1 Functions and Models

1.1 Four Ways to Represent a Function

Suggested Time and Emphasis

1 class   Essential material

Points to Stress

1. Understanding the interplay between the four ways of representing a function (verbally, numerically, visually, algebraically) perhaps using the concepts of increasing and decreasing functions as an example.
2. Finding the domain and range of a function, regardless of representation.
3. Investigating even and odd functions.
4. Working with piecewise defined functions.

Quiz Questions

- **TEXT QUESTION** Why does the author assert that "the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$"?

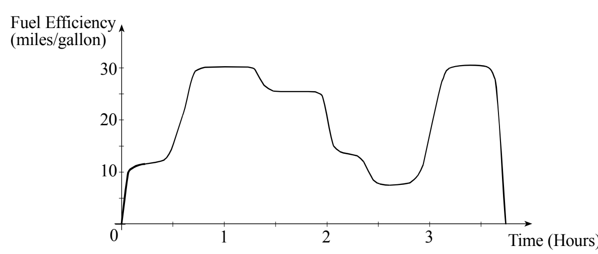
  ANSWER The calculator gives an approximation to the square root.

- **DRILL QUESTION** Fill in the blanks: $|x| = \begin{cases} \_\_\_ & \text{if } x \geq 0 \\ \_\_\_ & \text{if } x < 0 \end{cases}$

  ANSWER x, $-x$

Materials for Lecture

- Draw a graph of electrical power consumption in the classroom versus time on a typical weekday, pointing out important features throughout, and using the vocabulary of this section as much as possible.
- Draw a graph of fuel efficiency versus time on a trip, such as the one below. Lead a discussion of what could have happened on the trip.



- In 1984, United States President Ronald Reagan proposed a plan to change the United States personal income tax system. According to his plan, the income tax would be 15% on the first $19,300 earned, 25% on the next $18,800, and 35% on all income above and beyond that. Describe this situation to the class, and have them graph (marginal) tax rate and tax owed versus income for incomes ranging from $0 to $80,000. Then have them try to come up with equations describing this situation.

- In the year 2000, Presidential candidate Steve Forbes proposed a "flat tax" model: 0% on the first $36,000 and 17% on the rest. Have the students do the same analysis, and compare the two models. As an extension, perhaps have the students look at a current tax table and draw similar graphs.

Workshop/Discussion

- Present graphs of even and odd functions, such as $\sin x$, $\cos x + x^2$, and $\cos(\sin x)$, and check with the standard algebraic tests.
- Start with a table of values for the function $f(x) = \frac{1}{4}x^2 + x$:

x	0	1	2	3	4
$f(x)$	0	1.25	3	5.25	8

  First, have the class describe the behavior of the function in words, trying to elicit the information that the function is increasing, and that its rate of increase is also increasing. Then, have them try to extrapolate the function in both directions, debating whether or not the function is always positive and increasing. Plot the points and connect the dots, then have them try to concoct a formula (not necessarily expecting them to succeed).
- Discuss the domain and range of a function such as $f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

  Also talk about why f is neither increasing nor decreasing for $x > 0$. Stress that when dealing with new sorts of functions, it becomes important to know the precise mathematical definitions of such terms.
- Define "difference quotient" as done in the text. Define $f(x) = x^3$, and show that $\dfrac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2$. This example both reviews algebra skills and foreshadows future calculations.

Group Work 1: Every Picture Tells a Story

Put the students in groups of four, and have them work on the exercise. If there are questions, encourage them to ask each other before asking you. After going through the correct matching with them, have each group tell their story to the class and see if it fits the remaining graph.

ANSWERS

1. (b) **2.** (a) **3.** (c) **4.** The roast beef was cooked in the morning and put in the refrigerator in the afternoon.

Group Work 2: Finding a Formula

Make sure that the students know the equation of a circle with radius r, and that they remember the notation for piecewise-defined functions. Split the students into groups of four. In each group, have half of the students work on each problem first, and then have them check each other's work. If the students find these problems difficult, have them work together on each problem.

ANSWERS

1. $f(x) = \begin{cases} -x-2 & \text{if } x \le -2 \\ x+2 & \text{if } -2 < x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$ **2.** $g(x) = \begin{cases} x+4 & \text{if } x \le -2 \\ 2 & \text{if } -2 < x \le 0 \\ \sqrt{4-x^2} & \text{if } 0 < x \le 2 \\ x-2 & \text{if } x > 2 \end{cases}$

Homework Problems

CORE EXERCISES 3, 7, 15, 23, 25, 27, 31, 33, 39, 41, 43, 69, 73, 77

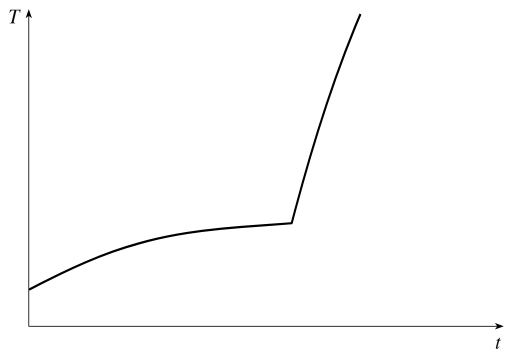
SAMPLE ASSIGNMENT 2, 3, 7, 9, 15, 23, 25, 27, 29, 31, 33, 39, 41, 43, 47, 49, 53, 61, 63, 64, 69, 73, 75, 77

EXERCISE	D	A	N	G
2	×	×		
3				×
7				×
9				×
15				×
23			×	×
25		×		
27		×		
29		×		
31		×		
33		×		
39		×		×

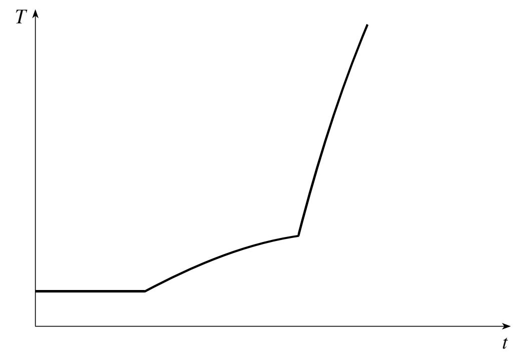
EXERCISE	D	A	N	G
41		×		×
43		×		×
47		×		×
49		×		×
53		×		
61		×		
63		×		
64		×		
69				×
73		×		×
75		×		×
77		×		×

Every Picture Tells a Story

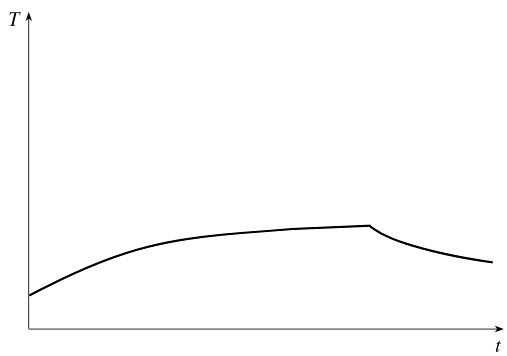
One of the skills you will be learning in this course is the ability to take a description of a real-world occurrence, and translate it into mathematics. Conversely, given a mathematical description of a phenomenon, you will learn how to describe what is happening in plain language. Here follow four graphs of temperature versus time and three stories. Match the stories with the graphs. When finished, write a similar story that would correspond to the final graph.



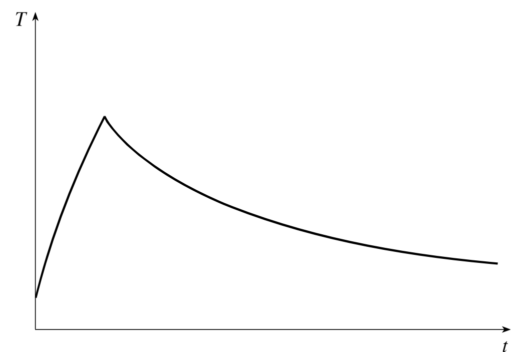
Graph 1



Graph 2



Graph 3



Graph 4

(a) I took my roast beef out of the freezer at noon, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

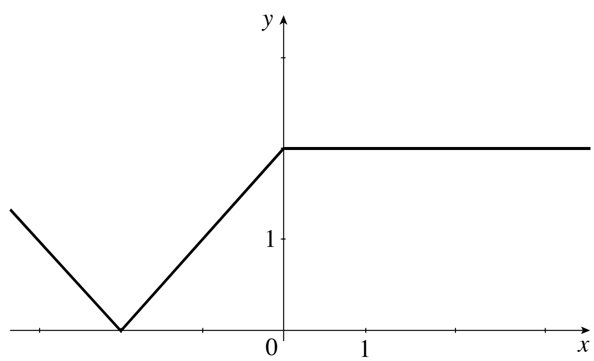
(b) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

(c) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. I forgot about it, and went out for Chinese food on my way home from work. I put it in the refrigerator when I finally got home.

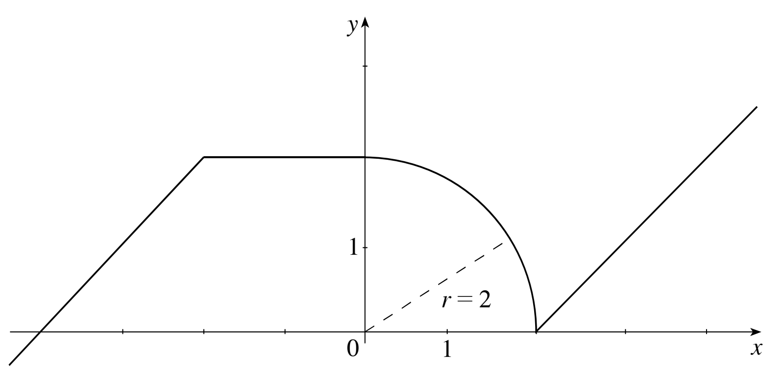
Finding a Formula

Find formulas for the following functions:

1.



2.



1.2 Mathematical Models: A Catalog of Essential Functions

Suggested Time and Emphasis

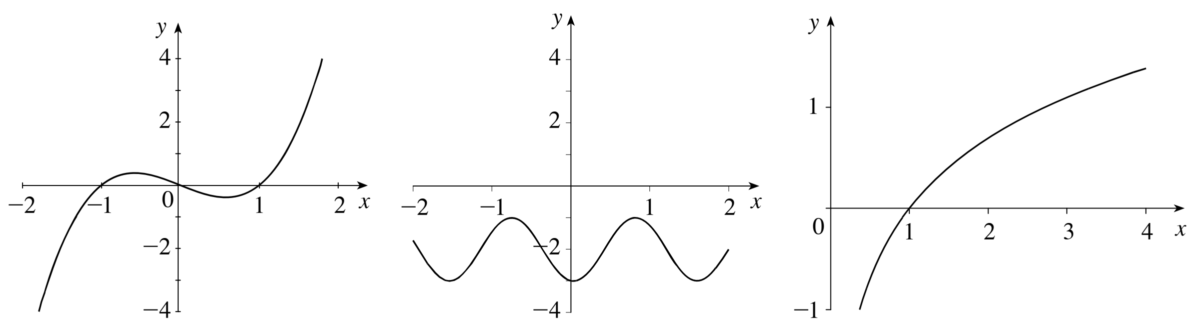
1 class Recommended material

Points to Stress

1. The modeling process: developing, analyzing, and interpreting a mathematical model.
2. Classes of functions: linear, power, rational, algebraic, trigonometric, and exponential functions. Include the special characteristics of each class of functions.

Quiz Questions

- **TEXT QUESTION** What is the difference between a power function x^n with $n = 3$ and a cubic function?
  ANSWER A cubic function can have lower order terms, whereas a power function has just one term.
- **DRILL QUESTION** Classify each function graphed below as a power function, root function, polynomial, rational function, algebraic function, trigonometric function, exponential function or logarithmic function. Explain your reasoning.



  ANSWER Polynomial, trigonometric, logarithmic

Materials for Lecture

- Show that linear functions have constant differences in y-values for equally spaced x-values, and that exponential functions have constant ratios in y-values for equally spaced x-values. These examples illustrate the point:

Linear function (difference=1.2)

x	$f(x)$
−2	−2.0
0	−0.8
2	0.4
4	1.6

Exponential function (ratio=1.25)

x	$f(x)$
3	1.000000
6	1.250000
9	1.562500
12	1.953125

- Discuss the shape, symmetries, and general "flatness" near 0 of the power functions x^n for various values of n. Similarly discuss $\sqrt[n]{x}$ for n even and n odd. A blackline master is provided at the end of this section, before the group work handouts.
- If Exercises 24–28 are to be assigned, Exercise 23 can be done in class, discussing part (c) in the context of the technology available to the students.

Workshop/Discussion

- Have the students graph 2^x, $\sin x$, $\sin 2^x$, and $2^{\sin x}$. Discuss why the latter two look the way that they do, then discuss the relationship among the graphs of $f(x) = 2^x$, $g(x) = (0.5)^x$, $h(x) = 2^{-x}$, and $k(x) = 4^x$.
- Figure 17 shows examples of a noncontinuous function and a nondifferentiable function, both expressible as simple formulas. Discuss these curves with the students, trying to get them to describe the ideas of a break in a graph and a cusp.

Group Work 1: Rounding the Bases

On the board, review how to compute the percentage error when estimating π by $\frac{22}{7}$. (Answer: 0.04%) Have them work on the problem in groups. If a group finishes early, have them look at $h(7)$ and $h(10)$ to see how fast the error grows.

ANSWERS **1.** 17.811434627, 17, 4.56% **2.** 220.08649875, 201, 8.67% **3.** 45.4314240633, 32, 29.56%

Group Work 2: The Small Shall Grow Large

If a group finishes early, ask them to similarly compare x^3 and x^4.

ANSWERS **1.** $x^6 \geq x^8$ for $-1 \leq x \leq 1$ **2.** $x^3 \geq x^5$ for $-\infty < x \leq -1, 0 \leq x \leq 1$ **3.** $x^3 \geq x^{105}$ for $-\infty < x \leq -1, 0 \leq x \leq 1$. If the exponents are both even, the answer is the same as for Problem 1, if the exponents are both odd, the answer is the same as for Problem 2.

Group Work 3: Fun with Fourier

This activity should be given before Fourier series are discussed in class. This activity will get students looking at combinations of sine curves, while at the same time foreshadowing the concepts of infinite series and Fourier series.

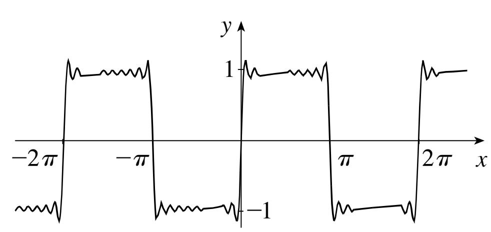
ANSWERS

1. No

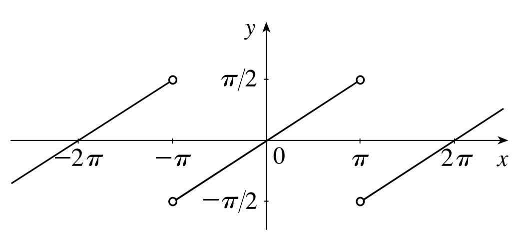
2. $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$

3. $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x\right)$

4. $$\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \frac{\sin 11x}{11} + \frac{\sin 13x}{13} + \frac{\sin 15x}{15} + \frac{\sin 17x}{17} + \frac{\sin 19x}{19}\right)$$



5.

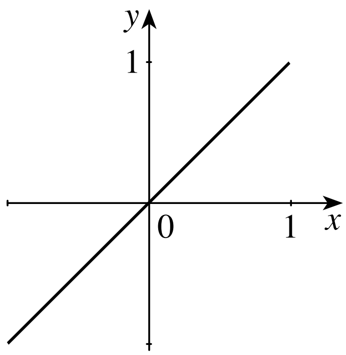


Homework Problems

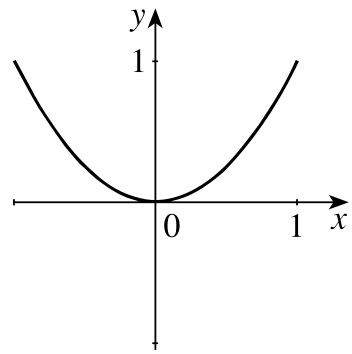
CORE EXERCISES 1, 3, 13, 15, 19, 21, 27

SAMPLE ASSIGNMENT 1, 3, 4, 5, 8, 11, 13, 15, 17, 19, 21, 27

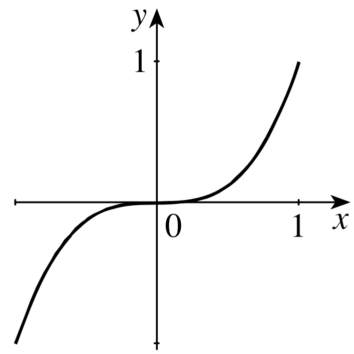
EXERCISE	D	A	N	G
1	×			
3				×
4				×
5	×	×		×
8		×		×
11	×	×		
13				×
15		×		
17		×		
19				×
21	×		×	×
27		×	×	



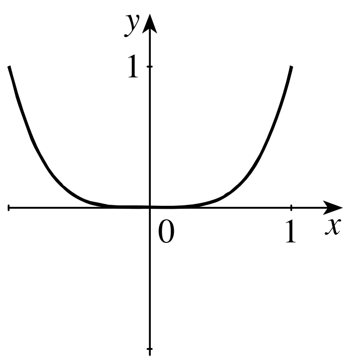
x



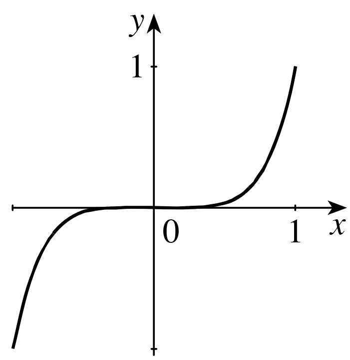
x^2



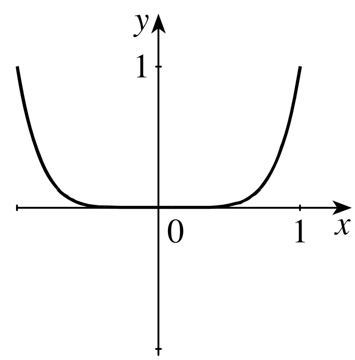
x^3



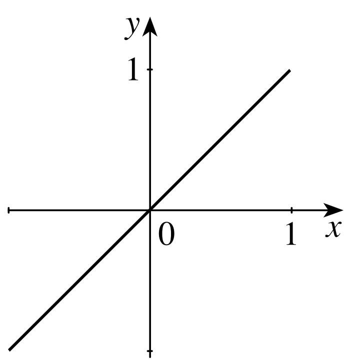
x^4



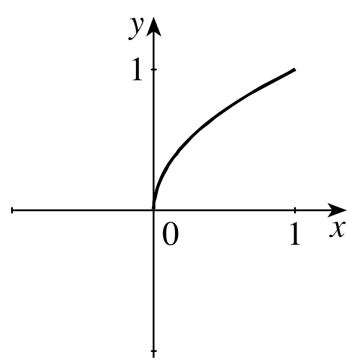
x^5



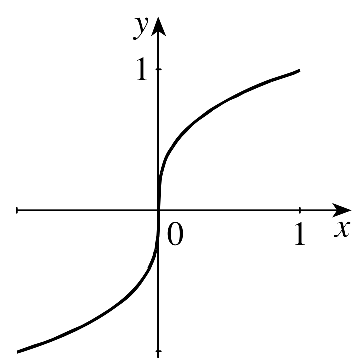
x^6



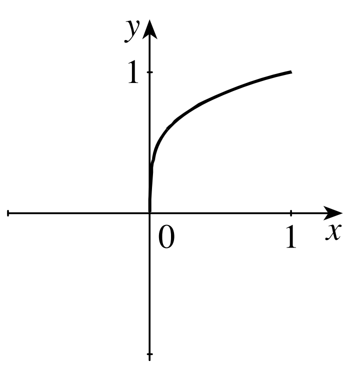
x



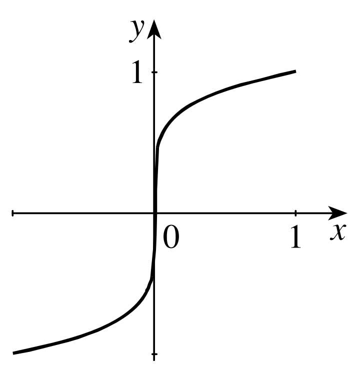
$\sqrt{x}$



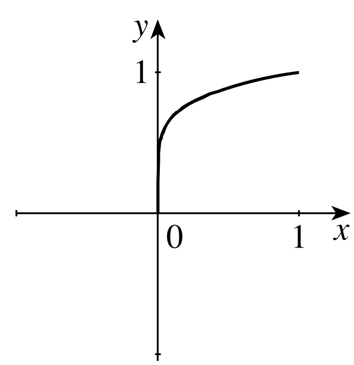
$\sqrt[3]{x}$



$\sqrt[4]{x}$



$\sqrt[5]{x}$



$\sqrt[6]{x}$

GROUP WORK 1, SECTION 1.2

Rounding the Bases

1. For computational efficiency and speed, we often round off constants in equations. For example, consider the linear function

$$f(x) = 3.137619523x + 2.123337012$$

In theory, it is very easy and quick to find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. In practice, most people doing this computation would probably substitute

$$f(x) = 3x + 2$$

unless a very accurate answer is called for. For example, compute $f(5)$ both ways to see the difference.

The actual value of $f(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

2. Now consider

$$g(x) = 1.12755319x^3 + 3.125694x^2 + 1$$

Again, one is tempted to substitute $g(x) = x^3 + 3x^2 + 1$.

The actual value of $g(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

3. It turns out to be very dangerous to similarly round off exponential functions, due to the nature of their growth. For example, let's look at the function

$$h(x) = (2.145217198123)^x$$

One may be tempted to substitute $h(x) = 2^x$ for this one. Once again, look at the difference between these two functions.

The actual value of $h(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

GROUP WORK 2, SECTION 1.2

The Small Shall Grow Large

1. For what values of x is $x^6 \geq x^8$? For what values is $x^6 \leq x^8$?

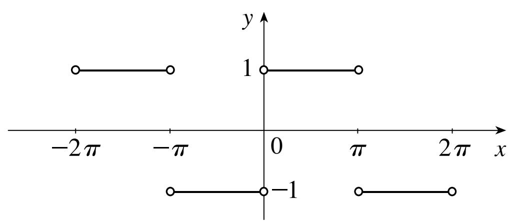
2. For what values of x is $x^3 \geq x^5$?

3. For what values of x is $x^3 \geq x^{105}$? Can you generalize your results?

GROUP WORK 3, SECTION 1.2

Fun with Fourier

The following function $S(x)$ is called a “square wave”.



1. Can you find a function on your calculator which has the given graph?

2. Select the function from among the following which gives the best approximation to the square wave: $\frac{4}{\pi}\sin x$, $\frac{4}{\pi}(\sin x + \sin 3x)$, and $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$.

3. Select the function from among the following which gives the best approximation to the square wave: $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$, $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$, $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x\right)$, and $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x\right)$.

4. A *Fourier approximation* of a function is an approximation of the form

$$F(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_n \cos nx + b_n \sin nx$$

You have just discovered the Fourier approximation to $S(x)$ with five terms. Find the Fourier approximation to $S(x)$ with ten terms, and sketch its graph.

5. The following expressions are Fourier approximations to a different function, $T(x)$:

$$\begin{aligned}
T(x) &\approx \sin x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \tfrac{1}{5}\sin 5x
\end{aligned}$$

Sketch $T(x)$.

1.3 New Functions from Old Functions

Suggested Time and Emphasis

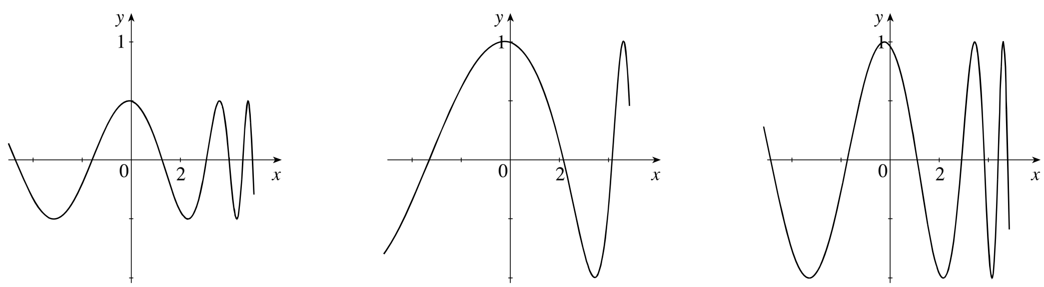
1 class   Essential material

Points to Stress

1. The mechanics and geometry of transforming functions.
2. The mechanics and geometry of adding, subtracting, multiplying, and dividing functions.
3. The mechanics and geometry of composing functions.

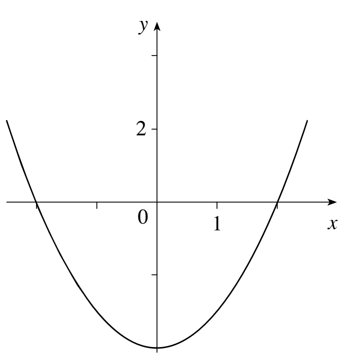
Quiz Questions

- **TEXT QUESTION** Label the following graphs: $f(x)$, $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$.

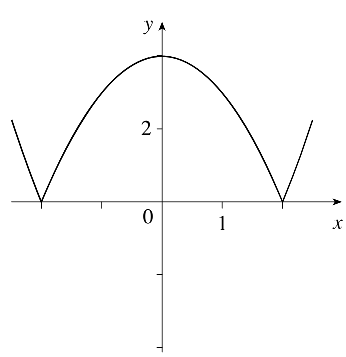


ANSWER $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$, $f(x)$

- **DRILL QUESTION** How can we construct the graph of $y = |f(x)|$ from the graph of $y = f(x)$? Explain in words, and demonstrate with the graph of $y = x^2 - 4$.
  ANSWER We leave the positive values of $f(x)$ alone, and reflect the negative values about the x-axis.



$y = x^2 - 4$

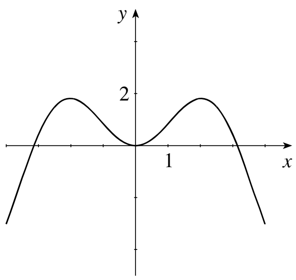


$y = \left|x^2 - 4\right|$

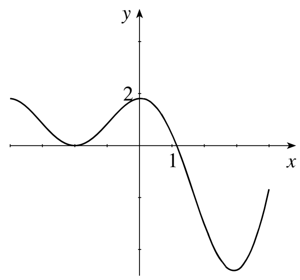
Materials for Lecture

- Using $f(x) = x \sin x$, explore graphs of $f(x+2)$, $f(x)+2$, $-f(x)$, $f(-x)$, $|f(x)|$. Note why $f(-x) = f(x)$.

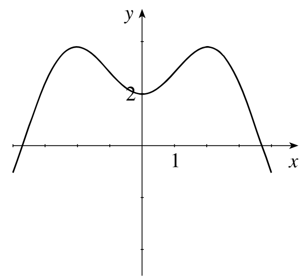
  ANSWER



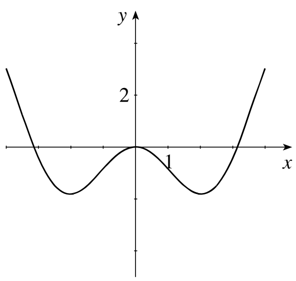
$y = f(x)$



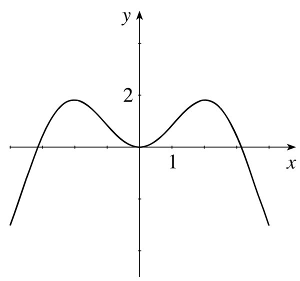
$y = f(x+2)$



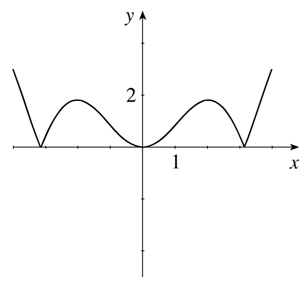
$y = f(x)+2$



$y = -f(x)$



$y = f(-x)$



$y = |f(x)|$

  $f(-x) = f(x)$ because f is even.

- Graph $f(x) = \sin(e^x)$ and $g(x) = e^{\sin x}$. Draw the relevant "arrow diagrams" and then write them in the forms $l \circ k$ and $k \circ l$. Then discuss reasons for the differences in their graphs.

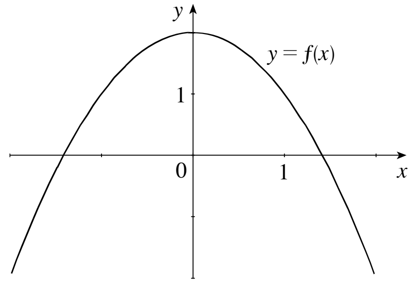
  ANSWER See the text for sample arrow diagrams. f is a sine function whose argument grows larger faster and faster as we move away from the origin. g is an exponential function whose argument oscillates, causing g to oscillate as well.

Workshop/Discussion

- Using $f(x) = 1/x^2$ and $g(x) = \cos x$, compute the domains of $f+g$, f/g, g/f, $f \circ g$, and $g \circ f$, and the range of $g \circ f$. Pay particular attention to the domain of g/f, as many students will think it is $\mathbb{R}$.

  ANSWER $f+g$ has domain $\{x \mid x \neq 0\}$, f/g has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, g/f has domain $\{x \mid x \neq 0\}$, $f \circ g$ has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, $g \circ f$ has domain $\{x \mid x \neq 0\}$, and $g \circ f$ has range $[-1, 1]$.

- Do the following problem with the students:



From the graph of $y = f(x) = -x^2 + 2$ shown above, compute $f \circ f$ at $x = -1$, 0, and 1. First do it graphically (as in Exercises 51 and 52), then algebraically.

- TEC Have the students graph the following functions, where $f(x) = x + x^2$, first guessing what each graph will look like, and then using a calculator to confirm their guesses.

1. $f(2x)$	**4.** $-2f(x)$	**7.** $f(x)+2$
2. $f\left(\frac{1}{2}x\right)$	**5.** $f(x-2)$	**8.** $2f(2x)$
3. $2f(x)$	**6.** $f(x)-2$	**9.** $2f\left(\frac{1}{2}x\right)$

- After doing a few basic examples of composition, it is possible to foreshadow the idea of inverses, which is covered in Section 1.6. Let $f(x) = 2x^3 + 3$ and $g(x) = x^2 - x$. Compute $f \circ g$ and $g \circ f$ for your students. Then ask them to come up with a function $h(x)$ with the property that $(f \circ h)(x) = x$. They may not be used to the idea of finding examples by themselves; important hints they might need are "Don't give up," "When in doubt, just try something and see what happens," and "I'm not expecting you to get it in fifteen seconds." If the class is really stuck, have them try $f(x) = 2x^3$ to get a feel for how the game is played. Once they have determined that $h(x) = \sqrt[3]{\dfrac{x-3}{2}}$, have them first compute $(h \circ f)(x)$, then conjecture whether $(f \circ g)(x) = x$ implies $(g \circ f)(x) = x$ in general.

Group Work 1: Which is the Original?

ANSWERS **1.** $2f(x+2)$, $2f(x)$, $f(2x)$, $f(x+2)$, $f(x)$ **2.** $2f(x)$, $f(x)$, $f(x+2)$, $f(2x)$, $2f(x+2)$

Group Work 2: Label Label Label, I Made It Out of Clay

Some of these transformations were not covered directly in the book. If the students are urged not to give up, and to use the process of elimination and testing individual points, they should be able to complete this activity.

ANSWERS **1.** (d) **2.** (a) **3.** (f) **4.** (e) **5.** (i) **6.** (j) **7.** (b) **8.** (c) **9.** (g) **10.** (h)

Group Work 3: It's More Fun to Compute

Each group gets one copy of the graph. During each round, one representative from each group stands, and one of the questions below is asked. The representatives write their answer down, and all display their answers at the same time. Each representative has the choice of consulting with their group or not. A correct solo answer is worth two points, and a correct answer after a consult is worth one point.

ANSWERS **1.** 0 **2.** 0 **3.** 1 **4.** 5 **5.** 1 **6.** 1 **7.** 1 **8.** 0 **9.** 2 **10.** 1 **11.** 1 **12.** 1

Homework Problems

CORE EXERCISES 3, 13, 15, 17, 19, 29, 31, 37, 41, 51, 53

SAMPLE ASSIGNMENT 2, 3, 5, 9, 11, 13, 15, 17, 19, 21, 29, 31, 35, 37, 39, 41, 50, 51, 53, 55

EXERCISE	D	A	N	G
2	×			
3				×
5				×
9				×
11				×
13				×
15				×
17				×
19				×
21				×

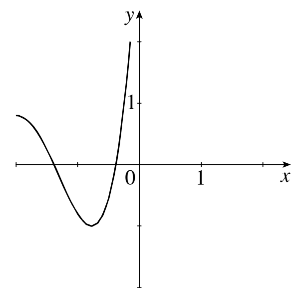
EXERCISE	D	A	N	G
29		×		
31		×		
35		×		
37		×		
39		×		
41		×		
50		×	×	
51				×
53	×	×		
55	×	×		

GROUP WORK 1, SECTION 1.3

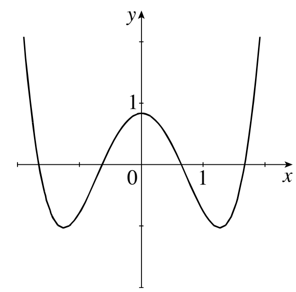
Which is the Original?

Below are five graphs. One is the graph of a function $f(x)$ and the others include the graphs of $2f(x)$, $f(2x)$, $f(x+2)$, and $2f(x+2)$. Determine which is the graph of $f(x)$ and match the other functions with their graphs.

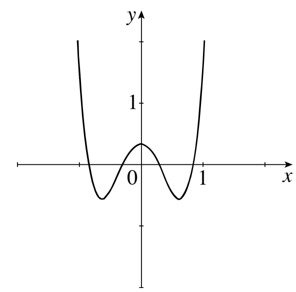
1.



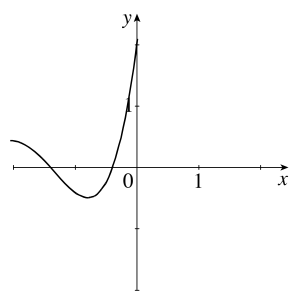
Graph 1



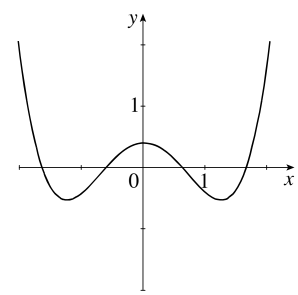
Graph 2



Graph 3

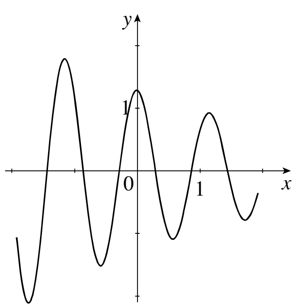


Graph 4

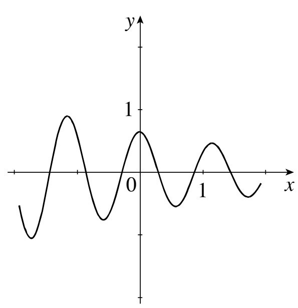


Graph 5

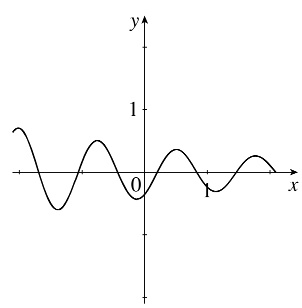
2.



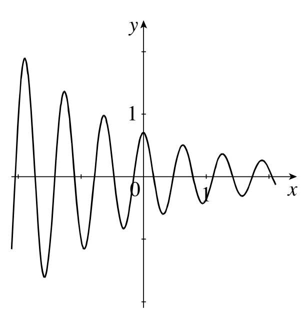
Graph 1



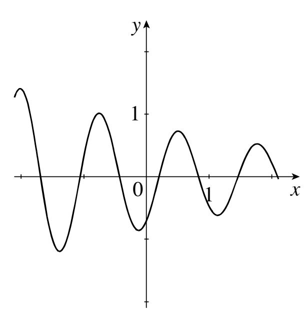
Graph 2



Graph 3



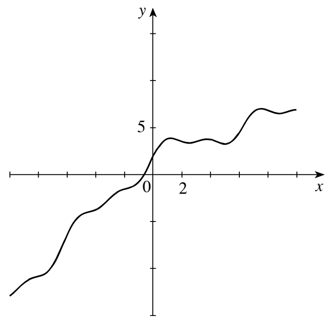
Graph 4



Graph 5

GROUP WORK 2, SECTION 1.3
Label Label Label, I Made it Out of Clay

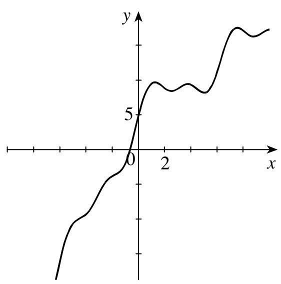
This is a graph of the function $f(x)$:



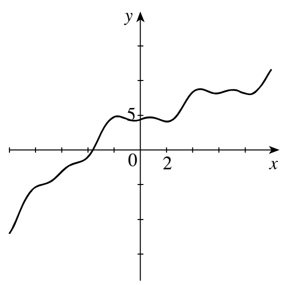
Give each graph below the correct label from the following:

(a) $f(x+3)$ (b) $f(x-3)$ (c) $f(2x)$ (d) $2f(x)$ (e) $|f(x)|$

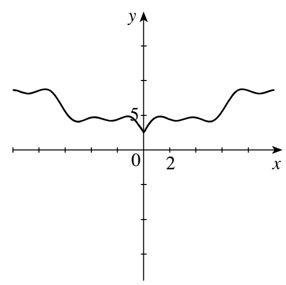
(f) $f(|x|)$ (g) $2f(x)-1$ (h) $f(2x)+2$ (i) $f(x)-x$ (j) $1/f(x)$



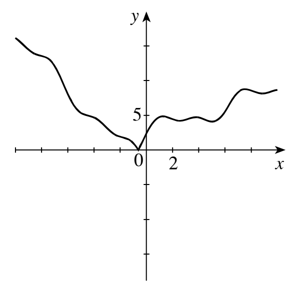
Graph 1



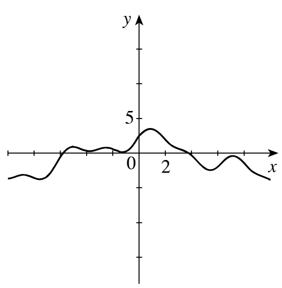
Graph 2



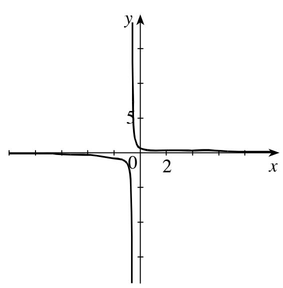
Graph 3



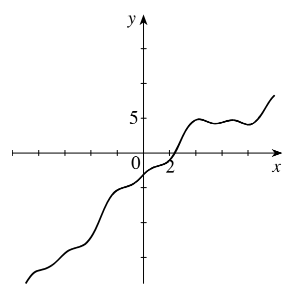
Graph 4



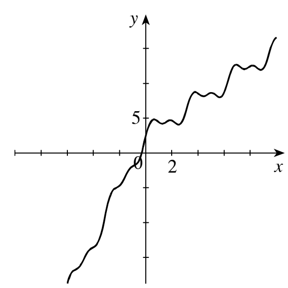
Graph 5



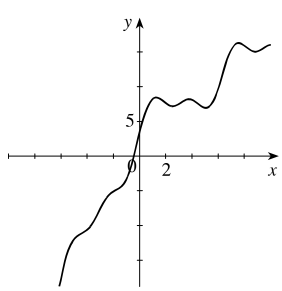
Graph 6



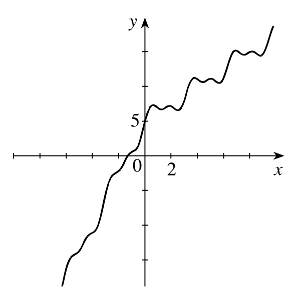
Graph 7



Graph 8



Graph 9



Graph 10

GROUP WORK 3, SECTION 1.3

It's More Fun to Compute

Using the graph below, find the following quantities.

1. $(f \circ g)(5)$

2. $(g \circ f)(5)$

3. $(f \circ g)(0)$

4. $(f \circ f)(5)$

5. $(g \circ g)(5)$

6. $(g \circ g)(-3)$

7. $(g \circ g)(-1)$

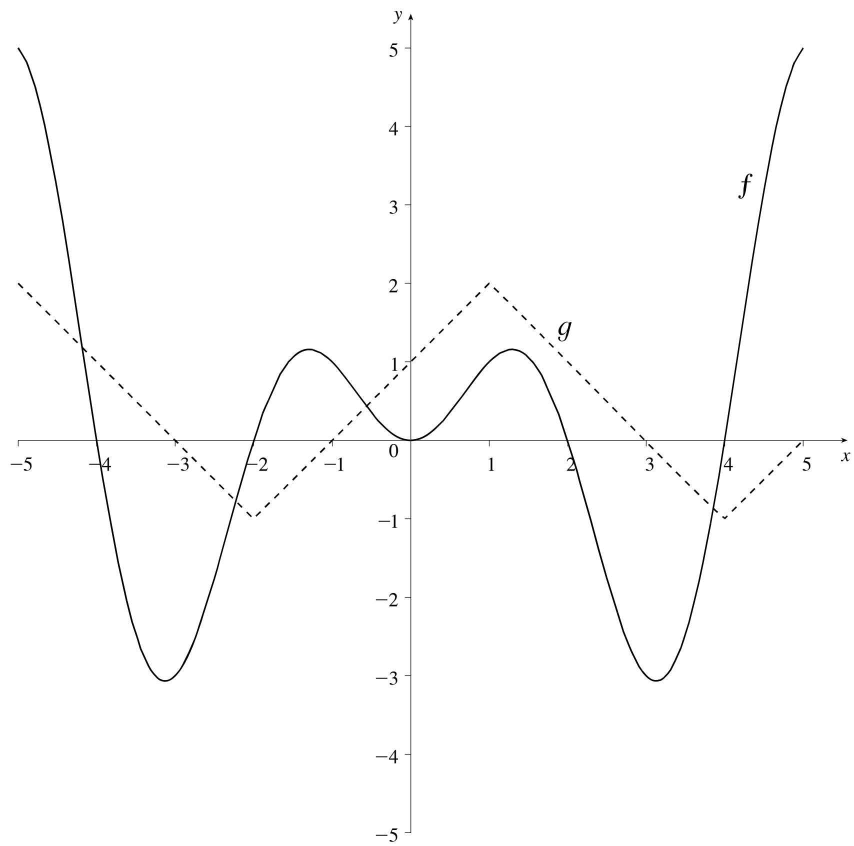
8. $(f \circ g)(1)$

9. $(g \circ f)(1)$

10. $(f \circ f \circ g)(4)$

11. $(g \circ f \circ f)(4)$

12. $(f \circ g \circ f)(4)$



1.4 Graphing Calculators and Computers

Suggested Time and Emphasis

$\frac{1}{2}$–1 class Optional material (If unassigned, students should be encouraged to read this self-contained section on their own.)

Points to Stress

1. When graphing an arbitrary function, some viewing windows are more appropriate than others, depending on the context of the inquiry and some analysis of the actual equation.
2. Some functions don't have any single viewing window that will give all the important details of the function.
3. One can use zoom and trace features to obtain estimates of solutions to difficult algebraic equations.
4. Graphing calculators can give misleading or wrong answers.

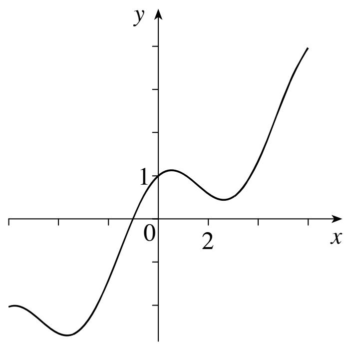
Quiz Questions

- **TEXT QUESTION** Why is it true that "The solutions of the equation $\cos x = x$ are the x-coordinates of the points of intersection of the curves $y = \cos x$ and $y = x$"?
  ANSWER If $y = \cos x$ and $y = x$, we can use substitution to set $x = y = \cos x$.
- **DRILL QUESTION** Find all solutions of the equation $x^3 - 10x^2 - 4 = 0$ correct to at least two decimal places.
  ANSWER $x \approx 10.0397$

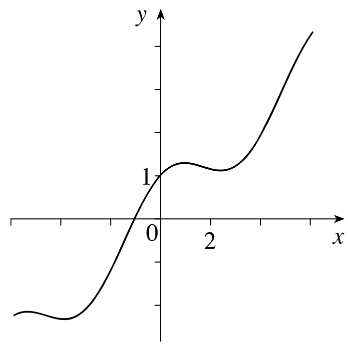
Materials for Lecture

- Caution students to take care when using their calculators, particularly when choosing a viewing window. The following are features of a function that are difficult to determine solely from computer graphics:
  - End behavior. For example, $f(x) = \ln x$
  - Certain asymptotes, such as oblique asymptotes. For example, $f(x) = 3x + 2 + \dfrac{100 \sin x}{\ln x}$, $x > 2$
  - The effects of parameters. For example, $f(x) = \sin\left(Ax^b\right)$
  - Hidden roots. For example, the roots of $f(x) = x^4 - 0.001x$
  - Domains and ranges. For example, the domain of $\sqrt{\sin(x + \cos x)}$, the range of $\ln(\ln x)$
- Have the class experiment with $f(x) = \cos\left(x + \dfrac{1}{x^2}\right)$ near $x = 0$, and also for large values of x.
- Attempt to graph $f(x) = \sqrt{x-2}\sqrt{x-4}$, and to find its domain. Certain packages or calculators (like the TI-89, Maple, and Mathematica) will graph it incorrectly, with domain $(-\infty, 2) \cup (4, \infty)$. Discuss why some software makes this error.
  ANSWER For $x < 2$, each of these algebra packages converts both multiplicands into imaginary numbers, multiplies them, and then converts the product back into a real number. This process is invalid in the context of real-valued functions.
- Demonstrate the use of a calculator as an investigative tool (for example, to explore a family of functions like $f(x) = Qx + \cos x$ by graphing with $Q = 0.5$, 0.75, 0.95, 1, and 1.1). Discuss the meaning of

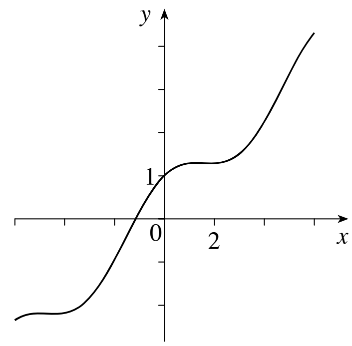
"parameter" using Q as an example. Be sure to note that when $Q < 1$, the function has "peaks and valleys" (local maxima and minima) and when $Q > 1$ the function is always increasing.
ANSWER



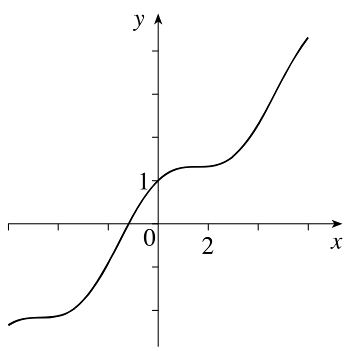
$Q = 0.5$



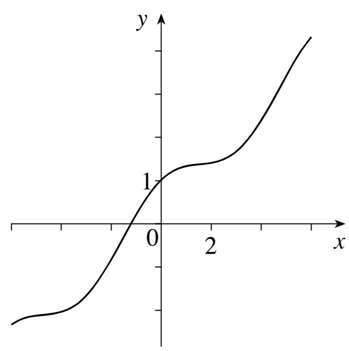
$Q = 0.75$



$Q = 0.95$



$Q = 1$



$Q = 1.1$

Workshop/Discussion

- Pose the question, "What happens to $\dfrac{\cos x - 1}{x}$ as x gets close to zero?" Show how the strategy of substituting $x = 0$ fails. Investigate this question numerically, and then by graphing. Follow up with the function $\dfrac{\cos x - 1}{x^2}$.

  ANSWER As x gets close to zero, the value of $\dfrac{\cos x - 1}{x}$ gets close to zero. as x gets close to zero, the value of $\dfrac{\cos x - 1}{x^2}$ gets close to $-\frac{1}{2}$.

- Have the students try to figure out what the graph of $f(x) = [x + 0.05\cos(20x)]^2$ looks like near $x = 0$ by experimenting with viewing windows. Then have them try to explain why their picture makes sense.
- Have the students first determine where $f(x) = x^3 - 50x^2$ is larger than x^2. Point out that it is easier to use some algebra to solve this problem than to use a calculator
  ANSWER $x > 51$
- Have the students first guess the shape of $f(x) = \cos\left(2^{\sin x}\right)$, and then graph it. Ask how they should have known it would be periodic.

Group Work 1: Short- and Long-Term Behavior

Before handing this exercise out, show what is meant by "long-term behavior".

Part of the idea of this exercise is getting the students to explore and play with functions (even unfamiliar ones) on their calculators. Encourage them to try varying the functions on the worksheet, and see what happens.

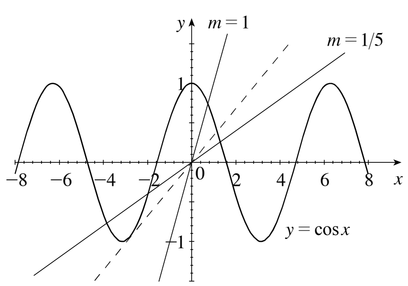
ANSWERS **1.** (a) $f(x) \to \infty$ as $x \to \infty$ (b) $x = 0$, $x \approx -0.877$ **2.** (a) $g(x) \to \infty$ as $x \to \infty$ (g is asymptotic to $y = 6x$) (b) None **3.** (a) $h(x) \to \infty$ as $x \to \infty$ (b) $x = 0, \pm\sqrt{3}$ **4.** (a) $j(x) \to 1$ as $x \to \infty$ (b) None **5.** (a) $k(x) \to -\infty$ as $x \to \infty$ (b) $x \approx 0.4027, 0.9952, 1.923$ **6.** (a) $l(x) \to -\infty$ as $x \to \infty$ (b) $x = 0$

Group Work 2: Just Two Solutions

This is an extension of Exercise 24. It is a difficult exercise, and the students may require some guidance. Before handing out the hint sheet, have the students try to solve this problem on their own: "Consider the equation $\cos x = mx$. Find a value of m such that this equation has exactly two solutions." After they have worked on the problem, and are clear on what they are trying to find, hand out the hint sheet. Don't let them hang too long! Conclude by discussing how the line $y = mx$ that yields exactly two solutions is called tangent to the curve $y = \cos x$. Explain that soon we will learn how to find the equation of such a line. This is a good example to bring back once they have covered derivatives of trigonometric functions in Section 3.3. At that point, they will wind up having to solve the equation $x = -\cot x$. Revisit the problem again after covering Newton's method in Section 4.8, at which point they will finally have a way of approximating m to any desired accuracy.

ANSWERS

1, 3.



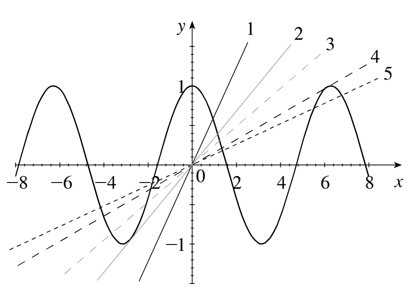
2. One solution for $m = 1$, three solutions for $m = \frac{1}{5}$.

4. Any number close to $\pm\frac{1}{3}$ is a good estimate.

5. $m \approx \pm 0.33651$

6. $\cos x$ is symmetric with respect to the y-axis, so the graphs of $y = -mx$ and $y = mx$ intersect it at the same number of points.

7.



Homework Problems

CORE EXERCISES 3, 9, 19, 21, 25

SAMPLE ASSIGNMENT 3, 5, 9, 15, 17, 19, 21, 23, 25, 31

EXERCISE	D	A	N	G
3				×
5				×
9				×
15	×	×		
17				×

EXERCISE	D	A	N	G
19				×
21				×
23				×
25				×
31	×			×

GROUP WORK 1, SECTION 1.4

Short- and Long-Term Behavior

For each of the following functions, (a) describe the long term behavior of the function, and (b) locate the zeros, if any.

1. $f(x) = \sin x + x^2$

2. $g(x) = 6x + \dfrac{1}{x} = \dfrac{6x^2 + 1}{x}$

3. $h(x) = \dfrac{x^3 - 3x}{x + 7}$

4. $j(x) = 2^{-1/x^6}$

5. $k(x) = x^5 - 2x^4 - x^2 + 3x - 1.01^x$

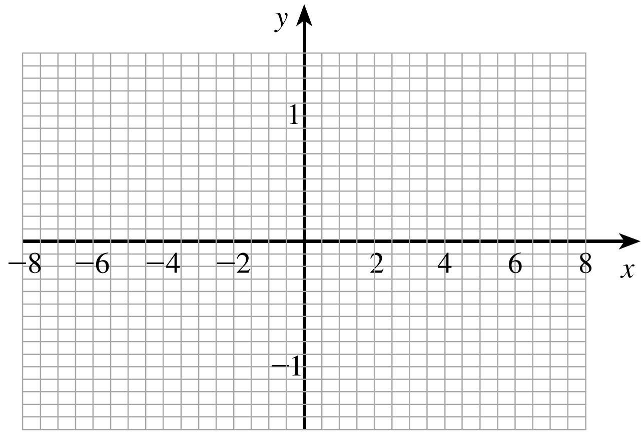
6. $l(x) = Qx + \sin x$ for $Q = 0.32$, 0.9, and 1.1 What are the differences among these three functions?

GROUP WORK 2, SECTION 1.4

Just Two Solutions (Hint Sheet)

PROBLEM Find a value of m such that the equation $\cos x = mx$ has exactly two solutions.

1. Very carefully, draw graphs of the following three functions below: $y = \cos x$, $y = x$, $y = \frac{1}{5}x$.



2. From the graphs, determine how many solutions the equation has for $m = 1$ and for $m = \frac{1}{5}$.

3. Add a sketch of $y = mx$ onto your picture for Problem 1, where m is chosen so that the equation $\cos x = mx$ has exactly two solutions. You can do this without actually computing m.

4. Use your sketch to estimate the correct value of m.

5. Now use your graphing calculator to refine your guess. Try to get an answer correct to three decimal places.

6. Why is it true that the value $-m$ will also give exactly two solutions to $\cos x = mx$?

7. Is it possible to find a value of m such that $\cos x = mx$ has exactly three solutions? Four? n? Explain your answer by sketching $y = \cos x$ and $y = mx$ for the relevant values of m.

1.5 Exponential Functions

Suggested Time and Emphasis

$\frac{1}{2}$–1 class   Essential or optional material, depending on student background

Points to Stress

1. Algebraic and geometric properties of exponential functions.
2. Translation and reflection of exponential functions, from both symbolic and geometric perspectives.
3. Exponential functions as models for population growth and decay.
4. Growth rates of exponential functions as compared to growth rates of polynomials.

Quiz Questions

- **TEXT QUESTION** $3^3 = 3 \cdot 3 \cdot 3$, $3^{3/4} = \sqrt[4]{3 \cdot 3 \cdot 3}$. How does one make sense of $3^{\sqrt{7}}$?
  ANSWER Answers will vary. One example: we can approximate $\sqrt{7}$ by a sequence of rational numbers, and can thus similarly approximate $3^{\sqrt{7}}$ to any degree of precision.
- **DRILL QUESTION** The half-life of Strontium 90 is 25 years. If you have 800 mg of Strontium 90, how much will be left after 100 years? Try to answer without reaching for a pencil or calculator.
  ANSWER About 50 mg

Materials for Lecture

- Start to draw a graph of 2^x vs x, using the scale of one inch per unit on both axes. Point out that after one foot, the height would be over 100 yards (the length of a football field). After two feet, the height would be 264 miles, after three feet it would be 1,000,000 miles (four times the distance to the moon), after three and a half feet it would be in the heart of the sun. If the graph extended five feet to the right, $x = 60$, then y would be over one light year up.
- Point out this contrast between exponential and linear functions: For equally spaced x-values, linear functions have constant *differences* in y-values, while pure exponential functions have constant *ratios* in y-values. Use this fact to show that the following table describes an exponential function, not a linear one.

x	y
−6.2	0.62000
−2.4	0.65100
1.4	0.68355
5.2	0.71773
9.0	0.75361
12.8	0.79129

- In 1985 there were 15,948 diagnosed cases of AIDS in the United States. In 1990 there were 156,024. Scientists said that if there was no research done, the disease would grow exponentially. Compute the number of cases this model predicts for the year 2000. The actual number was 774,467. Discuss possible flaws in the model with the students, and point out the dangers of extrapolation.

Workshop/Discussion

- Estimate where $3^x > x^3$ and where $2^x > x^8$ using technology. Notice that exponential functions start by growing *slower* than polynomial functions, and then wind up growing much *faster*. For example, if one were to graph x^2 vs x using one inch per unit, then when $x = 60$, y would be only 100 yards, as opposed to a light year for $y = 2^x$. (The sun is only 8 light minutes from the earth.)

- Estimate where $3^x > 10^7$ by use of technology. Then use algebra to find an exact answer. (Logarithms are briefly reviewed in Section 1.6.)

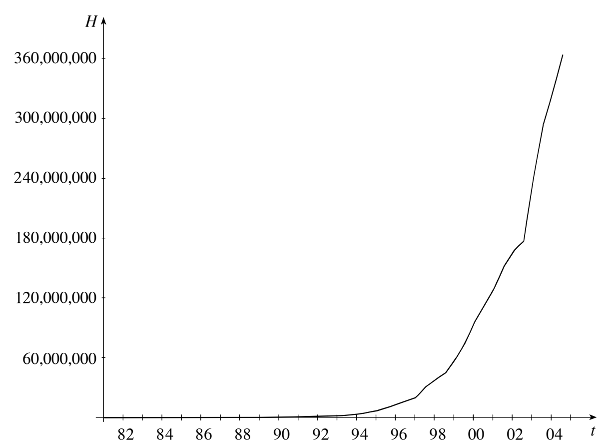
- One way to measure the growth of the internet is to measure the number of hosts. The following data show the number of internet hosts over approximately a quarter of a century. Try to determine with the students if this is exponential growth. (Note: Do not show the third column right away. Let them come up with the idea of finding growth rates between data points.)

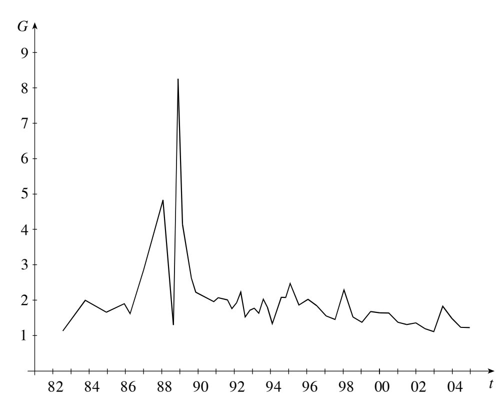
Month	Hosts	Growth
Aug 1981	213	–
May 1982	235	1.1400
Aug 1983	562	2.0088
Oct 1984	1024	1.6724
Oct 1985	1961	1.9150
Feb 1986	2308	1.6303
Nov 1986	5089	2.8699
Dec 1987	28,174	4.8534
Jul 1988	33,000	1.3113
Oct 1988	56,000	8.2927
Jan 1989	80,000	4.1649
Jul 1989	130,000	2.6406
Oct 1989	159,000	2.2378
Oct 1990	313,000	1.9686
Jan 1991	376,000	2.0824
Jul 1991	535,000	2.0246
Oct 1991	617,000	1.7690

Month	Hosts	Growth
Jan 1992	727,000	1.9275
Apr 1992	890,000	2.2461
Jul 1992	992,000	1.5434
Oct 1992	1,136,000	1.7198
Jan 1993	1,313,000	1.7846
Apr 1993	1,486,000	1.6407
Jul 1993	1,776,000	2.0403
Oct 1993	2,056,000	1.7961
Jan 1994	2,217,000	1.3520
Jul 1994	3,212,000	2.0990
Oct 1994	3,864,000	2.0943
Jan 1995	4,852,000	2.4862
Jul 1995	6,642,000	1.8739
Jan 1996	9,472,000	2.0337
Jul 1996	12,881,000	1.8493
Jan 1997	16,146,000	1.5712
Jul 1997	19,540,000	1.4646

Month	Hosts	Growth
Jan 1998	29,670,000	2.3056
Jul 1998	36,739,000	1.5333
Jan 1999	43,230,000	1.3846
Jul 1999	56,218,000	1.6911
Jan 2000	72,398,092	1.6585
Jul 2000	93,047,785	1.6518
Jan 2001	109,574,429	1.3868
Jul 2001	125,888,197	1.3199
Jan 2002	147,344,723	1.3699
Jul 2002	162,128,493	1.2107
Jan 2003	171,638,297	1.1208
Jul 2003	233,101,481	1.8444
Jan 2004	285,139,107	1.4963
Jul 2004	317,646,084	1.2410
Jan 2005	353,284,187	1.2370

ANSWER We can graph the data, and get a curve that looks like exponential growth. We can also graph growth rate and see (except for two spikes in the late 1980s) a relatively constant growth rate.





Group Work 1: I've Grown Accustomed to Your Growth

Before handing out this activity, it may be prudent to review the rules of exponentiation. This exercise enables students to discover for themselves the equal ratio property of exponential functions.

ANSWERS **1.** Yes ($m = 1$), no, yes ($m \approx 2.08$), yes ($m \approx 2.01$) **2.**Equally spaced changes in x-values result in equally spaced changes in y-values **3.** Equally spaced changes in x values result in equally proportioned changes in y-values with the same ratio. $b = 2$, $b = 0.9975$, $b = 2.25$, $b = 3$ **4.** The "$+C$" gets in the way when taking the ratio. However, the property is close to being true when A and b are large compared to C.

Group Work 2: Comparisons

The purpose of this group work is to give the students a bit of "picture sense." It is acceptable if they do this by looking at graphs on their calculators, setting the windows appropriately.

ANSWERS **1.** $0 < x < 1.374$ and $x > 9.940$ **2.** $0 < x < 1.052$ and $x > 95.7168$ **3.** $0 < x < 1.240$ and $x > 16$ **4.** $0 < x < 1.517$ and $x > 7.175$

Group Work 3: Estimating e

This activity requires technology that can be used to quickly compute the slope of a line tangent to the graph of a function. Alternatively, the difference quotient method of approximating a derivative near $x = 0$ can be shown to an advanced class at this time.

ANSWERS **1.** The slopes are approximately 0.956, 1.030 and 1.099. They increase as a increases. **2.** 2.6 and 2.8 **3.** 0.956, 0.993, 1.030; between 2.7 and 2.8 **4.** $e \approx 2.7183$

Homework Problems

CORE EXERCISES 1, 3, 9, 11, 19, 28, 29

SAMPLE ASSIGNMENT 1, 3, 6, 9, 11, 13, 19, 23, 28, 29, 32

EXERCISE	D	A	N	G
1		×		
3		×		
6	×			
9	×			×
11				×
13				×

EXERCISE	D	A	N	G
19		×		
23		×		
28				×
29		×		×
32		×	×	

GROUP WORK 1, SECTION 1.5
I've Grown Accustomed to Your Growth

1. Some of the following four tables of data have something in common: linear growth. Without trying to find complete equations of lines, determine which of them are linear growth, and determine their rate of change:

x	y
1	2
2	3
3	4
4	5

x	y
21.5	4.32
32.6	4.203
43.7	4.090
54.8	3.980

x	y
−3	1.1
−2.5	2.14
−2	3.18
−1.5	4.22

x	y
1	−5.00
3	−0.98
6	5.05
8	9.07

2. In a sentence, describe a property of linear growth that can be determined from a table of values.

3. The following four tables of data have something in common: exponential growth. Functions of the form $y = Ab^x$ (or Ae^{kx}) have a property in common analogous to the one you stated in Question 2. Find the property, and then find the value of b.

x	y
1	5
2	10
3	20
4	40

x	y
21.5	4.32
32.6	4.203
43.7	4.090
54.8	3.980

x	y
−3	1.1
−2.5	1.65
−2	2.475
−1.5	3.7125

x	y
1	0.8
3	7.2
6	194.4
8	1749.6

4. Unfortunately, the above property does not hold for functions of the form $y = Ab^x + C$. What goes wrong? For what kinds of values of A, b, and C does the property come close to being true?

GROUP WORK 2, SECTION 1.5

Comparisons

You have learned that an exponential function grows faster than a polynomial function. Find the values of $x > 0$ for which

1. $2^x \geq x^3$

2. $(1.1)^x \geq x^2$

3. $2^x \geq x^4$

4. $3^x \geq x^4$

GROUP WORK 3, SECTION 1.5

Estimating e

One definition of e is as the number a such that the slope of the tangent line to the graph of $y = a^x$ at $x = 0$ is 1.

1. Use technology to compute the slope of the line tangent to the graph of $y = a^x$ at $x = 0$ for $a = 2.6, 2.8,$ and 3.0. Does it seem that the slope increases as a increases?

2. Based on the results of Problem 1, and keeping in mind the definition given above, would you guess that e is between 2.6 and 2.8 or between 2.8 and 3.0? Why?

3. Compute the slope of the tangent line to the graph of $y = a^x$ at $x = 0$ for $a = 2.6, 2.7,$ and 2.8. Would you guess that e is between 2.6 and 2.7, or between 2.7 and 2.8?

4. Repeat Problem 3, each time narrowing the range for a, until you have an estimate of e you are confident is correct to three decimal places.

1.6 Inverse Functions and Logarithms

Suggested Time and Emphasis

1 class   Essential material

Points to Stress

1. The use of multiple representations (verbal, numeric, visual, algebraic) to understand inverse functions, always coming back to the central idea of reversing inputs and outputs.
2. Logarithmic functions and their algebraic and geometric properties.
3. Tests for one-to-one functions.

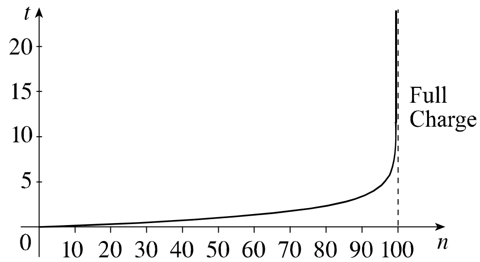
Quiz Questions

- **TEXT QUESTION** In this section, the author describes a technique to graph the inverse of a one-to-one function using the line $y = x$. Why does this technique work? Does it work for $y = x^2$?
  ANSWER It works because reflecting across the line $y = x$ is the same as reversing the inputs and outputs of a function. It doesn't work for $y = x^2$ because $y = x^2$ is not one-to-one.
- **DRILL QUESTION** What is the inverse function f^{-1} for $f(x) = 2\left(\log_3 x + 4\right)$?
  ANSWER $f^{-1}(x) = 3^{(x/2)-4}$

Materials for Lecture

- Starting with $f(x) = \sqrt[3]{x-4}$ compute $f^{-1}(-2)$ and $f^{-1}(0)$. Then use algebra to find a formula for $f^{-1}(x)$. Have the students try to repeat the process with $g(x) = x^3 + x - 2$. Note that facts such as $g^{-1}(-2) = 0$, $g^{-1}(0) = 1$, and and $g^{-1}(8) = 2$ can be found by looking at a table of values for $g(x)$ but that the algebraic approach fails to give us a general formula for $g^{-1}(x)$. Finally, draw graphs of f, f^{-1}, g, and g^{-1}.
- Pose the question: If f is always increasing, is f^{-1} always increasing? Give the students time to try prove their answer.
  ANSWER This is true. Proofs may involve diagrams and reflections about $y = x$, or you may try to get them to be more rigorous. This is an excellent opportunity to discuss concavity, noting that if f is concave up and increasing, then f^{-1} is concave down and increasing.
- Ask your students if they have ever had to deal with recharging the battery for a music device or a laptop. Assume that the battery is dead, and it takes an hour to charge it up half way. Ask them how much it will be charged in two hours. (The answer is 75% charged.) They may have noticed that, when charging a laptop battery, it takes a surprisingly long time for the monitor to change from "99% charged" to "100% charged." It turns out that the time it takes to charge the battery to n% is given by $t = -k \ln\left(1 - \frac{n}{100}\right)$; in our example $k = 1.4427$. Have students compute how long it would take the battery to get a 97% charge, a 98% charge, and a 99% charge. (Remind them that it took only an hour to go from 0% to 50%.)

Graph t versus n to demonstrate that the battery will never be fully charged according to this model.

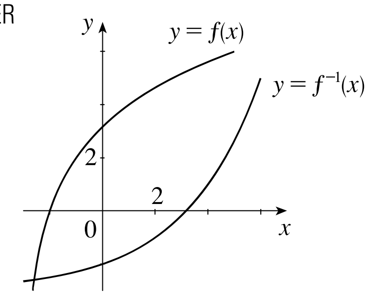


- Show students semilog graph paper (available at university bookstores, from your friendly neighborhood physics teacher, or from websites such as `http://www.csun.edu/science/ref/measurement/data/graph_paper.html`.) Point out how the distance between the y-axis lines is based on the logarithm of the y-coordinate, not on the y-coordinate itself. Have them graph $y = 2^x$ on semilog graph paper.

Workshop/Discussion

- Sketch a graph of $f(x) = \log_2\left((x+3)^2\right)$, sketch the inverse function, and then find an algebraic formula for the inverse.

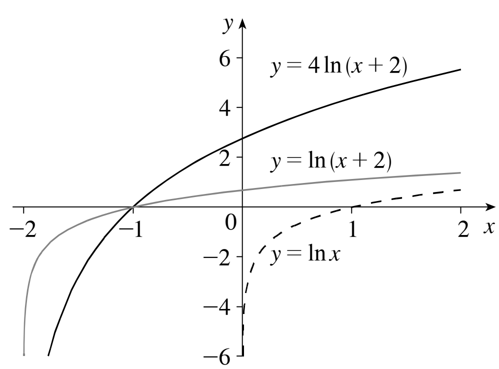
ANSWER



$$f^{-1}(x) = 2^{x/2} - 3$$

- Sketch a graph of $f(x) = 4\ln(x+2)$, starting with the graph of $y = \ln x$. Then show that this is also a graph of the function written as $\ln(x+2)^4$.

ANSWER



- Have the students use graphing technology to check if $f(x) = x^3 - 2x^2 - x + 2$ is one-to-one. Then show them how the domain can be restricted to get a 1-1 function.

Group Work 1: Inverse Functions: Domains and Ranges

While discussing the domains and ranges of inverse functions, this exercise will also foreshadow later excursions into the maximum and minimum values of functions.

If a group finishes early, ask them this question:

"Now consider the graph of $f(x) = \sqrt{2x-3}+2$. What are the domain and range of $f(x)$? Try to figure out the domain and range of $f^{-1}(x)$ by looking at the graph of f. In general, what information do you need to be able to compute the domain and range of $f^{-1}(x)$ from the graph of a function f?"

ANSWERS

1. It is one-to-one, because the problem says it climbs steadily.

2. a^{-1} is the time in minutes at which it achieves a given altitude.

3. Reverse the data columns in the given table to get the table for the inverse function. The domain and range of a are $0 \le t \le 30$ and $0 \le a \le 29{,}000$, so the domain and range of a^{-1} are $0 \le x \le 29{,}000$ and $0 \le a^{-1} \le 30$.

4. After approximately 8.5 minutes

5. a is no longer 1-1, because heights are now achieved more than once.

Bonus The domain of f^{-1} is the set of all y-values on the graph of f, and the range of f^{-1} is the set of all x-values on the graph of f.

Group Work 2: Functions in the Classroom

Before starting this one, review the definition of "function". Some of the problems can only be answered by polling the class after they are finished working. Don't forget to take leap years into account for the eighth problem. For an advanced class, follow up by defining "one-to-one" and "bijection", then determining which of the functions have these properties.

ANSWERS

Chairs: Function, one-to-one, bijection (if all chairs are occupied)

Eye color: Function, not one-to-one

Mom & Dad's birthplace: Not a function; mom and dad could have been born in different places

Molecules: Function, one-to-one (with nearly 100% probability); inverse assigns a number of molecules to the appropriate student.

Spleens: Function, one-to-one, bijection. Inverse assigns each spleen to its owner.

Pencils: Not a function; some people may have more than one or (horrors!) none.

Student number: Function, one-to-one; inverse assigns each number to its owner.

February birthday: Not a function; not defined for someone born on February 29.

Birthday: Function, perhaps one-to-one.

Cars: Not a function; some have none, some have more than one.

Cash: Function, perhaps one-to-one.

Middle names: Not a function; some have none, some have more than one.

Identity: Function, one-to-one, bijection. Inverse is the same as the function.

Calculus instructor: Function, not one-to-one.

Number of hairs: Function, not one-to-one. There are more people in New York City than there are possible values for this function. Therefore, at least two New Yorkers have the same number of hairs on their heads, and so the function does not have an inverse.

Group Work 3: The Column of Liquid

If the students need a hint, you can mention that the liquid in the mystery device was mercury.

ANSWERS **1.** The liquid is 1 cm high when the temperature is 32 °F. **2.** The liquid is 2 cm high when the temperature is 212 °F **3.** The inverse function takes a height in cm, and gives the temperature. So it is a device for measuring temperature. **4.** A thermometer

Group Work 4: Irrational, Impossible Relations

Before starting this activity, review the definitions of rational and irrational numbers. The hint sheet should be given out only after the students have tried to show that $\log_2 3$ is irrational, or at least discussed it enough to understand what they are trying to show. If a group finishes early, have them show that $\log_2 a$ is always irrational if a is an odd integer.

ANSWERS **1.** $-\gamma$ **2.** $-\gamma$ **3.** $\dfrac{\gamma}{2}$ **4.** $\log_2 3 = \dfrac{\ln 3}{\ln 2}$ **5.** The proof is outlined in the hint sheet.

ANSWERS (HINT SHEET) **1.** $2^{a/b} = 3 \quad \Rightarrow \quad \sqrt[b]{2^a} = 3 \quad \Rightarrow \quad 2^a = 3^b$ **2.** If $a = b = 0$, then $\log_2 3 = 0/0$, which is undefined. **3.** $2^a = 2 \cdot 2 \cdot \cdots \cdot 2$, and $3^b = 3 \cdot 3 \cdot \cdots \cdot 3$, so these numbers can never be equal, because the left is always divisible by two, and the right never is (unless $b = 0$). **4.** Irrational.

Homework Problems

CORE EXERCISES 1, 3, 5, 9, 15, 19, 21, 27, 35, 39, 47, 49, 51, 55, 63, 67, 71

SAMPLE ASSIGNMENT 1, 3, 5, 7, 9, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 38, 39, 41, 47, 49, 51, 53, 55, 63, 67, 71, 73

EXERCISE	D	A	N	G
1	×			
3			×	
5				×
7				×
9		×		
15		×		
17		×		
19	×	×		
21		×		
23		×		
25		×		
27		×		×
29				×

EXERCISE	D	A	N	G
31		×		×
38		×		
39		×		
41		×		
47				×
49		×		×
51		×		
53		×		
55		×		
63		×		
67		×		
71		×		
73				×

GROUP WORK 1, SECTION 1.6

Inverse Functions: Domains and Ranges

Let $a(t)$ be the altitude in feet of a plane that climbs steadily from takeoff until it reaches its cruising altitude after 30 minutes. We don't have a formula for a, but extensive research has given us the following table of values:

t	$a(t)$
0.1	50
0.5	150
1	500
3	2000
7	8000
10	12,000
20	21,000
25	27,000
30	29,000

1. Is $a(t)$ a one-to-one function on the given interval? How do you know?

2. What does the function a^{-1} measure in real terms? Your answer should be descriptive, similar to the way $a(t)$ was described above.

3. We are interested in computing values of a^{-1}. Fill in the following table for as many values of x as you can. What quantity does x represent?

x	$a^{-1}(x)$

What are the domain and range of a? What are the domain and range of a^{-1}?

4. You are allowed to turn on electronic equipment after the plane has reached 10,000 feet. Approximately when can you expect to turn on your laptop computer after taking off?

5. Suppose we consider $a(t)$ from the time of takeoff to the time of touchdown. Is $a(t)$ still one-to-one?

GROUP WORK 2, SECTION 1.6

Functions in the Classroom

Which of the following relations are functions?

DOMAIN	FUNCTION VALUES	FUNCTION
All the people in your classroom	Chairs	f (person) = his or her chair
All the people in your classroom	{blue, brown, green, hazel}	f (person) = his or her eye color
All the people in your classroom	Cities	f (person) = birthplace of his or her mom and dad
All the people in your classroom	$\mathbb{R}$, the real numbers	f (person) = number of molecules in his or her body
All the people in your classroom	Spleens	f (person) = his or her spleen
All the people in your classroom	Pencils	f (person) = his or her pencil
All the students in your classroom	Integers from 0–99999999	f (person) = his or her student number
All the living people born in February	Days in February, 2007	f (person) = his or her birthday in February 2007
All the people in your classroom	Days of the year	f (person) = his or her birthday
All the people in your classroom	Cars	f (person) = his or her car
All the people in your classroom	$\mathbb{R}$, the real numbers	f (person) = how much cash they have on them
All the people in your college	Names	f (person) = his or her middle name
All the people in your classroom	People	f (person) = theirself
All the students in your classroom	People	f (person) = his or her calculus instructor
All the people in New York City	$\mathbb{W}$, the whole numbers	f (person) = the number of hairs on his or her head

GROUP WORK 3, SECTION 1.6

The Column of Liquid

It is a fact that if you take a tube and fill it partway with liquid, the liquid will rise and fall based on the temperature. Assume that we have a tube of liquid, and we have a function $h(T)$, where h is the height of the liquid in cm at temperature T in $^\circ$F.

1. It is true that $h(32) = 1$. What does that mean in physical terms?

2. It is true that $h(212) = 10$. What does that mean in physical terms?

3. Describe the inverse function h^{-1}. What are its inputs? What are its outputs? What does it measure?

4. Hospitals used to use a device that measured the function h^{-1}. Some people used to have such a device in their homes. What is the name of this device?

GROUP WORK 4, SECTION 1.6

Irrational, Impossible Relations

1. If $\log_2 x = \gamma$, then what is $\log_{1/2} x$?

2. If $\log_b x = \gamma$, then what is $\log_{1/b} x$ (assuming $b > 1$)?

3. If $\log_b x = \gamma$, then what is $\log_{b^2} x$?

4. We are going to estimate $\log_2 3$. In pre-calculus, you memorized that $\log_2 3 \approx 1.584962501$. Suppose you didn't have this fact memorized. There is no $\log_2 3$ button on your calculator! How would you compute it?

5. Unfortunately, the calculator gives us only a finite number of digits. If $\log_2 3$ were a rational number, we would be able to express it as a fraction, giving us perfect accuracy. Do you think it is rational or irrational? Try to prove your result.

GROUP WORK 4, SECTION 1.6

Irrational, Impossible Relations (Hint Sheet)

So, you realize that it's not easy to determine whether $\log_2 3$ is rational!
One way to attempt to show that $\log_2 3$ is rational is to assume that it is, and try to find integers a and b such that $\log_2 3 = \frac{a}{b}$. If we can show that there are no such a and b, then $\log_2 3$ *cannot* be rational.

1. Assume that $\log_2 3 = \frac{a}{b}$. Show that a and b must then satisfy $2^a = 3^b$

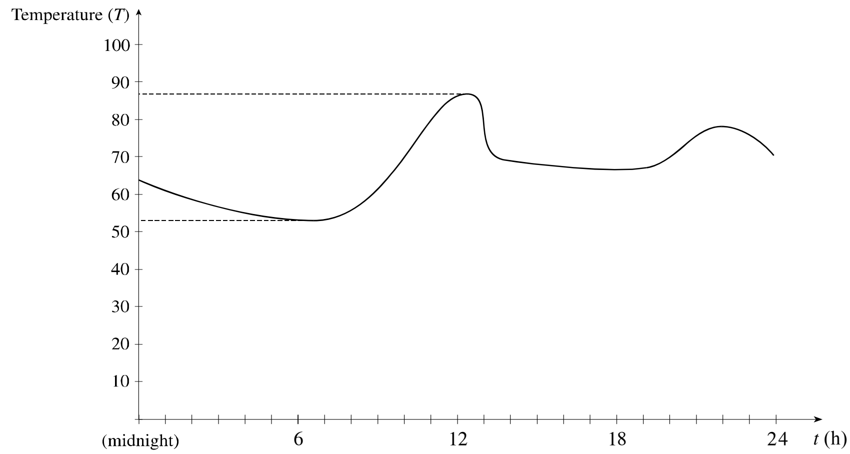
2. Notice that $a = 0$, $b = 0$ satisfies $2^a = 3^b$. Show that this fact doesn't help us.

3. Find $a \neq 0$ and $b \neq 0$ that satisfy $2^a = 3^b$, or show that no such $\{a, b\}$ exists.

4. Is $\log_2 3$ rational or irrational? Why?

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. The graph below shows the temperature of a room during a summer day as a function of time, starting at midnight.



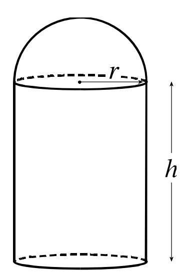
(a) Evaluate f (noon) and f (6 P.M.). State the range of f.

(b) Where is f increasing? Decreasing?

(c) Give a possible explanation for what happened at noon.

(d) Give a possible explanation why f attains its minimum value at 6 A.M.

2. A proposed new grain silo consists of a cylinder of height h and radius r, capped by a hemisphere.



Express its volume as a function of h and r.

3. The Slopps® trading card company has decided to put out its best line of trading cards ever: The "Famous Mathematicians" Series. Each pack of cards contains eight famous mathematicians, a mathematical puzzle, and a mathematics sticker. Naturally, you want a complete set, but you will have to buy a lot of cards because the really good ones (like the Galois, Sylvester, Hesse, Newton, and Leibniz cards) are very rare. Your local dealer will sell you an individual card, randomly selected, for 50 cents. Most people are interested in buying the packs of 8 for \$2.80. When you tell the dealer you want to buy a *lot* of them, he offers to sell you a box (containing 10 packs) for \$25, or a carton (containing 10 boxes) for \$230.

Let $c(x)$ be the (least) cost of buying x cards. Note that it is acceptable to buy more than x cards if it costs less than buying exactly x cards.

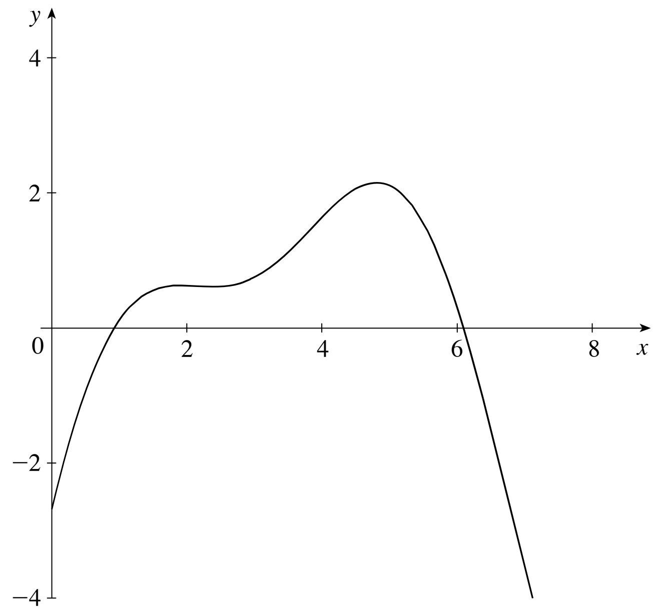
(a) Explain why the cheapest way to buy 6 cards is to buy a pack of 8. What is $c(6)$?

(b) Sketch a graph of $y = c(x)$ from $x = 0$ to $x = 24$.

(c) Find a formula for $c(x)$, valid for $x = 80$ to $x = 90$.

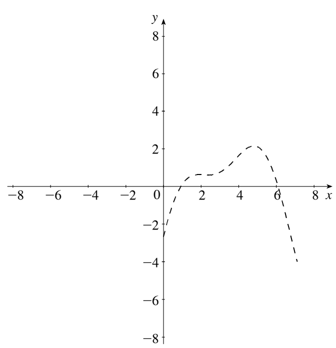
(d) If you wanted to buy 1005 cards, what is the least you would have to pay?

4. The following is a graph of $y = f(x)$.

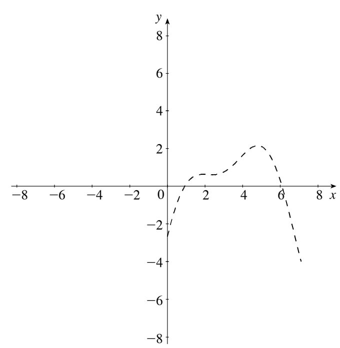


On the same axes, draw and label graphs of

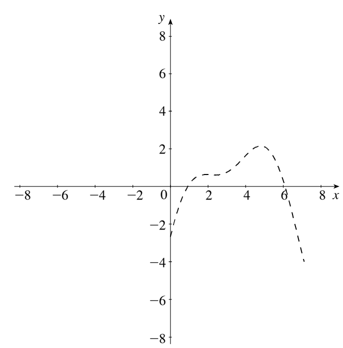
(a) $2f(x+2)$



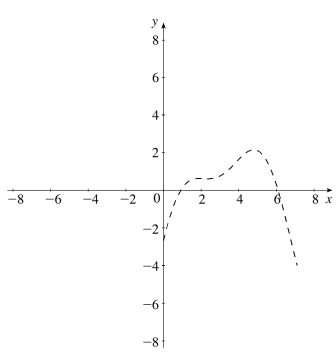
(b) $\frac{1}{2}f(-x)$



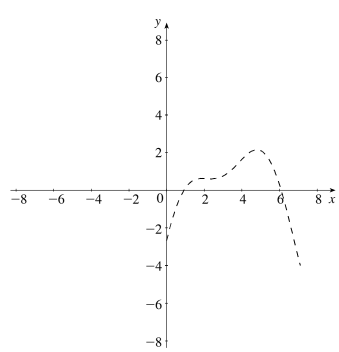
(c) $f(2x)$



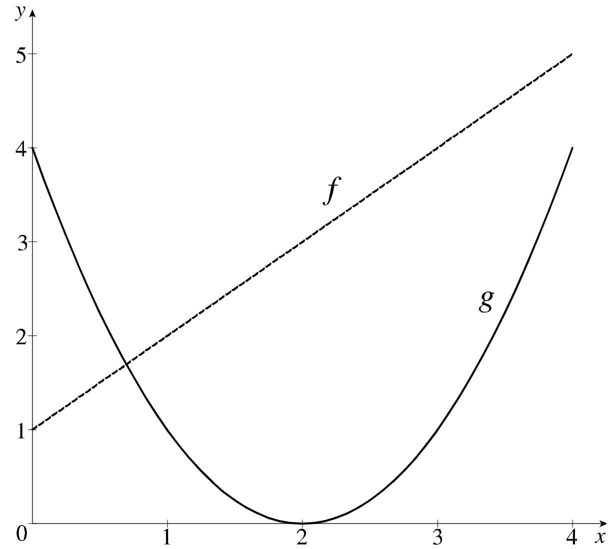
(d) $2f(2x-2)$



(e) $f(2x)-2$



5. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.



(a) $(f \circ g)(2)$

(b) $(g \circ f)(2)$

(c) $(f \circ f)(2)$

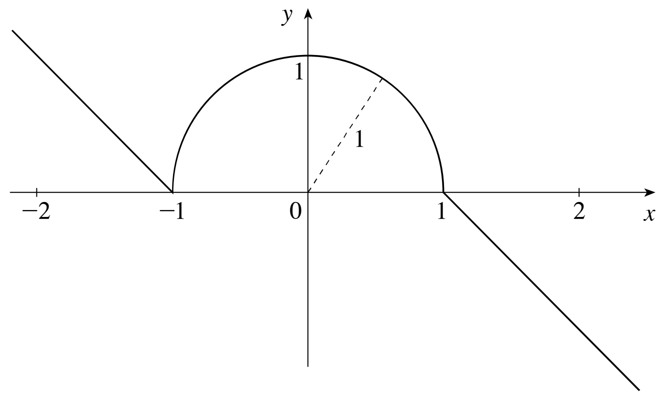
(d) $(g \circ g)(2)$

(e) $(f + g)(2)$

(f) $(f/g)(2)$

(g) $g^{-1}(2)$

6. Find a formula that describes the following function.



7. Let $f(x) = 2.912345x^2 + 3.131579x - 0.099999$

(a) To simplify approximation of f, write a quadratic function $g(x)$ with *integer* coefficients that closely models $f(x)$ for $-10 < x < 10$.

(b) Compute $g(4)$ and $f(4)$.

(c) Compute the error in using $g(4)$ to approximate $f(4)$ as a percentage of the correct answer $f(4)$.

(d) For larger values of x (say $x = 10$ or $x = 20$), would $g(x)$ be an overestimate or an underestimate of $f(x)$? Justify your answer without computing specific values of f and g.

8. Let f be a one-to-one function whose *inverse* function is given by the formula

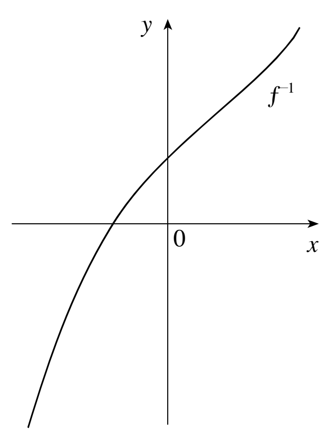
$$f^{-1}(x) = x^5 + 2x^3 + 3x + 1$$

(a) Compute $f^{-1}(1)$ and $f(1)$.

(b) Compute the value of x_0 such that $f(x_0) = 1$.

(c) Compute the value of y_0 such that $f^{-1}(y_0) = 1$.

(d) Below is a graph of f^{-1}. Draw an approximate graph of f.



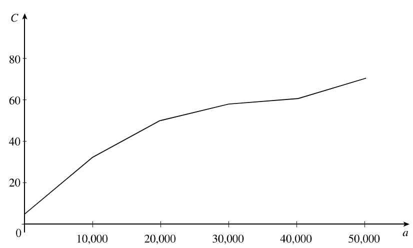
9. Find constants A, B, and k such that the equation $f(x) = A2^{kx} + B$ satisfies the following three conditions:

- $f(x)$ is always decreasing.
- $f(x)$ has a horizontal asymptote at $y = 1$, and
- $f(x)$ goes through the point $(0, 4)$.

10. Find constants A and k such that the equation $f(x) = A2^{kx}$ fits the following data as closely as possible:

x	y
1	1.8000
2	2.7000
4	6.0750
5	9.1125
7	20.503

11. A manufacturer hires a mathematician to come up with a function f that models the cost C of producing a alternative-music compact discs, where C is in thousands of dollars. The graph of f is given below.



(a) What does $f(50{,}000) - f(49{,}999)$ represent?

(b) What does $f^{-1}(10)$ represent?

(c) For what value or range of values is the cost per disc the least?

(d) Give a possible explanation for the sudden increase in the curve's slope at the end.

1 SAMPLE EXAM SOLUTIONS

1. Approximate answers are acceptable for this problem.

(a) $f\,(\text{noon}) = 87°$, $f\,(6\text{ P.M.}) = 67°$, range of f is $[53, 87]$.

(b) f is increasing on $(6, 12)$ and $(20, 22)$; f is decreasing on $(0, 6)$, $(12, 20)$ and $(22, 24)$.

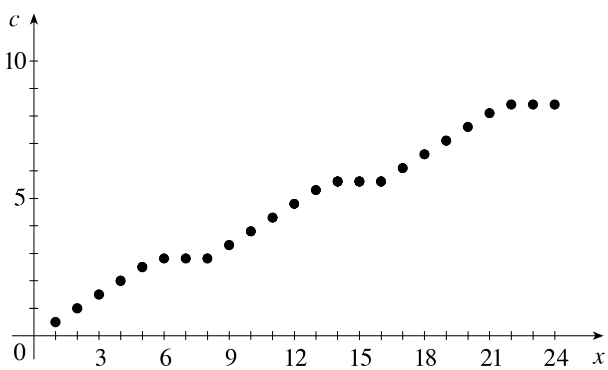
(c) Possible explanations for the drop in temperature at noon are a sudden thundershower, or an air conditioner being turned on.

(d) A possible explanation for f attaining its minimum value at 6 A.M. is that this is just before sunrise.

2. The total volume is the volume of a cylinder of height and radius r plus the volume of a hemisphere of radius r, that is, $V = \pi r^2 h + \frac{2}{3}\pi r^3$.

3. (a) If we buy 8 cards for \$2.80, then this costs less than buying 6 individual cards at \$0.50 apiece. Hence, $C\,(6) = \$2.80$.

(b)

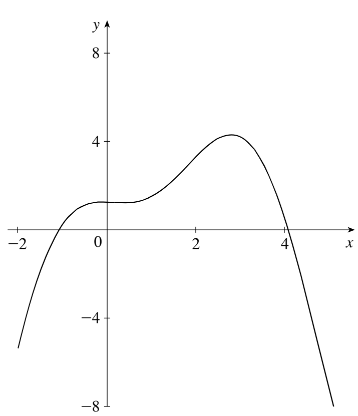


(c) $c\,(x) = \begin{cases} 25 + 0.5\,(x - 80) & \text{if } 80 \le x \le 85 \\ 27.8 & \text{if } 86 \le x \le 88 \\ 27.8 + 0.5\,(x - 88) & \text{if } 89 \le x \le 90 \end{cases}$

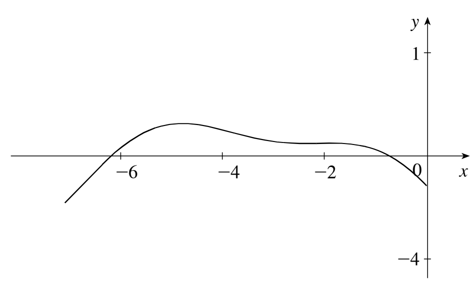
(d) To buy 1005 cards, the best deal is to buy one carton (800 cards), two boxes (160 cards), five packs (40 cards) and five individual cards. The total cost would be

$$230 + 2\,(25) + 5\,(2.80) + 5\,(0.50) = \$296.50$$

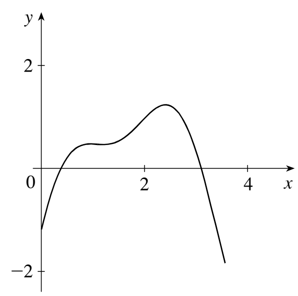
4. (a)



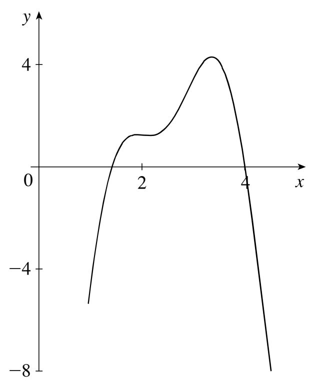
(b)



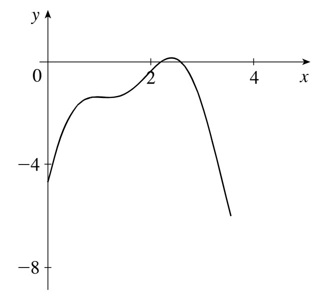
(c)



(d)



(e)



5. (a) $(f \circ g)(2) = f(0) = 1$

(b) $(g \circ f)(2) = g(3) = 1$

(c) $(f \circ f)(2) = f(3) = 4$

(d) $(g \circ g)(2) = g(0) = 4$

(e) $(f + g)(2) = f(2) + g(2) = 3 + 0 = 3$

(f) $\left(\dfrac{f}{g}\right)(2)$ is undefined because $g(2) = 0$.

(g) $g^{-1}(2)$ is undefined because $g(x)$ takes on the value 2 twice, for $x = 0.6$ and $x = 3.4$.

6. $f(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ \sqrt{1 - x^2} & \text{if } -1 \le x \le 1 \\ -x + 1 & \text{if } x > 1 \end{cases}$

7. (a) $g(x) = 3x^2 + 3x$

(b) $g(4) = 60$, $f(4) = 59.023837$

(c) The percentage error in using $g(4)$ as an approximation for $f(4)$ is $100\left|\dfrac{f(4) - g(4)}{g(4)}\right| = 1.63\%$.

(d) For larger values of x, $g(x)$ is an overestimate of $f(x)$ because the coefficient of the dominant term (x^2) is larger.

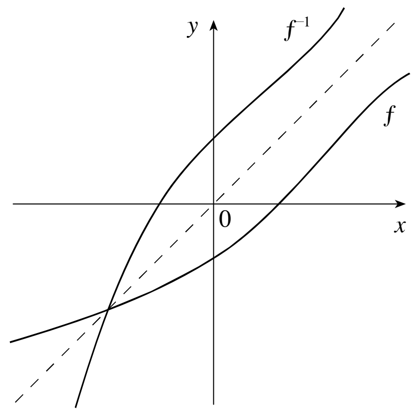
8. $f^{-1}(x) = x^5 + 2x^3 + 3x + 1$

(a) $f^{-1}(1) = 7$, $f(1) = 0$

(b) The value x_0 such that $f(x_0) = 1$ is $f^{-1}(1) = 7$.

(c) The value y_0 such that $f^{-1}(y_0) = 1$ is $f(1) = 0$.

(d) The graph of $f(x)$ is the graph of $f^{-1}(x)$ reflected about the line $y = x$.



9. Let $f(x) = 3(2)^{-x} + 1$. Then $f(x)$ is always decreasing, has a horizontal asymptote at $y = 1$, and $f(0) = 4$.

10. $f(x) = 1.2(2)^{0.585x}$

11. (a) $f(50,000) - f(49,999)$ represents the cost of producing the 50,000th disc.

(b) $f^{-1}(10)$ represents the number of discs that can be made for \$10,000.

(c) The cost per disc is cheapest for $30,000 < a < 40,000$. This is where the slope of f is the smallest.

(d) One possible explanation for the sudden increase in the curve's slope is scarcity of materials.