Solutions Manual

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for the book

Fundamentals of Complex Analysis, 3rd ed.

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CHAPTER 1: Complex Numbers

EXERCISES 1.1: The Algebra of Complex Numbers

1. $-i = a + bi \Longrightarrow a = 0$ and $b = -1 \Rightarrow$
   $(-i)^2 = (a^2 - b^2) + (2ab)i = -b^2 = -1$

2. The Commutative and Associative laws for addition follow directly from the real counterparts.
   Commutative law for multiplication:

$$\begin{aligned}(a+bi)(c+di) &= (ac-bd)+(bc+ad)i \\ &= (ca-db)+(da+cb)i \\ &= (c+di)(a+bi)\end{aligned}$$

   Associative law for multiplication:

$$\begin{aligned}&[(a+bi)(c+di)](e+fi) = [(ac-bd)+(bc+ad)i](e+fi) \\ &\quad = [(ac-bd)e-(bc+ad)f]+[(bc+ad)e+(ac-bd)f]i \\ &\quad = [a(ce-df)-b(de+cf)]+[b(ce-df)+a(de+cf)]i \\ &\quad = (a+bi)[(ce-df)+(de+cf)i] \\ &\quad = (a+bi)[(c+di)(e+fi)]\end{aligned}$$

   Distributive law:

$$\begin{aligned}&(a+bi)[(c+di)+(e+fi)] = (a+bi)[(c+e)+(d+f)i] \\ &\quad = [a(c+e)-b(d+f)]+[b(c+e)+a(d+f)]i \\ &\quad = [(ac-bd)+(bc+ad)i]+(ae-bf)+(be+af)i] \\ &\quad = (a+bi)(c+di)+(a+bi)(e+fi)\end{aligned}$$

3. a. $z_3 = z_2 - z_1 \iff$
   $e + fi = (c-a) + (d-b)i = (c+di) - (a+bi) \iff$
   $e = c - a$ and $f = d - b \iff$
   $e + a = c$ and $f + b = d \iff$
   $(e+fi) + (a+bi) = c + di \iff$

b. $(e+fi)(c+di) = a+bi \iff$
$ce - fd = a$ and $fc + ed = b \iff$

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i, \quad c+id \neq 0 \\ &= \frac{(ec-fd)c+(fc+ed)d}{c^2+d^2} \\ &\quad + \frac{(fc+ed)c-(ec-fd)d}{c^2+d^2}i \\ &= e+fi\end{aligned}$$

4. Suppose $z_1 \neq 0$. Then $z_2 = \dfrac{z_1 z_2}{z_1} = \dfrac{0}{z_1} = 0$.

5. a. $0 + \left(-\dfrac{3}{2}\right)i = \dfrac{-3}{2}i$
   b. $3 + 0i = 3$
   c. $0 + (-2)i = -2i$

6. a. $0 + (-2)i = -2i$
   b. $6 + (-3)i = 6 - 3i$
   c. $4 + \pi i$

7. a. $8 + 1i = 8 + i$
   b. $1 + 1i = 1 + i$
   c. $0 + \left(\dfrac{-8}{3}\right)i = -\dfrac{8i}{3}$

8. $\dfrac{33}{25} - \dfrac{19}{25}i$

9. $\dfrac{61}{185} - \dfrac{107}{185}i$

10. $-\dfrac{253}{4225} - \dfrac{204}{4225}i$

11. $2 + 0i = 2$

12. $-9 + (-7)i$

13. $6+5i$

14. $z=a+bi$. $\mathrm{Re}(iz)=\mathrm{Re}(ai-b)=-b=-\mathrm{Im}\,z$

15. $i^{4k}=(i^4)^k=1^k=1$
$i^{4k+1}=i^{4k}\cdot i=1\cdot i=i$
$i^{4k+2}=i^{4k}\cdot i^2=1\cdot(-1)=-1$
$i^{4k+3}=i^{4k}\cdot i^3=1\cdot(-i)=-i$

16. a. $-i$

b. -1

c. -1

d. $-i$

17. $3i^{2(4)+3}+6i^3+8i^{-5(4)}+i^{-1(4)+3}$
$=3(-i)+6(-i)+8(1)+(-i)=8-10i$

18. $(-1+i)^2+2(-1+i)+2=-2i+(-2+2i)+2=0$

19. The real equations are

$$\begin{aligned}\mathrm{Re}(z^3+5z^2) &= \mathrm{Re}(z+3i)\\ \mathrm{Im}(z^3+5z^2) &= \mathrm{Im}(z+3i).\end{aligned}$$

If $z=a+bi$ these can be rewritten as

$$a^3-3ab^2+5a^2-5b^2-a=0$$
$$3a^2b-b^3+10ab-b-3=0.$$

20. a. $z=\dfrac{4}{2i}=-2i$

b. $z=\dfrac{1-5i}{2-5i}=\dfrac{27}{29}-\dfrac{5i}{29}$

c. $z=0,\quad -\dfrac{1}{4}+\dfrac{i}{8}$

d. $z=\pm 4i$

21. $(-i)[(1-i)z_1+3z_2]+(1-i)[iz_1+(1+2i)z_2]$
$= -i(2-3i)+(1-i)(1)$
$\Longrightarrow z_2 = \dfrac{-2-3i}{3-2i} = -i \Longrightarrow z_1 = 1+i$

22. $0 = z^4-16 = (z-2)(z+2)(z-2i)(z+2i) \Longrightarrow z = 2, -2, 2i, -2i$

23. Suppose $z = a+bi$.

$$\operatorname{Re}\left(\frac{1}{z}\right) = \operatorname{Re}\left(\frac{a-ib}{a^2+b^2}\right) = \frac{a}{a^2+b^2} > 0$$

whenever $a > 0$.

24. Suppose $z = a+bi$.

$$\begin{aligned}\operatorname{Im}\left(\frac{1}{z}\right) &= \operatorname{Im}\left(\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i\right)\\ &= -\frac{b}{a^2+b^2} < 0 \text{ whenever } b > 0.\end{aligned}$$

25. Let $z_1 = a+bi$ and $z_2 = c+di$. The hypotheses specify that $a+c<0$, $b+d=0$, $ac-bd<0$, and $ad+bc=0$.

$b = 0 \Longrightarrow d = 0 \Longrightarrow z_1$ and z_2 are real.

$b \neq 0 \Longrightarrow d = -b$ and $ad+bc = a(-b)+bd = -b(a-c) = 0$

$\Longrightarrow a = c$, a contradiction of the fact that $z_1z_2 < 0$.

26. By induction: The case when $n = 1$ is obvious. Assume $\operatorname{Re}\left(\sum_{j=1}^{m} z_j\right) = \sum_{j=1}^{m} \operatorname{Re}(z_j)$ for all positive integers $m < n$

$$\begin{aligned}\operatorname{Re}\left(\sum_{j=1}^{n} z_j\right) &= \operatorname{Re}\left(\sum_{j=1}^{n-1} z_j + z_n\right)\\ &= \sum_{j=1}^{n-1}\operatorname{Re}(z_j)+\operatorname{Re}(z_n)\\ &= \sum_{j=1}^{n}\operatorname{Re}(z_j)\end{aligned}$$

The corresponding result for the imaginary parts follows by replacing "Re" by "Im" in the above proof.

Disprove: $\operatorname{Re}\left(\prod_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} \operatorname{Re}(z_j)$ and

$$\operatorname{Im}\left(\prod_{j=1}^{n} z_j\right) = \prod_{j=1}^{n} \operatorname{Im}(z_j).$$

$\operatorname{Re}[(a+bi)(c+di)] = ac - bd$
$\operatorname{Re}(a+bi)\operatorname{Re}(c+di) = ac$
These are not equal whenever $bd \neq 0$.

$\operatorname{Im}[(a+bi)(c+di)] = ad + bc$
$\operatorname{Im}(a+bi)\operatorname{Im}(c+di) = bd$
These are not equal whenever $ad + bc \neq bd$.
(For example, consider the pair 2 and i.)

27. By induction: The case when $n = 1$ is obvious. Assume

$$\begin{aligned}(z_1+z_2)^m &= z_1^m + \binom{m}{1} z_1^{m-1} z_2 + \cdots \\ &\quad + \binom{m}{k} z_1^{m-k} z_2^k + \cdots + z_2^m\end{aligned}$$

for all positive integers $m < n$. Recall that, for positive integers r and s with $r > s$,

$$\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1} \quad \text{and} \quad \binom{r}{0} = \binom{r}{r} = 1\cdot$$

$$\begin{aligned}(z_1+z_2)^n &= (z_1+z_2)^{n-1}(z_1+z_2) \\ &= z_1^{n-1}(z_1+z_2) + \binom{n-1}{1} z_1^{n-2} z_2 (z_1+z_2) \\ &\quad + \cdots + \binom{n-1}{k} z_1^{n-1-k} z_2^k (z_1+z_2) \\ &\quad + \cdots + z_2^{n-1}(z_1+z_2) \\ &= z_1^n + z_1^{n-1} z_2 + \binom{n-1}{1}(z_1^{n-1} z_2 + z_1^{n-2} z_2^2) \\ &\quad + \cdots + \binom{n-1}{k}(z_1^{n-k} z_2^k + z_1^{n-(k+1)} z_2^{k+1}) + \cdots + z_2^{n-1} z_1 + z_2^n\end{aligned}$$

$$
\begin{aligned}
&= z_1^n + \left[\binom{n-1}{0} + \binom{n-1}{1}\right] z_1^{n-1} z_2 \\
&\quad + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] z_1^{n-2} z_2^2 \\
&\quad + \cdots \left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] z_1^{n-k} z_2^k \\
&\quad + \cdots + z_2^n \\
&= z_1^n + \binom{n}{1} z_1^{n-1} z_2 + \binom{n}{2} z_1^{n-2} z_2^2 \\
&\quad + \cdots + \binom{n}{k} z_1^{n-k} z_2^k + \cdots + z_2^n.
\end{aligned}
$$

28. $2^5 + \binom{5}{1} 2^4(-i) + \binom{5}{2} 2^3(-i)^2 + \binom{5}{3} 2^2(-i)^3 + \binom{5}{4} 2(-i)^4 + (-i)^5$
$= 32 - 80i - 80 + 40i + 10 - i = -38 - 41i$

29. Suppose $x = \dfrac{p}{q}$, where p and q are relatively prime integers, and that $x^2 = 2$.
$\left(\dfrac{p}{q}\right)^2 = 2 \Longrightarrow p^2 = 2q^2 \Longrightarrow p^2 = 4k$ for some integer k and $q^2 = 2k$,
a contradiction (If p^2 is an even integer so is p.).

30. By contradiction. Suppose there is a nonempty subset P of the complex numbers satisfying (i), (ii), and (iii) and suppose i is in P.
Then, by (iii), $i^2 = -1$ and $(-1)i = -i$. This violates (i).
Similarly (i) is violated by assuming $-i$ belongs to P.

31. Purpose: to add, subtract, multiply and divide $z_1 = a + bi$ and $z_2 = c + di$.
Input a, b, c, d
Set sum= $(a + c, b + d)$
Print "$z1 + z2 =$ "; sum
Set diff=$(a - c, b - d)$
Print "$z1 - z2 =$ "; diff
Set prod=$(a * c - b * d, b * c + a * d)$

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Print "z1 * z2 = "; prod
Set denom = c^2 + d^2
If denom = 0, print "there is no quotient"
Else
   Set quot=((a * c + b * d)/(denom), (b * c - a * d)/(denom))
   Print "z1/z2 = "; quot
Endif
Stop
```

32. prod=$(a * c - b * d,\ (a + b) * (c + d) - a * c - b * d)$

EXERCISES 1.2: Point Representation of Complex Numbers; Absolute Value and Complex Conjugates

1. The real and imaginary parts of

$$\frac{z_1 + z_2}{2} = \frac{x_1 + x_2}{2} + i\frac{y_1 + y_2}{2}$$

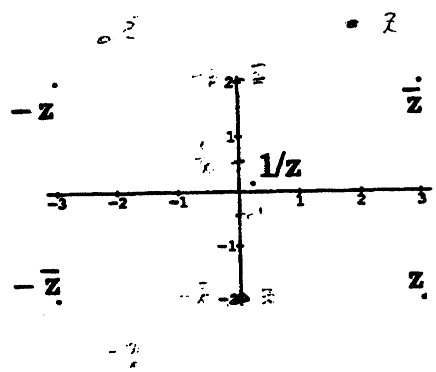
give the familiar algebra formula for the midpoint of the line segment joining two points in $\mathbf{R}^2$.

Alternatively, one could establish that $(z_1 + z_2)/2$ is a point on the line through z_1 and z_2 and that $|z_1 - (z_1 + z_2)/2| = |z_2 - (z_1 + z_2)/2|$.

2. $\hat{z} = \dfrac{2(1+i) + (-3i) + 3(1-2i) + 5(-6)}{2+1+3+5} = -\dfrac{25}{11} - \dfrac{7}{11}i$

3. -3

4. $\left(\dfrac{1}{z} = \dfrac{3}{13} + \dfrac{2}{13}i\right)$



5. The three side lengths are equal:

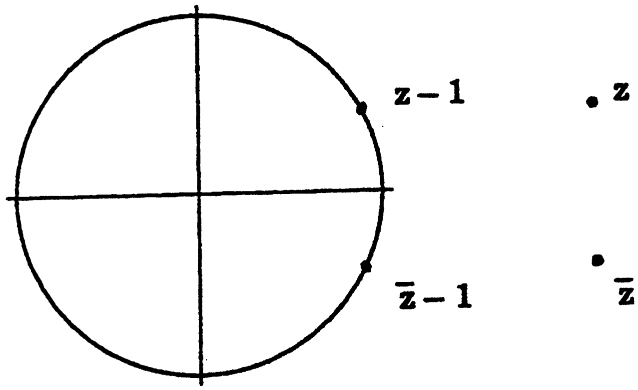
$$\left|1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right| = \left|1-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right|$$
$$= \left|\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right| = \sqrt{3}$$

6. The Pythagorean theorem is satisfied:

$$10+10 = |(3+i)-6|^2 + |(3+i)-(4+4i)|^2 = |6-(4+4i)|^2 = 20$$

7. a. All points on the horizontal line through $z=-2i$
   b. All points on the circle of radius 3 with center at $1-i$
   c. All points on the circle of radius 2 with center at $\frac{1}{2}i$
   d. The points must be equidistant from 1 and $-i$, thus lie on the perpendicular bisector of the line through 1 and $-i$.
   e. The equation can be written as $x = \frac{1}{4}y^2 - 1$. The points lie on this parabola.
   f. The points z have the property that their distance from 1 added to their distance from -1 is always 7, so the points lie on an ellipse with foci ± 1, with x intercepts $\pm\frac{7}{2}$ and y intercepts $\pm\frac{3}{2}\sqrt{5}$.
   g. All points on the circle of radius $\frac{3}{8}$ with center at $\frac{9}{8}$
   h. All points in the half plane $x \geq 4$
   i. All points inside the circle of radius 2 centered at i
   j. All points outside the circle of radius 6 centered at the origin

8. $|(a+bi)-1| = \sqrt{(a-1)^2+b^2}$
$= \sqrt{(a-1)^2+(-b)^2}$
$= |\overline{a+bi}-1|$



9. $|rz| = |r(a+bi)| = |ra+rbi| = \sqrt{(ra)^2+(rb)^2}$
$= \sqrt{r^2(a^2+b^2)} = r\sqrt{a^2+b^2} = r|z|$

10. $|\mathrm{Re}\, z| = |a| = \sqrt{a^2} \le \sqrt{a^2+b^2} = |z|$
$|\mathrm{Im}\, z| = |b| = \sqrt{b^2} \le \sqrt{a^2+b^2} = |z|$

11. $a = |a+bi| = \sqrt{a^2+b^2} \Longrightarrow a \ge 0$ and $b = 0$

12. a. $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{a_1+b_1 i}{a_2+b_2 i}\right)} = \overline{\left(\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}\right)}$

$$= \frac{(a_1a_2+b_1b_2)+(-a_2b_1+a_1b_2)i}{a_2^2+b_2^2}$$

$$= \frac{a_1-b_1 i}{a_2-b_2 i} = \frac{\overline{z_1}}{\overline{z_2}}.$$

b. $\frac{z+\overline{z}}{2} = \frac{(a+bi)+(a-bi)}{2} = a = \mathrm{Re}\, z$

c. $\frac{z-\overline{z}}{2i} = \frac{(a+bi)-(a-bi)}{2i} = b = \mathrm{Im}\, z$

13. $(\overline{z})^2 - z^2 = 0 \Longrightarrow (\overline{z} - z)(\overline{z} + z) = 0 \Longrightarrow$

either: $\overline{z} - z = 0 \Longrightarrow 2i\mathrm{Im}\, z = 0 \Longrightarrow z$ is real, or
$\overline{z} + z = 0 \Longrightarrow 2\mathrm{Re}\, z = 0 \Longrightarrow z$ is pure imaginary.

14. $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2$

15. By induction: The case when $k = 0$ is obvious. Assume $(\overline{z})^m = \overline{(z^m)}$ for all positive integers $m < k$.

$$(\overline{z})^k = (\overline{z})^{k-1}(\overline{z}) = (\overline{z^{k-1}})\overline{z} = \overline{z^{k-1} z} = \overline{z^k}$$

Also,

$$(\overline{z})^{-k} = \frac{1}{(\overline{z})^k} = \frac{1}{\overline{z^k}} = \overline{\left(\frac{1}{z^k}\right)} = \overline{z^{-k}}$$

16. Let $z = a + bi$. Since $|z|^2 = a^2 + b^2 = 1$,

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \mathrm{Re}\left(\frac{1}{(1-a) - bi}\right) = \mathrm{Re}\left(\frac{(1-a) + bi}{2 - 2a}\right) = \frac{1}{2}.$$

17. $\overline{z_0}^{\,n} + a_1 \overline{z_0}^{\,n-1} + \cdots + a_{n-1}\overline{z_0} + a_n$
$= \overline{z_0^n + a_1 z_0^{n-1} + \cdots + a_{n-1} z_0 + a_n} = \overline{0} = 0$

18. The roots of $z^2 + a_1 z + a_2 = 0$ are $z = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$.

$a_1^2 - 4a_2 \geq 0 \Longrightarrow$ Both roots are real
$\Longrightarrow$ Each root is its own conjugate
$a_1^2 - 4a_2 < 0 \Longrightarrow \pm\sqrt{a_1^2 - 4a_2} = \pm i\sqrt{4a_2 - a_1^2}$
$\Longrightarrow$ The roots are complex conjugates.

19. The line ax+by=c can be represented in the complex plane as $z = r\cos\theta + ir\sin\theta + c/a$ where $\theta = \tan^{-1}(-a/b)$ and $-\infty < r < \infty$. By working with triangles you can obtain

$\cos\theta = -b/\sqrt{(a^2 + b^2)}$ and $\sin\theta = a/\sqrt{(a^2 + b^2)}$. To get to point z write the equation from point c/a down the line and make a turn on the perpendicular as

$z = x+iy = r\cos\theta + ir\sin\theta + c/a - s\sin\theta + is\cos\theta$ with $-\infty < s < \infty$. Equating the real and imaginary parts

$x - c/a = r\cos\theta - s\sin\theta$; $y = r\sin\theta + s\cos\theta$

Solve for s as $s = (-\sin\theta(x-c/a) + y\cos\theta) = (-ax + c - by)/\sqrt{(a^2 + b^2)}$. The distance from the point z to the line ax + by = c is s. Denote the reflected point by z_r. The reflected point lies s units on the other side of the line.

$z_r = z - 2s(-a - ib)/)/\sqrt{(a^2 + b^2)} = x + iy - 2\{(-ax + c - by)/\sqrt{(a^2 + b^2)}\}(-a - ib)/\sqrt{(a^2 + b^2)}$

$= \{[(b^2 - a^2)x - 2aby + 2ac] + i[(a^2-b^2)y - 2abx + 2bc]\}/\sqrt{(a^2 + b^2)}$

$= [2ic + (b-ai)(x-iy)]/(b+ai)$

20. (a) Suppose $u^\dagger Au = 0$ for all n by 1 column vectors with complex entries. Let

$u = [0\ 0 \dots 1\ ..0]^T$ with the i^{th} entry being the only nonzero entry. Then $u^\dagger Au = (a_{ii}) = 0$ for i=1 to n.

Let u be all zeros except for $½ + i\sqrt{3}/2$ On the ith row and $½ - i\sqrt{3}/2$ on the jth row. Now

$u^\dagger Au = (a_{ij})(½ - i\sqrt{3}/2\ )^2 + (a_{ji})(½ + i\sqrt{3}/2)^2 = -(1/2 - i\sqrt{3}/2)(a_{ij}) - (½ + i\sqrt{3}/2)(a_{ji}) = 0.$

Setting the real and imaginary parts equal to zero yields $a_{ij} = 0$ and $a_{ji} = 0$ for all i,j= 1 to n. Consequently A = 0.

(b) Let A = [0 1; -1 0]. Now $u^\dagger Au = 0$ for all 2 by 1 real column vectors .

21. The matrix A is *Hermitian* $A^\dagger = A$. Observe $(Au)^\dagger = u^\dagger A^\dagger = u^\dagger A$.

(a) $(u^\dagger Au)^\dagger$ is the conjugate transpose of the matrix $u^\dagger Au$ which is a one by one matrix, so

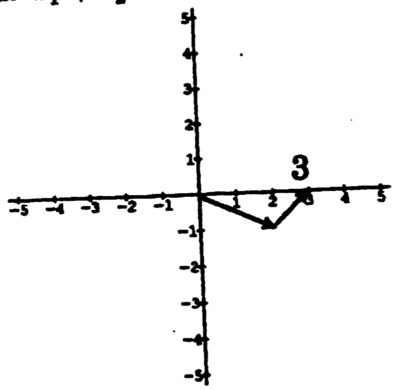
$(u^\dagger Au)^\dagger = u^\dagger A^\dagger u = u^\dagger Au$ because A is *Hermitian.* The conjugate is equal to the number only when the number is real.

(b) $(B^\dagger B)^\dagger = B^\dagger B$ and therefore is *Hermitian.*

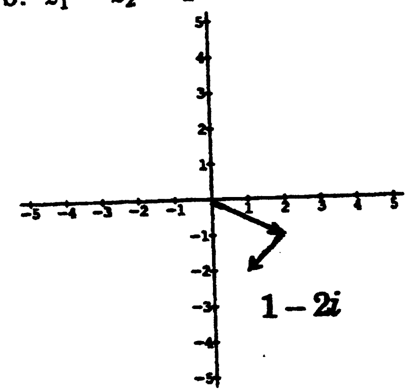
(c) $(u^\dagger B^\dagger Bu)^\dagger = (Bu)^\dagger(u^\dagger B^\dagger)^\dagger = u^\dagger B^\dagger Bu$ a real number.

EXERCISES 1.3: Vectors and Polar Forms

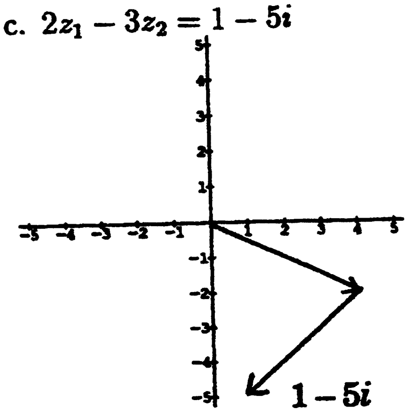
1. a. $z_1 + z_2 = 3$



b. $z_1 - z_2 = 1 - 2i$



c. $2z_1 - 3z_2 = 1 - 5i$



2. $|z_1z_2z_3| = |(z_1z_2)z_3| = |z_1z_2||z_3| = |z_1||z_2||z_3|$

3. $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

4. By induction: The case when $k = 0$ is obvious. Assume $|z^m| = |z|^m$ for all positive integers $m < k$.

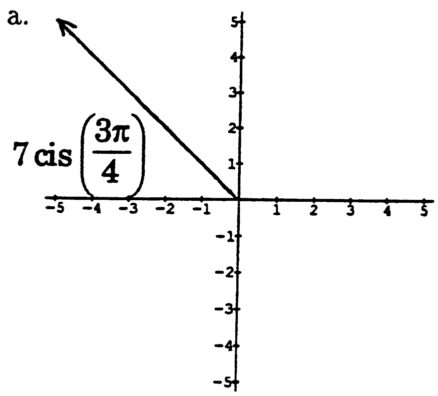
$$|z^k| = |z^{k-1}z| = |z^{k-1}||z| = |z|^{k-1}|z| = |z|^k$$

Also,

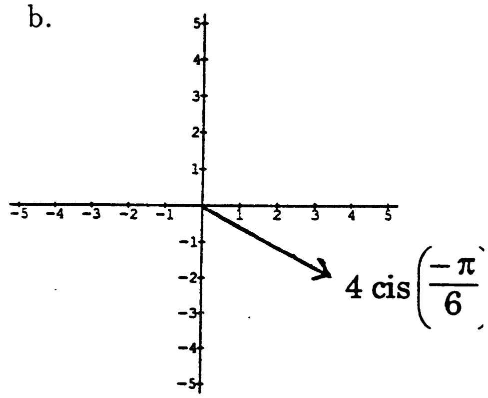
$$|z^{-k}| = \left|\frac{1}{z^k}\right| = \frac{1}{|z^k|} = \frac{1}{|z|^k} = |z|^{-k}$$

5. a. 1
   b. $5\sqrt{26}$
   c. $\dfrac{5\sqrt{5}}{2}$
   d. 1

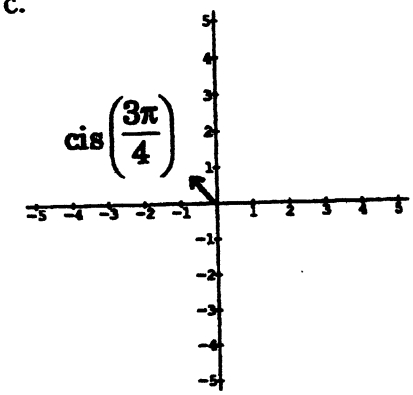
6. a.



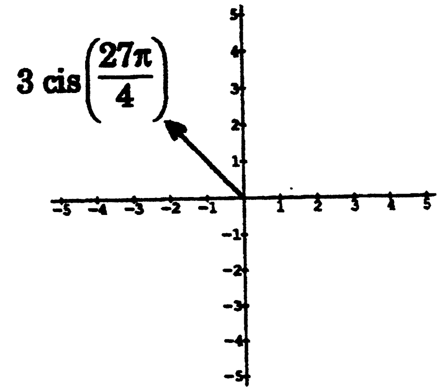
b.



c.



d.



7. (Only the value of Arg z is given for each of the following.)

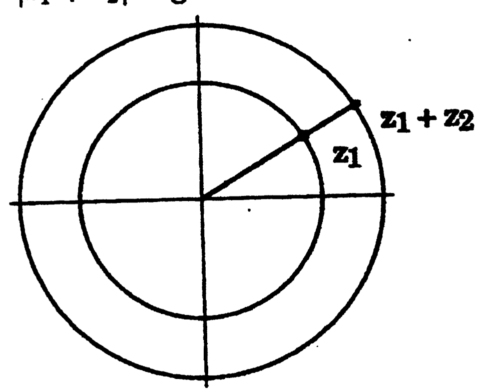
a. $\frac{1}{2}\text{cis}\,\pi$ b. $3\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

c. $\pi\text{cis}\left(-\frac{\pi}{2}\right)$ d. $4\text{cis}\left(-\frac{5\pi}{6}\right)$

e. $2\sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right)$ f. $4\text{cis}\left(-\frac{\pi}{3}\right)$

g. $\frac{1}{\sqrt{2}}\text{cis}\left(\frac{5\pi}{12}\right)$ h. $\frac{\sqrt{14}}{2}\text{cis}\left(\frac{-11\pi}{12}\right)$

8. Suppose $|z_2| = r$. Then $z_1 + z_2$ lies on the circle in the figure and $|z_1 + z_2|$ is greatest when $\arg z_1 = \arg z_2$



9. It is a vector of length $|z|$ and angle of inclination $\arg z + \phi$; it is obtained by rotating z by angle ϕ in the counterclockwise direction.

10a. $\arg(z_1z_2z_3) = \arg((z_1z_2)z_3) = \arg(z_1z_2) + \arg z_3 = \arg z_1 + \arg z_2 + \arg z_3$

10b. $\arg z_1\overline{z_2} = \arg z_1 + \arg \overline{z_2} = \arg z_1 - \arg z_2$

11. $(1 + i)(5 - i)^4 = \sqrt{2}\ \mathrm{cis}(\pi/4)\sqrt{(26)}\mathrm{cis}(-4\tan^{-1}(1/5)) = (1 + i)(24 - i10)^2$
$= (1 + i)(24^2 - 100 - i480) = 976 - i4$
$\arg(1 + i)(5 - i)^4) = \pi/4 - 4\tan^{-1}(1/5) = -\tan^{-1}(1/239)$
$\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$.

12. a. $-\dfrac{3\pi}{4}$
b. π
c. $\dfrac{\pi}{2}$
d. $-\dfrac{\pi}{6}$

13. b and d always true

Counterexample for part a:

$$z_1 = z_2 = \mathrm{cis}\,\frac{5\pi}{6} \Longrightarrow \mathrm{Arg}\, z_1z_2 = -\frac{\pi}{3}, \quad \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \frac{5\pi}{3}$$

Counterexample for part c:

$$z_1 = -i, \quad z_2 = i \Longrightarrow \mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \pi, \quad \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2 = -\pi$$

14. If $x > 0$ then $\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}(1 - 1) = \tan^{-1}\left(\frac{y}{x}\right)$, which corresponds to $\frac{-\pi}{2} < \arg z < \frac{\pi}{2}$.

If $x < 0$ then $\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}(1 + 1) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, which corresponds to $\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

If $x = 0$ and $y > 0$, then $\frac{\pi}{2}(1) = \arg z$.

If $x = 0$ and $y < 0$, then $\frac{\pi}{2}(-1) = \arg z$.

If $x = y = 0$ then $\arg z$ is undefined.

If $y > 0$ then $1 \cdot \cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$ corresponds to $0 < \mathrm{Arg}\, z < \pi$.

If $y < 0$ then $-\cos^{-1}\left(x/\sqrt{x^2 + y^2}\right)$ corresponds to $-\pi < \mathrm{Arg}\, z < 0$.

If $y = 0$ and $x > 0$ then $0 = \mathrm{Arg}\, z$.

15. $|z_1 - z_2| = |z_1 + (-z_2)| \le |z_1| + |-z_2| = |z_1| + |z_2|$

16. Apply Exercise 15 twice:

$|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \Longrightarrow$

$|z_1| - |z_2| \le |z_1 - z_2|$

Similarly (beginning with $|z_2|$),

$|z_2| - |z_1| \le |z_2 - z_1| = |z_1 - z_2|$

Thus,

$-|z_1 - z_2| \le |z_1| - |z_2| \le |z_1 - z_2|$, or

$||z_1| - |z_2|| \le |z_1 - z_2|$

17. If vector z_1 is parallel to vector z_2, then $z_2 = cz_1$ for some real number $c \ne 0$, and $z_1\overline{z_2}$ is real valued since $z_1\overline{z_2} = c|z_1|^2$.

Conversely if $z_1\overline{z_2}$ is real valued,

$\arg z_1 - \arg z_2 = \arg(z_1\overline{z_2}) = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots \Longrightarrow$

$\arg z_2 = \arg z_1 + k\pi \quad \Longrightarrow$ Vector z_2 is parallel to vector z_1.

18. By Example 1, the points z_2, z_1 and z lie on the same line if and only if $z - z_1 = c'(z_1 - z_2)$, which is true if and only if $z = z_1 + c(z_2 - z_1)$, where $c = -c'$. It follows that z lies strictly between z_1 and z_2 if and only if $0 < c < 1$.

19. $z_1 = cz_2$ with c real and $c > 0 \Longleftrightarrow$

$\arg z_1 = \arg c + \arg z_2 = 0 + \arg z_2 = \arg z_2$

20. The triangle with vertices z_1, z_2, and z_3 has sides represented by the vectors $z_2 - z_1$, $z_3 - z_1$, and $z_3 - z_2$. Let ϕ be the angle between $z_3 - z_1$ and $z_2 - z_1$. Then

$$
\begin{aligned}
\phi &= \arg(z_3 - z_1) - \arg(z_2 - z_1) \\
&= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)
\end{aligned}
$$

The result can now be recognized as the Law of Cosines.

21. $r_1\operatorname{cis}\theta_1 + r_2\operatorname{cis}\theta_2 = [r_1\cos\theta_1 + r_2\cos\theta_2] + i[r_1\sin\theta_1 + r_2\sin\theta_2]$

$$\begin{aligned}
\Longrightarrow\ r^2 &= [r_1\cos\theta_1 + r_2\cos\theta_2]^2 + [r_1\sin\theta_1 + r_2\sin\theta_2]^2 \\
&= r_1^2 + 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + r_2^2 \\
&= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) \\
\Longrightarrow\ r &= \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)} \\
\cos\theta &= \operatorname{Re}\left(\frac{r_1\operatorname{cis}\theta_1 + r_2\operatorname{cis}\theta_2}{r}\right) \\
&= \frac{r_1\cos\theta_1 + r_2\cos\theta_2}{\sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}} \\
\sin\theta &= \operatorname{Im}\left(\frac{r_1\operatorname{cis}\theta_1 + r_2\operatorname{cis}\theta_2}{r}\right) \\
&= \frac{r_1\sin\theta_1 + r_2\sin\theta_2}{\sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}} \\
\theta &= \tan^{-1}\left(\frac{r_1\sin\theta_1 + r_2\sin\theta_2}{r_1\cos\theta_1 + r_2\cos\theta_2}\right)
\end{aligned}$$

when $r_1\cos\theta_1 + r_2\cos\theta_2 > 0$.

See Exercise 14 to adjust θ for the other cases.

22. By induction. The case when $n = 2$ is the standard triangle inequality. Assume

$$\left|\sum_{k=1}^{m} z_k\right| \le \sum_{k=1}^{m} |z_k|$$

for all positive integers $m < n$. Then

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k\right| &= \left|\sum_{k=1}^{n-1} z_k + z_n\right| \\
&\le \left|\sum_{k=1}^{n-1} z_k\right| + |z_n| \\
&\le \sum_{k=1}^{n-1} |z_k| + |z_n| = \sum_{k=1}^{n} |z_k|.
\end{aligned}$$

23. $\left|\dfrac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3}\right|$

$$\leq \left|\frac{m_1z_1}{m_1+m_2+m_3}\right| + \left|\frac{m_2z_2}{m_1+m_2+m_3}\right| + \left|\frac{m_3z_3}{m_1+m_2+m_3}\right|$$

$$\leq \frac{m_1}{m_1+m_2+m_3} + \frac{m_2}{m_1+m_2+m_3} + \frac{m_3}{m_1+m_2+m_3} = 1$$

Physical interpretation: If three particles z_1, z_2, and z_3 lie inside or on the unit circle, then their center of mass also must be inside or on the unit circle.

24. (See Exercise 14)

```
      Input x,y
Step1 Set r = sqrt (x^2 + y^2)
Step2 If x ≤ 0 and y = 0, Set t = pi
Step3 Else Set t = sgn(y) * arccos(x/r)
Step4 Print "Polar coordinates are (r,t) = "; (r,t)
Step5 Stop
```

```
      Input r,t
Step1 Set x = r * cos(t), y = r * sin(t)
Step2 Print "Rectangular coordinates are (x,y) = "; (x,y)
Step3 Stop
```

25.

$\bar{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) = x_1x_2 + y_1y_2 + i(x_1y_2 - y_1x_2)$

$\text{Re}\,(\bar{z}_1 z_2) = x_1x_2 + y_1y_2$

26. $z_1 \bullet z_2 = x_1x_2 + y_1y_2 = 0 \Rightarrow y_2/x_2 = 1/(-y_1/x_1)$ and the vector z_1 is orthogonal to z_2. In other words z_1 leads z_2 $\pi/2$ radians so $z_1 = icz_2$.
If $z_1 = icz_2$ for some real c,
$z_1 \bullet z_2 = \text{Re}(\bar{z}_1 z_2) = \text{Re}(-ic(x_2 - iy_2)(x_2 + iy_2)) = -cx_2y_2 + cx_2y_2 = 0$
and z_1 is orthogonal to z_2.

27. (a) $\text{Im}(\bar{z}_1 z_2) = \text{Im}((x_1 - iy_1)(x_2 + iy_2)) = x_1y_2 - x_2y_2$

(b) If z_1 and z_2 are parallel $z_1 = cz_2 \Rightarrow \text{Im}(z_1z_2) = cx_2y_2 - x_2cy_2 = 0$
If $\text{Im}(z_1z_2) = 0$, $x_1y_2 - x_2y_1 = 0 \Rightarrow x_1/y_1 = x_2/y_2 \Rightarrow z_1 = cz_2$ for some real c.

EXERCISES 1.4: The Complex Exponential

1. a. $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

   b. $e^2 i$

   c. $e^{\cos 1}\cos(\sin 1) + ie^{\cos 1}\sin(\sin 1)$

2. a. $\sin 3$

   b. $e^3\sqrt{3} + e^3 i$

   c. $e^2\cos 2\sqrt{3} + ie^2\sin 2\sqrt{3}$

3. a. $\frac{\sqrt{2}}{3}e^{-i\pi/4}$

   b. $16\pi e^{-i2\pi/3}$

   c. $8e^{i3\pi/2}$

4. a. $e^{i2\pi/3}$

   b. $\frac{2\sqrt{2}e^{i\pi/4}}{2e^{i5\pi/6}} = \sqrt{2}e^{-i7\pi/12}$

   c. $\frac{2e^{i\pi/2}}{3e^4 e^i} = \frac{2}{3e^4}e^{i(\pi/2-1)}$

5. $|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x$

   $\arg(e^{x+iy}) = \arg e^x e^{iy} = \arg e^x + \arg e^{iy} = 0 + y + 2k\pi,\ k = 0, \pm 1, \ldots$

6. a. $\frac{\sin\theta}{\cos\theta} = \frac{(e^{i\theta} - e^{-i\theta})/2i}{(e^{i\theta} + e^{-i\theta})/2} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$

   b. $\frac{1}{\sin\theta} = \frac{2i}{e^{i\theta} - e^{-i\theta}} = \frac{2e^{i\pi/2}}{e^{i\theta} - e^{-i\theta}} = \frac{2}{e^{i(\theta-\pi/2)} - e^{-i(\theta+\pi/2)}}$

7. $$\begin{aligned} e^{z+2\pi i} &= e^{x+i(y+2\pi)} = e^x[\cos(y+2\pi) + i\sin(y+2\pi)] \\ &= e^x(\cos y + i\sin y) = e^{x+iy} = e^z \end{aligned}$$

8. a. $e^{z+\pi i} = e^x[\cos(y+\pi) + i\sin(y+\pi)] = -e^x[\cos y + i\sin y] = -e^z$

   b. $$\begin{aligned} \overline{e^z} = \overline{e^x \operatorname{cis} y} &= e^x(\cos y - i\sin y) \\ &= e^x(\cos(-y) + i\sin(-y)) \\ &= e^{\bar{z}} \end{aligned}$$

9. $$\begin{aligned} (e^z)^n = (e^x \operatorname{cis} y)^n &= e^{nx}(\operatorname{cis} y)^n \\ &= e^{nx}\operatorname{cis} ny \\ &= e^{n(x+iy)} = e^{nz} \end{aligned}$$

   $$(e^z)^{-n} = \frac{1}{(e^z)^n} = \frac{1}{e^{nz}} = e^{-nz}$$

10. $z = x + iy$ with $x < 0$. $|e^z| = e^x \le e^0 = 1$

11. a,c, and d are true. b is false because $e^{z+2\pi i} = e^z$.

12. a. $$\begin{aligned}\sin 3\theta &= \mathrm{Im}(\cos 3\theta + i\sin 3\theta) = \mathrm{Im}(\cos\theta + i\sin\theta)^3 \\ &= \mathrm{Im}[\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(-\sin\theta) - i\sin^3\theta] \\ &= 3\cos^2\theta\sin\theta - \sin^3\theta\end{aligned}$$

b. $$\begin{aligned}\sin 4\theta &= \mathrm{Im}(\cos 4\theta + i\sin 4\theta) = \mathrm{Im}(\cos\theta + i\sin\theta)^4 \\ &= \mathrm{Im}[\cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(-\sin^2\theta) \\ &\quad + 4\cos\theta(-i\sin^3\theta) + \sin^4\theta] \\ &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\end{aligned}$$

13. a. $$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 \\ &= -\frac{1}{4}(e^{i2\theta} - 2 + e^{-i2\theta}) + \frac{1}{4}(e^{i2\theta} + 2 + e^{-i2\theta}) = 1\end{aligned}$$

b. $$\begin{aligned}\cos(\theta_1 + \theta_2) &= \frac{e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)}}{2} \\ &= \frac{e^{i\theta_1}e^{i\theta_2} + e^{-i\theta_1}e^{-i\theta_2}}{2} \\ &= \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)}{2} \\ &\quad + \frac{[\cos(-\theta_1) + i\sin(-\theta_1)][\cos(-\theta_2) + i\sin(-\theta_2)]}{2} \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,\end{aligned}$$

since $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

14. Yes, because if $n > 0$ then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \frac{1}{(\cos\theta + i\sin\theta)^n} \\ &= \frac{1}{\cos n\theta + i\sin n\theta}\end{aligned}$$

$$
\begin{aligned}
&= \cos n\theta - i \sin n\theta \\
&= \cos(-n\theta) + i \sin(-n\theta)
\end{aligned}
$$

15. a. $e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2)$

$$
\begin{aligned}
&= e^{x_1}e^{x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i \cos y_1 \sin y_2 \\
&\quad + i \sin y_1 \cos y_2) \\
&= e^{x_1+x_2}(\cos(y_1+y_2) + i \sin(y_1+y_2)) \\
&= e^{z_1+z_2}
\end{aligned}
$$

b. $\dfrac{e^{z_1}}{e^{z_2}} = \dfrac{e^{x_1}(\cos y_1 + i \sin y_1)}{e^{x_2}(\cos y_2 + i \sin y_2)} \cdot \dfrac{\cos y_2 - i \sin y_2}{\cos y_2 - isiny_2}$

$$
\begin{aligned}
&= e^{x_1}e^{-x_2}(\cos y_1 \cos y_2 + \sin y_1 \sin y_2 + i \sin y_1 \cos y_2 \\
&\quad - i \cos y_1 \sin y_2) \\
&= e^{x_1-x_2}[\cos(y_1-y_2) + i \sin(y_1-y_2)] \\
&= e^{z_1-z_2}
\end{aligned}
$$

16. $\exp(\ln r + i\theta) = e^{\ln r}e^{i\theta} = re^{i\theta} = z$

17. The standard parametrization of the unit circle traversed in the counterclockwise direction is $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$, which gives $z = \cos t + i \sin t = e^{it}$.

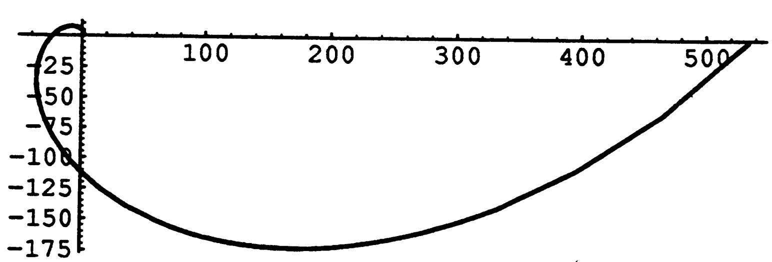
a. The circle $|z| = 3$ traversed counterclockwise.

b. The circle $|z - i| = 2$ traversed counterclockwise.

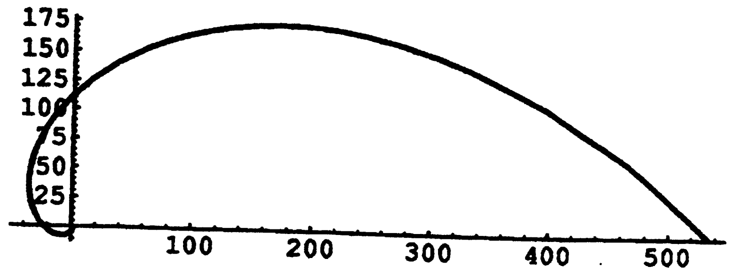
c. The upper half of the circle $|z| = 2$ traversed counterclockwise.

d. The circle $|z - (2 - i)| = 3$ traversed clockwise.

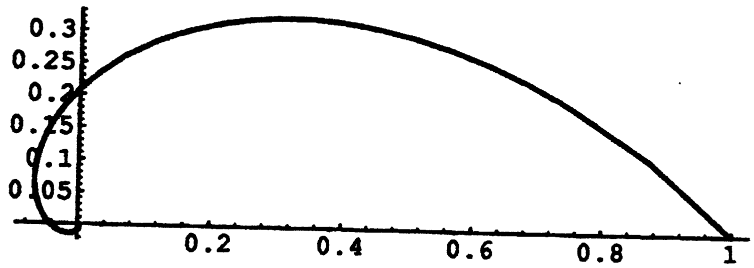
18. a.



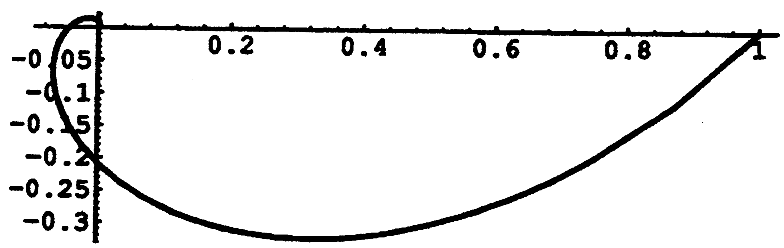
b.



c.



d.



19. $|e^{2k\pi i/n}| = 1$, $k = 0, 1, \ldots, n-1 \Longrightarrow$ The vertices lie on the unit circle.

$|e^{2k\pi i/n} - e^{2(k+1)\pi i/n}| = |1 - e^{2\pi i/n}| \Longrightarrow$ The n side lengths are equal.

20. $(z-1)(1+z+z^2+\cdots+z^n) = z^{n+1} - 1 \quad \Longrightarrow$

$$1+z+z^2+\cdots+z^n = \frac{z^{n+1}-1}{z-1} \quad \text{when} \quad z \neq 1$$

Suppose $z = e^{i\theta}$, $\theta \neq 0$. Then

$$\begin{aligned} 1 + z + z^2 + \cdots + z^n &= 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} \\ &= (1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta) \\ &\quad + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta) \end{aligned}$$

and

$$\begin{aligned} \frac{z^{n+1} - 1}{z - 1} &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{\cos(n+1)\theta - 1 + i\sin(n+1)\theta}{(\cos\theta - 1)^2 + \sin^2\theta}(\cos\theta - 1 - i\sin\theta) \\ &= \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta} \\ &\quad - i\frac{\sin n\theta - \sin(n+1)\theta + \sin\theta}{2 - 2\cos\theta} \\ &= \frac{\sin(n+1/2)\theta + \sin\theta/2}{2\sin\theta/2} + i\frac{\sin(n+1)\theta/2\,\sin(n\theta/2)}{\sin\theta/2} \end{aligned}$$

a) follows by equating the real parts of both equations and b) follows by equating the imaginary parts.

21. $$\left|\frac{1 - z^n}{1 - z}\right| = \left|\frac{1 - (\cos\theta + i\sin\theta)^n}{1 - (\cos\theta + i\sin\theta)}\right| = \left|\frac{(1 - \cos n\theta) + i(\sin n\theta)}{(1 - \cos\theta) + i(\sin\theta)}\right|$$

$$= \sqrt{\frac{(1 - \cos n\theta)^2 + \sin^2 n\theta}{(1 - \cos\theta)^2 + \sin^2\theta}} = \sqrt{\frac{2 - 2\cos n\theta}{2 - 2\cos\theta}}$$

$$= \sqrt{\frac{(1 - \cos n\theta)/2}{(1 - \cos\theta)/2}} = \sqrt{\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)}} = \left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right|$$

On the other hand,

$$\left|\frac{1 - z^n}{1 - z}\right| = |1 + z + z^2 + \cdots + z^{n-1}| \leq 1 + 1 + 1 + \cdots + 1 = n.$$

22.
$$\int_0^{2\pi} e^{in\theta}\,d\theta = \int_0^{2\pi} e^{0}\,d\theta = 2\pi,\ for \cdot n = 0$$
$$\int_0^{2\pi} e^{in\theta}\,d\theta = e^{i2\pi n} - 1 = 0,\ for \cdot n \neq 0$$

23 (a) $\int_0^{2\pi} \cos^8(\theta)d\theta = \int_0^{2\pi} \left((e^{i\theta} + e^{-i\theta})/2\right)^8 d\theta = \left(\frac{1}{256}\right)\int_0^{2\pi} \sum_{m=0}^{8} \left(C_m^8\right) e^{i(8-2m)\theta}\, d\theta$

$= 35\pi/64$

(b) $\int_0^{2\pi} \sin^6(2\theta)d\theta = \int_0^{2\pi} \left(\frac{e^{i2\theta} - e^{-i2\theta}}{2i}\right)^6 d\theta = -20(2\pi)/(i2)^6 = 5\pi/8$

EXERCISES 1.5: Powers and Roots

1. By induction: The case when $n = 1$ is obvious. Assume $z^m = r^m(\cos m\theta + i \sin m\theta)$ for all positive integers $m < n$.

$$\begin{aligned} z^n = z^{n-1}z &= r^{n-1}[(\cos(n-1)\theta + i\sin(n-1)\theta)][r(\cos\theta + i\sin\theta)] \\ &= r^n[\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta \\ &\quad + i\sin(n-1)\theta\cos\theta + i\sin\theta\cos(n-1)\theta] \\ &= r^n(\cos n\theta + i\sin n\theta) \end{aligned}$$

2. Let m be a positive integer. Then

$$\begin{aligned} z^{-m} &= \frac{1}{z^m} \\ &= \frac{1}{r^m(\cos m\theta + i\sin m\theta)} \\ &= \frac{1}{r^m}(\cos m\theta - i\sin m\theta) \\ &= r^{-m}(\cos(-m\theta) + i\sin(-m\theta)) \end{aligned}$$

3. By induction: The case when $n = 1$ is obvious. Assume $\arg(z^m) = m\text{Arg}\, z + 2k\pi$, $k = 0, \pm 1, \ldots$ for all positive integers $m < n$.

$$\begin{aligned} \arg(z^n) &= \arg(z^{n-1}z) = \arg(z^{n-1}) + \arg z \\ &= (n-1)\text{Arg}\, z + \arg z + 2k\pi \\ &= n\text{Arg}\, z + 2k\pi \end{aligned}$$

4. a. $(\sqrt{3} - i)^7 = 2^7\left(\cos -\frac{7\pi}{6} + i\sin -\frac{7\pi}{6}\right) = 2^7\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -64\sqrt{3} + 64i$

b. $(1+i)^{95} = (\sqrt{2})^{95}\left(\cos\frac{95\pi}{4} + i\sin\frac{95\pi}{4}\right)$

$= (\sqrt{2})^{95}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2^{47}(1-i)$

5. a. $(-16)^{1/4} = 2\exp\left(i\dfrac{\pi + 2k\pi}{4}\right)$, $k = 0,1,2,3$

b. $1^{1/5} = \exp\left(i\dfrac{2k\pi}{5}\right)$, $k = 0,1,2,3,4$

c. $i^{1/4} = \exp\left(i\dfrac{\pi/2 + 2k\pi}{4}\right)$, $k = 0,1,2,3$

d. $(1 - \sqrt{3}i)^{1/3} = \sqrt[3]{2}\exp\left(i\dfrac{-\pi/3 + 2k\pi}{3}\right)$, $k = 0,1,2$

e. $(i - 1)^{1/2} = \sqrt[4]{2}\exp\left(i\dfrac{3\pi/4 + 2k\pi}{2}\right)$, $k = 0,1$

f. $\left(\dfrac{2i}{1+i}\right)^{1/6} = (1+i)^{1/6} = \sqrt[12]{2}\exp\left(i\dfrac{\pi/4 + 2k\pi}{6}\right)$, $k = 0,1,2,3,4,5$

6. In each case one can find a root w, then construct the others as vertices of a regular pentagon inscribed in the circle $|z| = |w|$ by marking off arcs of length $|w|\dfrac{2\pi}{5}$.

   a. One root is -1.

   b. One root is $e^{i\pi/10}$.

   c. One root is $2^{1/10}e^{i\pi/20}$.

7. a. $z = -\dfrac{1}{4} \pm \dfrac{\sqrt{23}}{4}i$

   b. $z = 2 - i,\ 1 - i$

   c. $z = 1 \pm \sqrt{1-i} = 1 \pm 2^{1/4}\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)$

8. From the quadratic formula the two solutions

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are distinct and real when $b^2 - 4ac > 0$. When $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac} = i\sqrt{-(b^2 - 4ac)}$ so the solutions are non-real complex conjugates.

9. Note that $z^3 - 3z^2 + 6z - 4 = (z-1)(z^2 - 2z + 4)$, $z = 1, 1 \pm i\sqrt{3}$

10. $z = (-1)^{1/4} = \exp\left(i\dfrac{\pi + 2k\pi}{4}\right)$, $k = 0, 1, 2, 3$

$(z - e^{(\pi/4)i})(z - e^{(7\pi/4)i}) = z^2 - \sqrt{2}z + 1 \ (k = 0, 3)$

$(z - e^{(3\pi/4)i})(z - e^{(5\pi/4)i}) = z^2 + \sqrt{2}z + 1 \ (k = 1, 2)$

11. $\dfrac{(z+1)^5}{z^5} = \left(1 + \dfrac{1}{z}\right)^5 = 1 \Longrightarrow 1 + \dfrac{1}{z} = q^{1/5} = w$, where $w = e^{(2k\pi/5)i}$, $k = 0, 1, 2, 3, 4$. Therefore $z = \dfrac{1}{w-1}$, $k = 1, 2, 3, 4$.

12. $z_0^{1/n} = |z_0|^{1/n} \exp\left(i\dfrac{\theta_0 + 2k\pi}{n}\right)$, $k = 0, 1, \ldots, n-1$, where $\theta_0 = \operatorname{Arg} z_0$. For each k, $z_0^{1/n}$ is the constant distance $|z_0|^{1/n}$ from the origin, and the difference in the arguments of $z_0^{1/n}$ for consecutive k is the constant $\dfrac{2\pi}{n}$. Hence the n points $z_0^{1/n}$ are equally spaced on the circle $|z| = |z_0|^{1/n}$.

13. $\omega_3 = e^{(2\pi/3)i} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$1 + \omega_3 + \omega_3^2 = 1 + \left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) + \left(-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right) = 0$

$\omega_4 = e^{(\pi/2)i} = i$

$1 + \omega_4 + \omega_4^2 + \omega_4^3 = 1 + i + (-1) + (-i) = 0.$

14. $(z^m)^{1/n} = (|z|^m e^{im\theta})^{1/n}, \quad \theta = \arg z$

$$= |z|^{m/n} \exp\, i\left(\frac{m\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

$$= |z|^{m/n} \exp i\left(\frac{m\theta + 2km\pi}{n}\right) \quad \text{since } m \text{ and } n \text{ are relatively prime}$$

$$= |z|^{m/n} \exp\, im\left(\frac{\theta + 2k\pi}{n}\right) \quad (*)$$

$$= \left(|z|^{1/n} \exp\, i\left(\frac{\theta + 2k\pi}{n}\right)\right)^m = \left(z^{1/n}\right)^m$$

Expanding (*) gives

$$z^{m/n} = |z|^{m/n}\left(\cos\frac{m}{n}(\theta + 2k\pi) + i\sin\frac{m}{n}(\theta + 2k\pi)\right), \quad k = 0, 1, \ldots, n -$$

15. $(1-i)^{3/2} = (\sqrt{2})^{3/2} e^{(3i/2)(-\pi/4+2k\pi)}, \quad k = 0, 1$

$$= 2^{3/4} e^{i(-3\pi/8+3k\pi)}, \quad k = 0, 1$$

16. $(z+1)^{100} = (z-1)^{100} \Rightarrow (z+1) = (z-1)e^{2\pi ki/100} \Rightarrow z(1-e^{2\pi ki/100}) = -(1+e^{2\pi ki/100})$
$z = (e^{2\pi ki/100} + 1)/(e^{2\pi ki/100}-1) = (e^{\pi ki/100} + e^{-\pi ki/100})/(e^{\pi ki/100} - e^{\pi ki/100})$
$z = -i\cos(\pi k/100)/\sin(\pi k/100)$ for $k = 0,1,\ldots,99$. Because the cos and sin functions of a real variable are real z will have zero real part.

17. (Use Exercise 20 from Section 1.4)

$$1 + \omega_m^{\ell} + \omega_m^{2\ell} + \cdots + \omega_m^{(m-1)\ell} = \frac{\omega_m^{m\ell} - 1}{\omega_m - 1} = 0$$

18. Let $k = mn$. Then
$(\alpha\beta)^k = (\alpha\beta)^{mn} = \alpha^{mn}\beta^{mn} = (\alpha^n)^m(b^m)^n = 1^m 1^n = 1.$

19. (a) $F(z) = \left(1/|z - z_0|\right)e^{i\arg(z-z_0)} = \left(1/|z - z_0|e^{-i\arg(z-z_0)}\right) = 1/(\bar{z} - \bar{z}_0)$
(b) Solve $z_{o1} = 1+i$, $z_{o2} = -1+i$, $z_{o3} = 0$
$1/(\underline{z} - \underline{z}_{o1}) + 1/(\underline{z} - \underline{z}_{o2}) + 1/\underline{z} = 0 \Rightarrow z = (\pm\sqrt{2} + 2i)/3$

20. . Define subroutines called sum, diff, prod, and quot based on exercise 31, section 1.1. Also define subroutines called polar and rectangular based on exercise 24, section 1.3. Define compsqrt(x, y) as follows:

Input x, y
Set (r, t)=polar(x, y)
Set new$r$=sqrt(r), new$t$=$t/2$
Set (new$x$, new$y$)=rectangular(new$r$,new$t$)
Output (new$x$, new$y$)
Stop

Now the quadratic formula program can be written.

Input ar, ai, br, bi, cr, ci
Set (discrim r, discrim i) = prod(br, bi, br, bi)−4 $*$ prod (ar, ai, cr, ci)
Set (toproot$r$, toproot$i$)=compsqrt(discrim$r$, discrim$i$)
Set ($z1r$, $z1i$)=quot($-br$ + toproot$r$, $-bi$ + toproot$i$, 2 $*$ ar, 2 $*$ ai)
Set ($z2r$, $z2i$)=quot($-br$ − toproot$r$, $-bi$ − toproot$i$, 2 $*$ ar, 2 $*$ ai)
Print "One solution is (x, y) =";($z1r$, $z1i$); "which is (r, t) =";
    polar ($z1r$, $z1i$)
Print "The other solution is (x, y) =";($z2r$, $z2i$); "which is (r, t) =";
    polar($z2r$, $z2i$)
Stop

21. (a) ±(3+i) (b) ±(3+2i) (c) ±(5+i)
(d) ±(2-i) (e) ±(1+3i) (f) ±(3-i)

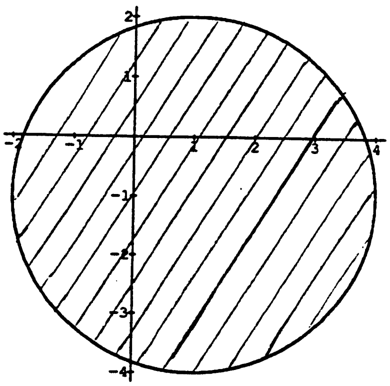
EXERCISES 1.6: Planar Sets

1. Let z_1 be in the neighborhood $|z - z_0| < \rho$ and let $R = \rho - |z_1 - z_0|$. Choose a point ω in $|z - z_1| < R$. Then

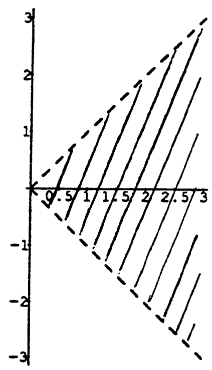
$$\begin{aligned} |z_0 - \omega| &= |z_0 - z_1 + z_1 - \omega| \\ &\le |z_0 - z_1| + |z_1 - \omega| \\ &< |z_0 - z_1| + R = \rho \end{aligned}$$

so z_1 is an interior pont of $|z - z_0| < \rho$ and the neighborhood is an open set.

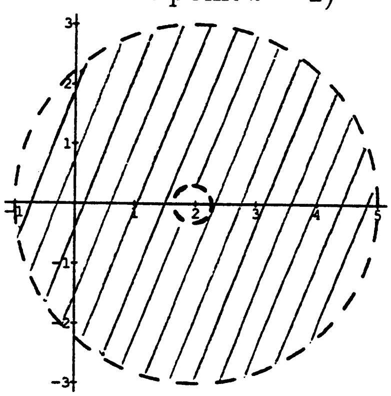
2. a. $|z - (1 - i)| \le 3$



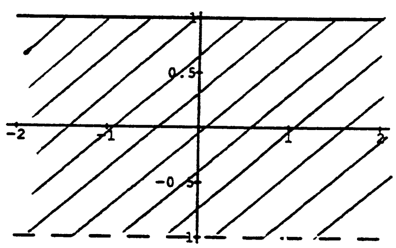
b. $|\mathrm{Arg}\, z| < \pi/4$



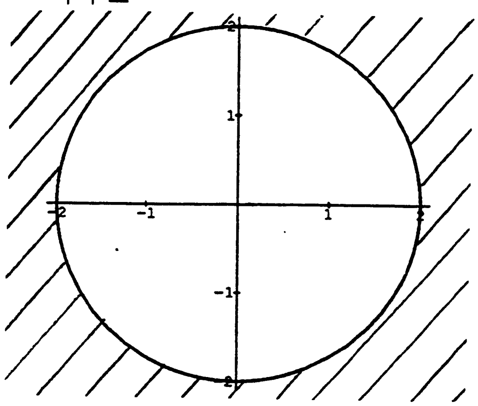
c. $0 < |z - 2| < 3$ (excludes the point $z = 2$)



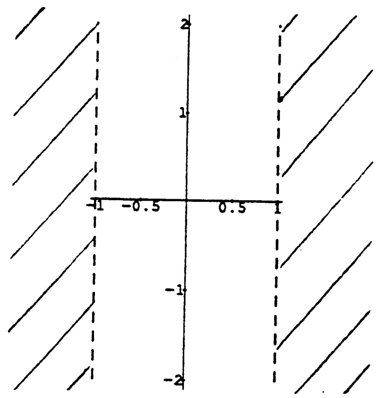
d. $-1 < \mathrm{Im}\, z \le 1$



e. $|z| \ge 2$



f. $(\mathrm{Re}\, z)^2 > 1$



3. b, c, f

4. b, c

5. a, c

6. a. $|z-(1-i)|=3$
   b. $z=re^{i\pi/4}$ and $z=re^{-i\pi/4}$
   c. $z=2$ and $|z-2|=3$
   d. $z=x+i$ and $z=x-i$ for all real x
   e. $|z|=2$
   f. $z=1+iy$ and $z=-1+iy$ for all real y

7. a, b, c, d, e

8. a, e

9. The set $S=\{z_1,z_2,\ldots,z_n\}$ is bounded by the neighborhood $|z|<\rho$, where $\rho>\max|z_j|$, $j=1,2,\ldots,n$.

10. Let $\rho_0=|z_0|$ and choose $R>\rho+\rho_0$. Then for z in $|z-z_0|\le\rho$

$$\begin{aligned}|z|=|z-z_0+z_0| &\le |z-z_0|+|z_0| \\ &\le \rho+\rho_0<R.\end{aligned}$$

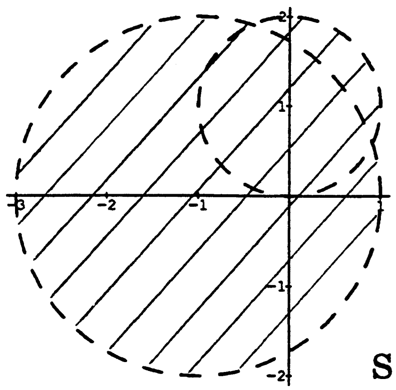
11. $S\cup\{0\}$

12. Since z_0 is not an interior point, every nighborhood of z_0 contains at least one point not in S. At the same time, every neighborhood of z_0 contains z_0, which is in S. Thus z_0 is a boundary point of S.

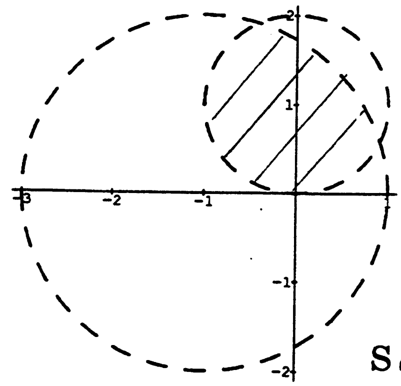
13. S is closed $\Longleftrightarrow$
S contains all of its boundary points. $\Longleftrightarrow$
No point of $\mathbf{C}\setminus S$ is a boundary point. $\Longleftrightarrow$
z_0 in $\mathbf{C}\setminus S$ implies that there exists a disk $|z-z_0|<\epsilon\subseteq\mathbf{C}\setminus S$. $\Longleftrightarrow$
$\mathbf{C}\setminus S$ is open.

14. By contradiction: Suppose z_0 is an accumulation point of S but that z_0 belongs to $\mathbf{C} \setminus S$. Then z_0 is a boundary point of S since each of its neighborhoods contains points in S. Because S is closed, z_0 is in $S \cap \mathbf{C} \setminus S \neq \emptyset$.

15.



S∪T



S∩T

16. Suppose z_0 is in $S \cup T$. If z_0 is in S, then there is a neighborhood $|z - z_0| < \rho$ that is contained in S, thus it is contained in $S \cup T$. Likewise if z_0 is in T there is a neighborhood $|z - z_0| < \rho$, (in T) that is contained in $S \cup T$. Hence z_0 is an interior point of $S \cup T$.

17. No. Counterexample:

$$S: \ 1 < |z| < 3$$
$$T: \ -1 < \operatorname{Im} z < 1$$

$S \cap T$ is not connected.

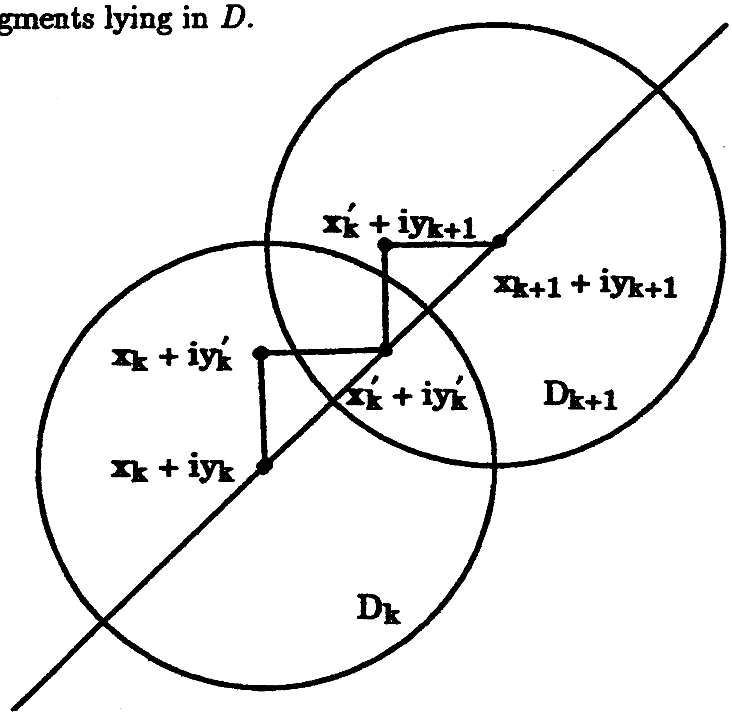
18. $S \cup T$ is open (by Exercise 16). To show that $S \cup T$ is connected, let z_0, z_1, and z be points in S, T, and $S \cap T$, respectively. Then z_0 and z can be joined by a polygonal path in S. Likewise z and z_1 can be joined by a polygonal path in T. Therefore z_0 and z_1 can be joined by a polygonal path in $S \cup T$.

19. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ in $\{z : |z| < 1\}$ because u is constant there and in $\{z : |z| > 2\}$ because u is constant there. Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ in D. Theorem 1 is not contradicted because D is not connected.

20. Let $v(x,y) = u(x,y) - xy$ at all points of D. Then $\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - y = 0$ and $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial y} - x = 0$. By Theorem 1, $v(x,y) = c$, a constant. Thus, $u(x,y) = xy + c$.

21. $\mu(x,y) = \log(x^2 + y^2) + C$ where C is a constant.

22. **Let ℓ be a line segment belonging to a polygonal path connecting two points in D. Let $z_k = x_k + iy_k$ for $k = 1, 2, 3, \ldots, K$ be the centers of open disks D_k in D that cover ℓ. Let $z_k' = x_k' + iy_k'$ be any point in $D_k \cap D_{k+1}$. Then the vertical segment from $x_k + iy_k$ to $x_k + iy_k'$ is in D_k, the horizontal segment from $x_k + iy_k'$ to $x_k' + iy_k'$ is in D_k, the vertical line segment from $x_k' + iy_k'$ to $x_k' + iy_{k+1}$ is in D_{k+1}, and the horizontal line segment from $x_k' + iy_{k+1}$ to $x_{k+1} + iy_{k+1}$ is in D_{k+1}. Thus, the line segment from $x_k + iy_k$ to $x_{k+1} + iy_{k+1}$ can be replaced by these horizontal and vertical segments without leaving $D_k \cup D_{k+1}$ (and without leaving D). In this manner one can replace ℓ by horizontal and vertical line segments lying in D, and one can replace the entire polygonal path connecting the pair of points by horizontal and vertical line segments lying in D.**



23. (a) The set is a continuum.
(b) The set is not a continuum.
(c) The set is not a continuum.
(d) The set is a continuum.

24. a. If $x_0 + iy_0$ and $x_1 + iy_1$ are the endpoints of the line segment then $x = (x_1 - x_0)t + x_0$, $y = (y_1 - y_0)t + y_0$ is such a parametrization.

b. $\dfrac{dU}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = 0 \cdot (x_1 - x_0) + 0 \cdot (y_1 - y_0) = 0.$

c. Any two points z_1, z_2 in D are connected by a polygonal path lying in D. u is constant on each line segment in this path, so u is constant on the path, and $u(x_1, y_1) = u(x_2, y_2)$.

Exercises 1.7

1. (a) $i \Rightarrow (x_1, x_2, x_3) = (0, 1, 0)$
   (b) $6 - 8i \Rightarrow (x_1, x_2, x_3) = (12/101, -16/101, 99/101)$
   (b) $-3/10 + 2i/5 \Rightarrow (x_1, x_2, x_3) = (-12/25, 16/25, -3/5)$
2. (a) $z = x+iy \Rightarrow (x_1, x_2, x_3) =$
   $[2x/(x^2 + y^2 +1), 2y/(x^2 + y^2 + 1), (x^2 + y^2 - 1)/(x^2 + y^2 + 1)]$
   $1/\underline{z}^* = x/(x^2 + y^2) + iy//(x^2 + y^2) \Rightarrow (\underline{x_1}, \underline{x_2}, \underline{x_3}) =$
   $[2x/(x^2 + y^2 +1), 2y/(x^2 + y^2 + 1),(1 - x^2 - y^2)/(x^2 + y^2 + 1)]$
   $(x_{n1}, xn_2, xn_3) = (x_1, x_2, -x_3)$
   (b) $-1/\underline{z} \Rightarrow (xx_1, xx_2, xx_3) = (-x_1, -x_2, -x_3)$
   $\text{dist}(Z,W) = 2|z + 1/\underline{z}|/\sqrt{(1+|z|^2)}\sqrt{(1 + |1/\underline{z}|^2)} = 2$
3. $Z = (x_1, x_2, x_3)$, $W = (w_1, w_2, w_3)$ and $(0, 0, 0)$ define a great circle because the distance from the point Z and 0 is unity. The great circle through Z and 0 must pass through $-1/\underline{z}$ as shown in Problem 2. Example 2 showed that all lines and circles in the z-plane correspond to circles on the Riemann sphere. In Problem 10 below it will be shown that circles on the Riemann sphere correspond to lines or circles in the z-plane. Therefore, the great circle corresponds to a line or circle in the z-plane that goes through points z, $-1/\underline{z}$, w, $-1/\underline{w}$.
4. The points w and $-1/\underline{w}$ correspond to many great circles that goes through W and the center of the Riemann sphere. One of these great circles also passes through the points z and $-1/\underline{z}$.
5. (a) The hemisphere $x_1 > 0$.
   (b) The bowl $x_3 < -3/5$
   (c) The slice $0 < x_3 < 3/5$
   (d) The dome $0.8 < x_3$
   (e) The great circle $x_1 = x_2$, $1 \geq x_3 \geq -1$ or longitude 45° and longitude 225°.
6. The point Z is away from the x_3 axis a distance
   $\{[2x/(1+|z|^2)]^2 + [2y/(1+|z|^2]^2\}^{.5} = 2|z|/(1+|z|^2)$.
   The right triangle formed by $x_3 = 1$ (the point ∞) and Z and back to the x_3 axis is similar to the right triangle formed by $x_3 = 1$ and the points z and 0 in the z-plane. This gives the ratio of sides: $\chi[z,\infty]/\sqrt{(1+|z|^2)} = \{2|z|/(1+|z|^2)\}/|z|$.
   Solving yields $\chi[z,\infty] = 2/\sqrt{(1+|z|^2)}$.
7. See Figure 1.21. $|z-w|$ is related to the triangle $x_3=1$,z,w by
   $|z-w|^2 = 1+|z|^2 + 1+|w|^2 - 2\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}\cos\alpha$.
   $\cos\alpha = [2+|z|^2+|w|^2 - |z-w|^2]/[2\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}]$

* ***In these solutions the complex conjugate of z is indicated by $\underline{z}$.***

   Applying the law of cosines again yields
   $|Z-W|^2 = (2/\sqrt{(1+|z|^2)})^2 + (2/\sqrt{(1+|w|^2)})^2 - 2\{(4)/[2\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}]\}\cos\alpha$

   Using the solution for $\cos\alpha$ in this equation gives
   $|Z-W| = 2|z-w|/\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}$.

8. $\chi[z,w] = 2|z-w|/[\sqrt{(1+|z|^2)}\sqrt{(1+|w|^2)}]$.
   $\chi[1/z,1/w] = 2|1/z-1/w|/[\sqrt{(1+1/|z|^2)}\sqrt{(1+1/|w|^2)}]$
   $= 2(|w-z|/|z||w|)/[\sqrt{(|z|^2+1)}\sqrt{(|w|^2+1)}/|z||w|]$
   $= 2(|w-z|)/[\sqrt{(|z|^2+1)}\sqrt{(|w|^2+1)}] = \chi[z,w]$
   $\chi[-z,-w] = \chi[z,w]$ Because the projection of $-1/\underline{z}$ is on the diameter starting at Z and the projection of $-1/\underline{w}$ is on the diameter starting at W, $\chi[-1/\underline{z},-1/\underline{w}] = \chi[z,w] = \chi[1/\underline{z},1/\underline{w}]$.
9. The chords $\chi[z_1,w]$, $\chi[z_2,w]$ and $\chi[z_1,z_2]$ form a triangle. The triangle inequality (11) holds.
10. A circle on the Riemann sphere satisfies the equations
    $x_1^2 + x_2^2 + x_3^2 = 1$ and $Ax_1 + Bx_2 + Cx_3 + D = 0$.

    $2xA/(1+|z|^2) + 2yB/(1+|z|^2)\ (|z|^2-1)C/(1+|z|^2) + D = 0$
    $2Ax + 2By + (x^2 + y^2 - 1)C + (1+x^2 + y^2)D = 0$
    $(C+D)(x^2 + y^2) + 2Ax + 2By + D-C = 0$
    Let $a=C+D$, $c = 2A$, $d = 2B$ and $e = D-C$ lets you write
    $a(x^2 + y^2) + cx + dy + e = 0$, an equation for a line or circle in the xy plane.