MODULE 2

Dynamic Programming

TEACHING SUGGESTIONS

Teaching Suggestion M2.1: *Overall Use of Dynamic Programming.*

Dynamic programming is a general approach that can be used to solve a number of different problems. The overall approach of breaking a larger problem into smaller stages is an important principle. In addition to being essential for the solution of a dynamic programming problem, this concept is a useful approach for general decision-making problems.

Teaching Suggestion M2.2: *Use of the Shortest-Route Problem.*

Dynamic programming can be a difficult topic for some students to understand. The shortest-route problem was used in this Module to show students how the principles of dynamic programming can be used to solve a familiar problem. Once students understand the use of dynamic programming to solve the shortest-route problem, more complex and difficult problems can be undertaken.

Teaching Suggestion M2.3: *QA in Action Box in This Module.*

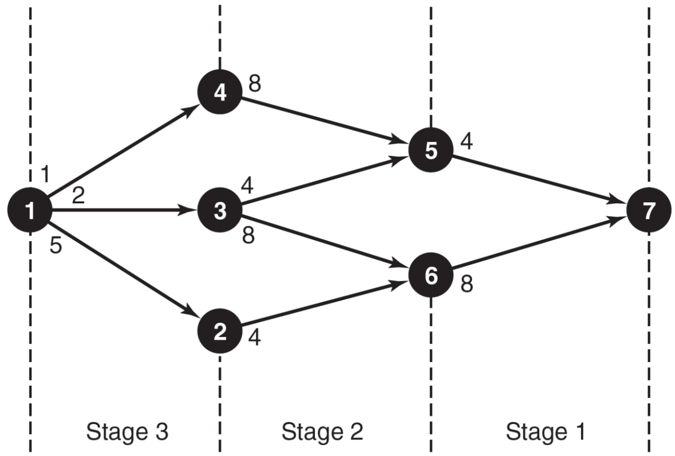
Because dynamic programming is a difficult and advanced topic, we selected an application that might interest the average student.

Teaching Suggestion M2.4: *Use of Terminology.*

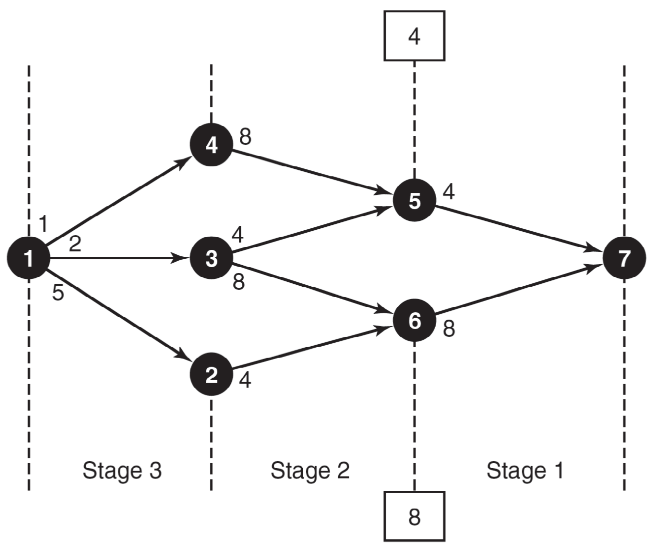
Understanding dynamic programming terminology is one approach to handling larger and more complex problems. Learning how the terminology of dynamic programming is applied to the shortest-route problem can help students understand larger and more complex dynamic programming problems.

ALTERNATIVE EXAMPLE

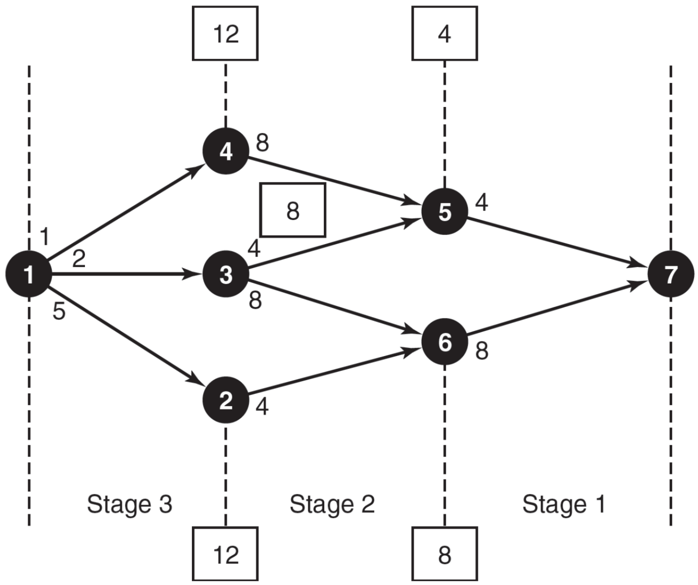
Alternative Example M2.1: Darrell Washington would like to use dynamic programming to solve the shortest-route problem shown in the following figure.



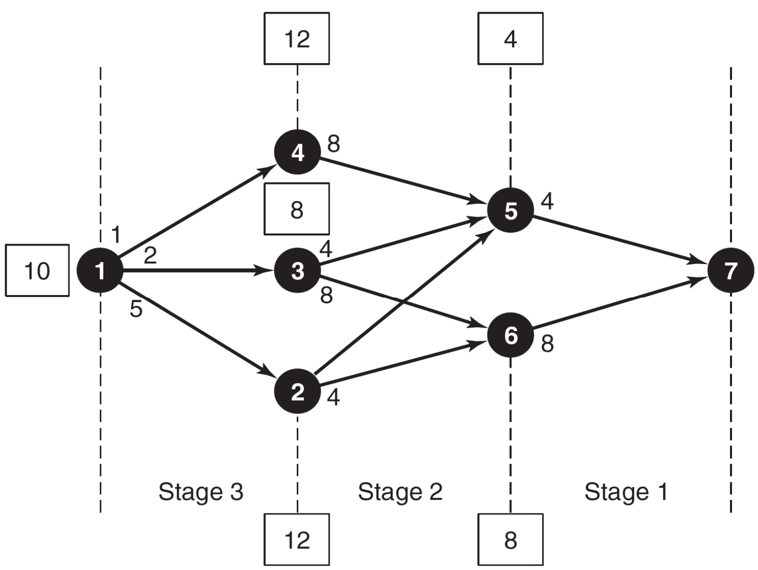
Beginning with stage 1, we begin to solve the problem. The distance from node 5 to node 7 is 4 and the distance from node 6 is 8. These values are put in boxes by the nodes. The results are shown in the following network.



Next, we solve stage 2. The minimum distances between nodes 2, 3, and 4 and the ending node 7 are 12, 8, and 12. These distances are also put in boxes by the nodes. The results for stage 2 are shown in the following network.



Finally, we solve stage 3. The minimum distance is through node 3. The distance from node 1 to node 3 is 2, and the minimum distance from node 3 to the end of the network is 8 as seen in the results for stage 2. Thus the shortest route through the network is 10.



SOLUTIONS TO QUESTIONS AND PROBLEMS

M2-1. A stage in dynamic programming is a period or a logical subproblem. Dynamic programming divides problems into a number of decision stages, whereby the outcome of a decision at one stage affects the decision at each of the next stages.

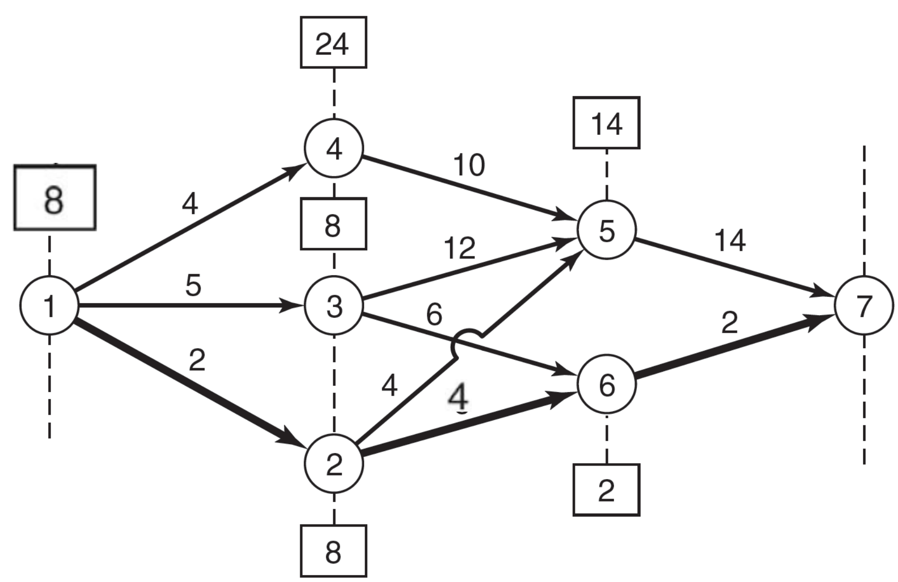
M2-2. State variables include all of the possible beginning situations or conditions of a stage. These have also been called the input variables. Decision variables, on the other hand, represent the alternatives or possible decisions that exist at each stage. Thus, state variables are possible existing situations or conditions at the beginning, while decision variables include the alternatives and possible actions or decisions that can exist at each stage.

M2-3. A decision criterion is a statement concerning the objective of the problem. In the shortest-route example in the module, the decision criterion was to minimize the total distance between two points or nodes.

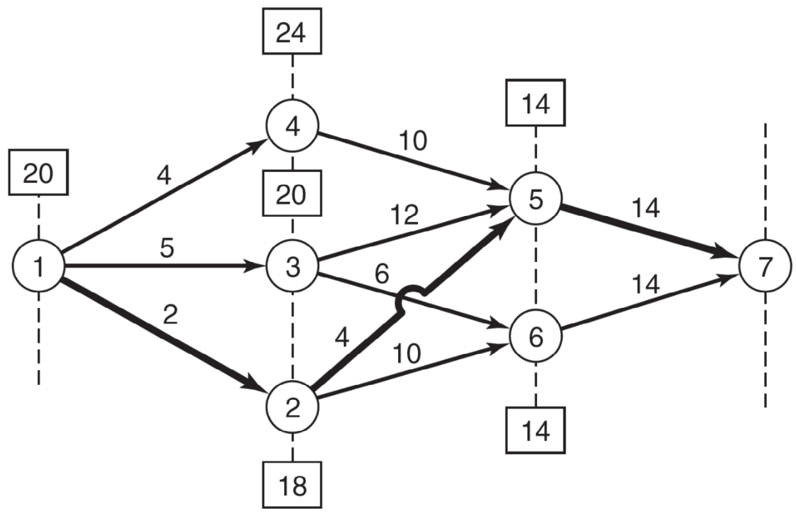
M2-4. The optimal policy is a set of decision rules, developed as a result of the decision criteria. The optimal policy is necessary for all dynamic programming problems to give those problems optimal decisions for any entering condition at any stage.

M2-5. A transformation is important for dynamic programming problems because it allows us to determine the relationship between stages. This permits us to go from one stage to the next in solving dynamic programming problems. In the shortest-route problem, the following transformation was used: the distance from the beginning of a given stage to the last node is equal to the distance from the beginning of the previous stage to the last node plus the distance from the given stage to the previous stage. This relationship is how we were able to go from one stage to the next in solving for the optimal solution to the shortest-route problem.

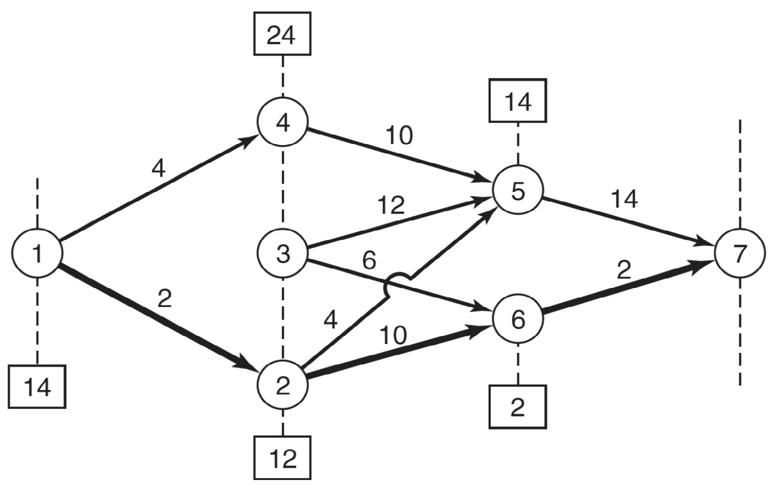
M2-6. The shortest route is 1–2–6–7 with a total distance of 8 miles. See the network below.



M2-7. The shortest route is 1–2–5–7 with a total distance of 20 miles. See the network below.



M2-8. The shortest route is 1–2–6–7 with a total distance of 14 miles. See the network below.

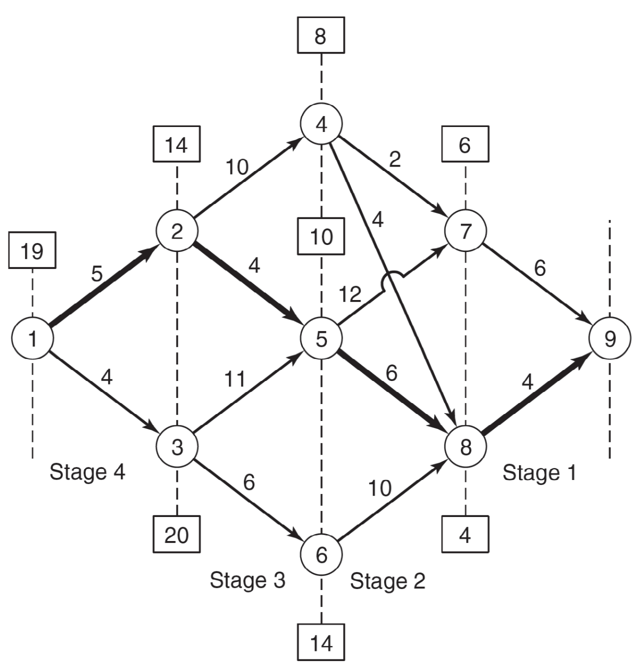


M2-9. The distances are summarized in Table M2.1. The stages are the same stages that were used to minimize the distance.

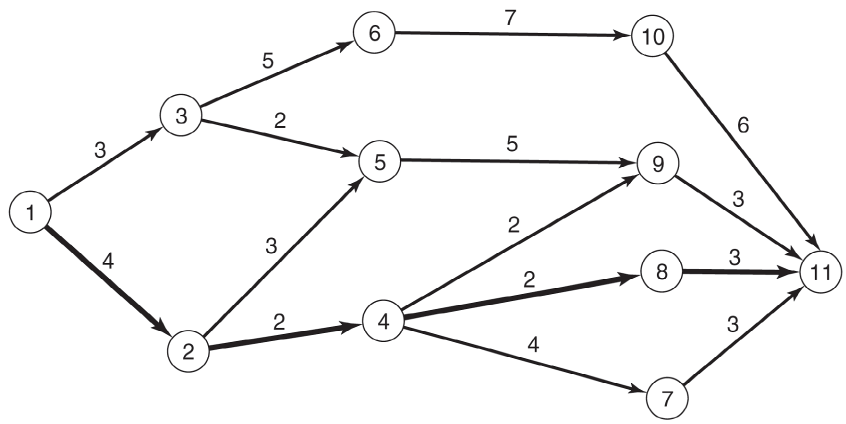
Stage 1		
Beginning node	**Longest distance to node 7**	**Arcs along this path**
5	14	5–7
6	2	6–7
Stage 2		
Beginning node	**Longest distance to node 7**	**Arcs along this path**
4	24	4–5
		5–7
3	26	3–5
		5–7
2	18	2–5
		5–7
Stage 3		
Beginning node	**Longest distance to node 7**	**Arcs along this path**
1	31	1–3
		3–5
		5–7

The longest distance to node 7 is 31. The shortest distance that was found in the module was 13. Thus, using the shortest-route method can potentially save 31 – 13 = 18 miles.

M2-10. The shortest route is 1–2–5–8–9 with a total distance of 19 miles. See the network below.

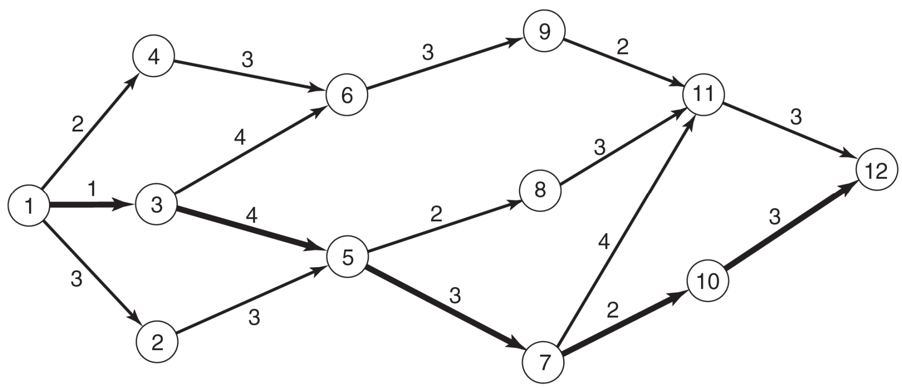


M2-11. The solution, route 1-2-4-8-11 with a total distance of 11, for this shortest-route problem can be seen in the following network:



M2-12. The optimal decision is to ship 4 units of item 1, 1 unit of item 2, and no units of items 3 and 4.

M2-13. Given the data presented in this problem, the shortest route for Leslie is the following: 1, 3, 5, 7, 10, and 12.



Other optimal solutions for Problem M2-13 are:

a) 1, 4, 6, 9, 11, 12

b) 1, 3, 6, 9, 11, 12

c) 1, 3, 5, 8, 11, 12

M2-14. Given the data presented in this problem, the following number of units should be shipped for each item:

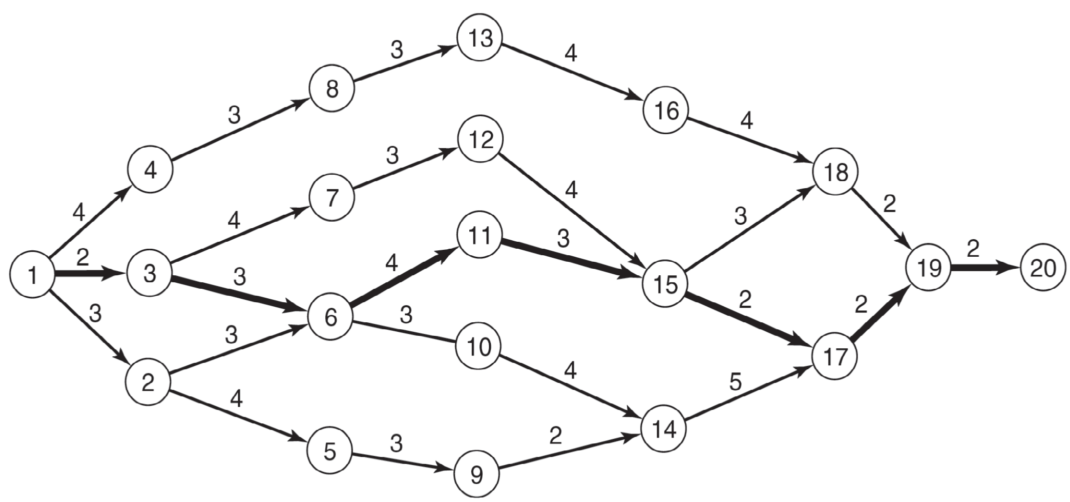
Item	Items to Ship	Optimal Return
1	6	$18
2	1	9
3	1	8
4	0	0
5	0	0
6	0	0
Total	8	$35

M2-15. With these changes, the new shipping pattern is:

Item	Items to Ship	Optimal Return
1	6	$18
2	1	9
3	1	8
4	0	0
5	0	0
6	1	2
Total	9	$37

As you can see, the shipping pattern is slightly different.

M2-16. The shortest route for this problem is 1, 3, 6, 11, 15, 17, 19, and 20 for a total distance of 18. The optimal solution is shown in the following network.



M2-17. The optimal solution is now 1, 3, 7, 12, 15, 17, 19, and 20 for a total distance of 19.

M2-18. The shortest route is 6, 11, 15, 17, 19, and 20. The total distance is 13. If the road from node 6 to node 11 is not available, the shortest route is 6, 10, 14, 17, 19, and 20. The total distance is 16.

M2-19. The optimal solution is to carry 1 unit of item A and 3 units of item C. The total nutritional value is 5,100. The total weight is 18 pounds.

SOLUTION TO UNITED TRUCKING CASE

1. The optimal shipping pattern is shown in the following table.

Item	Items to Ship	Optimal Return
1	2	$20
2	1	10
3	0	0
4	1	7
5	0	0
6	1	11
7	0	0
8	1	50
9	2	20
10	0	0
Total	8	$118

2. Increasing the total capacity to 20 tons has a dramatic impact on the optimal decision, as can be seen in the following table. This does include 11 items instead of 10, and assumes that the maximum number of items increased when the weight increased.

Item	Items to Ship	Optimal Return
1	2	$20
2	1	10
3	0	0
4	1	7
5	2	50
6	1	11
7	1	30
8	1	50
9	2	20
10	0	0
Total	11	$198

SOLUTION TO INTERNET CASE

Briarcliff Electronics

The apportionment of the $100,000 among the various models can be accomplished by means of dynamic programming. This can be viewed as a five stage process—at stage 1, an amount x_1 is invested in "Standard", at stage 2, an amount x_2 is invested in the "Micro" model, and so on through stage 5 where an amount x_5 is invested in the "Network" model. Even though in actuality the allocation is not made in stages, treating the situation as if it were multi-stage allows the use of the dynamic programming approach.

Define $f_n(x)$ as the maximum increased profit that can be realized over stages n through 5 given that the amount not yet invested at stage n is x. (Note that some texts number the stages backwards so that stage n would correspond to n allocations still to be made. Since the stages are an artificiality used in this problem, it makes no difference which sequence is used). Define $g_n(y)$ as the increased profit at stage n if y is invested at that stage. Then, if at the start of stage n the amount not yet invested is x,

$$f_n(y|x) = g_n(y) + f_{n+1}(x - y)$$

is the increased profit that would be realized over stages n through 5 if y is invested at stage n and the remaining x – y is invested optimally over stages n + 1 through 5. Then

$$f_n(x) = \max f_n(y|x) = \max [g_n(y) + f_{n+1}(x - y)]$$

is the recursive relationship that allows one to start at stage 5 and successively determine the optimum allocation for each stage—note that the maximization in this expression is taken over all $y \le x$.

For stage 5, the calculations are shown in Table 1. If there is still x_5 dollars not yet invested at this last stage, the optimum is clearly to invest as much as possible at that stage; either all of it or $50,000, whichever is less. Note that all profit figures in this and subsequent Tables represent thousands of dollars. The indicated optimum f values are the profit increases due to investing the indicated x_5 in the "Network" model.

Table 1 Stage 5 Calculations

x_5	y	$f_5(x_5)$
0	0	0
10	10	27
20	20	64
30	30	101
40	40	199
50	50	248
>50	50	248

Proceeding to stage 4, Table 2 summarizes the analysis. To illustrate, suppose $x_4 = 20$, there is $20,000 remaining after allocations at stages 1, 2, 3. If none is invested at stage 4 ($x_4 = 0$) the increased profit at stage 4 is zero and the surviving $20,000 carries over to stage 5 where it generates $64,000. On the other hand, if $10,000 is invested at stage 4, it will increase profit by $37,000 and the remaining $10,000 will increase the stage 5 profit by $27,000—a total of

$64,000 again. If all $20,000 is invested at stage 4, the increased profit is $59,000 at that stage with zero to invest at stage 5 yielding no profit at that stage. Since no other investment would be possible at stage 4, it is clear that if there is $20,000 available, either 0 or $10,000 should be invested on the "Extended" model since y = 0 or y = 10 yields the maximum $f_4(y|20)$. The entries in the other rows are obtained in a similar manner.

Table 2 Stage 4 Calculations

x_4|y	0	10	20	30	40	50	$f_4(y\|x_4)$	Optimal y
0	0	—	—	—	—	—	0	0
10	27	37	—	—	—	—	37	10
20	64	64	59	—	—	—	64	0 or 10
30	101	101	86	96	—	—	101	0 or 10
40	199	138	123	123	156	—	199	0
50	248	236	160	160	183	287	287	50
60	248	285	258	197	220	314	314	50
70	248	285	307	295	257	351	351	50
80	248	285	307	344	355	388	388	50
90	248	285	307	344	404	486	486	50
100	248	285	307	344	404	535	535	50

Table 3, on the previous page, shows the equivalent stage 3 calculations for the "Major" model and Table 4 shows the stage 2 calculations for the "Micro" model. At stage 1, all $100,000 is available. The calculations for the "Standard" model are shown in Table 5. This reveals that zero should be invested in the Standard model leaving $100,000 for stage 2. Table 4 shows that $10,000 should be invested in the Micro model leaving $90,000 for stage 3. Table 3 shows that zero should be invested in the Major model leaving $90,000 for stage 4. Table 2 shows that $50,000 should be invested in the Extended model leaving $40,000 for stage 5 investment in the Network model. The $10,000 investment in the Micro returns $71,000, the $50,000 investment in the Extended returns $287,000, and the $40,000 investment in the Network returns $199,000, a total return of $577,000.

Table 3 Stage 3 Calculations

x_3|y	0	10	20	30	40	50	$f_3(y\|x_3)$	Optimal y
0	0	—	—	—	—	—	0	0
10	37	60	—	—	—	—	60	10
20	64	97	119	—	—	—	119	20
30	101	124	156	151	—	—	156	20
40	199	161	183	188	183	—	199	0
50	287	259	220	215	220	243	287	0
60	314	347	318	252	247	280	347	10
70	351	374	406	350	284	307	406	20
80	388	411	433	438	382	344	438	30
90	486	448	470	465	470	442	486	0
100	535	546	507	502	497	530	546	10

Table 4 Stage 2 Calculations

x_2|y	0	10	20	30	40	50	$f_2(y\|x_2)$	Optimal y
0	0	—	—	—	—	—	0	0
10	60	71	—	—	—	—	71	10
20	119	131	92	—	—	—	131	10
30	156	190	152	112	—	—	190	10
40	199	227	211	172	134	—	227	10
50	287	270	248	231	194	188	287	0
60	347	358	291	268	253	248	358	10
70	406	418	379	311	290	307	418	10
80	438	477	439	399	333	344	477	10
90	486	509	498	459	421	387	509	10
100	546	557	530	518	481	475	557	10

Table 5 Stage 1 Calculations

x_1|y	0	10	20	30	40	50	$f_1(y\|x_1)$	Optimal y
100	557	525	533	498	552	530	557	0