**CHAPTER 3:**

**Preferences and Utility**

These problems provide some practice in examining utility functions by looking at indifference curve maps and at a few functional forms. The primary focus is on illustrating the notion of quasi-concavity (a diminishing MRS) in various contexts. The concepts of the budget constraint and utility maximization are not used until the next chapter.

**Comments on Problems**

**3.1** This problem requires students to graph indifference curves for a variety of functions, some of which are not quasi-concave.

**3.2** This problem introduces the formal definition of quasi-concavity (from Chapter 2) to be applied to the functions studied graphically in Problem 3.1.

**3.3** This problem shows that diminishing marginal utility is not required to obtain a diminishing *MRS*. All of the functions are monotonic transformations of one another, so this problem illustrates that diminishing *MRS* is preserved by monotonic transformations but diminishing marginal utility is not.

**3.4** This problem focuses on whether some simple utility functions exhibit convex indifference curves.

**3.5** This problem is an exploration of the fixed-proportions utility function. The problem also shows how the goods in such problems can be treated as a composite commodity.

**3.6** This problem asks students to use their imaginations to explain how advertising slogans might be captured in the form of a utility function.

**3.7** This problem shows how utility functions can be inferred from *MRS* segments. It is a very simple example of “integrability.”

**3.8**This problem offers some practice in deriving utility functions from indifference curve specifications.

**Analytical Problems**

**3.9** **Initial endowments.** This problem shows how initial endowments can be treated in simple indifference curve analysis.

**3.10** **Cobb–Douglas utility.** This problem provides some exercises with the Cobb–Douglas function, including how to integrate subsistence levels of consumption into the functional form.

**3.11** **Independent marginal utilities.** This problem shows how analysis can be simplified if the cross-partials of the utility function are zero.

**3.12 CES utility.** This problem shows how distributional weights can be incorporated into the CES form introduced in the chapter without changing the basic conclusions about the function.

**3.13** **The quasi-linear function.** This problem provides a brief introduction to the quasi-linear form, which (in later chapters) will be used to illustrate a number of interesting outcomes.

**3.14 Preference relations.** This problem provides a very brief introduction to how preferences can be treated formally with set-theoretic concepts.

**3.15 The benefit function.** This problem introduces Luenberger’s notion of reducing preferences to a cardinal number of replications of a basic bundle of goods.

**Solutions**

**3.1** Here we calculate the *MRS* for each of these functions:

a.  *MRS* is constant.

b.  Convex; *MRS* is diminishing.

c.  *MRS* is diminishing.

d.  *MRS* is increasing.

e.  Convex; *MRS* is diminishing.

**3.2** Because all of the first-order partials are positive, we must only check the second-order partials.

a.  Not strictly quasi-concave.

b.  Strictly quasi-concave.

c.  Strictly quasi-concave.

d. Even if we only consider cases where both of the second-order partials are ambiguous, and therefore the function is not necessarily strictly quasi-concave.

e.  Strictly quasi-concave.

**3.3** a. 

b. 

c.  This shows that monotonic transformations may affect diminishing marginal utility, but not the 

**3.4** a. In the range in which the same good is limiting, the indifference curve is linear. To see this, take the case in which both  and  Then , implying



as well.

In the range in which the limiting goods differ, we can show the indifference curve is strictly convex. Take the case  and  Then  and  implying



Hence, the indifference curve is convex.

b. Again, in the range in which the same good is maximum, the indifference curve can be shown to be linear. Consider a range in which different goods are maximum, specifically,  and  Then  and  implying

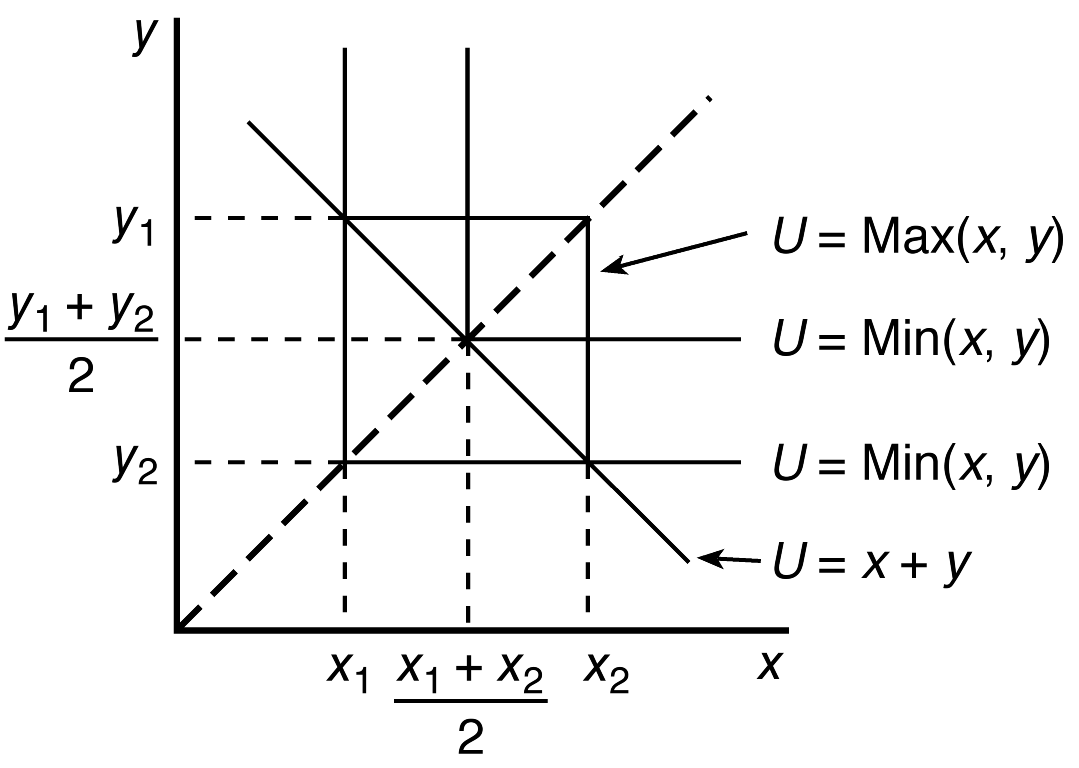


Hence, the indifference curve is concave.

c. Here,



Hence, the indifference curve is linear.



**3.5** a. All four are perfect compliments, 

b. A fully condimented hot dog.

c. $1.60.

d. $2.10, an increase of 31 percent.

e. Price would increase only to $1.725, an increase of 7.8 percent.

f. Raise prices so that a fully condimented hot dog rises in price to $2.60. This could be accomplished by raising all prices by 62.5 percent. But, because of the fixed proportions nature of the utility function, it could also be accomplished by any combination of increases in single good prices. For example, raising the price of the hot dog to $2 would accomplish this goal. Or, one could increase the hot dog price to $1.50 and the bun price to $1.50 and accomplish the same thing. Because of the fixed proportions utility function, all such increases would be equivalent to a lump-sum reduction in purchasing power of about 62 percent.

**3.6** For all the suggested utility functions, let *x* represent some other good and the good in question is represented by the appropriate letter:

a. 

b. Given .

c. Given 

d. .

e. 

**3.7** a.  at both points. Since both the points lie on the same indifference curve (as the utility at both points is the same), the slope of the indifference curve is constant (i.e., straight line). So the goods are perfect substitutes.

b. We know that for a Cobb–Douglas utility function,



Using this formula, yields:





Now use the fact that the two points yield equal utility:

 .

The utility function is of the form 

c. Yes, there was a redundancy. We never used the information about the second *MRS.* In fact, given that the function is assumed to be Cobb–Douglas, only the information about the first *MRS* was needed to get the ratio of the exponents. Since utility is invariant up to a monotonic transformation, any Cobb–Douglas for which  would yield the same behavior. For example, if the exponents sum to one, we have  and this function also satisfies the conditions of the problem.

**3.8** a. Exponentiate the function: 

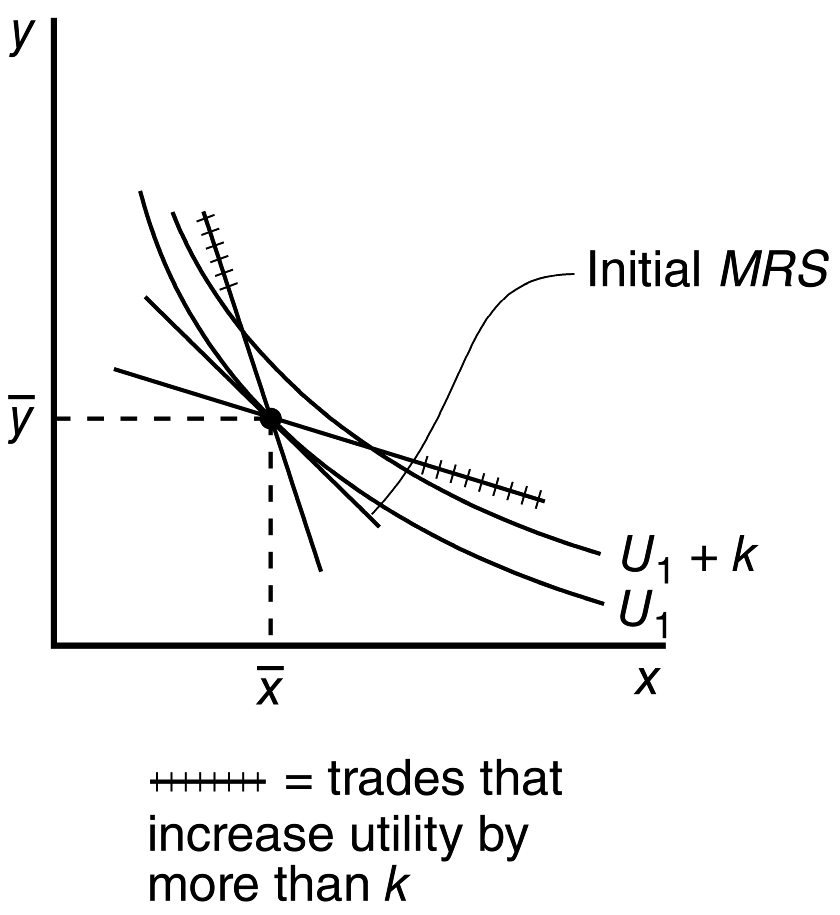
b. Move the term in *x* to the LHS: 

c. Multiply by 2*x*, move  to the LHS, square, and simplify: 

**Analytical Problems:**

**3.9** **Initial endowments**

a.



b. Any trading opportunities that differ from the *MRS* at  will provide the opportunity to raise utility (see figure).

c. A preference for the initial endowment will require that trading opportunities raise utility substantially. This will be more likely if the trading opportunities are significantly different from the initial *MRS* (see figure).

**3.10** **Cobb–Douglas utility**

a. 

This result does not depend on the sum  which, contrary to production theory, has no significance in consumer theory because utility is unique only up to a monotonic transformation.

b. The mathematics follow directly from part (a). If , the individual values  relatively more highly. Hence,  for 

c. The function is homothetic in  and , but not in  and 

**3.11** **Independent marginal utilities**

From Problem 3.2,  implies diminishing *MRS* providing  Conversely, the Cobb–Douglas not only has  and , but also has a diminishing *MRS* (see Problem 3.10a).

**3.12** **CES utility with weights**

a.  so this function is homothetic.

b. If   a constant. If 



This agrees with Problem 3.10.

c.  This is negative if and only if 

d. Follows from part (a). If  **

e. With ****

****

With 

****

Hence, the *MRS* changes more dramatically when  than when ****. The indifference curves are more sharply curved when  is lower. When , the indifference curves are L-shaped, implying fixed proportions.

**3.13** **The quasi-linear function**

a.  The *MRS* depends only on the amount of *y*. It is independent of *x.*

b. Check  We have











So,



c. , where *k* is the utility level.

d. Since the marginal utility of  is a constant at 1 while that of  is decreasing as  increases (as it is of the form ), we would expect consumers to shift more toward  when they buy more of both goods. We explore this in much more detail in the next chapter.

e. Refer to Example 3.4. This function is usually used to describe the consumption of one commodity with respect to all other commodities. So,  could represent the commodity of interest, while  could represent all the other goods consumed.

**3.14 Preference relations**

All of the suggested preference relations are complete, transitive, and continuous.

a. **Summation:**

Complete: Clearly all bundles are ranked by the sum of items contained.

Transitive: If bundle A has more items than B and B has more items than C, clearly A has more items than C.

Continuous: If bundle A contains more items than bundle B, then A is preferred to B and any bundle with slightly more items than B (but fewer than A) is also preferred to B.

b. **Lexicographic:**

Complete: All bundles can be ranked in this ordered way.

Transitive: If bundle A is preferred to bundle B with ties being broken at the *i*th good and B is preferred to C with ties broken at the (*i* + *j*)th good, then A will be preferred to C because it will break the tie at the *i*th good also.

Continuous: Suppose bundle A is preferred to B with the tie break occurring at the *i*th good. Then there exists a bundle C with slightly more of this good than B but less than A, which will be preferred to B. Note, however, that the idea of “closeness” here is being defined with respect to the first tie-break good only. The ranking is not continuous when more general notions of “closeness” are used.

c. **Bliss**

Complete: Clearly all bundles are ranked by the distance metric.

Transitive: The distance metric itself imposes a cardinal ranking, which is clearly transitive.

Continuous: If bundle A is any positive distance from bliss, there will exist another bundle slightly closer since any single good that is not at bliss can be made closer to it.

**3.15 The benefit function**

a. 

b. In this case, the benefit function cannot be computed because the Cobb–Douglas requires positive quantities of both goods to take a nonzero value.

c. In the graph below, the benefit associated with any initial endowment is the length of the vector from the initial endowment to the utility target where the direction of the vector is given by the composition of the elementary bundle.

d. In the graph below, two initial endowments are shown . The benefit for each endowment is also shown by the vectors in the graph. The benefit is also shown for an initial endowment given by . By completing the parallelogram, it is clear that the convexity of the indifference curve implies that



Hence the benefit function is concave in the initial endowments.

x

y

y0

x0

*E2*

*U\**



*E1*