

For  $n$  odd,  $x^n$  is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval  $[-\infty, \infty]$  is zero, we obtain  $E[X^n] = 0$  for  $n$  odd. Let  $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2) dx$ . Obviously  $I_n$  is a constant and its derivative with respect to  $x$  is zero, i.e.,

$$\frac{d}{dx} I_n = \int_{-\infty}^{\infty} \left[ nx^{n-1} e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2} x^{n+1} e^{-\frac{x^2}{2\sigma^2}} \right] dx = 0$$

which results in the recursion

$$I_{n+1} = n\sigma^2 I_{n-1}$$

This is true for all  $n$ . Now let  $n = 2k - 1$ , we will have  $I_{2k} = (2k - 1)\sigma^2 I_{2k-2}$ , with the initial condition  $I_0 = \sqrt{2\pi\sigma^2}$ . Substituting we have

$$\begin{aligned} I_2 &= \sigma^2 \sqrt{2\pi\sigma^2} \\ I_4 &= 3\sigma^2 I_2 = 3\sigma^4 \sqrt{2\pi\sigma^2} \\ I_6 &= 5 \times 3\sigma^2 I_4 = 5 \times 3\sigma^6 \sqrt{2\pi\sigma^2} \\ I_8 &= 7 \times \sigma^2 I_6 = 7 \times 5 \times 3\sigma^8 \sqrt{2\pi\sigma^2} \\ &\vdots \end{aligned}$$

and in general if  $I_{2k} = (2k - 1)(2k - 3)(2k - 5) \times \cdots \times 3 \times 1 \sigma^{2k} \sqrt{2\pi\sigma^2}$ , then  $I_{2k+2} = (2k + 1)\sigma^2 I_{2k} = (2k + 1)(2k - 1)(2k - 3)(2k - 5) \times \cdots \times 3 \times 1 \sigma^{2k+2} \sqrt{2\pi\sigma^2}$ . Using the fact that  $E[X^{2k}] = I_{2k}/\sqrt{2\pi\sigma^2}$ , we obtain

$$I_n = 1 \times 3 \times 5 \times \cdots \times (n - 1) \sigma^n$$

for  $n$  even.

### Problem 2.22

a. Since  $(X_r, X_i)$  are statistically independent :

$$p_{\mathbf{X}}(x_r, x_i) = p_X(x_r)p_X(x_i) = \frac{1}{2\pi\sigma^2}e^{-(x_r^2+x_i^2)/2\sigma^2}$$

Also :

$$\begin{aligned} Y_r + jY_i &= (X_r + X_i)e^{j\phi} \Rightarrow \\ X_r + X_i &= (Y_r + jY_i)e^{-j\phi} = Y_r \cos \phi + Y_i \sin \phi + j(-Y_r \sin \phi + Y_i \cos \phi) \Rightarrow \\ \begin{cases} X_r = & Y_r \cos \phi + Y_i \sin \phi \\ X_i = & -Y_r \sin \phi + Y_i \cos \phi \end{cases} \end{aligned}$$

The Jacobian of the above transformation is :

$$J = \begin{vmatrix} \frac{\partial X_r}{\partial Y_r} & \frac{\partial X_i}{\partial Y_r} \\ \frac{\partial X_r}{\partial Y_i} & \frac{\partial X_i}{\partial Y_i} \end{vmatrix} = \begin{vmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{vmatrix} = 1$$

Hence, by (2-1-55) :

$$\begin{aligned} p_{\mathbf{Y}}(y_r, y_i) &= p_{\mathbf{X}}((Y_r \cos \phi + Y_i \sin \phi), (-Y_r \sin \phi + Y_i \cos \phi)) \\ &= \frac{1}{2\pi\sigma^2}e^{-(y_r^2+y_i^2)/2\sigma^2} \end{aligned}$$

b.  $\mathbf{Y} = \mathbf{A}\mathbf{X}$  and  $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$

Now,  $p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}}e^{-\mathbf{x}'\mathbf{x}/2\sigma^2}$  (the covariance matrix  $\mathbf{M}$  of the random variables  $x_1, \dots, x_n$  is  $\mathbf{M} = \sigma^2\mathbf{I}$ , since they are i.i.d) and  $J = 1/|\det(\mathbf{A})|$ . Hence :

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{|\det(\mathbf{A})|} e^{-\mathbf{y}'(\mathbf{A}^{-1})'\mathbf{A}^{-1}\mathbf{y}/2\sigma^2}$$

For the pdf's of  $X$  and  $Y$  to be identical we require that :

$$|\det(\mathbf{A})| = 1 \text{ and } (\mathbf{A}^{-1})'\mathbf{A}^{-1} = \mathbf{I} \implies \mathbf{A}^{-1} = \mathbf{A}'$$

Hence,  $\mathbf{A}$  must be a unitary (orthogonal) matrix .

### Problem 2.23

Since we are dealing with linear combinations of jointly Gaussian random variables, it is clear that  $\mathbf{Y}$  is jointly Gaussian. We clearly have  $\mathbf{m}_Y = E[\mathbf{A}\mathbf{X}] = \mathbf{A}\mathbf{m}_X$ . This means that  $\mathbf{Y} - \mathbf{m}_Y = \mathbf{A}(\mathbf{X} - \mathbf{m}_X)$ . Also note that

$$C_Y = E[(\mathbf{Y} - \mathbf{m}_Y)(\mathbf{Y} - \mathbf{m}_Y)'] = E[\mathbf{A}(\mathbf{X} - \mathbf{m}_X)(\mathbf{X} - \mathbf{m}_X)\mathbf{A}']$$



resulting in  $C_Y = AC_X A'$ .

### Problem 2.24

a.

$$\psi_Y(jv) = E[e^{jvY}] = E[e^{jv \sum_{i=1}^n x_i}] = E\left[\prod_{i=1}^n e^{jvx_i}\right] = \prod_{i=1}^n E[e^{jvX}] = (\psi_X(e^{jv}))^n$$

But,

$$p_X(x) = p\delta(x-1) + (1-p)\delta(x) \Rightarrow \psi_X(e^{jv}) = 1 + p + pe^{jv}$$

$$\Rightarrow \psi_Y(jv) = (1 + p + pe^{jv})^n$$

b.

$$E(Y) = -j \frac{d\psi_Y(jv)}{dv} \Big|_{v=0} = -jn(1 - p + pe^{jv})^{n-1} jpe^{jv} \Big|_{v=0} = np$$

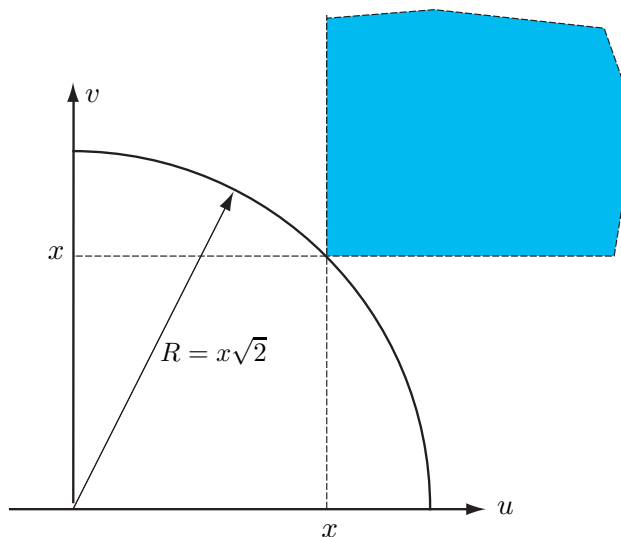
and

$$E(Y^2) = -\frac{d^2\psi_Y(jv)}{d^2v} \Big|_{v=0} = -\frac{d}{dv} [jn(1 - p + pe^{jv})^{n-1} jpe^{jv}] \Big|_{v=0} = np + np(n-1)p$$

$$\Rightarrow E(Y^2) = n^2p^2 + np(1-p)$$

### Problem 2.25

1. In the figure shown below



let us consider the region  $u > x, v > x$  shown as the colored region extending to infinity, call this region  $\mathcal{R}$ , and let us integrate  $e^{-\frac{u^2+v^2}{2}}$  over this region. We have

$$\begin{aligned} \iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}} du dv &= \iint_{\mathcal{R}} e^{-\frac{r^2}{2}} r dr d\theta \\ &\leq \int_{x\sqrt{2}}^{\infty} r e^{-\frac{r^2}{2}} dr \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{\pi}{2} \left[ -e^{-\frac{r^2}{2}} \right]_{x\sqrt{2}}^{\infty} \\ &= \frac{\pi}{2} e^{-x^2} \end{aligned}$$

where we have used the fact that region  $\mathcal{R}$  is included in the region outside the quarter circle as shown in the figure. On the other hand we have

$$\begin{aligned} \iint_{\mathcal{R}} e^{-\frac{u^2+v^2}{2}} du dv &= \int_x^{\infty} e^{-\frac{u^2}{2}} du \int_x^{\infty} e^{-\frac{v^2}{2}} dv \\ &= \left( \int_x^{\infty} e^{-\frac{u^2}{2}} du \right)^2 \\ &= \left( \sqrt{2\pi} Q(x) \right)^2 \\ &= 2\pi (Q(x))^2 \end{aligned}$$

From the above relations we conclude that

$$2\pi (Q(x))^2 \leq \frac{\pi}{2} e^{-x^2}$$

and therefore,  $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$ .

2. In  $\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2}$  define  $u = e^{-\frac{y^2}{2}}$  and  $dv = \frac{dy}{y^2}$  and use the integration by parts relation  $\int u dv = uv - \int v du$ . We have  $v = -\frac{1}{y}$  and  $du = -ye^{-\frac{y^2}{2}} dy$ . Therefore

$$\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} = \left[ -\frac{e^{-\frac{y^2}{2}}}{y} \right]_x^\infty - \int_x^\infty e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x)$$

Now note that  $\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} > 0$  which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) > 0 \Rightarrow Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

On the other hand, note that

$$\int_x^\infty e^{-\frac{y^2}{2}} \frac{dy}{y^2} < \frac{1}{x^2} \int_x^\infty e^{-\frac{y^2}{2}} dy = \frac{\sqrt{2\pi}}{x^2}Q(x)$$

which results in

$$\frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}Q(x) < \frac{\sqrt{2\pi}}{x^2}Q(x)$$

or,  $\sqrt{2\pi}\frac{1+x^2}{x^2}Q(x) > \frac{e^{-\frac{x^2}{2}}}{x}$  which results in

$$Q(x) > \frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}}$$

3. From

$$\frac{x}{\sqrt{2\pi}(1+x^2)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

we have

$$\frac{1}{\sqrt{2\pi}(\frac{1}{x}+x)}e^{-\frac{x^2}{2}} < Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

As  $x$  becomes large  $\frac{1}{x}$  in the denominator of the left hand side becomes small and the two bounds become equal, therefore for large  $x$  we have

$$Q(x) \approx \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$$

## Problem 2.26

1.  $F_{Y_n}(y) = P[Y_n \leq y] = 1 - P[Y_n > y] = 1 - P[x_1 > y, X_2 > y, \dots, X_n > y] = 1 - (P[X > y])^n$   
 where we have used the independence of  $X_i$ 's in the last step. But  $P[X > y] = \int_y^A \frac{1}{A} dy = \frac{A-y}{A}$ .  
 Therefore,  $F_{Y_n}(y) = 1 - \frac{(A-y)^n}{A^n}$ , and  $f_{Y_n}(y) = \frac{d}{dy} F_{Y_n}(y) = n \frac{(A-y)^{n-1}}{A^n}$ ,  $0 < y < A$ .

2.

$$\begin{aligned} f(y) &= \frac{n}{A} \left(1 - \frac{y}{A}\right)^{n-1} \\ &= \frac{\lambda}{1 - \frac{y}{A}} \left(1 - \frac{ny}{nA}\right)^n \\ &= \frac{\lambda}{1 - \frac{y}{A}} \left(1 - \frac{\lambda y}{n}\right)^n \rightarrow \lambda e^{-\lambda y} \quad y > 0 \end{aligned}$$

### Problem 2.27

$$\psi(jv_1, jv_2, jv_3, jv_4) = E \left[ e^{j(v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4)} \right]$$

$$E(X_1 X_2 X_3 X_4) = (-j)^4 \frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} \Big|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given rv's :

$$\psi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where  $\mathbf{v} = [v_1, v_2, v_3, v_4]'$ ,  $\mathbf{M} = [\mu_{ij}]$ .

We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that :

$$\frac{\partial \psi(j\mathbf{v})}{\partial v_i} = -\mu'_{i1} \mathbf{v} e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where  $\mu'_i = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$ . Also note that :

$$\frac{\partial \mu'_{j1} \mathbf{v}}{\partial v_i} = \mu_{ij} = \mu_{ji}$$

Hence :

$$\frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4} \Big|_{\mathbf{v}=0} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

**Problem 2.28**

1) By Chernov bound, for  $t > 0$ ,

$$P[X \geq \alpha] \leq e^{-t\alpha} E[e^{tX}] = e^{-t\alpha} \Theta_X(t)$$

This is true for all  $t > 0$ , hence

$$\ln P[X \geq \alpha] \leq \min_{t \geq 0} [-t\alpha + \ln \Theta_X(t)] = -\max_{t \geq 0} [t\alpha - \ln \Theta_X(t)]$$

2) Here

$$\ln P[S_n \geq \alpha] = \ln P[Y \geq n\alpha] \leq -\max_{t \geq 0} [tn\alpha - \ln \Theta_Y(t)]$$

where  $Y = X_1 + X_2 + \cdots + X_n$ , and  $\Theta_Y(t) = E[e^{X_1+X_2+\cdots+X_n}] = [\Theta_X(t)]^n$ . Hence,

$$\ln P[S_n \geq \alpha] = -\max_{t \geq 0} n [t\alpha - \ln \Theta_X(t)] = -nI(\alpha) \Rightarrow \frac{1}{n} P[S_n \geq \alpha] \leq e^{-nI(\alpha)}$$

$\Theta_X(t) = \int_0^\infty e^{tx} e^{-x} dx = \frac{1}{1-t}$  as long as  $t < 1$ .  $I(\alpha) = \max_{t \geq 0} (t\alpha + \ln(1-t))$ , hence  $\frac{d}{dt}(t\alpha + \ln(1-t)) = 0$  and  $t^* = \frac{\alpha-1}{\alpha}$ . Since  $\alpha \geq 0$ ,  $t^* \geq 0$  and also obviously  $t^* < 1$ .  $I(\alpha) = \alpha - 1 + \ln(1 - \frac{\alpha-1}{\alpha}) = \alpha - 1 - \ln \alpha$ , using the large deviation theorem

$$\ln P[S_n \geq \alpha] = e^{-n(\alpha-1-\ln \alpha)+o(n)} = \alpha^n e^{-n(\alpha-1)+o(n)}$$

**Problem 2.29**

For the central chi-square with  $n$  degree of freedom :

$$\psi(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}}$$

Now :

$$\begin{aligned} \frac{d\psi(jv)}{dv} &= \frac{jn\sigma^2}{(1 - j2v\sigma^2)^{n/2+1}} \Rightarrow E(Y) = -j \frac{d\psi(jv)}{dv} \Big|_{v=0} = n\sigma^2 \\ \frac{d^2\psi(jv)}{dv^2} &= \frac{-2n\sigma^4(n/2+1)}{(1 - j2v\sigma^2)^{n/2+2}} \Rightarrow E(Y^2) = -\frac{d^2\psi(jv)}{dv^2} \Big|_{v=0} = n(n+2)\sigma^2 \end{aligned}$$

The variance is  $\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 2n\sigma^4$

For the non-central chi-square with  $n$  degrees of freedom :

$$\psi(jv) = \frac{1}{(1 - j2v\sigma^2)^{n/2}} e^{jvs^2/(1-j2v\sigma^2)}$$

where by definition :  $s^2 = \sum_{i=1}^n m_i^2$ .

$$\frac{d\psi(jv)}{dv} = \left[ \frac{jn\sigma^2}{(1-j2v\sigma^2)^{n/2+1}} + \frac{js^2}{(1-j2v\sigma^2)^{n/2+2}} \right] e^{jvs^2/(1-j2v\sigma^2)}$$

Hence,  $E(Y) = -j \frac{d\psi(jv)}{dv} \big|_{v=0} = n\sigma^2 + s^2$

$$\frac{d^2\psi(jv)}{dv^2} = \left[ \frac{-n\sigma^4(n+2)}{(1-j2v\sigma^2)^{n/2+2}} + \frac{-s^2(n+4)\sigma^2 - ns^2\sigma^2}{(1-j2v\sigma^2)^{n/2+3}} + \frac{-s^4}{(1-j2v\sigma^2)^{n/2+4}} \right] e^{jvs^2/(1-j2v\sigma^2)}$$

Hence,

$$E(Y^2) = -\frac{d^2\psi(jv)}{dv^2} \big|_{v=0} = 2n\sigma^4 + 4s^2\sigma^2 + (n\sigma^2 + s^2)$$

and

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

### Problem 2.30

The Cauchy r.v. has :  $p(x) = \frac{a/\pi}{x^2+a^2}$ ,  $-\infty < x < \infty$

a.

$$E(X) = \int_{-\infty}^{\infty} xp(x)dx = 0$$

since  $p(x)$  is an even function.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{x^2+a^2} dx$$

Note that for large  $x$ ,  $\frac{x^2}{x^2+a^2} \rightarrow 1$  (i.e non-zero value). Hence,

$$E(X^2) = \infty, \sigma^2 = \infty$$

b.

$$\psi(jv) = E(jvX) = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2+a^2} e^{jvx} dx = \int_{-\infty}^{\infty} \frac{a/\pi}{(x+ja)(x-ja)} e^{jvx} dx$$

This integral can be evaluated by using the residue theorem in complex variable theory. Then, for  $v \geq 0$  :

$$\psi(jv) = 2\pi j \left( \frac{a/\pi}{x+ja} e^{jvx} \right)_{x=ja} = e^{-av}$$

For  $v < 0$  :

$$\psi(jv) = -2\pi j \left( \frac{a/\pi}{x-ja} e^{jvx} \right)_{x=-ja} = e^{av}$$

Therefore :

$$\psi(jv) = e^{-a|v|}$$

Note: an alternative way to find the characteristic function is to use the Fourier transform relationship between  $p(x)$ ,  $\psi(jv)$  and the Fourier pair :

$$e^{-b|t|} \leftrightarrow \frac{1}{\pi} \frac{c}{c^2 + f^2}, \quad c = b/2\pi, \quad f = 2\pi v$$

### Problem 2.31

Since  $R_0$  and  $R_1$  are independent  $f_{R_0, R_1}(r_0, r_1) = f_{R_0}(r_0)f_{R_1}(r_1)$  and

$$f_{R_0, R_1}(r_0, r_1) = \begin{cases} \frac{r_0 r_1}{\sigma^4} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2}{2\sigma^2}} e^{-\frac{r_1^2 + r_0^2}{2\sigma^2}}, & r_0, r_1 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} P(R_0 > R_1) &= \iint_{r_0 > r_1} f(r_0, r_1) dr_1 dr_0 \\ &= \int_0^\infty dr_1 \int_{r_1}^\infty f(r_0, r_1) dr_0 \\ &= \int_0^\infty f_{R_1}(r_1) \left( \int_{r_1}^\infty f_{R_0}(r_0) dr_0 \right) dr_1 \\ &= \int_0^\infty f_{R_1}(r_1) \left( \int_{r_1}^\infty \frac{r_0}{\sigma^2} e^{-\frac{r_0^2}{2\sigma^2}} dr_0 \right) dr_1 \\ &= \int_0^\infty f_{R_1}(r_1) \left[ -e^{-\frac{r_0^2}{2\sigma^2}} \right]_{r_1}^\infty dr_1 \\ &= \int_0^\infty e^{-\frac{r_1^2}{2\sigma^2}} f_{R_1}(r_1) dr_1 \\ &= \int_0^\infty \frac{r_1}{\sigma^2} I_0\left(\frac{\mu r_1}{\sigma^2}\right) e^{-\frac{\mu^2 + 2r_1^2}{2\sigma^2}} dr_1 \end{aligned}$$

Now using the change of variable  $y = \sqrt{2}r_1$  and letting  $s = \frac{\mu}{\sqrt{2}}$  we obtain

$$\begin{aligned} P(R_0 > R_1) &= \int_0^\infty \frac{y}{\sqrt{2}\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{2s^2 + y^2}{2\sigma^2}} \frac{dy}{\sqrt{2}} \\ &= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2 + y^2}{2\sigma^2}} dy \\ &= \frac{1}{2} e^{-\frac{s^2}{2\sigma^2}} \\ &= \frac{1}{2} e^{-\frac{\mu^2}{4\sigma^2}} \end{aligned}$$

where we have used the fact that  $\int_0^\infty \frac{y}{\sigma^2} I_0\left(\frac{sy}{\sigma^2}\right) e^{-\frac{s^2+y^2}{2\sigma^2}} dy = 1$  because it is the integral of a Rician pdf.

### Problem 2.32

1. The joint pdf of  $a, b$  is :

$$p_{ab}(a, b) = p_{xy}(a - m_r, b - m_i) = p_x(a - m_r)p_y(b - m_i) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(a-m_r)^2 + (b-m_i)^2]}$$

2.  $u = \sqrt{a^2 + b^2}$ ,  $\phi = \tan^{-1}b/a \Rightarrow a = u \cos \phi$ ,  $b = u \sin \phi$  The Jacobian of the transformation is  
 $: J(a, b) = \begin{vmatrix} \partial a / \partial u & \partial a / \partial \phi \\ \partial b / \partial u & \partial b / \partial \phi \end{vmatrix} = u$ , hence :

$$\begin{aligned} p_{u\phi}(u, \phi) &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[(u \cos \phi - m_r)^2 + (u \sin \phi - m_i)^2]} \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}[u^2 + M^2 - 2uM \cos(\phi - \theta)]} \end{aligned}$$

where we have used the transformation :

$$\left\{ \begin{array}{l} M = \sqrt{m_r^2 + m_i^2} \\ \theta = \tan^{-1}m_i/m_r \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} m_r = M \cos \theta \\ m_i = M \sin \theta \end{array} \right\}$$

3.

$$\begin{aligned} p_u(u) &= \int_0^{2\pi} p_{u\phi}(u, \phi) d\phi \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[-2uM \cos(\phi-\theta)]} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{uM \cos(\phi-\theta)/\sigma^2} d\phi \\ &= \frac{u}{\sigma^2} e^{-\frac{u^2+M^2}{2\sigma^2}} I_0(uM/\sigma^2) \end{aligned}$$



**Problem 2.33**

a.  $Y = \frac{1}{n} \sum_{i=1}^n X_i$ ,  $\psi_{X_i}(jv) = e^{-a|v|}$

$$\psi_Y(jv) = E \left[ e^{jv \frac{1}{n} \sum_{i=1}^n X_i} \right] = \prod_{i=1}^n E \left[ e^{j \frac{v}{n} X_i} \right] = \prod_{i=1}^n \psi_{X_i}(jv/n) = \left[ e^{-a|v|/n} \right]^n = e^{-a|v|}$$

b. Since  $\psi_Y(jv) = \psi_{X_i}(jv) \Rightarrow p_Y(y) = p_{X_i}(x_i) \Rightarrow p_Y(y) = \frac{a/\pi}{y^2+a^2}$ .

c. As  $n \rightarrow \infty$ ,  $p_Y(y) = \frac{a/\pi}{y^2+a^2}$ , which is not Gaussian ; hence, the central limit theorem does not hold. The reason is that the Cauchy distribution does not have a finite variance.

**Problem 2.34**

Since  $\mathbf{Z}$  and  $\mathbf{Z}e^{j\theta}$  have the same pdf, we have  $E[\mathbf{Z}] = E[\mathbf{Z}e^{j\theta}] = e^{j\theta}E[\mathbf{Z}]$  for all  $\theta$ . Putting  $\theta = \pi$  gives  $E[\mathbf{Z}] = \mathbf{0}$ . We also have  $E[\mathbf{Z}\mathbf{Z}^t] = E[\mathbf{Z}e^{j\theta}(\mathbf{Z}e^{j\theta})^t]$  or  $E[\mathbf{Z}\mathbf{Z}^t] = e^{2j\theta}E[\mathbf{Z}\mathbf{Z}^t]$ , for all  $\theta$ . Putting  $\theta = \frac{\pi}{2}$  gives  $E[\mathbf{Z}\mathbf{Z}^t] = \mathbf{0}$ . Since  $\mathbf{Z}$  is zero-mean and  $E[\mathbf{Z}\mathbf{Z}^t] = \mathbf{0}$ , we conclude that it is proper.

**Problem 2.35**

Using Equation 2.6-29 we note that for the zero-mean proper case if  $\mathbf{W} = e^{j\theta}\mathbf{Z}$ , it is sufficient to show that  $\det(\mathbf{C}_{\mathbf{W}}) = \det(\mathbf{C}_{\mathbf{Z}})$  and  $\mathbf{w}^H \mathbf{C}_{\mathbf{W}}^{-1} \mathbf{w} = \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z}$ . But  $\mathbf{C}_{\mathbf{W}} = [\mathbf{W}\mathbf{W}^H] = E[e^{j\theta}\mathbf{Z}e^{-j\theta}\mathbf{Z}^H] = E[\mathbf{Z}\mathbf{Z}^H] = \mathbf{C}_{\mathbf{Z}}$ , hence  $\det(\mathbf{C}_{\mathbf{W}}) = \det(\mathbf{C}_{\mathbf{Z}})$ . Similarly,  $\mathbf{w}^H \mathbf{C}_{\mathbf{W}}^{-1} \mathbf{w} = e^{-j\theta} \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z} e^{j\theta} = \mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1} \mathbf{z}$ . Substituting into Equation 2.6-29, we conclude that  $p(\mathbf{w}) = p(\mathbf{z})$ .

**Problem 2.36**

Since  $\mathbf{Z}$  is proper, we have  $E[(\mathbf{Z} - E(\mathbf{Z}))(\mathbf{Z} - E(\mathbf{Z}))^t] = \mathbf{0}$ . Let  $\mathbf{W} = \mathbf{A}\mathbf{Z} + \mathbf{b}$ , then

$$E[(\mathbf{W} - E(\mathbf{W}))(\mathbf{W} - E(\mathbf{W}))^t] = \mathbf{A}E[(\mathbf{Z} - E(\mathbf{Z}))(\mathbf{Z} - E(\mathbf{Z}))^t]\mathbf{A}^t = \mathbf{0}$$

hence  $\mathbf{W}$  is proper.

### Problem 2.37

We assume that  $x(t), y(t), z(t)$  are real-valued stochastic processes. The treatment of complex-valued processes is similar.

**a.**

$$R_{zz}(\tau) = E \{ [x(t + \tau) + y(t + \tau)] [x(t) + y(t)] \} = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

**b.** When  $x(t), y(t)$  are uncorrelated :

$$R_{xy}(\tau) = E [x(t + \tau)y(t)] = E [x(t + \tau)] E [y(t)] = m_x m_y$$

Similarly :

$$R_{yx}(\tau) = m_x m_y$$

Hence :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + 2m_x m_y$$

**c.** When  $x(t), y(t)$  are uncorrelated and have zero means :

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau)$$

### Problem 2.38

The power spectral density of the random process  $x(t)$  is :

$$\mathcal{S}_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau = N_0/2.$$

The power spectral density at the output of the filter will be :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f) |H(f)|^2 = \frac{N_0}{2} |H(f)|^2$$

Hence, the total power at the output of the filter will be :

$$R_{yy}(\tau = 0) = \int_{-\infty}^{\infty} \mathcal{S}_{yy}(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2} (2B) = N_0 B$$

### Problem 2.39

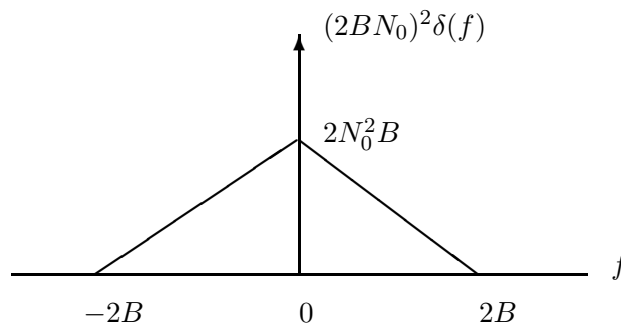
The power spectral density of  $X(t)$  corresponds to :  $R_{xx}(t) = 2BN_0 \frac{\sin 2\pi Bt}{2\pi Bt}$ . From the result of Problem 2.14 :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) = (2BN_0)^2 + 8B^2 N_0^2 \left( \frac{\sin 2\pi Bt}{2\pi Bt} \right)^2$$

Also :

$$\mathcal{S}_{yy}(f) = R_{xx}^2(0)\delta(f) + 2\mathcal{S}_{xx}(f) * \mathcal{S}_{xx}(f)$$

The following figure shows the power spectral density of  $Y(t)$  :



### Problem 2.40

$$\mathbf{M}_X = E[(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)'], \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mathbf{m}_x \text{ is the corresponding vector of mean values.}$$

Then :

$$\begin{aligned} \mathbf{M}_Y &= E[(\mathbf{Y} - \mathbf{m}_y)(\mathbf{Y} - \mathbf{m}_y)'] \\ &= E[\mathbf{A}(\mathbf{X} - \mathbf{m}_x)(\mathbf{A}(\mathbf{X} - \mathbf{m}_x))'] \\ &= E[\mathbf{A}(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)' \mathbf{A}'] \\ &= \mathbf{A} E[(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)'] \mathbf{A}' \\ &= \mathbf{A} \mathbf{M}_x \mathbf{A}' \end{aligned}$$

Hence :

$$\mathbf{M}_Y = \begin{bmatrix} \mu_{11} & 0 & \mu_{11} + \mu_{13} \\ 0 & 4\mu_{22} & 0 \\ \mu_{11} + \mu_{31} & 0 & \mu_{11} + \mu_{13} + \mu_{31} + \mu_{33} \end{bmatrix}$$

### Problem 2.41

$$Y(t) = X^2(t), \quad R_{xx}(\tau) = E[x(t+\tau)x(t)]$$

$$R_{yy}(\tau) = E[y(t+\tau)y(t)] = E[x^2(t+\tau)x^2(t)]$$

Let  $X_1 = X_2 = x(t)$ ,  $X_3 = X_4 = x(t+\tau)$ . Then, from problem 2.7 :

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

Hence :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

### Problem 2.42

$$p_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega}, \quad X = \frac{1}{\sqrt{\Omega}}R$$

$$\text{We know that : } p_X(x) = \frac{1}{1/\sqrt{\Omega}} p_R\left(\frac{x}{1/\sqrt{\Omega}}\right).$$

Hence :

$$p_X(x) = \frac{1}{1/\sqrt{\Omega}} \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \left(x\sqrt{\Omega}\right)^{2m-1} e^{-m(x\sqrt{\Omega})^2/\Omega} = \frac{2}{\Gamma(m)} m^m x^{2m-1} e^{-mx^2}$$

### Problem 2.43

The transfer function of the filter is :

$$H(f) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j2\pi fRC + 1}$$

a.

$$\mathcal{S}_{xx}(f) = \sigma^2 \Rightarrow \mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f) |H(f)|^2 = \frac{\sigma^2}{(2\pi RC)^2 f^2 + 1}$$

b.

$$R_{yy}(\tau) = F^{-1}\{\mathcal{S}_{xx}(f)\} = \frac{\sigma^2}{RC} \int_{-\infty}^{\infty} \frac{\frac{1}{RC}}{(\frac{1}{RC})^2 + (2\pi f)^2} e^{j2\pi f \tau} df$$

Let :  $a = RC$ ,  $v = 2\pi f$ . Then :

$$R_{yy}(\tau) = \frac{\sigma^2}{2RC} \int_{-\infty}^{\infty} \frac{a/\pi}{a^2 + v^2} e^{jv\tau} dv = \frac{\sigma^2}{2RC} e^{-a|\tau|} = \frac{\sigma^2}{2RC} e^{-|\tau|/RC}$$

where the last integral is evaluated in the same way as in problem P-2.9 . Finally :

$$E[Y^2(t)] = R_{yy}(0) = \frac{\sigma^2}{2RC}$$

### Problem 2.44

If  $\mathcal{S}_X(f) = 0$  for  $|f| > W$ , then  $\mathcal{S}_X(f)e^{-j2\pi fa}$  is also bandlimited. The corresponding autocorrelation function can be represented as (remember that  $\mathcal{S}_X(f)$  is deterministic) :

$$R_X(\tau - a) = \sum_{n=-\infty}^{\infty} R_X\left(\frac{n}{2W} - a\right) \frac{\sin 2\pi W \left(\tau - \frac{n}{2W}\right)}{2\pi W \left(\tau - \frac{n}{2W}\right)} \quad (1)$$

Let us define :

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{n}{2W}\right)}{2\pi W \left(t - \frac{n}{2W}\right)}$$

We must show that :

$$E\left[|X(t) - \hat{X}(t)|^2\right] = 0$$

or

$$E\left[\left(X(t) - \hat{X}(t)\right) \left(X(t) - \sum_{m=-\infty}^{\infty} X\left(\frac{m}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{m}{2W}\right)}{2\pi W \left(t - \frac{m}{2W}\right)}\right)\right] = 0 \quad (2)$$

First we have :

$$E\left[\left(X(t) - \hat{X}(t)\right) X\left(\frac{m}{2W}\right)\right] = R_X\left(t - \frac{m}{2W}\right) - \sum_{n=-\infty}^{\infty} R_X\left(\frac{n-m}{2W}\right) \frac{\sin 2\pi W \left(t - \frac{n}{2W}\right)}{2\pi W \left(t - \frac{n}{2W}\right)}$$

But the right-hand-side of this equation is equal to zero by application of (1) with  $a = m/2W$ . Since this is true for any  $m$ , it follows that  $E \left[ \left( X(t) - \hat{X}(t) \right) \hat{X}(t) \right] = 0$ . Also

$$E \left[ \left( X(t) - \hat{X}(t) \right) X(t) \right] = R_X(0) - \sum_{n=-\infty}^{\infty} R_X \left( \frac{n}{2W} - t \right) \frac{\sin 2\pi W \left( t - \frac{n}{2W} \right)}{2\pi W \left( t - \frac{n}{2W} \right)}$$

Again, by applying (1) with  $a = t$  and  $\tau = t$ , we observe that the right-hand-side of the equation is also zero. Hence (2) holds.

### Problem 2.45

$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt = P[N \geq x]$ , where  $N$  is a Gaussian r.v with zero mean and unit variance. From the Chernoff bound :

$$P[N \geq x] \leq e^{-\hat{v}x} E(e^{\hat{v}N}) \quad (1)$$

where  $\hat{v}$  is the solution to :

$$E(Ne^{vN}) - xE(e^{vN}) = 0 \quad (2)$$

Now :

$$\begin{aligned} E(e^{vN}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{vt} e^{-t^2/2} dt \\ &= e^{v^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t-v)^2/2} dt \\ &= e^{v^2/2} \end{aligned}$$

and

$$E(Ne^{vN}) = \frac{d}{dv} E(e^{vN}) = ve^{v^2/2}$$

Hence (2) gives :

$$\hat{v} = x$$

and then :

$$(1) \Rightarrow Q(x) \leq e^{-x^2} e^{x^2/2} \Rightarrow Q(x) \leq e^{-x^2/2}$$

### Problem 2.46

Since  $H(0) = \sum_{n=-\infty}^{\infty} h(n) = 0 \Rightarrow m_y = m_x H(0) = 0$

The autocorrelation of the output sequence is

$$R_{yy}(k) = \sum_i \sum_j h(i)h(j)R_{xx}(k-j+i) = \sigma_x^2 \sum_{i=-\infty}^{\infty} h(i)h(k+i)$$

where the last equality stems from the autocorrelation function of  $X(n)$  :

$$R_{xx}(k-j+i) = \sigma_x^2 \delta(k-j+i) = \begin{cases} \sigma_x^2, & j = k+i \\ 0, & o.w. \end{cases}$$

Hence,  $R_{yy}(0) = 6\sigma_x^2$ ,  $R_{yy}(1) = R_{yy}(-1) = -4\sigma_x^2$ ,  $R_{yy}(2) = R_{yy}(-2) = \sigma_x^2$ ,  $R_{yy}(k) = 0$  otherwise.

Finally, the frequency response of the discrete-time system is :

$$\begin{aligned} H(f) &= \sum_{-\infty}^{\infty} h(n)e^{-j2\pi fn} \\ &= 1 - 2e^{-j2\pi f} + e^{-j4\pi f} \\ &= (1 - e^{-j2\pi f})^2 \\ &= e^{-j2\pi f} (e^{j\pi f} - e^{-j\pi f})^2 \\ &= -4e^{-j\pi f} \sin^2 \pi f \end{aligned}$$

which gives the power density spectrum of the output :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)|H(f)|^2 = \sigma_x^2 [16 \sin^4 \pi f] = 16\sigma_x^2 \sin^4 \pi f$$

### Problem 2.47

$$R(k) = \left(\frac{1}{2}\right)^{|k|}$$

The power density spectrum is

$$\begin{aligned}
 \mathcal{S}(f) &= \sum_{k=-\infty}^{\infty} R(k) e^{-j2\pi f k} \\
 &= \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} e^{-j2\pi f k} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{-j2\pi f k} \\
 &= \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j2\pi f}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{-j2\pi f}\right)^k - 1 \\
 &= \frac{1}{1 - e^{j2\pi f}/2} + \frac{1}{1 - e^{-j2\pi f}/2} - 1 \\
 &= \frac{2 - \cos 2\pi f}{5/4 - \cos 2\pi f} - 1 \\
 &= \frac{3}{5 - 4 \cos 2\pi f}
 \end{aligned}$$

### Problem 2.48

We will denote the discrete-time process by the subscript  $d$  and the continuous-time (analog) process by the subscript  $a$ . Also,  $f$  will denote the analog frequency and  $f_d$  the discrete-time frequency.

a.

$$\begin{aligned}
 R_d(k) &= E[X^*(n)X(n+k)] \\
 &= E[X^*(nT)X(nT+kT)] \\
 &= R_a(kT)
 \end{aligned}$$

Hence, the autocorrelation function of the sampled signal is equal to the sampled autocorrelation function of  $X(t)$ .

b.

$$\begin{aligned}
 R_d(k) &= R_a(kT) = \int_{-\infty}^{\infty} \mathcal{S}_a(F) e^{j2\pi f k T} df \\
 &= \sum_{l=-\infty}^{\infty} \int_{(2l-1)/2T}^{(2l+1)/2T} \mathcal{S}_a(F) e^{j2\pi f k T} df \\
 &= \sum_{l=-\infty}^{\infty} \int_{-1/2T}^{1/2T} \mathcal{S}_a\left(f + \frac{l}{T}\right) e^{j2\pi F k T} df \\
 &= \int_{-1/2T}^{1/2T} \left[ \sum_{l=-\infty}^{\infty} \mathcal{S}_a\left(f + \frac{l}{T}\right) \right] e^{j2\pi F k T} df
 \end{aligned}$$



Let  $f_d = fT$ . Then :

$$R_d(k) = \int_{-1/2}^{1/2} \left[ \frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a((f_d + l)/T) \right] e^{j2\pi f_d k} df_d \quad (1)$$

We know that the autocorrelation function of a discrete-time process is the inverse Fourier transform of its power spectral density

$$R_d(k) = \int_{-1/2}^{1/2} \mathcal{S}_d(f_d) e^{j2\pi f_d k} df_d \quad (2)$$

Comparing (1),(2) :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \sum_{l=-\infty}^{\infty} \mathcal{S}_a\left(\frac{f_d + l}{T}\right) \quad (3)$$

c. From (3) we conclude that :

$$\mathcal{S}_d(f_d) = \frac{1}{T} \mathcal{S}_a\left(\frac{f_d}{T}\right)$$

iff :

$$\mathcal{S}_a(f) = 0, \quad \forall f : |f| > 1/2T$$

Otherwise, the sum of the shifted copies of  $\mathcal{S}_a$  (in (3)) will overlap and aliasing will occur.

### Problem 2.49

$$u(t) = X \cos 2\pi ft - Y \sin 2\pi ft$$

$$E[u(t)] = E(X) \cos 2\pi ft - E(Y) \sin 2\pi ft$$

and :

$$\begin{aligned} R_{uu}(t, t + \tau) &= E \{ [X \cos 2\pi ft - Y \sin 2\pi ft] [X \cos 2\pi f(t + \tau) - Y \sin 2\pi f(t + \tau)] \} \\ &= E(X^2) [\cos 2\pi f(2t + \tau) + \cos 2\pi f\tau] + E(Y^2) [-\cos 2\pi f(2t + \tau) + \cos 2\pi f\tau] \\ &\quad - E(XY) \sin 2\pi f(2t + \tau) \end{aligned}$$

For  $u(t)$  to be wide-sense stationary, we must have :  $E[u(t)] = \text{constant}$  and  $R_{uu}(t, t + \tau) = R_{uu}(\tau)$ . We note that if  $E(X) = E(Y) = 0$ , and  $E(XY) = 0$  and  $E(X^2) = E(Y^2)$ , then the above requirements for WSS hold; hence these conditions are necessary. Conversely, if any of the above

conditions does not hold, then either  $E[u(t)] \neq \text{constant}$ , or  $R_{uu}(t, t + \tau) \neq R_{uu}(\tau)$ . Hence, the conditions are also necessary.

### Problem 2.50

a.

$$\begin{aligned} R_a(\tau) &= \int_{-\infty}^{\infty} \mathcal{S}_a(f) e^{j2\pi f\tau} df \\ &= \int_{-W}^W e^{j2\pi f\tau} df \\ &= \frac{\sin 2\pi W\tau}{\pi\tau} \end{aligned}$$

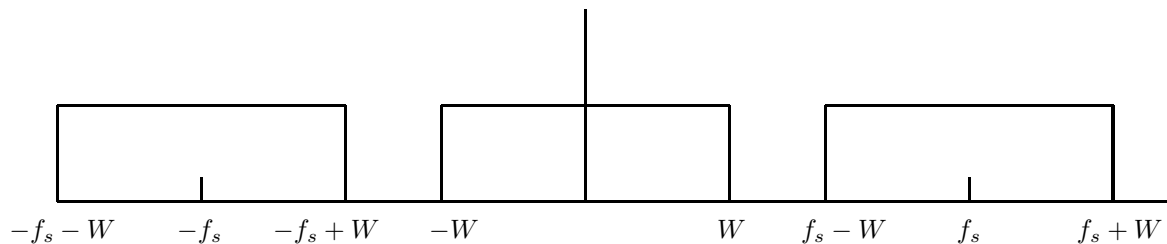
By applying the result in problem 2.21, we have

$$R_d(k) = f_a(kT) = \frac{\sin 2\pi W kT}{\pi kT}$$

b. If  $T = \frac{1}{2W}$ , then :

$$R_d(k) = \begin{cases} 2W = 1/T, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the sequence  $X(n)$  is a white-noise sequence. The fact that this is the minimum value of  $T$  can be shown from the following figure of the power spectral density of the sampled process:



We see that the maximum sampling rate  $f_s$  that gives a spectrally flat sequence is obtained when :

$$W = f_s - W \Rightarrow f_s = 2W \Rightarrow T = \frac{1}{2W}$$

c. The triangular-shaped spectrum  $\mathcal{S}(f) = 1 - \frac{|f|}{W}$ ,  $|f| \leq W$  may be obtained by convolving the rectangular-shaped spectrum  $\mathcal{S}_1(f) = 1/\sqrt{W}$ ,  $|f| \leq W/2$ . Hence,  $R(\tau) = R_1^2(\tau) = \frac{1}{W} \left( \frac{\sin \pi W \tau}{\pi \tau} \right)^2$ . Therefore, sampling  $X(t)$  at a rate  $\frac{1}{T} = W$  samples/sec produces a white sequence with autocorrelation function :

$$R_d(k) = \frac{1}{W} \left( \frac{\sin \pi W k T}{\pi k T} \right)^2 = W \left( \frac{\sin \pi k}{\pi k} \right)^2 = \begin{cases} W, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

### Problem 2.51

Let's denote :  $y(t) = f_k(t)f_j(t)$ . Then :

$$\int_{-\infty}^{\infty} f_k(t)f_j(t)dt = \int_{-\infty}^{\infty} y(t)dt = Y(f)|_{f=0}$$

where  $Y(f)$  is the Fourier transform of  $y(t)$ . Since :  $y(t) = f_k(t)f_j(t) \longleftrightarrow Y(f) = F_k(f) * F_j(f)$ . But :

$$F_k(f) = \int_{-\infty}^{\infty} f_k(t)e^{-j2\pi ft}dt = \frac{1}{2W}e^{-j2\pi fk/2W}$$

Then :

$$Y(f) = F_k(f) * F_j(f) = \int_{-\infty}^{\infty} F_k(a) * F_j(f - a)da$$

and at  $f = 0$  :

$$\begin{aligned} Y(f)|_{f=0} &= \int_{-\infty}^{\infty} F_k(a) * F_j(-a)da \\ &= \left( \frac{1}{2W} \right)^2 \int_{-\infty}^{\infty} e^{-j2\pi a(k-j)/2W} da \\ &= \begin{cases} 1/2W, & k = j \\ 0, & k \neq j \end{cases} \end{aligned}$$

### Problem 2.52

$$B_{eq} = \frac{1}{G} \int_0^{\infty} |H(f)|^2 df$$

For the filter shown in Fig. P2-12 we have  $G = 1$  and

$$B_{eq} = \int_0^{\infty} |H(f)|^2 df = B$$

For the lowpass filter shown in Fig. P2-16 we have

$$H(f) = \frac{1}{1 + j2\pi fRC} \Rightarrow |H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

So  $G = 1$  and

$$\begin{aligned} B_{eq} &= \int_0^\infty |H(f)|^2 df \\ &= \frac{1}{2} \int_{-\infty}^\infty |H(f)|^2 df \\ &= \frac{1}{4RC} \end{aligned}$$

where the last integral is evaluated in the same way as in problem P-2.9 .

### Problem 2.53

a.

$$\begin{aligned} E[z(t)z(t+\tau)] &= E[\{x(t+\tau) + jy(t+\tau)\} \{x(t) + jy(t)\}] \\ &= E[x(t)x(t+\tau)] - E[y(t)y(t+\tau)] + jE[x(t)y(t+\tau)] \\ &\quad + E[y(t)x(t+\tau)] \\ &= R_{xx}(\tau) - R_{yy}(\tau) + j[R_{yx}(\tau) + R_{xy}(\tau)] \end{aligned}$$

But  $R_{xx}(\tau) = R_{yy}(\tau)$  and  $R_{yx}(\tau) = -R_{xy}(\tau)$ . Therefore :

$$E[z(t)z(t+\tau)] = 0$$

b.

$$\begin{aligned} V &= \int_0^T z(t) dt \\ E(V^2) &= \int_0^T \int_0^T E[z(a)z(b)] da db = 0 \end{aligned}$$

from the result in (a) above. Also :

$$\begin{aligned} E(VV^*) &= \int_0^T \int_0^T E[z(a)z^*(b)] da db \\ &= \int_0^T \int_0^T N_0 \delta(a-b) da db \\ &= \int_0^T N_0 da = N_0 T \end{aligned}$$

**Problem 2.54**

$$\begin{aligned}
E[x(t+\tau)x(t)] &= A^2 E[\sin(2\pi f_c(t+\tau) + \theta) \sin(2\pi f_c t + \theta)] \\
&= \frac{A^2}{2} \cos 2\pi f_c \tau - \frac{A^2}{2} E[\cos(2\pi f_c(2t + \tau) + 2\theta)]
\end{aligned}$$

where the last equality follows from the trigonometric identity :

$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$ . But :

$$\begin{aligned}
E[\cos(2\pi f_c(2t + \tau) + 2\theta)] &= \int_0^{2\pi} \cos(2\pi f_c(2t + \tau) + 2\theta) p(\theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi f_c(2t + \tau) + 2\theta) d\theta = 0
\end{aligned}$$

Hence :

$$E[x(t+\tau)x(t)] = \frac{A^2}{2} \cos 2\pi f_c \tau$$

**Problem 2.55**

1) We have  $E[Z(t)] = E[X(t)] + jE[Y(t)] = 0 + j0 = 0$  and

$$\begin{aligned}
R_Z(t+\tau, t) &= E[(X(t+\tau) + jY(t+\tau))(X(t) - jY(t))] \\
&= R_X(\tau) + R_Y(\tau) \\
&= 2R_X(\tau)
\end{aligned}$$

because  $E[X(t+\tau)Y(t)] = E[Y(t+\tau)X(t)] = E[X(t+\tau)]E[Y(t)] = 0$  (by independence) and therefore  $Z(t)$  is obviously stationary. We also note that  $R_X(\tau) = R_Y(\tau) = \mathcal{F}^{-1} \left[ N_0 \Pi \left( \frac{f}{2W} \right) \right] = 2WN_0 \text{sinc}(2W\tau)$

2) To compute the power spectral density of  $Z(t)$ , we have  $\mathcal{S}_Z(f) = \mathcal{F}[2R_X(\tau)] = 2\mathcal{S}_X(f) = 2N_0 \Pi \left( \frac{f}{2W} \right)$ . Note that  $\Pi(t)$  is a rectangular pulse defined as

$$\Pi(t) = \begin{cases} 1, & |t| < 1 \\ \frac{1}{2}, & |t| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

3)  $E[Z_j] = E \left[ \int_{-\infty}^{\infty} Z(t) R_j^*(t) dt \right] = \int_{-\infty}^{\infty} E[Z(t)] R_j^*(t) dt = 0$  since  $Z(t)$  is zero-mean. For the

correlation we have

$$\begin{aligned}
 E[Z_j Z_k^*] &= E \left[ \int_{-\infty}^{\infty} Z(s) R_j^*(s) ds \int_{-\infty}^{\infty} Z^*(t) R_k(t) dt \right] \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) R_k(t) ds dt \\
 &= \int_{-\infty}^{\infty} R_k(t) \left[ \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds \right] dt \quad (**)
 \end{aligned}$$

Using Parseval's Theorem,  $\int_{-\infty}^{\infty} x(t)y^*(t) dt = \int_{-\infty}^{\infty} X(f)Y^*(f) df$ , we have  $(\mathcal{S}_j(f))$  is the Fourier transform of  $R_j(t)$ .

$$\begin{aligned}
 \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds &= \int_{-\infty}^{\infty} e^{-j2\pi ft} 2N_0 \Pi\left(\frac{f}{2W}\right) \mathcal{S}_j^*(f) df \\
 &\stackrel{a}{=} 2 \int_{-W}^W N_0 e^{-j2\pi ft} \mathcal{S}_j^*(f) df \\
 &\stackrel{b}{=} 2 \int_{-\infty}^{\infty} N_0 e^{-j2\pi ft} \mathcal{S}_j^*(f) df
 \end{aligned}$$

where (a) is due to the fact that  $\Pi\left(\frac{f}{2W}\right)$  is zero outside the  $[-W, W]$  interval and (b) follows from  $R_j(t)$  being bandlimited to  $[-W, W]$ . From above we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} R_Z(s-t) R_j^*(s) ds &= 2N_0 \left[ \int_{-\infty}^{\infty} e^{j2\pi ft} \mathcal{S}_j(f) df \right]^* \\
 &= 2N_0 R_j^*(t)
 \end{aligned}$$

Substituting this result in equation (\*\*) we have

$$\begin{aligned}
 E[Z_j Z_k^*] &= 2 \int_{-\infty}^{\infty} N_0 R_j^*(t) R_k(t) dt \\
 &= \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}
 \end{aligned}$$

This shows that  $Z_j$ 's are Gaussian random variables (since they are the result of linear operation on a Gaussian process) with mean zero and variance  $2N_0$ , i.e.,  $Z_j \sim \mathcal{N}(0, 2N_0)$ . Also note that for  $j \neq k$ ,  $Z_j$  and  $Z_k$  are independent since they are Gaussian and uncorrelated.

4) This is done similar to part 3 (lengthy but straightforward) and the result is that for any  $k$ ,  $Z_{kr}$  and  $Z_{ki}$  are zero-mean, independent Gaussian random variables with  $E(Z_{kr}^2) = E(Z_{ki}^2) = N_0$  and therefore the random vector  $(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \dots, Z_{nr}, Z_{ni})$  is a  $2n$ -dimensional Gaussian vector with independent zero-mean components each having variance  $N_0$ . In standard notation

$$(Z_{1r}, Z_{1i}, Z_{2r}, Z_{2i}, \dots, Z_{nr}, Z_{ni}) \sim \mathcal{N}(\mathbf{0}, N_0 \mathbf{I})$$

where  $\mathbf{0}$  is a  $2n$ -dimensional zero vector and  $\mathbf{I}$  is a  $2n \times 2n$  identity matrix.

5) We have

$$\begin{aligned} E[\hat{Z}(t)Z_k^*] &= E[(Z(t) - \sum_{j=1}^N Z_j R_j(t))Z_k^*] \\ &= E[Z(t)Z_k^*] - 2N_0 R_k(t) \end{aligned}$$

where we have used

$$E[Z_j Z_k^*] = \begin{cases} 2N_0, & j = k \\ 0, & j \neq k \end{cases}$$

Now we have

$$\begin{aligned} E[Z(t)Z_k^*] &= E \left[ Z(t) \int_{-\infty}^{\infty} Z^*(s) R_k(s) ds \right] \\ &= \int_{-\infty}^{\infty} R_Z(t-s) R_k(s) ds \\ &= \int_{-\infty}^{\infty} R_k(s) R_Z^*(s-t) ds \\ &= 2 \int_{-\infty}^{\infty} \mathcal{S}_k(f) e^{j2\pi f t} N_0 \Pi \left( \frac{f}{2W} \right) df \\ &= \int_{-W}^W 2N_0 \mathcal{S}_k(f) e^{j2\pi f t} dt \\ &\stackrel{a}{=} 2N_0 \int_{-\infty}^{\infty} \mathcal{S}_k(f) e^{j2\pi f t} df \\ &= 2N_0 R_k(t) \end{aligned}$$

(a): because  $R_k(t)$  is bandlimited to  $[-W, W]$ .

From above it follows that  $E[\hat{Z}(t)Z_k^*] = 0$  for all  $k = 1, 2, \dots, N$ . This means that the error term is independent of the projections.

### Problem 2.56

1.  $\mathcal{S}_{\hat{X}}(f) = |-j \operatorname{sgn}(f)|^2 \mathcal{S}_X(f) = \mathcal{S}_X(f)$ , hence  $R_{\hat{X}}(\tau) = R_X(\tau)$ .
2.  $\mathcal{S}_{X\hat{X}}(f) = \mathcal{S}_X(f)(-j \operatorname{sgn}(f))^* = j \operatorname{sgn}(f) \mathcal{S}_X(f)$ , therefore,  $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$ .
3.  $R_Z(\tau) = E \left[ \left( X(t+\tau) + j\hat{X}(t+\tau) \right) \left( X(t) - j\hat{X}(t) \right) \right]$ , expanding we have

$$R_Z(\tau) = R_X(\tau) + R_{\hat{X}}(\tau) - j [R_{X\hat{X}}(\tau) - R_{\hat{X}X}(\tau)]$$

Using  $R_{\hat{X}}(\tau) = R_X(\tau)$ , and the fact that  $R_{X\hat{X}}(\tau) = -\hat{R}_X(\tau)$  is an odd function (since it is the HT of an even signal) we have  $R_{\hat{X}X}(\tau) = R_{X\hat{X}}(-\tau) = -R_{X\hat{X}}(\tau)$ , we have

$$R_Z(\tau) = 2R_X(\tau) - j2R_{X\hat{X}}(\tau) = 2R_X(\tau) + j2\hat{R}_X(\tau)$$

Taking FT of both sides we have

$$\mathcal{S}_Z(f) = 2\mathcal{S}_X(f) + j2(-j\text{sgn}(f)\mathcal{S}_X(f)) = 2(1 + \text{sgn}(f))\mathcal{S}_X(f) = 4\mathcal{S}_X(f)u_{-1}(f)$$

4. We have

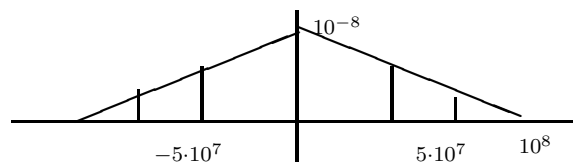
$$\begin{aligned} R_{X_l}(t + \tau, t) &= E \left[ Z(t + \tau) e^{-j2\pi f_0(t+\tau)} Z^*(t) e^{j2\pi f_0 t} \right] \\ &= e^{-j2\pi f_0 \tau} R_Z(\tau) \end{aligned}$$

This shows that  $X_l(t)$  is WSS (we already know it is zero-mean). Taking FT, we have  $\mathcal{S}_{X_l}(f) = \mathcal{S}_Z(f - f_0) = 4\mathcal{S}_X(f - f_0)u_{-1}(f - f_0)$ , this shows that  $X_l(t)$  is lowpass. Also from above  $R_X(\tau) = \frac{1}{2}\text{Re}[R_Z(t)] = \frac{1}{2}\text{Re}[R_{X_l}(\tau)e^{j2\pi f_0 \tau}]$ . This shows that  $R_{X_l}(\tau)$  is twice the LP equivalent of  $R_X(\tau)$ .

### Problem 2.57

1) The power spectral density  $\mathcal{S}_n(f)$  is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band  $49 \times 10^6 \leq |f| \leq 51 \times 10^6$ . The power content is

$$\begin{aligned} P &= \int_{-51 \times 10^6}^{-49 \times 10^6} 10^{-8} \left( 1 + \frac{f}{10^8} \right) df + \int_{49 \times 10^6}^{51 \times 10^6} 10^{-8} \left( 1 - \frac{f}{10^8} \right) df \\ &= 10^{-8} x \Big|_{-51 \times 10^6}^{-49 \times 10^6} + 10^{-16} \frac{1}{2} x^2 \Big|_{-51 \times 10^6}^{-49 \times 10^6} + 10^{-8} x \Big|_{49 \times 10^6}^{51 \times 10^6} - 10^{-16} \frac{1}{2} x^2 \Big|_{49 \times 10^6}^{51 \times 10^6} \\ &= 2 \times 10^{-2} \end{aligned}$$



2) The output process  $N(t)$  can be written as

$$N(t) = N_c(t) \cos(2\pi 50 \times 10^6 t) - N_s(t) \sin(2\pi 50 \times 10^6 t)$$



where  $N_c(t)$  and  $N_s(t)$  are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t) \cos(2\pi 50 \times 10^6 t) + \hat{N}(t) \sin(2\pi 50 \times 10^6 t) \\ N_s(t) &= \hat{N}(t) \cos(2\pi 50 \times 10^6 t) - N(t) \sin(2\pi 50 \times 10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2] \cos^2(2\pi 50 \times 10^6 t) + E[|\hat{N}(t)|^2] \sin^2(2\pi 50 \times 10^6 t) \\ &= E[|N(t)|^2] = 2 \times 10^{-2} \end{aligned}$$

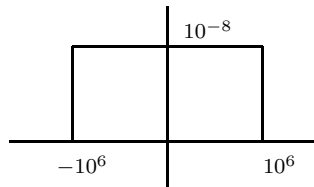
where we have used the fact that  $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$ . Similarly we find that  $E[|N_s(t)|^2] = 2 \times 10^{-2}$ .

3) The power spectral density of  $N_c(t)$  and  $N_s(t)$  is

$$\mathcal{S}_{N_c}(f) = \mathcal{S}_{N_s}(f) = \begin{cases} \mathcal{S}_N(f - 50 \times 10^6) + \mathcal{S}_N(f + 50 \times 10^6) & |f| \leq 50 \times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}_{N_c}(f)$  is depicted in the next figure. The power content of  $\mathcal{S}_{N_c}(f)$  can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2 \times 10^{-2}$$



4) The power spectral density of the output is given by

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f) |H(f)|^2 = 10^{-6} (|f| - 49 \times 10^6) (10^{-8} - 10^{-16} |f|) \quad \text{for } 49 \times 10^6 \leq |f| \leq 51 \times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= 10^{-6} \left( \int_{-51 \times 10^6}^{-49 \times 10^6} (-f - 49 \times 10^6) (10^{-8} + 10^{-16} f) df \right) \\ &\quad + 10^{-6} \left( \int_{49 \times 10^6}^{51 \times 10^6} (f - 49 \times 10^6) (10^{-8} - 10^{-16} f) df \right) \\ &= 10^{-6} \left( 2 \times 10^4 - \frac{4}{3} 10^2 \right) \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} \mathcal{S}_{Y_c}(f) = \mathcal{S}_{Y_s}(f) &= 10^{-6} ((f + 50 \times 10^6) - 49 \times 10^6) (10^{-8} - 10^{-16} (f + 50 \times 10^6)) \\ &\quad + 10^{-6} ((-f - 50 \times 10^6) - 49 \times 10^6) (10^{-8} + 10^{-16} (f - 50 \times 10^6)) \\ &= 10^{-6} (-2 \times 10^{-16} f^2 + 10^{-2}) \end{aligned}$$

for  $|f| \leq 10^6$  and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned}
 P_{Y_c} = P_{Y_s} &= 10^{-6} \int_{-10^6}^{10^6} (-2 \times 10^{-16} f^2 + 10^{-2}) df \\
 &= 10^{-6} \left( -2 \times 10^{-16} \frac{1}{3} f^3 \Big|_{-10^6}^{10^6} + 10^{-2} f \Big|_{-10^6}^{10^6} \right) \\
 &= 10^{-6} \left( 2 \times 10^4 - \frac{4}{3} 10^2 \right) = P_Y
 \end{aligned}$$